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DUALITY FOR A FAMILY OF NONLINEAR FRACTIONAL PROGRAMMING PROBLEMS

by

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Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT: Duality, an important concept in mathematical programming, is being investigated for a class of fractional programming problems. A dual is constrained as a function of the primal variables and the duality theorems are established.

1. INTRODUCTION

A group of functions for which local optimum is global optimum, and for which local optimum occurs at an extreme point of the constraint set are, as is well known, quasimonotonic in nature [4]. A family of objective functions of our primal problems is a subgroup of the group of quasimonotonic functions. The constraint set of the primal problem is linear in nature with a finite number of extreme points. Motivated by [1,3], we have formulated a dual, which is constrained as a function of the primal variables. The desired relationship between the primal and the dual problem is established with the help of duality theorems.

2. STATEMENT OF THE PROBLEMS AND ASSUMPTIONS.

We take the following problem as our primal problem:

$$\text{Minimize: } f(x) = \frac{\sqrt{Cx+C_0}}{\sqrt{Dx+D_0}} \equiv \frac{G(x)}{H(x)} \tag{1}$$

Subject to: $x \in L$, where $L = \{x | Ax \geq b,\ x \geq 0\}$

Here A is an mxn matrix, x and b are column vectors with n and m components, respectively, C and D are row vectors with n components; C_0 and D_0 are constants.

Furthermore, we assume that, problem (1) is nondegenerate; $Cx + C_0$ and $Dx + D_0$ are positive for x in L, and that $f(x)$ is differentiable. We present the following dual problem:

Maximize: t

Subject to: $uA + t^2 \frac{D}{H(x)} \le \frac{C}{H(x)}$ (2)

$$-ub + t^2 \frac{D_0}{H(x)} \le \frac{C_0}{H(x)}$$

$u, t \ge 0$. Let M denote the set of all feasible solutions of the dual problem.

We shall now establish the following lemmas:

Lemma 1. The function *f(x)* is quasimonotonic over the set L.

Proof: Suppose x_1 and x_2 are in L with $f(x_1) \ge f(x_2)$.

By Martos [4], it suffices to show that $f(x_1) \ge f(x_0) \ge f(x_2)$

for all $x_0 = \lambda x_1 + (1-\lambda)x_2$ where $0 \le \lambda \le 1$.

Now, $f(x_1) \ge f(x_2)$ implies that $\frac{\sqrt{Cx_1 + C_0}}{\sqrt{Dx_1 + D_0}} \ge \frac{\sqrt{Cx_2 + C_0}}{\sqrt{Dx_2 + D_0}}$, which means $\frac{Cx_1 + C_0}{Dx_1 + D_0} \ge \frac{Cx_2 + C_0}{Dx_2 + D_0}$ or $(Cx_1 + C_0)(Dx_2 + D_0) \ge (Cx_2 + C_0)(Dx_1 + D_0)$.

Now, $(Cx_1 + C_0)(Dx_0 + D_0) - (Cx_0 + C_0)(Dx_1 + D_0)$

$$= (Cx_1 + C_0)(D[\lambda x_1 + (1-\lambda)x_2] + D_0) - (C[\lambda x_1 + (1-\lambda)x_2] + C_0)(Dx_1 + D_0)$$

$$= \lambda(Cx_1 + C_0)(Dx_1 + D_0) + (1-\lambda)(Cx_1 + C_0)(Dx_2 + D_0) -$$

$$\lambda(Cx_1 + C_0)(Dx_1 + D_0) - (1-\lambda)(Cx_2 + C_0)(Dx_1 + D_0)$$

$$= (1-\lambda)[(Cx_1 + C_0)(Dx_2 + D_0) - (Cx_2 + C_0)(Dx_1 + D_0)] \ge 1-\lambda$$

$\ge$ 0. Thus, $(Cx_1 + Cx_0)(Dx_0 + D_0) \ge (Cx_0 + C_0)(Dx_1 + D_0)$

which implies that

$\frac{Cx_1 + C_0}{Dx_1 + D_0} \ge \frac{Cx_0 + C_0}{Dx_0 + D_0}$; or $\frac{\sqrt{Cx_1 + C_0}}{\sqrt{Dx_1 + D_0}} \ge \frac{\sqrt{Cx_0 + C_0}}{\sqrt{Dx_0 + D_0}}$; which proves that $f(x_1) \ge f(x_0)$.

An analogous argument will show that $f(x_2) \le f(x_0)$.

