Comptes rendus mathématiques
Mathematical Reports

DÉCEMBRE / DECEMBER 1999

21
No 4

IN THIS ISSUE / DANS CE NUMÉRO

C. R. Math. Rep. Acad. Sci. Canada Vol. **21** (4), 1999 pp. 97–126

FROM DEDEKIND'S GROUP DETERMINANT TO THE ISOMORPHISM PROBLEM

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1. Introduction. We shall try to give an account of some recent developments concerning the question asked from the very beginning of representation theory:

> Which properties of the group G can be derived from its representation theory?[1]

In Part I we report on Dedekind's and Frobenius' early work on the group determinant, culminating in Formanek's result that the group determinant determines the finite group up to isomorphism and Hoehnke's observation that the 3-characters already determine a finite group.

Part II reports on recent progress both positive and negative on the isomorphism problem, the Zassenhaus conjecture by Scott and the author[2] and on the automorphism problem. A main part here consists in indicating how Hertweck achieved his construction of the counterexample of the isomorphism problem for finite groups via a construction of groups by Mazur and a construction of an automorphism of a finite group which becomes inner in the semi-local group ring, thus giving a counterexample for the isomorphism problem for finite by infinite cyclic groups (A. Zimmermann and the author).

In Part III we report on the Čech cohomological obstruction theory for the isomorphism problem and the Zassenhaus conjecture (Scott, Kimmerle and the author). This theory is crucial for Hertweck to show that his two groups, which have isomorphic integral group rings, are not isomorphic. The automorphism problem gives some kind of theoretical background for Hertweck's construction (Marciniak and the author).

Finally in Part IV we will discuss some problems on Picard groups of module categories, derived categories and stable categories which are in close connection with the isomorphism problem and the Zassenhaus conjecture and which present a challenge to mathematicians working in this area.

[1] We omit, however, the results on modular representation theory. For this question we refer to [Bl-Ki-Ro-Wu; 97].

[2] Here I have omitted the numerous results on special classes of groups.

This research was partially supported by the Deutsche Forschungsgemeinschaft. Special thanks go to Ms. Stefanie Ide, who helped me prepare this paper.

AMS subject classification: 16G30.

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PART I: DEDEKIND'S GROUP DETERMINANT AND FROBENIUS' HIGHER CHARACTERS

2. Some historical remarks.

2.1. *Dedekind's contributions to representation theory prior to Frobenius.* I have learned about these historical connections—my interest in the group determinant and higher characters arose from a talk H. J. Hoehnke gave in Stuttgart in 1991—from the papers of Thomas Hawkins [Ha; 74], [Ha; 78] and K. W. Johnson [Jo; 91] and from Frobenius' papers [Fr; 1896].

It has been a long way from the definition of characters by Frobenius via the notion of the trace of a complex matrix representation and the question of which properties of a finite group G are reflected by its representations to the study of its integral group ring.

Next we shall briefly discuss the origins of representation theory.

Carl Friedrich Gauß (1777–1855) had introduced the word character in connection with groups of equivalence classes of binary integral quadratic forms to describe characteristic properties of these classes, properties which did lie in the nature of the integers represented by these forms. At the same time and independent from Gauß, Lejeune P. G. Dirichlet (1805–1859) had studied homomorphisms from abelian groups which took values ± 1. When Richard Dedekind (1831–1916), who had been working on ideal class groups—extending the research of his academic teachers Gauß and Dirichlet—edited the lectures of Dirichlet on number theory in 1879 he developed a general theory of characters of abelian groups, *i.e.*, homomorphisms from a finite abelian group into the multiplicative group of $\mathbb{C}$—thus merging Gauß' and Dirichlet's aspects into a solid theory.

It should be stressed that *eo ipso* these characters had nothing to do with traces of matrices.

Later in 1886 Dedekind studied Galois groups. For example, he asked the following question: when are all intermediate fields in the normal extension L of K normal—*i.e.*, when are all subgroups of a finite group—the Galois group—normal? As an answer he described the Hamiltonian groups.

The discriminant $\Delta(\omega_i)$ of a K-basis $\{\omega_i\}_{1\leq i\leq n}$ of the Galois extension L over K with group $G = \{g_i : 1 \leq i \leq n\}$ is an important arithmetical invariant:[3]

$$\Delta(\omega_i) = D^2(\omega_i) \quad \text{with} \quad D = \det(\omega_i g_j).$$

Dedekind studied the case of a normal basis:[4] Let $G := \mathrm{Gal}(L/K)$ and assume that there exists $\omega_0 \in G$ such that $\{\omega_0 g_i\}_{g_i\in G}$ is a K-basis for L. Then the discriminant arises from the determinant

$$D := \det\big(\omega_0(g_i \cdot g_j)\big).$$

[3] For the moment we write the action on the right.

[4] The existence of a normal basis was proved much later by Emmy Noether (1882–1935) around 1920.

Generically Dedekind thus investigated the determinant

$$D := \det(X_{g_i \cdot g_j})$$

with indeterminates $\{X_{g_i}\}_{g_i \in G}$. For the purpose of normalization, Dedekind studied the group determinant

$$\mathcal{D}_G := \det(X_{g_i \cdot g_j^{-1}}). \tag{1}$$

In the same year 1886 Dedekind derived an important formula, linking the group determinant to the theory of characters which he himself had recently developed:

THEOREM 2.1. *Let A be an abelian group and let*

$$\chi_s : A \longrightarrow \mathbb{C}^\times : 1 \leq s \leq |A|$$

be the different homomorphisms from A to the complex numbers. Then the group determinant can be described as:

$$\mathcal{D}_A = \prod_{1 \leq s \leq |A|} \Big(\sum_{a \in A} \chi_s(a) \cdot X_a \Big).$$

The values of the characters of the abelian group A and only these occur as coefficients of the linear factors; they occur with multiplicity one in the group determinant. Thus the decomposition of the group determinant into linear factors can be used to define characters—*i.e.*, homomorphisms to $\mathbb{C}$—of abelian groups.

Since the fact that the group determinant decomposes into linear factors characterizes abelian groups, an important aspect of "representation theory" is already apparent here:

PROBLEM 2.2. Which properties of the finite group G can be detected from its "representation theory"?

Dedekind further on made several calculations for non-abelian groups: for example he computed the group determinant for the symmetric group S_3 on three letters:

$$\mathcal{D}_{S_3} = (X_1 + X_2) \cdot (X_1 - X_2) \cdot (X_3 \cdot X_4 - X_5 \cdot X_6)^2, \tag{2}$$

where the last quadratic factor is irreducible over $\mathbb{C}$ and occurs with multiplicity 2. The other factors are linear and occur with multiplicity 1.

REMARK 2.3. Dedekind's aim was to extend the above Theorem 2.1 to non-abelian groups, trying to decompose the group determinant into irreducible factors by involving hyper-complex systems—*i.e.*, $\mathbb{C}$-algebras. The reason might have been that Dedekind had previously written several papers on the concept of hyper-complex systems, which was new at the time, and was thus very familiar with it.

2.2. *Frobenius enters the stage.* Georg Frobenius (1849–1917) was 18 years younger than Dedekind and had written his thesis in 1870 under Karl Weierstrass (1815–1897) on partial differential equations. In 1890 he turned to the theory of abstract finite groups. After accepting a chair at Berlin in 1892—at the age of 43—he also did research on determinants.

Dedekind and Frobenius had been in contact on their research since 1880. However, it was not until 1895 that Dedekind wrote to Frobenius about his work on group theory and on the group determinant. In 1896[5] Dedekind told Frobenius about his result on Hamiltonian groups and also about the character formula of the group determinant for abelian groups (*cf.* Theorem 2.1) and on the importance Dedekind attributed to hyper-complex systems in this connection.

Frobenius immediately picked up the subject enthusiastically. However, he was more interested in an ordinary factorization of the group determinant over $\mathbb{C}$, and he wrote to Dedekind (*cf.* [Ha; 78, p. 65]) "the entire subject is so new to me that I cannot yet see how the irreducible factors of the determinant are connected with the invariant subgroups ...". In response, Dedekind conjectured—he even suggested a line of a proof—the next result, which is again in the spirit of Problem 2.2.

THEOREM 2.4. *The number of linear factors in $\mathcal{D}_G$—G a finite group—over the complex numbers is equal to the index of the commutator subgroup and hence to the order of the abelian group G/G'.*

In this letter he invited Frobenius to investigate these matters since "I distinctly feel that I will not achieve anything here". To quote Hawkins [Ha; 78, p. 65]: "He undoubtedly felt as well that the study of $\mathcal{D}_G$ would take him too far afield from his principal interests, the theory of numbers. Time and energy should not be squandered at age sixty-five. The study of $\mathcal{D}_G$ was nevertheless a good research problem, one well suited to Frobenius' tastes, and he clearly wanted Frobenius to pursue it."

Frobenius now continued the work on the group determinant with great effort.

3. The group determinant and higher characters: From Frobenius to Formanek and Hoehnke.

Let $G = \{g_1, \ldots, g_n\}$ be a finite group, and let $\{X_{g_1} = X_1, \ldots, X_{g_n} = X_n\}$ be independent indeterminats over the field K. Let us denote by $K(X)$ the field of rational functions over K in these indeterminates.

Dedekind defined the group determinant (*cf.* above) as

$$\mathcal{D}_G = \mathcal{D}_G(X) := \det(X_{g_i \cdot g_j^{-1}}) \in K(X). \tag{3}$$

[5] We recall that Dedekind already started his work on the group determinant in 1886, *i.e.*, 10 years before.

Then $\mathcal{D}_G$ is a homogeneous polynomial of degree $|G|$ in the variables $\{X_g\}$. In modern terminology: Let

$$\mathcal{H}_G := \sum_{g\in G} X_g \cdot g \in K(X)G$$

be the generic element in the group ring $K[X]G$. Then $\mathcal{D}_G$ is the determinant under the regular representation ρ of the element $\mathcal{H}_G$.

Let us assume—as was done in the beginnings of representation theory—that $K = \mathbb{C}$ is the field of complex numbers. In $K[X]$ the group determinant decomposes uniquely after normalization into (homogeneous) irreducible factors:

$$\mathcal{D}_G = \prod_{1\le i\le h} \Phi_i^{m_i}.$$

Since $\mathbb{C}G$ is semi-simple we conclude that

$$\mathbb{C}(X)G = \bigoplus_{1\le i\le h} M_i^{m_i},$$

i.e., $\mathbb{C}(X)G$ decomposes into a direct sum of non-isomorphic irreducible modules M_i with multiplicity m_i, $1 \le i \le h$. Accordingly, the regular representation $\rho = \bigoplus_{1\le i\le h} \rho_i^{m_i}$ decomposes into the representations ρ_i on M_i, and so

$$\mathcal{D}_G = \prod_{1\le i\le h} \det\big(\rho_i(\mathcal{H}_G)\big)^{m_i}$$

and one has found a decomposition of $\mathcal{D}_G$. It turns out that this decomposition is even a decomposition into irreducible factors.

However, Frobenius did not know about semi-simple algebras in 1896, and some of his goals—he eventually reached all of them—were to get group theoretical interpretations of the following invariants in the flavor of Problem 2.2:

1. The number of irreducible factors in the decomposition of the group determinant.
   We know that this is the number of conjugacy classes.
2. The degree of these factors.
3. The multiplicity with which the different irreducible factors occur in the group determinant.
   We know that the multiplicity coincides with the degree.

Frobenius' intention was to generalize the linear characters from abelian groups—*i.e.*, homomorphisms to $\mathbb{C}$—to arbitrary finite groups as maps to $\mathbb{C}$. On the other hand, Dedekind's main goal was to generalize the above decomposition of the group determinant into linear characters (*cf.* Theorem 2.1). This was successful for the quaternion group,[6] but Dedekind did not find a general result.

The idea of Frobenius was to assign to the group determinant functions, the characters from the finite group G to $\mathbb{C}$ as follows:

[6] The reason is that over $\mathbb{Q}$ the quaternion skew field enters.

DEFINITION 3.1. Let Φ be an irreducible factor of $\mathcal{D}_G$.

1. The character[7] χ_Φ associated to Φ is defined as
   - $\chi_\Phi(1) = \text{degree}(\Phi) := f$.
   - For $1 \neq g \in G$ the value $\chi_\Phi(g)$ is defined as the coefficient of $X_1^{f-1} \cdot X_g$ in Φ.

   In Frobenius' opinion the most important property of these 'characters' was that they were constant on conjugacy classes; *i.e.*, $\chi_\Phi(g \cdot h) = \chi_\Phi(h \cdot g)$. On April 17, 1896 he wrote to Dedekind: "My feeling that the equation $\chi(a \cdot b) = \chi(b \cdot a)$ provided the key did not deceive me. I still have a long way to go, but I am certain I have chosen the right path."[8]
2. More generally, for a natural number $k \leq f = \text{degree}(\Phi)$ the k-character χ_Φ^k has value on a k-tuple of group elements, say on $(\gamma_1, \gamma_2, \dots, \gamma_k)$, defined as follows:
   - It is the coefficient of $X_1^{f-k} \cdot X_{\gamma_1} \cdot \dots \cdot X_{\gamma_k}$ in Φ, provided none of the $\gamma_i = 1$.
   - If $\gamma_i = 1$, then

$$\chi_\Phi^k(\gamma_1, \gamma_2, \dots, \gamma_{i-1}, 1, \gamma_{i+1}, \dots, \gamma_k) = \chi_\Phi^{k-1}(\gamma_1, \gamma_2, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_k).$$

It is then clear that for an irreducible factor Φ of the group determinant we have

$$\Phi = (1/f!) \cdot \Big(\sum_{\{\gamma_1, \gamma_2, \dots, \gamma_f\}} \chi_\Phi^f(\gamma_1, \gamma_2, \dots, \gamma_f) \cdot X_{\gamma_1} \cdot X_{\gamma_2} \cdot \dots \cdot X_{\gamma_f} \Big).$$

Moreover, Frobenius already noticed that the k-characters can inductively be derived from the ordinary characters via a formula closely related to the differential of the standard resolution for groups and also to the singular homology in topology.

The k-characters arise from the ordinary characters if one knows in addition to the character table $CT(G) := \big(\chi_i(K_j)\big)_{1 \leq i,j \leq h}$—$K_j$ are the conjugacy classes—the values of $\chi_i(\gamma_1 \cdot \dots \cdot \gamma_k)$ for k bounded by the maximum of the degrees of the irreducible representations, *i.e.*, one has to know the map

$$\Psi_k : G \times G \times \dots \times G \longrightarrow \cup K_j,$$
$$(\gamma_1, \dots, \gamma_k) \longrightarrow K_{\gamma_1 \cdot \dots \cdot \gamma_k}.$$

This is apparently not as strong as the knowledge of the multiplication table for G. However, it is nevertheless a very powerful condition.

[7] Frobenius asked Dedekind whether he knew of a good name for what Frobenius had discovered, whether this should not be termed 'characters', since for abelian groups they reduced by Dedekind's formula to the characters introduced by Gauß.

[8] Later, at the suggestion of Dedekind, Frobenius tried to link his characters with the theory of representations then developed by Schur and found out that his characters were the traces of the irreducible representations. At first—even for the orthogonality relations—representations did not enter into the picture.

Frobenius later derived the orthogonality relations for characters from his definition of the characters. He even obtained orthogonality relations for k-characters for $1 \leq k$.

There is an interesting connection with the theory of matrix invariants observed by Ed Formanek: let S_k be the symmetric group on k letters, and for $\sigma \in S_k$ we write

$$\sigma = \prod_{i=1,\dots,t} (\delta_i) \text{ as a product of disjoint cycles where 1-cycles are included.}$$

The trace function with respect to $\sigma \in S_k$

$$T_\sigma \colon M_n(\mathbb{C})^k \longrightarrow \mathbb{C}$$

is defined for $(x_1, \dots, x_k) \in M_n(\mathbb{C})^k$ as

$$T_\sigma(x_1, \dots, x_k) = Tr(x_{1_1} \cdot \dots \cdot x_{1_{k_1}}) \cdot \dots \cdot Tr(x_{t_1} \cdot \dots \cdot x_{t_{k_t}}),$$

according to the above cycle decomposition. Here Tr stands for the ordinary trace. Formanek showed in 1987 [Fo; 87] that all relations between the traces of tuples of $n \times n$-matrices are consequences of the single relation

$$\sum_{\sigma \in S_{n+1}} \operatorname{sig}(\sigma) \cdot T_\sigma(x_1, \dots, x_{n+1}) = 0.$$

Moreover, another description of the k-characters was given by K. W. Johnson in 1991 [Jo; 91] in connection with the above result of Formanek:

$$\chi_\Phi^k(\gamma_1, \gamma_2, \dots, \gamma_i, \dots, \gamma_k) = \sum_{\sigma \in S_k} \operatorname{sig}(\sigma) \cdot T_\sigma(X_{g_1}, \dots, X_{g_k}),$$

where X_g are the representing matrices. Let us recall that Frobenius had only defined the k-characters for $k \leq f$ where f is the degree of the representation. By the result of Formanek the $f+1$-character would be identically zero—another interesting coincidence.

The following surprising result is in the spirit of Dedekind (*cf.* Theorem 2.1 and 2.4) and Problem 2.2.

THEOREM 3.2 (E. FORMANEK AND D. SIBLEY, 1991 [FO-SI; 91][9]). *Let G and H be finite groups, such that $\mathcal{D}_G = \mathcal{D}_H$, then $G \simeq H$.*[10]

This means that the group determinant is an invariant which characterizes the underlying group. Since the group determinant can be recovered from the k-characters for the various values of k, the knowledge of all k-characters determines the underlying group up to isomorphism.

[10] A short proof of this result was later given by Mansfield in 1991 [Man; 91].

The next result shows that actually a much stronger connection holds. Let us recall from Definition 3.1 that a 2-character is given as follows

$$\chi^2(g,h) := \chi(g \cdot h) - \chi(g) \cdot \chi(h),$$

where χ is an ordinary character. Such a 2-character χ^2 measures the obstruction from χ being a homomorphism. In order to describe the 2-characters, it is thus necessary—in addition to the knowledge of the character values—to know explicitly the function

$$\Psi^2 : G \times G \longrightarrow \cup_{g \in G} K_g, \tag{4}$$

$$(g,h) \longrightarrow K_{g \cdot h}. \tag{5}$$

The knowledge of the function Ψ^2 in Equation 4 determines inductively the p-power map on the characters—*i.e.*, the class K_{g^p}. We thus get the next result:

NOTE 3.3. Assume that G and H have the same 2-characters—*i.e.*, there is a map $\beta : G \to H$ and a map σ from the irreducible characters of G to the irreducible characters of H—note that this induces a map on the 2-characters, also denoted by σ, such that

$$\chi(g) = \big(\sigma(\chi)\big)\big(\beta(g)\big) \quad \text{and} \tag{6}$$

$$\chi^2(g_1, g_2) = \big(\sigma(\chi^2)\big)\big(\beta(g_1), \beta(g_2)\big). \tag{7}$$

Then G and H form a Brauer pair—*i.e.*, they have the same spectral tables.[11]

K. W. Johnson and Surinder K. Sehgal [Jo-Se; 91] have shown in 1991 that a class of groups of which G. Cliff and Surinder K. Sehgal [Cl-Se; 81] proved in 1981 that they have isomorphic spectral tables[12] also have isomorphic 2-characters.

We have seen above that the 2-characters are not sufficient to distinguish two finite groups. The following result is therefore the more surprising:

PROPOSITION 3.4. (HOEHNKE AND JOHNSON [HO-JO; 91], AND ALSO [KI-RO; 91]) *The* $1-2-3$*-characters determine a group up to isomorphism, i.e., the character table and the map*

$$\Psi_3 : G \times G \times G \longrightarrow \cup K_j,$$

defined by

$$(g_1, g_2, g_3) \longrightarrow K_{g_1 \cdot g_2 \cdot g_3}$$

determine G *up to isomorphism.*[13]

This result answers in a strong way one of Richard Brauer's questions:

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[11] This connection was noted by K. W. Johnson in [Jo; 91].
[12] The first such example was given by E. C. Dade in 1964 [Da; 64].
[13] For a short proof see [RoI; 92].

PROBLEM 3.5. What, in addition to the character table, determines a finite group up to isomorphism?

Dr. W. Kimmerle and the author have been trying to find—in connection with the isomorphism problem for integral group rings—non-isomorphic groups which have isomorphic spectral tables and isomorphic Burnside matrices.[14] In this connection we tried to analyze the examples of G. Cliff and Surinder K. Sehgal [Cl-Se; 81] in such a way that we could give general criteria for when two groups

1. have isomorphic spectral tables,
2. have isomorphic Burnside matrices,
3. have isomorphic 2-characters.

We answered these questions to a large extent in 1992 (*cf.* [Ki-Ro; 92]). Using these results we constructed (*cf.* also [RoI; 92]) non-isomorphic groups which have

1. isomorphic spectral tables,
2. isomorphic 2-characters and
3. isomorphic Burnside matrices.

PART II: THE ISOMORPHISM PROBLEM

4. The problems. We shall restrict the attention here to the isomorphism problem and a variation by Hans Zassenhaus. For other problems on finite subgroups of the unit group of RG we refer to [Ro; 97].

Let K be a global number field and R its ring of integers. The most prominent example will be $K = \mathbb{Q}$ and $R = \mathbb{Z}$. For a group G we denote by RG its group ring. For many years there have been outstanding problems which dealt with ways of how can G be embedded into RG—this is tantamount to the questions on the automorphisms of RG.

