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VANISHING CARLESON TYPE MEASURE CHARACTERIZATION OF $Q_{p,0}$

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Presented by Vlastimil Dlab, FRSC

RÉSUMÉ. Dans cette note, nous introduisons le concept de mesure de p-Carleson nulle ainsi qu'une caractérisation de telles mesures. Nous montrons que, pour $Q_{p,0}$ non-trivial, $f \in Q_{p,0}$ si et seulement si $|\tilde{\nabla} f(z)|^2(1-|z|^2)^{np}\,d\lambda(z)$ est une mesure de p-Carleson nulle et que, pour $1 < p < \infty$, $f \in \mathcal{B}_0$ si et seulement si $|\tilde{\nabla} f(z)|^2(1-|z|^2)^{np}\,d\lambda(z)$ est une mesure de p-Carleson nulle.

1. Introduction. Let B denote the open unit ball in C^n, S denote the unit sphere and H(B) denote the collection of all holomorphic functions in B. Let $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$ denote the invariant gradient of f, see [3]. We say $f \in Q_{p,0}$ for $0 < p < \infty$ if

$$\lim_{|a| \to 1} \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)\,d\lambda(z) = 0,$$

and $f \in Q_p$ for $0 < p < \infty$ if

$$\sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)\,d\lambda(z) < \infty,$$

where $G(z,a) = g(\varphi_a(z))$ is the invariant Green's function(see [6] or [7]) and $d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$ is the invariant measure. In [4], it was proved that: (1) $Q_{1,0} =$ VMOA, the space of holomorphic functions of a vanishing mean oscillation on B; (2) when $1 < p < \frac{n}{n-1}$, $Q_{p,0} = \mathcal{B}_0$, the little Bloch space; (3) when $\frac{n-1}{n} < q < p \leq 1$, $Q_{q,0} \subset Q_{p,0}$; (4) when $\frac{n-1}{n} < p < \frac{n}{n-1}$, $Q_{p,0}$ is nontrivial but when $0 < p \leq \frac{n-1}{n}$ or $p \geq \frac{n}{n-1}$, $Q_{p,0}$ contains only constant functions.

The $Q_{p,0}$ and Q_p spaces in the unit disk were first introduced in [1], and now they have been studied by many colleagues, for example, see [2], although in C^n, the research about $Q_{p,0}$ and Q_p has just begun.

In Section 2, we introduce the concept of a vanishing p-Carleson measure and prove a characterization of a vanishing p-Carleson measure as a preparation Theorem. In Section 3, a vanishing p-Carleson measure characterization of $Q_{p,0}$ is given. A similar result about $\mathcal{B}_0$ is given too.

We refer the reader for unexplained notations to [5].

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2. Vanishing p-Carleson measure. For $a, b \in \overline{B}$, let $d(a,b) = |1 - \langle a, b\rangle|^{\frac{1}{2}}$, where $\langle a, b\rangle$ denotes the usual inner product on C^n. Then d satisfies the triangle inequality in $\overline{B}$, see [5].

For $\xi_0 \in S, \delta > 0$, let $\beta = \beta(\xi_0, \delta) = \{z \in B : |1 - \langle z, \xi_0\rangle| < \delta\}$.

For a Borel measure μ on B, let

$$\gamma(\mu, s) = \sup\left\{\frac{\mu(\beta)}{\delta^{np}} : \xi \in S, 0 < \delta \leq s\right\}.$$

We call μ a vanishing p-Carleson measure, if $\mu(B) < \infty$ and $\lim_{s\to 1}\gamma(\mu, s) = 0$.

THEOREM 1. *Let μ be a positive Borel measure on B. Then μ is a vanishing p-Carleson measure if and only if*

$$\lim_{|a|\to 1}\int_B \left(\frac{1-|a|^2}{|1-<z,a>|^2}\right)^{np} d\mu(z) = 0. \tag{1}$$

PROOF. ($\Rightarrow$): Suppose (1) is true. For arbitrary $\epsilon > 0$, there is an $r \in (0,1)$, so that when $a \in B \setminus B_r$, we have

$$\int_B \left(\frac{1-|a|^2}{|1-<z,a>|^2}\right)^{np} d\mu(z) < \frac{\epsilon}{4^{np}}. \tag{2}$$

Let $s \leq 1 - r$, then for $0 < \delta \leq s$, $z \in \beta(\xi, \delta)$, let $b = (1-\delta)\xi$. Then we have

$$\begin{aligned} |1 - \langle z, b\rangle| &= |1 - \langle z, \xi\rangle + \delta\langle z, \xi\rangle| \\ &\leq |1 - \langle z, \xi\rangle| + \delta \\ &\leq 2\delta \end{aligned}$$

and

$$1 - |b|^2 = 1 - (1-\delta)^2 = (2-\delta)\delta \geq \delta.$$

So,

$$\begin{aligned} \int_B \left(\frac{1-|b|^2}{|1-\langle z,b\rangle|^2}\right)^{np} d\mu(z) &\geq \int_{\beta(\xi,\delta)} \left(\frac{1-|b|^2}{|1-\langle z,b\rangle|^2}\right)^{np} d\mu(z) \\ &\geq \frac{1}{4^{np}\delta^{np}}\mu(\beta(\xi,\delta)). \end{aligned}$$

By (2),

$$\int_B \left(\frac{1-|b|^2}{|1-\langle z,b\rangle|^2}\right)^{np} d\mu(z) < \frac{\epsilon}{4^{np}},$$

thus

$$\frac{\mu(\beta(\xi,\delta))}{4^{np}\delta^{np}} < \frac{\epsilon}{4^{np}},$$

$$\gamma(\mu, s) = \sup_{0<\delta\leq s}\frac{\mu(\beta)}{\delta^{np}} \leq \epsilon \quad \text{for } 0 < s \leq 1 - r.$$

Since $r \to 1$ means $s \to 0$, by the definition, μ is a vanishing p-Carleson measure.

($\Leftarrow$): Suppose μ is a vanishing p-Carleson measure. By the definition, $\mu(B) < \infty$ and for an arbitrary $\epsilon > 0$, we can select a $\delta_0 > 0$, so that

$$\gamma(\mu, 8\delta_0) < \frac{(1-2^{-np})}{2^{10np+1}}\epsilon. \tag{3}$$

For the above δ_0, we can choose $r_0 \in (0,1)$, so that

$$\frac{(1-r_0^2)^{np}}{\delta_0^{2np}}\mu(B) < \frac{\epsilon}{2}. \tag{4}$$

For $a = r\xi \in B$, let $\delta = 2(1-r)$ and $\Omega = \{z \in B : |1-\langle z,a\rangle| < \delta_0\}$. Since in the following we will be concerned with the situation when $|a| \to 1$, we can set $1 - r < \delta_0$ and $r > r_0$.

$$\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{np} d\mu(z) = \left(\int_\Omega + \int_{B\setminus\Omega}\right)\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{np} d\mu(z). \tag{5}$$

On $B \setminus \Omega$, $|1-\langle z,a\rangle| \geq \delta_0$, by (4),

$$\int_{B\setminus\Omega} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{np} d\mu(z) \leq \frac{(1-r_0^2)^{np}}{\delta_0^{2np}}\mu(B) < \frac{\epsilon}{2}. \tag{6}$$

To estimate the integral over Ω, let $\beta_k = \beta(\xi, 2^k\delta)$, $k = 0, 1, \ldots, N$, where N satisfies

$$2^{N-1}\delta \leq 4\delta_0 < 2^N\delta.$$

If $z \in \Omega$, then $d(z,\xi) \leq d(z,a) + d(a,\xi) < 2\delta_0^{\frac{1}{2}}$, and $2^N\delta \leq 8\delta_0$, so

$$\Omega \subset \beta(\xi, 4\delta_0) \subset \beta_N \subset \beta(\xi, 8\delta_0). \tag{7}$$

Let $E_k = \beta_k - \beta_{k-1}$, $\beta_{-1} = \phi$, $k = 0, 1, \ldots, N$. Then by the definition of $\gamma(\mu, s)$, we get

$$\mu(E_k) \leq \mu(\beta_k) \leq 2^{knp}\gamma(\mu, 8\delta_0)\delta^{np}. \tag{8}$$

If $z \in E_0 = \beta_0$, then $(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2})^{np} \leq \frac{(1+r)^{np}}{(1-r)^{np}} \leq 2^{2np}\delta^{-np}$. If $z \in E_k$, $k \geq 1$, then

$$\begin{aligned} d(z,a) &\geq d(z,\xi) - d(\xi,a) \geq (2^{k-1}\delta)^{\frac{1}{2}} - (1-r)^{\frac{1}{2}} \\ &= \left(2^{\frac{k-1}{2}} - \frac{\sqrt{2}}{2}\right)\delta^{\frac{1}{2}} \geq 2^{\frac{k-5}{2}}\delta^{\frac{1}{2}}. \end{aligned}$$

So

$$\begin{aligned} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{np} &\leq 2^{-2(k-5)np}\delta^{-2np}(1-r^2)^{np} \\ &\leq 2^{10np}2^{-2knp}\delta^{-np}. \end{aligned}$$

By (8), for $k = 0, 1, \ldots, N$,

$$\int_{E_k} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} d\mu(z) \le 2^{10np} 2^{-2knp} \delta^{-np} \cdot 2^{knp} \gamma(\mu, 8\delta_0)\delta^{np} = 2^{10np} 2^{-knp} \gamma(\mu, 8\delta_0).$$

Therefore, by (7),

$$\begin{aligned} \int_{\Omega} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} d\mu(z) &\le \int_{\beta_N} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} d\mu(z) \\ &= \Sigma_{k=0}^{N} \int_{E_k} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} d\mu(z) \\ &\le 2^{10np}\gamma(\mu, 8\delta_0) \Sigma_{k=0}^{N} 2^{-knp} \\ &\le 2^{10np}\gamma(\mu, 8\delta_0) \frac{1}{1-2^{-np}}. \end{aligned} \tag{9}$$

By (5), (9), (3) and (6), we get for arbitrary $\epsilon > 0$, there exists r_0, so that when $a = r\xi \in B$, $r > r_0$, we have

$$\int_{B} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} d\mu(z) < \epsilon,$$

which means

$$\lim_{|a|\to 1} \int_{B} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} d\mu(z) = 0.$$

3. Characterizations of $Q_{p,0}$ and $\mathcal{B}_0$.

THEOREM 2. *For nontrivial $Q_{p,0}$ spaces (i.e., $\frac{n-1}{n} < p < \frac{n}{n-1}$), we have*

$f \in Q_{p,0} \Longleftrightarrow |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{np} d\lambda(z)$ *is a vanishing p-Carleson measure.*

PROOF. In Propositions 4.6 and 4.8 of [4], we have proved that for nontrivial $Q_{p,0}$ spaces,

$$f \in Q_{p,0} \Leftrightarrow \lim_{|a|\to 1} \int_{B} |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) = 0.$$

So

$$f \in Q_{p,0} \Leftrightarrow \lim_{|a|\to 1} \int_{B} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{np} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{np} d\lambda(z) = 0.$$

By Theorem 1, for $\frac{n-1}{n} < p < \frac{n}{n-1}$,

$f \in Q_{p,0} \Longleftrightarrow |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{np} d\lambda(z)$ is a vanishing p-Carleson measure.

