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is dedicated to

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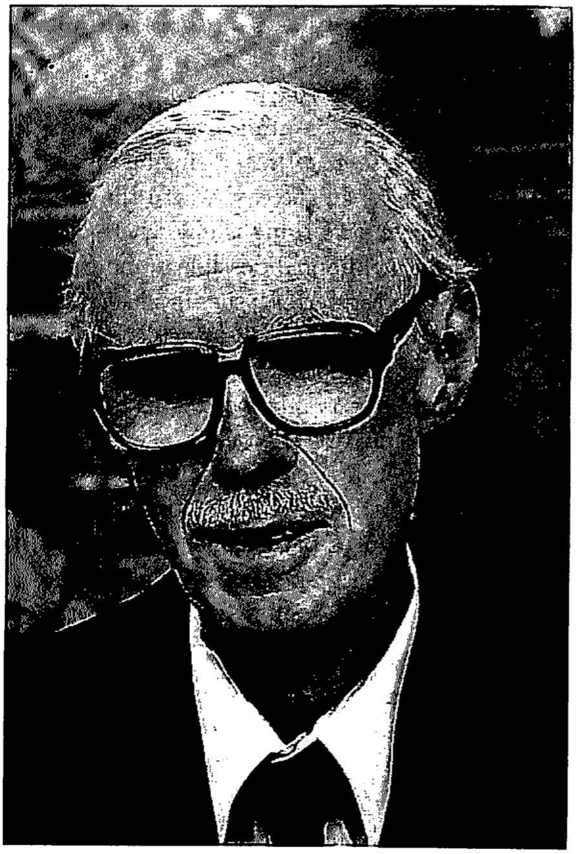
Professor Emeritus of Mathematics, University of Toronto



with grateful thanks for his vital contributions as
a member of the Editorial Board, as well as
the Production and Managing Editor of this journal for the past 10 years

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SPACE FILLING ZONOTOPES AND VORONOI'S CONJECTURE ON PARALLELOHEDRA

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In the last years of his life Georges Voronoi wrote two brilliant geometric memoirs which he titled *Nouvelles applications des paramètres continus à la théorie des formes quadratiques* [13], [14]. In the second of these, his *Deuxième Mémoire sur les paralléloèdres primitifs* [14], he considered polytopes with the property that a facet-to-facet tiling can be formed from a selection of translates. He referred to polytopes with this tiling property as *parallelohedra* to emphasize that each tile is a parallel translate of the original. Thus, a parallelohedron is a polytope that admits a facet-to-facet tiling of Euclidean space $\mathbb{E}^d$ by translates. Voronoi asked the fundamental question: *Is an arbitrary parallelohedron affinely equivalent to the Dirichlet domain for some lattice?* Using an ingenious construction he was able to give a positive answer in the case where the tiling is *primitive* ([14], *Première partie,* p. 273). A primitive tiling is one where the star at each vertex contains precisely $d+1$ tiles, the minimal possible number (the star is the collection of tiles that are attached to the vertex); the tiles in such tilings are also called primitive. Voronoi's question is now typically formulated as a conjecture, and in its most general form remains unresolved. The term Dirichlet domain which he used is now frequently replaced by *Voronoi polytope* as a tribute to this last paper. The Voronoi polytope at a point λ of a lattice is the set of points which are at least as close to λ as to any other lattice point. Any two Voronoi polytopes are related by a lattice translate, and the entire collection forms a facet-to-facet tiling. This metrical construction produces parallelohedra and, if Voronoi's conjecture is true, generates all possible species up to affine equivalence.

A more lengthy exposition of the contents of this paper, with additional references, can be found in [6].

1. Space Filling Zonotopes. A *zonotope* is a polytope in $\mathbb{E}^d$ which is either a parallelepiped, or the image under an orthogonal projection of a parallelepiped in a higher dimensional space. If centered at the origin a zonotope can be represented as the convex hull of the 2^r points

$$\frac{1}{2}(\theta_1 \mathbf{z}_1 + \theta_2 \mathbf{z}_2 + \cdots + \theta_r \mathbf{z}_r),$$

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where $r \geq d$, $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_r]$ is a (± 1)-vector, and where the vectors $\mathbf{z}_i \in \mathbb{E}^d$ are the *zone vectors.* If $r = d$ these points are the vertices of a parallelepiped with edge vectors $\mathbf{z}_i$, $i = 1, \ldots, d$. If $r > d$, the vectors $\mathbf{z}_i$ should be interpreted as the images of the edge vectors of a parallelepiped in $\mathbb{E}^r$.

The Voronoi polytope of any lattice in two or three dimensions is a zonotope, and zonotopes can appear as Voronoi polytopes in all other dimensions. Recently Voronoi's Conjecture has been resolved for the case of zonotopes [6].

THEOREM 1.1. *Any zonotope which admits a facet-to-facet tiling of Euclidean space* $\mathbb{E}^d$ *by translates is affinely equivalent to a Voronoi polytope for a lattice.*

Center a small sphere at a vertex $\mathbf{v} \in E^d$ of a Voronoi tiling, then expand it until it hits the lattice Λ. B. N. Delaunay referred to such a sphere $\mathcal{S}(\mathbf{v})$ as an *empty sphere* when he introduced his *empty sphere method* for lattices at the 1924 International Congress of Mathematicians held in Toronto[1] [3]. The sphere $\mathcal{S}(\mathbf{v})$ is empty since it has no lattice points in its interior. However, there are sufficiently many lattice points on its boundary so that their convex hull is a lattice polytope with full dimension. For each vertex $\mathbf{v}$ of the Voronoi tiling there is such a *Delaunay polytope*, and these fit together to form a second facet-to-facet tiling of $\mathbb{E}^d$, the *Delaunay tiling.* There are a finite number of translation classes of Delaunay polytopes, and the radii of the circumscribing empty spheres can vary with the translation class.

Lattices where the Voronoi polytope is a zonotope can be characterized in terms of their Delaunay tilings.

THEOREM 1.2. *The Voronoi polytope of a lattice is a zonotope if and only if the Delaunay tiling is a dicing.*

A *dicing* is an arrangement of hyperplanes where the vertices of the corresponding partition form a lattice Λ; the arrangement is required to be invariant under all lattice translates [5]. Dicings generalize the description of a lattice used by Gauss in his 1831 commentary on the thesis of Seeber on ternary quadratic forms [8]: d independent families of equi-spaced parallel hyperplanes partition Euclidean space into parallelepipeds, all translationally equivalent. The vertices form a d-dimensional lattice Λ, and the edge vectors a fundamental basis for this lattice. Gauss' construction, and earlier work by Seeber, mark the first appearance of lattices in a mathematical argument, and predate by over 80 years the fundamental experiment of M. von Laue that showed that lattices are part of nature. In 1911 M. von Laue and his collaborators [7] showed that crystals act as suitable gratings for diffracting X-rays. This fact was seized upon by W. L. Bragg [1] as

[1] The Russian delegation to the Toronto ICM was led by W. Steklov and consisted of 4 members. B. Delaunay was not among them, but was a corresponding delegate. He contributed two papers to the proceedings, the more famous being his *Sur la sphère vide* [3]; this included several excellent illustrations, and this is characteristic of his work. Delaunay's father N. Delaunay, also a corresponding delegate, contributed a single paper to the proceedings.

definitive evidence that the atoms or molecules in a crystal form a geometrical lattice. Dicings generalize the Gaussian construction by allowing more than d infinite families of parallel hyperplanes.

More formally, a dicing $\mathfrak{D}$ is an arrangement of hyperplanes formed by families of equi-spaced parallel hyperplanes that is *nondegenerate* and *vertex regular*, where these terms are defined as follows:

D1. *Among the families there must be d with linearly independent normal vectors.*

D2. *Through each vertex of the dicing there is one hyperplane from each family.*

It is vertex regularity that ensures that the vertex set Λ for a lattice forms a lattice.

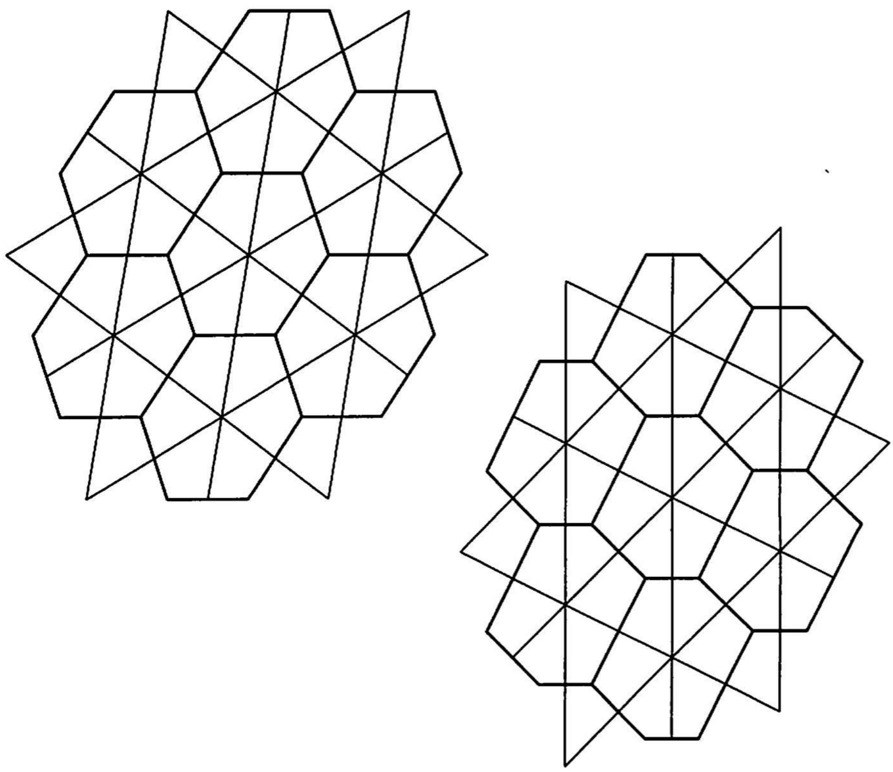

FIGURE 1. On the right is a portion of an hexagonal Voronoi tiling along with a portion of the dual Delaunay tiling which is superimposed (thin lines); on the left, an affine variant.

Some earlier work on space filling zonotopes was stimulated by H. S. M. Coxeter's paper *The classification of zonohedra by means of projective diagrams* [2]; the most important references in this direction are [11],[9].

Tiling the plane. For a polygon to admit an edge-to-edge tiling of the plane by translates it must either be a parallelogram or a centrally symmetric hexagon.

Both are zonotopes, the hexagon being the image of a 3-parallelepiped where two vertices are mapped to the interior. Both tilings are invariant under translation by center-to-center vectors, so the centers form a lattice Λ. The hexagonal tiling is primitive since the star of each vertex contains three hexagons.