For a normal subgroup N we have the augmentation sequence with respect to N

$$0 \longrightarrow I_R(N) \uparrow G \longrightarrow RG \xrightarrow{\epsilon_N} RG/N \longrightarrow 0, \tag{8}$$

where ϵ_N is induced by sending $g \in G$ to $g \cdot N \in G/N$. The kernel of the augmentation map, the augmentation ideal $I_R(N) \uparrow G$, is the RG-ideal generated by the elements $\{n - 1 : n \in N \setminus \{0\}\}$.

In case $N = G$, the map $\epsilon =: \epsilon_G$ is called the augmentation map of RG. By $U(RG)$ we denote the units in RG, and $V(RG)$ are the units of augmentation 1. Then, obviously, $U(RG) = V(RG) \cdot U(R)$. So we do not lose anything if we restrict ourselves to units of augmentation one. Similarly, it is no restriction if we assume that every ring homomorphism $\alpha: RG \to RH$ commutes with the augmentation, *i.e.*, is augmented.

[14] Sometimes called table of marks.

We shall briefly discuss some of the outstanding problems on integral group rings which are in the spirit of Dedekind's original approach (*cf.* Problem 2.2) of getting information about a finite group from its invariants and also in the spirit of R. Brauer's Problem 3.5.

PROBLEM 4.1.

1. Isomorphism problem (1939) [Hi; 39]: does $RG \simeq RH$ imply $G \simeq H$?
2. Zassenhaus conjecture (1976): let G be finite and assume that $RG = RH$ as augmented algebras; does this imply that there is a unit $a \in KG$ such that $a \cdot H \cdot a^{-1} = G$?
3. Zassenhaus conjecture for automorphisms: let G be finite and assume that $\alpha\colon RH \to RH$ is an augmented automorphism; does this imply that there is a group automorphism $\rho\colon G \to G$ and a central[15] automorphism $\gamma\colon RG \to RG$ such that $\alpha = \rho \cdot \gamma$?
4. Conjugacy problem: assume that G is finite and let K be a local number field with $|G| \cdot R \neq R$. If $RG = RH$, does there exist a unit $u \in RG$ with $u \cdot H \cdot u^{-1} = G$?
5. Outomorphism problem: since every group automorphism of G extends to a ring automorphism of RG, there is a natural homomorphism

$$\Phi_R(G)\colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(RG)$$

   of the outer automorphism groups. The problem is whether this map is always injective.

These problems are formulated here for the group rings. There are analogous problems formulated for various module categories to which we will turn in Part IV.

5. Some positive results. For a detailed account on the isomorphism problem as well as some more historical remarks we refer to [Ro-Ta,92].

THEOREM 5.1 (ROGGENKAMP AND SCOTT [RO-SC; 87]). *Let K be a local number field and let G be a finite p-group. Then the conjugacy problem 4.1 4. is true for RG. Moreover, the Zassenhaus conjectures (cf. Problems 4.1, 2., 3.) hold for nilpotent groups and $R = \mathbb{Z}$.*

With weaker hypotheses one obtains the conjugacy problem[16] for a wider class of groups:

THEOREM 5.2 (SCOTT [SC; 90]). *Let G be a finite solvable group such that for a prime p the group $O_{p'}(G) = 1$.*[17] *Assume that $\mathbb{Z}G = \mathbb{Z}H$. Then the conjugacy problem has a positive answer in the p-adic group ring $\widehat{\mathbb{Z}}_pG$.*[18]

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[15] *I.e.*, an automorphism leaving the center elementwise fixed.

[16] L. L. Scott proved this when collaborating with the author.

[17] Recall that $O_{p'}(G)$ is the largest normal subgroup of G of order prime to p.

[18] Dr. W. Kimmerle has filled in a gap (for $p = 2$) in the original proof.

Obviously both of the above theorems imply the Zassenhaus conjecture for $\mathbb{Q}$. Moreover, Theorem 5.2 allows for solvable groups to develop an obstruction theory for the Zassenhaus conjecture for automorphisms to hold. This gives rise to a Čech style cohomology (*cf.* [Ki-Ro; 93]). We shall touch on this in Section 11.

Although the Zassenhaus conjecture fails for solvable groups (*cf.* Section 6, Theorem 6.1) there are indications that it holds for simple groups (Bleher [Bl; 95], Bleher, Hiß and Kimmerle [Bl-Hi-Ki; 96], Bleher [Bl; 96], Peterson [Pe; 76], Bleher, Geck and Kimmerle [Bl-Ge-Ki; 96], Bleher [Bl; 96], [Bl; 97]). We have the following result:

THEOREM 5.3. *The Zassenhaus conjecture holds*

1. *for every finite Coxeter group (i.e., finite groups generated by reflections),*
2. *for the following list of finite simple groups—p is a rational prime and* $m \in \mathbb{N}$*:*

$$\begin{array}{lllll} \mathrm{SL}(2,p^m), & \mathrm{PSL}(2,p^m), & {}^2B_2(2^{2m+1}), & {}^2G_2(3^{2m+1}), & {}^2F_4(2^{2m+1}), \\ \mathrm{SL}(3,3^m), & \mathrm{PSL}(3,p^m), & \mathrm{SU}(3,3^{2m}), & \mathrm{PSU}(3,p^{2m}), & \mathrm{Sp}(4,2^m), \\ \mathrm{PSp}(4,p^m), & G_2(p^m), & {}^3D_4(p^{3m}), & & \end{array}$$

3. *for every minimal simple group,*
4. *for the Zassenhaus groups,*
5. *for every simple group with an abelian 2-Sylow-subgroup,*
6. *for the sporadic simple groups:*

$$M_{11}, M_{12,22}, M_{23}, M_{24}, Co_1, Co_2,$$
$$Co_3, J_1, J_2, J_3, H'S, H'N, Th, B, Ru.$$

Although the isomorphism problem is false for finite solvable groups (*cf.* Section 6, Theorem 6.2) the simple groups are much more rigid:

THEOREM 5.4. (KIMMERLE, LYONS, SANDLING AND TEAGUE [KI-LY-SA-TE; 90]) *The finite simple groups are determined by their integral group rings.*

We also have several classes of solvable groups for which the isomorphism problem has a positive solution (*cf.* Kimmerle [Ki;91], Kimmerle and Roggenkamp [Ki-Ro; 93], Roggenkamp and Scott [Ro-Sc; 86], Roggenkamp and Zimmermann [Ro-Zi; 92]).

THEOREM 5.5. *The isomorphism problem has a positive solution for the following classes of finite groups:*

1. *If G is nilpotent by abelian,*
2. *if G is abelian by nilpotent,*
3. *if G is a Frobenius group,*
4. *if G is a two-Frobenius group.*

6. Negative results. L. L. Scott and the author started out at the beginning of the 1980s to find a counterexample to the Zassenhaus conjecture, since we had the feeling that it was too strong. We could not; instead we succeeded in proving the conjugacy problem for p-groups (*cf.* Theorem 5.2) in 1986. We then turned again to the Zassenhaus conjecture and found:

THEOREM 6.1 (ROGGENKAMP AND SCOTT [Ro-Sc; 86]). *There exists a finite metabelian group G and an augmented automorphism $\alpha: \mathbb{Z}G \to \mathbb{Z}G$ which is a counterexample for the Zassenhaus conjecture for automorphisms (cf. Problem 4.1, 3.) and thus also for the Zassenhaus conjecture.*

The most spectacular result was recently obtained by Martin Hertweck in October 1996 [He; 97]:

THEOREM 6.2 (HERTWECK, 1996). *There are two finite 3-step abelian groups*[19] *such that*

1. *$G \not\simeq H$—to show this is not a triviality in this example (cf. below),*
2. *$\mathbb{Z}G \simeq \mathbb{Z}H$ as augmented algebras.*

The groups Hertweck constructs have order

$$|G| := 19.331.131.032.074.407.391.789.056 \sim 20 \times 10^{24},$$

and so it is not so easy to check for isomorphism. Here the results from Section 11 are used (*cf.* Remark 10.4).

This example—in particular the construction of the groups H and G—is not 'ad hoc' but is based upon previous work of Leonard L. Scott and the author [Ro-Sc; 86] (see also [Ro-Ta,92], [Kl; 91]), W. Kimmerle with the author [Ki-Ro; 93] and A. Zimmermann with the author [Ro-Zi I; 95], [Ro-Zi II; 95]. All the ingredients, namely M. Mazur's construction of the groups $G \rtimes_\alpha \mathbb{Z}$ (*cf.* below Section 8) and an example (*cf.* Section 7) where $\mathrm{Ker}(\Phi_R(G)) \neq 0$ (*cf.* Problem 4.1, 5.) did already exist in 1994.

It was a clever idea of Hertweck, though, together with skillful arrangement of the available ingredients, which eventually did lead to his construction of the counterexample. The next is the key result, which Hertweck modifies to construct the counterexample of his finite groups.

THEOREM 6.3 (ROGGENKAMP AND ZIMMERMANN [RO-ZI II; 95]). *There do exist two non-isomorphic finite by infinite cyclic groups $\mathcal{G}$ and $\mathcal{H}$, and a suitable algebraic number field K with ring of integers R such that*

$$R\mathcal{G} \simeq R\mathcal{H}.$$

19 This means that the group has an abelian normal subgroup such that the quotient is metabelian.

Here the construction of $\mathcal{G}$ and $\mathcal{H}$ is based on an example from finite groups in connection with the automorphism problem (*cf.* Problem 4.1, 5.) which involves invertible bi-modules and delicate calculations inside the group ring using Clifford theory for automorphisms (*cf.* [Ro-Ta,92]).

7. **The automorphism problem.** We write $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ for the group of outer automorphisms of the group G, and for a commutative integral domain R we write $\mathrm{Out}_n(RG) = \mathrm{Aut}_n(RG)/\mathrm{Inn}(RG)$ for the group of outer augmentation preserving automorphisms of the group ring. We also use the notation $N_X(Y)$ to denote the normalizer of Y in X and write $C_X(Y)$ for the centralizer in X of Y. $O_p(G)$ ($O'_p(G)$) denotes the largest normal p-subgroup of G (largest normal subgroup of order prime to p).

The automorphism problem (*cf.* Problem 4.1, 5.) deals with the determination of the kernel of the natural map

$$\Phi_R(G)\colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}_n(RG). \tag{9}$$

If there is no confusion we just write Φ. We recall some facts on $\mathrm{Ker}(\Phi)$:

NOTE 7.1.

1. Donald Coleman's result [Col; 64], [Ro-Sc] states that for a p-subgroup $P \leq G$ we have
$$N_{V(\mathbf{Z}G)}(P) = N_G(P) \cdot C_{V(\mathbf{Z}G)}(P).$$
In particular, for p-groups we have $\mathrm{Ker}\big(\Phi_{\mathbf{Z}}(G)\big) = 1$.
2. Jan Krempa [Kr; 87] has shown that $\mathrm{Ker}(\Phi_{\mathbf{Z}})$ is an elementary abelian 2-group.
3. Stefan Jackowski and Zbigniev Marciniak [Ja-Ma; 87] have shown that $\Phi_{\mathbf{Z}}$ is injective if the 2-Sylow-subgroup is normal.
4. F. Gross [Gr; 82] showed: Let G be a finite group and p be a prime number. If $C_G\big(O_p(G)\big) \leq O_p(G)$ then every automorphism of G which is the identity on a Sylow p-subgroup is inner on G.
5. From Gross' result it follows that if for some prime we have $O_{p'}(G) = 1$ in the solvable group G, then $\Phi_{\mathbf{Z}}$ is injective.
6. In [Marc-Ro; 97] it was shown that $\Phi_{\mathbf{Z}}$ is injective provided G is metabelian with abelian 2-Sylow subgroup.
7. For some further and more detailed results on $\mathrm{Ker}(\Phi)$ we refer the interested reader to [Marc-Ro; 97].

For any subgroup $H \leq G$ we define its Weyl group as the quotient

$$W_G(H) := N_G(H)/C_G(H).$$

Notice that this group acts naturally on H. The following statement indicates a construction of making groups with $\mathrm{Ker}(\Phi) = 1$ larger.

PROPOSITION 7.2 ([MARC-RO; 97]). *Let* $1 \to H \to G \to \overline{G} \to 1$ *be an exact sequence of finite groups. If* $\mathrm{Ker}(\Phi_{\mathbf{Z}}(\overline{G})) = 1$ *and the groups* H *and* $W_G(H)$ *have odd orders, then* $\mathrm{Ker}(\Phi_{\mathbf{Z}}(G)) = 1$.

REMARK 7.3.

1. The assumptions of the above proposition are satisfied in particular when the H-conjugacy classes of elements in H coincide with the G-conjugacy classes. This happens for example when H is central in G.
2. If G is a nilpotent group then its Sylow subgroups are normal and clearly $W_G(P) = 1$ for each of them. Since $\mathrm{Ker}(\Phi_{\mathbf{Z}}) = 1$ for p-groups, we obtain by an induction an easy proof that the same holds for all nilpotent groups.
3. If G is a minimal example of a group with $\mathrm{Ker}(\Phi_{\mathbf{Z}}) \neq 1$ then the center of G is a 2-group. Note that the trivial group is by definition a 2-group.

However, until 1994, when the following was proved, no example of a finite group G was known where this map is not injective for a global field K.

THEOREM 7.4 (ROGGENKAMP AND ZIMMERMANN [RO-ZI I; 95]). *There is a finite solvable group* G *and a non-inner automorphism* α *of* G *which becomes inner in* $\mathbb{Z}_\pi G$, *where* Z_π *is the semi-localization of* $\mathbb{Z}$ *at a finite set of primes containing all prime divisors of the order of* G. *In particular for a suitable algebraic number field with ring of integers* R *the map* $\Phi_R(G)$ *is not injective.*

8. Mazur's observation. Marcin Mazur knew the automorphism problem posed by Stefan Jackowski and Zbingniev Marciniak [Ja-Ma; 87] and dealt with the following construction of semi-direct products, using the automorphism Φ from above (*cf.* Equation 9):

DEFINITION 8.1 (MAZUR [MAZ; 95]). Let G be a group and let α be an automorphism of G of order n_0 for $n_0 \in \mathbb{N} \cup \{\infty\}$. Then we define the n_0-cyclic extension (simply, cyclic extension) of G with α as the group $G \rtimes_\alpha C_n$ for $n \in \mathbb{N} \cup \{\infty\}$, $n = \nu \cdot n_0$, $\nu \in \mathbb{N}$ as follows: as a set $G \rtimes_\alpha C_n$ is the product $G \times C_n$ where we denote the cyclic group of order n (if $n = \infty$, then this is the infinite cyclic group) by $C_n := \langle x : x^n \rangle$; the multiplication in $G \rtimes_\alpha C_n$ is defined for

$$g, g' \in G, \quad x^z, x^{z'} \in C_n \quad \text{as} \quad (g, x^z) \cdot (g', x^{z'}) = (g \cdot {}^{\alpha^z} g', x^{z+z'}).$$

Whenever we use the symbol $G \rtimes_\alpha C_n$ we automatically imply that n is a multiple of the order of α in case this is finite or that $n = \infty$. A direct consequence of this construction is the crucial observation of Mazur:

LEMMA 8.2 (MAZUR [MAZ; 95]).

1. $G \rtimes_\alpha C_\infty \simeq G \rtimes_\beta C_\infty$ *for a group* G *if and only if* $\gamma := \beta^{-1} \circ \alpha$ *is inner on* G.

2. *Moreover, for an integral domain R of characteristic zero we have for the group rings* $R(G \rtimes_\alpha C_\infty) \simeq R(G \rtimes_\beta C_\infty)$, *provided* $\gamma = \beta^{-1} \circ \alpha$ *is inner on RG, i.e.,* $\Phi_R(G)(\beta^{-1} \circ \alpha) = 1$.

Although the proof for the groups and also for group rings is straightforward and can be found in Mazur's paper [Maz; 95] we nevertheless want to point out the construction of the isomorphism: assume that G is finite and that

$$\gamma = \beta^{-1} \circ \alpha \Longleftrightarrow \alpha = \beta \circ \gamma,$$

where γ is conjugation with a unit $u \in RG$. We note that u is not unique—it is only unique modulo the center of RG.

Although γ has the same finite order as $\beta^{-1} \circ \alpha$, say ν, the conjugating element u has in general an order which is larger than the order of γ; as a matter of fact the element u will in general have infinite order, since u^ν is a unit in the center of RG and the units in the center of RG are in general of infinite order. The above relation means—applied to elements $g \in G$—

$$(10) \qquad {}^{\beta}u \cdot {}^{\beta}g \cdot {}^{\beta}u^{-1} = {}^{\beta}(u \cdot g \cdot u^{-1}) = {}^{\alpha}g, g \in G \quad \text{and}$$

$$(11) \qquad u \cdot {}^{\alpha^{-1}}g \cdot u^{-1} = {}^{\beta^{-1}}g \quad \text{since} \quad \gamma \circ \alpha^{-1} = \beta^{-1}.$$

We have used the symbol $\cdot_\beta$ to stress that the multiplication takes place in $G \rtimes_\beta C_\infty$. We now define a map

$$(12) \qquad \phi: G \rtimes_\alpha C_\infty \longrightarrow G \rtimes_\beta C_\infty \quad \text{by}$$

$$(13) \qquad \phi: (g, x^z) \longrightarrow (g, 1) \cdot_\beta \left({}^{\beta}u \cdot_\beta (1, x)\right)^z.$$

We have to make sure that this is an isomorphism. For this it is enough to check the multiplicativity on products of the form $(1, x^z) \cdot (g, 1)$. This is easily done by induction using Equation (10).

REMARK 8.3. Let $\alpha, \beta \in \mathrm{Aut}(G)$; then from the above (*cf.* Lemma 8.2) we conclude: assume that $\gamma = \beta^{-1} \circ \alpha$ is inner in RG, then

$$R(G \rtimes_\alpha C_\infty) \simeq R(G \rtimes_\beta C_\infty) \quad \text{and} \quad R(G \rtimes_{\beta^{-1}\circ\alpha} C_\infty) \simeq R(G \times C_\infty).$$

9. **Hertweck's observation.** Together with A. Zimmermann (*cf.* Theorem 7.4) we have constructed a finite group G and an automorphism α of G which is not inner but which lies in the kernel of the natural homomorphism Φ (*cf.* Equation 9).

$$\Phi: \mathrm{Out}(G) \to \mathrm{Out}(\mathbb{Z}_\pi G) \quad (\textit{cf. } \text{Equation } 9);$$

here $\mathbb{Z}_\pi$ is the semi-localization of $\mathbb{Z}$ at a finite set of rational primes which contains all the prime divisors of $|G|$.

REMARK 9.1.

1. Using the automorphism α from [Ro-Zi I; 95] we see:
   (a) $G \rtimes_\alpha C_\infty \not\simeq G \times C_\infty$ and
   (b) $R(G \rtimes_\alpha C_\infty) \simeq R(G \times C_\infty)$ where $R = \mathbb{Z}_\pi$.
2. One might be tempted to ask why this construction does not work for finite groups, *i.e.*, why we cannot replace C_∞ with a finite group C_n? An analysis of the above arguments shows that then u—the automorphism α on RG is conjugation by u—must have an order dividing n. On the other hand one knows that for finite groups F_1 and F_2 an isomorphism $\mathbb{Z}F_1 \simeq \mathbb{Z}F_2$ implies that $F_1 = G_1 \times H_1$ is a direct product of two groups if and only if $F_2 = G_2 \times H_2$ is a direct product of two groups with $|G_1| = |G_2|$ and $|H_1| = |H_2|$.
   A consequence of these considerations is that the conjugating element u cannot have finite order.
3. This did "convince" Zimmermann and me that this construction could not be used to construct a counterexample to the isomorphism problem for finite groups.
4. Hertweck was not so "naïve"!

Martin Hertweck did look carefully at the above proof:

1. In our example the unit u must have infinite order and the observation in Remark 8.3 shows that whenever a non-inner automorphism α of G becomes inner in RG we always have
   (a) $G \rtimes_\alpha C_\infty \not\simeq G \times C_\infty$ and
   (b) $R(G \rtimes_\alpha C_\infty) \simeq R(G \times C_\infty)$ where $R = \mathbb{Z}_\pi$.
   Hence, no matter what representative $u \in RG$ we choose to represent γ, the unit u never has finite order. This means by using only one automorphism there is no hope of constructing a counterexample to the isomorphism problem for finite groups.
2. However, if we look again at the isomorphism ϕ from Equation 12,
   $$\phi\colon (g, x^z) \longrightarrow (g,1) \cdot_\beta \left({}^\beta u \cdot_\beta (1,x)\right)^z,$$
   we see that the situation here is completely different: the generating element x of C_∞ is sent to ${}^\beta u \cdot_\beta (1,x)$. Let us assume that x has finite order ν. Although u has infinite order the element ${}^\beta u \cdot_\beta (1,x)$ may very well have finite order. Let us assume that $x^2 = 1$, then $\beta^2 = 1$ and we have
   (14) $$\left({}^\beta u \cdot_\beta (1,x)\right)^2 = {}^\beta u \cdot {}^{\beta^2} u \cdot (1,x^2) = {}^\beta u \cdot u.$$
   If u were now an element which is inverted by β, then ${}^\beta u \cdot_\beta (1,x)$ would be an involution, and the group rings $R(G \rtimes_\alpha C_2)$ and $R(G \rtimes_\beta C_2)$ would be isomorphic.
3. This is how Hertweck proceeded.

4. If we do not have the semi-direct product with an infinite cyclic group it is not so easy to show that the two groups constructed are non-isomorphic. However, in [Ki-Ro; 93] we have given a description of H in terms of G, provided $\mathbb{Z}G \simeq \mathbb{Z}H$ (*cf.* Section 11).