THEOREM 3. *$f \in H(B)$, $1 < p < \infty$, then*

$f \in \mathcal{B}_0 \Longleftrightarrow |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{np} d\lambda(z)$ *is a vanishing p-Carleson measure.*

PROOF. In Proposition 4.6 of [4], we have proved that for $1 < p < \infty$,

$$f \in \mathcal{B}_0 \Leftrightarrow \lim_{|a|\to 1} \int_B |\tilde{\nabla} f(z)|^2 (1 - |\varphi_a(z)|^2)^{np} \, d\lambda(z) = 0.$$

So

$$f \in \mathcal{B}_0 \Leftrightarrow \lim_{|a|\to 1} \int_B \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{np} |\tilde{\nabla} f(z)|^2 (1 - |z|^2)^{np} \, d\lambda(z) = 0.$$

By Theorem 1,

$$f \in \mathcal{B}_0 \iff |\tilde{\nabla} f(z)|^2 (1 - |z|^2)^{np} \, d\lambda(z) \text{ is a vanishing } p\text{-Carleson measure for } 1 < p < \infty.$$

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UNE CONGRUENCE POUR LES COEFFICIENTS BINOMIAUX

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Presented by M. Ram Murty, FRSC

ABSTRACT. We compute the smallest non-zero digit in the p-adic development of a factorial number and of a binomial or multinomial coefficient in terms of the digits of its arguments; and we explicit the relation of these results with the notion of divided powers.

RÉSUMÉ. Nous calculons le plus petit chiffre non-nul dans le développement p-adique d'un factoriel et d'un coefficient binomial ou multinomial en fonction des chiffres de ses arguments; et nous explicitons le lien entre ces résultats et la notion de puissances divisées.

1. Introduction. Fixons-nous un nombre premier p, et, pour tout entier $m \geq 0$, notons m_i ses chiffres en base p: $m = \sum_{i \geq 0} m_i p^i$, $0 \leq m_i < p$.

Les coefficients binomiaux vérifient la classique congruence de Lucas (*cf.* [4, p. 52])

$$\binom{n}{a} \equiv \prod_{i \geq 0} \binom{n_i}{a_i} \pmod{p},$$

où l'on a convenu que $\binom{m}{k} = 0$ lorsque $0 \leq m < k$.

On en déduit aisément que pour tous entiers $a, b, \ldots, l \geq 0$ le coefficient multinomial $((a, b, \ldots, l)) = \frac{(a+b+\cdots+l)!}{a!\,b!\cdots l!}$ vérifie la congruence

$$((a, b, \ldots, l)) \equiv \prod_{i \geq 0} ((a_i, b_i, \ldots, l_i)) \pmod{p}$$

et donc que

$$((a, \ldots, l)) \not\equiv 0 \pmod{p} \iff a_i + \cdots + l_i < p \quad \text{pour tout } i.$$

Cette équivalence est à rapprocher de l'interprétation par Kummer (*cf.* [3, p. 116], ou [5, p. 24]) de la p-valuation du coefficient binomial $((a, b))$ comme étant le nombre de retenues dans l'addition de a et b en base p.

Notre résultat essentiel généralise et précise les énoncés précédents:

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THÉORÈME 1. *Soient $a, b, \ldots, l$ des entiers ≥ 0. La valuation p-adique du coefficient multinomial $((a, b, \ldots, l))$ est égale au nombre r de retenues dans l'addition de $a, b, \ldots, l$ en base p, et, plus précisément, on a la congruence*

$$((a, b, \ldots, l)) \equiv (-p)^r \prod_{i \geq 0} \frac{(a+b+\cdots+l)_i}{a_i!\, b_i! \cdots l_i!} \pmod{p^{r+1}}.$$

Il s'agit de "retenues avec multiplicités", au sens que si, par exemple, $3p \leq a_0 + b_0 + \cdots + l_0 < 4p$, alors on considère qu'il y a 3 retenues au titre des unités.

2. **Puissances divisées.** Le théorème 1 résulte des trois propositions suivantes (et de la congruence de Wilson $(p-1)! \equiv -1 \pmod p$):

PROPOSITION 1. *Si $\nu_p(m!)$ désigne la p-valuation du factoriel d'un entier $m \geq 0$, le produit $(p!)^{\nu_p(m!)} \prod_{i \geq 0} m_i!$ divise $m!$ et le quotient $u_p(m)$ est congru à 1 selon le module p.*

Rappelons que $\nu_p(m!) = \sum_{i \geq 1} \frac{p^i - 1}{p-1} m_i = \sum_{i \geq 1} [m/p^i]$ (parties entières).

COROLLAIRE 1. *Si p^s divise $m!$ (s entier ≥ 0), alors $(p!)^s$ divise $m!$.*

PROPOSITION 2. *Soit γ un système de puissances divisées définies sur un idéal A_+ d'un anneau commutatif A (cf. [6] ou [1]).*

Pour tout $x \in A_+$ et tout entier $m \geq 0$, on a $m_{[p]} \cdot \gamma_m x = \prod_{i \geq 0} (\gamma_p^i x)^{m_i}$, où l'on a posé

$$m_{[p]} = u_p(m) \prod_{i \geq 0} m_i! \quad \left(= m!/(p!)^{\nu_p(m!)}\right).$$

Ici γ_p^i désigne le i^{eme} itéré de la puissance divisée $\gamma_p \colon A_+ \longrightarrow A_+$. Si l'anneau A est de caractéristique p, on retrouve un théorème de Cartan: *cf.* [2].

PROPOSITION 3. *Si r est le nombre de retenues dans l'addition de $a, b, \ldots, l$ en base p (cf. supra), on a l'égalité $((a, b, \ldots, l)) = (p!)^r \frac{(a+b+\cdots+l)_{[p]}}{a_{[p]} b_{[p]} \cdots l_{[p]}}$.*

Pour de telles assertions, la seule difficulté est de les découvrir. Il est ensuite facile d'en trouver de multiples démonstrations. Par exemple, on pourrait donner une interprétation combinatoire de l'entier $u_p(m) = \frac{m!}{(p!)^{\nu_p(m!)} \prod m_i!}$: c'est le nombre de suites $\mathcal{P}_1, \ldots, \mathcal{P}_i, \ldots$ de partitions $\mathcal{P}_i$ d'un ensemble de cardinal m, $\mathcal{P}_i$ étant plus fine que $\mathcal{P}_{i+1}$, et toutes les parties de $\mathcal{P}_i$ ayant p^i éléments, à l'exception peut-être d'une seule, de cardinal $< p^i$.

Ici nous déduirons nos propositions (et donc le théorème 1) de [6], où il est démontré qu'il existe un entier $\equiv 1 \pmod p$, ne dépendant que de p et de m, également noté $u_p(m)$, et pour lequel on a les conclusions de la proposition 2, sauf peut-être la dernière égalité entre parenthèses. Faisant $x = X$ dans l'anneau de polynômes $A = \mathbf{Q}[X]$, muni des puissances divisées standard $\gamma_n t = t^n/n!$ (sur A tout entier), on voit que cet entier vérifie aussi l'égalité entre parenthèses, ce qui établit les propositions 1 et 2.

Démontrons maintenant la proposition 3 (sans utiliser le résultat de Kummer). Dans l'anneau divisé $\mathbf{Q}[X]$, défini plus haut, on a $\gamma_a X.\gamma_b X \cdots \gamma_l X = ((a,b,\ldots,l))\gamma_m X$ avec $m = a+b+\cdots+l$. Traduite à l'aide de la proposition 2 cette égalité s'écrit

$$\frac{\prod_i (\gamma_p^i X)^{a_i+b_i+\cdots+l_i}}{a_{[p]} b_{[p]} \cdots l_{[p]}} = ((a,b,\ldots,l)) \frac{\prod_i (\gamma_p^i X)^{m_i}}{m_{[p]}}.$$

Puis, en suivant pas à pas l'algorithme usuel de l'addition en base p, on diminue de p, dans le membre de gauche de cette dernière égalité, les exposants qui sont $\geq p$, en remplaçant $(\gamma_p^i X)^p$ par $p!\,\gamma_p^{i+1} X$. On effectue ces remplacements autant de fois qu'il y a de retenues, et l'on obtient ainsi les exposants qui figurent au membre de droite. Ensuite, en égalant les coefficients des monômes obtenus aux deux membres, on trouve l'égalité de la proposition 3.

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EVERY APPROXIMATELY TRANSITIVE AMENABLE ACTION OF A LOCALLY COMPACT GROUP IS A POISSON BOUNDARY

Dedicated to Alain Connes on the occasion of his fiftieth birthday, and to Jim Woods on the occasion of his sixtieth birthday.

GEORGE A. ELLIOTT, FRSC, AND THIERRY GIORDANO

ABSTRACT. In this note, we answer a question of A. Connes and E. J. Woods by proving that if G is a separable, locally compact group, then any amenable and approximately transitive G-space is the Poisson boundary of a group-invariant, time-dependent Markov random walk.

RÉSUMÉ. Dans cette note, nous résolvons une question de A. Connes et E. J. Woods en démontrant que tout G-espace moyennable et approximativement transitif est le bord de Poisson d'une marche aléatoire G-invariante et dépendante du temps.

0. Introduction. In 1978, R. J. Zimmer introduced the notion of amenability for an ergodic action of a separable locally compact group, or for an equivalence relation, on a standard Borel space with a probability measure [Z].

In 1985, A. Connes and E. J. Woods introduced the notion of an approximately transitive (AT) action [CW1], in order to provide a necessary and sufficient condition for an approximately finite dimensional von Neumann factor to be an infinite tensor product of factors of type I (ITPFI). In 1989, they pointed out that the Poisson boundary of a group-invariant, time-dependent Markov random walk on a separable, locally compact group G is an AT, amenable G-space [CW2]. They showed in the case of $\mathbb{Z}$ or $\mathbb{R}$ that every AT standard Borel G-space can be represented as a Poisson boundary.

In [EG] (in the discrete case) and [AEG] (in the general case), it was proved that any amenable G-space can be represented as the Poisson boundary of a group-invariant, time-dependent, matrix-valued Markov random walk on G.