A portion of an hexagonal tiling is shown on the left in Figure 1, along with a portion of a second *dual tiling* by triangles. The dual tiling is formed by partitioning the plane by lines through the centers of opposite sides of the hexagons. These lines can be grouped into three infinite families of equi-spaced parallel lines which have the property that one line from each family passes through each vertex of the triangulation. So by the conditions stated above for a dicing, the dual tiling is a dicing. The tiles of the triangular tiling belong to two distinct translation classes. The vertices of the triangulation are the centers of the hexagons, and the vertices of the hexagons are the centroids of the triangles. This relationship is similar to that between the Voronoi and Delaunay tilings for a hexagonal lattice as pictured on the right in Figure 1. The hexagonal tiling is the Voronoi tiling for the lattice of centers Λ, and the triangles are generated by Delaunay's metrical construction. The vertices of each triangle lie on an *empty circle* centered at a vertex of the hexagonal tiling. The vertices of the Voronoi hexagons are the centers of the circumscribing empty circles, and the vertices of the Delaunay triangles are the centers of the Voronoi hexagons. Consistent with this metrical construction, the edges of the Voronoi hexagons cross perpendicularly those of the Delaunay triangles.

It is apparent that the two tilings of Figure 1 are affinely equivalent, and it is a simple geometrical exercise to prove it. This is of course consistent with Voronoi's conjecture on parallelohedra.

2. Venkov Arrangements. Building on the work of Minkowski, B. A. Venkov [12] was able to show that a polytope P is a parallelohedron if and only if: (1) P is centrally symmetric with centrally symmetric facets; (2) the image of P under every orthogonal projection along a $(d-2)$-face must be either a parallelogram or a centrally symmetric hexagon. Thus, the natural images along $(d-2)$-faces are required to be two dimensional parallelohedra. All zonotopes satisfy the first of these conditions, so it is the second that characterizes zonotopes with the tiling property. Unaware of the work of Venkov, and at a somewhat later time, P. McMullen reproduced this characterization and published an interesting account of the issues involved [10].

For a zonotope Z there is a correspondence between the k-faces and the *natural k-spaces* generated by k linearly independent zone vectors. Each k-face is parallel to a natural k-space, and for each natural k–space there are parallel k-faces of Z. An orthogonal projection along a natural $(d-k)$-space will be called a *natural k-projection.* For a two dimensional image of a zonotope to be a parallelogram or a hexagon, the images of the zone vectors must belong to either two or three co-linearity classes. Hence, a zonotope satisfies the Venkov conditions if and only if the images of the zone vectors under each natural 2-projection fall into either 2 or 3 co-linearity classes.

The Venkov conditions for a zonotope can also be formulated in terms of hyperplanes perpendicular to the zone vectors. Let $\mathfrak{A}$ be an arrangement of distinct hyperplanes at the origin. Then the intersection of $d-k$ independent hyperplanes is the range of the natural k-projection along the normals to these hyperplanes, and will be called a *natural k-section.*

DEFINITION. An arrangement $\mathfrak{A}$ of hyperplanes at the origin is Venkov if each natural 2-section contains either 2 or 3 natural 1-sections. Equivalently, the images of the normal vectors under each natural 2-projection fall into either 2 or 3 co-linearity classes.

Hence a zonotope Z admits a facet-to-facet tiling of $\mathbb{E}^d$ by translates if and only if the zone vectors are normal to a Venkov arrangement.

The link to dicings. Now consider a dicing $\mathfrak{D}$ with r hyperplane families. Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ be normal vectors for the hyperplane families with lengths adjusted so that hyperplanes in a family have the equations

$$\mathbf{d}_i \cdot (\mathbf{x} - \boldsymbol{\lambda}) = z, \quad z \in \mathbb{Z} \text{ (the integers)},$$

where $\boldsymbol{\lambda}$ is a fixed point of the lattice of vertices $\Lambda(\mathfrak{D})$ (if $\mathfrak{D}$ has a vertex at the origin $\boldsymbol{\lambda}$ can be taken to be $\mathbf{0}$). The dicing can then be *represented* by the centrally symmetric set $D=\{\mathbf{d}_1, -\mathbf{d}_1, \mathbf{d}_2, -\mathbf{d}_2, \ldots, \mathbf{d}_r, -\mathbf{d}_r\}$ of $2r$ normal vectors where both $\mathbf{d}_i$ and $-\mathbf{d}_i$ represent the i-th hyperplane family. A second dicing $\mathfrak{D}'$ has the same *representation* D if and only if $\mathfrak{D}'$ is a translate of the original $\mathfrak{D}$. If one normal vector is selected from each co-linear pair $\mathbf{d}, -\mathbf{d} \in D$, an *oriented representation* D^+ is formed.

THEOREM 2.1. *Consider an arrangement $\mathfrak{A}$ of hyperplanes at the origin with normal vectors $\mathbf{a}_i, 1 \le i \le r$. Then $\mathfrak{A}$ is Venkov if and only if there are positive scale factors α_i so that with $\mathbf{d}_i = \alpha_i \mathbf{a}_i$ the hyperplane families $\mathbf{d}_i \cdot \mathbf{x} = z_i \in \mathbb{Z}$ dice $\mathbb{E}^d$.*

Thus a zonotope Z has the tiling property if and only if the zone vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots,$ $\mathbf{z}_r$, can be scaled to obtain a representation $\{\alpha_1\mathbf{z}_1, -\alpha_1\mathbf{z}_1, \alpha_2\mathbf{z}_2, \ldots, \alpha_r\mathbf{z}_r, -\alpha_r\mathbf{z}_r\}$ for a dicing.

3. From Dicings to Zonotopes and Back Again. To further explore the link between space filling zonotopes and dicings, an affine variant of the standard Voronoi and Delaunay tilings for lattices is useful. Suppose that Λ is a lattice, and suppose that φ is a positive quadratic form. If $\mathbf{c} \in \mathbb{E}^d$ and if R is a positive constant, then the equation

$$\varphi(\mathbf{x} - \mathbf{c}) = R^2$$

determines a $(d-1)$-ellipsoid $\mathcal{E}$ centered at $\mathbf{c}$. This ellipsoid is *empty* if it contains no elements of Λ in its interior. If in addition $\mathcal{E}$ circumscribes a lattice polytope P with full dimension, and with vertex set given by $\mathcal{E} \cap \Lambda$, then P is a *Delaunay polytope* for Λ *with metrical form* φ. The collection of all such Delaunay polytopes gives a facet-to-facet tiling of $\mathbb{E}^d$, and is uniquely determined by φ. This is the *Delaunay tiling* $\mathcal{D}_\varphi$ *for* Λ *with metrical form* φ.

Keeping the metric φ fixed, the Voronoi polytope at $\boldsymbol{\lambda} \in \Lambda$ is given by

$$V_\varphi(\boldsymbol{\lambda}) = \{\mathbf{x} \in \mathbb{E}^d \mid \varphi(\mathbf{x}-\boldsymbol{\lambda}) \leq \varphi(\mathbf{x}-\boldsymbol{\lambda}'), \boldsymbol{\lambda}' \in \Lambda, \boldsymbol{\lambda}' \neq \boldsymbol{\lambda}\}$$

and is that portion of $\mathbb{E}^d$ which is at least as close to $\boldsymbol{\lambda}$ as to any other point of Λ; $V_\varphi(\boldsymbol{\lambda})$ is also equal to the convex hull of the centers of the empty ellipsoids determined by φ that circumscribe each of the Delaunay polytopes in the star at $\boldsymbol{\lambda}$ (that is, the Delaunay polytopes with a vertex at $\boldsymbol{\lambda}$). The Voronoi tiling $\mathcal{V}_\varphi$ is generated by taking all lattice translates of $V_\varphi(\boldsymbol{\lambda})$.

Suppose that $\mathcal{D}$ is a Delaunay tiling for the lattice Λ. Then the set of positive quadratic forms

$$\Phi(\mathcal{D}) = \{\varphi \text{ positive } | \mathcal{D}_\varphi = \mathcal{D}\}$$

is called the *domain of the tiling* $\mathcal{D}$. These domains are relatively open polyhedral cones in the $\binom{d+1}{2}$-dimensional linear space of quadratic forms. The cone $\Phi(\mathcal{D})$ has full dimension $\binom{d+1}{2}$ if and only if the Delaunay tiling is simplicial. Simplicial tilings are the generic Delaunay tilings, and remain fixed under small perturbations of the metrical form.

Delaunay and Voronoi tilings for dicings. Consider a dicing $\mathfrak{D}$ with oriented representation D^+, and a positive quadratic form

$$\varphi(\mathbf{x}) = \sum_{\mathbf{d} \in D^+} \omega_\mathbf{d}(\mathbf{d} \cdot \mathbf{x})^2, \quad \omega_\mathbf{d} > 0;$$

$\varphi(\mathbf{x})$ is a positive linear combination of the squares of the linear forms $\mathbf{d} \cdot \mathbf{x}$, where $\mathbf{d} \in D^+$ is a normal vector to a hyperplane family. If C is an arbitrary d-cell of $\mathfrak{D}$, choose $\boldsymbol{\lambda}$ in the associated lattice $\Lambda(\mathfrak{D})$ to be a vertex of C, and choose an orientation for D^+ so that C lies in each of the half spaces $\mathbf{d} \cdot (\mathbf{x}-\boldsymbol{\lambda}) \geq 0$. Then for each $\mathbf{d} \in D^+$, C lies between the adjacent hyperplanes $\mathbf{d} \cdot (\mathbf{x}-\boldsymbol{\lambda}) = 0$ and $\mathbf{d} \cdot (\mathbf{x}-\boldsymbol{\lambda}) = 1$. As a consequence the function $p_\mathbf{d}(\mathbf{x}) = [\mathbf{d} \cdot (\mathbf{x}-\boldsymbol{\lambda})] \times [\mathbf{d} \cdot (\mathbf{x}-\boldsymbol{\lambda}) - 1]$ is non-positive on C and non-negative on $\Lambda(\mathfrak{D})$. The same is true for the function

$$f(\mathbf{x}) = \sum_{\mathbf{d} \in D^+} \omega_\mathbf{d} p_\mathbf{d}(\mathbf{x})$$

since $\omega_\mathbf{d} > 0$. Therefore the surface $f(\mathbf{x}) = 0$ is an empty ellipsoid $\mathcal{E}_f$ circumscribing the d-cell C so that the vertices of C are given by $\mathcal{E}_f \cap \Lambda(\mathfrak{D})$. By completing the square the equation for $\mathcal{E}_f$ can be rewritten as

$$\varphi(\mathbf{x}-\mathbf{c}) = R^2$$

where $\mathbf{c}$ is the center. This shows that C is a Delaunay polytope for the lattice $\Lambda(\mathfrak{D})$ with metrical form φ. Since C was chosen arbitrarily, it follows that $\mathfrak{D}$ is the Delaunay tiling $\mathcal{D}_\varphi$ for $\Lambda(\mathfrak{D})$ with metrical form φ. More precisely:

THEOREM 3.1. *Any dicing $\mathfrak{D}$ is a Delaunay tiling, and the domain of the dicing $\Phi(\mathfrak{D})$ is the cone of all positive forms*

$$\varphi(\mathbf{x}) = \sum_{\mathbf{d}\in D^+} \omega_{\mathbf{d}}(\mathbf{d}\cdot x)^2,$$

where $\omega_{\mathbf{d}} > 0$, and is a simplicial open cone.