Remark: For a differentiable function, $f(x)$, it has been shown by Martos [4] that the above criterion for quasimonotonicity has the following simple form:

$f(x_1) \ge f(x_2)$ implies $f_x(x_0)(x_1 - x_2) \ge 0$, for all $x_0 = \lambda x_1 + (1-\lambda)x_2,\ 0 \le \lambda \le 1$;

$f_x(x_0) = \left[\frac{\delta f(x_0)}{\delta x_i}\right]$ is the gradient row vector of $f(x)$ at x_0.

Lemma 2: Let $x = \left(x_B^T, 0\right)^T$ be an optimal solution to the primal problem. Then for all columns a_j of the matrix $(A,-I)$ the following holds:

$$(D_B x_B + D_0)(C_j - C_B B^{-1} a_j) \ge (C_B x_B + C_0)(D_j - D_B B^{-1} a_j) \tag{3}$$

Here C_j and D_j are the jth components of the vectors C and D respectively; B, C_B, and D_B are the submatrices of A, C, and D corresponding to the optimal basic feasible solution x. T is used to denote the transpose of a matrix.

Proof: In light of Lemma 1 and [4], it follows that the optimum solution of the primal problem will occur at a basic feasible solution of the set L. Let that basic feasible solution be $x = \left(x_B^T, 0\right)^T$ with $B = (a_1, a_2, \ldots, a_m)$ being the corresponding basic matrix and $x_B = B^{-1}b$. An improved basic feasible solution $\bar{x}$, formed by replacing column a_r with column a_j, is given by [2]

$$\bar{x}_B = x_B - \theta\alpha_j \text{ and } x_j = \theta.\ \alpha_j = B^{-1}a_j \text{ and } \theta = \frac{x_{Br}}{\alpha_{rj}}. \tag{4}$$

Inserting the 0-components corresponding to the nonbasic variables, (4) is the same as

$\bar{x} = x - \theta\bar{\alpha}_j + \theta e_j$ where $\bar{\alpha}_j = \left(\alpha_j^T, 0\right)^T$ and e_j is a unit column vector of n components with 1 in the j^{th} component. Because $f(x)$ is quasimonotonic and differentiable, it follows that, $f(\bar{x}) \ge f(x)$ implies that

$$0 \le f_x(x)(\bar{x}-x) = f_x(x_B)\theta(e_j-\alpha_j)^T = f_{x_j}(x_B) - f_x(x_B)\alpha_j^T .$$

Thus, under the non-degeneracy assumption, if for some j; $f_{x_j}(x_B) - f_x(x_B)\alpha_j^T < 0$ then $f(x) > f(\bar{x})$, showing that the insertion of column a_j into the basis will decrease the value of the objective function. Consequently, if $x = \left(x_B^T, 0\right)^T$ is an optimal solution to the primal problem, then inequality (3) will be true.

3. DUALITY THEOREMS.

Theorem 1. If x is any feasible solution of the primal problem and (u,t) is any feasible solution of the dual problem, then $t \le f(x)$.

Proof: Consider any (u,t) of M and any x of L.

By (2) and the fact that $x \ge 0$, it follows that $uAx + \frac{t^2 Dx}{H(x)} \le \frac{Cx}{H(x)}$.

Now, $Ax \ge b$ and $u \ge 0$ imply that $uAx \ge ub$. Substituting in the above inequality yields

$$ub + \frac{t^2 Dx}{H(x)} \le \frac{Cx}{H(x)} . \tag{5}$$

By (2), $-ub + \frac{t^2 D_0}{H(x)} \le \frac{C_0}{H(x)}$ and therefore $ub \ge \frac{t^2 D_0}{H(x)} - \frac{C_0}{H(x)}$. (6)

Combining (5) and (6) we get $\frac{t^2 D_0}{H(x)} - \frac{C_0}{H(x)} + \frac{t^2 Dx}{H(x)} \le \frac{Cx}{H(x)}$,

which yields $t^2 \le \frac{Cx+C_0}{Dx+D_0}$ and $t \le \frac{\sqrt{Cx+C_0}}{\sqrt{Dx+D_0}}$, which proves the theorem.

Theorem 2. If for any x^* of L and for any (u^*,t^*) of M, $f(x^*) = g(u^*,t^*)$; then x^* minimizes the primal problem and (u^*,t^*) maximizes the dual problem.

Proof: From Theorem 1, $g(u,t) \le f(x)$, for all (u,t) of M and all x of L. Therefore,

$g(u,t) \le f(x^*)$ for all (u,t) of M. Since $f(x^*) = g(u^*,t^*)$, we have $g(u,t) \le g(u^*,t^*)$ for all (u,t) of M implying that (u^*,t^*) maximizes the dual problem. The other part follows similarly.