If G is a separable, locally compact group, we shall show in this note that any amenable and approximately transitive G-space is the Poisson boundary of a group-invariant, time-dependent Markov random walk.

The main tool we use in the proof is T. Hamachi's measure-theoretical proof of the Connes-Woods theorem on AT flows [H]. More precisely, we shall use

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the applicability of this in a more general setting, observed by Golodets and Sokhet [GoSo].

1. Mackey Range. In this section, we first recall the notion of a cocycle defined on an measured equivalence relation with values in a separable locally compact group, and we prove a technical result we will need in Section 4.

Let (Ω, μ) be a non-atomic Lebesgue space and let $\mathcal{R}$ be a discrete measured equivalence relation on (Ω, μ). If G is a separable, locally compact group, a cocycle α is a measurable homomorphism from $\mathcal{R}$ to G, *i.e.*, a measurable map such that

$$\alpha(e_1, e_3) = \alpha(e_1, e_2)\alpha(e_2, e_3) \quad \text{if} \quad e_1\mathcal{R}e_2 \quad \text{and} \quad e_2\mathcal{R}e_3.$$

We will denote the group of all cocycles from $\mathcal{R}$ to G by $\mathrm{Z}^1(\mathcal{R}, G)$. (Strictly speaking, Z^1 is the group of cohomology classes; *cf.* Section 3.)

Choose a probability measure m_G on G equivalent to the (right) Haar measure. To a cocycle $\alpha \in \mathrm{Z}^1(\mathcal{R}, G)$, we associate the equivalence relation $\mathcal{R}_\alpha$ on $(\Omega\times G, \mu\times m_G)$, called the skew-product relation, given by

$$(x, g)\mathcal{R}_\alpha(y, h) \quad \text{if} \quad x\mathcal{R}y \quad \text{and} \quad g\alpha(x, y) = h.$$

Note that the G-action $\lambda_g, g \in G$, defined on $\Omega \times G$ by $\lambda_g(x, k) = (x, gk)$, $(x, k) \in \Omega \times G$, commutes with $\mathcal{R}_\alpha$ and therefore induces a G-action on $(\Omega \times G, \mu \times m_G)/\mathcal{R}_\alpha$. The Mackey range $\mathrm{MR}(\Omega, \mathcal{R}, \alpha)$ associated to (Ω, μ), $\mathcal{R}$, and the cocycle α is the G-space $(\Omega \times G, \mu \times m_G)/\mathcal{R}_\alpha$.

On the space $X = \prod_{n\geq 1}\{0, 1\}$, denote by $\mathcal{R}_X$ the equivalence relation given by tail equivalence, *i.e.*,

$$x\mathcal{R}_X y \quad \text{if} \quad \exists n \geq 1 \quad \text{such that } x_k = y_k \quad \text{for} \quad k \geq n+1.$$

Choose an $\mathcal{R}_X$-ergodic probability measure $\mu_X = \bigotimes_{n\geq 1} \mu_n^X$ on X of type III_1.

On the space $(\Omega \times X, \mu \times \mu_X)$, consider the equivalence relation $\mathcal{R}'$ given by

$$(e, x)\mathcal{R}'(f, y) \quad \text{if} \quad e\mathcal{R}f \quad \text{and} \quad x\mathcal{R}_X y.$$

Let π denote the canonical projection from $\Omega \times X$ onto Ω. Consider the Radon-Nikodym cocycle Δ' of $\mu \times \mu_X$. By construction, $\Delta'\big((e, x), (f, y)\big) = \Delta_\mu(e, f)\Delta_X(x, y)$, where Δ_μ and Δ_X denote the Radon-Nikodym cocycles of μ and μ_X.

Let us still denote by $\alpha: \mathcal{R}' \to G$ the cocycle defined by $\alpha\big((e, x), (f, y)\big) = \alpha(e, f)$, and denote by $\alpha \times \Delta': \mathcal{R}' \to G \times \mathbb{R}_+^*$ the cocycle given by

$$(\alpha \times \Delta')\big((e, x), (f, y)\big) = \Big(\alpha(e, f), \Delta'\big((e, x), (f, y)\big)\Big).$$

Choose a probability measure $m_{\mathbb{R}_+^*}$ on $\mathbb{R}_+^*$ equivalent to the Haar measure. Then $\mathcal{R}'_{\alpha\times\Delta'}$, $\mathcal{R}'_\alpha$, and $\mathcal{R}_\alpha$ will denote the skew-product equivalence relations on respectively $\Omega \times X \times G \times \mathbb{R}_+^*$, $\Omega \times X \times G$, and $\Omega \times G$.

Let us still denote by π the projections from $\Omega \times X \times G \times \mathbb{R}_+^*$ to $\Omega \times G$ and from $\Omega \times X \times G$ to $\Omega \times G$ defined by $\pi(e, x, g, t) = (e, g)$ and $\pi(e, x, g) = (e, g)$.

LEMMA 1. *Let $S' \subset \mathcal{R}'$ denote the subequivalence relation* $\mathrm{id} \times \mathcal{R}_X$ *on* $(\Omega \times X, \mu \times \mu_X)$, *and* $S'_{\alpha\times\Delta'} \subset \mathcal{R}'_{\alpha\times\Delta'}$ *the corresponding skew-product equivalence relation on* $\Omega \times X \times G \times \mathbb{R}_+^*$.

Then π induces an isomorphism of G-spaces between the quotient $(\Omega \times X \times G \times \mathbb{R}_+^*)/S'_{\alpha\times\Delta'}$ *and* $\Omega \times G$.

PROOF. As π is $(S'_{\alpha\times\Delta'})$-invariant and $\pi(\mu \times \mu_X \times m_G \times m_{\mathbb{R}_+^*}) = \mu \times m_G$, we only have to show that π induces a surjective map from $L^\infty(\Omega \times G, \mu \times m_G)$ onto $L^\infty(\Omega \times X \times G \times \mathbb{R}_+^*, \mu \times m_G \times m_{\mathbb{R}_+^*})^{S'_{\alpha\times\Delta'}}$.

If $f \in L^\infty(\Omega \times X \times G \times \mathbb{R}_+^*, \mu \times m_G \times m_{\mathbb{R}_+^*})^{S'_{\alpha\times\Delta'}}$, we denote by $F \in L^\infty(\Omega \times G, \mu \times m_G)$ the function

$$F(e,g) = \int_{X\times\mathbb{R}_+^*} f(e,x,g,t)\, d\mu_X(x)\, dm_{\mathbb{R}_+^*}(t).$$

As $(X, \mu_X, \mathcal{R}_X)$ is of type III_1 (in other words, the skew-product equivalence relation defined on $X \times \mathbb{R}_+^*$ by $\mathcal{R}_X$ and Δ_X is ergodic),

$$\mu_X \times m_{\mathbb{R}_+^*}\big(\{(x,t); F(e,g) = f(e,x,g,t)\}\big) = 1$$

and therefore $F \circ \pi = f$. ■

LEMMA 2. *For any $(e,g), (f,h) \in \Omega \times G$, the following are equivalent:*

(i) $(e,g)\ \mathcal{R}_\alpha\ (f,h)$.

(ii) For all measurable subsets E, E' of $\Omega \times X \times G \times \mathbb{R}_+^*$, such that

$\mu_X \times m_{\mathbb{R}_+^*}\big(\{(x,t); (e,x,g,t) \in E\}\big)$ *and* $\mu_X \times m_{\mathbb{R}_+^*}\big(\{(y,s); (f,y,h,s) \in E'\}\big)$

are nonzero, there exist $(e,x,g,t) \in E$, $(f,y,h,s) \in E'$ such that

$$(e,x,g,t)\mathcal{R}_{\alpha\times\Delta'}(f,y,h,s).$$

PROOF. For $(e,x,g,t), (f,y,h,s) \in \Omega \times X \times G \times \mathbb{R}_+^*$, we have:

$$(e,x,g,t)\mathcal{R}_{\alpha\times\Delta'}(f,y,h,s) \iff (e,g)\mathcal{R}_\alpha(f,h) \quad \text{and} \quad \big(x, t\Delta_\mu(e,f)\big)\mathcal{R}_X(y,s).$$

Therefore, we need only show that (i) implies (ii).

If $A \subset \Omega\times X\times G\times\mathbb{R}_+^*$, let $A_{(e,g)}$ denote the subset $\{(x,t) \in X\times\mathbb{R}_+^*; (e,x,g,t) \in A\}$. Let E, E' be two measurable subsets of $\Omega \times X \times G \times \mathbb{R}_+^*$, such that

$$\mu_X \times m_{\mathbb{R}_+^*}(E_{(e,g)}) > 0 \quad \text{and} \quad \mu_X \times m_{\mathbb{R}_+^*}(E'_{(f,h)}) > 0.$$

Set $\tilde{E} = \big\{(f,x,h,u); \big(e,x,g,u\Delta(e,g)^{-1}\big) \in E\big\}$. As $m_{\mathbb{R}_+^*}$ is equivalent to the Haar measure on $\mathbb{R}_+^*$, we have $\mu_X \times m_{\mathbb{R}_+^*}(\tilde{E}_{(f,h)}) > 0$.

As the skew-product equivalence relation defined on $X \times \mathbb{R}_+^*$ by $\mathcal{R}_X$ and Δ_X is ergodic, the lemma is proved. ■

By the appendix of [GiSa], it follows that the quotient $(\Omega\times X\times G\times\mathbb{R}_+^*)/\mathcal{R}'_{\alpha'\times\Delta'}$ is equal to the quotient of $\Omega \times G$ by $\mathcal{R}_\alpha$.

By definition, π intertwines the left regular representation of G on $\Omega' \times G \times \mathbb{R}_+^*$ (resp. $\Omega \times X \times G$) and on $\Omega \times G$. It intertwines also the regular representation of $\mathbb{R}_+^*$ on $\Omega' \times G \times \mathbb{R}_+^*$ and the trivial representation of $\mathbb{R}_+^*$ on $\Omega \times G$. Therefore we get:

COROLLARY 2.

(i) The Mackey range $\mathrm{MR}(\Omega, \mathcal{R}, \alpha)$ *is isomorphic as a G-space to* $\mathrm{MR}(\Omega \times X, \mathcal{R}', \alpha)$.

(ii) If $\mathrm{MR}(\Omega, \mathcal{R}, \alpha)$ *is considered as a trivial $\mathbb{R}_+^*$-space, then it is isomorphic as a $(G \times \mathbb{R}_+^*)$-space to* $\mathrm{MR}(\Omega \times X, \mathcal{R}', \alpha \times \Delta')$.

