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ON A NON-SELF-ADJOINT PROBLEM WITH AN INDEFINITE WEIGHT FOR ELLIPTIC SYSTEMS WITH A PARAMETER

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RÉSUMÉ. Nous établissons la complétude et la summabilité par la méthode d'Abel-Lidskii des vecteurs propres généralisés d'un problème élliptique avec un poids indéfini et la distribution angulaire des valeurs propres du problème.

1. Introduction. In a bounded region $\Omega \subset \mathbf{R}^n$ with a $(n-1)$-dimensional boundary Γ, we consider the boundary value problem

$$(1) \qquad (A - \lambda\omega E)u = 0 \text{ in } \Omega; \quad B_k u = 0 (k = 1, \dots, r) \text{ on } \Gamma$$

where $A = A(x, D)$ is a square matrix of dimension N consisting of differential operators of order $2m$ with complex coefficients, $B_k = B_k(x, D)$ $(k = 1, \dots, r = Nm)$ are N-dimensional rows whose components are differential operators of order $m_k \leq 2m - 1$ with complex coefficients ω is a real-valued function which assumes both positive and negative values and E is the unit matrix.

In this note we establish some results on the completeness and the summability by Abel's method (see Lidskii 1962, Kostyuchenko-Razdievskij 1974, for details) of the root vectors of problem (1), and the angular distribution of its eigenvalues. The completeness of root vectors and the angular distribution of eigenvalues have been obtained in (Faierman 1990) for regular scalar elliptic problems (when $N = 1$). Here our results, which are a generalization of Faierman's to elliptic systems, are derived through the existence theory of an auxilliary transmission problem under weaker conditions. This approach makes it possible to extend our results to L_p spaces and to more general problems; for example, to systems elliptic in the sense of Douglis-Nirenberg with a parameter. We refer to (Agranovich 1990, Kozhevnikov 1973), where these problems have been investigated in the case when $\omega(x) \equiv 1$. We also note the important contributions of (Agmon 1962) and (Grisvard and Geymonat 1967) to the L_p-theory.

2. Basic assumptions. Let $x = (x_1, \dots, x_n)$, $D_j = -i\frac{\partial}{\partial x_j}$, $D = (D_1, \dots, D_n)$, $D^\alpha = D_1^{\alpha_1}, \dots, D_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ ($\mathbf{Z}_+$ is the set of non-

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negative integers) and $|\alpha| = \sum_{j=1}^{n} \alpha_j$,

$$A(x,D) = \Big\{ \sum_{|\alpha|\le 2m} a_\alpha^{ij}(x) D^\alpha \Big\}_{i,j=1}^{N},$$

$$B_k(x,D) = \Big\{ \sum_{|\beta|\le m_k} b_\beta^{kj}(x) D^\beta \Big\}_{j=1,\dots,N} (k = 1,\dots,Nm).$$

We assume throughout that the operators A, B_k $(k = 1,\dots,Nm)$ and the domain Ω satisfy the following smoothness conditions: the region Ω is of class $\mathbf{C}^{2m}$, the coefficients $a_\alpha^{ij}(x)$ in A are continuous in $\bar{\Omega}$ for $|\alpha| = 2m$ and bounded for $|\alpha| \le 2m-1$, the coefficients $b_\beta^{kj}(x)$ in B_k belong to $\mathbf{C}^{2m-m_k}(\Gamma)$ for $|\beta| = m_k$ and bounded together with their derivatives of order up to $2m - m_k$ for $|\beta| \le m_k - 1$.

Let $H_l(\Omega, N)$ (l is an integer), be the direct product of N Sobolev spaces $W_2^l(\Omega)$ when $l = 0$ we write $H_0(\Omega, N) = L_2(\Omega, N)$. We denote by $H_{l-\frac{1}{2}}(\Gamma)$ $(l \ge 1)$ the space of boundary values of functions from $W_2^l(\Omega)$ and by $H_{l-\frac{1}{2}}(\Gamma, N)$ the direct product of N such spaces.

Next we turn to the assumptions concerning the weight function $\omega(x)$. They will be closely related to a certain partition of the domain Ω into appropriate subdomains.

Assumptions 1. Let there be given some $(n-1)$-dimensional manifolds $\Gamma_1,\dots,\Gamma_s$ each of class C^{2m}, lying inside Ω and having no point in common with Γ and such that $\Gamma_l \cap \Gamma_k \ne 0$ for $l \ne k$. They divide Ω into subdomains

$$\Omega_1,\dots,\Omega_{s+1}.$$

We assume that the weight function $\omega(x)$ is continuous in each $\bar{\Omega}_l$, can pertain a discontinuity of first kind and change sign while crossing any Γ_p and

$$|\omega(x)| > 0 \text{ a.e. in } \Omega.$$

Since the function $\omega(x)$ is assumed to be discontinuous across Γ_p, the solution of (1) may not belong to the functional space $H_{2m}(\Omega, N)$ which is of importance to us. Thus, in order to preserve the membership of the solution to this space, we impose, among others, the following natural conjugation conditions:

$$D_n^j u_{(l)}(x) = D_n^j u_{(l')}(x) (j = 0,\dots,2m-1) \quad \text{on each } \Gamma_p \tag{2}$$

where $u_{(l)}$ and $u_{(l')}$ are the restrictions of the function u to Ω_l and $\Omega_{l'}$ respectively, Γ_p separates Ω_l from $\Omega_{l'}$ and D_n is the derivative along the inward normal to Γ_p.

Definition 1. A complex number λ will be called an eigenvalue of the boundary problem (1) if the problem (1) with the transmission conditions (2) admits at least a non-trivial solution $u \in H_{2m}(\Omega, N)$; this solution is referred to as the eigenfunction of (1) corresponding to λ; otherwise the number λ is called regular point of (1).

Next, let $A_0(x,\xi)$ and $B_{0k}(x,\xi)$ $(k=1,\dots,Nm)$ be respectively the principal parts of the operators A and B_k $(k=1,\dots,Nm)$. Let also

$$\Xi(\theta_1,\theta_2)=\{\lambda\in\mathbf{C}:\theta_1\le\arg\lambda\le\theta_2\}$$

be an angular sector in the complex plane and λ an element of $\Xi(\theta_1,\theta_2)$.

ASSUMPTIONS 2. (i) We require the matrix

$$A_0(x,\xi)-\lambda\omega(x)E \tag{3}$$

to be invertible for all $\xi\in\mathbf{R}^n$; $|\xi|+|\lambda|\neq 0$ and all $x\in\bar{\Omega}_p$ $(p=1,\dots,s+1)$.

(ii) Let x_0 be any point on Γ. We shall turn the coordinates axes such that, the axis x_n takes the direction of the inward normal to Γ at x_0. For simplicity, we suppose that the operators A, B_k are written in the system of coordinates connected with x_0. We consider the following problem on the ray.

$$(A_0(x_0,\xi',D_t)-\lambda\omega(x_0)E)v(t)=0,\quad t>0 \tag{4}$$

$$B_{0k}(x_0,\xi',D_t)v(t)|_{t=0}=g_k\quad (k=1,\dots,Nm), \tag{5}$$

where $\xi'=(\xi_1,\dots,\xi_{n-1})$ and $D_t=-i\frac{d}{dt}$.

For any $\xi'\in\mathbf{R}^{n-1}$, $|\xi'|+|\lambda|\neq 0$, the space of solutions of system (4), exponentially decreasing in modulus when $t\to\infty$, is Nm-dimensional and the problem (4)–(5) is uniquely solvable for any g_k in this space.

(iii) Let $A_l(A_{l'})$ and $\omega_l(\omega_{l'})$ be respectively the restriction of the matrix A and the function ω to $\Omega_l(\Omega_{l'})$ and assume that Γ_p separates Ω_l and $\Omega_{l'}$. We take a point $x_0\in\Gamma_p$ and turn the coordinate axes such that, the axis x_n takes the direction of the inward normal to Γ_p at x_0. We consider the following transmission problem on the line.

$$(A_{l0}(x_0,\xi',D_t)-\lambda\omega_l(x_0)E)v_l(t)=0,\quad t>0 \tag{6}$$

$$(A_{l'0}(x_0,\xi',D_t)-\lambda\omega_{l'}(x_0)E)v_{l'}(t)=0,\quad t<0 \tag{7}$$

$$D_t^{\mu}v_l(0)-D^{\mu}v_{l'}(0)=h_{\mu p},\quad \mu=0,\dots,2m-1. \tag{8}$$

For any $\xi'\in\mathbf{R}^{n-1}$; $|\xi'|+|\lambda|\neq 0$, the space of solutions of the system (6)–(7), exponentially decreasing in modulus when $|t|\to\infty$, is $2Nm$-dimensional and the problem (6)–(8) is uniquely solvable in this space for any N-dimensional column $h_{\mu p}$.

We define an exponentially decreasing solution of (6)–(7) as a vector $v=(v_l,v_{l'})$, where v_l and $v_{l'}$ are respectively solutions of (6) and (7) and $v_l(t)\to 0$ when $t\to+\infty$ while $v_{l'}(t)\to 0$ when $t\to-\infty$.

Let $\mathcal{A}$ be the unbounded operator with the domain

$$\mathcal{D}(\mathcal{A})=\{u\in H_{2m}(\Omega,N):B_ku=0(k=1,\dots,Nm)\}$$

acting by $\mathcal{A}u = A(x, D)u$ for all $u \in \mathcal{D}(\mathcal{A})$ and T the bounded operator induced by the multiplication by ω in $L_2(\Omega, N)$. Clearly $\mathcal{D}(\mathcal{A})$ is dense in $L_2(\Omega, N)$. We rewrite problem (1) in the form

$$T^{-1}\mathcal{A}u = \lambda u, \quad u \in \mathcal{D}(\mathcal{A}) \tag{9}$$

and call λ an eigenvalue of problem (1) if λ is an eigenvalue of (9) and refer to the root vectors of $T^{-1}\mathcal{A}$ corresponding to λ as those of (1) corresponding to λ. Lastly let $S(\lambda) = \mathcal{A} - \lambda T$ be the pencil acting in $L_2(\Omega, N)$ with the domain $\mathcal{D}(S) = \mathcal{D}(\mathcal{A})$ and $\rho(S)$ the set of its regular points.