In [Marc-Ro; 97] we looked more carefully at Hertweck's construction and asked ourselves: What is it that makes Hertweck's construction work (*cf.* Equation 14):

DEFINITION 9.2. For an element x in the ring Λ and $\beta \in \mathrm{Aut}(\Lambda)$ we define the β-norm of x.

$$^{N_n(\beta)}x = x \cdot {}^{\beta}x \cdot {}^{\beta^2}x \cdot \dots \cdot {}^{\beta^{n-1}}x.$$

With this definition the construction of Hertweck appears in a new light:

PROPOSITION 9.3 ([MARC-RO; 97]). *Let $\alpha, \beta \in \mathrm{Aut}(G)$ be such that $\alpha \circ \beta^{-1}$ is inner, say a conjugation with g_0. The map $\phi_n : G \rtimes_\alpha C_n \longrightarrow G \rtimes_\beta C_n$ induced by*

$$(g, x) \mapsto (g, 1) \cdot (g_0, 1) \cdot (1, x)$$

is a homomorphism if and only if $^{N_n(\beta)}g_0 = 1$. A similar statement holds for group rings if $\alpha \circ \beta^{-1} \in \mathrm{Ker}(\Phi)$.

PROPOSITION 9.4. *Suppose we have a ring isomorphism*

$$\Psi : \mathbb{Z}(G \rtimes_\alpha C_n) \longrightarrow \mathbb{Z}(G \rtimes_\beta C_n)$$

such that $\Psi|_G = \mathrm{id}$ and $\Psi(h)h^{-1} \in \mathbb{Z}G$ for all $h \in H := G_\alpha/G$. If the order of H is odd, then $G \rtimes_\alpha C_n \simeq G \rtimes_\beta C_n$.

PART III:
THE OBSTRUCTION TO THE ZASSENHAUS CONJECTURE AND TO THE AUTOMORPHISM PROBLEM VIA ČECH COHOMOLOGY

10. Projective limits and isomorphisms of groups. Let G be a finite solvable group.

DEFINITION 10.1. $N_i = O_{p_i'}(G)$, where $\{p_i\} = \pi(G)$ are the prime divisors of $|G|$, $G_i := G/N_i$ and $G_{ij} := G/(N_i \cdot N_j)$. We then have an isomorphism (*cf.* [Ki-Ro; 93]) $G \simeq \lim.\mathrm{proj}.(G/N_i) \simeq_{\natural} \{(g_1, \dots, g_n) \mid g_i \in G_i : g_i \equiv g_j \text{ in } G_{ij}\}$.

The basic questions around the isomorphism problem, the Zassenhaus conjecture, and the automorphism problem should also be seen as a comparison between the category of groups and the category of group rings. The behavior of projective limits is quite different in both cases. Although G is the projective limit of the quotients $G_i := G/N_i$, the projective limit

$$\Gamma(G) := \lim.\mathrm{proj}.(\mathbb{Z}G_i)$$

of the group rings $\mathbb{Z}G_i$ does not coincide with $\mathbb{Z}G$. As a matter of fact,

$$\Gamma(G) := \{(\gamma_1, \ldots, \gamma_n) \mid \gamma_i \in \mathbb{Z}G_i : \gamma_i \equiv \gamma_j \text{ in } \mathbb{Z}G_{ij}\}$$

is a comparatively small quotient of $\mathbb{Z}G$.

The induced map $\sigma: \mathbb{Z}G \longrightarrow \Gamma(G)$ [20] has kernel

$$\mathrm{Ker}(\sigma) = \cap_{1\leq i\leq n} I_{\mathbb{Z}}(N_i) \uparrow G. \tag{15}$$

We should point out that G embeds into $\Gamma(G)$, and this projective limit reflects many properties of the integral group ring and seems to be an interesting substitute for the integral group ring. For example, the cohomology rings of $\mathbb{Z}G$ and $\Gamma(G)$ are isomorphic [Ki-Ro; 93].

The main result gives an explicit description of H in terms of G provided $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras and it also provides an obstruction theory for the Zassenhaus conjecture and the automorphism problem. Before we can state it we have to introduce some more notation:

DEFINITION 10.2. A central automorphism cocycle is defined as

$$\rho = (\rho_{ij}); \quad \text{with} \quad \rho_{ii} = 1 \quad \text{and} \quad \rho_{ij} = \rho_{ji}^{-1},$$

where $\rho_{ij} \in \mathrm{Aut}_c(G_{ij})$, the group of conjugacy class preserving automorphisms of G_{ij}. Then

$$G(\rho) = \left\{(g_i) \in \prod_{1\leq i\leq n} G_i \mid (g_i \cdot G_{ij})\rho_{ij} = (g_j \cdot G_{ij})\right\}$$

is a group.

The main ingredient in the proof of the next essential result as well in the proofs of Proposition 11.3 and Proposition 11.5 is Theorem 5.1.

THEOREM 10.3 ([KI-RO; 93]).

1. *Assume that $\Gamma(G) = \Gamma(H)$ as augmented algebras and that G is soluble. Then there exists a central automorphisms cocycle ρ such that $H \simeq G(\rho)$.*
2. *In particular if $\mathbb{Z}G = \mathbb{Z}H$, then $H \simeq G(\rho)$.*
3. *Moreover, $G(\rho) \simeq G$ if and only if there are automorphisms $\rho_i \in \mathrm{Aut}(G_i)$— not necessarily in $\mathrm{Aut}_c(G_i)$—with*

$$\rho_{ij} = \rho_i \cdot \rho_j^{-1};$$

i.e., ρ is an automorphism coboundary. Hence the obstruction is a question of lifting automorphisms.

Let us point out that the above is a purely group theoretical description.

[20] I do not know whether this map is surjective.

REMARK 10.4. Let us recall that Hertweck did construct two groups G and H—of very large order—such that $\mathbb{Z}G \simeq \mathbb{Z}H$; however, he did not know offhand whether the groups are isomorphic.

Because of Theorem 10.3 he knows that $H \simeq G(\rho)$ for a central automorphism cocycle ρ. From the construction of the groups this cocycle can be computed, and now it is not so hard to show that ρ cannot be an automorphism coboundary, thus showing that G and H cannot be isomorphic.

In the next section we shall report on some general observations concerning Čech cohomology.

11. Čech cohomology. We shall set up the Čech cohomology in a general setting: Let $\mathcal{K}$ be either the category of groups, of rings, or of modules.

DEFINITION 11.1. Assume that for each $X \in \text{ob}(\mathcal{K})$ we are given a 'natural' subgroup $\text{Aut}_*(X) \leq \text{Aut}(X)$—*e.g.* inner automorphisms or central automorphisms, *etc.* Let us be given objects and epimorphisms

$$\{X_i, X_{ij} = X_{ji}, X_{ii} = X_i \mid \phi_{ij} : X_i \longrightarrow X_{ij}, 1 \leq i \neq j \leq n\}.$$

Then we can form the projective limit

$$X = \{(x_i) : x_i \in X_i \mid x_i\phi_{ij} = x_j\phi_{ji}\}.$$

In addition, we assume that the kernels of the maps ϕ_{ij} are '$*$-characteristic'.

We shall use $\underline{X}$ to stress that we are working with a projective limit.

1. The $*$-automorphism cocycles are defined as:

$$Z(\underline{X}, \text{Aut}_*(\underline{X})) = \{(\rho) = (\rho_{ij}) : \rho_{ij} \in \text{Aut}_*(X_{ij}) \mid \rho_{ij} = \rho_{ji}^{-1}, \rho_{ii} = \text{id}_{X_i}\}.$$

2. The $*$-automorphism coboundaries are defined as

$$B(\underline{X}, \text{Aut}_*(\underline{X})) \\ = \{(\rho_{ij}) \in Z(\underline{X}, \text{Aut}_*(\underline{X})) \mid \rho_{ij} = \rho_i \cdot \rho_j^{-1} \text{ for some } \rho_i \in \text{Aut}_*(X_i)\}.$$

3. Two $*$-automorphism cocycles are said to be $*$-equivalent,

$$(\rho_{ij}) \sim (\rho'_{ij}) \Longleftrightarrow \rho_i \cdot \rho_{ij} \cdot \rho_j^{-1} = \rho'_{ij} \quad \text{for} \quad \rho_i \in \text{Aut}_*(X_i).$$

(Here we mean the maps induced on the quotients.)

4. The equivalence classes of the $*$-automorphism cocycles are denoted by

$$\check{H}(\underline{X}, \text{Aut}_*(\underline{X})).$$

This is a pointed set, with point the class of the $*$-automorphism coboundaries.

5. For $\rho = (\rho_{ij}) \in Z\big(\underline{X}, \mathrm{Aut}_*(\underline{X})\big)$ we define

$$X(\rho) := \{(x_i) : x_i \in X_i \mid x_i\phi_{ij} \cdot \rho_{ij} = x_j\phi_{ji}\}.$$

This definition should be compared with the Čech cohomology of a topological space with respect to an open covering. We point out that these definitions depend strongly on the specially chosen subgroup $\mathrm{Aut}_*(-) \leq \mathrm{Aut}(-)$. Direct calculations show:

LEMMA 11.2 ([KI-RO; 93]). *There is an isomorphism of projective limits*

$$\sigma = (\sigma_i): X(\rho) \longrightarrow X(\rho') \quad \textit{with} \quad \sigma_i \in \mathrm{Aut}_*(X_i)$$
$$\textit{provided} \quad \rho \sim \rho' \quad \textit{in} \quad Z\big(\underline{X}, \mathrm{Aut}_*(\underline{X})\big).$$

The converse is not true in general.

11.1. *Isomorphism problem.* Assume that $G = \lim.\mathrm{proj}_{\cdot 1 \leq i \leq n}(G_i)$ with common quotients G_{ij} where $N_i = \mathrm{Ker}(G \longrightarrow G_i)$ are characteristic subgroups. We assume that the isomorphism problem holds for the groups G_i. We denote by $\Gamma(G)$ the corresponding projective limit of the group rings $\mathbb{Z}G_i$. We note that $\Gamma(G)$ is an augmented algebra.

Assume that $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras. Then the hypothesis implies that we obtain isomorphisms

$$\beta_i: G_i \longrightarrow H_i \quad \text{and hence automorphisms}$$
$$\beta_i \cdot \beta_j^{-1} =: \rho_{ij} \in \mathrm{Aut}(G_{ij}),$$

which naturally gives rise to an automorphisms cocycle $\rho \in Z\big(\underline{G}, \mathrm{Aut}(\underline{G})\big)$. The above Lemma 11.2 now reads (here the converse is also true):

PROPOSITION 11.3 ([KI-RO; 93]).
1. $H \simeq G(\rho)$.
2. $H \simeq G \Longleftrightarrow \rho \in B\big(\underline{G}, \mathrm{Aut}(\underline{G})\big)$.
3. $\Gamma(G) \simeq \Gamma(H) \Longleftrightarrow \rho \in B\Big(\underline{\Gamma(G)}, \mathrm{Aut}_{\mathrm{augm}}\big(\underline{\Gamma(G)}\big)\Big)$.

We can formulate this more conceptually for the projective limit $\Gamma(G)$ as follows: We have a natural map

$$\Phi: Z\big(\underline{G}, \mathrm{Aut}(\underline{G})\big) \longrightarrow Z\Big(\underline{\Gamma(G)}, \mathrm{Aut}\big(\underline{\Gamma(G)}\big)\Big).$$

LEMMA 11.4 ([KI-RO; 93]). *Given*

$$\rho \in B\Big(\underline{\Gamma(G)}, \mathrm{Aut}_{\mathrm{augm}}\big(\underline{\Gamma(G)}\big)\Big) \cap \Im(\Phi), \quad \textit{then}$$

$$\Gamma(G) \simeq \Gamma\big(G(\rho)\big) \quad \textit{and} \quad G \simeq G(\rho) \Longleftrightarrow \rho \in \Phi\Big(B\big(\underline{G}, \mathrm{Aut}(\underline{G})\big)\Big).$$

11.2. *Zassenhaus conjecture.* Assume that $G = \lim.\text{proj}_{.1\leq i\leq n}(G_i)$ with $G_i = G/O_{p_i'}(G)$ and common quotients G_{ij}. Then $N_i = O_{p_i'}(G)$ are characteristic subgroups. The Zassenhaus conjecture is true for the groups G_i (*cf.* Theorem 5.2, [Sc; 87], [Sc; 90]). We denote by $\Gamma(G)$ the corresponding projective limit of the group rings $\mathbb{Z}G_i$.

Let $\alpha: \mathbb{Z}G \longrightarrow \mathbb{Z}G$ be an augmented automorphism. Then we obtain from the validity of the Zassenhaus conjecture a family of isomorphisms

$$\alpha_i = \gamma_i \cdot \rho_i : \mathbb{Z}G_i \longrightarrow \mathbb{Z}G_i$$

where the maps γ_i are central automorphisms of $\mathbb{Z}G_i$ and $\rho_i \in \text{Aut}(G_i)$. Since α_i and α_j are induced from the automorphism α on $\mathbb{Z}G$ the maps α_i and α_j must coincide on $\mathbb{Z}G_{ij}$:

$$\gamma_i \cdot \rho_i = \gamma_j \cdot \rho_j \quad \text{on} \quad \mathbb{Z}G_{ij},$$

i.e., the map $\rho_{ij} := \rho_i \cdot \rho_j^{-1} = \gamma_i^{-1} \cdot \gamma_j$ is a central automorphism of G_{ij}. Hence we have associated to an automorphism α of $\mathbb{Z}G$ an element

$$\rho := (\rho_{ij}) \in \check{H}\big(\underline{G}, \text{Aut}_c(\underline{G})\big). \tag{16}$$

We note that $\text{Aut}_c(-)$ is a very small subgroup of $\text{Aut}(-)$.

It should be noted that $\check{H}\big(\underline{G}, \text{Aut}_c(\underline{G})\big)$ is a purely group theoretical invariant.

Similar arguments as above hold if we start with an automorphism $\alpha: \Gamma(G) \longrightarrow \Gamma(G)$. The above Lemma 11.2 here translates into:

PROPOSITION 11.5 ([KI-RO; 93]). *The Zassenhaus conjecture holds for our automorphism $\alpha: \Gamma(G) = \Gamma(G)$ if and only if ρ is a coboundary. (I.e., there are $\rho_i \in \text{Aut}_c(G_i)$ with $\rho_{ij} = \rho_i \cdot \rho_j^{-1}$.)*

Note again that this is a purely group theoretical question.

This can now be used to construct a semi-local counterexample for the Zassenhaus conjecture for $\Gamma(G)$.

We have a natural map

$$\Phi: Z\big(\underline{G}, \text{Aut}(\underline{G})\big) \longrightarrow Z\Big(\underline{\Gamma(G)}, \text{Aut}\big(\underline{\Gamma(G)}\big)\Big).$$

We also have a natural map

$$\Xi: Z\big(\underline{G}, \text{Aut}_c(\underline{G})\big) \longrightarrow Z\big(\underline{G}, \text{Aut}(\underline{G})\big).$$

LEMMA 11.6 ([KI-RO; 93]). *Given $\rho \in Z\big(\underline{G}, \text{Aut}_c(\underline{G})\big)$ (cf. Equation 16).*

1. *Then $\Gamma(G) \simeq \Gamma(G(\rho)) \Longleftrightarrow \rho\Xi \cdot \Phi \in B\Big(\underline{\Gamma(G)}, \text{Aut}_{\text{augm}}\big(\underline{\Gamma(G)}\big)\Big)$.*
2. *$G \simeq G(\rho) \Longleftrightarrow \rho\Xi \in B\big(\underline{G}, \text{Aut}(\underline{G})\big)$.*
3. *The Zassenhaus conjecture holds for $\Gamma(G) \Longleftrightarrow \rho \in B\big(\underline{G}, \text{Aut}_c(\underline{G})\big)$.*

11.3. *The automorphism problem.* Interpreting a finite solvable group G as a projective limit $G = \lim.\mathrm{proj}.(G/O_{p_i'}(G))$ has been very successfully applied in dealing with the Zassenhaus conjecture and the isomorphism problem (*cf.* above). A similar theory as above does exist for $\mathrm{Ker}(\Phi_R(G))$ (*cf.* [Marc-Ro; 97]). As above, $\mathcal{K}$ is either the category of groups or of rings and X is the projective limit of $\{X_i\}_{1\leq i\leq n}$ (*cf.* Definition 11.1). We write $Z(X)$ for the center of X.

DEFINITION 11.7.
1. The central cocycles[21] are defined as:

$$Z(\underline{X},\mathbb{Z}) = \{(\gamma) = (\gamma_{ij}) : \gamma_{ij} \in Z(X_{ij}) \mid \gamma_{ij} = \gamma_{ji}^{-1}, \gamma_{ii} = 1\}.$$

2. The central coboundaries are defined as

$$B_c(\underline{X},\mathbb{Z}) = \{(\gamma_{ij}) \in Z(\underline{X},\mathbb{Z}) \mid \gamma_{ij} = \gamma_i \cdot \gamma_j^{-1} \text{ for } \gamma_i \in Z(X_i)\}.$$

3. The coboundaries are defined as

$$B(\underline{X},\mathbb{Z}) = \{(\gamma_{ij}) \in Z(\underline{X},\mathbb{Z}) \mid \gamma_{ij} = \gamma_i \cdot \gamma_j^{-1} \text{ for } \gamma_i \in X_i\}.$$

   Every central coboundary is then a coboundary.
4. In case X is a group we define the locally central coboundaries as

$$B_{lc}(\underline{X},\mathbb{Z}) = \{(\gamma_{ij}) \in Z(\underline{X},\mathbb{Z}) \mid \text{for each } x = (x_i) \in X \text{ we have } \gamma_{ij} = \gamma_i(x) \cdot \gamma_j^{-1}(x) \text{ for } \gamma_i(x) \in C_{X_i}(x_i)\}.$$

5. Two cocycles are said to be centrally equivalent provided

$$(\gamma_{ij}) \sim (\gamma'_{ij}) \iff \gamma_i \cdot \gamma_{ij} \cdot \gamma_j^{-1} = \gamma'_{ij} \quad \text{for} \quad \gamma_i \in \mathbb{Z}(X_i).$$

6. The equivalence classes of the central cocycles modulo central coboundaries are denoted by

$$\check{H}_c(\underline{X},\mathbb{Z}(\underline{X})).$$

   This is a pointed set with the class of the central coboundaries being the distinguished point.

We now assume that $X = \lim.\mathrm{proj}.(X_i) \in \mathcal{K}$ where $\mathcal{K}$ is either the category of groups or of rings. We assume that $N_i = \mathrm{Ker}(X \to X_i)$ are characteristic sub-objects in X.

DEFINITION 11.8. We denote by $\mathrm{Aut}_{li}(X)$ the group of locally inner automorphisms, *i.e.*, of those automorphisms $\rho \in \mathrm{Aut}(X)$ which induce inner automorphisms $\rho_i = \mathrm{conj}(\gamma_i(\rho))$ on all of X_i. These give rise to coboundaries

$$\gamma(\rho) = (\gamma_{ij}(\rho)) = (\gamma_i(\rho) \cdot \gamma_j^{-1}(\rho)) \quad \text{with} \quad \gamma_{ij} \in Z(X_{ij}).$$

[21] This should not be confused with the central automorphism cocycles from Definition 10.2.

LEMMA 11.9. *Let us be given a central cocycle* $\gamma = (\gamma_{ij}) \in Z(\underline{X}, \mathbb{Z})$.

1. *There exists* $\rho \in \mathrm{Aut}_{li}(X)$ *with* $\gamma(\rho) = \gamma$ *if and only if* γ *is a coboundary. We will denote this automorphism by* $\rho(\gamma)$.
2. *If* γ *is a coboundary, then* $\rho = \rho(\gamma)$ *is inner if and only if* γ *is a central coboundary.*
3. *For groups we have in addition: Let* γ *be a coboundary. Then* $\rho = \rho(\gamma) \in \mathrm{Aut}_c(X)$ *if and only if* γ *is a locally central coboundary.*
4. *Let* γ *and* δ *be two coboundaries. Then the corresponding automorphisms* $\rho(\gamma)$ *and* $\rho(\delta)$ *differ by an inner automorphism if and only if* γ *and* δ *are centrally equivalent.*

We shall assume from now on that G is a finite solvable group and we shall write $\Phi = \Phi_Z(G)$. We assume that $G = \mathrm{lim.proj.}(G_i)$. Let us be given an automorphism $\rho \in \mathrm{Ker}(\Phi)$. We assume that ρ is inner on G_i. From Note 7.1 and Equation 10.1 it follows that this is the case, provided $G_i = G/O_{p_i'}(G)$. The homomorphism ρ then gives rise to conjugating elements γ_i on G_i and the cocycle $\gamma_{ij} = \gamma_i \cdot \gamma_j^{-1}$. From Note 7.1 we also know that ρ is central and that ρ^2 is inner.

PROPOSITION 11.10. *Let* $G = \mathrm{lim.proj.}(G_i)$ *and assume that* $\rho \in \mathrm{Ker}(\Phi)$ *has the property that it is inner on each* G_i*—this holds for example when* $G_i = G/O_{p_i'}(G)$. *Then* ρ *can be modified by an inner automorphism so that* $\gamma_{ij}(\rho) \in Z(G_{ij})$ *are involutions. In particular, if for all pairs* i, j *the center of* G_{ij} *is odd, then* $\mathrm{Ker}(\Phi) = 1$.