Notice that the domain $\Phi(\mathfrak{D})$ does not depend on the orientation of the representation.

The center $\mathbf{c}$ of the empty ellipsoid $\mathcal{E}_f$ described above is easily computed and has the following formula

$$\mathbf{c} - \boldsymbol{\lambda} = \frac{1}{2}\sum_{\mathbf{d}\in D^+} \mathbf{z}_{\mathbf{d}}, \quad \mathbf{z}_{\mathbf{d}} = \Big(\sum_{\mathbf{d}\in D^+} \omega_{\mathbf{d}}\mathbf{d}\mathbf{d}^T\Big)^{-1}\omega_{\mathbf{d}}\mathbf{d}.$$

(In the second equality $\mathbf{d}$ should be interpreted as a column vector so that $\mathbf{d}\mathbf{d}^T$ is a square matrix.) The center is a vertex of the Voronoi polytope $V_\varphi(\boldsymbol{\lambda})$ centered at $\boldsymbol{\lambda} \in \Lambda(\mathfrak{D})$, and has the form of a vertex of a zonotope. In fact, *the Voronoi polytope $V_\varphi(\boldsymbol{\lambda})$ is a zonotope with zone vectors $\mathbf{z}_{\mathbf{d}} = (\sum_{\mathbf{d}\in D^+} \omega_{\mathbf{d}}\mathbf{d}\mathbf{d}^T)^{-1}\omega_{\mathbf{d}}\mathbf{d}$, $\mathbf{d} \in D^+$.*

THEOREM 3.2. *For any dicing $\mathfrak{D}$, and positive quadratic form $\varphi \in \Phi(\mathfrak{D})$, the tiles of the Voronoi tiling $\mathcal{V}_\varphi$ are zonotopes. Conversely, if Z is a zonotope admitting a facet-to-facet tiling $\mathcal{T}_Z$ by translates there is a dicing $\mathfrak{D}$ and form $\varphi \in \Phi(\mathfrak{D})$ so that the Voronoi tiling $\mathcal{V}_\varphi = \mathcal{T}_Z$.*

The second statement of this theorem completes the cycle, going from zonotopes back to dicings. A key step in the proof requires an application of Theorem 2.1.

The main theorems. This last theorem supplies the main step to prove Theorems 1.1 and 1.2. If Z is a zonotope with the tiling property, then by Theorem 3.2 there is a dicing $\mathfrak{D}$ and metrical form $\varphi \in \Phi(\mathfrak{D})$ so that the tiling by Z and the Voronoi tiling $\mathcal{V}_\varphi$ coincide. Each Delaunay polytope in the dual tiling $\mathcal{D}_\varphi$ can be circumscribed by an empty ellipsoid determined by φ, all homothetic copies of some original ellipsoid. Hence, any affine transformation mapping these ellipsoids onto spheres, also maps the Delaunay tiling $\mathcal{D}_\varphi$ onto a standard Delaunay tiling where the tiles are circumscribed by empty spheres. In the process the Voronoi tiling $\mathcal{V}_\varphi$ is mapped to a standard Voronoi tiling determined by the Euclidean metric; the vertices of the Voronoi tiling are the centers of the empty spheres for the Delaunay tiling. This supplies a proof for Theorem 1.1. The proof of Theorem 1.2 is *almost* a direct application of Theorem 3.2; there are a few technical details that must be taken into account.

4. Zonotopes with the Tiling Property. By Theorem 3.2 the problem of classifying the zonotopes in $\mathbb{E}^d$ that have the tiling property is equivalent to the problem of classifying the dicings in that dimension. Thus a complete classification is achieved for a given dimension d if all the *maximal dicings are enumerated up to affine equivalence*; all other dicings are simply obtained by deleting hyperplane families from a maximal one. A maximal dicings is one where a hyperplane family cannot be added without violating the regularity condition D2.

The classification of the dicings in $\mathbb{E}^2$ is straightforward. Two families of equispaced parallel lines dice the plane into parallelograms, all translationally equivalent. There are only two ways to add a third family so that vertex regularity is maintained, and these correspond to the two ways the parallelogram can be partitioned into two triangles. But each of these triangular tilings are affinely equivalent. No additional family of lines can be added without destroying vertex regularity, and as a consequence there are exactly two dicings of the plane. The triangular dicing is maximal, and up to affine equivalence is the only maximal dicing in the plane (see Figure 1). This triangular dicing is the first in an infinite sequence of simplicial dicings $\mathfrak{D}^1(d) \subset \mathbb{E}^d, d = 1, 2, \ldots$, all maximal. These dicings have $\binom{d+1}{2}$ hyperplane families, and are the only dicings where the cells are all simplexes. The dicing $\mathfrak{D}^1(d)$ is affinely equivalent to the Delaunay tiling for the weight lattice A_d^*, and the domain $\Phi(\mathfrak{D}^1(d))$ is referred to as the *first perfect domain* by geometers.

For $d = 3$, $\mathfrak{D}^1(3)$ is the only maximal dicing. For dimension $d = 4$ there is one additional maximal dicing, $\mathfrak{D}^2(4)$, with nine hyperplane families; for $d = 5$ there are three additional maximal dicings $\mathfrak{D}^2(5), \mathfrak{D}^3(5), \mathfrak{D}^4(5)$, with 12, 12, and 10 hyperplane families respectively [5]. If the sets of columns of the following matrices are augmented with the d standard unit vectors they form oriented representations for the maximal dicings in $\mathbb{E}^4$ and $\mathbb{E}^5$.

$$\mathbf{D}^1(4) = \begin{bmatrix} 1&0&0&1&0&1 \\ 1&1&0&1&1&1 \\ 0&1&1&1&1&1 \\ 0&0&1&0&1&1 \end{bmatrix}; \quad \mathbf{D}^2(4) = \begin{bmatrix} 1&1&0&0&1 \\ 0&0&1&1&1 \\ 1&0&1&0&1 \\ 0&1&0&1&1 \end{bmatrix}$$

$$\mathbf{D}^1(5) = \begin{bmatrix} 1&0&0&0&1&0&0&1&0&1 \\ 1&1&0&0&1&1&0&1&1&1 \\ 0&1&1&0&1&1&1&1&1&1 \\ 0&0&1&1&0&1&1&1&1&1 \\ 0&0&0&1&0&0&1&0&1&1 \end{bmatrix}; \quad \mathbf{D}^2(5) = \begin{bmatrix} 1&0&0&0&1&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&1&0&0&0&1 \\ 0&0&1&1&1&0&1 \\ 0&0&0&1&1&1&1 \end{bmatrix}$$

$$\mathbf{D}^3(5) = \begin{bmatrix} 1&0&0&0&1&0&1 \\ 1&1&0&0&0&0&1 \\ 0&1&1&0&0&1&1 \\ 0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&1 \end{bmatrix}; \quad \mathbf{D}^4(5) = \begin{bmatrix} 1&1&0&0&1 \\ 0&0&1&1&1 \\ 1&0&1&0&1 \\ 0&1&0&1&1 \\ 1&1&1&1&1 \end{bmatrix}$$

If D^+ is an oriented representation for a maximal dicing, and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r\} \subset D^+$ is a non-degenerate subset of normal vectors, then the set $\{\alpha_1\mathbf{d}_1, \alpha_2\mathbf{d}_2, \ldots, \alpha_r\mathbf{d}_r\}$ are zone vectors for a zonotope with the tiling property; the scale factors $\alpha_i, i = 1, \ldots, r$, are positive but otherwise arbitrary. And conversely, if Z is a zonotope in $\mathbb{E}^d$ with the tiling property, then the zone vectors of Z have the form $\{\alpha_1\mathbf{d}_1, \alpha_2\mathbf{d}_2, \ldots, \alpha_r\mathbf{d}_r\}$, where $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r\}$ is non-degenerate subset of normal vectors for an oriented representation D^+ of a maximal dicing. More specifically, let $D^{1+}(4), D^{2+}(4), D^{1+}(5), D^{2+}(5), D^{3+}(5), D^{4+}(5)$, be the oriented representations for the six maximal dicings in $\mathbb{E}^4$ or $\mathbb{E}^5$ quoted above. Then up to affine equivalence, these determine all the zonotopes in these dimensions with the tiling property. If Z is such a zonotope with r zone vectors, then there is an affinely equivalent zonotope Z' with zone vectors $\{\alpha_1\mathbf{d}_1, \alpha_2\mathbf{d}_2, \ldots, \alpha_r\mathbf{d}_r\}$, where $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r\}$ is a subset of one of these representations.

5. Voronoi's Theory of Lattice Types. Through his investigation of primitive parallelohedra Voronoi[2] was led to his geometric *theory of lattice types*, introduced in the second part of his *Deuxième Mémoire.* Voronoi died at the height of his powers in 1908 at the age of 40, with this famous *Seconde partie* in draft form. The draft of *Domaines de formes quadratiques correspondant aux différents types de paralléloèdres primitifs* was edited by H. Minkowski, then published posthumously in 1909 [14].

Voronoi's theory of lattice types was his most striking contribution to the geometry of numbers, providing a link between the geometry of Delaunay and Voronoi tilings on the one hand, and the arithmetic theory of positive quadratic forms on the other. He considered the cone of positive quadratic forms, and the fixed lattice $\mathbb{Z}^d$. Each positive quadratic form φ determines a Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^d$, and each Delaunay tiling $\mathcal{D}$ determines an *L-type domain* of positive quadratic forms given by

$$\Phi(\mathcal{D}) = \{\varphi \text{ positive} \mid \mathcal{D}_\varphi = \mathcal{D}\}.$$

This partition into L-type domains was the central geometrical object studied in the second part of his *Deuxième Mémoire.* The L-type domains are relatively open polyhedral cones in the linear space of quadratic forms. The L-type domains vary in dimension, those with full dimension $\binom{d+1}{2}$ fitting together facet-to-facet; the domains with lesser dimension appear as faces. More precisely, Voronoi showed that L-type domains have the following properties.

1. L-type domains are arithmetically equivalent if and only if the corresponding Delaunay tilings are affinely equivalent.

[2] Delaunay described in ([4], p. 227) an interesting encounter he had with Voronoi in 1904 when he was 14 years old. At the time Voronoi was completely absorbed in developing his theory of lattice types. Delaunay's father was a professor at the Warsaw Polytechnic Institute, and Voronoi a professor at Warsaw University. Delaunay's empty sphere method, introduced in 1924, added a vivid geometric picture to many of the constructions of Voronoi.

2. The L-type domains which have full dimension in the linear space of quadratic forms have simplicial Delaunay tilings. These are called general L-type domains, and form a facet-to-facet tiling of the cone of positive quadratic forms.
3. All other L-type domains are special and appear as faces of varying dimension of the general L-type domains.
4. For each dimension d there are a finite number of arithmetically inequivalent L-type domains.

Two forms φ, φ' are arithmetically equivalent if there is a matrix $\mathbf{U} \in \mathrm{GL}_d(\mathbb{Z})$ so that $\varphi(\mathbf{x}) = \varphi'(\mathbf{Ux})$.