3. **Results.** Now we can formulate our first result which plays a central role in our investigations. We have

THEOREM 1. *Let Assumptions 1 and 2 be satisfied. Then there exists a positive number C such that $\lambda \in \rho(S)$ and*

$$\|S^{-1}(\lambda)\|_{L_2(\Omega,N)\to L_2(\Omega,N)} \leq C|\lambda|^{-1} \tag{10}$$

for $\lambda \in \Xi(\theta)$ and sufficiently large in modulus.

IDEA OF THE PROOF. We consider the following transmission problem induced by the boundary value problem (1) and the conjugation conditions (2):

$$L_l u = \big(A_l(x, D) - \lambda\omega_l(x)E\big)u_l(x) = 0 \quad \text{in } \Omega_l (l = 1, \dots, s+1), \tag{11}$$

$$[D^\mu u]_p = D^\mu u_l(x) - D^\mu u_{l'}(x) = 0 \text{on } \Gamma_p (\mu = 0, \dots, 2m-1; p = 1, \dots, s), \tag{12}$$

$$B_k(x, D)u(x) = 0 \quad \text{on } \Gamma (k = 1, \dots, Nm). \tag{13}$$

This problem is elliptic with a parameter under the assumptions 2. Following (Agranovich and Vishik, 1964) and (Roitberg and Serdyuk, 1991), we prove that for $|\lambda|$ sufficiently large, the operator $\mathcal{U}$ defined by

$$\begin{aligned} \mathcal{U}u = \big\{&L_1u_1, \dots, L_{s+1}u_{s+1}, [u]_1, \dots, \\ &[D^{2m-1}u]_1, \dots, [u]_s, \dots, [D^{2m-1}u]_s, B_1u, \dots, B_{Nm}u\big\}, \end{aligned}$$

connected with the problems (11)–(13) and acting from

$$H_{2m}(\Omega_1, N) \times \cdots \times H_{2m}(\Omega_{s+1}, N)$$

to

$$\prod_{l=1}^{s+1} L_2(\Omega_l, N) \times \prod_{p=1}^{s} \prod_{\mu=0}^{2m-1} H_{2m-\mu-\frac{1}{2}}(\Gamma_p, N) \times \prod_{k=1}^{Nm} H_{2m-m_k-\frac{1}{2}}(\Gamma),$$

establishes an isomorphism between these two spaces. Hence, thanks to the homogeneous boundary conditions (12), we obtain that the operator $\big(A(x, D) - \lambda\omega(x)\big)$ establishes an isomorphism between $H_{2m}(\Omega, N)$ and $L_2(\Omega, N)$ and the estimate

$$\|u\|_{2m,\Omega} + |\lambda|\,\|u\|_{0,\Omega} \leq C\big\|\big(A(x, D) - \lambda\omega(x)\big)u\big\|_{0,\Omega}$$

holds for all solutions $u \in H_{2m}(\Omega, N)$ of (11)–(13) with $|\lambda|$ sufficiently large; C is a positive constant independent of u and λ. Furthermore the operator $\mathcal{A}$ is closed. Now (10) immediately follows from this inequality and the closed graph theorem.

Let $0 \in \rho(S)$. We are now in the position to establish our main results. We have

THEOREM 2. *Let assumptions 1 be satisfied. Suppose that assumptions 2 are also satisfied along certain rays* $\Xi(\theta_j)$ $(j = 1, \dots, k)$ *in the complex plane emanating from the origin and making an angle* θ_j *with the positive real axis, and in this connection let the maximal angle between successive rays not exceed* $2m\pi/n$.

Then the spectrum of problem (1) is discrete and its root vectors are complete in $L_2(\Omega, N)$.

For the proof, we show that under the conditions of the theorem the $\Xi(\theta_j)$ are rays of minimal growth of $(T^{-1}\mathcal{A} - \lambda E)^{-1}$, *i.e.*, this operator exists for $\lambda \in \Xi(\theta_j)$ and

$$\|(T^{-1}\mathcal{A} - \lambda E)^{-1}\| \leq \text{const}\,|\lambda|^{-1} \tag{14}$$

for $|\lambda|$ sufficiently large. The first assertion of the theorem follows from the compactness of $(T^{-1}\mathcal{A} - \lambda E)^{-1}$ in $L_2(\Omega, N)$. Furthermore $(T^{-1}\mathcal{A} - \lambda E)^{-1}$ belongs to the Von Neuman-Schatten class $\mathbf{C}_{\frac{n}{2m}+\varepsilon}$; $\varepsilon > 0$. Thus, the completeness of the root vectors of problem (1) follows from the inequality (14) and (Dunford and Schwartz 1963, Chapter XI, Sect. 9, Corollary 31).

THEOREM 3. *Under the assumptions of Theorem 2, the system of root vectors of problem (1) is summable by the method* $\mathcal{A}(L_2(\Omega, N), \alpha_j, \beta_j)$ $(\alpha_j = \frac{\theta_{j+1}+\theta_j}{2}$, $j = 1, \dots, k-1)$ *for* $\beta_j \in (\frac{n}{2m}, \frac{\pi}{\alpha_j})$, *where* $\sigma_j = |\theta_{j+1} - \theta_j|$.

Since $(T^{-1}\mathcal{A} - \lambda E)^{-1}$ is of class $\mathbf{C}_{\frac{n}{2m}+\varepsilon}$; $\varepsilon > 0$ and Inequality (14) holds on the rays $\Xi(\theta_j)$ $(j = 1, \dots, k)$, the affirmation the theorem is an immediate consequence of (Kostyuchenko and Razdievskij, 1974, Theorem 1).

The following result deals with the angular distribution of the eigenvalues of problem (1).

THEOREM 4. *Let Assumptions 1 be satisfied and suppose that there exist the rays* $\{\Xi(\theta_j)\}$ $(j = 1, 2)$ *in the complex plane such that*

(i) $0 < \theta_2 - \theta_1 < \min\{2\pi, 2m\pi/n\}$

(ii) Assumptions 2 are satisfied for $\theta = \theta_j$ $(j = 1, 2)$.

Suppose also that for some θ' $(\theta_1 < \theta' < \theta_2)$ *at least one of the assumptions 2 (i), (ii) or (iii) is violated for* $\theta = \theta'$.

Then there are infinitely many eigenvalues of the problem (1) in the sector $\theta_1 < \arg \lambda < \theta_2$.

EXAMPLE. Suppose that the operator $A(x, D)$ is strongly elliptic and the boundary conditions are of Dirichlet type then the assumptions 2 (i), (ii) are immediately satisfied (see Agranovich and Vishik 1964, Chap. 6). Following the same arguments as in this paper, one can show that the assumption 2 (iii) is satisfied only when $\arg\lambda = \pm\frac{\pi}{2}$. Thus the revolvent set of the pencil $S(\lambda)$ is located on the imaginary axis in the complex plane and the rays $\arg\lambda = \pm\frac{\pi}{2}$ are the rays of minimal growth of $\big(S(\lambda)\big)^{-1}$. Furthermore from Theorem 2 the spectrum of $S(\lambda)$ is discrete and lies in the half-planes $\operatorname{Im}\lambda < 0$ and $\operatorname{Im}\lambda > 0$. We obtain also that when $2m > n$, the root vectors are complete in $L_2(\Omega, N)$ and summable by the method $\mathcal{A}\big(L_2(\Omega, N), \alpha_j, \beta_j\big)$ for $\beta_j \in (\frac{n}{2m}, 1)$ $(j = 1, 2)$; here $\alpha_j = 0, \pi$.

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NOTE ADDED IN PROOF. The extension of the completeness results from this paper to systems elliptic in the sense of Agmon-Douglis-Nirenberg in L_p Sobolev spaces $(1 < p < \infty)$, without any restriction on the orders of the operators involved, has been announced in the author's paper: C. R. Acad. Sci. Paris, t. 329, Série I, pp. 703–708, 1998. A detailed version of our results will be published elsewhere.

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GORDON'S ϵ-CONJECTURE ON THE LACUNARITY OF MODULAR FORMS

KEN ONO

Presented by M. Ram Murty, FRSC

ABSTRACT. In this note we prove B. Gordon's ϵ-conjecture regarding the lacunarity of modular forms. We show that if $f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(N,\chi)$ has the property that there exists an $\epsilon > 0$ for which

$$\#\{n < X \mid a(n) \neq 0\} = O(X^{1-\epsilon}),$$

then $f(z)$ is a finite linear combination of theta series of weight $1/2$ or $3/2$.

RÉSUMÉ. Ici, on démontre une conjecture de B. Gordon qui s'agit de la lacunarité des formes modulaire. Soit $f = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N,\chi)$ une forme modulaire. On montre que si il existe un $\epsilon > 0$ pour que

$$\#\{n < X \mid a(n) \neq 0\} = O(X^{1-\epsilon}),$$

puis f est une combination des series theta du poids $1/2$ ou $3/2$.