The kernel of our homomorphism $\Phi\colon \mathrm{Out}(G) \to \mathrm{Out}(\mathbb{Z}G)$ has an interesting interpretation in terms of $\widehat{G} = \prod_i G_i$, inside of which the group $G = \mathrm{lim.proj.}(G_i)$ is naturally embedded.

LEMMA 11.11. $\mathrm{Ker}(\Phi) \neq 1$ *if and only if there exist a group element* $\widehat{g} \in N_{\widehat{G}}(G) \setminus \big(C_{\widehat{G}}(G) \cdot G\big)$ *and a central unit* $u \in \mathbb{Z}\widehat{G}$ *such that* $u \cdot \widehat{g} \in \mathbb{Z}G$.

For further application of this Čech cohomology set up we refer the reader to the paper [Marc-Ro; 97].

PART IV: EXTENDING THE ISOMORPHISM PROBLEM TO MODULE CATEGORIES

12. The categories and variations on the isomorphism problem. Here we shall deal with the following categories, their isomorphisms and automorphisms.

DEFINITION 12.1 (THE CATEGORIES).

1. $\mathcal{K}_1^R(G)$ stands for the category of groups and isomorphisms. Of particular interest besides $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$ and $\mathrm{Out}(G)$ is the group $\mathrm{Aut}_c(G)$ of conjugacy class preserving automorphisms, and also $\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\mathrm{Inn}(G)$.

2. $\mathcal{K}_2^R(G)$ stands for the category of group rings and isomorphisms which commute with the augmentation. Apart from the natural groups $\mathrm{Aut}_n(RG)$, $\mathrm{Inn}(RG)$ and $\mathrm{Out}_n(RG)$, the group $\mathrm{Aut}_c(RG)$ of automorphisms which leave the center elementwise fixed, and also $\mathrm{Out}_c(RG) = \mathrm{Aut}_c(RG)/\mathrm{Inn}(G)$, play an exceptional role in connection with the Zassenhaus conjecture.
3. $\mathcal{K}_3^R(G)$ stands for the category of finitely generated R-projective RG-modules ${}_{RG}\mathcal{M}^0$ and Morita equivalences. The auto-equivalences are given by invertible bi-modules. For a subring $R \subseteq S \subseteq Z(RG)$ we write $\mathrm{Pic}_S(RG)$ for those isomorphism classes of invertible bi-modules M with $s \cdot m = m \cdot s$ for $s \in S$, $m \in M$ (*cf.* [Fro; 73]); in particular we write $\mathrm{Piccent}(RG) = \mathrm{Pic}_{\mathrm{cent}}(RG)$ for the isomorphism classes of central invertible bi-modules.
4. $\mathcal{K}_4^R(G)$ stands for the derived category of bounded complexes and Rickard equivalences (*cf.* [Ri; 89]).
5. $\mathcal{K}_5^R(G)$ stands for the stable category of ${}_{RG}\mathcal{M}^0$ together with stable equivalences of Morita type (*cf.* [Li; 91]).

REMARK 12.2.
1. It is obvious that for $i = 1, \ldots, 4$ an isomorphism $\mathcal{K}_i^R(G) \simeq \mathcal{K}_i^R(G)$ implies an isomorphism $\mathcal{K}_{i+1}^R(G) \simeq \mathcal{K}_{i+1}^R(G)$.
2. The isomorphism problem and its various modifications deal with the converse implications.
3. For finite groups and $\mathrm{char}(R)$ prime to $|G|$ the group $\mathrm{Aut}_c(RG)$ consists, by the theorem of Skolem Noether, of the conjugations with $a \in KG$ normalizing RG. In case the group G is not finite these two concepts do not necessarily coincide.
4. Let us briefly recall the notion of a stable equivalence of Morita type: Let $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ be two module categories of connected noetherian rings A and B. These categories are said to be stably equivalent of Morita type, provided there are bi-modules ${}_AM_B$ and ${}_BN_A$ and an isomorphism of bi-modules

$$ {}_AM_B \otimes_B {}_BN_A \simeq {}_AA_A \oplus {}_AP_A \quad \text{and} \quad {}_BN_A \otimes_A {}_AM_B \simeq {}_BB_B \oplus {}_BQ_B $$

where ${}_AP_A$ and ${}_BQ_B$ are projective bi-modules. The stable equivalence is then induced by tensoring with ${}_AM_B$ and ${}_BN_A$ resp.

Before we come to the generalizations of the isomorphism problem we would like to point out one problem, a positive solution of which would be of great importance in constructing automorphisms from local data.

Let us assume that G is a finite group and let $R = \mathrm{alg.int.}(K)$ for a global number field K. Then we have Fröhlich's commutative diagram with exact rows and columns (*cf.* [Fro; 73]).

$$
\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Cl}_{\mathbb{Z}G}(Z(\mathbb{Z}G)) & \longrightarrow & \mathrm{Out}_c(\mathbb{Z}G) & \longrightarrow & \mathrm{Out}_c(\mathbb{Z}_{\pi(G)}G) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \| & \\
0 \longrightarrow & \mathrm{Cl}(Z(\mathbb{Z}G)) & \longrightarrow & \mathrm{Piccent}(\mathbb{Z}G) & \longrightarrow & \prod_{p\in\pi(G)} \mathrm{Piccent}(\mathbb{Z}_{\pi(G)}G) &
\end{array} ,
$$

where $\mathrm{Cl}(Z(\mathbb{Z}G)$ is the class group—*i.e.*, the group of the isomorphism classes of invertible ideals—of the center of $\mathbb{Z}G$, and $\mathrm{Cl}_{\mathbb{Z}G}(Z(\mathbb{Z}G)$ is the subgroup of the classes of those ideals $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathbb{Z}G$ is a principal two-sided ideal. Moreover, $\mathbb{Z}_{\pi(G)}$ is the semi-localization at the prime divisors of $|G|$.

The point now is that for invertible bi-modules we have a "local-global" principle, *i.e.*, the map

$$\mathrm{Piccent}(\mathbb{Z}G) \longrightarrow \mathrm{Piccent}(\mathbb{Z}_{\pi(G)}G)$$

is surjective. It is by no means clear that we have a local-global principle for automorphisms, too.

PROBLEM 12.3 (LOCAL-GLOBAL). Given a central automorphism $\alpha_{\pi(G)}$, does there exist a central automorphism α which localizes to $\alpha_{\pi(G)}$?

Let us interpret this problem differently, then it will seem more likely to be true: since $\alpha_{\pi(G)}$ corresponds bijectively to a central invertible $\mathbb{Z}_{\pi(G)}$ bi-module $M_{\pi(G)}$ there exists a central invertible $\mathbb{Z}G$ bi-module M which is mapped to $M_{\pi(G)}$. But then all the bi-modules in $\mathrm{Cl}\big(Z(\mathbb{Z}G)\big) \cdot M$ are also mapped to $M_{\pi(G)}$. A positive answer to the above problem is tantamount to the fact that in the fiber $\mathrm{Cl}\big(Z(\mathbb{Z}G)\big) \cdot M$ there exists an element M_0 such that M_0 is free of rank one on one side.

We now turn to the isomorphism problems for the various module categories for $i = 2, \ldots, 5$:

PROBLEM 12.4 (PROBLEM(I,R)). Are there two groups G and H such that we have for the categories

$$\mathcal{K}_i^R G \simeq \mathcal{K}_i^R H \quad \text{but} \quad \mathcal{K}_{i-1}^R G \not\simeq \mathcal{K}_{i-1}^R H \quad 2 \le i \le 5.$$

REMARK 12.5.

1. Hertweck's counterexample (*cf.* Theorem 6.3) shows that Problem(2,$\mathbb{Z}$) is false.
2. Little is known for the Problem(i,$\mathbb{Z}$) for $i > 2$.
3. J. Rickard [Ri; 89] (*cf.* also [Li; 91]) has shown that blocks of cyclic defect are derived equivalent provided their centers coincide. Hence p-adically Problem(4,R) has a negative answer since the module category of a block corresponding to a star is not equivalent to the module category of a block corresponding to a stem.
4. I myself do not know—even over the p-adic integers—an example where Problem(5,R) has a negative answer.

There is one instance, however, which shows that p-groups are not only rigid but even very rigid:

THEOREM 12.6 ([Li; 96], [Ro-Zi, 95], [Zi; 97]). *Let P and Q be finite p-groups, and assume that there is a stable equivalence of Morita type between the categories of $\widehat{\mathbb{Z}}_pP$-lattices and of $\widehat{\mathbb{Z}}_pQ$-lattices. Then P and Q are isomorphic. (This holds even more for $\mathbb{Z}P$ and $\mathbb{Z}Q$.)*

Also the Zassenhaus conjecture has analogues for the automorphisms of the various categories $\mathcal{K}_i^R(G)$. First we have to generalize the notion of "augmented automorphism". We are assuming that R is an integral domain of characteristic zero.

DEFINITION 12.7.

1. An auto-equivalence of the module category, the derived category, or the stable category of RG respectively is said to be augmented, provided it maps the trivial module to the trivial module.

NOTE 12.8. As an arbitrary automorphism can be modified to become augmented, the same holds also for auto-equivalences:

1. If the module M is an invertible bi-module such that $M \otimes_{RG} R \simeq R_0$, then R_0 is invertible and we replace M with $R_0^{-1} \otimes_R M$.
2. If the complex $M^\bullet$ induces an equivalence of the derived category such that $M^\bullet \otimes_{RG}^\bullet R \simeq R_0^\bullet$, then $R_0^\bullet$ is an invertible complex and we replace $M^\bullet$ by $R_0^{\bullet\,-1} \otimes_R^\bullet M^\bullet$. Here we have identified R with the complex concentrated in degree 0.
3. If the module M is a bi-module, inducing a stable equivalence of Morita type such that $M \otimes_{RG} R \simeq R_0 \oplus P$ for some projective P, then R_0 is invertible and we replace M by $R_0^{-1} \otimes_R M$—note that $X \otimes_R P$ is projective for every R-free RG-module X.

DEFINITION 12.9.

1. $\mathrm{Out}_n(RG)$ stands for the group of classes of augmented automorphisms, and $\mathrm{Out}_c(RG)$ is the group of classes of automorphisms leaving the center elementwise fixed.
2. $\mathrm{Pic}_n(RG)$ denotes the classes of invertible augmented bi-modules, and $\mathrm{Piccent}(RG)$ denotes the isomorphism classes of invertible bi-modules where the center acts in the same way on both sides.
3. $\mathrm{Pic}_n^\bullet(RG)$ denotes the isomorphism classes of augmented invertible complexes, and $\mathrm{Piccent}^\bullet(RG)$ denotes the isomorphism classes of invertible complexes consisting of those bi-modules where the center acts in the same way on both sides.
4. $\mathrm{Pic}_n(\underline{RG})$ are the isomorphism classes of augmented bi-modules which induce a stable equivalence of Morita Type, and $\mathrm{Piccent}(\underline{RG})$ denotes the isomorphism classes of those bi-modules where the center acts in the same way on both sides.

PROBLEM 12.10 (ZASSENHAUS PROBLEM).
Zass 2: For which finite groups does the equation

$$\mathrm{Out}_n(\mathbb{Z}G) = \mathrm{Out}(G) \cdot \mathrm{Out}_c(\mathbb{Z}G) \text{ hold?}$$

(Note that this is the Zassenhaus conjecture.)
Zass 3: For which finite groups does the equation

$$\mathrm{Pic}_n(\mathbb{Z}G) = \mathrm{Out}(G) \cdot \mathrm{Piccent}(\mathbb{Z}G) \text{ hold?}$$

Zass 4: For which finite groups does the equation

$$\mathrm{Pic}_n^\bullet(\mathbb{Z}G) = \mathrm{Out}(G) \cdot \mathrm{Piccent}^\bullet(\mathbb{Z}G) \text{ hold?}$$

Zass 5: For which finite groups does the equation

$$\mathrm{Pic}_n(\underline{\mathbb{Z}G}) = \mathrm{Out}(G) \cdot \mathrm{Piccent}(\underline{\mathbb{Z}G}) \text{ hold?}$$

REMARK 12.11. Very little is known about the answers.[22] A special case is treated in [Rou-Zi; 96].

We can weaken these problems by asking for which groups the augmented auto-equivalences of the category $\mathcal{K}_i^{\mathbb{Z}}(G)$ are induced from the central auto-equivalences and the auto-equivalences induced from $\mathcal{K}_{i-1}^{\mathbb{Z}}$ for $i < 1$.

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C. R. Math. Rep. Acad. Sci. Canada Vol. 21 (4), 1999 pp. 127–131

A NEW CLASS OF HOMOGENEOUS MANIFOLDS WITH LIOUVILLE-INTEGRABLE GEODESIC FLOWS

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ABSTRACT. The existence of a family of nilmanifolds which possess riemannian metrics whose cogeodesic flow is Liouville-integrable is demonstrated. These homogeneous spaces are of the form $D \setminus H$, where H is a connected, simply-connected and two-step nilpotent Lie group and D is a discrete, cocompact subgroup of H. The metric on these homogeneous spaces is obtained from a left-invariant metric on H. The topology of these nilmanifolds is quite rich; the first example of a Liouville-integrable geodesic flow on a manifold whose fundamental group possesses no commutative subgroup of finite index is obtained. It is shown that the conclusions of Taĭmanov's theorem [6] do not obtain in the category of Liouville-integrable geodesic flows with smooth first integrals.

RÉSUMÉ. On construit une famille de nilvariétés qui possède une métrique riemannienne avec un flot cogéodésique qui est intégrable [au sens de Liouville]. Ces espaces homogènes sont des quotients $D \setminus H$ où H est un groupe de Lie connexe et simplement connexe et D est un sousgroupe discret et cocompact de H. La métrique sur $D \setminus H$ vient d'une métrique sur H invariante à gauche. La topologie de ces espaces est très riche; en particulier, ils sont les premiers examples des variétés avec un group fondamentale non-presque-abelien qui possède un flot cogéodésique intégrable. On démontre que les conclusions du théorème de Taĭmanov [6], concernant la topologie d'un tel espace, sont fausses si les intégrales du flot cogéodésique ne sont que C^∞.

1. Introduction. In [4], Gabriel Paternain conjectures that

CONJECTURE 1. If (M, g) is a compact riemannian manifold whose cogeodesic flow is completely integrable, then $\pi_1(M)$ has polynomial growth; if $\pi_1(M)$ is finite then M is rationally elliptic.

The Euler-Lagrange equations for the metric g induce a hamiltonian vector field on T^*M. This vector field preserves the energy $2H(v) = g(v, v)$; if it possesses $n - 1 = \dim M - 1$ additional first integrals that are pairwise in involution and are functionally independent on an open, dense subset of T^*M, then

Received by the editors 28 October, 1998.
The author would like to thank Oleg Bogoyavlenskij, Leo Jonker and Dan Offin for their comments; and the National Sciences and Engineering Research Council of Canada for its financial support. All errors and omissions are solely the responsibility of the author.
AMS subject classification: Primary: 58F17, 58F07; secondary, 58F05.
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the cogeodesic flow is said to be completely or Liouville integrable (*e.g.* [1]). Liouville-integrable hamiltonian vector fields (flows) are especially nice because the generic trajectory is a quasi-periodic, translation-type flow on an embedded torus $\mathbb{T}^k \hookrightarrow T^*M$, where $k \le n$.

In the case where the $n = \dim M$ first integrals of the cogeodesic flow are constrained to be real analytic, Taĭmanov [6] has proven that the fundamental group of M must be almost abelian. This theorem implies that

THEOREM 1 (TAĬMANOV[6]). *$\pi_1(M)$ has polynomial word growth of degree less than or equal to $\dim M$ if M possesses a metric whose geodesic flow is Liouville integrable with real-analytic first integrals. Moreover, there is an injection of algebras $H^*(\mathbb{T}^d;\mathbb{Q}) \hookrightarrow H^*(M;\mathbb{Q})$ where $d = \dim H^1(M;\mathbb{Q})$.*

This paper presents the first example of compact riemannian manifolds which admit an integrable cogeodesic flow with C^∞ first integrals for which the two conclusions of Taĭmanov's theorem are false. Section 2.1 discusses these manifolds and their topologies, while Section 2.2 demonstrates the Liouville integrability of the cogeodesic flows of certain riemannian metrics.

2. Results.

2.1. *The Groups $\mathcal{H}_p$.*

Define the groups $\mathcal{H}_p$ by putting following multiplication law on the set $\mathbb{R}^1 \times \mathbb{R}^p \times \mathbb{R}^p$

$$(x,y,z)\cdot(x',y',z') := (x+x', y+y', z+z'+xy'). \tag{1}$$

The case $p = 1$ is the classical Heisenberg group. The coordinates (x,y,z) are known as coordinates of the second kind; coordinates of the first kind will not be needed for our purposes.

2.1.1. Subgroups of $\mathcal{H}_p$.

LEMMA 1. *Let $D_p(k_1,\dots,k_p)$ be the subgroup of $\mathcal{H}_p$ that is generated by the elements $a = (1,0,0)$, $b_i = (0,e_i,0)$ and $c_i = (0,0,k_i^{-1}e_i)$ where $k_i \ge 1$ are integers, $i = 1,\dots,p$, and e_i is the i-th standard basis vector of $\mathbb{R}^p$. Then $D_p(k_1,\dots,k_p)$ is a discrete, cocompact subgroup of $\mathcal{H}_p$. Moreover, if D is a discrete, cocompact subgroup of $\mathcal{H}_p$ then D is isomorphic to some $D_p(k_1,\dots,k_p)$.*

PROOF. ($\Rightarrow$) The additive group in Lie($\mathcal{H}_p$) that is generated by the logarithm (= the inverse of the exponential map) of the group $D_p(k_1,\dots,k_p)$ is a lattice isomorphic to $\mathbb{Z}^{2p+1}$. It is known that the compactness of Lie$(\mathcal{H}_p)/\log D_p(k_1,\dots,k_p)$ is equivalent to $D_p(k_1,\dots,k_p)$ being discrete and cocompact.

($\Leftarrow$) If D is a discrete, cocompact subgroup of $\mathcal{H}_p$, then there is a rank $2p$ normal abelian subgroup N of D such that $D/N \simeq \mathbb{Z}$. Let $x \in D$ be an element such that xN generates the quotient group D/N. The subgroup $N = Z(D) \oplus M$

where $Z(D)$ is the center of D and M is a commutative rank p subgroup of N. The element x commutes with only the identity in M. Let $y_1, \ldots, y_p$ be generators of M, so that the commutators $[x, y_1], \ldots, [x, y_p]$ generate a rank p subgroup of $Z(D)$. Let $z_i = k_i^{-1}[x, y_i]$ be elements in $Z(D)$ such that $(k_i + 1)^{-1}[x, y_i] \notin Z(D)$ and $k_i \geq 1$. By the discreteness of D, such k_i exist. The elements $x, y_1, \ldots, y_p$, $z_1, \ldots, z_p$ form a generating set of D and the obvious map makes D isomorphic to $D_p(k_1, \ldots, k_p)$. ∎

REMARK. A system of coordinates of the second kind (x, y, z) on $\mathcal{H}_p$ will be said to be *compatible* with the discrete, cocompact group D if, in these coordinates, there are generators of D that give D a presentation as some $D_p(k_1, \ldots, k_p)$. It is apparent that the above lemma permits the construction of compatible coordinates for any such D.

THEOREM 2. *The fundamental groups of $D_p(k_1, \ldots, k_p)\backslash\mathcal{H}_p$ have polynomial word growth of degree $3p+1$; they also possess no abelian subgroup of finite index.*

PROOF. In [7] it is shown that if Γ is a finitely generated 2-step nilpotent group with lower central series $1 = \Gamma_2 \trianglelefteq \Gamma_1 \trianglelefteq \Gamma = \Gamma_0$ such that $\Gamma_{i+1} = [\Gamma, \Gamma_i]$ and $\Gamma_i/\Gamma_{i+1} \simeq A_i \times T_i$ where A_i is free abelian with rank n_i and T_i is the torsion subgroup, then the word growth function is asymptotically $W_\Gamma(\lambda) \sim \lambda^{n_0+2n_1}$. In our case, the lower central series of $D_p(k_1, \ldots, k_p)$ has $n_0 = p+1$ and $n_1 = p$. ∎

2.1.2. The cohomology ring of $\mathcal{H}_p$.

LEMMA 2. *Let $D_p(k_1, \ldots, k_p) =: D$ be as above. Then there is a frame of left-invariant 1-forms on $\mathcal{H}_p$, α, $\beta_1, \ldots, \beta_p$, $\gamma_1, \ldots, \gamma_p$ such that*

$$d\alpha = d\beta_i = 0, \quad d\gamma_i = -\alpha \wedge \beta_i.$$

Furthermore, let $N \trianglelefteq D$ be the rank $2p$ abelian subgroup generated by $b_1, \ldots, b_p$, $c_1, \ldots, c_p$. Then, the real cohomology ring of $N\backslash\mathcal{H}_p \simeq \mathbb{R} \times \mathbb{T}^{2p}$ is generated by the 1-forms $dy_i, dz_i : i = 1, \ldots, p$ and the first real cohomology group of $D\backslash\mathcal{H}_p$ is generated by $\alpha, \beta_1, \ldots, \beta_p$.

PROOF. Take the logarithm of the generators of D in the above lemma; these are left-invariant vector fields on $\mathcal{H}_p$ that form a basis of $\mathrm{Lie}(\mathcal{H}_p)$: $X := \log a$, $Y_i := \log b_i$, $Z_i := k_i \log z_i$. The basis α, β_i, γ_i of $\mathrm{Lie}(\mathcal{H}_p)^*$ that is dual to this frame of vector fields furnishes the required 1-forms. ∎

The lemma implies that a second conclusion of Taĭmanov's theorem is violated, too.

