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ON THE GEOMETRY OF PLANAR VECTOR FIELDS

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In this article we survey results and new developments on century old open problems about planar vector fields $D = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$ or equivalently differential equations of the form

$$\text{(S)} \qquad \frac{dx}{dt} = P(x,y), \frac{dy}{dt} = Q(x,y)$$

where P and Q are real polynomial, analytic (or sometimes just smooth) functions in x, y.

Systems (S) appear often in applications. Problems concerning these systems are stated in simple terms but progress has been slow and the area could perhaps get the prize for the number of errors in the published literature, some made by famous mathematicians.

This survey is as selfcontained as possible. There is no single text to which we can refer the reader because although the basic notions are encountered in various articles and books not all authors use the terminology in the same way. For example the term "polycycle" is used in three different ways in [P81], [IY], [DRR]. For this reason we give here in a coherent way definitions of the necessary notions and state results in as precise terms as possible with a minimum of technical detail. Our goal is to make this survey easily accessible and at the same time to describe in simple ways some of the newest and most interesting developments in this area.

The theory of these systems was initiated by Poincaré. His motivations came from celestial mechanics and more specifically from the problem of the stability of the solar system. This is the problem of determining whether in the very long distant future, the solar system will behave in a similar way or whether collisions will occur in the future or perhaps one of the celestial bodies will escape from the system. In the mathematical problem (or more generally the N-body problem), Poincaré was interested in an infinite interval of time. As this problem is very complicated, Poincaré decided to concentrate first on simpler ones and in his two

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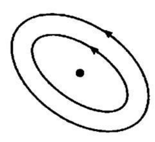
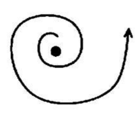
memoirs of 1881 and 1885 (*cf.* [P81], [P85]) he focused on planar polynomial systems. In these two memoirs, Poincaré initiated the theory of dynamical systems. He defined a number of notions, among them the notions of center, focus, limit cycle, first return map, *etc.*

DEFINITION. A *solution* of a system (S) is a map $s\colon (a,b) \to \mathbb{R}^2$, $a, b \in \mathbb{R}$, $a < b$, $s(t) = \big(x(t), y(t)\big)$, such that $dx(t)/dt = P\big(x(t), y(t)\big)$, $dy(t)/dt = Q\big(x(t), y(t)\big)$, for all t in (a,b). The set $\bar{s} = \big\{\big(x(t), y(t)\big) \mid t \in (a,b)\big\}$ is called the *phase curve* (or *trajectory*) of the solution s. Let $p \in \mathbb{R}^2$ and let (α_p, β_p) be the maximal interval of existence of the unique solution s of (S) such that $s(0) = p$. The *flow* of a system (S) is the function $(t,p) \mapsto \Phi_t(p)$ which associates to any point p in the plane and any $t \in (\alpha_p, \beta_p)$ the value $\Phi_t(p) = s(t)$ of the unique solution s such that $s(0) = p$. The set $\bar{s} = \{\phi_t(p) \mid t \in (\alpha_p, \beta_p)\}$ is called the *orbit* of p under the flow ϕ. The *phase portrait* of (S) is the partition of $\mathbb{R}^2$ into orbits or global phase curves, oriented according to increasing t in (α_p, β_p).

DEFINITION. A point (x_0, y_0) such that $P(x_0, y_0) = 0 = Q(x_0, y_0)$ is called a *singular point* of (S). A *center* is an isolated singular point (x_0, y_0) which is surrounded by closed trajectories (Fig. 1a). A *focus* is an isolated singular point possessing a neighborhood U such that for every point p of U, $\Phi_t(p)$ is defined for all $t \geq 0$ (or for all $t \leq 0$) and tends to p when t tends to $+\infty$ (or when t tends to $-\infty$) such that the tangent at $\Phi_t(p)$ to the curve $\bar{s}$ has no limit as t tends to $+\infty$ (or when t tends to $-\infty$), the curve spiraling towards p (Fig. 1b).



(a) Center



(b) focus

Figure 1

DEFINITION. Let p be a singular point of a system (S) with P, Q, C^1-functions defined in a neighbourhood of p. The eigenvalues λ_1, λ_2 of the Jacobian matrix at p of P, Q are called *the eigenvalues of the linearized system at* p (or simply *the eigenvalues at* p) of (S). A singular point p of a system (S) is called *elementary* (respectively *hyperbolic*) if at least one of the two eigenvalues at p is not zero (respectively both eigenvalues have non-zero real parts).

EXAMPLE 1. The origin in the linear system $dx/dt = -y$, $dy/dt = x$ is a center. Indeed, for any solution $\big(x(t), y(t)\big)$ we have $d(x^2 + y^2)/dt = 2(xdx/dt + ydy/dt) = 2(-xy + yx) = 0$, so $x^2 + y^2$ is a constant along a phase curve and the phase curves are circles. We note that 0 is an elementary but not a hyperbolic singular point since the eigenvalues at 0 are $\pm i$.

EXAMPLE 2.

$$\frac{dx}{dt} = -y + x(x^2 + y^2), \quad \frac{dy}{dt} = x + y(x^2 + y^2)$$

In this example we have the same linear part as in the linear Example 1 but the trajectories behave differently and the origin is a focus. Indeed, passing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we obtain the system $dr/dt = r^2$, $d\theta/dt = 1$. Since $dr/dt = 2r > 0$ and $d\theta/dt > 0$, both $r(t)$ and $\theta(t)$ are increasing functions and hence we have a focus.

EXAMPLE 3 (MOUSSU [M.R]). $dx/dt = y^3$, $dy/dt = -x^3 + (x^2y^2)/2$. As will be later shown, in this quartic example the origin is a center (Fig. 2). This will result from a symmetry which the system has with respect to the x-axis. More precisely we have: $P(x,-y) = -P(x,y)$ and $Q(x,-y) = Q(x,y)$ where $P(x,y) = y^3$ and $Q(x,y) = -x^3 + (x^2y^2/2)$. We note that in this example both the linear part as well as the quadratic part at the origin are zero. We say that the origin is here a degenerate center.

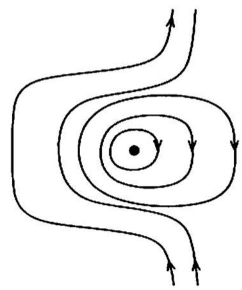


Figure 2: Moussu's example

THE PROBLEM OF THE CENTER. Determine and classify the systems (S) which have a center.

The first result on this problem was obtained by Poincaré in [P85].

DEFINITION. Let P, Q be C^1-functions in a open set U of $\mathbb{R}^2$. A *first integral* of (S) in U is a C^1-function $F: U \to \mathbb{R}^2$ such that $F(x(t), y(t))$ is constant for all t for all solutions $(x(t), y(t))$ of (S) with values in U. The system is *integrable on* U if it admits a first integral on U which is nonconstant on any open subset of U.

THEOREM (POINCARÉ [P85]). *Let p be a singular point of a polynomial system (S). Let us assume that the eigenvalues at p are purely imaginary. Then the system has a center at p if and only if the system admits a nonconstant first integral in a neighbourhood of p which is analytic.*

IDEA OF PROOF. Without loss of generality we may suppose that the singular point is placed at the origin. A rotation of axes and time rescaling brings the system to the form:

$$\frac{dx}{dt} = -y + P_2(x,y) + \cdots + P_n(x,y), \quad \frac{dy}{dt} = x + Q_2(x,y) + \cdots + Q_n(x,y)$$

with $P_i(x,y)$, $Q_i(x,y)$ homogeneous polynomials in x, y of degree i. For the necessity we look for an analytic function $F = F_1 + \cdots + F_i + \cdots$ (F_i homogeneous polynomial of order i) which is a first integral in a neighbourhood of 0. Identifying with zero the coefficients in $dF/dt = (\partial F/\partial x)P + (\partial F/\partial y)Q = 0$ yields equations of the form: $y(\partial F_i/\partial x) - x(\partial F_i/\partial y) = H_i$, H_i a polynomial in x, y. Passing to polar coordinates, Poincaré then shows that if 0 is a center these equations yield an inductive definition of F_i, H_i being completely determined by F_j with $j < i$. Convergence is then proved. The sufficiency is immediate: in view of $dF/dt = (\partial F/\partial x)P + (\partial F/\partial y)Q = 0$ calculations show that $F_1 = 0$ and F_2 must be $a(x^2 + y^2)$, with a a constant and hence an analytic change of coordinates brings the first integral to the form $F^*(x', y') = x'^2 + y'^2$. ∎

The following algebraic lemma is useful:

LEMMA (SHI [SH2]). *Let us consider a polynomial system (S) with* $P(x,y) = -y + \sum_{i+j=2}^{n} a_{ij}x^iy^j$, $Q(x,y) = x + \sum_{i+j=2}^{n} b_{ij}x^iy^i$. *Let us consider the ring* $K = \mathbb{Q}[a_{20}, \ldots, a_{0n}, b_{20}, \ldots, b_{0n}]$ *of polynomials in* $a_{20}, \ldots, a_{0n}$, $b_{20}, \ldots, b_{0n}$ *with rational coefficients. Then there exists a formal series* $F(x,y) = F_1(x,y) + F_2(x,y) + \cdots + F_n(x,y) + \cdots$ *with* $F_2(x,y) = (x^2 + y^2)/2$, $F_p(x,y)$ *a homogeneous polynomial in* x, y *with rational coefficients and* $V_i \in K$ *such that the following holds in* $K[x,y]$: $(\partial F/\partial x)P(x,y) + (\partial F/\partial y)Q(x,y) = \Sigma V_i(x^2+y^2)^{i+1}$.

PROOF. Staightforward calculation and linear algebra. ∎

The problem of determining all cases of quadratic systems (*i.e.*, systems (S) with $\max(\deg(P), \deg(Q)) = 2$) with a center was considered by Dulac in [Du]. Actually Dulac considered complex differential equations and he defined a center (*cf.* [Du, p. 5]) as being a singular point p whose eigenvalues λ_1, λ_2 are both non-zero with λ_1/λ_2 a rational negative number and such that the system has a nonconstant analytic first integral in the neighbourhood of p. He considered the particular case when $\lambda_1/\lambda_2 = -1$, case which can be brought to the normal form:

$$\frac{dx}{dt} = x + P_2(x,y), \quad \frac{dy}{dt} = -y + Q_2(x,y).$$

THEOREM (DULAC [DU1]). *Let* P, Q *be polynomials with real or complex coefficients with* $\max(\deg P, \deg Q) = 2$. *Suppose a system (S) has a singular point which is a center in the above sense. Then the system can be integrated in finite terms, i.e., by using a finite number of transformations on the system, we can integrate it obtaining an elementary expression* F *such that* $DF = 0$ *where* $D = P\,\partial/\partial x + Q\,\partial/\partial y$.