Besides determining a Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^d$, a form φ also determines a linear mapping which converts $\mathcal{D}_\varphi$ into a standard Delaunay tiling. Let $\mathbf{B}$ be a real square matrix so that $\varphi(\mathbf{x}) = \mathbf{x}^T\mathbf{B}^T\mathbf{Bx}$. If $\mathbf{x}$ is taken to be a point in $\mathbb{E}^d$, then $\varphi(\mathbf{x})$ is the square of the length of $\mathbf{x}$ in the metric determined by φ. But this quantity can also be interpreted as the Euclidean length of the vector $\mathbf{Bx}$. In this second interpretation $\mathbf{x}$ is a coordinate vector for a point referred to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_d\}$ of columns of $\mathbf{B}$. The linear mapping $\mathcal{L}^{\mathbf{B}}$ with matrix $\mathbf{B}$ maps $\mathcal{D}_\varphi$ onto a standard Delaunay tiling for the image lattice $\mathcal{L}^{\mathbf{B}}(\mathbb{Z}^d)$; the empty ellipsoids for $\mathcal{D}_\varphi$ are mapped to empty spheres for the image tiling $\mathcal{L}^{\mathbf{B}}(\mathcal{D}_\varphi)$. With this construction the cone of positive quadratic forms serves as a convenient parameterization for geometric lattices. This is the *method of continuous parameters* mentioned in the title of Voronoi's two memoirs.

The edges of the 1-skeleton of a Delaunay tiling $\mathcal{D}_\varphi$ provide information on the Voronoi tiling $\mathcal{V}_\varphi$. Any edge is a translate of an edge at the origin, and the edges at the origin can be represented by a centrally symmetric edge set Ξ of primitive integer vectors. Each edge vector pierces the center of a facet of the Voronoi polytope at the origin, orthogonally in the metric determined by φ. Therefore, the number of edge vectors $|\Xi|$ is equal to the number of facets of the Voronoi polytope. The elements of the edge set can be classified according to their *parity class*; there are 2^d cosets of $2\mathbb{Z}^d$ in $\mathbb{Z}^d$, and I will refer to these as the parity classes. The following theorem appears on the first page of the *Seconde partie* ([14], *Seconde partie*, p. 67).

THEOREM 5.1. *A Delaunay edge set Ξ has no elements in the parity class equivalent to* $\mathbf{0}$. *If the tiling is simplicial all other parity classes contain precisely two elements, an edge and its negative, so that* $|\Xi| = 2(2^d-1)$. *For other Delaunay tilings each of the parity classes inequivalent to* $\mathbf{0}$ *contains at most two elements.*

A Delaunay tiling is simplicial if and only if the corresponding Voronoi tiling is primitive. It follows that primitive Voronoi polytopes, and more generally primitive parallelohedra, have precisely $2(2^d - 1)$ facets.

Passage from a point in a general L-type domain, to a point in the relative interior of a facet can be visualized as follows. Initially, as the metrical form is perturbed, the empty ellipsoids change shape but still circumscribe the same simplicial Delaunay polytopes. However, at the moment the point arrives at the facet one of the empty ellipsoids pushes against an additional lattice point, creating a Delaunay polytope with $d+2$ vertices. The empty ellipsoids circumscribing two or more simplexes coalesce into a single empty ellipsoid, and the corresponding simplexes glue together to form this new Delaunay polytope. By Vornoi's Theorem no edges are gained by this action, but some may be converted to *diagonals*; these lattice vectors run between a pair of vertices of a Delaunay cell, but not along an edge. New diagonals may also appear by other geometric processes. The edge set Ξ and the diagonal set Δ satisfy the relations $|\Xi| = 2(2^d - 1)$, $|\Delta| = 0$, for the initial simplicial tiling, but for the final special tiling satisfy the relations $|\Xi| \leq 2(2^d - 1)$, $|\Xi \cup \Delta| \geq 2(2^d - 1)$.

Passage from any face to a sub-face involves similar action where the edge set Ξ shrinks by transfering elements to Δ, and the diagonal set Δ grows. As a consequence the following general statements hold for Delaunay tilings.

1. The set $\Xi \cup \Delta$ includes an even number of lattice vectors in each of the parity classes inequivalent to $\mathbf{0}$: $|\Xi \cup \Delta| \geq 2(2^d - 1)$.
2. Some of the parity classes may be missing from the edge set so that $|\Xi| \leq 2(2^d - 1)$.

The set $\Xi \cup \Delta$ can be characterized as the set of all lattice vectors that run vertex to vertex within some Delaunay cell.

Unimodular representations of dicings. Every dicing is affinely equivalent to a dicing $\mathfrak{D}$ where the dicing lattice is equal to $\mathbb{Z}^d$. The domains of such dicings, as described in Theorem 3.1, provide examples of L-type domains. The *edge forms* $(\mathbf{d} \cdot \mathbf{x})^2$ that generate the extreme rays of a dicing domain are rank 1 quadratic forms, making dicing domains very special. Typically, the edge forms for L-type domains vary in rank. The relationship of dicing domains to more general L-type domains is completely characterized by the following theorem.

THEOREM 5.2. *An L-type domain is a dicing domain if and only if all the edge forms are rank one forms.*

There are examples of L-type domains where none of the edge forms are rank one forms, and examples where the edge forms are a mixture of rank one forms and higher rank forms. For more details see [5].

Suppose that $\mathfrak{D}$ is a dicing with dicing lattice equal to $\mathbb{Z}^d$, and that $D = \{\mathbf{d}_1, -\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_r, -\mathbf{d}_r\}$ is the corresponding representation by normal vectors. Then it is easy to see that the elements of D must be primitive integer vectors. More precisely:

THEOREM 5.3. *Assume that D is a centrally symmetric set of integer vectors with full rank. Then D is a representation of a dicing with lattice $\mathbb{Z}^d$ if and only if D is unimodular.*

A set of vectors $S \subset \mathbb{Z}^d$ with full rank is *unimodular* if every d-subset has determinant 0 or ± 1.

A dual operation on sets of integer vectors. If $S \subset \mathbb{Z}^d$ is centrally symmetric with full rank, define the *dual* S° by the formula

$$S^\circ = \{\mathbf{w} \in \mathbb{Z}^d | \mathbf{w} \neq \mathbf{0}, \mathbf{w} \cdot \mathbf{s} \in \{0, \pm 1\} \text{ for all } \mathbf{s} \in S\}.$$

Then $S \subset (S^\circ)^\circ$, and S° is centrally symmetric. The elements of S° are the integer vectors lying in the convex set bounded by the hyperplanes $\mathbf{s} \cdot \mathbf{x} = \pm 1$ for each vector $\mathbf{s} \in S$.

Suppose that D is a representation of a dicing with lattice $\mathbb{Z}^d$ so that the hyperplane families are given by $\mathbf{d} \cdot \mathbf{x} = z \in \mathbb{Z}$, $\mathbf{d} \in D$. If $\mathbf{w} \in D^\circ$, and if $\mathbf{d} \cdot \mathbf{x} = z \in \mathbb{Z}$ is a hyperplane familiy, then $\mathbf{w}$ is either parallel to the family, or runs between lattice points on adjacent hyperplanes of the family. The elements of D° are therefore the lattice vectors entirely contained within a cell of the dicing. That is, D° is the union of the edge set and diagonal set for the dicing: $D^\circ = \Xi \cup \Delta$. By the discussion following Theorem 5.1, D° has elements in each of the $2^d - 1$ parity classes inequivalent to $\mathbf{0}$. This statement can be strenghtened to yield a characterization for representations of dicings.

1. Suppose that $D \subset \mathbb{Z}^d$ is centrally symmetric with full rank. Then D is a representation of a dicing if and only if D° has elements in each of the $2^d - 1$ parity classes inequivalent to $\mathbf{0}$.

Other interesting statements can be made about the relationship between parity classes and representations of dicings.

2. Suppose that $D \subset \mathbb{Z}^d$ is a representation for a dicing $\mathfrak{D}$. Then $\mathfrak{D}$ is maximal if and only if $D = \Xi^\circ$.
3. Suppose that $D \subset \mathbb{Z}^d$ is a representation for a dicing. Then $\mathbf{e} \in D^\circ$ is an edge (and a member of Ξ) if and only if its parity class in D° includes only the two elements $\mathbf{e}, -\mathbf{e}$.

If $D \subset \mathbb{Z}^d$ is a representation of a dicing then the elements of D° can be grouped into the $2^d - 1$ parity classes that are inequivalent to $\mathbf{0}$, each class containing a non-zero, even number of elements. Let $c_{2k}(D)$ be the number of parity classes that contain exactly $2k$ elements. Then $c_2(D)$ is the number of classes that contain a pair of edges $\mathbf{e}, -\mathbf{e} \in \Xi$, and for $k > 1, c_{2k}(D)$ is the number of classes containing $2k$ diagonals. The numbers c_{2k} are affine invariants of the dicing, and satisfy the equality $c_2 + c_4 + c_6 + \cdots = 2^d - 1$. These numbers are easily calculated for the maximal dicings in $\mathbb{E}^4$ and $\mathbb{E}^5$ using the representations quoted in Section 4, and are displayed in the following table (columns have been added for the affine invariants $|D|, |D^\circ|, |\Xi|$).

dicing	c_2	c_4	c_6	$\lvert D\rvert$	$\lvert D^\circ\rvert$	$\lvert E\rvert$
$\mathfrak{D}^1(4)$	15	0	0	10	30	30
$\mathfrak{D}^2(4)$	15	0	0	9	30	30
$\mathfrak{D}^1(5)$	31	0	0	15	62	62
$\mathfrak{D}^2(5)$	29	2	0	12	66	58
$\mathfrak{D}^3(5)$	28	3	0	12	68	56
$\mathfrak{D}^4(5)$	30	0	1	10	66	60

6. Some Concluding Remarks. I started by considering space filling zonotopes, but have ended up by showing how these space fillers relate to a variety of mathematical topics. The following theorem illustrates an interesting connection with the theory on unimodular matrices.

THEOREM 6.1. *Let* $\mathbf{M} = [\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r]$ *be an integer* $d \times r$ *matrix with full rank. Then* $\mathbf{M}$ *is unimodular if and only if the dual set of vectors*

$$M^\circ = \{\mathbf{z} \in \mathbb{Z}^d | \mathbf{z} \cdot \mathbf{m}_i \in \{0, \pm 1\}, i = 1, \dots, r\}$$

includes elements in each of the $2^d - 1$ *parity classes that are inequivalent to* $\mathbf{0}$.

The proof follows directly from the characterization of representations in terms of determinants (Theorem 5.3), and the characterization in terms of the parity classes of D°.

Several of the topics we have touched on have associated classification problems. In particular *the classification of the following are equivalent:*

1. The affine classes of dicings in $\mathbb{E}^d$;
2. The affine classes of Venkov arrangements in $\mathbb{E}^d$;
3. The arithmetic classes of $d \times r$ integer unimodular matrices with full rank (two $d \times r$ unimodular matrices $\mathbf{U}$, $\mathbf{U}'$ are arithmetically equivalent if there is a matrix $\mathbf{Q} \in GL_d(\mathbb{Z})$ such that $\mathbf{U}' = \mathbf{QU}$);
4. The L-type domains in Voronoi's theory of lattice types with rank one edge forms;
5. The zonotopes which admit facet-to-facet tilings of $\mathbb{E}^d$.