2. **AF-cocycles and amenable G-actions.** Recall that a Bratteli diagram $B = (V, E)$ consists of a vertex set $V = \coprod_{n \geq 0} V_n$ and an edge set $E = \coprod_{n \geq 1} E_n$ such that:

(i) $1 \leq |V_n|, |E_n| < \infty$, $V_0 = \{v_0\}$ and $V_n = \{v_1^n, \ldots, v_{k(n)}^n\}$;

(ii) there exist a source map s and a range map r from E to V such that

$$r(E_n) \subseteq V_n \quad \text{and} \quad s(E_n) \subseteq V_{n-1}.$$

Let us assume also that $s^{-1}(v) \neq \emptyset$, for all $v \in V$ and $r^{-1}(v) \neq \emptyset$, for all $v \in V \setminus V_0$.

We will denote by Ω_B the infinite path space associated to $B = (V, E)$:

$$\Omega_B = \{(e_1, e_2, \ldots); e_i \in E_i, r(e_i) = s(e_{i+1}), i \geq 1\}.$$

Let us consider the topology on Ω_B which has the family of cylinder sets

$$U(e_1, e_2, \ldots, e_n) = \{(f_1, f_2, \ldots) \in \Omega_B; f_i = e_i, 1 \leq i \leq n\}$$

as a basis. Each cylinder set is clopen and so Ω_B becomes a compact Hausdorff space with a countable basis of clopen sets. If B is a simple diagram, then Ω_B does not have isolated points and therefore is a Cantor set.

An AF-measure μ on Ω_B is a measure determined by a system of transition probabilities p (*i.e.*, maps $p\colon E \to [0,1]$ with $p(e) \geq 0$ and $\sum_{\{e; s(e) = v\}} p(e) = 1$ for every vertex v), given by

$$\mu(U(e_1, e_2, \ldots, e_n)) = \prod_{k=0}^{n} p(e_k).$$

Note that Ω_B carries a canonical equivalence relation $\mathcal{R}_B$ defined by

$$e \mathcal{R}_B f \quad \text{if} \quad \exists n, e_k = f_k \quad \text{for all } k \geq n.$$

DEFINITION 3. Let Ω_B, $\mathcal{R}_B$ be as above and let G be a separable, locally compact group.

A cocycle $\alpha \in Z^1(\mathcal{R}_B, G)$ will be called an AF-cocycle if there exists a sequence $(b_n)_{n\geq 1}$ of maps $b_n : E(n) \to G$ such that

$$\alpha(e,f) = b_1(e_1)\cdots b_n(e_n)b_n(f_n)^{-1}\cdots b_1(f_1)^{-1}$$

whenever $e\mathcal{R}_B f$ and $e_k = f_k$ for $k \geq n+1$.

Let (X,ν) be an amenable G-space. By [AEG] (*cf.* [GoSi]), there exists a Bratteli diagram B, an AF-measure μ on Ω_B, and an AF-cocycle α on $\mathcal{R}_B$ such that (X,ν) is G-isomorphic to the Mackey range $\mathrm{MR}(\Omega_B, \mathcal{R}_B, \alpha)$.

Let $B = (V,E)$ be a UHF Bratteli diagram (*i.e.*, $V = \coprod_{n\geq 0} V_n$, $|V_n| = 1$ for each $n \geq 0$ and $E = \coprod_{n\geq 1} E_n$, $E_n = \{e_1^n, \ldots, e_{k_n}^n\}$). The space Ω_B of infinite paths is the Cantor set

$$\prod_{n\geq 1} E_n \cong \prod_{n\geq 1} \{1,2,\ldots,k_n\}.$$

Following Renault ([R], p. 128), the tail equivalence relation $\mathcal{R}_B$ on Ω_B will be referred to as the UHF (or Glimm) equivalence relation on Ω_B.

For $n \geq 1$, let $\mathcal{R}_n \subset \mathcal{R}_B$ denote the sub equivalence relation defined on Ω_B by

$$\mathcal{R}_n = \{(e,f) \in \mathcal{R}; e_k = f_k \text{ for } k \geq n+1\}.$$

Clearly,

$$\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3 \subset \cdots \subset \mathcal{R}_n \subset \cdots \subset \bigcup_{n\geq 1} \mathcal{R}_n = \mathcal{R}_B.$$

For $n \geq 1$, let $\mathcal{T}_n$ denote the transitive equivalence relation on $E^n = \prod_{k=1}^n E_k$ (*i.e.*, any two points of E^n are equivalent).

DEFINITION 4. Let $B = (V,E)$ be a UHF Bratteli diagram and denote by $\mathcal{R}_B$ the corresponding UHF equivalence relation on the space Ω_B of infinite paths in B. Let G be a separable, locally compact group and let $\alpha \in Z^1(\mathcal{R}_B, G)$.

We shall say that α is a product cocycle if for each $n \geq 1$, $\alpha|\mathcal{R}_n \in Z^1(\mathcal{T}_n, G)$.

One checks easily that α is a product cocycle if and only if it is an AF-cocycle defined on a UHF Bratteli diagram.

3. The Property AT. Let G be a separable, locally compact group and let (S,μ) be a standard measured G-space (*i.e.*, μ is a quasi-invariant probability measure on S). Denote by $L^1(S,\mu)$ the space of complex measures absolutely continuous with respect to μ and by $P_1(S,\mu)$ the subset of probability measures. For $g \in G$ and $\nu \in P_1(S,\mu)$, let us write $g\nu$ for the measure $g\nu(A) = \nu(g^{-1}A)$, $A \subset S$.

The action of G on (S,μ) is approximately transitive (AT) if for every pair $\nu_1, \nu_2 \in P_1(S,\mu)$ and every $\varepsilon > 0$, there exists $\nu \in P_1(S,\mu)$ such that the distance with respect to the total variation norm from each of ν_1 and ν_2 to the convex hull $\operatorname{conv}(G\nu)$ is less than ε [CW1].

Let $B = (V, E)$ be a UHF Bratteli diagram and μ be a product measure on Ω_B. If Δ_μ denotes the Radon-Nikodym cocycle of μ and if $\alpha \in Z^1(\mathcal{R}_B, G)$ is a product cocycle, then the Mackey range $\mathrm{MR}(\Omega_B, \mathcal{R}_B, \alpha\times\Delta_\mu)$ is an AT, amenable G-space [CW2].

For each $i = 1, 2$, let $\mathcal{R}_i$ be a discrete measured equivalence relation on (Ω_i, μ_i) and let $\beta_i \colon \mathcal{R}_i \to H$ be a cocycle with values in a separable locally compact group H. The two pairs $(\mathcal{R}_i, \beta_i)$ are said to be weakly equivalent if there exists an orbit equivalence $F\colon \Omega_1 \to \Omega_2$ (*i.e.*, F and F^{-1} preserve null sets, and $F(\omega_1)\mathcal{R}_2 F(\omega_2)$ if and only if $\omega_1 \mathcal{R}_1 \omega_2$) and a measurable function $g\colon \Omega_2 \to H$ such that $\beta_1 \circ F(e,f) = g^{-1}(e)\beta_2(e,f)g(f)$, $e\mathcal{R}_2 f$ (*i.e.*, $\beta_1 \circ F$ and β_2 are cohomologous). If this holds, the Mackey ranges are isomorphic H-spaces (see *e.g.* [BG]).

Noticing that Hamachi's measure-theoretical proof of the Connes-Woods theorem on AT-flows can be naturally extended, Golodets and Sokhet prove in [GoSo] the following result:

THEOREM 5. *Let $\mathcal{R}$ be a type III amenable ergodic discrete equivalence relation on (Ω, ν) and let $\Delta_\nu \colon \mathcal{R} \to \mathbb{R}_+^*$ denote its Radon-Nikodym cocycle. If $\beta \in Z^1(\mathcal{R}, G)$ is a cocycle, then there exist a UHF Bratteli diagram B, a product measure μ on Ω_B, and a product cocycle $\alpha \in Z^1(\mathcal{R}_B, G)$ such that $(\mathcal{R}, \beta \times \Delta_\nu)$ is weakly equivalent to $(\mathcal{R}_B, \alpha \times \Delta_\mu)$ if, and only if, the Mackey range associated to $(\mathcal{R}, \beta \times \Delta_\nu)$ is an AT $(G \times \mathbb{R}_+^*)$-space.*

4. The main result.

The main result of this section is the following:

THEOREM 6. *Let G be a separable, locally compact group and let (S, ν) be a standard measured space.*

If (S, ν) is an AT and amenable G-space, then it is isomorphic as a G-space to the Poisson boundary of a group-invariant Markov random walk on G.

PROOF. By [AEG], Theorem A, there exists a Bratteli diagram C, an AF-measure μ_C on Ω_C, and an AF-cocycle α on $\mathcal{R}_C$ such that (S,ν) is G-isomorphic to the Mackey range $\mathrm{MR}\,\Omega_C, \mathcal{R}_C, \alpha)$.

Let (X, μ_X) and $\mathcal{R}_X$ denote the type III_1 hyperfinite equivalence relation introduced in Section 1.

On $(\Omega_C \times X, \mu_C \times \mu_X)$, let $\mathcal{R}'$ denote the equivalence relation $\mathcal{R}_C \times \mathcal{R}_X$. Since both $\mathcal{R}_C$ and $\mathcal{R}_X$ are hyperfinite, the relation $\mathcal{R}'$ is also hyperfinite. In particular, $\mathcal{R}'$ is also hyperfinite. In particular, $\mathcal{R}'$ is amenable. Since $\mathcal{R}_X$ is of type III_1, $\mathcal{R}'$ is of type III_1.

Note that $\mathrm{MR}(\Omega_C, \mathcal{R}_C, \alpha)$ is AT as a G-space and therefore also as a $(G\times\mathbb{R}_+^*)$-space with the trivial $\mathbb{R}_+^*$-action. By Corollary 2 (which uses the notation of

Section 1), it is isomorphic as a $(G\times \mathbb{R}_+^*)$-space to $\mathrm{MR}(\Omega_C\times X, \mathcal{R}', \alpha\times\Delta_{\mu_C\times\mu_X})$. In particular, this space is also AT (as a $(G\times\mathbb{R}_+^*)$-space).

By Theorem 5, there exist a UHF Bratteli diagram B, a product measure μ_B on Ω_B and a product cocycle $\beta\in Z^1(\mathcal{R}_B, G)$ such that the Mackey range $\mathrm{MR}(\Omega_C\times X, \mathcal{R}', \alpha\times\Delta_{\mu_C\times\mu_X})$ is isomorphic as a $G\times\mathbb{R}_+^*$-space to $\mathrm{MR}(\Omega_B, \mathcal{R}_B, \beta\times\Delta_{\mu_B})$. In particular, $\mathrm{MR}(\Omega_B, \mathcal{R}_B, \beta\times\Delta_{\mu_B})$ is a trivial $\mathbb{R}_+^*$-space and is isomorphic as a G-space to (S,ν).