DULAC'S PROOF. Assume $F(x, y)$ is an analytic first integral of the system in a neighbourhood of the center which we may assume to be placed at the origin. Imposing the condition that $dF/dt = 0$ yields $(\partial F/\partial x)P + (\partial F/\partial y)Q = 0$ and identifying with zero the corresponding coefficients of the terms in x and y, yields conditions on the coefficients of P, Q, the conditions to have a center. When these conditions are satisfied, Dulac shows that the system can be integrated via various *ad hoc* methods. ∎

For a real system with a center Dulac's result says that the local condition on the system to have a center at a singular point p in $\mathbb{R}^2$ implies "global" integrability, *i.e.*, integrability on an open dense subset U of $\mathbb{R}^2$ which turns out to be the complement of an algebraic curve (*cf.* [S1]). The conditions on the coefficients of P, Q to have a center are algebraic and they form global integrability conditions for the system in the above sense. A note of warning: Although conditions for a center appear in [Dul], they are not readily accessible being scattered throughout the article and they do not appear in a compact form in [Dul]. For references on more compact forms for these conditions for real systems and for the history of the subject consult [S1].

It is interesting to note that the global integrability result obtained by Dulac is due to the presence of algebraic curves $f(x, y) = 0$ which are formed by phase curves.

DEFINITION. A curve $f(x, y) = 0$ with f a polynomial with complex coefficients, is a solution of a polynomial system (S) if (i) f is irreducible over $\mathbb{C}$ and (ii) $Df = fK$ where $D = P\partial/\partial x + Q\partial/\partial y$ and K is a polynomial in x, y with complex coefficients.

In [Da] Darboux gave a beautiful geometric method of integration of systems (S) when they possess sufficiently many algebraic solutions.

THEOREM (DARBOUX [DA]). *Assume the polynomial system (S) with $m = \max(\deg P, \deg Q)$ has $n > m(m+1)/2$ algebraic solutions $f_i = 0$. Then there exist complex numbers $\lambda_1, \dots, \lambda_n$ such that $F = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}$ is a first integral of the system in the sense that $DF = 0$.*

PROOF. We have $Df_i = f_i K_i$ for $i = 1, \dots, n$. It follows that $\deg(K_i) \leq m-1$. Hence K_i has at most $m(m+1)/2$ coefficients and since $n > m(m+1)/2$ we must have a linear dependence $\lambda_1 K_1 + \cdots + \lambda_n K_n = 0$ over $\mathbb{C}$. So $DF = F(\lambda_1 Df_1/f_1 + \cdots + \lambda_n Df_n/f_n) = F(\lambda_1 K_1 + \cdots + \lambda_n K_n) = 0$. ∎

Instead of applying the various *ad hoc* methods used by Dulac to integrate the systems, for the various possible values of the parameters one can apply the geometric method of integration of Darboux via algebraic curves obtaining a unified proof for all values of the parameters (*cf.* [S2], [S3]). In the case of cubic systems the situation is much more complex: some of those which have a center are integrable due to the presence of algebraic solutions but others do not have such curves. There are many works in this direction but only partial results were obtained. Żołądek and his student Sokulski have worked on this problem and Żołądek formulated the conjecture that the cubic systems are of two kinds: those integrable via Darboux's method and those which are reversible in a sense he defines in [Z] but so far this work is not completely understood. Although we have numerous results on cubic systems, the problem of the center remains open in the general cubic case. Obtaining conditions for center along the lines of Dulac's proof is much more difficult in this case due to the increased number of coefficients. Attempts by researchers to calculate a basis of the ideal I generated by the polynomials V_i (of the Shi's Lemma) in the ring $K = \mathbb{Q}[a_{20}, \dots, a_{0n}, b_{20}, \dots, b_{0n}]$ via computer algebra methods were unsuccessful in the general case. The number of terms in the polynomials V_i grows very fast with i, for example V_2 has over a hundred terms. In view of Hilbert's basis theorem, this ideal I is finitely generated so there exists an integer m such that I is generated by $V_1, \dots, V_m$. For $n = \max(\deg P, \deg Q)$ we are interested in determining the smallest such m, which is called the Bautin index. Work on the various series (A series) which could be used for defining the Bautin index is being done by M. Briskin, J. P. Francoise and Y. Yomdin.

DEFINITION (POINCARÉ [P85]). A *limit cycle* of a system (S) is a closed phase curve (cycle) which is isolated (Fig. 3).

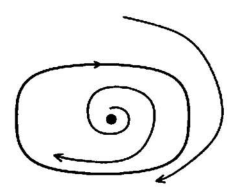


Figure 3: Limit cycle

In quadratic systems centers cannot coexist with limit cycles but they can coexist in cubic systems as the following interesting cubic system given by Dolov [Do] (see Fig. 4) shows:

EXAMPLE 4 (DOLOV [DO]).

$$\frac{dx}{dt} = x^2(-1 + \varepsilon + y) - y + y^2, \quad \frac{dx}{dt} = x - x^3$$

with ε positive and $\varepsilon \ll 1$ (Fig. 4).

THE POINCARÉ FIRST RETURN MAP. Let Γ be the trajectory of a non-trivial periodic solution s of a differential system (S) and let $T > 0$ be the smallest period of s. Let Σ be a line segment passing through a point A of Γ, transversal to Γ. Since the solution $\Phi_t(A)$ with initial condition A returns to A after the period T, by the continuity of solutions with respect to the initial conditions, we

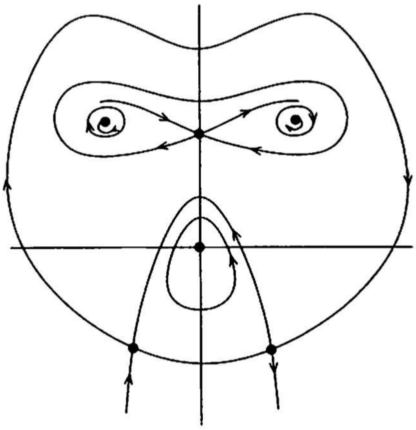


Figure 4: Dolov's example

can show that for an initial condition A' situated on Σ and sufficiently close to A, the solution will return to Σ at a time τ close to T, *i.e.*, there exists a time t such that $\Phi_t(A') \in \Sigma$. The *Poincaré first return map* for a small neighbourhood σ in Σ of A is the map $P: \sigma \to \Sigma$, with $P(A') = \Phi_\tau(A')$ where τ is the first return time of $\Phi_t(A')$ to Σ.

The notion of the Poincaré first return map can analogously be defined for a periodic orbit of a flow on a manifold. In particular for a C^1 flow on the cylinder $S^1 \times \mathbb{R}$ for which $S^1 \times \{0\}$ is an orbit, we have a return map for any section $\{\theta_0\} \times [-1,1]$. As an application of the Poincaré first return map we now show that in the Example 3 the origin is a center: Passing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ the system becomes $(S_{r,\theta}) : dr/dt = r^3\psi(\theta)$, $d\theta/dt = -r^2\varphi(\theta)$ with $\psi(\theta) = (\sin\theta\cos\theta)(\sin^2\theta - \cos^2\theta + \frac{\sin^2\theta\cos\theta}{2})$, $\varphi(\theta) = \cos^4\theta + \sin^4\theta - \frac{(\sin^2\theta\cos^3\theta)}{2}$. A simple calculation yields $\varphi(\theta) = \cos^2 2\theta + \frac{\sin^2 2\theta}{2}(4 - \cos\theta)$ and hence $\varphi(\theta) > 0$ for all θ. Thus, for $r \neq 0$ the system $(S_{r,\theta})$ has no singularity and clearly its phase portrait coincides for $r \neq 0$ with that of the system $S'_{r,\theta} : dr/d\tau = r\psi(\theta)$, $d\theta/d\tau = -\varphi(\theta)$. Both systems have right hand sides which are periodical in θ and yield flows on the cylinder $S^1 \times \mathbb{R}$. Any solution $\theta(t)$ of $d\theta/d\tau = -\varphi(\theta)$ yields the periodic solution $r = 0$, $\theta(\tau)$ of $(S'_{r,\theta})$. Hence, for any transversal section $\Sigma = \{0\} \times (-\varepsilon, \varepsilon) \subseteq S^1 \times \mathbb{R}$ we have a return map P. Since the map $S^1 \times \mathbb{R}^+ \to \mathbb{R}^2 \setminus \{0\}$, $(\theta, r) \mapsto (r\cos\theta, r\sin\theta)$ is a diffeomorphism, the return map P^+ on Σ^+ in $S^1 \times \mathbb{R}$, yields a return map P' on the section induced by $\theta = 0$, *i.e.*, on $(0,\varepsilon) \times \{0\}$ of the flow on $\mathbb{R}^2$. In view of the symmetry of the quartic system with respect to the x-axis, we must have $P'(x) = x$ for $x \in (0,\delta)$, $\delta \leq \varepsilon$ and hence the origin is a center.

REMARK. We observe that in the Example 3 discussed above, the quartic system on $\mathbb{R}^2 \setminus \{(0,0\}$ is brought via the diffeomorphism $S^1 \times (0,\infty) \to \mathbb{R}^2 \setminus \{0\}$

to a system on the upper half of the cylinder. This last system has the same phase portrait as $(S'_{\theta,r})$ for $r \neq 0$ which admits an extension on the whole cylinder and in particular on $S^1 \times [0,\infty)$, which has a periodic solution $\theta(t)$, $r = 0$. Thus the origin in $\mathbb{R}^2$ is replaced in this construction ("the blow-up construction via polar coordinates") by the circle $S^1 \times \{0\}$ on the cylinder. On $S^1 \times [0,\infty)$ the system $(S^1_{\theta,r})$ no longer has singularities. This is a very simple example of the technique of "desingularisation" of vector fields. For more information regarding this technique see articles by Dumortier, Il'yashenko and Roussarie in [S6].