A formal power series $P(q) := \sum_{n \geq N_0} a(n)q^n$ is called *lacunary* if

$$\lim_{X \to \infty} \frac{\#\{n < X \mid a(n) = 0\}}{X} = 1.$$

These power series have the property that "almost all" of their coefficients are zero. Many important q-series in the theory of partitions are lacunary. For instance the following well known identities are examples of lacunary power series:

$$\text{(Euler)} \qquad \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}},$$

$$\text{(Jacobi)} \qquad \prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}.$$

For each $k \in \frac{1}{2}\mathbb{Z}$, let $M_k(N,\chi)$ be the space of modular forms of weight k on $\Gamma_0(N)$ (if k is half-integral then $4|N$) with Nebentypus character χ, and let

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$S_k(N,\chi)$ denote its subspace of cusp forms. In this note we are interested in those $f(z) \in M_k(N,\chi)$ whose Fourier expansions $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ (throughout $q := e^{2\pi i z}$) are lacunary. Serre [S] proved a "basis theorem" for lacunary integral weight forms. He proved that an integral weight $f(z)$ is lacunary if and only if it is a finite linear combination of forms with complex multiplication. Using this description, Serre [S2] and Gordon, Hughes and Robins [G-H, G-R] have classified all the lacunary integer weight modular forms in certain special families of forms whose Fourier expansions are given by infinite products. V. K. Murty [M] has obtained an intriguing alternative description of the lacunary integer weight forms.

The characterization of lacunary half-integral weight modular forms remains open. Elementary theta functions serve as convenient examples of lacunary half-integral weight forms. If $i = 0$ or 1, $0 \le r < t$, and $a \ge 1$, then the elementary theta function $\theta_{a,i,r,t}(z)$ is given by

$$\theta_{a,i,r,t}(z) := \sum_{n \equiv r \pmod t} n^i q^{an^2}.$$

Each function $\theta_{a,i,r,t}(z)$ is a holomorphic form of weight $i + \frac{1}{2}$, and any $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ that is a finite linear combination of such series is called *superlacunary*. In particular every superlacunary form has weight $1/2$ or $3/2$. By a theorem of Serre and Stark [S-Sta], it is well known that every weight $1/2$ modular form is superlacunary. Clearly every superlacunary $f(z)$ is lacunary since there exists a non-zero constant c_f for which

$$\#\{n < X \mid a(n) \ne 0\} \sim c_f \sqrt{X}.$$

Recalling Dedekind's eta-function $\eta(z) := q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$, we find that the identities above are examples by Euler and Jacobi obtained from

$$\eta(24z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2},$$

$$\eta^3(8z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2}.$$

It is widely believed that every lacunary half-integral weight modular form is superlacunary, *i.e.*, is a finite linear combination of elementary theta series. A proof of this conjecture seems to be well beyond current methods. In view of these technical difficulties, Gordon posed the following unpublished conjecture.

GORDON'S ϵ-CONJECTURE. *If $f(z) = \sum a(n)q^n$ belongs to $M_k(N,\chi)$ and has the property that there exists an $\epsilon > 0$ for which*

$$\#\{n < X \mid a(n) \ne 0\} = O(X^{1-\epsilon}),$$

then $f(z)$ is superlacunary.

In this note we prove:

THEOREM 1. *If $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N,\chi)$ is not superlacunary, then*

$$\#\{n < X \mid a(n) \neq 0\} \gg_f X/\log X.$$

COROLLARY 1. *Gordon's ϵ-conjecture is true.*

It is well known that Lehmer speculated that Ramanujan's function $\tau(n)$ is non-zero for every positive integer n. Recall that $\tau(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

In view of Lehmer's conjecture and Serre's paper on the lacunarity of even powers of the eta-function, we record an elementary corollary that contains estimates on the number non-zero coefficients of all the powers of the eta-function. Although one can make better estimates in many cases, we have sacrificed this for a clear and comprehensive statement.

COROLLARY 2. *If r is a positive integer, then define $\tau_r(n)$ by*

$$\sum_{n=0}^{\infty} \tau_r(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^r.$$

If $r \neq 1$ or 3, then

$$\#\{n < X \mid \tau_r(n) \neq 0\} \gg_r \begin{cases} X & \text{if } r \neq 2,4,6,8,10,14,26 \text{ is even,} \\ X/\log X & \text{if } r \text{ odd or } r = 2,4,6,8,10,14,26. \end{cases}$$

PROOF OF RESULTS. If $f \in S_{k+\frac{1}{2}}(N,\chi)$ has the property that for every prime $p \nmid N$ there exists a complex number $\lambda(p)$ for which

$$T(p^2) \mid f = \lambda(p)f,$$

then we shall refer to f as an "eigenform". The author and C. Skinner [O-S] proved the following key lemma. For each positive integer r let $P(r)$ denote the set

$$P(r) := \{D \mid D > 1 \text{ square-free with exactly } r \text{ prime factors}\}.$$

LEMMA 1. *Let $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$ be an eigenform for which*
(i) $b(m) \neq 0$ for at least one square-free $m > 1$ coprime to N,
(ii) the coefficients $b(n)$ are algebraic integers contained in a number field K. Let v be a place of K over 2, and for each s let

$$B_s := \left\{m \mid m > 1 \text{ square-free, } (m,N) = 1, \text{ and } \operatorname{ord}_v(b(m)) = s\right\}.$$

Let s_0 be the smallest integer for which $B_{s_0} \neq \emptyset$. If $B_{s_0} \cap P(r) \neq \emptyset$, then

$$\#\{m \in B_{s_0} \cap P(r) \mid m \leq X\} \gg \frac{X}{\log X}(\log\log X)^{r-1}.$$

PROOF OF THEOREM 1. In view of Serre's work [S] it is well known that every integral weight modular form $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ has the property that

$$\#\{n < X \mid a(n) \neq 0\} \gg_f X/\log X.$$

Moreover amongst the half-integral weight forms it is well known that we can without loss of generality assume that $f(z)$ is a cusp form, and by the theorem of Serre and Stark we may assume that its weight $\geq 3/2$.

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$ be an eigenform. If $f(z)$ is not superlacunary, then the conclusion of the lemma shows that the number of $n < X$ for which $a(n) \neq 0$ is $\gg_f \frac{X}{\log X}$. It suffices to show that the hypotheses of the lemma are satisfied for a suitable non-trivial scalar multiple of $f(z)$. Since f is in the orthogonal complement of the elementary theta series, its Shimura lift is a weight $2k$ cuspidal eigenform. Hence by Waldspurger theory there exists an arithmetic progression with the property that for every square-free n coprime to N the number $a(n)^2$ is the "algebraic part" of the central critical value of the modular L-function of the Shimura lift of $f(z)$ twisted by a quadratic character. (See [Wal, Corollary 2]).

Verifying (i) now follows from a theorem of Friedberg and Hoffstein [F-H] that guarantees that infinitely many such values are non-zero. To show that $f(z)$ satisfies (ii) one may consult the theory of modular symbols [G-S, M-T-T], *i.e.*, the existence of uniform periods of modular L-functions of twists so that the "algebraic parts" of these twisted values are algebraic integers in some number field K. Therefore Theorem 1 holds for every non-superlacunary eigenform.

Now we consider the case where f is not an eigenform. This argument is similar to the integral weight argument employed in [S,M]. If $g = \sum b(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$, then define $M_g(X)$ by

$$M_g(X) := \#\{n < X \mid b(n) \neq 0\}.$$

It is easy to see that

$$M_{g_1+g_2}(X) \leq M_{g_1}(X) + M_{g_2}(X). \tag{1}$$

Suppose that $f(z) \in S_{k+\frac{1}{2}}(N,\chi)$ has the property that $M_f(X) = O(X^{1-\epsilon})$ for some $\epsilon > 0$. If $f = f_\theta + f_1$ where f_θ is superlacunary or trivial, and f_1 is orthogonal to the elementary theta series, then by (1) we find that $M_{f_1(X)} \leq M_f(X) + M_{f_\theta}(X)$. In particular there exists an $\epsilon_1 > 0$ for which $M_{f_1}(X) = O(X^{1-\epsilon_1})$.

Recall that if p is prime and $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$, then

(2)

$$g(z) \mid T_{p^2} := \sum_{n=1}^{\infty} \left(b(p^2n) + \chi(p) \left(\frac{(-1)^k n}{p} \right) p^{k-1} b(n) + \chi(p^2) p^{2k-1} b(n/p^2) \right) q^n.$$

By a quick examination of (2) one finds that

$$M_{T(p^2)|g}(X) \leq M_g(p^2X) + 2M_g(X). \tag{3}$$

Now let $\mathbb{T}$ be the Hecke algebra and let $\mathbb{X} := \mathbb{T}f_1(z)$. By (1) and (3) we see that for every $h(z) \in \mathbb{X}$ that $M_h(X) = O(X^{1-\epsilon_1})$. Since $\mathbb{T}$ is commutative, every simple $\mathbb{T}$ submodule of $\mathbb{X}$ is of the form $\mathbb{C}h(z)$, but on the other hand $h(z)$ is an eigenform. Therefore by the eigenform case we find that $M_h(X) \gg \frac{X}{\log X}$, and this is a contradiction. ■

PROOF OF COROLLARY 2. Serre [S2] proved that the only even r for which $\sum_{n=0}^{\infty} \tau_r(n)q^n$ is lacunary are $r = 2, 4, 6, 8, 10, 14, 26$. The result follows immediately from Theorem 1. ■

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DISCRIMINANTS AND ELECTROSTATICS OF GENERAL ORTHOGONAL POLYNOMIALS

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Presented by P. G. Rooney, FRSC

ABSTRACT. This is an announcement of our forthcoming results where we evaluate the discriminants of general orthogonal polynomials and provide an electrostatic model where the zeros of the orthogonal polynomials identify the equilibrium position of an N particle system.

RÉSUMÉ. Il s'agit d'annoncer un résultat qui paraîtra dans un futur proche et qui consiste à évaluer le discriminant d'un polynôme orthogonal générale et donner un modèle éléctrostatique identifiant les zéros du polynôme orthogonal avec la position d'équilibre d'un système à N particules.

1. Introduction. Stieltjes [11], [12] considered the electrostatic model of two fixed charges $\alpha + 1$ and $\beta + 1$ at $x = \pm 1$ and n movable unit charges at distinct points in $(-1, 1)$. The question is to determine the equilibrium position of the movable charges when the interaction forces obey a logarithmic potential. He proved that the equilibrium position is attained at the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. He stated the value of the discriminant of Jacobi polynomials. Hilbert [6] and much later Schur [10] gave proofs of the evaluation of the discriminant. For details see [13].