By Section II of [AEG], because of the triviality of the $\mathbb{R}_+^*$-action, $\mathrm{MR}(\Omega_B, \mathcal{R}_B, \beta\times\Delta_{\mu_B})$ is the Poisson boundary of a group-invariant Markov random walk on G, as desired. ∎

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NONEXISTENCE OF SPHERICALLY SYMMETRIC SOLUTIONS FOR p-LAPLACIAN IN THE BALL

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RÉSUMÉ. Nous étudions un problème de non-existence des solutions symétrie-sphériques pour l'équation de p-Laplacian, sur une boule B de R^N, qui joue un rôle essentiel dans l'étude des équations elliptiques quasi-linéaires de Leray-Lions. On va demontrer que il n'y a pas une solution sphérique dans $C^2(B\setminus\{0\})\cap C(\overline{B})$ à condition q'un coefficient f_0 soit assez grand. De plus, dans ce cas, nous allons construire une sequence non-bornée des sous-solutions de l'équation considerée ici.

1. Introduction. We consider the following quasilinear elliptic equation:

$$(1)\qquad \begin{cases} -div(\alpha \mid \nabla v \mid^{p-2} \nabla v) = f_0 + g_0 \mid \nabla v \mid^p, \text{ in } B \\ v \in W_0^{1,p}(B)\bigcap L^\infty(B), \\ v \text{ is positive, spherically symmetric, and decreasing function,} \end{cases}$$

where $B = B(0)$ is an open ball in R^N $(N \geq 1)$ centered at the origin 0, $1 < p < \infty$, and α, f_0 and g_0 are arbitrary given positive constants.

It is very well known that equation (1) plays important role in the investigation of quasilinear elliptic equations of Leray-Lions type, particularly in recently years, see for example [1], [5] and [12]–[13]. Next, in [6] and [13] it was remarked that probably the equation (1) permits a solution in $C^2(B\setminus\{0\})\cap C(\overline{B})$ only provided that the constant f_0 is "small enough", that is to say, it was asked: what do we have in (1) in the case when f_0 is "large enough"? Our aim in this work is to give a precise answer to this question. We will prove that if f_0 is "large enough" then equation (1) doesn't allow any solution in $C^2(B\setminus\{0\})\cap C(\overline{B})$. Moreover, in this case we are able to construct an unbounded sequence of subsolutions of (1).

Let us note that if $f_0 \leq 0$ and $g_0 \in R$ then (1) has no solutions (for much more general results in this case see Corollary 6 in [7]).

2. Statement of problem. In author's note [13] it was proved via Schwarz symmetrisation technique that equation (1) is closely related to the following nonlinear ordinary differential equation with singular point in $t = 0$:

$$(2)\qquad \begin{cases} d\omega/dt = L + M \cdot t^{-\gamma}\omega^\beta(t), \text{ a.e. } t \in (0,T), \\ \omega(0) = 0, \\ \omega \in AC^+([0,T]), \end{cases}$$

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where the constants T, L, M, β and γ satisfy: $T > 0$, $L > 0$, $M > 0$, $\beta > 1$ and $\gamma \in [0, \beta)$, and $AC^+([0,T])$ denotes the cone of all absolutely continuous and positive functions on $[0,T]$.

This connection was described as follows. For each solution $v \in C^2(B \setminus \{0\}) \cap C(\overline{B})$ of (1) the function

$$(3) \qquad \omega(t) = e^{-\lambda v^*(t)} \int_0^t e^{\lambda v^*(\sigma)}\, d\sigma, \quad \lambda = g_0/\alpha,$$

is a solution of (2) in particular for

$$(4) \quad T = |B|, \quad L = 1, \quad M = \frac{g_0 f_0^{p'-1}}{(\alpha N C_N^{1/N})^{p'}}, \quad \beta = p' \quad \text{and} \quad \gamma = p'(1 - 1/N),$$

where v^* is the spherically symmetric rearrangement of v, $|B|$ is the Lebesgue measure of B, $1/p + 1/p' = 1$ and C_N is the measure of the unit ball in R^N. Conversely, there is a solution $\omega \in AC^+([0,T])$ of (2) with (4), such that the function v defined by

$$(5) \quad v(x) = V(C_N|x|^N) \quad \text{and} \quad V(s) = \frac{M \cdot \alpha}{g_0} \int_s^{|B|} t^{-p'(1-1/N)} \omega^{p'/p}(t)\, dt,$$

is a solution of (1). On the sphericaly symmetric rearrangement of real functions see for example [2], [4], [15] and [16].

According to the preceding, instead of considering partial differential equation (1) it is enough to investigate ordinary differential equation (2).

Now, what do we know about equation (2)? Firstly, it is not difficult to check, using the classical technique of the Banach fixed point theorem, see for example in [13], that problem (2) has a unique solution provided the constant M is "small enough". We can take for example that $M < [-\gamma + \beta]/[(\beta + 1)(L + 1)^\beta T^{-\gamma+\beta}]$. Next, in particular for $L = 1$, $\beta = 2$ and $\gamma = 0$, we have in (2) very well known "tang" equation with parameter M:

$$(6) \qquad \begin{cases} d\omega/dt = 1 + M\omega^2(t),\ t \in (0,T), \\ \omega(0) = 0, \\ \omega \in AC^+([0,T]), \end{cases}$$

where $M > 0$ and $T > 0$ are two arbitrary given real numbers. It is easy to see that the function $\omega(t) = [tg(M^{1/2}t)]/M^{1/2}$, $t \in (0,T)$, satisfies the following property: if M is "small enough", that is to say if $M < \pi^2/(4T^2)$, then ω is a solution of (6), and if M is "large enough", that is to say if $M > \pi^2/(4T^2)$, then ω is not a function from $AC^+([0,T])$ and so ω is not a solution of (6). Thus, in this particular case of (2) we can find a principal motivation for the next considerations.

3. Auxiliary results. Let us start with some basic results concerning equation (2).

LEMMA 1. *For arbitrary given constants $T > 0$, $L > 0$, $\beta > 1$ and $\gamma \in [0, \beta)$, let M^* be a number defined by*

$$M^* = \frac{\beta^{(2\beta-1)/(\beta-1)}}{(\beta-1)T^{-\gamma+\beta}L^{\beta-1}}. \tag{7}$$

If $M > M^*$ then there is no any solution of (2). Moreover, there is an unbounded sequence $(\omega_n) \subseteq C([0,T])$ of sub-solutions of (2).

Suppose for the moment that we have proved the following two results:

PROPOSITION 1. *For each solution ω of (2) we have:*

$$\omega(t) \geq L \cdot t + \sum_{m=1}^{n} z_m(t), \quad t \in [0,T], \quad n \in N, \tag{8}$$

where

$$z_m(t) = \frac{L^{\beta^m} M^{\sum_{k=0}^{m-1} \beta^k} t^{-\gamma \sum_{k=0}^{m-1} \beta^k + \sum_{k=0}^{m} \beta^k}}{\prod_{k=1}^{m} \left(-\gamma \sum_{j=0}^{k-1} \beta^j + \sum_{j=0}^{k} \beta^j\right)^{\beta^{m-k}}}. \tag{9}$$

PROPOSITION 2. *Let $z_m(t)$ be defined in (9). Then*

$$z_m(t) \longrightarrow \infty, \quad as \quad m \longrightarrow \infty, \quad t \in (t^*, T), \tag{10}$$

where t^* denotes

$$t^* = \left(\frac{\beta^{(2\beta-1)/(\beta-1)}}{M(\beta-1)L^{\beta-1}}\right)^{1/(-\gamma+\beta)}. \tag{11}$$

PROOF OF LEMMA 1. Remark that $t^* \in (0,T)$ for any $M > M^*$. Now, suppose that $M > M^*$ and assume that there is a solution ω of problem (2). Then, from (8) we have that the sequence of functions $s_n(t) = \sum_{m=1}^{n} z_m(t)$, $t \in [0,T]$, $n \in N$, is uniformly bounded and increasing. It implies that for each $t \in [0,T]$ there exists $s(t) \in R$ such that there holds

$$\lim_n s_n(t) = s(t) \leq \omega(t) - L \cdot t \leq \omega(t) < \infty, \quad t \in [0,T]. \tag{12}$$

On the other hand, let t^* be defined by (11). Then from (10) we have that $\lim_n s_n(t) = \infty$, $t \in (t^*, T)$, which is a contradiction with (12). Thus, for $M > M^*$ problem (2) has no any solution. This is the first claim of Lemma 1.

Now, we shall prove the second claim of Lemma 1. In this direction, let (ω_n) be a sequence defined by: $\omega_n(t) = (K\omega_{n-1})(t)$, $t \in [0,T]$, $n \in N$, and $\omega_0(t) = L \cdot t$, $t \in [0,T]$, where the operator K is defined by

$$(K\varphi)(s) = Ls + M \int_0^s r^{-\gamma} \varphi^\beta(r)\, dr, \quad s \in (0,T).$$

Obviously, K is a nonlinear operator from $W(0,T)$ into $C([0,T])$, where $W(0,T) = \{\varphi \in C([0,T]) : 0 \leq \varphi(s) \leq (L+1)s, s \in [0,T]\}$. Since $\omega_0(t) \leq (K\omega_0)(t) = \omega_1(t)$, $t \in [0,T]$ and K is a monotone operator, we find that (ω_n) is a sequence of sub-solutions of (2) and $\omega_n(t) \geq L \cdot t + \sum_{m=1}^{n} z_m(t)$, $t \in [0,T]$, $n \in N$, where $z_m(t)$ was defined in (9). From the previous inequality and from (10) for $M > M^*$ we see that the sequence (ω_n) is unbounded. So, we have obtained the second claim of Lemma 1. ∎

Thus, in order to prove Lemma 1 it is enough to prove Proposition 1 and Proposition 2.