THE POINCARÉ COMPACTIFICATION. A real planar polynomial system (S) can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space $\mathbb{R}^3$ with coordinates X, Y, Z. The central projection of the vector field $X = P\partial/\partial x + Q\partial/\partial y$ on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. There exists (for a proof *cf.* [GV]) an analytic vector field $\pi(X)$ on the whole sphere such that its restriction on the upper hemisphere has "the same phase" portrait as the one induced by the polynomial vector field projected on the upper hemisphere.

By "the same phase portrait" we mean that the two portraits are orbitally topologically equivalent, *i.e.*, there exists a homeomorphism of the upper hemisphere into itself carrying the orbits of one phase portrait into orbits of the other and preserving their orientation.

DEFINITION. A *stable (unstable) characteristic orbit* (or *characteristic phase curve*) of a vector field $P\partial/\partial x + Q\partial/\partial y$ *at a singular point* p is a trajectory of a solution $s(t) = \big(x(t), y(t)\big)$, defined for all $t \geq 0$ (or for all $t \leq 0$) with $s(t)$ tending to p as t tends to $+\infty$ (or to $-\infty$) becoming tangent to a fixed straight line, *i.e.*, such that the function $t \mapsto \big(s(t)-p\big)/ \parallel s(t) - p \parallel$ tends to a limit as t tends to $+\infty$ (or to $-\infty$). A *graphic* in $\mathbb{R}^2$ (or S^2), is a subset formed by singular points p_i (not necessarily distinct) and characteristic orbits $\bar{s}_i$, $i = 1, 2, \ldots, n$ so that $\bar{s}_i$ is an unstable characteristic orbit of p_i and a stable characteristic orbit of p_{i+1} with $p_{n+1} = p_1$. A graphic is oriented in the sense of the flow.

DEFINITION. Let st: $[0,\varepsilon] \to \mathbb{R}^2$ be a *smooth semitransversal* to a graphic Γ of a smooth vector fields (S) on the plane at a nonsingular point p of Γ, *i.e.*, st is a smooth map, $\mathrm{st}(0) = p$ and $\big\langle \big(d(\mathrm{st})/dt\big)(0), \big(P(p), Q(p)\big)\big\rangle \neq 0$ (we sometimes just say that $\mathrm{st}([0,\varepsilon])$ is the semitransversal). We say that the graphic Γ has a Poincaré first return map for (S) with respect to st if $\exists\delta > 0$, $\delta < \varepsilon$ such that for any $u \in (0,\delta)$ and for $q = \mathrm{st}(u)$, there exist $t > 0$ such that $\phi_t(q) \in \mathrm{st}(0,\varepsilon)$. Let T be the first such $t > 0$. The map $P_{\mathrm{st}}\colon \sigma \to \Sigma$ where $\sigma = \mathrm{st}\big((0,\delta)\big)$, $\Sigma = \mathrm{st}\big((0,\varepsilon)\big)$ and $P_{\mathrm{st}}(q) = \phi_T(q)$ is called the *Poincaré first return map* of the graphic Γ with respect to st (Fig. 5.)

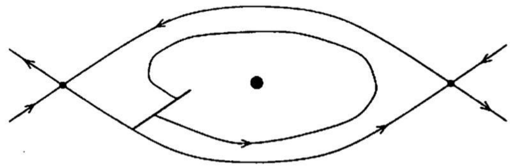


Figure 5: A graphic and its first return map

DEFINITION. A *polycycle* is a graphic with $k \geq 1$ singular points and with a return map.

EXAMPLE 5.

$$\frac{dx}{dt} = -y - x^2 - y^2, \quad \frac{dy}{dt} = x + 3xy.$$

In this system we have two polycycles, each with two singularities. We indicate the phase portrait of the system in Fig. 6. In this example both polycycles have return maps which are identities since the polycycles surround centers.

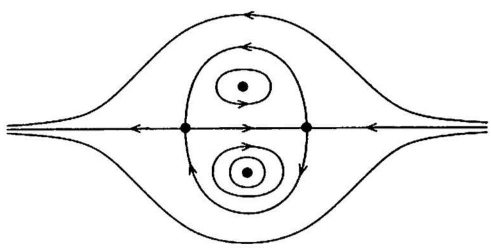


Figure 6

At the International Congress of Mathematicians in 1900 David Hilbert gave a list of 23 open problems [H]. The most famous problem on planar polynomial vector fields is the second part of Hilbert's 16th problem.

DEFINITION. Let n be a positive integer and let $H(n)$ be the maximum of the number of limit cycles occuring in a system (S) defined by real polynomials P, Q of maximum degree n, *i.e.*, if $K(n, P, Q)$ is the number of limit cycles in a real polynomial system (S) with $n = \max(\deg P, \deg Q)$, then $H(n)$ is the maximum of the numbers $K(n, P, Q)$ when we fix n and vary P, Q.

THE SECOND PART OF HILBERT'S 16TH PROBLEM. Find $H(n)$ and describe the relative positions of limit cycles of systems (S) defined by polynomials P, Q with $\max(\deg P, \deg Q) \leq n$.

What makes this problem so hard is that it is a doubly global problem: we want the behaviour of the systems in the whole plane, and this for the whole class of such systems with maximum degree n.

In 1923 Dulac claimed to have proved the following theorem in [Du2]:

THE FINITENESS THEOREM. Any given real planar polynomial vector field has a finite number of limit cycles.

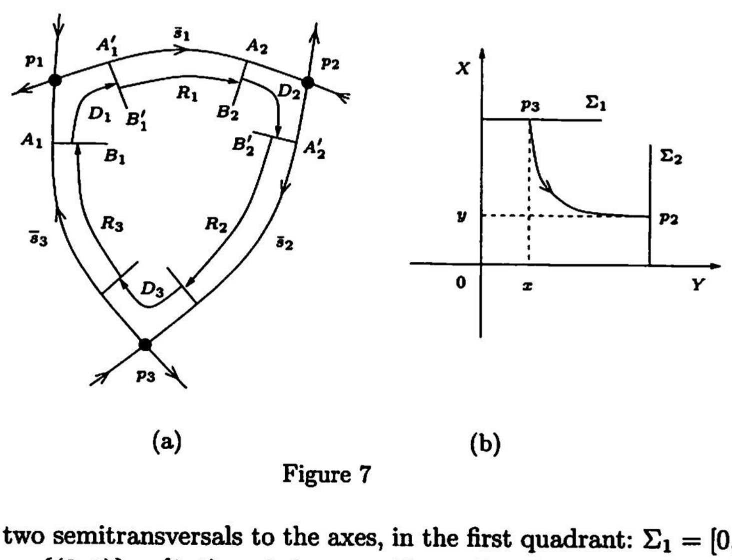
In the 1970's Dumortier noticed a gap in this proof and later Il'yashenko discovered that there was also an error (*cf.* [I]). In the late 1980's, independently of one another, J. Ecalle and Yu. Il'yashenko gave proofs of the finiteness theorem. An outline of one of the proofs appeared in [EMMR]. Ecalle and Il'yashenko wrote books [I], [E], containing the proofs. Actually the theorem which they proved is more general.

THEOREM (ECALLE [E], IL'YASHENKO [I]). *An analytic vector field on a closed two-dimensional surface has only finitely many limit cycles.*

From this theorem the Finiteness Theorem follows, via the Poincaré compactification of systems (S) on the sphere.

Dulac's idea for proving the finiteness theorem was to construct the first return map of a polycycle as a composition of transit maps D_i around its "corners" determined by the singular points p_i and regular transit maps R_i along the phase curves $\bar{s}_i$ (Fig. 7(a)).

While the maps R_i's are analytic also at A_i''s, D_i's in general have singularities at A_i. To see this consider the simple example of the linear system $dx/dt = x$, $dy/dt = -ry$ with $r > 0$ whose flow is $\phi_t(x_0, y_0) = (e^t x_0, e^{-rt} y_0)$ and a corresponding transit map (Fig. 7(b)).



(a) (b)

Figure 7

Consider two semitransversals to the axes, in the first quadrant: $\Sigma_1 = [0,1) \times \{(0,1)\}$, $\Sigma_2 = \{(0,1)\} \times [0,1)$ and the map $\Sigma_1 \to \Sigma_2$ which associates to the point $(x,1)$ on Σ_1 its image $(1,y)$ in Σ_2 under the flow, *i.e.*, $\phi_t(x,1) = (1,y)$. This yields $1 = e^t x$ and $y = e^{-rt}$. Hence $y = x^r$ for $x > 0$. The map $D\colon \Sigma_1 \setminus \{(0,1)\} \to \Sigma_2 \setminus \{(1,0)\}$ which is $D(x,1) = (1,x^r)$ is determined by the map $x \mapsto x^r$. Making an abuse of notation we may denote this last map by D, *i.e.*, $D\colon (0,1) \to (0,1)$,

$D(x) = x^r$. This map is analytic for $r \in \mathbb{N}$. If $r \in (k, k+1)$, $k \geq 0$, $k \in \mathbb{N}$ then $y = x^r$ is C^k in $[0,1)$ and k-flat at zero, *i.e.*, $(d^{(i)}y/dx^i)(0^+) = 0$ for $0 \leq i \leq k$, but it is not C^{k+1} at 0^+. In this example, the system is linear and the constructed transit map D has a simple expression $D(x) = x^r$. Nevertheless, the map D may not even be C^1 at 0^+ (*e.g.* for $r < 1$). If $0 < r < 1$, $\lim_{x \to 0^+} \frac{y(x)-y(0)}{x} = +\infty$ and we say that the function is vertical at $x = 0$. If the vector field is not linear the expression for the transit map D becomes much more complicated.

The Poincaré first return map P of a periodic orbit is analytic and so is $P(x) - x$ and hence isolated zeroes of this function cannot accumulate on a zero x_0 of $P(x) - x$. As isolated zeroes of $P(x) - x$ correspond to limit cycles, we conclude that limit cycles cannot accumulate on a limit cycle.