The purpose of this note is to announce the results obtained in our forthcoming papers [7], [8] concerning discriminants and electrostatic models of general orthogonal polynomials.

The discriminant of a polynomial f_n, with zeros $x_{1n}, x_{2n}, \dots, x_{nn}$, is defined by

$$(1.1)\quad D(f_n) = \gamma^{2n-2} \prod_{1\le j<k\le n} (x_{jn} - x_{kn})^2, \quad f_n(x) := \gamma x^n + \text{ lower order terms.}$$

Recently I. M. Gelfand and his school [5] used the concepts of discriminants and resultants in several variables to construct a theory of multivariate special functions.

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Let $\{p_n(x)\}$ be orthonormal with respect to a weight function $w(x) = e^{-v(x)}$ supported on an interval $[a, b]$, finite or infinite, that is

$$\int_a^b p_m(x)p_n(x)w(x)\,dx = \delta_{m,n}. \tag{1.2}$$

The initial values and three term recurrence relation of $\{p_n(x)\}$ will take the form

$$p_0(x) = 1, \quad p_1(x) = (x - b_0)/a_1, \tag{1.3}$$

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n > 0. \tag{1.4}$$

The annihilation operator for $\{p_n\}$ is given by [3], [2] and [1].

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \tag{1.5}$$

where $A_n(x)$ and $B_n(x)$ are defined by

$$\begin{aligned} A_n(x) = {}& \frac{a_nw(b^-)p_n^2(b^-)}{b-x} + \frac{a_nw(a^+)p_n^2(a^+)}{x-a} \\ &+ a_n\int_a^b \frac{v'(x)-v'(y)}{x-y}p_n^2(y)w(y)\,dy, \end{aligned} \tag{1.6}$$

$$\begin{aligned} B_n(x) = {}& \frac{a_nw(a^+)p_n(a^+)p_{n-1}(a^+)}{x-a} + \frac{a_nw(b^-)p_n(b^-)p_{n-1}(b^-)}{b-x} \\ &+ a_n\int_a^b \frac{v'(x)-v'(y)}{x-y}p_n(y)p_{n-1}(y)w(y)\,dy. \end{aligned} \tag{1.7}$$

In (1.6) and (1.7) it is assumed that

$$y^n\frac{v'(x)-v'(y)}{x-y}\,w(y), \quad n = 0, 1, \dots, \tag{1.8}$$

is integrable and the boundary terms in (1.6) and (1.7) exist.

2. Discriminants and functions of the second kind. Our first result is:

THEOREM 2.1 ([7]). *Let $\{p_n(x)\}$ be orthonormal with respect to $w(w) = \exp(-v(x))$ on $[a, b]$ and let it be generated by (1.3) and (1.4). Assume that*

$$x_{1n} > x_{2n} > \cdots > x_{nn},$$

are the zeros of $p_n(x)$. Then the discriminant of $p_n(x)$ is given by

$$D_n = \left\{\prod_{j=1}^n \frac{A_n(x_{jn})}{a_n}\right\}\left[\prod_{k=1}^n a_k^{2k-2n+2}\right]. \tag{2.1}$$

Observe that Theorem 2.1 only assumes the existence of a lowering operator of the form (1.5) with continuous $A_n(x)$ and $B_n(x)$.

If the terms in (1.8) are integrable then the orthonormal polynomials p_n's satisfy the differential equation [1], [2], [3]

$$p_n''(x) + R_n(x)p_n'(x) + S_n(x)p_n(x) = 0, \tag{2.2}$$

where

$$R_n(x) := -\left[v'(x) + \frac{A_n'(x)}{A_n(x)}\right], \tag{2.3}$$

$$\begin{aligned} S_n(x) := B_n'(x) - B_n(x)\frac{A_n'(x)}{A_n(x)} - B_n(x)\left[v'(x) + B_n(x)\right] \\ + \frac{a_n}{a_{n-1}}A_n(x)A_{n-1}(x). \end{aligned} \tag{2.4}$$

Given $\{p_n(x)\}$ the function of the second kind is

$$Q_n(z) = \frac{1}{w(z)}\int_{-\infty}^{\infty} \frac{p_n(y)}{z-y}w(y)\,dy, \quad n \geq 0, z \notin \operatorname{supp}\{w\}. \tag{2.5}$$

THEOREM 2.2 ([7]). *Let $\{p_n(x)\}$ are orthonormal with respect to $w(x) = e^{-v(x)}$ on $[a,b]$. Assume further that the functions in (1.8) are integrable for all n, $n \geq 0$. Then for $n \geq 0$ both p_n and Q_n have the same raising and lowering operators and satisfy (2.2). Furthermore, $p_n(x)$ and $Q_n(x)$ form a basis of solutions of the differential equation (2.2) for $n \geq 0$.*

3. Electrostatics. The conventional wisdom in potential theory is that a weight function $w(x) = \exp(-v(x))$ introduces an external field whose potential is $v(x)$. We propose that a weight function $w(x)$ creates two external fields. One is a long range field whose potential at a point x is $v(x)$, as conventional wisdom dictates. In addition w produces a short range field whose potential is $\ln(A_n(x)/a_n)$. Thus the total external potential $V(x)$ is the sum of the short and long range potentials, that is

$$V(x) = v(x) + \ln(A_n(x)/a_n). \tag{3.1}$$

THEOREM 3.1. *Let $v(x)$ and $V(x)$ be twice continuously differentiable functions whose second derivative is nonnegative on (a,b). Then the equilibrium position of n movable unit charges in (a,b) in the presence of the external potential $V(x)$ of (3.1) is unique and attained at the zeros of $p_n(x)$, provided that the particle interaction obeys a logarithmic potential.*

Observe that finding the equilibrium distribution of the charges in Theorem 3.1 is equivalent to finding the maximum of $T(\mathbf{x})$,

$$T(\mathbf{x}) := \left[\prod_{j=1}^{n} \frac{\exp(-v(x_j))}{A_n(x_j)/a_n}\right] \prod_{1\leq l<k\leq n} (x_l - x_j)^2, \tag{3.2}$$

where

$$\mathbf{x} := (x_1, x_2, \dots, x_n). \tag{3.3}$$

THEOREM 3.2 ([8]). *Let $T_{\max}$ and E_n be the maximum value of $T(\mathbf{x})$ and the equilibrium energy of the n particle system and let $x_{1n} > x_{2n} > \cdots > x_{nn}$ be the zeros of $p_n(x)$. Then*

$$T_{\max} = \exp\Big(-\sum_{j=1}^{n} v(x_{jn})\Big) \prod_{k=1}^{n} a_k^{2k}, \tag{3.4}$$

$$E_n = \sum_{j=1}^{n} v(x_{jn}) - 2\sum_{j=1}^{n} j \ln a_j, \tag{3.5}$$

Let $\{p_n(x)\}$ be a family of orthonormal polynomials generated by (1.3) and (1.4). The numerator polynomials $\{p_n^*(x)\}$ satisfy the recurrence relation (1.4) and are

$$p_0^*(x) = 0, \quad p_1^*(x) = 1/a_1. \tag{3.6}$$

The polynomials $\{p_n(x)\}$ and $\{p_n^*(x)\}$ form a basis of solutions of the second order difference equation (1.4). The polynomials $\{p_{n+1}^*(x)\}$ also form a set of orthogonal polynomials. The numerators $\{p_{n+1}^*(x)\}$ are multiples of associated polynomials with association parameter equal to 1. Wimp [14] showed that the associated Jacobi polynomials satisfy a linear fourth order differential equation. This motivated us to extend Wimp's result to general numerators of general orthogonal polynomials.

THEOREM 3.3. *Under the assumptions in Theorem 2.2 the numerator polynomials satisfy the differential equation*

$$\begin{vmatrix} \frac{w(z)}{A_0(z)} L_n\left(\frac{p_n^*(z)}{w(z)}\right) & \alpha_{11} & \alpha_{12} \\ \frac{d}{dz}\left[\frac{w(z)}{A_0(z)} L_n\left(\frac{p_n^*(z)}{w(z)}\right)\right] & \alpha_{21} & \alpha_{22} \\ \frac{d^2}{dz^2}\left[\frac{w(z)}{A_0(z)} L_n\left(\frac{p_n^*(z)}{w(z)}\right)\right] & \alpha_{31} & \alpha_{32} \end{vmatrix} = 0, \tag{3.7}$$

where L_n is the differential operator

$$L_n := \frac{d^2}{dz^2} + R_n(z)\frac{d}{dz} + S_n(z), \tag{3.8}$$

and the α's are

$$\alpha_{11} = 2, \quad \alpha_{12} = \frac{A_0'(z)}{A_0(z)} - \frac{A_n'(z)}{A_n(z)} = \frac{d}{dz}\ln\big(A_0(z)/A_n(z)\big), \tag{3.9}$$

(3.10)

$$\alpha_{21} = 2v'(z) + \frac{A_0'(z)}{A_0(z)} + \frac{A_n'(z)}{A_n(z)}, \quad \alpha_{22} = -2S_n(z) + \frac{d^2}{dz^2}\ln\big(A_0(z)/A_n(z)\big),$$

$$\alpha_{31} = -\alpha_{21}R_n(z) + \alpha_{22} + \alpha_{21}', \quad \alpha_{32} = \alpha_{22}' - \alpha_{21}S_n(z). \tag{3.11}$$

Finally we state a corollary which gives two additional linear independent solutions of the differential equation (3.7).

COROLLARY 3.4. *The functions* $w(z)p_n(z)$, $w(z)Q_n(z)$ *and* $p_n^*(z)$ *are linear independent solutions of the differential equation (3.7) provided that* w *can be extended to a continuous function in a horizontal strip* S *with the* x *axis in its interior and* $z^n w(z) \to 0$ *as* $x \to \infty$ *for all* $n \geq 0$ *and* z *in the strip* S.