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FACE-TRANSITIVE POLYHEDRA WITH RECTANGULAR FACES

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Résumé. On décrit plusieurs polyèdres de l'espace euclidien de dimension 3, dont les faces sont des rectangles non-carrés, et dont toutes les faces sont équivalentes par symétrie du polyèdre. Deux de ces polyèdres ont 24 faces chacun et la symétrie complète du group octaédrique. Les faces d'un des polyèdres sont rectangles avec le rapport des côtés $\sqrt{8/3} = 1.632993\ldots$, tandis que les faces des autres sont des rectangles de rapport 2. Les densités des deux polyèdres sont 5 et 2. D'autres polyèdres remarquables peuvent être obtenus par subdivision des faces, ou par réciprocation.

Polyhedra endowed with symmetry properties have attracted attention since antiquity. In more modern times the investigations extended to polyhedra with self-intersections, and also to isohedral and to isogonal polyhedra, in which the symmetries are assumed only to act transitively on faces or on vertices. A culmination of one direction was the determination of all uniform polyhedra, that is, possibly self-intersecting isogonal polyhedra in which the faces are regular polygons. The enumeration itself was given by Coxeter, Longuet-Higgins and Miller [2], and its completeness was established by Sopov [5] and Skilling [4]. Wenninger [6] presents photos of cardboard models of these uniform polyhedra, and instructions for building them; Har'El [3] describes methods for calculating and drawing the uniform polyhedra and their isohedral reciprocals. Among the latter are several rhombic isohedra (see Coxeter [1]), that is, isohedral polyhedra in which the faces are congruent rhombi. However, although various other isohedra have been described as well, isohedra having nonsquare rectangles as faces seem not to have been considered. (The polyhedron shown in Figure 1 and mentioned in the literature does not contradict this statement: each of its faces is a pentagon, the polyhedron is equivalent to the regular dodecahedron, and the apparent rectangular shape is accidental.) We shall describe here two isohedral polyhedra having as faces non-square rectangles. Several other polyhedra of this kind exist, but due to their complicated structure they will be described separately.

The polyhedron P_1 shown in Figure 2 (and, partially disassembled, in Figure 3) has 24 rectangles as faces, and its symmetry group is the full octahedral group [3, 4] in the Coxeter notation and $m3m$ or O_h in the crystallographic notation.

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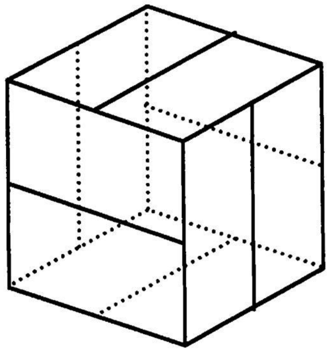

FIGURE 1. An isohedral polyhedron with faces that have rectangular shape but in fact are pentagons with pairs of collinear edges. Regarded as a map, it is equivalent to the regular dodecahedron $\{5,3\}$.

The density at the center of P_1 is 5. P_1 has two orbits of vertices, one coinciding with the vertices of a cube, the other with those of a regular octahedron. The faces themselves are rectangles with ratio of sides $\sqrt{8/3} = 1.632993\ldots$. The interpenetration of the faces of P_1 is quite complex; it is shown in Figure 4.

Another aspect of the complexity of P_1 is indicated by its reciprocal P_1^*, which is isogonal and is shown in Figure 5. Each of the 24 short edges of P_1 joins a vertex of the octahedron to a vertex of the cube. Of the 24 long edges, 12 are edges of the octahedron, while 12 are diagonals of the cube. In the sense that the 6 faces of a cube are arranged in 3 "zones" of 4, the faces of P_1 are arranged in 8 narrow zones of 3 and in 6 wide zones of 4. The map corresponding to P_1 has genus 6; it is shown in Figure 6.

The stabilizer of each face of P_1 contains a reflection; by subdividing each face along this mirror we obtain an isohedral polyhedron with 48 rectangular faces.

The second isohedral polyhedron P_2 has 24 2-by-1 rectangles as faces; they lie by fours in the faces of a cube of edge 2. This orientable polyhedron of density 2 is illustrated in Figure 7. Its symmetry group is the full octahedral group, and its vertices form two orbits; eight coincide with the vertices of a cube, and twelve are at midpoints of the cube's edges. Each face of P_2 has one long edge coinciding with an edge of the cube, and the other long edge coincides with a midline of a face of the cube. Each of the two short edges of a face of P_2 connects a vertex of the cube with a vertex at the midpoint of an edge of the cube. There is a lot of overlap among the elements of P_2, but this leads to no problems.

A map equivalent to P_2 has genus 3 and is shown in Figure 8; it may as well be used as a net for the polyhedron, if the near-rectangles are replaced by rectangles. The faces of P_2 are arranged in 6 narrow zones of 4 and in 3 wide zones of 8.

If the faces of P_2 are subdivided into two squares each, there results an ori-

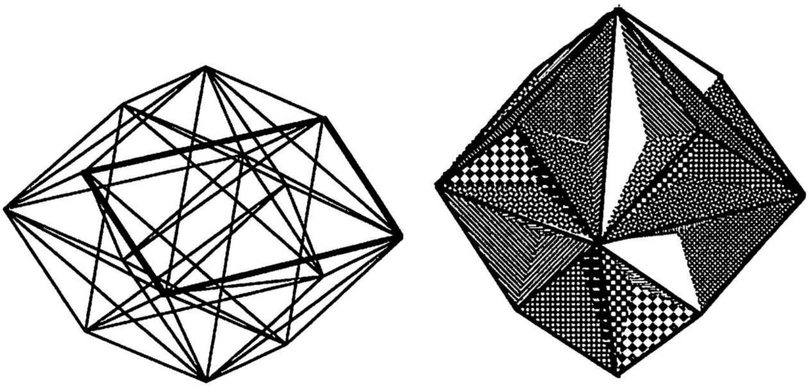

FIGURE 2. (a) A skeletal view of the isohedral polyhedron P_1 with 24 rectangular faces. One of the faces is emphasized by heavy lines. (b) A perspective view of a cardboard model of the same polyhedron P_1.

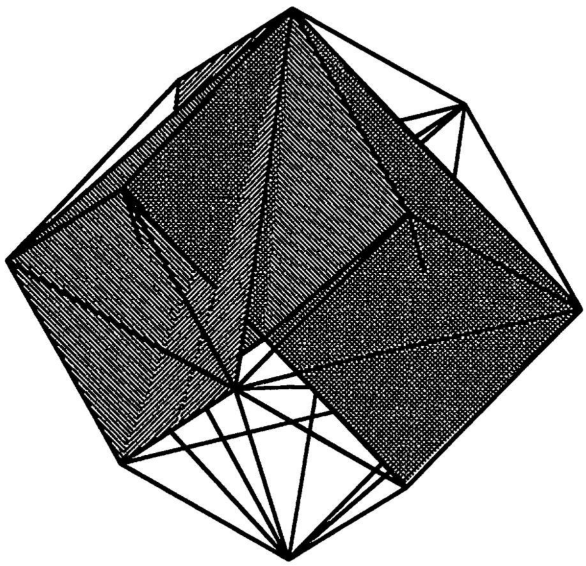

FIGURE 3. The skeleton and several faces of the polyhedron P_1, which illustrate their interpenetration.

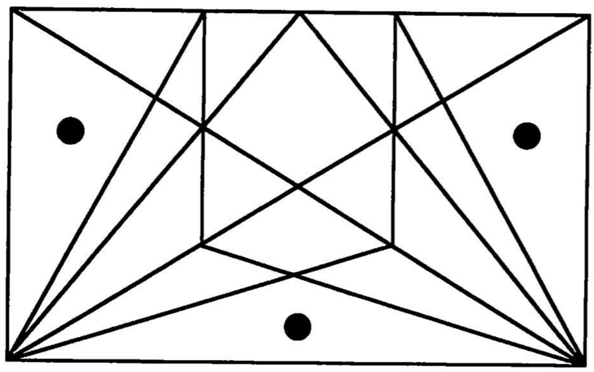

FIGURE 4. One face of the polyhedron P_1 with the traces of the faces that intersect it. The top edge is "hidden" in the cardboard model, in which only the three triangles marked by bullets are visible from the "outside".

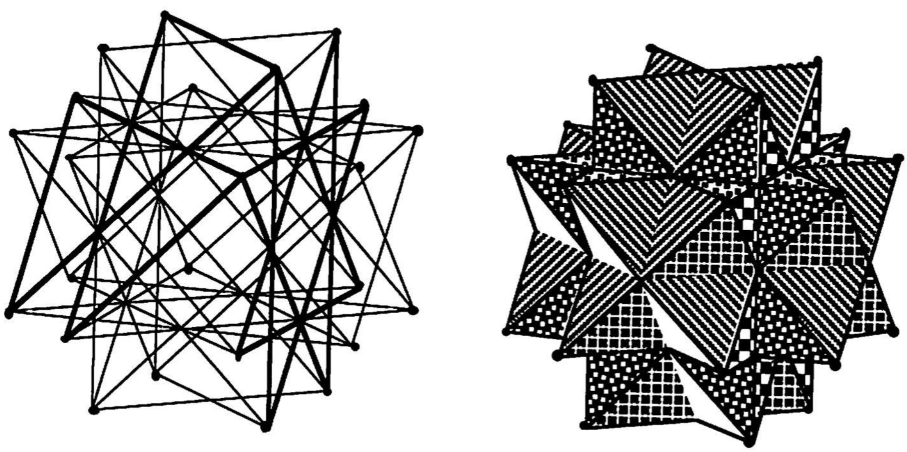

FIGURE 5. (a) A skeletal view of the isogonal polyhedron P_1^* which is reciprocal to P_1. Two faces, one hexagonal and one octagonal, are emphasized. (b) A perspective view of a cardboard model of the same polyhedron P_1^*.

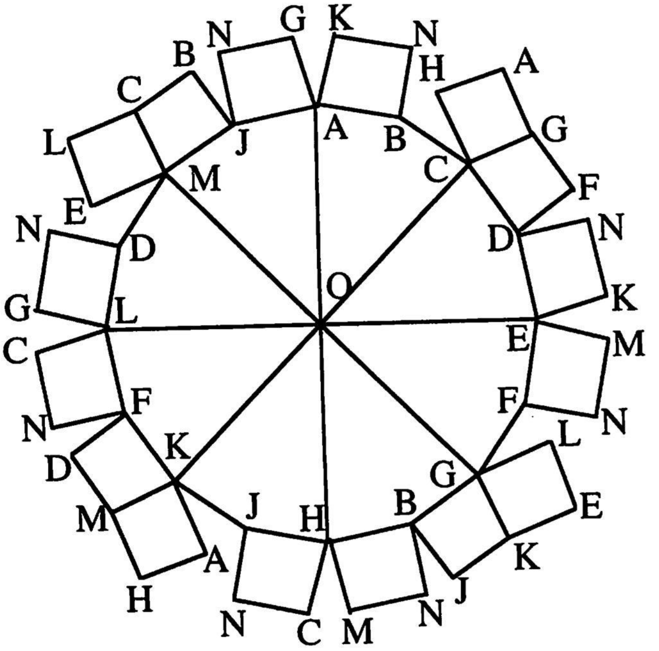

FIGURE 6. The map corresponding to the polyhedron P_1. The lettering defines the identification of vertices.