This argument does not automatically extend for polycycles because $P = R_n \circ D_n \circ \cdots \circ R_1 \circ D_1$ and the maps D_i's are in general not even smooth at A_i. To be able to conclude that isolated zeroes of the map $P(x) - x$ cannot accumulate at A_1 a subtle analysis of the map P (and hence of D_i's) needs to be made. It was in a Lemma of this analysis that Dulac made his error. In Dulac's work we find however the proof (checked by Il'yashenko as indicated in [I, p. 11]) of the following theorem:

DULAC'S THEOREM. We can find a semitransversal to a polycycle of an analytic vector field such that the corresponding return map P is either flat, or vertical or admits an asymptotic expansion

$$\tilde{P}(x) := a_0 x^{v_o} + \sum_{i=1}^{\infty} x^{v_i} P_i(\ell n x)$$

with $0 < v_0 < v_1 < \cdots < v_n < \cdots$, $\lim_{i \to \infty} v_i = \infty$ and $P_i \in \mathbb{R}[x]$ (in other words, for any natural number N there exists a partial sum $S(x)$ of the above series such that $P(x) - S(x) = o(x^N)$). (The series above may be divergent.)

The proofs of Il'yashenko and Ecalle of the finiteness theorem are very different: while Il'yashenko's proof is based on complex methods and in particular on a generalization of the Phragmen-Lindelöf principle, Ecalle's proof is based on his resummation techniques of divergent series.

Ecalle and Il'yashenko's proofs of the finiteness theorem were obtained more than sixty years after its announcement by Dulac in 1923. Another theorem whose proof came many years after its statement was Fermat's Last Theorem (FLT) which says that the equation $x^n + y^n = z^n$ has no solution in integers for $n > 2$. It is believed that this theorem was stated by Fermat around 1657. Its proof was obtained in 1994 by Andrew Wiles with the help of Richard Taylor. This theorem and Hilbert's 16th problem have something in common: they both appeal by their simplicity of statement. Because of its very simple statement and the long time it took to prove it, the announcement by A. Wiles in 1993 of the proof of FLT aroused a tremendous amount of interest and the proof has been under much scrutiny by outstanding specialists since 1993. First, the referees for the work, discovered an error which was corrected by A. Wiles together with R. Taylor in 1994. Later, other specialists studied the proof. We cannot say that Ecalle and Il'yashenko's proofs of the Finiteness Theorem were scrutinized in a similar way by the mathematical community.

An important step in proving FLT was the statement in 1922 of a conjecture of Mordell.

MORDELL'S CONJECTURE. An algebraic curve $f(x, y) = 0$ over $\mathbb{Q}$ (*i.e.*, f is a polynomial in x, y with rational coefficients) of genus $g \geq 2$ has a finite number of rational points (*i.e.*, points (a, b) in $\mathbb{Q}^2$).

Mordell's conjecture was proved by Faltings in 1983 and in 1986 Faltings received the Fields' medal.

There is a parallelism in dates for the two finiteness conjectures and proofs shown below:

Fermat's Last Theorem (FLT):	Hilbert's 16th problem (H16):
Statement of (FLT): 1637	Statement of (H16): 1900
Mordell's conjecture (MC): 1922	Dulac's conjecture (DC): 1923
Falting's proof of (MC): 1983	Ecalle/Il'yashenko's proofs of (DC): 1987
Wiles's proof of (FLT): 1994	Solution of (H16)? $H(n) = ?$

We do not have a solution for Hilbert's 16th problem even in its simplest case, *i.e.*, we do not know what is $H(2)$.

A. Wiles' lecture at the ICM 98 showed clearly that this achievement was obtained by a truly collective work. The broad view and deep connections used for proving FLT is more than impressive. There was the 1955 Taniyama-Shimura conjecture (TSC) which connected elliptic curves over the rationals and modular functions. K. Ribet said that the two subjects seemed so far apart that he never considered working on TSC. Yet these subjects turned out to be related as was shown by A. Wiles who in 1994, with the help of R. Taylor, proved a weaker version of TSC: semi-stable elliptic curves are modular. G. Frey was the first person to point out that TSC must imply FLT. In 1986 Ribet showed that the weaker version of TSC implies FLT. The amazing unity of mathematics is evident in these strikingly deep connections.

The subject of planar vector fields is not an isolated area of research. In this study analysis plays an important part and it has had the upper hand, this in spite of the geometrical ideas which have intervened starting with the work of Poincaré. Bifurcation theory of dynamical systems, which describes sudden qualitative changes in the behaviour of systems when parameters change smoothly, is very much at the core of the subject. Computer algebra is very helpful in performing some calculations and numerical analysis and computer simulations are useful in extending our intuition. Commutative and differential algebra also intervene in the subject. Nevertheless, the depth of connections and the broad view which intervened in the proof of FLT are a great deal more impressive. Of course, number theory is an old and very fundamental subject, the queen of mathematics as Gauss said. By comparison, the subject of dynamical systems founded by Poincaré in the 1880's (in particular the study of planar vector fields) is a much newer area of research. Broader viewpoints as well as deeper connections could only be helpful and recent research points in this direction as we shall later see.

In view of the Finiteness Theorem, we know that $H(n) \leq \aleph_0$. A first question is:

QUESTION. Is $H(n)$ finite?

We are far from having an answer to this question even in the case of quadratic systems but work which involves the notion of finite cyclicity of graphics is in progress.

DEFINITION. A graphic Γ in the phase portrait of a vector field $X = P\partial/\partial x + Q\partial/\partial y$ has *finite cyclicity* in a family of vector fields $X_\lambda = P_\lambda\partial/\partial x + Q_\lambda\partial/\partial y$, analytic (smooth) for (x, y) in an open subset of $\mathbb{R}^2$ and λ in an open set of $\mathbb{R}^m$, with $X_{\lambda_0} = X$, if there exists a natural number N, a neighbourhood A of Γ and a neighbourhood V of λ_0 such that the number of limit cycles of X_λ in A is less than N, for all λ in V.

This notion can be extended for graphics Γ and families X_λ of vector fields on a surface, in particular on the sphere S^2.

An important step for answering in the affirmative the question mentioned above is to prove that every graphic of a vector field X (or system (S)) of maximum degree n has finite cyclicity within the family of planar polynomial systems (S) with $\max(\deg P, \deg Q) \leq n$.

The first result in this direction was obtained by Bautin [B] who proved that if p is a singular point with purely imaginary eigenvalues of a quadratic vector field X_0 then the graphic $\Gamma = \{p\}$ has finite cyclicity within the family of quadratic systems. Actually Bautin's result says more: it says that three is the maximum number of limit cycles which could arise near p in any quadratic perturbation of X_0. To put it briefly, we say that the "cyclicity" of Γ in the family of quadratic systems is three.

The notion of cyclicity of a graphic Γ within a family of vector fields X_λ where λ is a parameter, was introduced by R. Roussarie [Rou]. We consider below the definition of this notion. To specify what is meant by "limit cycles close to a graphic" we use the Hausdorff distance $d_H(A, B)$, for two nonempty compact sets A, B of S^2 or of a submanifold M of S^2:

$$d_H(A, B) = \sup_{(x,y)\in A\times B} \{d(x, B), d(y, A)\}.$$

DEFINITION. Let $\{X_\lambda\}_{\lambda\in I}$ be a family of vector fields $I \subset R^k$ on a submanifold M of S^2 and let Γ be a nonempty compact subset of M which is the union of phase curves for some X_{λ_0}. Let us suppose that there exists a sequence $\{\lambda_i\}_{i\geq 1}$, $\lambda_i \in I$ and $\lambda_0 \in I$ with $\lim_{i\to\infty} \lambda_i = \lambda_0$ and for each $i \geq 1$ a limit cycle γ_i of the vector field X_{λ_i} such that $\lim_{i\to\infty} d_H(\gamma_i, \Gamma) = 0$. In this case we say that Γ is a *limit periodic set* of the family $\{X_\lambda\}_{\lambda\in I}$ [FP].

DEFINITION. Let us suppose that Γ is a graphic of a vector field X_{λ_0} on a submanifold M of S^2 which has *finite cyclicity* within a family $\{X_\lambda\}_{\lambda\in I}$ on M, *i.e.*, there exists $N < \infty$, A a neighbourhood of Γ in S^2 and V a neighbourhood of λ_0 in S^2 such that for every λ in V, the number of limit cycles of X_λ in A is less than N. Let V be a small neighbourhood of λ_0 in I, *i.e.*, the diameter of V is less than a small $\delta > 0$. For sufficiently small $\varepsilon > 0$ and $\lambda \in V$. Then the *cyclicity* of Γ is the number

$$\mathrm{Cycl}(\Gamma, \{X_\lambda\}_{\lambda\in I}) = \inf_{\substack{\varepsilon\to\{0\},\delta\to\{0\}\\ \lambda\in V,\, \mathrm{diam}\, V<\delta}} N.$$

THEOREM (BAUTIN [B]). *Within the family of quadratic systems, the cyclicity of a graphic $\Gamma = \{p\}$ is three when p is a singular point with pure imaginary eigenvalues.*

DEFINITION. An *elementary* (resp. *hyperbolic*) *graphic* is a graphic having only elementary (resp. hyperbolic) singular points. A *homoclinic loop* of a vector field X is a graphic $\Gamma = \{p, \bar{s}\}$ where p is a singular point of X and $\bar{s}$ is the trajectory of a solution $\phi_t(q)$ such that $\lim_{t\to+\infty} \phi_t(q) = p = \lim_{t\to-\infty} \phi_t(q)$ (Fig. 8(b)).

THEOREM (LEONTOVICH [L]). *Let $\Gamma = \{p, \bar{s}\}$ be a homoclinic loop of a planar analytic vector field X with p a hyperbolic saddle, i.e., $\lambda_1 \cdot \lambda_2 < 0$, where λ_1, λ_2 are real eigenvalues at p and $\lambda_1 + \lambda_2 \neq 0$. Then the cyclicity of Γ within any one parameter analytic family of planar analytic vector fields is at most one.*

A number of other results on finite cyclicity of graphics of vector fields were obtained, some of them are mentioned in the articles of Roussarie or Rousseau in [S6]. In particular the above theorem was generalized by Roussarie (see article by Roussarie in [S6]).

Results proving finite cyclicity for smooth families of vector fields were obtained for generic cases by Il'yashenko and Yakovenko (*cf.* [IY]).