PROOF OF PROPOSITION 1. Firstly, it is easy to see that $z_m(t)$ from (9) is a sequence of positive functions. Next, from (2) we have in particular that $\omega(t) \geq L \cdot t$ and therefore $d\omega/dt \geq L + L^\beta M \cdot t^{-\gamma+\beta}$. Integrating the preceding inequality over $(0,t)$ we obtain:

$$\omega(t) \geq L \cdot t + \frac{L^\beta M \cdot t^{-\gamma+\beta+1}}{-\gamma+\beta+1} = L \cdot t + z_1(t). \tag{13}$$

Now, we start with the assumption that estimate (8) holds true for some $n \in N$ and prove that (8) holds true for $n+1$ too. Indeed, from (8) and from relation (2), and using $(a+b)^\beta \geq a^\beta + b^\beta$, $a \geq 0$, $b \geq 0$, $\beta \geq 1$, we obtain:

$$\begin{aligned}
\frac{d\omega}{dt} &= L + M \cdot t^{-\gamma}\omega^\beta(t) \geq L + M \cdot t^{-\gamma}\big(L \cdot t + \sum_{m=1}^{n} z_m(t)\big)^\beta \\
&\geq L + M \cdot t^{-\gamma}\big(L^\beta t^\beta + \sum_{m=1}^{n} z_m^\beta(t)\big) \\
&= L + L^\beta M \cdot t^{-\gamma+\beta} \\
&\qquad + M \cdot t^{-\gamma} \sum_{m=1}^{n} \frac{L^{\beta^{m+1}} M^{\sum_{k=0}^{m-1}\beta^{k+1}} t^{-\gamma\sum_{k=0}^{m-1}\beta^{k+1} + \sum_{k=0}^{m}\beta^{k+1}}}{\prod_{k=1}^{m}(-\gamma\sum_{j=0}^{k-1}\beta^j + \sum_{j=0}^{k}\beta^j)^{\beta^{m-k+1}}} \\
&= L + L^\beta M \cdot t^{-\gamma+\beta} + \sum_{m=1}^{n} \frac{L^{\beta^{m+1}} M^{\sum_{k=0}^{m}\beta^k} t^{-\gamma\sum_{k=0}^{m}\beta^k + \sum_{k=0}^{m}\beta^{k+1}}}{\prod_{k=1}^{m}(-\gamma\sum_{j=0}^{k-1}\beta^j + \sum_{j=0}^{k}\beta^j)^{\beta^{m-k+1}}}.
\end{aligned}$$

Integrating the preceding inequality over $(0,t)$, $t \in [0,T]$ we obtain

$$\begin{aligned}
&\omega(t) \\
&\geq L \cdot t + \frac{L^\beta M \cdot t^{-\gamma+\beta+1}}{-\gamma+\beta+1} \\
&+ \sum_{m=1}^{n} \frac{L^{\beta^{m+1}} M^{\sum_{k=0}^{m}\beta^k} t^{-\gamma\sum_{k=0}^{m}\beta^k + \sum_{k=0}^{m}\beta^{k+1} + 1}}{(-\gamma\sum_{k=0}^{m}\beta^k + \sum_{k=0}^{m}\beta^{k+1} + 1)\prod_{k=1}^{m}(-\gamma\sum_{j=0}^{k-1}\beta^j + \sum_{j=0}^{k}\beta^j)^{\beta^{m-k+1}}}
\end{aligned}$$

$$= L \cdot t + \frac{L^{\beta} M \cdot t^{-\gamma+\beta+1}}{-\gamma+\beta+1}$$

$$+ \sum_{m=1}^{n} \frac{L^{\beta^{m+1}} M^{\sum_{k=0}^{m}\beta^k} t^{-\gamma\sum_{k=0}^{m}\beta^k + \sum_{k=0}^{m+1}\beta^k}}{(-\gamma\sum_{k=0}^{m}\beta^k + \sum_{k=0}^{m+1}\beta^k)\prod_{k=1}^{m}(-\gamma\sum_{j=0}^{k-1}\beta^j + \sum_{j=0}^{k}\beta^j)^{\beta^{m-k+1}}}$$

$$= L \cdot t + \frac{L^{\beta} M \cdot t^{-\gamma+\beta+1}}{-\gamma+\beta+1} + \sum_{m=1}^{n} \frac{L^{\beta^{m+1}} M^{\sum_{k=0}^{m}\beta^k} t^{-\gamma\sum_{k=0}^{m}\beta^k + \sum_{k=0}^{m+1}\beta^k}}{\prod_{k=1}^{m+1}(-\gamma\sum_{j=0}^{k-1}\beta^j + \sum_{j=0}^{k}\beta^j)^{\beta^{m-k+1}}}$$

$$= L \cdot t + \sum_{m=1}^{n+1} \frac{L^{\beta^{m}} M^{\sum_{k=0}^{m-1}\beta^k} t^{-\gamma\sum_{k=0}^{m-1}\beta^k + \sum_{k=0}^{m}\beta^k}}{\prod_{k=1}^{m}(-\gamma\sum_{j=0}^{k-1}\beta^j + \sum_{j=0}^{k}\beta^j)^{\beta^{m-k}}} = L \cdot t + \sum_{m=1}^{n+1} z_m(t).$$

Thus, from the assumption that estimate (8) is true for some $n \in N$, we have proved (8) for $n+1$, which together with (13) implies the desired conclusion of Proposition 1. ■

PROOF OF PROPOSITION 2. Since $-\beta < -\gamma \leq 0$ and $\sum_{k=0}^{m} q^k = (q^{m+1}-1)/(q-1)$, we have:

$$z_m(t) \geq \frac{L^{\beta^m} M^{\frac{\beta^m-1}{\beta-1}} t^{-\gamma\frac{\beta^m-1}{\beta-1} + \frac{\beta^{m+1}-1}{\beta-1}}}{\prod_{k=1}^{m}(\frac{\beta^{k+1}-1}{\beta-1})^{\beta^{m-k}}}$$

$$= \frac{L^{\beta^m}\left[(M \cdot t^{-\gamma+\beta})^{\beta^m}\right]^{1/(\beta-1)}}{(M \cdot t^{-\gamma+1})^{1/(\beta-1)}} \frac{\prod_{k=1}^{m}(\beta-1)^{\beta^{m-k}}}{\prod_{k=1}^{m}(\beta^{k+1}-1)^{\beta^{m-k}}}$$

$$\geq \frac{L^{\beta^m}\left[(M \cdot t^{-\gamma+\beta})^{\beta^m}\right]^{1/(\beta-1)}}{(M \cdot t^{-\gamma+1})^{1/(\beta-1)}} \frac{\prod_{k=1}^{m}(\beta-1)^{\beta^{m-k}}}{\prod_{k=1}^{m}\beta^{(k+1)\beta^{m-k}}} = \left(\prod A^{b_k} = A^{\sum b_k}\right)$$

$$= \frac{L^{\beta^m}\left[(M \cdot t^{-\gamma+\beta})^{\beta^m}\right]^{1/(\beta-1)}}{(M \cdot t^{-\gamma+1})^{1/(\beta-1)}} \frac{(\beta-1)^{\sum_{k=1}^{m}\beta^{m-k}}}{\beta^{\sum_{k=1}^{m}(k+1)\beta^{m-k}}}$$

$$= \Bigg(\sum_{k=1}^{m}\beta^{m-k} = \frac{\beta^m-1}{\beta-1},$$

$$\sum_{k=1}^{m}(k+1)\beta^{m-k} = \frac{2\beta-1}{(\beta-1)^2}\beta^m - \frac{1}{\beta-1}m - \frac{2\beta-1}{(\beta-1)^2}\Bigg)$$

$$= L^{\beta^m}\left(\frac{\beta^{\frac{2\beta-1}{\beta-1}}}{(\beta-1)M \cdot t^{-\gamma+1}}\right)^{1/(\beta-1)}$$

$$(\beta^m)^{1/(\beta-1)}\left[\left(\frac{(\beta-1)M \cdot t^{-\gamma+\beta}}{\beta^{\frac{2\beta-1}{\beta-1}}}\right)^{\beta^m}\right]^{1/(\beta-1)}$$

$$= L^B (d_{M,t,\beta,\gamma} B \cdot A^B)^{1/(\beta-1)},$$

where the numbers $d_{M,t,\beta,\gamma}$, B and A satisfy:

$$d_{M,t,\beta,\gamma} = \frac{\beta^{\frac{2\beta-1}{\beta-1}}}{(\beta-1)M \cdot t^{-\gamma+1}}, \quad B = \beta^m \quad \text{and} \quad A = \frac{(\beta-1)M \cdot t^{-\gamma+\beta}}{\beta^{\frac{2\beta-1}{\beta-1}}}.$$

From the preceding we derive $z_m(t) \geq (d_{M,t,\beta,\gamma}B)^{1/(\beta-1)}(A^{1/(\beta-1)}L)^B$, $t \in [0,T]$, $m \in N$. Now, this inequality and (7), (11) and $t^* \in (0,T)$ for any $M > M^*$ imply the desired conclusion of this proposition using the following easy fact: $A^{1/(\beta-1)}L > 1$ for all $t > t^*$ and $\lim_{m\to\infty} \beta^m = \infty$ imply $\lim_{m\to\infty}[A^{1/(\beta-1)}L]^{\beta^m} = \infty$, $t > t^*$. ∎

4. Main result. Summarizing relations (3)–(5) and Lemma 1 above we can immediately conclude:

THEOREM 1. *For arbitrary given constants* $\alpha > 0$ *and* $g_0 > 0$, *let* f^* *be a number defined by*

$$f^* = \left(\frac{\alpha N C_N^{1/N}}{|B|^{1/N}}\right)^p \left(\frac{p\cdot(p')^p}{g_0}\right)^{1/(p'-1)}. \tag{14}$$

If $f_0 > f^*$ *then there is no any solution* $v \in C^2(B\backslash\{0\}) \cap C(\overline{B})$ *of the equation (1). Moreover, if* ω_n *is a sequence of functions defined in Lemma 1 above then*

$$v_n(x) = V_n(C_N|x|^N), \quad \text{where} \quad V_n(s) = \frac{M\cdot\alpha}{g_0}\int_s^{|B|} t^{-p'(1-1/N)}\omega_n^{p'/p}(t)\,dt,$$

is an unbounded sequence of subsolutions of the equation (1).

Let us mention that using the classical Banach fixed point theorem, see for example [13], we can easily obtain the existence result for equation (1) for each $f_0 < f_*$, where we can take

$$f_* = \left(\frac{\alpha N C_N^{1/N}}{2|B|^{1/N}}\right)^p \left(\frac{p'}{g_0 N(p'+1)}\right)^{1/(p'-1)}. \tag{15}$$

5. In conclusion. Thus, from (14) and (15) we have that the intervals $(0, f_*)$ and (f^*, ∞) are two corresponding domains of the existence and nonexistence of solutions of equation (1) in the space $C^2(B \setminus \{0\}) \cap C(\overline{B})$. Regarding [3], it remains to find a positive real number $f_{cr} \in (f_*, f^*)$, the so called *critical explosion parameter* such that $(0, f_{cr})$ and (f_{cr}, ∞) are two maximal intervals of the existence and nonexistence of solutions of equation (1). Finally, we can ask for the nonexistence of solutions of equation (1) without the regularity assumption $v \in C^2(B \setminus \{0\}) \cap C(\overline{B})$ as in Theorem 1.