THEOREM 3. *There is no injection $H^*(\mathbb{T}^{p+1}; \mathbb{Q}) \hookrightarrow H^*(\mathcal{Q}_{p,k}; \mathbb{Q})$.*

In the sequel, we will suppose that the discrete, cocompact group D and its maximal abelian subgroup N are fixed and denote by $\mathcal{Q}_{p,k} := D\backslash\mathcal{H}_p$ and $\mathcal{C}_p := N\backslash\mathcal{H}_p$.

2.2. *The cogeodesic vector field.* The metric $ds^2 = \alpha\otimes\alpha+\sum_{i=1}^{p}\beta_i\otimes\beta_i+\gamma_i\otimes\gamma_i$ is left-invariant and diagonal in coordinates on $\mathcal{H}_p$ compatible with D. The Euler-Lagrange equations for ds^2 induce the following hamiltonian vector field on $T^*\mathcal{H}_p$

$$X_H = \begin{cases} \dot{p}_x = -\sum_{i=1}^{p}(p_{y_i}+xp_{z_i})p_{z_i}, & \dot{x} = p_x, \\ \dot{p}_{y_i} = 0, & \dot{y}_i = p_{y_i}+xp_{z_i}, \\ \dot{p}_{z_i} = 0, & \dot{z}_i = p_{z_i}+x(p_{y_i}+xp_{z_i}). \end{cases} \tag{2}$$

Where (x, y, z) are coordinates of the second kind that are compatible with D and (p_x, p_y, p_z) are induced by the frame of 1-forms dx, dy, dz. The hamiltonian of this vector field is

$$2H = p_x^2 + \sum_{i=1}^{p}(p_{y_i}+xp_{z_i})^2 + p_{z_i}^2, \tag{3}$$

which possesses the $2p$ real-valued first integrals $f_i = p_{y_i}$, $g_i = p_{z_i}$.

2.2.1. Liouville integrability of X_H. Let $(x, y, z, p_\alpha, p_\beta, p_\gamma)$ be coordinates of the second kind that are compatible with D and induced by the frame of 1-forms α, β, γ. These coordinates are non-canonical, but they descend to the quotient $T^*(\mathcal{Q}_{p,k}) \simeq \mathcal{Q}_{p,k} \times \mathrm{Lie}(\mathcal{H}_p)^*$. In these coordinates the first integrals are $f_i = p_{\beta_i} - xp_{\gamma_i}$, $g_i = p_{\gamma_i}$. Because $\alpha = dx$ descends to the quotient $\mathcal{Q}_{p,k}$ as a closed, non-exact 1-form, the function x is a $\mathbb{T}^1$-valued, smooth function on the compact quotient. Let $G(u, v) := \phi_\epsilon(u)\sin 2\pi\frac{v}{u}$, where $\phi_\epsilon \in C^\infty(\mathbb{R}, \mathbb{R})$, ϕ_ϵ is unity on the complement of $(-\epsilon, \epsilon)$, is $O(\exp(-1/x^2))$ at zero, zero is the unique minimum value of ϕ_ϵ and this occurs at 0. Then, the functions $h_i := G(g_i, f_i)$, $g_i \in C^\infty(T^*(\mathcal{Q}_{p,k}), \mathbb{R})$; moreover, they are pairwise in involution and functionally independent on an open subset of the phase space. This proves that

THEOREM 4. *the vector field X_H is Liouville integrable on the cotangent bundle $T^*(C_p)$ with first integrals $f_i, g_i : i = 1, \ldots, p$ and H. The vector field X_H descends to $T^*(\mathcal{Q}_{p,k})$ and is Liouville integrable with first integrals $h_i, g_i : i = 1, \ldots, p$ and H.*

2.2.2. Quasiperiodicity of the cogeodesic flow. It is common for integrable cogeodesic flows on homogeneous manifolds to possess a non-commutative algebra of first integrals; the cogeodesic flow of a left-invariant metric on SO(3) is such an example. In this example,

THEOREM 5. *The cogeodesic flow φ_t is, generically, a quasi-periodic winding on invariant tori $\mathbb{T}^{2p+1} \subset T^*(\mathcal{Q}_{p,k})$ for $p \geq 2$. For $p = 1$ the flow is generically a quasi-periodic winding on 2-dimensional invariant tori.*

PROOF. Consider a point $q \in T^*(\mathcal{Q}_{p,k})$ and the flow of X_H through q, $\varphi_t(q)$. By the path-lifting property of the covering $\pi: T^*(C_p) \to T^*(\mathcal{Q}_{p,k})$, for each $Q \in \pi^{-1}q$ there is a trajectory $\vartheta_t(Q)$; it is elementary to show that the flow

ϑ_t is the solution to the differential equation $\dot{Q} = X_H$. If q is a regular point for the first-integral mapping $x \to (h_1(x), \ldots, h_p(x), g_1(x), \ldots, g_p(x), H(x))$ then q must lie on an φ_t-invariant torus in $T^*(\mathcal{Q}_{p,k})$; Q must lie on a ϑ_t-invariant torus in $T^*(\mathcal{C}_p)$, too. Moreover, the latter torus covers the former. Hence, if the trajectory $\vartheta_t(Q)$ is dense in the invariant torus, the same must be true for $\varphi_t(q)$.

The equations of motion of the cogeodesic flow ϑ_t on $T^*(\mathcal{C}_p)$ (2) can be readily integrated by introducing canonical action-angle coordinates [2]. A second computation reveals that the transformation from action coordinates to the frequencies is a local diffeomorphism almost everywhere when $p \geq 2$. For $p = 1$, the cogeodesic flow φ_t enjoys the additional first integral $h = \phi_\epsilon(p_\gamma) \sin 2\pi(\frac{p_\beta}{p_\gamma} + y)$. Similar computations show that the cogeodesic flow on the common level set of the four first integrals H, f_1, g_1, h is a quasi-periodic winding on two-dimensional invariant tori. ∎

Using the Liouville integrability of the cogeodesic flow on $T^*(\mathcal{Q}_{p,k})$, along with the techniques of symplectic reduction, it is possible to show [2].

THEOREM 6 (VANISHING OF THE TOPOLOGICAL ENTROPY). *The topological entropy of the cogeodesic flow φ_t is zero [5].*

This is of interest because it is currently unknown if all integrable geodesic flows have non-vanishing topological entropy.

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C. R. Math. Rep. Acad. Sci. Canada Vol. **21** (4), 1999 pp. 132–138

GENERALIZED SUBDIFFERENTIALS: A BAIRE CATEGORICAL APPROACH

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ABSTRACT. In this report we describe how to use Baire categorical arguments to construct certain pathological locally Lipschitz functions. The origins of this approach can be traced back to Banach and Mazurkiewicz (1931) who independently used a similar categorical approach to show that "almost every continuous real-valued function defined on [0, 1] is nowhere differentiable". As with the results of Banach and Mazurkiewicz, it appears that it is easier to show that almost every function possesses a certain property than to construct a single concrete example. After describing the basic framework of the construction the paper goes on to give numerous applications. Perhaps among the most surprising results is the following: "Almost every 1-Lipschitz function defined on a Banach space has a Clarke subdifferential mapping that is identically equal to the dual ball".

RÉSUMÉ. Dans ce compte rendu, nous décrivons comment utiliser des arguments de type catégorie de Baire pour construire certaines fonctions localement Lipschitz pathologiques. Les origines de cette approche remontent à Banach et Mazurkiewicz (1931) qui, de manière indépendante, utilisèrent une approche semblable pour montrer que "presque toutes les fonctions réelles continues définies sur [0, 1] sont nulle part différentiables". Comme dans les résultats de Banach et Mazurkiewicz, il s'avère qu'il est plus facile de montrer que presque toutes les fonctions possèdent une certaine propriété que de construire un seul exemple concret. Après une description du cadre de base de la construction, l'article continue en donnant de nombreuses applications. Parmis les résultats les plus surprenants, citons le suivant : "Presque toutes les fonctions 1-Lipschitz définies sur un espace de Banach admettent une application sous-différentielle de Clarke qui est identiquement égale à la boule unité dans l'espace dual".

1. Introduction. In this paper we report on a general method for constructing examples and counter-examples for the differentiability theory of Lipschitz functions. In 1981 R. T. Rockafellar noted an example of a (badly) nowhere C^1-Lipschitz function on $\mathbb{R}$, while here we demonstrate, *à la* Banach and Mazurkiewicz, that this result is typical for locally Lipschitz functions defined on Banach

Received by the editors January 25, 1999.

The first author's research was supported by NSERC and Simon Fraser University. The second author's research was supported by a Marsden fund grant, VUW 703, administered by the Royal Society of New Zealand.

AMS subject classification: 49J52, 26A27.

Key words and phrases: subdifferentials, differentiability, Baire category, Banach spaces, upper semi-continuous set-valued map.

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spaces.

The first and most crucial step towards achieving this result is to produce a candidate complete metric space on which we may apply the Baire category theorem. Here we consider the class of T-Lipschitz functions. Loosely speaking, for a weak* cusco $T: X \to 2^{X^*}$ from a Banach space X into its dual X^*, we say that a locally Lipschitz function f on X is *T-Lipschitz* if $\nabla f(x) \in T(x)$ whenever $\nabla f(x)$ exists. We show that for a fixed weak* cusco T the set of all T-Lipschitz functions form a complete metric space (under an appropriately defined metric). This simple result provides the basis which enables us to derive a plentiful supply of both examples and counter-examples.

NOTATION. For a normed linear space $(X, \|\cdot\|)$, we denote by X^* its dual space;

$$B_X := \{x \in X : \|x\| \le 1\}; \quad S_X := \{x \in X : \|x\| = 1\};$$
$$B_{X^*} := \{x^* \in X^* : \|x^*\| \le 1\}; \quad B_\delta(x) := \{y \in X : \|x - y\| < \delta\}.$$

For a non-empty subset E of X^* we denote by $\overline{E}^{w^*}$ the weak* closure of E; $\overline{\text{co}}^{w^*} E$ the weak* closed convex hull of E.

We begin by recalling some preliminary definitions and properties regarding subdifferentials and upper semi-continuity.

USCOS AND CUSCOS. Let T be a set-valued mapping from a topological space A into the dual of a normed linear space X. We say that T is *weak* upper semi-continuous* on A if for each weak* open subset W of X^*, $\{x \in A : T(x) \subseteq W\}$ is open in A. When the images of T are non-empty and compact we call T a *weak* usco* and if, in addition, the images of T are also convex then we call T a *weak* cusco*. We call T a *minimal* weak* cusco (usco) if its graph does not properly contain the graph of any other weak* cusco (usco) on A. By the *graph* of T we mean the set $\text{Gr}(T) := \{(x, x^*) : x^* \in T(x)\}$, which is closed whenever T is an usco. The following result from [1] enables us to generate cuscos and uscos from densely defined set-valued mappings.

LEMMA 1 ([1]). *Let T be a densely defined set-valued mapping that maps from a topological space A into the dual of a Banach space X. If T is locally bounded on A then there exists a unique smallest weak* usco (weak* cusco) containing T, denoted* USC(T) *(*CSC(T)*) and given by;*

$$\text{USC}(T)(x) := \bigcap\{\overline{T(V)}^{w^*} : V \text{ is an open neighbourhood of } x\},$$
$$\text{CSC}(T)(x) := \bigcap\{\overline{\text{co}}^{w^*} T(V) : V \text{ is an open neighborhood of } x\}.$$

SUBDERIVATIVES AND SUBDIFFERENTIALS. Let $f\colon A \subseteq X \mapsto \mathbb{R}$ be a locally Lipschitz function defined on a non-empty open set A of a Banach space X. The *Clarke derivative* of f at $x \in A$, [6] is given by:

$$f^0(x,v) := \limsup_{t\downarrow 0, y\to x} \frac{f(y+tv)-f(y)}{t}.$$

The *lower Dini-derivatives* of f at x are given by:

$$f^-(x,v) := \liminf_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

The corresponding generalized subdifferentials are defined by:

$$\partial_\sharp f(x) := \{x^* \in X^* \mid \langle x^*, v\rangle \le f^\sharp(x,v) \text{ for all } v \in X\},$$

where $\sharp$ is one of 0, $-$. When X is a smooth Banach space, *the approximate subdifferential* [2] is given by: $\partial_a f(x) := \mathrm{USC}(\partial_- f)(x)$. If in addition (X^*, weak^*) is angelic (for example when X is weakly countably determined) then,

$$\partial_a f(x) = \{x^* : x^* = \text{w}^*\text{-}\lim_{x_n\to x} x_n^*, \text{ and } x_n^* \in \partial_- f(x_n)\}.$$

In all cases $\partial_0 f$ is a weak* cusco on A and $\partial_a f(x)$ is a weak* usco on A.

2. **Basic properties of the space of T–Lipschitz functions.** Let T be weak* cusco that maps from a non-empty open subset A of a Banach space X into its dual space X^*. For such a cusco mapping one may consider the following (possibly empty) set of locally Lipschitz functions defined on A, called the *T-Lipschitz* functions on A;

$$\mathcal{X}_T := \{f \in \mathbb{R}^A : f \text{ is locally Lipschitz and } \partial_0 f(x) \subseteq T(x) \text{ for all } x \in A\}.$$

On $\mathcal{X}_T$, we may define a metric ρ by: $\rho(f,g) := \min\{1, d(f,g)\}$, where $d(f,g) := \sup_{x\in A} |f(x) - g(x)|$. If T is identically equal to some non-empty weak* compact, convex subset C of X^* then we may simply write $\mathcal{X}_C$ in place of $\mathcal{X}_T$. Some basic properties of T and $\mathcal{X}_T$ are:

LEMMA 2. *Let X be a Banach space and let A be a metric space, then each weak* usco from A into X^* is locally bounded on A.*

With a little extra effort one can show that each weak* usco mapping from a so called q-space into X^* is locally bounded. However, we have no need for this extra generality here.

LEMMA 3. *Let A be a non-empty open subset of a normed linear space X and let $T\colon A \to 2^{X^*}$ be a weak* cusco on A, then $\mathcal{X}_T$ is a convex sub-lattice of the locally Lipschitz functions defined on A.*

LEMMA 4. *Let A be a non-empty open subset of a Banach space X and let $T: A \to 2^{X^*}$ be a weak* cusco on A, then $(\mathcal{X}_T, \rho)$ is a complete metric space.*

The following result follows from Lemmas 1 and 2.

PROPOSITION 1. *Let $\mathcal{F}$ be a family of real-valued locally Lipschitz functions defined on a non-empty open subset A of a Banach space X. Then the functions in $\mathcal{F}$ are T-Lipschitz for some weak* cusco $T: A \to 2^{X^*}$ if and only if the family of functions $\mathcal{F}$ is locally equi-Lipschitz on A.*

3. Results for separable Banach spaces.

LEMMA 5. *Let X be a separable Banach space and let f be a locally Lipschitz function on a non-empty open subset A of X, then there exists a countable set $C \subseteq \mathrm{Gr}(\partial_- f)$ such that $\mathrm{Gr}(\partial_a f) = \overline{C}$, where the closure is taken with respect to the product topology on $X \times X^*$ and X^* is endowed with weak* topology.*

LEMMA 6. *Let Y be a finite dimensional subspace of a smooth Banach space X and let $h: X \to (-\infty, \infty]$ be a proper lower semi-continuous function. Suppose that $\varepsilon, \delta > 0$ and $x_0 \in X$ are given and h satisfies*
(i) h is bounded below on $B_\delta(x_0)$;
(ii) $h(x) - h(x_0) > -\delta \cdot \varepsilon$ for all $x \in x_0 + \delta B_Y$.
Then there exists a point $x \in B_\delta(x_0)$ and $x^* \in \partial_- h(x)$ with $\|x^*|_Y\| < 2\varepsilon$.

Using a Baire category argument in $(\mathcal{X}_T, \rho)$, together with Lemmas 5 and 6, we obtain:

THEOREM 1 (THE APPROXIMATE SUBDIFFERENTIAL CASE). *Let A be a non-empty open subset of a separable Banach space X and let $T: A \mapsto 2^{X^*}$ be a weak* cusco on A, then for each $f \in \mathcal{X}_T$, $\{g \in \mathcal{X}_T : \partial_a f(x) \subseteq \partial_a g(x)$ for all $x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$.*

We observe that Theorem 1 fails if T is only assumed to be a weak* usco.

COROLLARY 1. *Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of locally equi-Lipschitz functions defined on a non-empty open subset A of a separable Banach space X. If we define $T: A \to 2^{X^*}$ by $T(x) := \bigcup_{n\in\mathbb{N}} \partial_a f_n(x)$, then $\{g \in \mathcal{X}_{\mathrm{CSC}(T)} : \mathrm{USC}(T)(x) \subseteq \partial_a g(x) \subseteq \partial_0 g(x) = \mathrm{CSC}(T)(x)$ for every $x \in A\}$ is residual in $(\mathcal{X}_{\mathrm{CSC}(T)}, \rho)$.*

THEOREM 2 (THE CLARKE SUBDIFFERENTIAL CASE). *Let A be a non-empty open subset of a separable Banach space X and let $T: A \to 2^{X^*}$ be a weak* cusco on A, then for each $f \in \mathcal{X}_T$, $\{g \in \mathcal{X}_T : \partial_0 f(x) \subseteq \partial_0 g(x)$ for all $x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$.*

A few consequences are obtained by making judicious choices of the map T:

COROLLARY 2. *Let $\{f_n : n \in \mathbb{N}\}$ be a locally equi-Lipschitz family of real-valued functions defined on a non-empty open subset A of a separable Banach space X. If we define $T: A \to 2^{X^*}$ by $T(x) := \bigcup_{n\in\mathbb{N}} \partial_0 f_n(x)$, then $\{g \in \mathcal{X}_{\mathrm{CSC}(T)} : \partial_0 g(x) = \mathrm{CSC}(T)(x)$ for all $x \in A\}$ is residual in $(\mathcal{X}_{\mathrm{CSC}(T)}, \rho)$.*

COROLLARY 3. *Let $f_1, f_2, \ldots, f_n$ be real-valued locally Lipschitz functions defined on a non-empty open subset A of a separable Banach space X. If $T\colon A \to 2^{X^*}$ is defined by,*

$$T(x) := \text{co}\{\partial_0 f_1(x), \partial_0 f_2(x), \ldots, \partial_0 f_n(x)\}$$

then $\{g \in \mathcal{X}_T : \partial_0 g(x) = T(x) \text{ for all } x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$. In particular, the Clarke subdifferential is closed under the operation of taking finite convex hulls.

Corollary 3 improves the main result of [5], where the minimality of each $\partial_0 f_j$ was required.

COROLLARY 4. *Let f be a real-valued locally Lipschitz function defined on a non-empty open connected subset A of a separable Banach space X. Then the following conditions are equivalent:*

(i) f is "strongly integrable" that is, for each real-valued locally Lipschitz function g defined on A with $\partial_0 g(x) \subseteq \partial_0 f(x)$ for all $x \in A$, $f - g \equiv$ constant.
(ii) f is "weakly integrable that is, for each real-valued locally Lipschitz function g defined on A with $\partial_0 g(x) = \partial_0 f(x)$ for all $x \in A$, $f - g \equiv$ constant.

Let I be an open interval in $\mathbb{R}$ and let $f\colon I \mapsto \mathbb{R}$. We say that f is *robustly lower (upper) semi-continuous* if, $f(x) = \liminf_{\substack{y \to x \\ y \notin N}} f(y)$ $\left(f(x) = \limsup_{\substack{y \to x \\ y \notin N}} f(y)\right)$ for each Lebesgue null set N of I.

COROLLARY 5. *Let I be an open interval in $\mathbb{R}$ and let α and β be functions on I such that $\alpha \leq \beta$. Then the following are equivalent:*

(i) α is robustly lower semi-continuous and β is robustly upper semi-continuous;
(ii) there exists a locally Lipschitz function f on I such that $\partial_0 f(x) = [\alpha(x), \beta(x)]$ for all $x \in I$;
(iii) if $T\colon I \to 2^{\mathbb{R}}$ is defined by, $T(x) := [\alpha(x), \beta(x)]$, then

$$\{g \in \mathcal{X}_T : \partial_0 g(x) = [\alpha(x), \beta(x)] \text{ for all } x \in I\} \text{ is residual in } \mathcal{X}_T.$$

This has recovered and improved the main result in [3].

4. Results concerning general Banach spaces.

In general Banach spaces we obtain weaker, but still highly useful results.

LEMMA 7. *Let A be a non-empty open subset of a Banach space X and let $T\colon A \to 2^{X^*}$ be a weak* cusco on A. If $f, g \in \mathcal{X}_T$ and $E \subseteq \mathbb{R}$ is Lebesgue measurable then the function $h\colon A \to \mathbb{R}$ defined by $h(x) := \lambda_E\big((f-g)(x)\big) + g(x) \in \mathcal{X}_T$ and $\rho(h, g) \leq \lambda(E)$, where $\lambda_E\colon \mathbb{R} \to \mathbb{R}$ is defined by $\lambda_E(x) := \int_{[0,x]} \mathcal{X}_E(t)\, dt$.*

By applying Lemmas 6 and 7 and the Baire category theorem we obtain:

THEOREM 3. (THE APPROXIMATE SUBDIFFERENTIAL IN SMOOTH BANACH SPACE). *Let A be a non-empty open subset of a smooth Banach space X and let $T\colon A \to 2^{X^*}$ be a weak* cusco on A. If $f \in \mathcal{X}_T$ and $x \to \partial_a f(x)$ is a minimal weak* usco, then $\{g \in \mathcal{X}_T : \partial_a f(x) \subseteq \partial_a g(x) \text{ for all } x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$.*

While minimality of $x \to \partial_a f(x)$ is quite restrictive, it does hold when f is smooth or concave on A. Even this allows for some nice applications:

COROLLARY 6. *Let A be a non-empty open subset of a smooth Banach space X and suppose $C \subseteq X^*$ is non-empty, weak* compact, convex and weak* separable, then $\{g \in \mathcal{X}_C : \partial_a g(x) = C$ for all $x \in A\}$ is residual in $(\mathcal{X}_C, \rho)$.*

Note that when X is a separable Banach space every weak* compact set is weak* separable.