THEOREM 1 (IL'YASHENKO AND YAKOVENKO [IY]). *An elementary graphic in a generic smooth finite parameter family of vector fields $X_\lambda = f_\lambda(x,y)\partial/\partial x + g_\lambda(x,y)\partial/\partial y$ can generate in this family at most a finite number of limit cycles.*

THEOREM 2 (IL'YASHENKO AND YAKOVENKO [IY]). *For any natural k, the number of limit cycles which can be born from an elementary graphic occuring* *in a generic k-parameter smooth family of planar vector fields, admits an upper estimate by a certain number* $E(k) < \infty$ *which depends only on the number of parameters. The function* $k \mapsto E(k)$ *is an effectively computable primitive recursive function.*

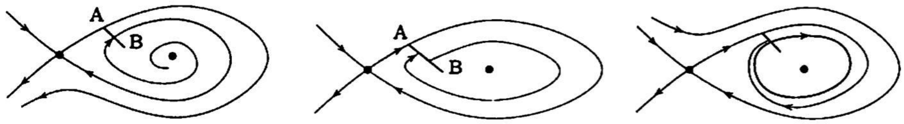
The meaning of the word "generic" in the above two theorems is not easily accessible in the proofs. For example, specialists working on this problem for the particular case of quadratic systems affirm about [IY] in [GR]: "the genericity conditions are not explicitly given". In [K] V. Kaloshin improved the theorems above and using some new ideas, he obtained the first explicit bound for $E(k)$, *i.e.*, for $k \in \mathbb{Z}_+$, $E(k) \leq 2^{25k^2}$. Furthermore, his proof is simpler easing the access to the genericity conditions.

The proofs of the theorems of Il'yashenko and Yakovenko involve calculations of the first return map for families which are smooth perturbations of graphics, in a sense to be explained below:

When an elementary graphic Γ of a vector field $X = X_{\lambda_0}$ is smoothly perturbed in a family X_λ ($\lambda, \lambda_0 \in \mathbb{R}$), it may yield a generalized Poincaré first return map in X_λ for λ close to λ_0. To understand this we consider here a simple example: a homoclinic loop $\Gamma = \{p, \bar{s}\}$ where p is a hyperbolic point with eigenvalues $\lambda_1 < 0 < \lambda_2, \lambda_1 + \lambda_2 \neq 0$. We suppose the graphic has one characteristic orbit $\bar{s}$ corresponding to a solution $s(t)$ such that $s(t)$ tends to p when t tends to both plus and minus infinity and the other two characteristic orbits of p lie outside the loop. It can be shown that in this case the graphic is a polycycle (*cf.* [ALGM]). Let us suppose that an analytic (smooth) family of vector fields X_λ, $\lambda \in \mathbb{R}$ is such that for a value λ_0 of the parameter we have $X = X_{\lambda_0}$. Let us consider a line segment AB (semitransversal st) belonging to the interior of the loop formed by $\bar{s}$ with A on the characteristic orbit $\bar{s}$ associated to s. Since hyperbolic points are preserved under small smooth perturbations and since their characteristic orbits vary smoothly with these perturbations, one can show (*cf.* [ALGM]) that there exist sufficiently small $\varepsilon > 0$ and $\delta > 0$ such that for any A' in $\sigma = \mathrm{st}((0, \varepsilon))$ and any λ with $\|\lambda - \lambda_0\| < \delta$, for $\lambda > \lambda_0$ (or respectively $\lambda < \lambda_0$) there is a time $t_\lambda > 0$ such that $\Phi_{t_\lambda}(A') \in \Sigma = \mathrm{st}([0, 1])$. Let τ_λ be the smallest such t_λ. For each λ we put $P_\lambda : \sigma \to \Sigma$ with $P_\lambda(A') = \Phi_{\tau_\lambda}(A')$. The map P is a Poincaré first return map for the family X_λ (Fig. 8(b) or (c)). (We observe that the segment AB in Fig. 8(a) has no return map.) We point out that we may have graphics which are not necessarily polycycles which smoothly perturbed possess a Poincaré first return map for the family. It is this more general situation which occurs in the theorems of Il'yashenko and Yakovenko.

The theorems of Il'yashenko and Yakovenko were obtained by adapting to this problem the beautiful geometric method of A. Khovansky (*cf.* [Kh1], [Kh2]).

Briefly put, the method of Khovansky when applied to the problem of bounding the number of limit cycles in a family of vector fields in the neighbourhood



(a) $\lambda < 0$ (b) $\lambda = 0$ (c) $\lambda > 0$

Figure 8

of a polycycle, simplifies the analytic arguments by introducing some geometric arguments in the picture. To give a hint of this method here let us consider a smooth family $\{X_\lambda\}_{\lambda \in I}$ of vector fields in the neighbourhood of a graphic Γ of $X = X_{\lambda_0}$, which we may here suppose to be hyperbolic. Let us assume we have return maps P_λ. In evaluating P_λ we use Dulac transit maps $D_{\lambda,i}$, for each singularity p_i of Γ and regular transit maps $R_{\lambda,i}$ (as in Fig. 7(a)) to obtain $P_\lambda = R_{\lambda,k} \circ D_{\lambda,k} \circ \cdots \circ R_{\lambda,1} \circ D_{\lambda,1}$. We then look for isolated zeroes of $P_\lambda(x) - x$ (fixed points of P_λ). Equivalently we may look for isolated solutions of the functional equations:

$$\text{(FE)} \qquad y_1 = D_{\lambda,1}(x_1), R_{\lambda,1}(y_1) = x_2, \ldots, y_k = D_{\lambda,k}(x_k), R_{\lambda,k}(y_k) = x_1.$$

The next step in Khovansky's method is to replace these functional equations with a mixed Pfaff-functional system of equations. To see this we consider here a very simple example (for more substantial applications see [MR], [IY], [RSZ], [GR]). Let us consider the case where Γ is a homoclinic loop with a nondegenerate hyperbolic singular point which we may assume placed at the origin. Assume $\{X_\lambda\}_{\lambda \in I}$ to be analytic. Let us suppose that $D_\lambda(x) = x^{r(\lambda)}$ is obtained from the system which in the vicinity of 0 is $\dot{x} = x$, $\dot{y} = -r(\lambda)y$. Then $y = D_\lambda(x) = x^{r(\lambda)}$ turns out to be is an integral curve of the Pfaff equation $xdy - r(\lambda)ydx = 0$. Indeed, by differentiating $y = x^{r(\lambda)}$, after multiplication by x we obtain the Pfaff differential equation $xdy - r(\lambda)ydx = 0$ of which $y = x^{r(\lambda)}$ is an integral curve. We have thus obtained a mixed Pfaff-functional system of equations:

$$\text{(Pf-F)} \qquad xdy - r(\lambda)ydx = 0, \quad R_\lambda(y) = x$$

and we look for intersections of the curve $R_\lambda(y) = x$ with an integral curve of the Pfaff equation. Considering the curve $y = D_\lambda(x)$ as one of the integral curves of the Pfaff equation presents advantages. Suppose we have two "adjacent" intersection points of the system $y = D_\lambda(x)$, $R_\lambda(y) = x$. Then on $R_\lambda(y) = x$ between these two points it can be shown that there must be a point of contact of this curve with an integral curve of the Pfaff equation, *i.e.*, the two curves have the same tangent at that point. This is analogous to the theorem of Rolle and in [Kh2] Khovansky proves such analogues of the theorem of Rolle. It follows that the number of intersection points of the two functional equations $y = D_\lambda(x)$, $R_\lambda(y) = x$ is bounded by one plus the number of contact points of $R_\lambda(y) = x$ with integral curves of the Pfaff equation. Hence this number is one plus the number of contact points of the Pfaff system obtained by differentiating the second equation:

$$\text{(Pf)} \qquad xdy = r(\lambda)ydx = 0, \quad R'_\lambda(y)dy - dx = 0$$

and situated on $R_\lambda(y) = x$.

This is the same as the number of zeroes for the equation

$$\begin{vmatrix} x & -r(\lambda)y \\ R'_\lambda(y) & -1 \end{vmatrix} = 0$$

situated on the curve $x = R_\lambda(y)$. Replacing $x = R_\lambda(\lambda)$ in the above equation we just need to find the solutions of $-R_\lambda(y) + r(\lambda)yR'_\lambda(y) = 0$. Since $R_\lambda(y)$ is analytic we obtain:

$$\operatorname{Cycl}(\Gamma, X_\lambda) \leq \operatorname{ord}\big(-R_\lambda(y) + r(\lambda)yR'_\lambda(y)\big) + 1$$

and since $R_\lambda(y)$ is a homeomorphism we thus obtain:

$$\operatorname{Cycl}(\Gamma, X_\lambda) \leq 2.$$

For an analytic family X_λ, the result of Leontovich gives $\operatorname{Cycl}(\Gamma, X_\lambda) \leq 1$. We thus traded here the lower bound 1, for an easy and more geometric proof of the result in this instance. The method is helpful in proving finiteness results and was used in the proofs of the theorems of Il'yashenko and Yakovenko. It was Moussu (and Roche) who first used it in such applications (*cf.* [MR]) and it was in a lecture by Moussu that the author first heard about this method. Later other authors lectured about it and used it in their work (*e.g.* [IY], [RSZ], [GR]).[1] We point out that in the general case of a graphic with several singular points, we must use several times the last reduction from the Pfaff system to obtain results.

A recent interesting work concerning the finiteness part of Hilbert's 16th problem is A. Mourtada's work [M.A] in which a geometric and at the same time differential algebraic approach is taken. Algebras of quasi-analytic maps are constructed and one of the goals is to show that the problem considered in the work is described by these algebras. For this, certain Weierstrass type preparation arguments and differential algebraic methods for these more general algebras are used. Part of these are based on work of Tougeron [T].

DEFINITION. A *connection* between two singular points p and q of a system (S) is a characteristic orbit approaching p if t tends to $+\infty$ (or to $-\infty$) and approching q as t tends to $-\infty$ (resp. to $+\infty$). (Here p need not be different from q.)

[1] The author thanks Anna Guzman who in lively mathematical discussions explained to her applications of Khovansky's method. She also thanks C. Rousseau for reprints and references.