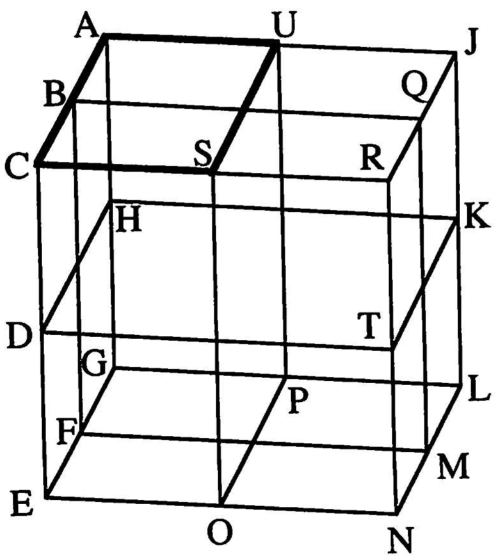

FIGURE 7. The polyhedron P_2 which has as faces 24 rectangles of size 2-by-1. One face is emphasized by heavy lines.

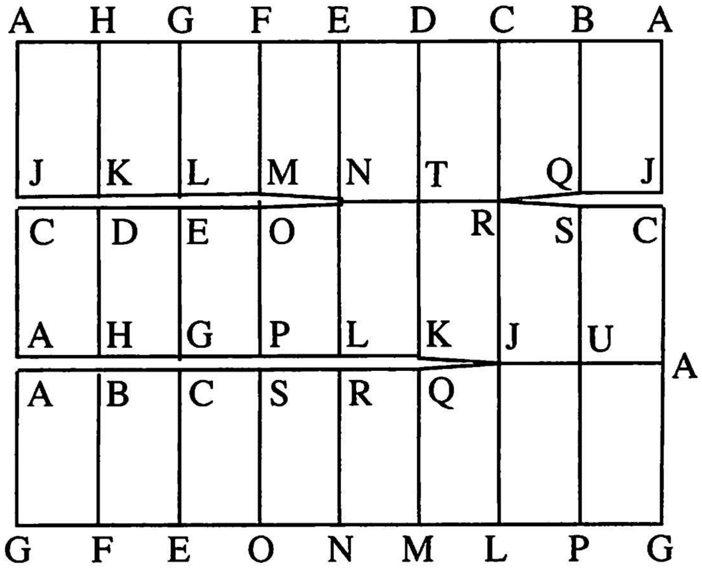

FIGURE 8. A map equivalent to the polyhedron P_2.

entable isohedral polyhedron of genus 3 and density 2 with 48 faces. Its appearance is just like Figure 7, but all elements except the vertices of the cube are doubled up. By a slight modification of the construction one may obtain orientable isohedral polyhedra with $24k$ square faces, of density k and genus $3k-3$, for every $k \geq 1$.

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STRATIFIED ALGEBRAS

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RÉSUMÉ. Le concept de l'algèbre stratifiée est en relation directe avec la théorie des représentations des algèbres de Lie complexes semisimples de dimension finie et est une généralisation naturelle du concept de l'algèbre quasi-héréditaire: les algèbres quasi-héréditaires ne sont autres que les algèbres stratifiées de dimension globale finie. De plus, les autres caractérisations des algèbres quasi-héréditaires, aussi bien homologiques que numeriques, résultent immediatement des caractérisations respectives des algèbres stratifiées. Bien que la motivation originale pour l'étude des algèbres stratifiées vienne de la généralisation de la catégorie $\mathcal{O}$ de Bernstein-Gelfand-Gelfand qui a été introduite dans le travail de Futorny et Mazorchuk, ce concept entre dans le contexte générale présenté de façon indépendante par Cline, Parshall et Scott. Un exposé préliminaire sur les résultats de cet article a été présenté au Séminaire d'Algèbre à Paris en Mai, 1997.

Let A be a basic connected finite dimensional K-algebra and $A_A = \bigoplus_{i=1}^n P(i) = \bigoplus_{i=1}^n e_iA$ its (right) regular representation. Denote by $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ the complete sequence of its indecomposable orthogonal idempotents and put $\varepsilon_i = \sum_{t=i}^n e_t$, $1 \le i \le n$, $\varepsilon_{n+1} = 0$. Thus the symbol $(A, \mathbf{e})$ will denote the algebra A with the fixed order $\mathbf{e}$ of the idempotents e_i, or equivalently, with the fixed order $\big(S(1), S(2), \ldots, S(n)\big)$ of the (right) simple A-modules $S(i) = P(i)/\operatorname{rad} P(i) = e_iA/e_i \operatorname{rad} A$. Let us remark that this way we have also fixed the order of the left indecomposable projective A-modules $P^\circ(i) = Ae_i$ and the left simple A-modules $S^\circ(i) = Ae_i/\operatorname{rad} Ae_i$.

DEFINITION 1.1. For a given algebra $(A, \mathbf{e})$, the sequence of the *right standard A-modules*

$$\Delta = \big(\Delta(1), \Delta(2), \ldots, \Delta(n)\big)$$

and the sequence of *right proper standard A-modules*

$$\bar{\Delta} = \big(\bar{\Delta}(1), \bar{\Delta}(2), \ldots, \bar{\Delta}(n)\big)$$

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is defined by

$$\Delta(i) = e_iA/e_i \operatorname{rad} A\varepsilon_{i+1}A, \quad 1 \le i \le n, \quad \text{and}$$
$$\bar{\Delta}(i) = e_iA/e_i \operatorname{rad} A\varepsilon_iA, \quad 1 \le i \le n,$$

respectively. The sequences of *left standard modules* Δ° and *left proper standard modules* $\bar{\Delta}^\circ$ are defined similarly. Moreover, define the *right costandard A-modules* $\nabla(i)$ and the *right proper costandard A-modules* $\bar{\nabla}(i)$ by:

$$\nabla(i) = \operatorname{Hom}_K(\Delta^\circ(i), K), \quad 1 \le i \le n, \quad \text{and}$$
$$\bar{\nabla}(i) = \operatorname{Hom}_K(\bar{\Delta}^\circ(i), K), \quad 1 \le i \le n.$$

Although most of our statements will have their dual counterparts, formulated for the opposite algebra, we shall usually refrain from stating both statements explicitly.

Let us observe that, for any K-algebra A and any order e, $\Delta(n)$ is projective and $\bar{\Delta}(1)$ is a simple A-module. Clearly, $[\bar{\Delta}(i) : S(i)] = 1$ and thus $D_i = \operatorname{End}_A \bar{\Delta}(i) \simeq \operatorname{End}_A S(i) \simeq e_iAe_i/e_i \operatorname{rad} Ae_i \simeq \operatorname{End}_A S^\circ(i)$ is a division algebra for all $1 \le i \le n$. Let us write $d_i = [D_i : K]$. In fact, one can see immediately that $\Delta(i) = \bar{\Delta}(i)$ (and consequently, $\Delta^\circ(i) = \bar{\Delta}^\circ(i)$) if and only if $\operatorname{End}_A \Delta(i) \simeq e_iAe_i/e_iA\varepsilon_{i+1}Ae_i = D_i$, *i.e.*, if and only if $e_iA\varepsilon_{i+1}Ae_i = e_i \operatorname{rad} Ae_i$.

DEFINITION 1.2. Let Φ be a fixed set of (right) A-modules from mod-A, the category of all right A-modules of finite length. A module $X \in$ mod-A is said to be *Φ-filtered* if there is a chain of submodules

$$X = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_t \supseteq X_{t+1} = 0$$

such that $X_s/X_{s+1} \in \Phi$ for all $1 \le s \le t$. Denote by $\mathcal{F}(\Phi)$ the full subcategory of mod-A of all Φ-filtered A-modules.

We will consider mainly the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\Delta})$ of mod-A. Every $X \in \mathcal{F}(\Delta)$ has a canonical Δ-filtration given by a refinement of the *trace filtration*

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \cdots \supseteq X^{(i)} = \operatorname{trace}_{\varepsilon_iA} X = X\varepsilon_iA \supseteq X^{(n)} \supseteq X^{(n+1)} = 0;$$

here, for each $1 \le i \le n$, the module $X^{(i)}/X^{(i+1)}$ is a direct sum of $\Delta(i)$'s (*cf.* [DK]). Similarly, any $X \in \mathcal{F}(\bar{\Delta})$ has a canonical $\bar{\Delta}$-filtration given by a refinement of the *proper trace filtration* which itself is a refinement of the trace filtration of X:

$$\cdots X^{(i)} = X^{(i,0)} \supseteq X^{(i,1)} \supseteq \cdots \supseteq X^{(i,j)} = \operatorname{trace}_{\varepsilon_iA} \operatorname{rad} X^{(i,j-1)}$$
$$= X(\varepsilon_i \operatorname{rad} A\varepsilon_i)^jA \supseteq \cdots \supseteq X^{(i,h_i)} = X^{(i+1)} \supseteq \cdots;$$

here, for each $1 \le i \le n$, the module $X^{(i,j)}/X^{(i,j+1)}$ is a direct sum of $\bar{\Delta}(i)$'s for every $0 \le j \le h_i - 1$.

DEFINITION 1.3. The algebra (A, e) is said to be *stratified of type* $\mathrm{s} = (s_1, s_2, \ldots, s_n)$, where $s_i = \pm 1$, if each factor $A\varepsilon_iA/A\varepsilon_{i+1}A$ of the trace filtration of A_A belongs to $\mathcal{F}(\Delta^{-s_i})$; here $\Delta^{+1} = \Delta$ and $\Delta^{-1} = \bar{\Delta}$. In particular, $(A, \mathbf{e})$ is *standardly stratified* if it is a stratified algebra of type $\mathbf{1} = (+1, +1, \dots, +1)$ (*cf.* [CPS]).

REMARK. Note that the type of a stratified algebra is not uniquely defined; in fact, s_1 can be both $+1$ and -1 for any stratified algebra. Moreover, those algebras which are both of type $\mathbf{s}$ and $-\mathbf{s}$, and consequently both of type $+\mathbf{1}$ and $-\mathbf{1}$ are said to be *fully standardly stratified algebras* and are of particular interest.

1. **Stratified algebras.** The first part of the following lemma is related to a characterization of hereditary ideals in [DR].

LEMMA 2.1. *Let e be an indecomposable idempotent and $_A(AeA)$ be projective. Then the multiplication map*

$$Ae \underset{eAe}{\otimes} eA \xrightarrow{\text{mult.}} AeA$$

is bijective, $_{eAe}eA$ is projective and $_A(AeA) \in \mathcal{F}(Ae)$. Observe that $_{eAe}(eA)$ is projective if and only if $eA \in \mathcal{F}(\bar{\Delta}(e))$ and $[AeA : Ae] = \text{length}\,\bar{\Delta}(e)$, where $\bar{\Delta}(e) = eA/e\,\mathrm{rad}\,AeA$.