COROLLARY 7. *Let A be a non-empty open subset of a smooth Banach space X, then $\{g \in \mathcal{X}_{B_{X^*}} : \partial_a g(x) = B_{X^*}$ for all $x \in A\}$ is residual in $(\mathcal{X}_{B_{X^*}}, \rho)$.*

THEOREM 4. (THE CLARKE SUBDIFFERENTIAL IN ARBITRARY BANACH SPACE). *Let A be a non-empty open subset of a Banach space X and let $T: A \to 2^{X^*}$ be a weak* cusco on A, then for each $f \in \mathcal{X}_T$, $\{g \in \mathcal{X}_T : \partial_0 g(x) \cap \partial_0 f(x) \neq \emptyset$ for all $x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$. In particular, if $\partial_0 f$ is a minimal cusco, then $\{g \in \mathcal{X}_T : \partial_0 f(x) \subseteq \partial_0 g(x)$ for all $x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$.*

COROLLARY 8. *Let A be a non-empty open subset of a Banach space X and let $\{f_n : n \in \mathbb{N}\}$ be a sequence of locally equi-Lipschitz real-valued functions. If $T: A \to 2^{X^*}$ is defined by $T(x) := \bigcup_{n \in \mathbb{N}} \partial_0 f_n(x)$ and each $\partial_0 f_n$ is a minimal weak* cusco then $\{g \in \mathcal{X}_{\mathrm{CSC}(T)} : \partial_0 g(x) = \mathrm{CSC}(T)(x)$ for each $x \in A\}$ is residual in $(\mathcal{X}_{\mathrm{CSC}(T)}, \rho)$.*

Note that there are Clarke subdifferentials that can not be expressed as the cusco generated by a countable family of minimal cuscos.

A striking application of Theorem 4 or Corollary 8 is:

COROLLARY 9 (THE DUAL BALL CASE). *Let A be a non-empty open subset of a Banach space X, then $\{g \in \mathcal{X}_{B_{X^*}} : \partial_0 g(x) = B_{X^*}$ for all $x \in A\}$ is residual in $(\mathcal{X}_{B_{X^*}}, \rho)$.*

COROLLARY 10. *Let A be a non-empty open subset of a Banach space X, then*

$$\{g \in \mathcal{X}_{B_{X^*}} : \text{ for each } v \in S_X, \{x \in A : g'(x; v) \text{ exists}\} \text{ is first category}\}$$

is residual in $(\mathcal{X}_{B_{X^*}}, \rho)$.

Since each maximal cyclically monotone operator defined on a non-empty open convex subset of a Banach space is the Clarke subgradient of some convex function we have:

COROLLARY 11. *Let A be a nonempty open convex subset of a Banach space X and let $\{T_1, T_2, \ldots, T_n\}$ be a finite family of maximal cyclically monotone operators from A into non-empty subsets of X^*. Then there exists a real-valued locally Lipschitz function f defined on A such that,*

$$\partial_0 f(x) = \mathrm{co}\{T_1(x), T_2(x), \ldots, T_n(x)\} \text{ for every } x \in A.$$

This generalizes Corollary 2 in [5] to nonseparable spaces.

The details and proofs of the results contained in this report may be found in [4].

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C. R. Math. Rep. Acad. Sci. Canada Vol. **21** (4), 1999 pp. 139–145

MULTIPLIERS FOR GENERALIZED RIEMANN INTEGRALS

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Presented by Vlastimil Dlab, FRSC

RÉSUMÉ. La présente note traite d'une intégrale de Riemann généralisée dans l'espace euclidien $\mathbf{R}^m$. Les résultats suivants sont annoncés. (1) Une fonction g est un multiplicateur (c'est-à-dire que fg est intégrable dès que f l'est) si et seulement si g est bornée et à variation bornée. (2) Le dual de l'espace des fonctions intégrables dans un ensemble A à variation bornée est isomorphe à l'espace des fonctions bornées à variation bornée s'annulant hors de A. En outre, chaque élément du dual est représenté par une formule intégrale usuelle. (3) Une formule d'intégration par parties est valide sur des domaines lipschitziens et pour des champs continus qui sont ponctuellement lipschitziens partout sauf éventuellement sur un ensemble de mesure de Hausdorff $\mathcal{H}^{m-1}$ σ-finie.

In dimension one, the problem of recovering derivative by integration has been solved a long time ago by Denjoy and Perron [8, Chapters 6 and 8]. In higher dimensions, however, a substantial progress was made only in the eighties. Following the ideas of Henstock, Kurzweil, and McShane, several authors suggested various Riemann type extensions of the Lebesgue integral which provide the Gauss-Green theorem for any differentiable (not necessarily continuously) vector field. Among these generalized Riemann integrals, the most viable appear those introduced in [6, Chapters 12 and 13]: they are defined over bounded BV sets, are invariant with respect to lipeomorphic changes of coordinates, and integrate functions with large, topologically unrestricted, sets of singularities. They have a common extension, referred to as the R_*-integral below, and as the continuous integral in [5, Section 9].

Our main result is a complete solution of the multiplier problem for the R_*-integral. We show the family of all functions, called *multipliers*, whose product with every locally R_*-integrable function is again locally R_*-integrable is the algebra of all locally bounded locally BV functions. Using this, we obtain the usual integral representation of continuous linear functionals on the space of all functions that are R_*-integrable in a bounded BV set.

1. Preliminaries. The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this note is $\mathbb{R}^m$ where $m \geq 1$ is a fixed integer. By $x \cdot y$ we

Received by the editors January 28, 1999.

The first author was supported in part by the Hungarian National Foundation for Scientific Research, Grant No. T019476 and FKFP B07/1997.

AMS subject classification: 28A75, 49Q15.

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denote the standard inner product of $x, y \in \mathbb{R}^m$, and let $|x| = \sqrt{x \cdot x}$. The closed ball of radius $r > 0$ about the origin of $\mathbb{R}^m$ is denoted by B_r. The symbols $\operatorname{cl} E$, ∂E, and $d(A)$ denote, respectively, the closure, boundary, and diameter of a set $E \subset \mathbb{R}^m$.

The Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$. The *essential closure* $\operatorname{cl}_* E$ and *essential boundary* $\partial_* E$ of a measurable set $E \subset \mathbb{R}^m$ are the sets of all points in $\mathbb{R}^m$ at which the density of E lies in $(0,1]$ and $(0,1)$, respectively [4, Section 1.7.1]. We say E is *essentially closed* whenever $\operatorname{cl}_* E$ is a closed set. A *thin set* is a set $S \subset \mathbb{R}^m$ whose $(m-1)$-dimensional Hausdorff measure $\mathcal{H}^{m-1}$ is σ-finite.

The linear space of all locally bounded locally BV functions in $\mathbb{R}^m$ is denoted by $\mathrm{BV}^\infty_{\mathrm{loc}}$, and the linear space of all bounded BV functions in $\mathbb{R}^m$ with compact support is denoted by BV^∞_c. If $g \in \mathrm{BV}^\infty_c$, we denote by Dg and $\|g\|$ the distributional gradient and total variation of g, respectively. For $n = 1, 2, \ldots$, topologize the family

$$\mathrm{BV}_n = \{g \in \mathrm{BV}^\infty_c : \operatorname{supp} g \subset B_n \text{ and } \|g\| + |g|_\infty \le n+1\}$$

by the L^1 norm $|\cdot|_1$. Each BV_n is a compact set, and the largest topology τ in BV^∞_c for which all inclusion maps $\mathrm{BV}_n \hookrightarrow \mathrm{BV}^\infty_c$ are continuous is a non-metrizable locally convex Hausdorff topology that is sequential and sequentially complete. We view the family $\mathcal{BV}$ of all bounded BV subsets of $\mathbb{R}^m$ as a closed subspace of $(\mathrm{BV}^\infty_c, \tau)$. The *perimeter* and *unit exterior normal* of $A \in \mathcal{BV}$ are denoted by $\|A\|$ and ν_A, respectively. Moreover, we let

$$r(A) = \begin{cases} \frac{\mathcal{L}^m(A)}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. **Charges.** A *charge* is an additive τ-continuous function defined on $\mathcal{BV}$. If F is a charge and E is a locally BV set, define a charge $F \llcorner E$ by letting

$$(F \llcorner E)(A) = F(A \cap E)$$

for each $A \in \mathcal{BV}$. We say F is a *charge in* E whenever $F = F \llcorner E$. The linear spaces of all charges and of all charges in a locally BV set E are denoted by CH and $\mathrm{CH}(E)$, respectively. The following are canonical examples of charges.

(1) If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $F(A) = \int_A f \, d\mathcal{L}^m$ for each bounded BV set A, then F is a charge by the absolute continuity of the Lebesgue integral. It is called the *indefinite Lebesgue integral* of f, denoted by $\int f \, d\mathcal{L}^m$.

(2) If $v \in C(\mathbb{R}^m; \mathbb{R}^m)$ and $F(A) = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}$ for each bounded BV set A, then F is a charge by [6, Proposition 11.2.8]. It is called the *flux* of v, denoted by $\int v \cdot \nu \, d\mathcal{H}^{m-1}$.

For a charge F and $n = 1, 2, \ldots$, let

$$\|F\|_n = \sup\{|F(B)| : B \in \mathcal{BV} \cap \mathrm{BV}_n\},$$

and observe $\{\|\cdot\|_n\}$ is an increasing sequence of seminorms, which defines a Fréchet topology σ in CH. If $A \in \mathcal{BV}$, then $\|F\|_n$ are equivalent *Banach norms* on CH(A) for all sufficiently large n.

PROPOSITION 2.1. *Let F be a charge, and let A be a bounded* BV *subset of $\mathbb{R}^{m+1}$. For $\mathcal{L}^1$-almost all $t \in \mathbb{R}$, the section $A^t = \{x \in \mathbb{R}^m : (x,t) \in A\}$ belongs to $\mathcal{BV}$, and the function $t \mapsto F(A^t)$ belongs to $L^1(\mathbb{R})$. Letting*

$$(F \times \mathcal{L}^1)(A) = \int_{\mathbb{R}} F(A^t)\,dt$$

for each bounded BV *set $A \subset \mathbb{R}^{m+1}$ defines a charge $F \times \mathcal{L}^1$ in $\mathbb{R}^{m+1}$. The map $F \mapsto F \times \mathcal{L}^1$ is a continuous linear map from $\big(\mathrm{CH}(\mathbb{R}^m), \sigma\big)$ to $\big(\mathrm{CH}(\mathbb{R}^{m+1}), \sigma\big)$.*

Denote by CH* and $(\mathrm{BV}_c^\infty)^*$ the dual spaces of (CH, σ) and $(\mathrm{BV}_c^\infty, \tau)$, respectively. We endow $(\mathrm{BV}_c^\infty)^*$ with a Fréchet topology ν induced by the uniform convergence on each BV_n. For a nonnegative function $g \in \mathrm{BV}_c^\infty$,

$$\Sigma_g = \{(x,t) \in \mathbb{R}^m \times \mathbb{R} : 0 < t < g(x)\}$$

is a bounded BV subset of $\mathbb{R}^{m+1}$. In view of Proposition 2.1, the number

$$\langle F, g\rangle = (F \times \mathcal{L}^1)(\Sigma_{g^+}) - (F \times \mathcal{L}^1)(\Sigma_{g^-})$$

is defined for each $F \in$ CH and each $g \in \mathrm{BV}_c^\infty$.

THEOREM 2.2. *The pairing $(F,g) \mapsto \langle F, g\rangle$ is a nondegenerate separately continuous bilinear functional on $(\mathrm{CH} \times \mathrm{BV}_c^\infty, \sigma \times \tau)$. For $T \in (\mathrm{BV}_c^\infty)^*$ and $A \in \mathcal{BV}$, let $\Lambda(T)(A) = T(\chi_A)$. Then $\Lambda(T)$ is a charge, and the map*

$$\Lambda\colon \big((\mathrm{BV}_c^\infty)^*, \nu\big) \to (\mathrm{CH}, \sigma)\colon T \mapsto \Lambda(T)$$

is a linear homeomorphism with $\Lambda^{-1}\colon F \mapsto \langle F, \cdot\rangle$.

3. The R-integral. A charge F is *derivable* at $x \in \mathbb{R}^m$ if $\lim[F(B_n)/|B_n|] \neq \pm\infty$ exists for each $\{B_n\}$ in $\mathcal{BV}$ with $\lim d(B_n \cup \{x\}) = 0$ and $\inf r(B_n \cup \{x\}) > 0$. If all these limits exist, they have the same value called the *derivate* of F at x, denoted by $F'(x)$.

A charge F is called AC if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|F(A)| < \varepsilon$ whenever $A \subset B_{1/\varepsilon}$ is a BV set with $\mathcal{L}^m(A) < \delta$. It is well-known each AC charge F is derivable almost everywhere with $F' \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $\int F'\,d\mathcal{L}^m = F$.

A nonnegative function δ defined on a set $E \subset \mathbb{R}^m$ is called a *gage* on E whenever its null set is thin. A *partition* is a collection (possibly empty) $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are disjoint bounded BV sets, and $x_1, \dots, x_p$ are points of $\mathbb{R}^m$. Given $\varepsilon > 0$ and a gage δ on $E \subset \mathbb{R}^m$, the partition P is called

- *ε-regular* if $r(A_i \cup \{x_i\}) > \varepsilon$ for $i = 1, \dots, p$,

• *δ-fine* if $x_i \in E$ and $d(A_i \cup \{x_i\}) < \delta(x_i)$ for $i = 1, \dots, p$.

The *critical variation* of a charge F on a set $E \subset \mathbb{R}^m$ is the number

$$V_* F(E) = \sup_{\varepsilon > 0} \inf_{\delta} \sup_{P} \sum_{i=1}^{p} |F(A_i)|$$

where δ is a gage on E, and $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ is an ε-regular δ-fine partition. The extended real-valued function $V_* F: E \mapsto V_* F(E)$ is a Borel regular measure in $\mathbb{R}^m$, which coincides with the classical variation of F on every bounded BV set. The following proposition was proved in [2, Proposition 3.3].

PROPOSITION 3.1. *A charge F is* AC *if and only if $V_* F$ is absolutely continuous and locally finite.*

A charge F is called AC$_*$ whenever the measure $V_* F$ is absolutely continuous. The linear space of all AC$_*$ charges or all AC$_*$ charges in a bounded BV set A is denoted by $\mathrm{CH}_{\mathrm{AC}_*}$ or $\mathrm{CH}_{\mathrm{AC}_*}(A)$, respectively. While neither of the spaces $(\mathrm{CH}_{\mathrm{AC}_*}, \sigma)$ and $(\mathrm{CH}_{\mathrm{AC}_*}(A), \sigma)$ is complete, both are barrelled [3, Theorem 4.1]. It turns out many important properties of AC charges remain valid for AC$_*$ charges.

DEFINITION 3.2. A function f defined on $\mathbb{R}^m$ is *locally R-integrable* whenever there is an AC$_*$ charge F such that $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^m$. The charge F, uniquely determined by f, is called the *indefinite R-integral* of f. If A is a bounded BV set, the number $F(A)$ is called the *R-integral* of f over A.

The linear space of all locally R-integrable functions is denoted by $R_{\mathrm{loc}}(\mathbb{R}^m)$. Clearly $L^1_{\mathrm{loc}}(\mathbb{R}^m) \subset R_{\mathrm{loc}}(\mathbb{R}^m)$, and the indefinite R-integral F of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ coincides with the indefinite Lebesgue integral of f. Thus without danger of confusion, we denote F by $\int f \, d\mathcal{L}^m$, and $F(A)$ by $\int_A f \, d\mathcal{L}^m$ for each bounded BV set A. The next theorem, proved in [2, Theorem 4.6], shows the R-integral is akin to the generalized Riemann integrals of [1, Definition 5.1].

THEOREM 3.3. *A function f defined on $\mathbb{R}^m$ is locally R-integrable if and only if there is a charge F satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on $B_{1/\varepsilon}$ so that*

$$\sum_{i=1}^{p} \big| f(x_i)|A_i| - F(A_i) \big| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$. In this case F is the indefinite R-integral of f.

A *multiplier* for a family X of functions defined on a set E is a function g defined on E such that $fg \in X$ for each $f \in X$. The collection of all multipliers for the family X is denoted by MX. It follows from [7, Proposition 3.1] that $MR_{\mathrm{loc}}(\mathbb{R}^m) \subset \mathrm{BV}^{\infty}_{\mathrm{loc}}$.

THEOREM 3.4. *Let F be the indefinite R-integral of an $f \in R_{\mathrm{loc}}(\mathbb{R}^m)$, and let $g \in \mathrm{BV}^{\infty}_{\mathrm{loc}}$. Then $fg \in R_{\mathrm{loc}}(\mathbb{R}^m)$ and*

$$\int_A fg\, d\mathcal{L}^m = \langle F, g\chi_A \rangle$$

for each bounded BV *set A. In particular, $MR_{\mathrm{loc}}(\mathbb{R}^m) = \mathrm{BV}^{\infty}_{\mathrm{loc}}$.*

Theorem 3.4 is proved by showing the function $G: B \mapsto \langle F, g\chi_B \rangle$ is an AC_* charge such that $G' = F'g$ almost everywhere.

For a function f defined on a set $E \subset \mathbb{R}^m$, let $\overline{f}$ be the extension of f to $\mathbb{R}^m$ by zero. If $A \in \mathcal{BV}$, we denote by $\mathrm{BV}^{\infty}(A)$ the family of all functions g defined on A whose zero extension $\overline{g}$ belongs to BV^{∞}_c.

DEFINITION 3.5. A function f defined on a bounded BV set A is called *R-integrable* in A whenever the zero extension $\overline{f}$ of f is locally R-integrable. The indefinite R-integral F of $\overline{f}$ is called the *indefinite R-integral* of f, and the number $F(A)$ is called the *R-integral* of f over A.

The linear space of all R-integrable functions in a bounded BV set A is denoted by $R(A)$. The indefinite integral F of a function $f \in R(A)$ is a charge in A, and the map

$$F \mapsto F' \restriction A : \mathrm{CH}_{AC_*}(A) \to R(A)$$

is a linear bijection, which induces in $R(A)$ a normed barrelled topology. For f in $R(A)$, we denote by $\int_A f\, d\mathcal{L}^m$ the R-integral of f over A. The R-integral has the usual properties associated with the word "integral", including the common convergence theorems. Moreover, the R-integral is invariant with respect to lipeomorphisms (*i.e.*, bi-Lipschitzian maps). The term "locally R-integrable", introduced in Definition 3.2, is now justified: a function f defined on $\mathbb{R}^m$ is locally R-integrable if and only if $f \restriction A$ is R-integrable in A for each bounded BV set A.

THEOREM 3.6. *Let A be a bounded* BV *set. Then $MR(A) = \mathrm{BV}^{\infty}(A)$, and for each $g \in \mathrm{BV}^{\infty}(A)$, the map*

$$T_g : f \mapsto \int_A fg\, d\mathcal{L}^m$$

is a continuous linear functional on $R(A)$. The map $g \mapsto T_g$ is a continuous linear bijection from $(\mathrm{BV}^{\infty}(A), \tau)$ onto the dual space of $R(A)$ equipped with the w^*-topology.

One of the principal reasons for introducing the R-integral is the divergence theorem for discontinuously differentiable vector fields. If $E \subset \mathbb{R}^m$, we say a vector field $v: E \to \mathbb{R}^m$ is *Lipschitz at $x \in E$* whenever there are positive numbers c and r, depending on x, such that

$$|v(y) - v(x)| \leq c|y - x|$$

for each $y \in E$ with $|y-x| < r$. Let $E \subset \mathbb{R}^m$ be a closed set, and let $v \in C(E; \mathbb{R}^m)$ be Lipschitz at almost all $x \in E$. In view of Whitney's and Stepanoff's theorems, $\operatorname{div} v$ is uniquely defined almost everywhere in E.

THEOREM 3.7. *Let A be a bounded* BV *set that is essentially closed, and let $v \in C(\mathrm{cl}_* A; \mathbb{R}^m)$. Suppose there is a thin set $S \subset \mathrm{cl}_* A$ such that v is Lipschitz at each $x \in \mathrm{cl}_* A - S$. Then $\operatorname{div} v$ is R-integrable in A and*

$$\int_A \operatorname{div} v \, d\mathcal{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}.$$

If Ω is a *Lipschitz domain* [4, Section 4.2.1] and f is a BV function in Ω, we can define the *trace* of f, which is a function in $L^1(\partial\Omega, \mathcal{H}^{m-1})$, denoted by $\operatorname{Tr} f$.