THEOREM (MOURTADA [M.A]). *In any analytic deformation which preserves the connections, the number of limit cycles close to a hyperbolic polycycle and their multiplicity are uniformly bounded by a natural number.*

Mourtada modifies the geometrical picture pushing it one step forward, to the construction of a vector field (a derivation) which he then uses to obtain his result. We only give here a hint of his approach. For simplicity, let us consider a hyperbolic graphic Γ of $X = X_{\lambda_0}$ with three singular points p_1, p_2, p_3 (Fig. 7(a)). In this case the functional equations are:

$$y_1 = D_{\lambda,1}(x_1), \quad y_2 = D_{\lambda,2}(x_2), \quad y_3 = D_{\lambda,3}(x_3)$$
$$R_{\lambda,1}(y_1) = x_2, \quad R_{\lambda,2}(y_2) = x_3, \quad R_{\lambda,3}(y_3) = x_1.$$

The maps $R_{\lambda,i}$'s are regular homeomorphisms. We consider their inverses $y_i = f_{\lambda,i}(x_{i+1}), i = 1,2,3,$ (with the convention that $x_4 = x_1$) and we have $f_{\lambda,i}(x_{i+1}) = \alpha_i(\lambda) + a_i(\lambda)x_{i+1} + o(x_{i+1}, \lambda)$ with $a_i(0) > 0$. These yield coordinate transformations $X_{i+1} = a_i(\lambda)x_{i+1} + o(x_{i+1}, \lambda)$. Using the coordinates X_i's and replacing the equations $R_{\lambda,i}(y_i) = x_{i+1}$ by $y_i = f_{\lambda,i}(x_{i+1})$ we obtain the new system of functional equations:

$$y_i = D_{\lambda,i}(X_i), \quad y_i = \alpha_i(\lambda) + X_{i+1}, \quad i = 1,2,3.$$

Eliminating y_i, $i = 1,2,3$ in the above equations we obtain just three equations in the variables X_i, $i = 1,2,3$:

$$(\mathrm{F_i}) \qquad \alpha_i(\lambda) = D_{\lambda,i}(X_i) - X_{i+1}, \ i = 1,2,3.$$

We note that for λ fixed, the functions on the right hand side are constant. Let $G = (g_1, g_2)$ where $g_1 = D_{\lambda,1}(X_1) - X_2$, $g_2 = D_{\lambda,2}(X_2) - X_3$. A level set $G = G_o$ where $(g_{10}, g_{20}) = G_0$ is a constant, is a curve for which we may take as a parametrization $X_1 = X_1, X_2 = D_{\lambda,1}(X_1) - g_{10}, X_3 = D_{\lambda,2}(D_{\lambda_1}(X_1) - g_{10}) - g_{20}$. Differentiating these three equations with respect to X_1, we obtain a vector field χ (a derivation). In constructing this vector field we only used the first two of the three equations $(\mathrm{F_i})$. A solution of the system of three equations $(\mathrm{F_i})$ is obtained by intersecting the above integral curve of the vector field with the third functional equation (F_3) which we may write $f = 0$, where $f = D_{\lambda,3}(x_3) - x_1 - \alpha_3(\lambda)$.

In the general case we have a derivation $\chi = \partial/\partial X_1 + \sum_{j=2}^{k} \chi_j \partial/\partial X_j$ which is studied and made to act on algebras of quasi-analytic functions with asymptotic developments which Mourtada calls of "Hilbert type" since they appear in Hilbert's 16th problem. Using this derivation Mourtada gives an interpretation of the coefficients of the Bautin ideal in the context of a polycycle. The derivation is then used to reinterpret "finite cyclicity" as the finiteness of the degree of a certain projection map π_χ, restricted to $f = 0$ (*cf.* [M.A]). These new developments bring differential algebraic methods into the subject. We point out that differential algebraic techniques have recently become present in other areas of mathematics, in particular in diophantine geometry (*cf.* [B1], [B2], [B3], [B4]). In [B1], A. Buium gave a new and very short proof, using differential algebraic techniques, to a finiteness conjecture, the Manin-Mumford conjecture in diophantine geometry, initially proved by Raynaud [R] with algebro-geometric methods. As in Buium's work, where the basic building blocs are derivations on rings, in Mourtada's approach to Hilbert's 16th problem the key elements are derivations, this time on quasi-analytic algebras "of Hilbert type". Thus the differential algebraic techniques, along with geometry form here a common ground in these recent works in two different mathematical disciplines.

Let us now consider the much more limited question:

QUESTION. Is $H(2)$ finite?

There is a program under way to solve this problem (*cf.* [DRR]) which reduces, via the Poincaré compactification of planar vector fields on the sphere, the finiteness of $H(2)$ to the problem of proving the finite cyclicity for the 121 cases of possible graphics which appear in the family of all quadratic vector fields. About half of the 121 possible cases were settled but the remaining ones and especially those which suround centers are considerably more difficult to treat. While work on these cases is in progress, up to now NO example of a quadratic system was found for which we can prove that more than four limit cycles exist. (For an example with at least four limit cycles *cf.* [Sh1].) For this reason we have the following:

CONJECTURE. $H(2) = 4$.

It would be impossible to prove this conjecture along the lines described above for proving that $H(2)$ is finite. Assuming we had the proof that $H(2)$ is finite, to solve this conjecture we would still need to do the global analysis of this large class with numerous phase portraits (perhaps over two thousand). Several studies of specific subclasses of quadratic systems were done (Hamiltonian systems [AL1], systems with a center [S4] and [PS], systems with a weak focus of order three [AL2] and many others) but except for very limited special classes such as the systems with a center, these studies do not give us a clear idea of the global geometry, the results often being a catalogue of phase portraits (some incomplete), and done in terms of coefficients of global normal forms. As coefficients change under affine transformations, most of these classifications are not intrinsic. Work in this area with interdisciplinary methods which combine geometric vision with algebraic and analytic tools is needed. New geometric structure needs to be defined for better organizing the information contained in numerous calculations and making it more accessible and transparent. A program in this direction is under way. Studies involving the concept of multiplicity of intersection of projective curves and its use in organizing global informations in an intrinsic way for some integrable quadratic or cubic systems were done, for example in [PS], [S5] and [RS1].

QUESTION. The affine group acts on the class of quadratic systems and hence this class depends essentially on five parameters. Is there a way to interpret geometrically these five parameters?

In [RS] the authors view the class of quadratic systems as a fiber bundle over pencils of conics, via the action of the affine group and show that this is a principal fiber bundle. This work yields a nice geometric interpretation of the five parameters: two of them move the singular points and another three are rotation parameters. It is expected that this work will help in summing up the information contained in numerous articles regarding the problem of classification of quadratic systems and also in further research on this problem.

To complete this survey, we also mention the work of the logicians [vDS] in the direction of Hilbert's 16th problem, using a model-theoretic approach. In [vDS] the authors construct an expansion $\mathbb{R}_{an*}$ of the ordered real field ($\mathbb{R}$, <, 0, 1, +, −, ·) and they prove that it has two model-theoretic properties: model completeness and *o*-minimality. The model theorists hope that "in a sufficiently rich *o*-minimal expansion of $\mathbb{R}$ one would have all of the Poincaré return maps for polynomial vector fields in $\mathbb{R}^2$ (cf. [Ma]). This approach brings to mind Ecalle's construction of the trialgebra of transseries $\mathbb{R}[[[x]]]$ and of the trialgebra $\mathbb{R}\{\{\{x\}\}\}$ of transseries which are accelero-summable. Ecalle's approach to the finiteness theorem (Dulac's conjecture) is in constructing the formal counterpart $\tilde{P}$ of the return map, studying it and then resumming it. "All properties of the return map P, beginning with its non-oscillating character, can be detected, understood and proven on the formal object $\tilde{P}$" (*cf.* [E1]). These two approaches have quite a bit in common and their more constructive nature makes them both very appealing.

As we have seen in this survey, this area of research is very much alive and broader connections are beginning to appear, motivating new investigations.

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EULER POLYNOMIALS AT RATIONAL NUMBERS

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Presented by M. Ram Murty, FRSC

ABSTRACT. We prove that $s^n\left(E_n(r/s) - (-1)^{rs}E_n(0)\right) \in \mathbf{Z}$ whenever $r, s \in \mathbf{Z}$, $s \neq 0$, where $E_n(t)$ is the n-th Euler polynomial. This result is similar to a result of Almkvist and Meurman concerning Bernoulli polynomials.

RÉSUMÉ. Nous démontrons que $s^n\left(E_n(r/s) - (-1)^{rs}E_n(0)\right) \in \mathbf{Z}$ quand $r, s \in \mathbf{Z}$, $s \neq 0$, et quand $E_n(t)$ est l'$n^{\text{ième}}$ polynôme d'Euler. Ce résultat correspond au résultat d'Almkvist et Meurman en ce qui concerne les polynômes de Bernoulli.

Introduction. For $n \in \mathbf{Z}$, $n \geq 0$, the Euler polynomials, $E_n(t)$, are defined by the generating function

$$\frac{2e^{tx}}{e^x+1} = \sum_{n=0}^{\infty} E_n(t)\frac{x^n}{n!}, \quad |x| < \pi. \tag{1}$$

The following well-known property of Euler polynomials,

$$E_n(t+1) + E_n(t) = 2t^n, \tag{2}$$

derives from (1), as does the property

$$E_n(1-t) = (-1)^n E_n(t), \tag{3}$$

each of which holds for all $n \in \mathbf{Z}$, $n \geq 0$. From (2), we see that for $m, n \in \mathbf{Z}$, $m \geq 1$, $n \geq 0$,

$$E_n(m) - (-1)^m E_n(0) = 2\sum_{j=0}^{m-1}(-1)^{m-1-j}j^n.$$

This implies that $E_n(m) - (-1)^m E_n(0) \in \mathbf{Z}$ for these values of m and n. By analyzing this quantity for $m \leq 0$, utilizing both (2) and (3), we obtain $E_n(m) - (-1)^m E_n(0) \in \mathbf{Z}$ for all $m, n \in \mathbf{Z}$, $n \geq 0$.

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The Bernoulli polynomials, $B_n(t)$, $n \in \mathbf{Z}$, $n \geq 0$, are similarly defined by

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}, \quad |x| < 2\pi.$$

We can then define the Bernoulli numbers, B_n, according to $B_n = B_n(0)$, from which we obtain $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6, \ldots$, and $B_n = 0$ for n odd, $n \geq 3$. For n even, $n \geq 2$, we have

$$B_n = -\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m}B_m.$$

Bernoulli polynomials can be written in terms of Bernoulli numbers by means of the relation

$$B_n(t) = \sum_{m=0}^{n}\binom{n}{m}B_m t^{n-m}.$$

The Bernoulli numbers are rational numbers whose denominators are square-free. In fact, the von Staudt-Clausen theorem states that

$$B_n + \sum_{\substack{p\,\text{prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}$$

for even n. The Euler polynomials have an expansion in terms of Bernoulli numbers according to

$$E_n(t) = \sum_{m=0}^{n}\binom{n}{m}2(1-2^{m+1})\frac{B_{m+1}}{m+1}t^{n-m}. \tag{4}$$

Additional information concerning both the Bernoulli and the Euler polynomials can be found in [1].