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A NOTE ON LJUNGGREN'S THEOREM ABOUT THE DIOPHANTINE EQUATION $aX^2 - bY^4 = 1$

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Presented by David Boyd, FRSC

RÉSUMÉ. W. Ljunggren a montré que l'équation diophantienne $aX^2 - bY^4 = 1$ possède au plus une solution en entiers positifs, et que cette solution, si elle existe, provient d'une puissance d'une certaine unité d'un corps de nombres quadratique ou bi-quadratique. Le but de cet article est de déterminer exactement quelles puissances de cette unité conduisent à une solution.

ABSTRACT. W. Ljunggren showed that the Diophantine equation $aX^2 - bY^4 = 1$ has at most one solution in positive integers, and that a solution must come from a power of a certain unit in a quadratic or a biquadratic number field. The purpose of this paper is to determine exactly which powers of this unit can lead to a solution.

1. Introduction. In [7] Ljunggren proved some remarkable results on the solvability of Diophantine equations of the form $aX^2 - bY^4 = c$, for $c = 1, 2, 4$. In this paper we consider the case $c = 1$. For this case, Ljunggren's proof gives the following precise statement on the solvability of the equation of the title.

THEOREM 1. *Let a and b be coprime positive integers, with $a \neq 1$, such that not both a and b are perfect squares and such that the equation $aX^2 - bY^2 = 1$ is solvable in positive integers. Let (u, v) be the solution in positive integers of $aX^2 - bY^2 = 1$ with u minimal, and put $\eta = u\sqrt{a} + v\sqrt{b}$. Let $v = k^2 l$ with l odd and squarefree. The Diophantine equation*

$$aX^2 - bY^4 = 1 \tag{1}$$

has at most one solution in positive integers. If a solution (x, y) to (1) exists, then

$$x\sqrt{a} + y^2\sqrt{b} = \eta^l. \tag{2}$$

This result has recently been rediscovered independently by Chen and Voutier [3], although their proof is based on using the hypergeometric method for solving families of Thue equations, whereas Ljunggren's proof relies on properties

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of units in biquadratic fields. We gratefully acknowledge the authors of [3] for providing a preliminary version of their work. In [3] the authors ask which values of l in (2) lead to a solution of (1). In this paper we will give a partial answer to this question. In particular, we prove

THEOREM 2. *With notation as above,*

1. *For $l = 3$ and 5 there are infinitely many pairs a, b for which (1) is solvable.*
2. *If $l > 5$, then there are finitely many pairs a, b for which (1) is solvable. In particular, for $l = 7$ there are no pairs a, b for which (1) is solvable.*

It seems difficult to prove results for large values of l in Theorem 1. On the other hand, heuristics indicate that the following statement is true. We will provide some justification for this in the final section.

CONJECTURE 1. Let $a \neq 1$ and b be positive integers, not both perfect squares, such that the equation $aX^2 - bY^2 = 1$ is solvable in positive integers, and let $\eta = u\sqrt{a} + v\sqrt{b}$ be the minimal solution, with $v = k^2 l$, l odd and squarefree. If $l \geq 7$, then there are no solutions to $aX^2 - bY^4 = 1$.

For the related equation

$$aX^4 - bY^2 = 1, \tag{3}$$

the situation here is much different, as very little has been proved. In the case that a is a perfect square, (3) has been completely solved in [1]. In the case that a is a nonsquare integer, then aside from the fact that there are only finitely many integer solutions to (3), the only general result is by Le [6], who showed that a solution exists only if the minimal solution $\eta = u\sqrt{a} + v\sqrt{b}$ of $aX^2 - bY^2 = 1$ satisfies $u = x^2$ for some integer x. We note that Bumby [2] has proved that the only positive integer solutions to $3X^4 - 2Y^2 = 1$ are given by $(X, Y) = (1, 1)$ and $(3, 11)$. Heuristics similar to those for Conjecture 1 indicate that the following more general results holds.

CONJECTURE 2. *Let a and b be positive integers such that $aX^2 - bY^2 = 1$ is solvable, and let $\eta = u\sqrt{a} + v\sqrt{b}$ be the minimal solution. Then there are at most 2 solutions to (3). If (3) is solvable, then one solution comes from η, i.e., u is a perfect square, and a possible second solution comes from η^3.*

2. Solutions to $aX^2 - bY^2 = 1$.

Let a and b be positive integers as in the statement of Theorem 1. The Diophantine equation

$$aX^2 - bY^2 = 1 \tag{4}$$

has been well studied in the literature. A good reference for this is Walker [10]. In this section we state a theorem describing the integer solutions to (4).

PROPOSITION 1. *Assume that (4) is solvable in positive integers. Let $\eta = u\sqrt{a} + v\sqrt{b}$ denote the positive integer solution of (4) with u minimal. Then all positive integer solutions of (4) are given by*

$$u_{2k+1}\sqrt{a} + v_{2k+1}\sqrt{b} = \eta^{2k+1}$$

for $k \geq 0$.

EXAMPLE. Let $a = 3$ and $b = 2$, then

$$\eta = \sqrt{3} + \sqrt{2}$$

is the minimal solution to $3X^2 - 2Y^2 = 1$, and all solutions in positive integers to this equation are given by

$$\eta^{2k+1} = u_{2k+1}\sqrt{3} + v_{2k+1}\sqrt{2} \quad (k \geq 0).$$

In the notation of Theorem 1, $l = 1$, and so the only positive integer solution to $3X^2 - 2Y^4 = 1$ is given by $X = 1$ and $Y = 1$. Note that $\eta^3 = 9\sqrt{3} + 11\sqrt{2}$, so that both η and η^3 yield solutions to Equation (3).

3. Proof of Theorem 2. In the paper we will make reference to certain polynomials. In the definition of η in Theorem 1, let $M = au^2$ so that $\eta = \sqrt{M} + \sqrt{M-1}$. Define 2 sequences of polynomials $\{P_{2t+1}(M)\}$ and $\{Q_{2t+1}(M)\}$ by

$$\eta^{2t+1} = P_{2t+1}(M)\sqrt{M} + Q_{2t+1}(M)\sqrt{M-1},$$

for $t \geq 0$. It is evident that a solution to (1) is equivalent to an integer M for which $Q_l(M) = lz^2$ for some integer z, with $M - 1 = l^2 x$ for some integer x.

We remark that the polynomial $Q_{2t+1}(x)$ is of degree t with the distinct roots $\sin^2\left(\frac{(2i+1)\pi}{4t+2}\right)$ for $i = 0, \ldots, t-1$. In fact, we have more precisely that, for $t \geq 0$,

$$Q_{2t+1}(x) = 4^t \prod_{i=0}^{t-1}\left(x - \sin^2\left(\frac{(2i+1)\pi}{4t+2}\right)\right).$$

We now proceed with the proof of Theorem 2. For $l = 3$, $Q_3(M) = 4M - 1$, and so a solution to (1) is equivalent to values M for which $4M - 1 = 3z^2$, with $M - 1 = 9x$. Let $z > 1$ be an integer such that $z \equiv \pm 1, \pm 5 \pmod{12}$, so that $z^2 \equiv 1 \pmod{12}$, and choose z so that $3z^2 + 1$ and $\frac{z^2-1}{12}$ are not both perfect squares. Let $b = \frac{z^2-1}{12}$ and $a = 9b + 1$, then a and b are positive integers, $a > 1$, ab is not a square, $\eta = \sqrt{a} + 3\sqrt{b}$ is the minimal solution to

$$aX^2 - bY^2 = 1,$$

and

$$\eta^3 = (4a - 3)\sqrt{a} + (3z)^2\sqrt{b}.$$

There are clearly infinitely many choices for the integer z, resulting in infinitely many pairs a, b with solutions to (1) for $l = 3$.

Let $l = 5$, then by the comments above we want to solve $Q_5(a) = 16a^2 - 12a + 1 = 5y^2$ with $a - 1 = 25b$ for some positive integers a, b, and y, with ab not a square. Substituting $25b + 1$ for a and simplifying, we arrive at

$$(2y)^2 - 5(40b + 1)^2 = -1. \tag{5}$$

It is well known that the minimal solution of $X^2 - 5Y^2 = -1$ is $2 + \sqrt{5}$, and that all solutions in positive integers are given by $(2 + \sqrt{5})^{2k+1} = T_{2k+1} + U_{2k+1}\sqrt{5}$, with $k \geq 0$. It is easy to prove by induction that the sequence $\{U_{2k+1}\}$ is periodic modulo 40, and that $U_{2k+1} \equiv 1 \pmod{40}$ if and only if $2k + 1 \equiv 1 \pmod{20}$. Thus, for each integer $k > 1$ satisfying this congruence, we get a solution to (5). Any of these choices for k will lead to a solution of (1).

Let l be an odd integer with $l \geq 7$. In this case, $Q_l(x)$ is a polynomial of degree at least 3, with only simple roots. Therefore, by Siegel's theorem [9], the curve $ly^2 = Q_l(x)$ has finitely many integer points (x, y). This results in only finitely many pairs of integers (a, b) for which (1) solvable for this value of l.

We complete the proof of Theorem 2 by considering the particular case of $l = 7$. As noted above, a solution to (1) with $l = 7$ is given by an integer solution to $Q_7(M) = 7y^2$, with M of the form $M = 1 + 49n$, and $n > 0$. Since $Q_7(M) = 64M^3 - 80M^2 + 24M - 1$, substituting $1 + 49n$ for M, and then $x = 28n$, we find that x and y satisfy

$$y^2 = 49x^3 + 49x^2 + 14x + 1. \tag{6}$$

By the transformation $Y = 49y$, $X = 49x + 16$, we obtain the minimal Weierstrauss model

$$Y^2 = X^3 + X^2 - 114X - 127,$$

which has been well studied. From the tables in [4], we find that the group of rational points on this curve is of rank 0, and with torsion isomorphic to Z_3. Since the curve as given in (6) has the obvious points $(x, y) = (0, \pm 1)$, it follows that these are all of the finite rational points satisfying (6), forcing $n = 0$. Thus, (1) is not solvable if $l = 7$. ∎

EXAMPLE. Consider the case $l = 3$. From the proof of Theorem 2, the smallest choice for z is $z = 5$, which results in $b = 2$, $a = 19$, and $\eta = \sqrt{19} + 3\sqrt{2}$. In this case, $\eta^3 = 73\sqrt{19} + 225\sqrt{2}$, and so $X = 73$, $Y = 15$ is a solution to $19X^2 - 2Y^4 = 1$.