THEOREM 3.8. *Let Ω be a Lipschitz domain, and let $v \in C(\mathrm{cl}\,\Omega; \mathbb{R}^m)$. Suppose there is a thin set $S \subset \Omega$ such that v is Lipschitz at each $x \in \Omega - S$. Then*

$$\int_\Omega g \operatorname{div} v \, d\mathcal{L}^m = \int_{\partial\Omega} (\operatorname{Tr} g)\, v \cdot \nu_\Omega \, d\mathcal{H}^{m-1} - \int_\Omega v \cdot Dg$$

for each $g \in BV^\infty(\Omega)$.

4. **The R_*-integral.** While the R-integral introduced in Section 3 is an averaging process of appreciable appeal, it is not adequate for integration over arbitrary bounded BV sets: Theorem 3.7 is false when A is not essentially closed [5, Example 5.21]. Fortunately, a simple extension of the R-integral eliminates this shortcoming and retains all desirable properties of the R-integral.

If f is a function defined on a bounded BV set A and F is a charge, denote by $\mathcal{R}(f, F; A)$ the family of all $B \in \mathcal{BV}(A)$ for which $f \restriction B$ belongs to $R(B)$ and $F \llcorner B$ is the indefinite R-integral of $f \restriction B$. The closure operator in the space $(\mathcal{BV}, \tau)$ is denoted by cl_τ.

DEFINITION 4.1. A function f defined on a BV set A is called *R_*-integrable* in A if there is a charge F in A such that $A \in \mathrm{cl}_\tau \mathcal{R}(f, F; A)$. The charge F, uniquely determined by f, is called the *indefinite R_*-integral* of f, and the number $F(A)$ is called the *R_*-integral* of f over A.

The linear space of all R_*-integrable functions in a bounded BV set A is denoted by $R_*(A)$. Clearly $R(A) \subset R_*(A)$, and arguments similar to those of [5, Sections 8 and 9] show the R_*-integral has the usual "integral properties", including the invariance with respect to lipeomorphic changes of coordinates. The validity of Theorems 3.6 for the R_*-integral is also easy to verify. With no danger of confusion, for $f \in R_*(A)$ we denote by $\int_A f \, d\mathcal{L}^m$ the R_*-integral of f over A. As τ is a sequential topology, it is easy to see no new integral is produced by a further iteration of Definition 4.1.

LEMMA 4.2. *Let* $A \in \mathcal{BV}$ *and* $v \in C(\operatorname{cl} A; \mathbb{R}^m)$. *Suppose there is a family* $\mathcal{E} \subset \mathcal{BV}(A)$ *such that for each* $E \in \mathcal{E}$, *the restriction* $v \restriction \operatorname{cl} E$ *is Lipschitz at almost all* $x \in E$. *If* $A \in \operatorname{cl}_\mathcal{T} \mathcal{E}$, *then there is a function* f, *defined on* A *and unique up to equivalence, such that given* $E \in \mathcal{E}$,

$$f(x) = \operatorname{div}(v \restriction \operatorname{cl} E)(x)$$

for almost all $x \in E$. *In particular,* $f(x) = \operatorname{div} v(x)$ *for almost all* $x \in A$ *whenever* $A \in \mathcal{E}$.

Under the assumptions of Lemma 4.2, we call the function f the *divergence* of v, still denoted by $\operatorname{div} v$. The last claim of Lemma 4.2 shows this is consistent with our previous definition of $\operatorname{div} v$ introduced prior to Theorem 3.7.

THEOREM 4.3. *Let* A *be any bounded* BV *set, and let* $v \in C(\operatorname{cl} A; \mathbb{R}^m)$. *Suppose there is a family* $\mathcal{E} \subset \mathcal{BV}(A)$ *with the following property: given* $E \in \mathcal{E}$, *we can find a thin set* $S_E \subset \operatorname{cl}_* E$ *so that* $v \restriction \operatorname{cl} E$ *is Lipschitz at each* $x \in \operatorname{cl}_* E - S_E$. *If* $A \in \operatorname{cl}_\mathcal{T} \mathcal{E}$, *then* $\operatorname{div} v$ *is* R_**-integrable in* A *and*

$$\int_A \operatorname{div} v \, d\mathcal{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}.$$

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C. R. Math. Rep. Acad. Sci. Canada Vol. **21** (4), 1999 pp. 146–152

EXTENDED DIRICHLET SPACES

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ABSTRACT. This note builds on the main result of [Sch] which says roughly that a Dirichlet form on $L^2(E;m)$ automatically has an extension to $L^0(E;m)$. This result has been known since the beginning of Dirichlet form theory [FOT], [S] and has applications to the analysis of time changed Markov processes. However, earlier proofs of the extension used unnecessarily strong topological assumptions. Here we offer consequences of the general result, which allow us to prove well-known results of Dirichlet form theory under weakened hypotheses.

RÉSUMÉ. La présente communication porte sur le résultat principal de [Sch] qui stipule essentiellement qu'une forme de Dirichlet sur $L^2(E;m)$ possède automatiquement une extension à $L^0(E;m)$. Ce résultat est connu depuis l'aube de la théorie des formes de Dirichlet [FOT], [S] et possède des applications à l'analyse des processus de Markov avec changement de temps. Les preuves précédentes de l'existence d'une telle extension nécessitent toutefois des hypothèses topologiques inutilement fortes. Nous offrons ici, comme conséquence d'un résultat plus général, des preuves de quelques résultats bien connus de la théorie des formes de Dirichlet, mais sous des hypothèses affaiblies.

Let $(E;m)$ be a measure space and $L^0(E;m)$ the vector space of real-valued measurable functions on E, where functions that agree m-almost everywhere are identified. We define a (positive semi-definite) *form* to be the pair $(\mathcal{E},\mathcal{F})$ where $\mathcal{F}$ is a linear subspace of $L^0(E;m)$ and $\mathcal{E}$ is a bilinear mapping from $\mathcal{F}\times\mathcal{F}$ to $\mathbb{R}$ such that $\mathcal{E}(u,u)\geq 0$ for every $u\in\mathcal{F}$.

DEFINITION 1. The form $(\mathcal{E},\mathcal{F})$ satisfies the *strong sector condition* if there is a constant $K\geq 1$, so that $|\mathcal{E}(u,v)|\leq K\ \mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2}$ for all $u,v\in\mathcal{F}$.

DEFINITION 2. The form $(\mathcal{E},\mathcal{F})$ has the *Fatou property* if $\mathcal{E}(u,u)\leq \liminf_n \mathcal{E}(u_n,u_n)$ for every $\mathcal{E}$-bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ that converges m-almost everywhere to u in $\mathcal{F}$.

1. L^0 Theory.

In this section we assume that $(\mathcal{E},\mathcal{F})$ satisfies the strong sector condition and has the Fatou property.

LEMMA 1. *If $(u_n)_{n\in\mathbb{N}}$ is $\mathcal{E}$-Cauchy and $u_n\to 0$ m-almost everywhere, then $\mathcal{E}(u_n,u_n)\to 0$.*

Received by the editors March 23, 1999.
AMS subject classification: 31C25.
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PROOF. As $l \to \infty$ we have $u_n - u_l \to u_n$ m-almost everywhere. Therefore the Fatou property gives $\mathcal{E}(u_n, u_n) \le \liminf_l \mathcal{E}(u_n - u_l, u_n - u_l)$, and we get the result by letting $n \to \infty$. ■

For sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, the strong sector condition gives

$$\begin{aligned} |\mathcal{E}(u_n, v_n) - \mathcal{E}(u_l, v_l)| &\le |\mathcal{E}(u_n, v_n - v_l)| + |\mathcal{E}(u_n - u_l, v_l)| \\ &\le K\mathcal{E}(u_n, u_n)^{1/2}\mathcal{E}(v_n - v_l, v_n - v_l)^{1/2} \\ &\quad + K\mathcal{E}(u_n - u_l, u_n - u_l)^{1/2}\mathcal{E}(v_l, v_l)^{1/2}. \end{aligned}$$

If the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are $\mathcal{E}$-Cauchy, it follows that the sequence $\mathcal{E}(u_n, v_n)$ of numbers is Cauchy and so $\lim_n \mathcal{E}(u_n, v_n)$ exists. In addition, if $(u_n)_{n\in\mathbb{N}}$, $(\widehat{u}_n)_{n\in\mathbb{N}}$, and $(v_n)_{n\in\mathbb{N}}$ are $\mathcal{E}$-Cauchy sequences with $u_n - \widehat{u}_n \to 0$ m-almost everywhere as $n \to \infty$, then we have

$$|\mathcal{E}(u_n, v_n) - \mathcal{E}(\widehat{u}_n, v_n)| \le K\mathcal{E}(u_n - \widehat{u}_n, u_n - \widehat{u}_n)^{1/2}\mathcal{E}(v_n, v_n)^{1/2}.$$

Lemma 1 says that the right hand side goes to zero as $n \to \infty$ so that $\lim_n \mathcal{E}(u_n, v_n) = \lim_n \mathcal{E}(\widehat{u}_n, v_n)$. Thus the extended form given below is well-defined.

DEFINITION 3. We let the *extended form* $(\mathcal{E}, \mathcal{F}_e)$ be given by

$$\mathcal{F}_e = \{u \in L^0(E; m) \mid u_n \to u \text{ } m\text{-a.e. for an } \mathcal{E}\text{-Cauchy sequence } (u_n)_{n\in\mathbb{N}} \in \mathcal{F}\}.$$
$$\mathcal{E}(u, v) = \lim_n \mathcal{E}(u_n, v_n).$$

NOTE. In connection with Definition 3, the classical work [AS] of Aronszajn and Smith should be noted.

We would like to establish that $(\mathcal{E}, \mathcal{F}_e)$ inherits some of the properties of $(\mathcal{E}, \mathcal{F})$. It is not hard to show that $\mathcal{F}_e$ is a linear subspace of $L^0(E; m)$, and that $\mathcal{E}$ is a positive semi-definite bilinear mapping from $\mathcal{F}_e \times \mathcal{F}_e$ to $\mathbb{R}$. Let $u, v \in \mathcal{F}_e$ and $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be approximating $\mathcal{E}$-Cauchy sequences in $\mathcal{F}$. The strong sector condition carries over since

$$\begin{aligned} |\mathcal{E}(u, v)| &= \lim_n |\mathcal{E}(u_n, v_n)| \le \lim_n K\mathcal{E}(u_n, u_n)^{1/2}\mathcal{E}(v_n, v_n)^{1/2} \\ &= K\mathcal{E}(u, u)^{1/2}\mathcal{E}(v, v)^{1/2}. \end{aligned}$$

We note as well that $\mathcal{E}(u - u_n, u - u_n) = \lim_l \mathcal{E}(u_l - u_n, u_l - u_n) \to 0$ as $n \to \infty$. Combined with the strong sector condition, this implies that $\mathcal{E}(u_n, v) \to \mathcal{E}(u, v)$.

LEMMA 2. *If $(u_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-bounded sequence in $\mathcal{F}$, and $u_n \to u$ m-almost everywhere, then $u \in \mathcal{F}_e$ and* $\liminf_n \mathcal{E}(u_n, v) \le \mathcal{E}(u, v) \le \limsup_n \mathcal{E}(u_n, v)$ *for every $v \in \mathcal{F}_e$. In particular, we have* $\mathcal{E}(u, u) \le \liminf_n \mathcal{E}(u_n, u_n)$.

PROOF. Since $(\mathcal{E},\mathcal{F})$ is an inner product space, the $\mathcal{E}$-bounded sequence $(u_n)_{n\in\mathbb{N}}$ admits a subsequence, that we denote $(u_{n_i})_{i\in\mathbb{N}}$, whose Cesàro means $w_N = (1/N)\sum_{i=1}^N u_{n_i}$ are $\mathcal{E}$-Cauchy. Since $u_n \to u$ m-almost everywhere, we have $w_N \to u$ m-almost everywhere and so $u \in \mathcal{F}_e$. For $v \in \mathcal{F}_e$ we have,

$$\mathcal{E}(u,v) = \lim_N \mathcal{E}(w_N, v) = \lim_N \frac{1}{N}\sum_{i=1}^N \mathcal{E}(u_{n_i}, v),$$

and

$$\liminf_n \mathcal{E}(u_n, v) \le \lim_N \frac{1}{N}\sum_{i=1}^N \mathcal{E}(u_{n_i}, v) \le \limsup_n \mathcal{E}(u_n, v).$$

Positivity implies $0 \le \mathcal{E}(u-u_n, u-u_n)$ or $\mathcal{E}(u,u_n)+\mathcal{E}(u_n,u)-\mathcal{E}(u,u) \le \mathcal{E}(u_n,u_n)$. Taking lim sup on the left and lim inf on the right gives $\mathcal{E}(u,u) \le \liminf_n \mathcal{E}(u_n,u_n)$. ■

LEMMA 3. *Suppose that each function in $\mathcal{F}$ has m-σ-finite support. Then $(\mathcal{E},\mathcal{F}_e)$ has the Fatou property and $(\mathcal{F}_e)_e = \mathcal{F}_e$.*

PROOF. Let $(u_n)_{n\in\mathbb{N}}$ be an $\mathcal{E}$-bounded sequence in $\mathcal{F}_e$ that converges m-almost everywhere to a function u. For each $n \in \mathbb{N}$, let $(u_{n,j})_{j\in\mathbb{N}}$ be an $\mathcal{E}$-Cauchy sequence in $\mathcal{F}$ that converges m-almost everywhere to u_n as $j \to \infty$. The set $F := \bigcup_{n,j}\{|u_{n,j}| > 0\}$ is m-σ-finite, and also supports the functions u_n and u. Let $(F_n)_{n\in\mathbb{N}}$ satisfy $m(F_n) < \infty$ for each $n \in \mathbb{N}$ and $F_n \uparrow F$ as $n \to \infty$.

For $n \in \mathbb{N}$, choose j_n so that $m(F_n \cap \{|u_{n,j_n} - u_n| > n^{-1}\}) \le 2^{-n}$ and $\mathcal{E}(u_n - u_{n,j_n}, u_n - u_{n,j_n}) \le n^{-1}$. Then $u_{n,j_n} \to u$ m-almost everywhere as $n \to \infty$ and $(u_{n,j_n})_{n\in\mathbb{N}}$ is $\mathcal{E}$-bounded. By Lemma 2, $u \in \mathcal{F}_e$ and $\mathcal{E}(u,u) \le \liminf_n \mathcal{E}(u_{n,j_n}, u_{n,j_n}) = \liminf_n \mathcal{E}(u_n,u_n)$ which gives the Fatou property. The argument above shows that the pointwise limit of an $\mathcal{E}$-bounded sequence in $\mathcal{F}_e$ belongs to $\mathcal{F}_e$, so that $(\mathcal{F}_e)_e = \mathcal{F}_e$. ■

EXAMPLE 1. The conclusion $(\mathcal{F}_e)_e = \mathcal{F}_e$ is false without the σ-finiteness assumption. Let $E = \mathbb{R}$ equipped with the σ-algebra of all subsets of E, and the measure

$$m(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ +\infty & \text{otherwise.} \end{cases}$$

This way every function is measurable, and m-almost everywhere convergence coincides with pointwise everywhere convergence. Set $\mathcal{E} \equiv 0$ and $\mathcal{F}$ to be the space of continuous real-valued functions on E. Then $\mathcal{F}_e$ is the first Baire class of functions, and $(\mathcal{F}_e)_e$ is the second Baire class. But the second Baire class is strictly larger than the first; for example the indicator function $1_{\mathbb{Q}}$ belongs to $(\mathcal{F}_e)_e$, but not $\mathcal{F}_e$.

DEFINITION 4. A form $(\mathcal{E},\mathcal{F})$ is called *positivity preserving* if $u \in \mathcal{F}$ implies $u^+ \in \mathcal{F}$ and $\mathcal{E}(u^+, u-u^+) \ge 0$. A form $(\mathcal{E},\mathcal{F})$ is called *Markov* if $u \in \mathcal{F}$ implies $u \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u\wedge 1, u - u\wedge 1) \ge 0$.

NOTE. These definitions date back at least to the seminal paper of Beurling and Deny [BD, Définition 1, Remarque 2] but with different terminology. In the L^2 case, these properties of $\mathcal{E}$ correspond to properties of the associated semigroup $(T_t)_{t\geq 0}$: a form is positivity preserving if and only if $0 \leq u$ implies $0 \leq T_t u$ [MR2, Theorem 1.5], while a form is Markov if and only if $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$ [MR1, Proposition 4.3 and Theorem 4.4].

LEMMA 4. *The positivity preserving and Markov properties extend to* $(\mathcal{E}, \mathcal{F}_e)$.

PROOF. Suppose that $(\mathcal{E}, \mathcal{F})$ is positivity preserving. Let $u \in \mathcal{F}_e$ and $(u_n)_{n\in\mathbb{N}} \in \mathcal{F}$ so that $\mathcal{E}(u-u_n, u-u_n) \to 0$ and $u_n \to u$ m-almost everywhere as $n \to \infty$. The positivity preserving property shows that $(u_n^+)_{n\in\mathbb{N}}$ is $\mathcal{E}$-bounded;

$$\mathcal{E}(u_n^+, u_n^+) \leq \mathcal{E}(u_n^+, u_n^+) - 2\mathcal{E}(u_n^+, u_n^-) + \mathcal{E}(u_n^-, u_n^-) = \mathcal{E}(u_n, u_n).$$

Since $u_n^+ \to u^+$ m-almost everywhere, Lemma 2 shows that $u^+ \in \mathcal{F}_e$ and

$$\mathcal{E}(u^+, u^+) \leq \liminf_n \mathcal{E}(u_n^+, u_n^+) \leq \liminf_n \mathcal{E}(u_n^+, u_n) = \liminf_n \mathcal{E}(u_n^+, u) \leq \mathcal{E}(u^+, u).$$

The proof when $(\mathcal{E}, \mathcal{F})$ has the Markov property is similar. ∎

DEFINITION 5. A form $(\mathcal{E}, \mathcal{F})$ has a *square field*[1] if there is a bilinear map $田: \mathcal{F} \times \mathcal{F} \to L^1(E; m)$ such that $田(u,u) \geq 0$, $|田(u,v)| \leq K田(u,u)^{1/2}田(v,v)^{1/2}$ for $u, v \in \mathcal{F}$, and

$$\mathcal{E}(u,v) = \frac{1}{2}\int_E 田(u,v)(z)\, m(dz).$$

If $(\mathcal{E}, \mathcal{F})$ has a square field, then Lemma 1 says that $田(u_n, u_n) \to 0$ in $L^1(E; m)$ for any $\mathcal{E}$-Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ that converges to zero m-almost everywhere as $n \to \infty$. So the arguments used for the form $\mathcal{E}$ up to Lemma 2 also apply to the square field 田. This gives the following result.

LEMMA 5. *If* $(\mathcal{E}, \mathcal{F})$ *has a square field, then it extends uniquely to* $\mathcal{F}_e$. *If* $(u_n)_{n\in\mathbb{N}}$ *is an* $\mathcal{E}$*-bounded sequence in* $\mathcal{F}$, *and* $u_n \to u$ m*-almost everywhere, then* $u \in \mathcal{F}_e$ *and* $\liminf_n 田(u_n, v) \leq 田(u,v) \leq \limsup_n 田(u_n, v)$ *for every* $v \in \mathcal{F}_e$. *In particular, we have* $田(u,u) \leq \liminf_n 田(u_n, u_n)$.

2. L^2 Theory.

In this section we assume that $(\mathcal{E}, \mathcal{F})$ satisfies the strong sector condition and that $\mathcal{F} \subseteq L^2(E; m)$. We are first interested in whether or not the Fatou property holds. By way of comparison, we say that the form $(\mathcal{E}, \mathcal{F})$ is *closed* if $\mathcal{F}$ is complete with respect to the norm $\|u\|_1 = \{\mathcal{E}(u,u) + (u,u)_{L^2}\}^{1/2}$, and that $(\mathcal{E}, \mathcal{F})$ is *closable* if it has a closed extension. These properties depend on the behaviour of the associated quadratic function q,

$$q(u) = \begin{cases} \mathcal{E}(u,u) & \text{if } u \in \mathcal{F}, \\ \infty & \text{if } u \in L^2(E;m) \setminus \mathcal{F}. \end{cases}$$

[1] The notation 田 is based on the Chinese character Tián, which means 'field'.

It is known [D, Theorem 4.12] that the form $(\mathcal{E}, \mathcal{F})$ is closed [closable] if and only if q is L^2-lower semicontinuous on $L^2(E; m)$ [on $\mathcal{F}$].

Comparing this result with Definition 2 shows us that the Fatou property is something like an $L^0(E; m)$ version of closability. But we should be cautious with this analogy; almost everywhere convergence cannot in general be topologized, and since we do not assume that m is σ-finite, convergence in measure is not a suitable substitute.

However, since L^2 convergence implies almost everywhere convergence along a subsequence, we find that if $(\mathcal{E}, \mathcal{F})$ has the Fatou property, then it must be closable. The converse is false, as the following example demonstrates.

EXAMPLE 2. Let $\mathcal{F} = L^2(E; m)$ where $E = (0, 1)$ and m is Lebesgue measure. Define a form on $\mathcal{F}$ by $\mathcal{E}(u, v) = (u, 1)_{L^2}(v, 1)_{L^2}$. Letting $u_n = (1 - n)1_{(0,1/n)} + 1_{(1/n,1)}$ so that $(u_n, 1)_{L^2} = 0$ but $u_n \to u = 1$ pointwise, we have $1 = \mathcal{E}(u, u) > \liminf_n \mathcal{E}(u_n, u_n) = 0$.