It has recently been shown by Almkvist and Meurman [2] that the Bernoulli polynomials have the property that whenever $r, s \in \mathbf{Z}$, $s \neq 0$, then $s^n\big(B_n(r/s) - B_n(0)\big) \in \mathbf{Z}$, for any $n \geq 0$. Other proofs of this result are given in [3], [4], and [5]. We wish to show that the Euler polynomials share a similar property. We shall prove the following

THEOREM. *Let* $r, s \in \mathbf{Z}$, $s \neq 0$. *Then for any* $n \in \mathbf{Z}$, $n \geq 0$, *we have* $s^n\big(E_n(r/s) - (-1)^{rs}E_n(0)\big) \in \mathbf{Z}$.

This theorem is obtained by making use of the above classical results on Euler polynomials and Bernoulli numbers, and the method of proof can, in a similar manner, be applied to prove the result of Almkvist and Meurman concerning Bernoulli polynomials (see [4]).

THE PROOF OF THE THEOREM. We shall use the following lemma.

LEMMA. *Let p be prime and let $r, s \in \mathbf{Z}$ be such that $p \nmid s$. If $f(x) \in \mathbf{Q}[x]$, then there exists some $k \in \mathbf{Z}$, $k \geq 1$, such that*

$$f\left(\frac{r}{s}\right) \equiv f(rs^{\phi(p^k)-1}) \pmod{p}.$$

PROOF. For $f(x) \in \mathbf{Q}[x]$ there must exist some $k \in \mathbf{Z}$, $k \geq 1$, such that $p^{k-1}f(x) \in \mathbf{Z}_p[x]$, where $\mathbf{Z}_p$ is the set of p-adic integers. Since $p \nmid s$, we see that $s^{\phi(p^k)} \equiv 1 \pmod{p^k}$. Therefore, since the coefficients of $p^{k-1}f(x)$ are in $\mathbf{Z}_p$,

$$p^{k-1}f\left(\frac{r}{s}\right) = p^{k-1}f(rs^{-1}) \equiv p^{k-1}f(rs^{\phi(p^k)-1}) \pmod{p^k}.$$

Dividing through by p^{k-1}, we have

$$f(rs^{-1}) \equiv f(rs^{\phi(p^k)-1}) \pmod{p},$$

completing the proof. ∎

We can now prove the theorem.

PROOF. Let $r, s \in \mathbf{Z}$, $s \neq 0$, and let p be prime.

Suppose that $p \nmid s$. From (4) we see that $E_n(t) \in \mathbf{Q}[t]$. Thus, by the lemma, there is some $k \in \mathbf{Z}$, $k \geq 1$, such that

$$E_n\left(\frac{r}{s}\right) \equiv E_n(rs^{\phi(p^k)-1}) \pmod{p}.$$

Note that rs has the same parity as $rs^{\phi(p^k)-1}$, thus

$$E_n\left(\frac{r}{s}\right) - (-1)^{rs}E_n(0) \equiv E_n(rs^{\phi(p^k)-1}) - (-1)^{rs^{\phi(p^k)-1}}E_n(0) \pmod{p}.$$

Since $rs^{\phi(p^k)-1} \in \mathbf{Z}$, we must then have $E_n(r/s) - (-1)^{rs}E_n(0) \in \mathbf{Z}_p$.

If $p \mid s$, then we consider

$$\begin{aligned} &s^n E_n\left(\frac{r}{s}\right) - s^n(-1)^{rs}E_n(0) \\ &= \sum_{m=0}^{n} \binom{n}{m} 2(1-2^{m+1})\frac{B_{m+1}}{m+1} r^{n-m}s^m - (-1)^{rs}2(1-2^{n+1})\frac{B_{n+1}}{n+1}s^n, \end{aligned}$$

which follows from (4). Using $v_p(a)$ to denote the exponent of p in the rational number a, we see that for each m with $0 \leq m \leq n$, we have

$$v_p\left(\binom{n}{m} 2(1-2^{m+1})\frac{B_{m+1}}{m+1} r^{n-m}s^m\right) \geq m + v_p(B_{m+1}) - v_p(m+1) + v_p(2) \geq 0,$$

with the second of these two inequalities utilizing the von Staudt-Clausen theorem by means of the relation $v_p(B_{m+1}) \geq -1$. Thus $s^n\left(E_n(r/s) - (-1)^{rs}E_n(0)\right) \in \mathbf{Z}_p$.

Therefore, if p is prime, we must have $s^n\left(E_n(r/s) - (-1)^{rs}E_n(0)\right) \in \mathbf{Z}_p$. Since this is true for every prime, this same quantity must be in $\mathbf{Z}$. ∎

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ON THE IRRATIONAL QUARTIC ALGEBRA

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RÉSUMÉ. Dans cette note, on présente un compte rendu détaillé des résultats récents sur l'automorphisme d'ordre de quatre de la C*-algèbre de rotation. Si A_θ est une C*-algèbre de rotation engendrée par des unitaires U, V tels que $VU = e^{2\pi i\theta}UV$ et si σ est un automorphisme de Fourier de A_θ défini par $\sigma(U) = V$, $\sigma(V) = U^{-1}$, alors on démontre que, pour un ensemble G_δ dense du paramètre réel θ dans $[0,1]$ (qui contient les rationnels), il y a un isomorphisme $K_0(A_\theta \rtimes_\sigma \mathbb{Z}_4) \cong \mathbb{Z}^9$ donné explicitement par neuf modules canoniques sur la C*-algèbre $A_\theta \rtimes_\sigma \mathbb{Z}_4$ (le produit croisé de groupe cyclique d'ordre quatre $\mathbb{Z}_4$).

1. Introduction. This paper reports on recent work on the order four automorphism of the rotation algebra and its associated crossed product C^*-algebra by the finite cyclic group $\mathbb{Z}_4$. Let A_θ be the rotation algebra, generated by canonical unitaries U and V such that $VU = \lambda UV$, where $\lambda = e^{2\pi i\theta}$. Consider the *Fourier* automorphism σ of A_θ given by $\sigma(U) = V$, $\sigma(V) = U^{-1}$, let $B_\theta = A_\theta \rtimes_\sigma \mathbb{Z}_4$ denote the associated crossed product by the cyclic group $\mathbb{Z}_4$, and let W denote the canonical unitary of this crossed product. The algebra B_θ is called a *quartic* algebra in [5], and we shall adopt that terminology here. The reason for calling σ the Fourier automorphism is that it corresponds to the Fourier transform for an important module used in the construction below. (Not to mention that it has order *four*!)

The following questions were raised by George Elliott for irrational θ:

(1) What are the K-groups of B_θ?

(2) Is B_θ an AF-algebra?

The purpose of this note is to address the first question, as a first step in attempting the second. The difficulty of this problem lies in finding a way to compute the K-groups of crossed products by finite cyclic groups (for which there are no known exact sequences similar to the Pimsner-Voiculescu sequences for crossed products by the free groups $\mathbb{Z}$ or $\mathbb{F}_n$). Questions (1) and (2) have been settled for the flip automorphism in [2], [3], [7], [9], [10].

For the rational case, the K-groups are known:

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THEOREM A [5]. *For rational θ, $K_0(B_\theta) \cong \mathbb{Z}^9$ and $K_1(B_\theta) = 0$.*

One would expect that this would hold for all θ. However, the techniques that have worked for the flip automorphism ($U \mapsto U^{-1}$, $V \mapsto V^{-1}$) do not work in this case. For example, Kumjian's computation [7] of the K-groups for the associated crossed product $A_\theta \rtimes \mathbb{Z}_2$ made use of Natsume's exact sequence for K-groups of crossed products by amalgamated products of groups, and one of the key steps is the use of the isomorphism $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ which enabled him to express $A_\theta \rtimes \mathbb{Z}_2$ as the crossed product of $C(\mathbb{T})$ by the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. However, it is not clear that anything similar to this can be done in the Fourier case and the group $\mathbb{Z}_4$. (It is still an open question, pointed out by Chris Phillips to the author, whether one can write B_θ as the crossed product of $C(\mathbb{T})$ by some group—hopefully, an amenable group.) In [12, Example 2 of Section 7] the author provides a method for obtaining the same result as Kumjian's for a dense G_δ set of θ's, but without use of Natsume's sequence nor the fact $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$. This technique, although less strong and giving the result for a dense G_δ set of the parameter, seems a little less *ad hoc* since it also works for the irrational quartic algebra, as will be outlined here, and one also obtains a specific basis for K_0. Detailed proofs are contained in [11] and [12].

2. The modules. There is a canonical set of nine modules over B_θ. Canonical in the sense that the modules come from $K_0(\Gamma)$, where Γ is the C^*-algebra of the continuous field of C^*-algebras $\{B_t : t \in [0,1]\}$. Six of these modules can be written down as projections in B_θ:

$$P_1(\theta) = \tfrac{1}{2}(1 + W^2)$$
$$P_2(\theta) = \tfrac{1}{2} + \left(\tfrac{1}{+}i4\right) W + \left(\tfrac{1}{-}i4\right) W^3$$
$$P_3(\theta) = \tfrac{1}{4}(1 + W + W^2 + W^3)$$
$$P_4(\theta) = \tfrac{1}{2}(1 + \lambda^{1/2} UVW^2)$$
$$P_5(\theta) = \tfrac{1}{2} + \left(\tfrac{1}{+}i4\right) \lambda^{1/4} UW + \left(\tfrac{1}{-}i4\right) \lambda^{-1/4} VW^3$$
$$P_6(\theta) = \tfrac{1}{4}\left(1 + \lambda^{1/4} UW + \lambda^{1/2} UVW^2 + \lambda^{-1/4} VW^3\right).$$