EXAMPLE. Consider the case $l = 5$. From the proof of Theorem 2, the smallest choice for k is $k = 21$. We therefore compute

$$(2 + \sqrt{5})^{21} = 7331474697802 + 3278735159921\sqrt{5} = T_{21} + U_{21}\sqrt{5}.$$

Letting $b = (U_{21} - 1)/40 = 81968378998$ and $a = 1 + 25b = 2049209474951$, we find that the minimal solution to $aX^2 - bY^2 = 1$ is

$$\tau = \sqrt{2049209474951} + 5\sqrt{81968378998},$$

and that

$$\tau^5 = 671881515556222653537394 01\sqrt{2049209474951} + (18328686744505)^2\sqrt{81968378998}.$$

Therefore, the equation $2049209474951X^2 - 81968378998Y^4 = 1$ is solvable, and comes from the fifth power of the minimal solution of $2049209474951X^2 - 81968378998Y^2 = 1$.

4. **A heuristic for large values of l.** Based on a refined version of a consequence of the ABC conjecture recently proved by Langevin, we will show heuristical evidence for Conjecture 1. In [8] Masser made the following

THE ABC CONJECTURE. Let $\epsilon > 0$. Then there is $K > 0$ depending on ϵ such that for all triples (a, b, c) of positive integers with $c = a + b$ and $\gcd(a, b, c) = 1$,

$$c < K\,\mathrm{rad}(abc)^{1+\epsilon},$$

where $\mathrm{rad}(abc)$ is the product of the distinct primes dividing abc.

There have been many results proved concerning consequences of the ABC conjecture. Recently, Langevin [5] has proved the following very interesting and powerful consequence of the ABC conjecture. In this theorem, $\mathrm{rad}(n)$ denotes the largest squarefree factor of the positive integer n.

PROPOSITION 2. *Let $P(x)$ denote a polynomial of degree $d > 1$, with integer coefficients, and with no multiple roots. The ABC conjecture implies that for all $\epsilon > 0$,*

$$\mathrm{rad}\big(P(n)\big) > n^{d-1-\epsilon},$$

for all sufficiently large positive integers n.

This result is not quite sharp enough to deduce results on the solvability of (1) for larger values of l. In order to do this we require an explicit form of Proposition 1 for a specific set of polynomials.

Define a sequence of polynomials $\{q_{2t+1}(x)\}$ for $t \geq 0$ by

$$q_{2t+1}(x) = Q_{2t+1}(x/4).$$

It is easy to prove that these polynomials are squarefree, of degree t, and have integer coefficients. Thus, for $t > 2$, these polynomials satisfy the hypothesis of Langevin's theorem. For the purpose of providing a heuristic for Conjecture 1, we assume the following explicit form of Langevin's theorem for the sequence of polynomials $\{q_{2t+1}(x)\}$.

HYPOTHESIS A. Let $t \geq 2$. If n is an integer satisfying $n \geq 4(2t+1)^2$, then $\mathrm{rad}(q_{2t+1}(n)) > n^{t-\frac{3}{2}}$.

The lower bound for n in this hypothesis is somewhat arbitrary, but sufficient for our purpose. The truth on this matter is likely that the conclusion of Hypothesis A holds for all n in a range which is somewhat larger than stated above.

HYPOTHESIS A IMPLIES CONJECTURE 1. We need to show, for $l \geq 7$, that there is no solution in integers x, z to the equation $q_l(4x) = lz^2$, with x of the form $x = 1 + l^2 n$, for n a positive integer. The case $l = 7$ has been proved unconditionally, so we assume that $l \geq 11$. Note that if $q_l(4x) = lz^2$, then $z > l$. It is easy to prove that for $x > 1$, $x^t > q_{2t+1}(x)$, and so because $4x > 4l^2$, we can apply Hypothesis A to get

$$lz \geq \mathrm{rad}(lz^2) = \mathrm{rad}(q_l(4x)) > q_l(4x)^{\frac{l-4}{l-1}} = (lz^2)^{\frac{l-4}{l-1}} > l^{\frac{2l-11}{l-1}} z,$$

forcing $l < 10$. ∎

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QUANTUM IMAGINARY VERMA MODULES FOR AFFINE LIE ALGEBRAS

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Presented by Vlastimil Dlab, FRSC

ABSTRACT. Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra and $M(\lambda)$ an imaginary Verma module for $\mathfrak{g}$ with S-highest weight $\lambda \in P$. We construct quantum imaginary Verma modules $M^q(\lambda)$ over the quantum group $U_q(\mathfrak{g})$, investigate their properties and show that $M^q(\lambda)$ is a true quantum deformation of $M(\lambda)$ in the sense that the weight structure is preserved under the deformation.

RÉSUMÉ. Soit $\mathfrak{g}$ une algèbre Kac-Moody affine non-tordue et $M(\lambda)$ un module Verma imaginaire pour $\mathfrak{g}$ avec un poids S-plus haut $\lambda \in P$. Nous construisons des modules Verma imaginaires quantiques $M^q(\lambda)$ au-dessus du groupe quantique $U_q(\mathfrak{g})$, examinons leurs propriétés et montrons que $M^q(\lambda)$ est une vraie déformation quantique de $M(\lambda)$ dans le sens que la structure de poids est préservée sous la déformation.

1. Let $\dot{\mathfrak{g}}$ be a finite-dimensional simple complex Lie algebra with root system $\dot{\Delta}$. Denote by $\dot{\Delta}_+$ and $\dot{\Delta}_-$ the positive and negative roots of $\mathfrak{g}$. Let $\mathfrak{g}$ be the untwisted affine Kac-Moody algebra associated to $\dot{\mathfrak{g}}$, with Cartan subalgebra $\mathfrak{h}$. The root system Δ of $\mathfrak{g}$ is given by

$$\Delta = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}, k \neq 0\},$$

where δ is the indivisible imaginary root. Let I be the indexing set for the simple roots.

Consider the partition $\Delta = S \cup -S$ of the root system of $\mathfrak{g}$ where $S = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\}$. This is a non-standard partition of the root system Δ in the sense that S is not Weyl equivalent to the set Δ_+ of positive roots.

The algebra $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{g}_{-S} \oplus \mathfrak{h} \oplus \mathfrak{g}_S$, where $\mathfrak{g}_S = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$. Let $U(\mathfrak{g}_S)$ (resp. $U(\mathfrak{g}_{-S})$) denote the universal enveloping algebra of $\mathfrak{g}_S$ (resp. $\mathfrak{g}_{-S}$).

Let $\lambda \in P$, the weight lattice of $\mathfrak{g}$. A weight (with respect to $\mathfrak{h}$) $U(\mathfrak{g})$-module V is called an S-highest weight module with highest weight λ if there is some nonzero vector $v \in V$ such that

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(i) $u^+ \cdot v = 0$ for all $u^+ \in U(\mathfrak{g}_S)$;

(ii) $V = U(\mathfrak{g}) \cdot v$.

Let $\lambda \in P$. We make $\mathbb{C}$ into a 1-dimensional $U(\mathfrak{g}_S \oplus \mathfrak{h})$-module by picking a generating vector v and setting $(x + h) \cdot v = \lambda(h)v$, for all $x \in \mathfrak{g}_S$, $h \in \mathfrak{h}$. The induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_S \oplus \mathfrak{h})} \mathbb{C}v = U(\mathfrak{g}_{-S}) \otimes \mathbb{C}v$$

is called the *imaginary Verma module* with S-highest weight λ. Imaginary Verma modules are in many ways similar to ordinary Verma modules except they contain both finite and infinite-dimensional weight spaces. They were studied in [Fu], from which we summarize (*cf.* [Fu, Proposition 1, Theorem 1]).

PROPOSITION 1. *Let $\lambda \in P$, and let $M(\lambda)$ be the imaginary Verma module of S-highest weight λ. Then $M(\lambda)$ has the following properties.*

(i) The module $M(\lambda)$ is a free $U(\mathfrak{g}_{-S})$-module of rank 1 generated by the S-highest weight vector $1 \otimes 1$ of weight λ.

(ii) $M(\lambda)$ has a unique maximal submodule.

(iii) Let V be a $U(\mathfrak{g})$-module generated by some S-highest weight vector v of weight λ. Then there exists a unique surjective homomorphism $\phi: M(\lambda) \mapsto V$ such that $\phi(1 \otimes 1) = v$.

(iv) $\dim M(\lambda)_\lambda = 1$. *For any $\mu = \lambda - k\delta$, k a positive integer,* $0 < \dim M(\lambda)_\mu < \infty$. *If $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and* $\dim M(\lambda)_\mu \neq 0$, *then* $\dim M(\lambda)_\mu = \infty$.

(v) Let $\lambda, \mu \in P$. Any non-zero element of $\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M(\mu))$ *is injective.*

(vi) The module $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.

2. The quantized universal enveloping algebra of $\mathfrak{g}$, $U_q(\mathfrak{g})$, is an associative algebra with 1 over $\mathbb{C}(q)$ with generators E_i, F_i, $K_i^{\pm 1}$ $(i \in I)$ and $D^{\pm 1}$. Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by E_i (resp. F_i), $i \in I$, and let $U_q^0(\mathfrak{g})$ denote the subalgebra generated by $K_i^{\pm}$ $(i \in I)$ and $D^{\pm}$.