Following an idea of Section 2 in [D], we can derive the following results.

THEOREM 2.2. *An algebra $(A, \mathbf{e})$ is stratified of type $\mathbf{s}$ if and only if the opposite algebra $(A^{opp}, \mathbf{e})$ is stratified of type $-\mathbf{s}$. In particular, every right projective A-module is $\bar{\Delta}$-filtered if and only if every left projective A-module is Δ-filtered.*

THEOREM 2.3. *If $(A, \mathbf{e})$ is a stratified algebra of type $\mathbf{s} = (s_1, s_2, \dots, s_n)$, then $(A, \mathbf{e})$ has a stratification in the sense of [CPS]. In particular,*

$$\dim_K A = \sum_{i=1}^{n} \frac{1}{d_i} \dim_K \Delta^{\circ s_i}(i) \cdot \dim_K \Delta^{-s_i}(i),$$

where $\Delta^{+1}(i) = \Delta(i)$, $\Delta^{\circ +1}(i) = \Delta^{\circ}(i)$, $\Delta^{-1}(i) = \bar{\Delta}(i)$, $\Delta^{\circ -1}(i) = \bar{\Delta}^{\circ}(i)$, if and only if $(A, \mathbf{e})$ is stratified of type $(s_1, s_2, \dots, s_n)$.

THEOREM 2.4 (*cf.* [D]). *Let $(A, \mathbf{e})$ be a stratified algebra. Then the following statements are equivalent:*

(*i*) *gl.dim $A < \infty$;*
(*ii*) *$\Delta(i) = \bar{\Delta}(i)$ for all $1 \le i \le n$;*
(*iii*) *$(A, \mathbf{e})$ is quasi-hereditary.*

The following is a general form of the Bernstein-Gelfand-Gelfand law.

THEOREM 2.5. *Let (A, e) be a stratified algebra of type* $\mathrm{s} = (s_1, s_2, \dots, s_n)$. *Then*

$$d_j \left[P^\circ(i) : \Delta^{\circ s_j}(j)\right] = d_i \left[\Delta^{-s_j}(j) : S(i)\right] \quad \text{for all } 1 \leq i, j \leq n.$$

In particular, if (A, e) is a standardly stratified algebra, then

$$d_j \left[P^\circ(i) : \Delta^\circ(j)\right] = d_i \left[\bar{\Delta}(j) : S(i)\right] \quad \text{for all } 1 \leq i, j \leq n.$$

Moreover, if all $d_i = d_j = 1$ (as it is the case when K is algebraically closed) then we get the "classical" BGG-reciprocity relation

$$\left[P^\circ(i) : \Delta^\circ(j)\right] = \left[\bar{\Delta}(j) : S(i)\right]$$

and the Cartan matrix $C(A)$ of A has a product decomposition

$$C(A) = \bar{\Delta}^{tr} \cdot \Delta^\circ.$$

Let us conclude this section with a canonical construction of the shallow algebra over a given K-species $\mathcal{S} = (D_i, {}_iW_j \mid 1 \leq i, j \leq n)$ (*cf.* [ADL]) and a numerical characterization of shallow algebras.

DEFINITION 2.6. A basic algebra (A, e) is said to be a *shallow stratified algebra* of type $\mathrm{s} = (s_1, s_2, \dots, s_n)$ if, for every $1 \leq i \leq n$, $\operatorname{rad} \Delta^{\circ s_i}(i)$ and $\operatorname{rad} \Delta^{-s_i}(i)$ are semisimple.

REMARK. A shallow stratified algebra (A, e) is always lean (in the sense of [ADL]) and clearly satisfies $\operatorname{rad}^3 A = 0$.

Now, given an ordered K-species $\mathcal{S}$, let us define the canonical shallow stratified algebra $S_{\mathcal{S},\mathrm{s}}$ of type $\mathrm{s} = (s_1, s_2, \dots, s_n)$ by $S_{\mathcal{S},\mathrm{s}} = T\mathcal{S}/I$, where I is generated by all ${}_iW_j \otimes {}_jW_k$ with $j < \max\{i, k\}$ or $i + k + s_j(k - i) = 2j$. Thus, the canonical shallow algebra of type $(+1, +1, \dots, +1)$ and of type $(-1, -1, \dots, -1)$ are $T\mathcal{S}/\langle {}_iW_j \otimes {}_jW_k \mid j < \max\{i, k\} \text{ or } j = k\rangle$ and $T\mathcal{S}/\langle {}_iW_j \otimes {}_jW_k \mid j < \max\{i, k\} \text{ or } i = j\rangle$.

THEOREM 2.7. *Let $\mathcal{S} = \mathcal{S}_A$ be the ordered species of a stratified algebra (A, e) of type* s. *Then*

$$\dim_K A \geq \dim_K S_{\mathcal{S},\mathrm{s}}.$$

Moreover, if $\dim_K A = \dim_K S_{\mathcal{S},\mathrm{s}}$, then A is shallow.

There is a similar construction of a canonical fully standardly stratified shallow algebra $S_{\mathcal{S}}$ over an ordered species $\mathcal{S}$:

$$S_{\mathcal{S}} = T\mathcal{S}/\langle {}_iW_j \otimes {}_jW_k \mid j < \max i, k \text{ or } i = j = k\rangle.$$

Here, $\operatorname{rad}^4 S_{\mathcal{S}} = 0$ and an analogue of Theorem 2.7 holds.

As an illustration of the preceding constructions, let us remark that the shallow algebras over a complete quiver on n vertices (*i.e.*, an oriented graph with an arrow from i to j for all pairs $1 \leq i, j \leq n$, including the loops from i to i) have the following dimensions:

$$\dim S_{Q,\mathbf{s}} = \frac{n(n+1)(n+2)}{3} \quad \text{for all types } \mathbf{s} \text{ and}$$
$$\dim S_Q = \frac{n(n+1)(2n+1)}{3}.$$

2. Standardly stratified algebras. In this section we sketch the proof of the main result characterizing standardly stratified algebras. For situations where $\Delta = \bar{\Delta}$, this result reduces to a characterization of quasi-hereditary algebras (*cf.* for example Theorem A.2.6 of [DK]).

THEOREM 3.1. *Let $(A, \mathbf{e})$ be a K-algebra. Then the following statements are equivalent:*

(*i*) *$(A, \mathbf{e})$ is standardly stratified.*
(*ii*) $\mathrm{Ext}_A^2(\bar{\Delta}(i), \nabla(j)) = 0$ *for all* $1 \leq i, j \leq n$.
(*iii*) $\mathcal{F}(\bar{\Delta}) = \{X \mid \mathrm{Ext}_A^1(X, \nabla(j)) = 0$ *for all* $1 \leq j \leq n\}$.
(*ii*)′ $\mathrm{Ext}_A^k(\bar{\Delta}(i), \nabla(j)) = 0$ *for all* $1 \leq i, j \leq n$ *and all* $k \geq 1$.
(*iii*)′ $\mathcal{F}(\bar{\Delta}) = \{X \mid \mathrm{Ext}_A^k(X, \nabla(j)) = 0$ *for all* $1 \leq j \leq n$ *and all* $k \geq 1\}$.

In the proof we will need the following lemma (*cf.* Proposition A.2.3 of [DK]).

LEMMA 3.2. *The category $\mathcal{F}(\bar{\Delta})$ is closed under kernels of epimorphisms.*

PROOF. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence with $Y, Z \in \mathcal{F}(\bar{\Delta})$. First of all, it is clearly enough to prove that $X \in \mathcal{F}(\bar{\Delta})$ if $Y \in \mathcal{F}(\bar{\Delta})$ and $Z \simeq \bar{\Delta}(j)$ for some $1 \leq j \leq n$; the repeated application of this result will yield the general result. To prove this statement, we shall proceed by induction on the length of the $\bar{\Delta}$-filtration of Y, the initial steps being trivial. (Note that the length of the $\bar{\Delta}$-filtration is an invariant of any module in $\mathcal{F}(\bar{\Delta})$.)

Thus assume that Y has a $\bar{\Delta}$-filtration $0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_\ell$ which is the refinement of the proper standard filtration of the module Y. In particular, if $Y/X \simeq \bar{\Delta}(j)$ and $Y_1 \simeq \bar{\Delta}(k)$ then $j \leq k$.

If $Y_1 \subseteq X$ then we get the following exact sequence: $0 \to X/Y_1 \to Y/Y_1 \to \bar{\Delta}(j) \to 0$, and here the length of the $\bar{\Delta}$-filtration of Y/Y_1 is smaller than that of Y. Hence, by induction, we get that $X/Y_1 \in \mathcal{F}(\bar{\Delta})$ and this immediately yields that $X \in \mathcal{F}(\bar{\Delta})$.

If $Y_1 \not\subseteq X$ then the natural projection $Y \to Y/X = \bar{\Delta}(j)$ gives a non-zero map $\varphi: Y_1 = \Delta(k) \to \bar{\Delta}(j)$. But the existence of such a map implies that $j = k$ and in this case φ is an isomorphism. Hence $X \cap Y_1 = 0$, thus X embeds into Y/Y_1. A simple dimension argument shows that this embedding is an isomorphism, too, *i.e.*, $X \simeq Y/Y_1 \in \mathcal{F}(\bar{\Delta})$. ∎

PROOF OF THEOREM 3.1. In what follows, we shall freely use the following simple facts: $\operatorname{Ext}_A^1(\bar{\Delta}(i), \nabla(j)) = 0$ for $1 \leq i, j \leq n$, and furthermore $\operatorname{Ext}_A^1(S(i), \nabla(j)) = 0$, whenever $i \leq j$.

Let us observe first that the implications $(ii)' \Rightarrow (ii)$, $(iii)' \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are trivial, while the implications $(i) \Rightarrow (ii)'$ and $(iii) \Rightarrow (iii)'$ can be proved using Lemma 3.2 and a clear induction.

Thus, we shall concentrate on proving $(ii) \Rightarrow (iii)$. We would like to show that $\mathcal{F}(\bar{\Delta}) \supseteq \{X \mid \operatorname{Ext}_A^1(X, \nabla(j)) = 0 \text{ for all } 1 \leq j \leq n\} = \mathcal{F}$, the opposite inclusion being trivial.

Let Y be a module from $\mathcal{F}$. We shall prove by induction on the composition length of the module Y that $Y \in \mathcal{F}(\bar{\Delta})$.