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THE DIOPHANTINE EQUATIONS $x^4 - y^4 = z^p$ and $x^4 - 1 = dy^q$

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RÉSUMÉ. Dans ce document, nous allons prouver que les équations $x^4 - y^4 = z^p$ $\left(x, y, z \in \mathbb{Z}, xyz \neq 0, \gcd(x, y) = 1\right)$ et $x^4 - 1 = 2^m y^q$ $(x, y \in \mathbb{N}, q > 2, m \in \{1, \ldots, q-1\})$ n'ont pas respectivement de solutions, et l'équation $x^4 - 1 = 2^m p^n y^q$ $(x, y \in \mathbb{N}, p > 2, q > 2, m, n \in \{1, \ldots, q-1\})$ a pour unique solution $(p, q, m, n, x, y) = (5, 3, 1, 1, 3, 2)$. C'est suivant que si $p \equiv 1 \pmod 4$, alors $Q_p(x) = y^q$ $(q > 2, y \in \mathbb{N})$ n'a pas de solutions, et $Q_p(x) = 2^m y^q$ $(q > 2, x, y \in \mathbb{N}, m \in \{1, \ldots, q-1\})$ a pour unique solution $(p, q, m, x, y) = (5, 3, 1, 3, 2)$, où $Q_p(x) = (x^{p-1} - 1)/p$ est le quotient de Fermat.

1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{P}$ be the sets of integers, positive integers and primes respectively, $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$ and throughout this paper, we assume that $p, q \in \mathbb{P}$.

In [11], Powell proved that the Diophantine equation

$$x^4 - y^4 = z^p, \quad xyz \neq 0, \quad \gcd(x, y) = 1 \tag{1}$$

has no solutions $x, y, z \in \mathbb{Z}$ if $p > 2$ and $p \nmid xyz$. In [3], Henri Darmon proved that if Taniyama-Shimura conjecture holds, $p \geq 11$, and $p \equiv 1 \pmod 4$ or $2 \mid z$, then equation (1) has no solutions $x, y, z \in \mathbb{Z}$. In [15], K. Wu and M. Le proved that if $2 \mid z$ and $p \nmid z$, then equation (1) has no solutions $x, y, z \in \mathbb{Z}$.

In this paper, we first prove the following general result by the recent result of Ribet [12], and Darmon and Merel [4].

THEOREM 1. *The Diophantine equation* (1) *has no solutions* $x, y, z \in \mathbb{Z}$.

Next, we consider the Diophantine equation

$$x^4 - 1 = dy^q, \quad x, y \in \mathbb{N}, \tag{2}$$

where $d \in \mathbb{N}$. When $q = 2$, the solvability of (2) was discussed by Ljunggren, Cohn, Chao Ko and Qi Sun, and the author [1], *etc.* For example, Ljunggren [9] proved that if $p > 2$, then the equation $x^4 - 1 = py^2$ has no solutions $x, y \in \mathbb{N}$,

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except $(p,x,y) = (5,3,4)$, $(29,99,1820)$. Chao Ko and Qi Sun [6] proved that if $p > 2$, then the equation $x^4 - 1 = 2py^2$ has no solutions $x, y \in \mathbb{N}$, except $(p,x,y) = (3,7,20)$. In this paper, we discuss the solvability of equation (2) in the case $q > 2$.

THEOREM 2. *If $q > 2$, then the Diophantine equation*

$$x^4 - 1 = p^n y^q, \quad x,y \in \mathbb{N}, \quad p > 2, \quad n \in \mathbb{Z}_q^* \tag{3}$$

has no solutions.

THEOREM 3. *If $q > 2$, then the Diophantine equation*

$$x^4 - 1 = 2^m p^n y^q, \quad x,y \in \mathbb{N}, \quad p > 2, \quad m,n \in \mathbb{Z}_q^* \tag{4}$$

has only the solution $(p,q,m,n,x,y) = (5,3,1,1,3,2)$.

The Fermat's quotient $Q_p(x)$ is defined by

$$Q_p(x) = (x^{p-1} - 1)/p, \quad p > 2, \quad x \in \mathbb{N}, \quad \gcd(x,p) = 1.$$

Lucas proved that $Q_p(2)$ is a square only for $p = 3$ and 7. In [2], we proved that if $p > 3$, then the equation $Q_p(x) = y^2 (y \in \mathbb{N})$ has only the solutions $(p,x,y) = (5,3,4)$ and $(7,2,3)$, and the equation $Q_p(x) = 2y^2 (y \in \mathbb{N})$ has no solutions. In [10], [14], Osada and Terai studied the equation

$$Q_p(x) = y^q, \quad q > 2, \quad y \in \mathbb{N} \tag{5}$$

and they proved that (i) equation (5) has only the solution $p = 3$ with $x = 2$, (ii) if $p > 3$ and x is even, then equation (5) has no solutions, and (iii) if x is odd ≥ 3, $x \equiv 3,5 \pmod 8$ and $x < 50$, then equation (5) has only the solution $(p,q,x,y) = (3,3,5,2)$. In [7], Le proved that if $p \equiv 1 \pmod 4$ and $p > 4 \cdot 10^{176}$, then equation (5) has no solutions.

Now, from Theorems 2 and 3 we have

COROLLARY. *If $p \equiv 1 \pmod 4$, then equation (5) has no solutions, and if $p \equiv 1 \pmod 4$ then the equation*

$$Q_p(x) = 2^m y^q, \quad q > 2, \quad y \in \mathbb{N}, \quad m \in \mathbb{Z}_q^* \tag{6}$$

has only the solution $(p,q,m,x,y) = (5,3,1,3,2)$.

2. Preliminaries.

LEMMA 1 (RIBET [12] AND DARMON AND MEREL [4]). *If $p > 2$, then the equation*

$$x^p + 2^n y^p + z^p = 0, \quad n \in \mathbb{Z}_p^*$$

has no solutions $x, y, z \in \mathbb{Z}$ with $|xyz| > 1$.

LEMMA 2 (LEBESGUE [8] AND CHAO KO [5]). *The equation*

$$x^2 + 1 = y^q, \quad x, y \in \mathbb{N}$$

has no solutions, and the equation

$$x^2 - 1 = y^q, \quad x, y \in \mathbb{N}, \quad q > 2$$

has only the solution $(q, x, y) = (3, 3, 2)$.

LEMMA 3 (STÖRMER [13]). *The equation*

$$x^2 + 1 = 2y^q, \quad x, y \in \mathbb{N}, \quad q > 2$$

has no solutions with $y > 1$.

3. Proof of the theorems.

PROOF OF THEOREM 1. When $p = 2$, Theorem 1 is well known. Now we assume that $p > 2$. Suppose (p, x, y, z) is an integral solution of (1). Then

$$(x + y)(x - y)(x^2 + y^2) = z^p, \quad xyz \neq 0, \quad \gcd(x, y) = 1. \tag{7}$$

If $2 \nmid z$, then $\gcd(x^2 - y^2, x^2 + y^2) = \gcd(x + y, x - y) = 1$. So (7) gives

$$x + y = a^p, \quad x - y = b^p, \quad x^2 + y^2 = c^p, \quad z = abc, \tag{8}$$

where $a, b, c \in \mathbb{Z}$, $abc \neq 0$, and $\gcd(a, b) = \gcd(ab, c) = 1$. From (8), we have

$$a^{2p} + b^{2p} = 2c^p.$$

Hence $|a| = |b| = c = 1$ by Lemma 1. So (8) gives $xy = 0$.

If $2 \mid z$, then $\gcd(x^2 - y^2, x^2 + y^2) = \gcd(x + y, x - y) = 2$. Notice that $2 \parallel (x^2 + y^2)$. So (7) gives

$$x \pm y = 2^{p-2}a^p, \quad x \mp y = 2b^p, \quad x^2 + y^2 = 2c^p, \quad z = 2abc, \tag{9}$$

where $a, b, c \in \mathbb{Z}$, $abc \neq 0$, and $\gcd(a, b) = \gcd(ab, c) = 1$. From (9), we have

$$2^{2(p-3)}a^{2p} + b^{2p} = c^p, \quad abc \neq 0,$$

which is impossible by Lemma 1. This completes the proof of Theorem 1. ∎

PROOF OF THEOREM 2. Suppose (p, q, n, x, y) is a solution of (3). Then

$$(x^2 - 1)(x^2 + 1) = p^n y^q, \quad x, y \in \mathbb{N}, \quad p > 2, \quad n \in \mathbb{Z}_q^*. \tag{10}$$

If $2 \mid x$, then $\gcd(x^2 - 1, x^2 + 1) = 1$ and so (10) gives

$$x^2 \pm 1 = p^n y_1^q, \quad x^2 \mp 1 = y_2^q, \quad y = y_1 y_2, \tag{11}$$

where $y_1, y_2 \in \mathbb{N}$, $\gcd(y_1, y_2) = 1$. By Lemma 2, the equations $x^2 \mp 1 = y_2^q$ have no solutions with $2 \mid x$. Thus, (11) is impossible.

If $2 \nmid x$, then $\gcd(x^2-1, x^2+1) = 2$. Since $2 \parallel (x^2+1)$, from (10) we get that

$$x^2 + 1 = 2p^n y_1^q, \quad x^2 - 1 = 2^{q-1} y_2^q, \quad y = 2y_1y_2, \tag{12}$$

or

$$x^2 + 1 = 2y_1^q, \quad x^2 - 1 = 2^{q-1} p^n y_2^q, \quad y = 2y_1y_2, \tag{13}$$

where $y_1, y_2 \in \mathbb{N}$, $\gcd(y_1, y_2) = 1$. By Lemma 3, we see that (13) is impossible. For (12), we show that the equation

$$x^2 - 1 = 2^{q-1} y_2^q, \quad x, y_2 \in \mathbb{N}, \quad q > 2 \tag{14}$$

has no solutions. Otherwise, (14) gives

$$x \pm 1 = 2a^q, \quad x \mp 1 = 2^{q-2} b^q, \quad y_2 = ab,$$

where $a, b \in \mathbb{N}$, $\gcd(a, b) = 1$. It implies that

$$a^q - 2^{q-3} b^q = \pm 1. \tag{15}$$

Since $q > 2$ $q \in \mathbb{P}$, $a, b, \in \mathbb{N}$, we see from Lemma 1 that (15) is impossible. The theorem is proved. ∎

PROOF OF THEOREM 3. Suppose (p, q, m, n, x, y) is a solution of (4). Then

$$(x^2 - 1)(x^2 + 1) = 2^m p^n y^q, \quad x, y \in \mathbb{N}, \quad p > 2, \quad m, n \in \mathbb{Z}_q^*. \tag{16}$$

Clearly, $\gcd(x^2-1, x^2+1) = 2$. Since $2 \parallel (x^2+1)$, from (16) we get that

$$x^2 + 1 = 2p^n y_1^q, \quad x^2 - 1 = 2^{m-1} y_2^q, \quad y = y_1y_2, \tag{17}$$

or

$$x^2 + 1 = 2y_1^q, \quad x^2 - 1 = 2^{m-1} p^n y_2^q, \quad y = y_1y_2, \tag{18}$$

where $y_1, y_2 \in \mathbb{N}$, $\gcd(y_1, y_2) = 1$ and $2 \nmid y_1$. By Lemma 3, we see that (18) is impossible. For (17), if $m = 1$ then we get by Lemma 2 that $x = 3$, $q = 3$, $p^n = 5$, $y = 2$. If $m > 1$, then from

$$x^2 - 1 = 2^{m-1} y_2^q, \quad x, y_2 \in \mathbb{N}, \quad q > 2$$

and Lemma 1 we see that (17) is impossible. The theorem is proved. ∎

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LINEAR INDEPENDENCE IN THE SELBERG CLASS

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ABSTRACT. We prove that distinct functions in the Selberg class $\mathcal{S}$ are linearly independent over the ring $\mathcal{F}$ of p-finite Dirichlet series. As a consequence, $\mathcal{S}$ has unique factorization if and only if primitive functions of $\mathcal{S}$ are algebraically independent over $\mathcal{F}$.