Beck [Be1, Be2] has given a total ordering of the root system Δ and a PBW like basis for $U_q(\mathfrak{g})$. Here we follow the construction in [BK] and let E_β denote the root vectors for each $\beta \in \Delta$, counting with multiplicity for the imaginary roots.

Let $U_q(\pm S)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta \mid \beta \in \pm S\}$, and let B_q denote the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta \mid \beta \in S\} \cup U_q^0(\mathfrak{g})$.

Let $\lambda \in P$. A $U_q(\mathfrak{g})$ weight module V^q is called an S-highest weight module with highest weight λ if there is a non-zero vector $v \in V^q$ such that:

(i) $u^+ \cdot v = 0$ for all $u^+ \in U_q(S) \setminus \mathbb{C}(q)^*$;

(ii) $V^q = U_q(\mathfrak{g}) \cdot v$.

Note that, in the absence of a general quantum PBW theorem for non-standard partitions, we cannot immediately claim that an S-highest weight module V^q is generated by $U_q(-S)$. This is in contrast to the classical case.

Let $\mathbb{C}(q) \cdot v$ be a 1-dimensional vector space. Let $\lambda \in P$, and set $E_\beta \cdot v = 0$ for all $\beta \in S$, $K_i^{\pm 1} \cdot v = q^{\pm\lambda(h_i)} v$ $(i \in I)$ and $D^{\pm 1} \cdot v = q^{\pm\lambda(d)} v$. Define $M^q(\lambda) = U_q(\mathfrak{g}) \otimes_{B_q} \mathbb{C}(q)v$. Then $M^q(\lambda)$ is an S-highest weight U_q-module called the *quantum imaginary Verma module* with highest weight λ.

Although we cannot say in general that an S-highest weight module is generated by $U_q(-S)$, in the case of imaginary Verma modules, we can make use of Beck's explicit ordered basis and the grading given in [BK, Proposition 1.8]to obtain the following result.

THEOREM 2. *As a vector space over $\mathbb{C}(q)$, $M^q(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta} \dots E_{-k\delta} \dots E_{-\alpha+k\delta}$, $\alpha \in \dot{\Delta}_+$, $n \geq 0$, $k > 0$.*

Let G be the subalgebra of $U_q(\mathfrak{g})$ generated by the imaginary root vectors. Let $M^q(\lambda)$ be the quantum imaginary Verma module over $U_q(\mathfrak{g})$ with S-highest weight $\lambda \in P$ and generating vector v. Consider the G-submodule of $M^q(\lambda)$ generated by v, $H^q(\lambda) = G \cdot v$. The G-module $H^q(\lambda)$ is irreducible iff $\lambda(c) \neq 0$. Denote by $H_0^q(\lambda)$ the unique maximal submodule of $H^q(\lambda)$. Denote $M_0^q(\lambda) = U_q(\mathfrak{g}) H_0^q(\lambda)$ and set $\widetilde{M^q(\lambda)} = M^q(\lambda)/M_0^q(\lambda)$.

THEOREM 3. *For any $\lambda \in P$,*

(i) $M^q(\lambda)$ is irreducible iff $\lambda(c) \neq 0$;

(ii) $\widetilde{M^q(\lambda)}$ is irreducible iff $\lambda(h_i) \neq 0$, $i \in I$;

(iii) let $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$ for all $i \in I$. Then $M^q(\lambda)$ has an infinite filtration with irreducible quotients $\widetilde{M^q(\lambda + k\delta)}$, $k \geq 0$;

(iv) let $\lambda(c) = 0$, $\lambda(h_i) \neq 0$, $i \in I$ and N be a submodule of $M^q(\lambda)$. Then N is generated by $N \cap H^q(\lambda)$.

3. We have constructed quantum imaginary Verma modules and determined some of their properties. Now we show that these quantum imaginary Verma modules are quantum deformations of imaginary Verma modules defined over the affine algebra. That is, the weight-space structure of a given module $M^q(\lambda)$ is the same as that of its classical counterpart $M(\lambda)$ for any $\lambda \in P$. To do this, we construct an intermediate module, called an $\mathbb{A}$-form.

Following [Lu], for each $i \in I$, $s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we define the *Lusztig elements*

$$\begin{bmatrix} K_i\,;\,s \\ n \end{bmatrix} = \prod_{r=1}^{n} \frac{K_i q_i^{s-r+1} - K_i^{-1} q_i^{-(s-r+1)}}{q_i^r - q_i^{-r}}$$

and

$$\begin{bmatrix} D\,;\,s \\ n \end{bmatrix} = \prod_{r=1}^{n} \frac{D q^{s-r+1} - D^{-1} q^{-(s-r+1)}}{q^r - q^{-r}}$$

in $U_q(\mathfrak{g})$. Let $\mathbb{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]_{q_i}}, i \in I, n > 0]$. Define the $\mathbb{A}$-form, $U_{\mathbb{A}}(\mathfrak{g})$, of $U_q(\mathfrak{g})$ to be the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements E_i, F_i, $K_i^{\pm 1}$,

$\begin{bmatrix} K_i\,;\,s \\ n \end{bmatrix}$, $i \in I$, $D^{\pm 1}$, $\begin{bmatrix} D\,;\,s \\ n \end{bmatrix}$. Let $U_{\mathbb{A}}^+$ (resp. $U_{\mathbb{A}}^-$) denote the subalgebra of $U_{\mathbb{A}}$ generated by the E_i, (resp. F_i), $i \in I$, and let $U_{\mathbb{A}}^0$ denote the subalgebra of $U_{\mathbb{A}}$ generated by the elements $K_i^{\pm}$, $\begin{bmatrix} K_i\,;\,s \\ n \end{bmatrix}$, $D^{\pm 1}$, $\begin{bmatrix} D\,;\,s \\ n \end{bmatrix}$.

The algebra $U_{\mathbb{A}}$ inherits the standard triangular decomposition of $U_q(\mathfrak{g})$. In particular, any element u of $U_{\mathbb{A}}$ can be written as a sum of monomials of the form $u^-u^0u^+$ where $u^{\pm} \in U_{\mathbb{A}}^{\pm}$ and $u^0 \in U_{\mathbb{A}}^0$. In fact, we can say rather more. From the construction of Beck's basis, it follows that all the root vectors E_β, $\beta \in \Delta$ are in $U_{\mathbb{A}}$, too.

Let $\lambda \in P$, and let $M^q(\lambda)$ be the imaginary Verma module over $U_q(\mathfrak{g})$ with S-highest weight λ and highest weight vector v_λ. The $\mathbb{A}$-form of $M^q(\lambda)$, $M^{\mathbb{A}}(\lambda)$, is defined to be the $U_{\mathbb{A}}$ submodule of $M^q(\lambda)$ generated by v_λ. That is, we set $M^{\mathbb{A}}(\lambda) = U_{\mathbb{A}} \cdot v_\lambda$.

PROPOSITION 4. *As an $\mathbb{A}$-vector space, $M^{\mathbb{A}}(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta}\cdots E_{-k\delta}\cdots E_{-\alpha+k\delta}$, $\alpha \in \dot{\Delta}_+$, $n \geq 0$, $k > 0$.*

PROOF. This essentially follows from Theorem 2 and the fact that all the root vectors are in $U_{\mathbb{A}}$. One must also check the action of the Lusztig elements. ∎

Define a weight structure on $M^{\mathbb{A}}(\lambda)$ by setting $M^{\mathbb{A}}(\lambda)_\mu = M^{\mathbb{A}}(\lambda) \cap M^q(\lambda)_\mu$ for each $\mu \in P$. Then $M^{\mathbb{A}}(\lambda)$ is a weight module with the weight decomposition $M^{\mathbb{A}}(\lambda) = \bigoplus_{\mu \in P} M^{\mathbb{A}}(\lambda)_\mu$, and, for each $\mu \in P$, $M^{\mathbb{A}}(\lambda)_\mu$ is a free $\mathbb{A}$-module such that $\operatorname{rank}_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu = \dim_{\mathbb{C}(q)} M^q(\lambda)_\mu$.

4. Next, we take the classical limits of the $\mathbb{A}$-forms of the quantum imaginary Verma modules, and show that they are isomorphic to the imaginary Verma modules of $U(\mathfrak{g})$.

Let $\mathbb{J}$ be the ideal of $\mathbb{A}$ generated by $q - 1$. Then there is an isomorphism of fields $\mathbb{A}/\mathbb{J} \cong \mathbb{C}$ given by $f + \mathbb{J} \mapsto f(1)$ for any $f \in \mathbb{A}$. Set $U' = (\mathbb{A}/\mathbb{J}) \otimes_{\mathbb{A}} U_{\mathbb{A}}$. Then $U' \cong U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}}$. Denote by u' the image in U' of an element $u \in U_{\mathbb{A}}$. Then $(D')^2 = 1$ and $(K_i')^2 = 1$ for all $i \in I$ [DK]. If K' denotes the ideal of U' generated by $D' - 1$ and $\{K_i' - 1 \mid i \in I\}$, then $\overline{U} = U'/K' \cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

For $\lambda \in P$, let $M'(\lambda) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)$. Then $M'(\lambda) \cong M^{\mathbb{A}}(\lambda)/\mathbb{J}M^{\mathbb{A}}(\lambda)$ and $M'(\lambda)$ is a U'-module. For $\mu \in P$, let $M'(\lambda)_\mu = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu$. Since $M^{\mathbb{A}}(\lambda) = \bigoplus_{\mu \in P} M^{\mathbb{A}}(\lambda)_\mu$, we must have $M'(\lambda) = \bigoplus_{\mu \in P} M'(\lambda)_\mu$. For $\mu \in P$, $\dim_{\mathbb{A}/\mathbb{J}} M'(\lambda)_\mu = \operatorname{rank}_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu$.