(Here, of course, the unitaries depend on θ.) The remaining three modules arise from the Fourier module constructed as follows. Let $S(\mathbb{R})$ denote the Schwartz space on the reals. Consider the Heisenberg module over A_θ (of Connes and Rieffel) given by $S(\mathbb{R})$ with the right actions

$$(fU)(t) = f(t + \theta), \quad (fV)(t) = e^{2\pi i t} f(t).$$

It turns out that the classical Fourier transform extends this to a (finite projective) module $\mathcal{F}_\theta$ over B_θ according to action

$$(fW)(t) = \frac{1}{\sqrt{\theta}} \widehat{f}(-t/\theta),$$

where $\widehat{f}$ is the Fourier transform of $f \in S(\mathbb{R})$ (and $\theta > 0$). It yields two other ("rotated") modules: $\mathcal{F}_\theta(i)$, where the action of W is replaced by iW, and $\mathcal{F}_\theta(-1)$ where $W \to -W$, and in both cases the actions of U, V remain the same. Thus we have three more modules

$$P_7(\theta) = \mathcal{F}_\theta, \quad P_8(\theta) = \mathcal{F}_\theta(i), \quad P_9(\theta) = \mathcal{F}_\theta(-1).$$

3. The Chern map. There is a Chern map

$$\mathbf{T}: K_0(B_\theta) \to \mathbb{R} \times \mathbb{Z} \times \mathbb{C}^2 \times \mathbb{R}^3, \quad \mathbf{T} = (\tau; C_1; T_{10}, T_{11}; T_{20}, T_{21}, T_{22})$$

where τ is the canonical trace of B_θ, C_1 the canonical second order Chern character on A_θ (of Connes [4, III.2.β]) appropriately extended to B_θ, and T_{jk} are the unbounded traces on B_θ given, on the canonical smooth subalgebra of B_θ, by

$$\begin{aligned} T_{20}(U^m V^n W^2) &= \lambda^{-mn/2}\,\delta'_{m,0}\delta'_{n,0} & T_{10}(U^m V^n W^3) &= \lambda^{(m-n)^2/4}\,\delta'_{m,n} \\ T_{21}(U^m V^n W^2) &= \lambda^{-mn/2}\,\delta'_{m,1}\delta'_{n,1} & T_{11}(U^m V^n W^3) &= \lambda^{(m-n)^2/4}\,\delta'_{m,n+1} \\ T_{22}(U^m V^n W^2) &= \lambda^{-mn/2}\,\delta'_{m,n+1} \end{aligned}$$

where at other generic elements $U^m V^n W^k$ they vanish, and $\delta'_{r,s}$ is 1 if r and s have the same parity and is 0 otherwise. (See [11, Section 2].)

(One also has the 0-dimensional cyclic cohomology group isomorphism $HC^0(B_\theta) \cong \mathbb{C}^8$ for irrational θ [11, Theorem 2.3].) Calculating the Chern map on each of the nine modules $P_j(\theta)$, which was the main task of [11], one gets the following table of values:

CHERN CHARACTER TABLE

Projection	τ	C_1	T_{10}	T_{11}	T_{20}	T_{21}	T_{22}
$P_1(\theta)$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0
$P_2(\theta)$	$\frac{1}{2}$	0	$\frac{1-i}{4}$	0	0	0	0
$P_3(\theta)$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0
$P_4(\theta)$	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	0
$P_5(\theta)$	$\frac{1}{2}$	0	0	$\frac{1-i}{4}$	0	0	0
$P_6(\theta)$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0
$P_7(\theta)$	$\frac{\theta}{4}$	-1	$\frac{1+i}{8}$	$\frac{1+i}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
$P_8(\theta)$	$\frac{\theta}{4}$	-1	$\frac{i-1}{8}$	$\frac{i-1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$
$P_9(\theta)$	$\frac{\theta}{4}$	-1	$-\frac{1+i}{8}$	$-\frac{1+i}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

From this one immediately obtains:

THEOREM B [11, Theorem 2.2]. *The classes of the modules $P_1(\theta), \dots, P_9(\theta)$ are independent in $K_0(B_\theta)$ for each $0 < \theta \le 1$. For rational θ, the Chern map* $\mathbf{T}$ *is injective.*

The latter result holds by the previous theorem since in the rational case $K_0(B_\theta) \cong \mathbb{Z}^9$ and the nine modules are independent in it. The computation of the unbounded traces for the Fourier module $\mathcal{F}_\theta$ turned out to involve the classical theta functions of Jacobi. More specifically, they arose when one looks more closely at Rieffel's formula relating traces on strongly Morita equivalent C^*-algebras [8, Proposition 2.2]. Such a formula reduces to one of Jacobi's equations relating three theta functions. (An example is given in the introduction to [11].) In fact, Boca [1] has recently been able to use theta function theoretic techniques to construct a projection of trace θ in the fixed point algebra of A_θ under σ.

4. The rational case. The main step is to show that when θ is rational the nine modules generate, and hence (by Theorem B) form a basis for, $K_0(B_\theta)$. This is done by constructing an auxiliary basis for $K_0(B_\theta)$ using the realization of B_θ, where $\theta = p/q$, as a sphere with singularities obtained by Farsi-Watling [5, Theorem 6.2.1].[1] (That is, it consists of continuous $M_q(\mathbb{C})$-valued functions on the 2-sphere that commute with certain finite order unitaries at three points.) Our approach consisted not of computing $\mathbf{T}$ but rather its 0-dimensional component $\mathbf{T}' = (\tau; T_{10}, T_{11}; T_{20}, T_{21}, T_{22})$ on this auxiliary basis. We show that when $\mathbf{T}'$ is computed for each auxiliary basis element the result is in the $\mathbb{Z}$-span of $\mathbf{T}'$ of the nine modules. Thus

$$\mathbf{T}'(K_0(B_\theta)) = \sum_{j=1}^{9} \mathbb{Z}\mathbf{T}'(P_j(\theta)),$$

(see [12, Proposition 4-D]) and one is left with showing that the kernel of $\mathbf{T}'$ is in the $\mathbb{Z}$-span of the nine modules. Using the aforementioned realization of B_θ, one shows that the kernel of $\mathbf{T}'$ (on K_0) has rank one and is in fact generated by a Bott element $\mathfrak{b}$ whose second order Chern character is computed to be $C_1(\mathfrak{b}) = 4qn$ for some integer n (which turns out to be ± 1) [12, Lemma 5-C]. The bounded and unbounded traces of $\mathfrak{b}$ turn out to vanish hence its Chern value is $\mathbf{T}(\mathfrak{b}) = (0; 4qn; 0, 0; 0, 0, 0)$. Now in terms of the nine modules one can write down a class whose Chern value is $(0; 4q; 0, 0; 0, 0, 0)$. Let

$$\kappa_{p,q} = (p+q)([P_1]-2[P_3]+[P_4]-2[P_6])+p([P_2]+[P_5]+[P_7])-2q[P_8]-(2q+p)[P_9]$$

which is in the $\mathbb{Z}$-span of the nine modules (where θ is suppressed). It is straightforward to check using the above Table that $\mathbf{T}(\kappa_{p,q}) = (0; 4q; 0, 0; 0, 0, 0)$. Hence,

[1] In the Appendix of [12] some corrections are made to two of the proofs in [5] which are needed in an essential way in [12, Section 3].

$\mathbf{T}(\mathfrak{b}) = n\mathbf{T}(\kappa_{p,q})$ and using the injectivity of $\mathbf{T}$ (in the rational case) one concludes that $\mathfrak{b} = n\kappa_{p,q}$ in $K_0(B_\theta)$. But since the Bott element generates the kernel and $\kappa_{p,q}$ is already in it, it follows that $n = \pm 1$, so that $\mathfrak{b} = \pm\kappa_{p,q}$ is in the span of the nine modules. It turns out that in order to obtain the latter result without too much complication we had to first prove it for a special set of dense rationals p/q where $q = 4^d$ and p is odd of a certain form [12, Section 2.3]. Once this is done one obtains the range of the Chern map on $K_0(B_\theta)$ to be the span of the rows in the above Table for *all* θ, *i.e.*, it is spanned by $\mathbf{T}(P_j(\theta))$, $j = 1, \dots, 9$ (see [12, Theorem 6-B]). This leads, in the end, to the following result (again using the injectivity of $\mathbf{T}$ in the rational case).

THEOREM C [12, Corollary 6-C]. *For rational $\theta \in (0,1)$, the modules $P_1(\theta)$, $\dots, P_9(\theta)$ form a basis for $K_0(B_\theta)$.*

5. The irrational case. Before one can carry out a Baire category argument, one needs a way of parametrizing the elements of $K_0(B_\theta)$ without reference to the parameter θ. For this we use [12, Corollary 7.3-E] which says that under certain hypotheses on the continuous field $\{B_t\}$ (which are satisfied in our case), the canonical map induced by evaluation $K_0(\Gamma) \to K_0(B_t)$ is surjective for each t, where Γ is the C^*-algebra of the field. That hypothesis states: there are positive classes $[P_1], \dots, [P_N]$ in $K_0(\Gamma)$ and a dense subset Q of $[0,1]$ such that for each $t \in Q$ the classes $[P_1(t)], \dots, [P_N(t)]$ generate $K_0(B_t)$. Once this is shown for Q being the rationals, one can use a Baire category argument and deduce the following from Theorem C.

THEOREM D [12, Corollary 6-E]. *There is a dense G_δ set of the parameter θ (containing the rationals) such that $K_0(B_\theta) \cong \mathbb{Z}^9$ and the modules $P_1(\theta), \dots,$ $P_9(\theta)$ form a basis for it. Furthermore, $K_1(B_\theta) = 0$ for a dense G_δ.*

REMARK. The K_1 result does not depend on the K_0 result, but follows easily from [12, Theorem 7.2-B] (using a Baire category argument). Since for parameters in $(0,1)$ other than $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ the crossed product B_θ is strongly Morita equivalent to the fixed point subalgebra A_θ^σ, one gets that for a dense G_δ set of such parameters, $K_0(A_\theta^\sigma) \cong \mathbb{Z}^9$ and $K_1(A_\theta^\sigma) = 0$. To get generators for the latter K_0, one uses the projection that Boca [1] constructed in the fixed point subalgebra—but for this, one needs to compute its Chern value.

Though the above gives $K_0(B_\theta) \cong \mathbb{Z}^9$ for many irrational θ, one expects it to hold for all irrational θ. There is enough known to show, as a result, that the positive cone of $K_0(B_\theta)$ for irrational θ (in the dense G_δ) is equal to the set of elements of positive (bounded) trace. Jeong and Osaka [6, Proposition 6.2] proved a result that implies that B_θ has cancellation for θ irrational. It now follows that two projections in $M_n(B_\theta)$ are unitarily equivalent if and only if they have the same Chern value (for θ in a dense G_δ).

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