RÉSUMÉ. Nous montrons que les fonctions dans la classe de Selberg $\mathcal{S}$ sont linéairement indépendantes sur l'anneau $\mathcal{F}$ des séries de Dirichlet p-finies. Comme conséquence, $\mathcal{S}$ a une factorization unique si et seulement si les fonctions primitives de $\mathcal{S}$ sont algébriquement indépendantes sur $\mathcal{F}$.

Roughly speaking, the main conjecture about the Selberg class $\mathcal{S}$ (see the survey paper [4] for the basic definitions, conjectures and properties of $\mathcal{S}$) is that $\mathcal{S}$ can be essentially identified with the class of suitably normalized automorphic L-functions. Based on such conjecture, one may guess that $\mathcal{S}$ contains only countably many essentially different functions, although $\mathcal{S}$ itself is certainly not countable since it contains all *shifts* $F_\theta(s) = F(s+i\theta)$, $\theta \in \mathbb{R}$, of any entire function $F \in \mathcal{S}$. However, one may guess that $\mathcal{S}$ becomes a countable set provided shifts are not counted. This is the *countability problem* for $\mathcal{S}$. This problem was communicated to us by Peter Sarnak, and we now understand that it was independently raised also by Ram Murty and Eduard Wirsing.

In order to state the problem in a rigorous form, we consider the equivalence relation in $\mathcal{S}$

$$F(s) \approx G(s) \iff G(s) = F_\theta(s) \quad \text{for some} \quad \theta \in \mathbb{R}.$$

Denoting by $\mathcal{P}$ the set of all *primitive* functions in $\mathcal{S}$, a formulation of Sarnak's problem is the *countability conjecture*

$$\mathcal{P}/\approx \text{ is countable.}$$

Observe that we need to consider $\mathcal{P}/\approx$, since $\mathcal{S}/\approx$ is still an *uncountable* set. In fact, for a given primitive character $\chi \pmod q$ with $q \geq 2$, the functions $\zeta(s)L_\theta(s,\chi)$, $\theta \in \mathbb{R}$, are all in $\mathcal{S}$ and non-equivalent under $\approx$. We refer to Section 9 of [4] for another version of the countability conjecture, involving a suitable

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normalization of the whole class $\mathcal{S}$. Such normalization, and hence the statement of the other version of the conjecture, require the assumption of Selberg orthonormality conjecture (see Section 4 of [4]). However, the two versions of the countability conjecture are equivalent under Selberg's conjecture.

An approach to the countability conjecture may run as follows. We believe that, modulo shifts, there are only countably many possibilities for the functional equations satisfied by the functions in $\mathcal{S}$, and in fact several conjectures (for example the degree and modulus conjectures, see Section 9 of [4]) point to this direction. Moreover, we believe that the vector space formed by the meromorphic Dirichlet series satisfying a functional equation of such type and the Ramanujan hypothesis has at most countable dimension. This is in fact known when the degree d satisfies $0 \leq d \leq 1$, see Sections 3 and 5 of [4], in which case the dimension is actually finite. We remark that Brian Conrey, David Farmer and Amit Ghosh have brought to our attention an interesting example, due to Hecke, of a vector space of meromorphic Dirichlet series with functional equation; see Chapter 2 of [2]. Hecke proved that such a vector space is infinite dimensional, and it is not difficult to show that in fact it has an uncountable basis. However, we believe that such Dirichlet series do not satisfy Ramanujan hypothesis.

If the above assumptions are correct, the countability conjecture would then follow from a suitable linear independence property of functions in $\mathcal{S}$, ensuring that each functional equation has in fact at most countably many solutions in $\mathcal{S}$.

As a small step towards the countability conjecture, in this note we prove the required linear independence property. Although the proof is short and simple, we believe that the result is of some independent interest as well. In order to state our result in a general form, we need to define the *p-finite* Dirichlet series. A Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} c(n)n^{-s},$$

absolutely convergent in some right half plane, is called p-finite if there exists a positive integer M such that $c(n) = 0$ whenever n has a prime factor p such that $p \nmid M$. In this case, the arithmetical function $c(n)$ is called p-finite as well. We denote by $\mathcal{F}$ both the ring of p-finite Dirichlet series and the ring of p-finite arithmetical functions. Observe that $\mathcal{F}$ contains all Dirichlet polynomials. We have

THEOREM 1. *Distinct functions in $\mathcal{S}$ are linearly independent over $\mathcal{F}$.*

In particular, Theorem 1 implies

COROLLARY 1. *Distinct functions $F_1, \ldots, F_N \in \mathcal{S}$ are linearly independent over $\mathbb{C}$.*

We note that Selberg [6] and Bombieri-Hejhal [1] already observed the result in Corollary 1, under the assumption that $F_1(s), \ldots, F_N(s)$ are pairwise *orthogonal*, see Section 7 of [4]. Moreover, special cases of Theorem 1 are already known in the literature, see for example Lemma 8.1 of [3] where Dirichlet L-functions are considered. However, in such special cases one usually exploits special properties of the involved functions. For instance, the proof of Lemma 8.1 of [3] is based on the fact that Dirichlet L-functions formed with distinct primitive characters are pairwise orthogonal. We also note that axiom *(iv)* of Selberg class, *i.e.*, Ramanujan hypothesis, plays no role in the proof of Theorem 1.

We remark that Theorem 1 is basically a result on multiplicative arithmetical functions. In fact, calling *equivalent* two multiplicative arithmetical functions $f(n)$ and $g(n)$ if $f(p^m) = g(p^m)$ for all integers $m \geq 1$ and all but finitely many primes p, we have

THEOREM 2. *Pairwise non-equivalent multiplicative arithmetical functions are linearly independent over $\mathcal{F}$.*

PROOF. By contradiction, assume that there exist pairwise non-equivalent multiplicative arithmetical functions $f_1(n), \dots, f_N(n)$ and non-identically vanishing p-finite arithmetical functions $c_1(n), \dots, c_N(n)$ such that

$$\sum_{j=1}^{N} (c_j * f_j)(n) = 0 \quad \text{identically.} \tag{1}$$

Moreover, let N be the minimal value required in order (1) to be satisfied. Clearly, $N > 1$ since the ring of arithmetical functions has no zero-divisors.

Let M be such that $c_j(n) = 0$, $j = 1, \dots, N$, if n has a prime factor p with $p \nmid M$. Moreover, let $p_0 > M$ and $k_0 \geq 1$ be such that

$$f_1(p_0^{k_0}) \neq f_j(p_0^{k_0}) \quad \text{for } j = 2, \dots, N.$$

Write

$$\mathbb{N}_0 = \{m \in \mathbb{N} : (m, p_0) = 1\}$$

and consider (1) with $n = p_0^{k_0} m$, $m \in \mathbb{N}_0$, thus getting

$$\sum_{j=1}^{N} f_j(p_0^{k_0}) \sum_{d|m} c_j(d) f_j\left(\frac{m}{d}\right) = 0 \quad \text{for every } m \in \mathbb{N}_0. \tag{2}$$

Multiplying (1), with $n = m$, by $f_1(p_0^{k_0})$ and subtracting (2) we get

$$\sum_{j=2}^{N} \left(f_1(p_0^{k_0}) - f_j(p_0^{k_0})\right) \sum_{d|m} c_j(d) f_j\left(\frac{m}{d}\right) = 0 \quad \text{for every } m \in \mathbb{N}_0. \tag{3}$$

For $j = 2, \dots, N$ define

$$\tilde{c}_j(n) = \left(f_1(p_0^{k_0}) - f_j(p_0^{k_0})\right) c_j(n)$$

and

$$\tilde{f}_j(n) = \begin{cases} f_j(n) & \text{if } n \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the $\tilde{f}_j(n)$ are pairwise non-equivalent multiplicative arithmetical functions and the $\tilde{c}_j(n)$ are non-identically vanishing p-finite arithmetical functions. Moreover, in view of (3) such functions satisfy

$$\sum_{j=2}^{N} (\tilde{c}_j * \tilde{f}_j)(n) = 0 \quad \text{identically,}$$

which contradicts the minimality of N. Theorem 2 is therefore proved. ∎

Theorem 1 follows immediately from Theorem 2 by means of the following result of Murty-Murty [5], see also Section 2 of [4].

PROPOSITION (MURTY-MURTY [5]). *Let $F, G \in \mathcal{S}$. If $F_p(s) = G_p(s)$ for all but finitely many primes p, then $F = G$.*

Here $F_p(s)$ and $G_p(s)$ denote the p-th Euler factors of $F(s)$ and $G(s)$, respectively, *i.e.*,

$$F(s) = \prod_p F_p(s) \quad \text{and} \quad G(s) = \prod_p G_p(s), \quad \sigma > 1.$$

Therefore, the Proposition asserts that the coefficients of distinct functions in $\mathcal{S}$ are pairwise non-equivalent, and hence Theorem 1 follows from Theorem 2 since such coefficients are multiplicative.

We further remark that it is not difficult to construct examples showing that the non-equivalence is a necessary assumption in Theorem 2.

We recall that a function $F \in \mathcal{S}$ is *primitive* if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$ implies that $F_1 = 1$ or $F_2 = 1$. Moreover, every function in $\mathcal{S}$ can be factored into primitive functions. It is conjectured that such a factorization is unique, and this would follow from Selberg orthonormality conjecture, see Section 4 of [4]. Our final remark is that the unique factorization is equivalent to a suitable *algebraic* independence property in $\mathcal{S}$. In fact, from Theorem 1 we easily get

COROLLARY 2. *$\mathcal{S}$ has unique factorization if and only if distinct primitive functions are algebraically independent over $\mathcal{F}$.*

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