Let i be the maximal index for which $[Y : S(i)] \neq 0$, and let $X \subseteq Y$ be a minimal submodule containing a composition factor isomorphic to $S(i)$. Then clearly, X is local, containing $S(i)$ in its top only, with all composition factors having smaller index. Thus we have the following exact sequences:

$$0 \to X \to Y \to Z \to 0, \tag{3.1.1}$$

$$0 \to X' \to \bar{\Delta}(i) \to X \to 0. \tag{3.1.2}$$

We want to show first that both Z and X belong to $\mathcal{F}$. Let us consider the long exact sequence obtained from (3.1.1) by applying $\operatorname{Hom}_A(-, \nabla(j))$:

$$\begin{array}{cccccccc} & & & \cdots & \to & \operatorname{Hom}_A(X, \nabla(j)) & \to \\ \to & \operatorname{Ext}_A^1(Z, \nabla(j)) & \to & \operatorname{Ext}_A^1(Y, \nabla(j)) & \to & \operatorname{Ext}_A^1(X, \nabla(j)) & \to \\ \to & \operatorname{Ext}_A^2(Z, \nabla(j)) & \to & \cdots & & & \end{array}$$

Here $\operatorname{Ext}_A^1(Z, \nabla(j)) = 0$ is automatically satisfied for $j \geq i$ since all the composition factors of Z have index at most i, and $\operatorname{Ext}_A^1(S(k), \nabla(j)) = 0$ for $k \leq j$. On the other hand, if $j < i$, then $\operatorname{Hom}_A(X, \nabla(j)) = 0$ (since $\operatorname{top} X \simeq S(i)$), which together with the assumption that $\operatorname{Ext}_A^1(Y, \nabla(j)) = 0$ yields that $\operatorname{Ext}_A^1(Z, \nabla(j)) = 0$. Thus, $Z \in \mathcal{F}$. Since $\ell(Z) < \ell(Y)$, by induction we get that $Z \in \mathcal{F}(\bar{\Delta})$. Furthermore, this implies, using condition (ii), that $\operatorname{Ext}_A^2(Z, \nabla(j)) = 0$ for every j, hence we obtain that $X \in \mathcal{F}$, as well.

To finish the proof, we have to show that $X \in \mathcal{F}(\bar{\Delta})$. It is clearly enough to show, that $X \simeq \bar{\Delta}(i)$, that is, $X' = 0$. But if $X' \neq 0$, then we would get a non-split extension of X with $S(j)$ for some $j < i$. The pushout of this extension along the natural embedding map $S(j) \to \nabla(j)$ would result in a non-split extension of X with $\nabla(j)$, a contradiction. (The fact that the resulting sequence does not split can be seen as follows: the middle term of the original non-split extension of X with $S(j)$ has a simple top $S(i)$ with $i > j$, hence it has no non-trivial homomorphisms into $\nabla(j)$). This shows that $X \in \mathcal{F}(\bar{\Delta})$, and hence that $Y \in \mathcal{F}(\bar{\Delta})$. ∎

REMARK. There is an analogous characterization of stratified algebras of type $s = (s_1, s_2, \ldots, s_n)$; these are the algebras satisfying:

$$\mathrm{Ext}_A^2(\Delta^{-s_i}(i), \nabla^{s_i}(j)) = 0 \quad \text{for all } 1 \le i, j \le n.$$

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WIEFERICH PRIMES AND HALL'S CONJECTURE

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RÉSUMÉ. Nous montrons que la Conjecture de Hall pour courbes elliptiques implique l'existence d'une infinité de nombres premiers p satisfaisant $a^{p-1} \not\equiv 1 \pmod{p^{16}}$, pour chaque nombre entier $a > 1$.

In the long and eventful history of Fermat's last theorem, the following criterion stands out:

If a prime p satisfies

$$2^{p-1} \not\equiv 1 \pmod{p^2},$$

then the first case of Fermat's last theorem is true for the exponent p.

This criterion, proven by Wieferich [1] in 1909, was subsequently generalized by a number of authors to allow 2 to be replaced by any prime $q \leq 89$. (See [2].) Computational evidence shows that the only primes $p \leq 4 \times 10^{12}$ satisfying $2^{p-1} \equiv 1 \pmod{p^2}$ are $p = 1093$ and 3511. Furthermore, for any integer a, the primes satisfying $a^{p-1} \equiv 1 \pmod{p^2}$ are very few. (Tables listing all solutions of $a^{p-1} \equiv 1 \pmod{p^2}$ for $2 \leq a \leq 99$, $3 \leq p \leq 2^{32}$ are listed in [3], pp. 347–349. The smallest known example where this congruence is satisfied is the case of $3^{10} \equiv 1 \pmod{11^2}$.) However, the following conjecture remains outstanding:

CONJECTURE 1. For any $a \in \mathbb{Z}$, $a > 1$, there exist infinitely many primes p such that

$$a^{p-1} \not\equiv 1 \pmod{p^2}.$$

It is not known whether the assertion holds for any fixed integer a. Neither is the truth of the following weaker assertion.

CONJECTURE 2. For a given $a \in \mathbb{Z}$, $a > 1$, there exists a natural number n such that there exist infinitely many primes p such that

$$a^{p-1} \not\equiv 1 \pmod{p^n}.$$

Silverman [4] has shown that the ABC-conjecture of Masser and Oesterlé implies Conjecture 1. More precisely, he showed that the ABC-conjecture implies that

$$\#\{p \leq x \mid a^{p-1} \not\equiv 1 \pmod{p^2}\} \gg_a \log x \text{ as } x \to \infty.$$

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Another consequence of the ABC-conjecture is the following conjecture of Hall [5]:

CONJECTURE 3 (HALL). Let $\epsilon > 0$ be given. Then for any nonzero integer D, every solution $x, y \in \mathbb{Z}$ to the equation

$$y^2 = x^3 + D$$

satisfies

$$|x| \ll_\epsilon |D|^{2+\epsilon}.$$

We will show in this note that Conjecture 3 (which is weaker than the ABC-conjecture) implies Conjecture 2.

THEOREM. *If Hall's Conjecture (Conjecture 3) is true, then*

$$\#\{p \leq x \mid a^{p-1} \not\equiv 1 \pmod{p^{16}}\} \gg_a \frac{\log x}{\log\log x} \text{ for any integer } a > 1.$$

We require two elementary lemmas.

DEFINITION. Let $n \in \mathbb{Z}$. We will call n d^{th}-powerful if, for every $p|n$, we have $p^d|n$.

LEMMA 1. Let $n \in \mathbb{Z}$ be d^{th}-powerful. Then n may be written in the form

$$n = \begin{cases} \alpha^2\beta^d & \text{if } d \text{ is odd} \\ \alpha^2\beta^{d+1} & \text{if } d \text{ is even,} \end{cases}$$

for some integers α, β, with β squarefree.

PROOF. Let $n = \prod_{i \in S} p_i^{e_i}$ be the canonical factorization of n. By hypothesis, each e_i is greater than or equal to d.

If d is odd, we may write

$$n = \prod_{e_i \text{ even}} p_i^{e_i} \prod_{e_j \text{odd}} p_j^{e_j} = \prod_{e_i \text{even}} p_i^{e_i} \prod_{e_j \text{odd}} p_j^{e_j - d} \prod_{e_j \text{odd}} p_j^d.$$

$$= \Big(\prod_{e_i \text{even}} p_i^{\frac{e_i}{2}} \prod_{e_j \text{odd}} p_j^{\frac{e_j - d}{2}}\Big)^2 \Big(\prod_{e_j \text{odd}} p_j\Big)^d$$

If d is even, write

$$n = \prod_{e_i = d} p_i^{e_i} \cdot m,$$

where $m = \prod_{e_i > d} p_i^{e_i}$ is $(d+1)^{th}$-powerful. By the above, $m = \alpha^2\beta^{d+1}$, for some $\alpha, \beta \in \mathbb{Z}$, β squarefree. Therefore,

$$n = \Big(\prod_{e_i = d} p_i^{d/2}\alpha\Big)^2 \beta^{d+1}.$$

LEMMA 2. *Let $a > 1$ be an integer. Suppose that $p^t \| a^n - 1$ with $t \geq 1$. (i.e., p^t is the highest power of p dividing $a^n - 1$.) Then $a^{p-1} \not\equiv 1 \pmod{p^{t+1}}$.*

PROOF. Let d denote the multiplicative order of $a \bmod p$. Then $a^n \equiv 1 \pmod{p} \Rightarrow d|n \Rightarrow a^d - 1|a^n - 1$ so that $a^d \not\equiv 1 \pmod{p^{t+1}}$.

Let $p^s \| a^d - 1$ (so that $1 \leq s \leq t$) and write $a^d = 1 + kp^s$, where $p \not| k$. Now, $d|p-1$, say $p - 1 = df$. Thus,

$$\begin{aligned} a^{p-1} &= (a^d)^f = (1 + kp^s)^f \\ &= \sum_{i=0}^{f} \binom{f}{i} (kp^s)^i \\ &\equiv 1 + fkp^s \pmod{p^{s+1}} \\ &\not\equiv 1 \pmod{p^{t+1}} \end{aligned}$$

PROOF (OF THEOREM). We show that, for any $a > 1$, $a^{3k+1} - 1$ is not 16^{th}-powerful, for all k sufficiently large. By the previous lemma, this will establish that, for all $p|a^{3k+1} - 1, a^{p-1} \not\equiv 1 \pmod{p^{16}}$.

Suppose that $a^{3k+1} - 1 = \alpha_k^2 \beta_k^d$, where d is odd. (By Lemma 1, any integer that is at least $(d-1)^{th}$-powerful may be written in this form.) Then, multiplying through by $a^2\beta_k^3$, we obtain

$$a^{3k+3}\beta_k^3 - a^2\beta_k^3 = a^2\alpha_k^2\beta_k^{d+3}$$

or

$$(a^{k+1}\beta_k)^3 - a^2\beta_k^3 = (a\alpha_k\beta_k^{\frac{d+3}{2}})^2.$$

Applying Hall's Conjecture, we have

$$a^{k+1}\beta_k \ll_\epsilon (a^2\beta_k^3)^{2+\epsilon} \quad \text{or} \quad a^{k+1} \ll_{\epsilon,a} \beta_k^{5+\epsilon}.$$

Then

$$\beta_k^d < \alpha_k^2\beta_k^d < a^{3k+1} - 1 < (a^{k+1})^3 \ll_{\epsilon,a} \beta_k^{15+3\epsilon}.$$

Now, for fixed β, the curve $\beta^d y^2 = ax^3 - 1$ has but finitely many integral points. Therefore, $\beta_k \to \infty$ as $k \to \infty$. We can thus conclude from the above inequality that $d \leq 15$, for all sufficiently large k. Thus, for all sufficiently large k, $a^{3k+1} - 1$ is not 16^{th}-powerful $\Rightarrow$ for all $p|a^{3k+1} - 1$, $a^{p-1} \not\equiv 1 \pmod{p^{16}}$. Since for $\gcd(3k+1, 3l+1) = 1$, $\gcd(a^{3k+1} - 1, a^{3l+1} - 1) = a - 1$, every sufficiently large prime $q \equiv 1 \pmod 3$ yields a distinct prime $p|\frac{a^q - 1}{a-1}$. There are thus, infinitely many primes p such that $a^{p-1} \not\equiv 1 \pmod{p^{16}}$. In fact,

$$\begin{aligned} \#\{p \leq x \mid a^{p-1} \not\equiv 1 \pmod{p^{16}}\} &\gg \#\{p \equiv 1 \pmod 3 \mid a^p - 1 \leq x\} \\ &\gg \#\{p \equiv 1 \pmod 3 \mid p \leq \frac{\log x}{\log a}\} \\ &\gg_a \frac{\log x}{\log\log x} \text{ as } x \to \infty, \end{aligned}$$

by the prime number theorem for arithmetic progressions.

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