A natural question, then, is which closable forms have the Fatou property? Fatou's lemma shows that the inner product does, and a simple comparison argument shows that if $c\|u\|_{L^2}^2 \leq q(u)$ for some $c > 0$, then $(\mathcal{E}, \mathcal{F})$ has the Fatou property. In [Sch] it is shown[2] that if $(\mathcal{E}, \mathcal{F})$ is a closed positivity preserving form, then $q(u) \leq \liminf_n q(u_n)$ whenever $u_n \to u$ m-almost surely for $u_n, u \in L^2(E; m)$. In particular, $(\mathcal{E}, \mathcal{F})$ has the Fatou property, and $\mathcal{F} = \mathcal{F}_e \cap L^2(E; m)$.

DEFINITION 6. If $(\mathcal{E}, \mathcal{F})$ has the Fatou property and if μ and m are mutually absolutely continuous measures, we define $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(E; \mu)$.

LEMMA 6. *Let $(\mathcal{E}, \mathcal{F})$ be a closed form on $L^2(E; m)$ that has the Fatou property. If μ and m are mutually absolutely continuous, then $(\mathcal{E}, \mathcal{F}^\mu)$ is a closed form on $L^2(E; \mu)$. If $(\mathcal{E}, \mathcal{F})$ is either positivity preserving or Markov, then so is $(\mathcal{E}, \mathcal{F}^\mu)$.*

We must show that $\mathcal{F}^\mu$ is complete with respect to $\|u\|_1^\mu = \{\mathcal{E}(u, u) + (u, u)_{L^2(\mu)}\}^{1/2}$. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}^\mu$ that is $\mathcal{E}$-Cauchy and also Cauchy in $L^2(E; \mu)$. Then there exists $u \in L^2(E; \mu)$ and a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ that converges to u μ-almost everywhere and in $L^2(E; \mu)$. Since μ and m are mutually absolutely continuous, $u_{n_i} \to u$ m-almost everywhere, and so by Lemma 2 we see that $u \in \mathcal{F}_e$ and that $\mathcal{E}(u_{n_i} - u, u_{n_i} - u) \to 0$ as $i \to \infty$. Combined with the $L^2(E; \mu)$ convergence, this gives the closedness. Lemma 4 guarantees that $(\mathcal{E}, \mathcal{F}^\mu)$ inherits the positivity preserving or Markov property. ∎

For $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}^\mu)$ as above, and any measurable subset F of E, define

$$\mathcal{F}_F = \{u \in \mathcal{F} \mid u = 0 \text{ } m\text{-almost everywhere on } F^c\},$$
$$\mathcal{F}_F^\mu = \{u \in \mathcal{F}^\mu \mid u = 0 \text{ } \mu\text{-almost everywhere on } F^c\}.$$

[2] Under the additional, but unnecessary, assumption that $\mathcal{F}$ is dense in $L^2(E; m)$.

LEMMA 7. *Let $(\mathcal{E},\mathcal{F})$ be a closed positivity preserving form on $L^2(E;m)$. Define a measure μ by $d\mu = g\,dm$ for a strictly positive and bounded function g, and consider the corresponding form $(\mathcal{E},\mathcal{F}^\mu)$ on $L^2(E;\mu)$. If $(F_k)_{k\in\mathbb{N}}$ is an increasing sequence of sets in E, then $\bigcup_{k\in\mathbb{N}}\mathcal{F}_{F_k}$ is $\| \, \|_1$-dense in $\mathcal{F}$ if and only if $\bigcup_{k\in\mathbb{N}}\mathcal{F}^\mu_{F_k}$ is $\| \, \|^\mu_1$-dense in $\mathcal{F}^\mu$.*

PROOF. We first show that the inclusion $\mathcal{F} \subseteq \mathcal{F}^\mu$ is dense. If $u \in \mathcal{F}^\mu = \mathcal{F}_e \cap L^2(E;\mu)$, then we can choose $u_n \in \mathcal{F}$ so that $\mathcal{E}(u_n - u, u_n - u) \to 0$ and $u_n \to u$ m-almost everywhere as $n \to \infty$. Define $v_n = (-|u|) \vee (u_n \wedge |u|)$, so that $v_n \in \mathcal{F}$, $(v_n)_{n\in\mathbb{N}}$ is $\mathcal{E}$-bounded, and $v_n \to u$ in $L^2(E;\mu)$. By taking a subsequence and then Cesàro means we obtain a sequence $(w_N)_{N\in\mathbb{N}}$ in $\mathcal{F}$ so that $\|w_N - u\|^\mu_1 \to 0$ as $N \to \infty$.

Since $\bigcup_k \mathcal{F}_{F_k} \subseteq \bigcup_k \mathcal{F}^\mu_{F_k}$, it follows that if $\bigcup_k \mathcal{F}_{F_k}$ is $\| \, \|_1$-dense in $\mathcal{F}$, then $\bigcup_k \mathcal{F}^\mu_{F_k}$ is $\| \, \|^\mu_1$-dense in $\mathcal{F}^\mu$.

To prove the other direction, suppose that $\bigcup_{k=1}^\infty \mathcal{F}^\mu_{F_k}$ is a dense subset of $\mathcal{F}^\mu$ and let $u \in \mathcal{F}$. Since $u \in \mathcal{F}^\mu$ there is a sequence $(u_n)_{n\in\mathbb{N}}$ from $\bigcup_{k=1}^\infty \mathcal{F}^\mu_{F_k}$ such that $\|u_n - u\|^\mu_1 \to 0$. Setting $v_n = (-|u|) \vee (u_n \wedge |u|)$, we have $v_n \in \bigcup_{k=1}^\infty \mathcal{F}_{F_k}$ and $(v_n)_{n\in\mathbb{N}}$ is $\mathcal{E}$-bounded. By taking a subsequence and then Cesàro means we obtain a sequence $(w_N)_{N\in\mathbb{N}}$ so that $w_N \in \bigcup_{k=1}^\infty \mathcal{F}_{F_k}$, and $\|w_N - u\|_1 \to 0$ as $N \to \infty$. ∎

Suppose that E is a topological space, and m is a σ-finite Borel measure. A sequence $(F_k)_{k\in\mathbb{N}}$ of closed sets is called an *$\mathcal{E}$-nest* whenever $\bigcup_{k=1}^\infty \mathcal{F}_{F_k}$ is dense in $\mathcal{F}$, and a function $u\colon E \to \mathbb{R}$ is called *$\mathcal{E}$-quasi-continuous* if there exists a $\mathcal{E}$-nest so that the restriction $u|_{F_k}$ is continuous for each k. We suppose that $(\mathcal{E},\mathcal{F})$ is a closed form with the Markov property, that is, a *Dirichlet form*.

LEMMA 8. *If $(\mathcal{E},\mathcal{F})$ is a Dirichlet form and $(u_n)_{n\in\mathbb{N}}$ an $\mathcal{E}$-bounded sequence of $\mathcal{E}$-quasi-continuous functions in $\mathcal{F}$ that converges everywhere to a function u, then u is $\mathcal{E}$-quasi-continuous.*

PROOF. This result is proved similarly by Kuwae [K, Proposition 3.2] when E is a complete separable metric space.

Let $(u_n)_{n\in\mathbb{N}}$ and u as described above. We define an auxiliary function g on E so that $0 < g \le 1$ m-almost everywhere and $\int u_n^2 g\,dm \le 1$ for all $n \in \mathbb{N}$. To do so, choose subsets F_N of $F := \{z \in E \mid 0 < \sup_n u_n^2(z) < \infty\}$ so that $F_N \uparrow F$ and $m(F_N) < \infty$. Define

$$g(z) := \begin{cases} 1 \wedge \left(2^N \sup_n u_n^2(z) m(F_N)\right)^{-1} & \text{if } z \in F_N \setminus F_{N-1},\ N = 1,2,\dots \\ 1 & \text{if } z \notin F. \end{cases}$$

Setting $d\mu = g\,dm$, Lemma 6 shows us that $(\mathcal{E},\mathcal{F}^\mu)$ is a Dirichlet form on $L^2(E;\mu)$. Since $\sup_n \|u_n\|^\mu_1 < \infty$, the usual proof [MR1 Chapter III, Proposition 3.5] shows that u is $\mathcal{E}^\mu$-quasi-continuous. Finally we note that Lemma 7 shows that every $\mathcal{E}^\mu$-quasi-continuous function is also $\mathcal{E}$-quasi-continuous. ∎

ACKNOWLEDGEMENTS. Thanks to Jean Vaillancourt for the abstract in French.

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C. R. Math. Rep. Acad. Sci. Canada Vol. **21** (4), 1999 pp. 153–159

A TAUBERIAN ESTIMATE FOR SUMS OF COEFFICIENTS OF DIRICHLET SERIES

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ABSTRACT. Given a Dirichlet series whose coefficients satisfy some growth conditions, we deduce a Tauberian estimate for the partial sums of the coefficients weighted by a linear polynomial factor.

RÉSUMÉ. Étant donné une série de Dirichlet dont les coefficients satisfont á une condition de croissance, on déduit une estimation taubérienne pour les sommes partielles des coefficients avec un poids polynômial linéaire.

1. Introduction. In studying L-functions defined by a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

it is often more convenient to treat a dampened sum such as

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \exp(-n/X)$$

for a suitable parameter X. One can then try to deduce estimates for the partial sums

$$\sum_{n \leq X} \frac{a(n)}{n^s}$$

in order to recover information on the L-function itself. The Tauberian theorem discussed in this note is about such estimates. It was suggested by the considerations of [MM, Chapter 6] where it was used in an implicit way.

Tauberian theorems get their name from Tauber's converse to Abel's theorem. Abel showed that if the series

$$(*) \qquad \sum_{n=0}^{\infty} a_n$$

converges to the sum s, then

$$(**) \qquad \sum_{n=0}^{\infty} a_n t^n \longrightarrow s$$

Received by the editors May 3, 1999.
Research partially supported by an NSERC grant.
AMS subject classification: Primary: 11M45; secondary: 30B50.
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as $t \longrightarrow 1$ through real values. Tauber showed that the converse holds (namely that (**) implies that (*) converges and to the limit s) provided that

$$na_n = \mathbf{o}(1).$$

Littlewood significantly strengthened this by replacing the above condition with

$$na_n = \mathbf{O}(1).$$

(A good exposition of these results may be found in the book of Titchmarsh [Ti, Section 7.6].)

Hardy and Littlewood reconsidered this problem after making the change of variable

$$t = \exp(-1/X).$$

In particular, they studied how the growth of

$$\sum_{n=1}^{\infty} a_n \exp(-n/X)$$

is related to

$$\sum_{n \leq X} a_n.$$

For example, they showed [HL, Theorem 11] that for $\{a_n\}$ a sequence of real numbers satisfying

$$na_n \geq -c \tag{1.1}$$

for some constant $c > 0$, and

$$\sum_{n=1}^{\infty} a_n \exp(-n/X) = \mathbf{o}(1) \quad \text{as } X \longrightarrow \infty \tag{1.2}$$

we have

$$\sum_{n \leq X} a_n = \mathbf{o}(1). \tag{1.3}$$

Their result is actually more general. In particular, (1.1) may be replaced by

$$a_n > -n^{\alpha-1}L(n) \tag{1.4}$$

where $\alpha \geq 0$ and $L(x)$ is a *slowly oscillating* function, that is a positive continuous function defined for $x \geq 1$ and such that for every $a > 0$, we have

$$L(ax)/L(x) \longrightarrow 1 \quad \text{as } x \longrightarrow \infty.$$

Examples of slowly oscillating functions are $L(x) = (\log x)^{\beta}$, $(\log x)^{\beta_1}$, $(\log\log x)^{\beta_2}, \ldots$.

Suppose that

$$F(X) = \sum_{n=1}^{\infty} a_n \exp(-n/X)$$

satisfies

$$F(X) = \big(A_1 + \mathbf{o}(1)\big)X^{\alpha}L(X)$$

as $X \longrightarrow \infty$ for a slowly oscillating function L. Suppose also that (1.4) holds. Then, Hardy and Littlewood prove that

$$\sum_{n\le X} a_n = \left(\frac{A_1}{\Gamma(\alpha+1)} + \mathbf{o}(1)\right) X^{\alpha}L(X).$$

Several authors have tried to make explicit the relationship between the two o-terms. In particular, Freud [F] proved that if we replace (1.2) by

$$\sum_{n=1}^{\infty} a_n \exp(-n/X) \ll X^{-\delta} \tag{1.5}$$

for some $\delta > 0$, then

$$\sum_{n\le X} a_n = \mathbf{O}\big((\log X)^{-1}\big)$$

as $X \longrightarrow \infty$. Korevaar [K] showed that this is best possible. Further work on explicit remainder terms for Tauberian theorems can be found in the work of Ganelius [Ga].

In this note, we begin by proving a Tauberian theorem of the above kind where we introduce an additional smoothing.

Let us set

$$F(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w} \exp(-n/X).$$

THEOREM 1.6. *Let $\{a_n\}$ be a sequence of complex numbers satisfying*

$$\sum_{n\le X} na_n = \mathbf{O}(X^{1-\delta}) \tag{1.7}$$

and

$$|a_n| \ll n^{-\delta} \tag{1.8}$$

for some $0 < \delta < 1$. Then

$$\frac{1}{X}\int_X^{2X}\Big(\sum_{n\le x} a_n\Big)\,dx = F(0)\exp(1/X) + \mathbf{O}(X^{-\delta}\log X).$$

In particular, if (1.5) holds, the left hand side is $\mathbf{O}(X^{-\delta}\log X)$.

We remark that the restriction $\delta < 1$ is only for convenience and is not essential.

As an application of the above, we consider a Dirichlet series

$$L(s) = \sum \frac{b_n}{n^s}$$

which converges absolutely for $\mathrm{Re}(s) > 3/2$ and which has an analytic continuation as an entire function of order 1. We do not assume that $L(s)$ satisfies a functional equation. We suppose that

$$\sum_{n \le X} b_n \ll X^{1-\delta}$$

and

$$|b_n| \ll n^{1-\delta}.$$

Note that all of these assumptions are satisfied if $L(s)$ is the L-function associated to a modular form of weight 2.

THEOREM 1.9. *If* $L(1) = 0$, *then*

$$\frac{1}{X}\int_X^{2X}\left(\sum_{n\le x}\frac{b_n}{n}\right)dx \ll X^{-\delta}\log X$$

for some $\delta > 0$.

PROOF. Let $a_n = b_n/n$. With

$$F(w) = \sum \frac{a_n}{n^w}\exp(-n/X)$$

as before, we have

$$F(0) = \frac{1}{2\pi i}\int_{(2)} L(1+w)X^w\Gamma(w)\,dw$$

where the integration is on the line $\mathrm{Re}\,w = 2$. Moving the line of integration to the line $\mathrm{Re}(w) = -\eta$ with $0 < \eta < 1$, and using the fact that $L(1) = 0$, we deduce that

$$F(0) \ll X^{-\eta}.$$

Now applying Theorem (1.6), the result follows.

2. Proof of Theorem (1.6). We need some lemmas and notation. Let us set

$$S(x) = \sum_{n\le x} na_n$$

$$A(x) = \sum_{n\le x} a_n \exp(-n/X).$$

LEMMA 2.1. *Assume (1.7) holds. For* $\operatorname{Re} w = -c$, $c > \delta$, *we have*

$$|F(w)| \ll X^{c-\delta}(|w|+1)\Gamma(c-\delta).$$

PROOF. This follows from (1.7) by partial summation. Indeed,

$$\sum n a_n n^{-1+c} \exp(-n/X) = \int_{1-}^{\infty} t^{-1+c} \exp(-t/X)\, dS(t).$$

By (1.7), this is

$$\begin{aligned}&\left|\sum n a_n n^{-1-w} \exp(-n/X)\right| \\ &\quad \ll 1 + \int_1^{\infty} t^{1-\delta} \exp(-t/X) t^{-1+c} \left\{\frac{1}{X} + \frac{(|w|+1)}{t}\right\} dt \\ &\quad \ll X^{c-\delta}(|w|+1)\Gamma(c-\delta).\end{aligned}$$

This proves the Lemma.

LEMMA 2.2. *Suppose (1.7) and (1.8) hold. Let g be a differentiable function and suppose that*

$$|g(x)| \ll 1, \quad |g'(x)| \ll U^{-1}$$

on the interval $[1, U]$ *(where the implied constants are absolute). Suppose that for some* $0 < c < 1$ *(say),*

$$\frac{1}{U}\int_1^U x^w g(x)\, dx$$

is analytic for $\operatorname{Re} w \geq -c$. *Then for* $U \in [X, 2X]$,

$$\frac{1}{U}\int_1^U A(x) g(x)\, dx = F(0)\left(\frac{1}{U}\int_1^U g(x)\, dx\right) + O\big(X^{-\delta}(\log X)\Gamma(c-\delta)\big).$$

PROOF. We start with the identity

$$\frac{1}{U}\int_1^U A(x) g(x)\, dx = \frac{1}{2\pi i}\int_{(\lambda)} F(w)\left(\frac{1}{U}\int_1^U x^w g(x)\, dx\right)\frac{dw}{w} \tag{2.3}$$

valid for any $\lambda > 0$. We have

$$\frac{1}{U}\int_1^U x^w g(x)\, dx = \frac{U^{w+1}g(U) - g(1)}{(w+1)U} - \frac{1}{w+1}\frac{1}{U}\int_1^U x^{w+1} g'(x)\, dx. \tag{2.4}$$

Estimating this on the line $\operatorname{Re} w = -c$, we find

$$\begin{aligned}\left|\frac{1}{U}\int_1^U x^w g(x)\, dx\right| &\leq \frac{(U^{1-c}+1)}{U(|w+1|)} + \frac{1}{|w+1|}\frac{1}{U^2}\frac{|U^{2-c}-1|}{|2-c|} \\ &\ll \frac{1}{|w+1|}U^{-c}.\end{aligned}$$

The integral over w in (2.3) may be truncated to $|\operatorname{Im} w| \leq T$ introducing an error which is

$$\ll T^{-1}(U^{\lambda}X^{1-\delta-\lambda} + U^{1-\delta}) \ll T^{-1}X^{1-\delta}.$$

(Here, we have used (1.8).) Moving the line of integration to $\operatorname{Re} w = -c$, we find that the horizontal integrals contribute an amount

$$\ll U^{-1}X^{c-\delta}.$$

Now, using Lemma (2.1), and choosing $T = X$, we deduce the result.

LEMMA 2.5. *Suppose (1.7) and (1.8) hold. Let g be a differentiable function satisfying*

$$|g(x)| \ll 1, \quad |g'(x)| \ll U^{-1}$$

on the interval $[U, 2U]$ where the implied constants are absolute. Suppose that for some $0 < c < 1$

$$\frac{1}{U}\int_U^{2U} x^w g(x)\,dx$$

is analytic for $\operatorname{Re} w \geq -c$. Then for $U \in [X, 2X]$,

$$\frac{1}{U}\int_U^{2U} A(x)g(x)\,dx = F(0)\left(\frac{1}{U}\int_U^{2U} g(x)\,dx\right) + \mathbf{O}\big(X^{-\delta}(\log X)\Gamma(c-\delta)\big).$$

PROOF. The argument is similar to the proof of Lemma (2.2) and is omitted. With these preparations, we are now ready to prove Theorem (1.6).

PROOF OF THEOREM 1.6. We have

$$\sum_{n\leq x} a_n = \int_{1^-}^{x} \exp(t/X)\,dA(t) = \exp(x/X)A(x) - \frac{1}{X}\int_1^x \exp(t/X)A(t)\,dt.$$

Integrating over x, we deduce that

$$\frac{1}{X}\int_X^{2X}\Big(\sum_{n\leq x} a_n\Big)\,dx = \frac{1}{X}\int_X^{2X} \exp(x/X)A(x)\,dx$$

$$- \frac{1}{X^2}\int_X^{2X}\int_1^x A(t)\exp(t/X)\,dt\,dx. \tag{2.6}$$

By Lemma (2.5), we have (with $g(x) = \exp(x/X)$, $U = X$)

$$\frac{1}{X}\int_X^{2X} \exp(x/X)A(x)\,dx = F(0)(e^2 - e) + \mathbf{O}\big(\Gamma(c-\delta)X^{-\delta}(\log X)\big).$$

Let $x \in (X, 2X)$ and $0 < c < 1$. By Lemma (2.2), (with $g(t) = \exp(t/X)$, $U = x$) we have

$$\frac{1}{x}\int_1^x A(t)\quad \exp(t/X)\,dt$$

$$= F(0)\frac{X}{x}\big(\exp(x/X) - \exp(1/X)\big) + \mathbf{O}\big(X^{-\delta}(\log x)\Gamma(c-\delta)\big).$$

Hence,

$$\begin{aligned}\frac{1}{X^2}\int_X^{2X}\int_1^x A(t)\exp(t/X)\,dt\,dx \\ &= \frac{1}{X^2}\int_X^{2X} F(0)X\big(\exp(x/X)-\exp(1/X)\big)+\mathbf{O}\big(xX^{-\delta}(\log x)\Gamma(c-\delta)\big)\,dx \\ &= F(0)\big(e^2-e-\exp(1/X)\big)+\mathbf{O}\big(X^{-\delta}(\log X)\Gamma(c-\delta)\big).\end{aligned}$$

Thus,

$$\frac{1}{X}\int_X^{2X}\Big(\sum_{n\le x} a_n\Big)\,dx = F(0)\exp(1/X)+\mathbf{O}\big(X^{-\delta}(\log X)\Gamma(c-\delta)\big). \tag{2.7}$$

Choosing c appropriately (relative to δ) proves the result.

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