PROPOSITION 5. *The elements D' and K_i' $(i \in I)$ in U' act as the identity on the U' module $M'(\lambda) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)$.*

Since $M'(\lambda)$ is a U'-module, $\overline{M}(\lambda) = M'(\lambda)/K'M'(\lambda)$ is a $\overline{U} = U'/K'$-module. But K' was the ideal generated by $D' - 1$ and the $K_i' - 1$, and D' and each K_i' acts as the identity on $M'(\lambda)$, so $\overline{M}(\lambda) = M'(\lambda)$. Since $\overline{U} \cong U(\mathfrak{g})$, this means $\overline{M}(\lambda)$ has a $U(\mathfrak{g})$-structure. The module $\overline{M}(\lambda)$ is called the classical limit of $M^{\mathbb{A}}(\lambda)$. For $v \in M^{\mathbb{A}}(\lambda)$, let $\overline{v}$ denote the image of v in $\overline{M}(\lambda)$.

PROPOSITION 6. *Let v_λ be the generating vector for $M^{\mathbb{A}}(\lambda)$. Then as a $U(\mathfrak{g})$-module, $\overline{M}(\lambda)$ is an S-highest weight $U(\mathfrak{g}_{-S})$-module generated by $\overline{v_\lambda}$.*

PROOF. The crucial part of the proof is observing that the images in $\overline{U}$ of the ordered monomials in the root vectors E_β, $\beta \in -S$, form a basis for $U(\mathfrak{g}_{-S})$. ∎

Assembling these ingredients, we obtain the following result.

THEOREM 7. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Let $\lambda \in P$. Then the imaginary Verma module $M(\lambda)$ admits a quantum deformation to the quantum imaginary Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ in such a way that the weight space decomposition is preserved.*

Details of the proofs and additional results will be given in a later paper.

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WHENCE DOES AN ELLIPSE LOOK LIKE A CIRCLE?

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ABSTRACT. In Euclidean 3-space, a system of confocal quadrics contains two degenerate members: a *focal ellipse* and a *focal hyperbola*, lying in perpendicular planes in such a way that the foci of each coincide with the vertices of the other. George Salmon discovered that, when viewed from any point on the focal hyperbola, the focal ellipse looks like a circle! It seems worth while, quite apart from the consideration of quadrics, to give an elementary solution to the problem of finding the locus of viewpoints from which a given ellipse looks like a circle.

RÉSUMÉ. Dans l'espace euclidien de trois dimensions un système de quadriques confocales contient deux éléments dégénérés: une *ellipse focale* et un *hyperbole focale*, appartenant a des plans perpendiculaires de telle sorte que les foyers de chacune de ces courbes coincident avec les sommets de l'autre. George Salmon a trouvé que, lorsqu'elle est vue à partir de n'importe quel point sur l'hyperbole focale, l'ellipse focale ressemble à un cercle! Il semble donc intéressant, pour des raisons autres que purement rattachées aux quadriques, de donner une solution élémentaire au problème de déterminer la courbe tracée par les points de vue desquels une ellipse spécifiée resemble à un cercle.

1. The focal conics.

THEOREM 1. *The locus of apices of right circular cones containing the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

is the hyperbola

$$\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2} = 1, \quad y = 0.$$

We assume two lemmas which follow easily from Salmon's discussion of quadrics of revolution [3, p. 82], [4, pp. 113, 187]:

LEMMA 1. *If the cone*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

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is right-circular and one of f, g, h is zero, then another one also must be zero.

LEMMA 2. *The cone*

$$ax^2 + by^2 + cz^2 + 2gzx = 0$$

is right-circular if and only if

$$g^2 = (a-b)(c-b).$$

PROOF OF THEOREM 1. A fixed point (x_1, y_1, z_1) is joined to a variable point $(x_0, y_0, 0)$ on the given ellipse by a cone which may be described as the locus of points (x, y, z) such that

$$x_0 = \frac{\lambda x + \mu x_1}{\lambda + \mu}, \quad y_0 = \frac{\lambda y + \mu y_1}{\lambda + \mu}, \quad 0 = \frac{\lambda z + \mu z_1}{\lambda + \mu}$$

for various values of λ/μ. The third relation reveals that $\lambda/\mu = -z_1/z$, so that we have, more precisely,

$$x_0 = \frac{z_1 x - z x_1}{z_1 - z}, \quad y_0 = \frac{z_1 y - z y_1}{z_1 - z}$$

and the equation that puts $(x_0, y_0, 0)$ on the ellipse yields, for the cone,

$$\frac{(z_1 x - x_1 z)^2}{a^2} + \frac{(z_1 y - y_1 z)^2}{b^2} = (z_1 - z)^2.$$

When the origin is shifted to (x_1, y_1, z_1), the term $(z_1 - z)^2$ on the right becomes simply z^2 so that

$$\frac{z_1^2}{a^2}x^2 + \frac{z_1^2}{b^2}y^2 + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)z^2 - \frac{2y_1 z_1}{b^2}\, yz - \frac{2z_1 x_1}{a^2}\, zx = 0.$$

In this case, the a, b and c of Lemma 1 are

$$\frac{z_1^2}{a^2}, \quad \frac{z_1^2}{b^2} \quad \text{and} \quad \frac{x_1^2}{a^2} + \frac{y^2}{b^2} - 1$$

while $f = -y_1 z_1/b^2$, $g = -z_1 x_1/a^2$, and $h = 0$. By that Lemma, either x_1 or y_1 must be zero. It is geometrically obvious that the viewpoint cannot lie in the plane $x = 0$, so $x_1 \neq 0$ and

$$y_1 = 0.$$

Omitting the terms involving y_1, we are left with

$$\frac{z_1^2}{a^2}x^2 + \frac{z_1^2}{b^2}y^2 + \left(\frac{x_1^2}{a^2} - 1\right)z^2 - \frac{2z_1 x_1}{a^2}\, zx = 0$$

so that again, $g = -z_1x_1/a^2$. By Lemma 2,

$$\begin{aligned} 0 &= \left(\frac{z_1x_1}{a^2}\right)^2 - \left(\frac{z_1^2}{a^2} - \frac{z_1^2}{b^2}\right)\left(\frac{x_1^2}{a^2} - 1 - \frac{z_1^2}{b^2}\right) \\ &= z_1^2\left(\frac{1}{a^2} - \frac{1}{b^2} + \frac{x_1^2}{a^2b^2} + \frac{z_1^2}{a^2b^2} - \frac{z_1^2}{b^4}\right) \\ &= \frac{(a^2-b^2)z_1^2}{a^2b^2}\left(\frac{x_1^2}{a^2-b^2} - \frac{z_1^2}{b^2} - 1\right). \end{aligned}$$

Replacing x_1, y_1, z_1 by x, y, z, we see that the desired locus of viewpoints is the hyperbola

$$\frac{x^2}{c^2} - \frac{z^2}{b^2} = 1, \quad y = 0 \quad (c^2 = a^2 - b^2),$$

whose foci $(\pm a, 0, 0)$ and vertices $(\pm c, 0, 0)$ are the vertices and foci of the given ellipse.

REMARK. It is just as easy to show that the locus of centres of right circular cones containing the above hyperbola is the ellipse

$$\frac{x^2}{b^2+c^2} + \frac{y^2}{b^2} = 1, \quad z = 0.$$

2. **Prolate Spheroids.** A somewhat related theorem, discovered by Nils Abramson [1], [2], [5] and proved by Björn Bonnevier, concerns the prolate spheroid constructed by rotating an ellipse about its major axis. Naturally, the spheroid

$$\frac{z^2}{a^2} + \frac{x^2+y^2}{b^2} = 1 \quad (a > b)$$

is said to have two *foci*: $(0, 0, \pm c)$ $(c^2 = a^2 - b^2)$. It will be shown that any plane section of the prolate spheroid looks like a circle when viewed from either focus. This requires another lemma [3, p. 82], [4, p. 113]:

LEMMA 3. *The cone of Lemma 1 is right circular if $fgh \neq 0$ and*

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}.$$

THEOREM 2. *The cone joining any plane section of a prolate spheroid to either focus is right circular.*

PROOF. When the origin is shifted to a focus, the equation becomes

$$\frac{(z-c)^2}{a^2} + \frac{x^2+y^2}{b^2} = 1 \quad (c^2 = a^2 + b^2).$$

The section of this spheroid by the plane

$$Xx + Yy + Zz = 1$$

is joined to the origin by the cone

$$\frac{\{z - c(Xx + Yy + Zz)\}^2}{a^2} + \frac{x^2 + y^2}{b^2} = (Xx + Yy + Zz)^2$$

or

$$\begin{aligned} 0 &= a^2(x^2 + y^2) + b^2\{z^2 - 2cz(Xx + Yy) - 2cZz^2 - b^2(Xx + Yy + Zz)^2\} \\ &= (a^2 - b^4X^2)x^2 + (a^2 - b^4Y^2)y^2 + b^2(1 - 2cZ - b^2Z^2)z^2 \\ &\quad - 2b^2(c + b^2Z)(Yyz + Xzx) - 2b^4XYxy. \end{aligned}$$

The a, b, c of Lemma 3 are now

$$a^2 - b^4X^2, \quad a^2 - b^4Y^2, \quad b^2(1 - 2cZ - b^2Z^2),$$

while

$$f = -b^2(c + b^2Z)Y, \quad g = -b^2(c + b^2Z)X \quad \text{and} \quad h = -b^4XY.$$

Thus the criterion for the cone to be right circular becomes

$$a^2 - b^4X^2 + b^4X^2 = a^2 - b^4Y^2 + b^4Y^2 = b^2(1 - 2cZ - b^2Z^2) + (c + b^2Z)^2.$$

This is clearly true, since $a^2 = b^2 + c^2$.

REMARK. It is just as easy to prove that the cone joining any plane section of the *oblate* spheroid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

to any point on the "focal circle"

$$x^2 + y^2 = c^2, \quad z = 0$$

is right circular.

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