Theorem 3: If $x_0 = \left(x_B^T, 0\right)^T$ is the optimal solution to the primal problem, then the corresponding optimal solution of the dual problem is given by

$$u_0 = \left[\frac{C_B}{\sqrt{D_B B^{-1} b + D_0}} - \frac{C_B B^{-1} b + C_0}{D_B B^{-1} b + D_0} \cdot \frac{D_B}{\sqrt{D_B B^{-1} b + D_0}}\right] \cdot B^{-1}$$

$$t_0 = \frac{\sqrt{C_B x_B + C_0}}{\sqrt{D_B x_B + D_0}}$$

Proof: Since x_0 is the optimal solution to the primal problem, it follows from Lemma 2 that

$(D_B x_B + D_0)\left(C - C_B B^{-1} A\right) - (C_B x_B + C_0)\left(D - D_B B^{-1} A\right) \ge 0$. This is the same as

$C \ge C_B B^{-1} A + \left(\frac{C_B x_B + C_0}{D_B x_B + D_0}\right)\left(D - D_B B^{-1} A\right)$, or

$C \ge \left(\frac{C_B x_B + C_0}{D_B x_B + D_0}\right) D + \left[C_B - \left(\frac{C_B x_B + C_0}{D_B x_B + D_0}\right) D_B\right] B^{-1} A$, or

$C \ge t_0^2 D + \left[C_B - \left(\frac{C_B x_B + C_0}{D_B x_B + D_0}\right) D_B\right] B^{-1} A$. Dividing by $\sqrt{D_B x_B + D_0}$ yields

$$\left[\frac{C_B}{\sqrt{D_B x_B + D_0}} - \left(\frac{C_B x_B + C_0}{D_B x_B + D_0}\right) \frac{D_B}{\sqrt{D_B x_B + D_0}}\right] B^{-1} A + \frac{t_0^2 D}{\sqrt{D_B x_B + D_0}} \le \frac{C}{\sqrt{D_B x_B + D_0}};$$

and $x_B = B^{-1} b$ gives $u_0 A + t_0^2 \frac{D}{H(x_0)} \le \frac{C}{H(x_0)}$.

Using $D_B x_B = D x_0$, $C_B x_B = C x_0$ and $x_B = B^{-1} b$ we can write u_0 as

$$u_0 = \left[\frac{C_B}{\sqrt{D x_0 + D_0}} - \left(\frac{C x_0 + C_0}{D x_0 + D_0}\right) \frac{D_B}{\sqrt{D x_0 + D_0}}\right] B^{-1}$$

Therefore, $u_0 b = \frac{Cx_0}{H(x_0)} - t_0^2 \frac{Dx_0}{H(x_0)}$.

$$-u_0 b + \frac{t_0^2 D_0}{H(x_0)} = \frac{-Cx_0}{H(x_0)} + t_0^2\left(\frac{Dx_0 + D_0}{H(x_0)}\right) = \frac{-Cx_0}{H(x_0)} + \left(\frac{Cx_0 + C_0}{Dx_0 + D_0}\right)\left(\frac{Dx_0 + D_0}{H(x_0)}\right) = \frac{C_0}{H(x_0)}.$$

Thus, $-ub + \frac{t^2 D_0}{H(x)} \le \frac{C_0}{H(x)}$ holds as an equality.

Finally, the constraints of the primal problem can be written as $Ax - x_s = b$. The optimality of x_0 requires that the condition of Lemma 2 holds for all columns e_i associated with the surplus variables x_{s_i}. This reduces to the condition that u_0 defined above is indeed non-negative. Thus the solution (u_0, t_0), as defined above, is feasible for the dual problem and in light of Theorems 1 and 2 is indeed optimal.

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EXAMPLES OF NON-KÄHLER, ALMOST KÄHLER SYMMETRIC SPACES

Presented by P.A. Fillmore, F.R.S.C.

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ABSTRACT. Concerning the Goldberg conjecture, it is worthwhile to study integrability of almost Kähler structures on a symmetric space. Few examples of non-Kähler almost Kähler symmetric spaces are known. In the present paper, we construct uncountably many almost Kähler structures on the Riemannian product of the hyperbolic space and the Euclidean space.

1. INTRODUCTION

An almost Hermitian manifold $M = (M, J, g)$ is called an *almost Kähler* manifold if the corresponding Kähler form Ω is closed, and is a *Kähler* manifold if the almost complex structure J is parallel with respect to the Levi-Civita connection ∇ of the Riemannian metric g. The almost Kähler condition $d\Omega = 0$ is equivalent to $\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0$ for all vector fields X, Y, $Z \in \mathfrak{X}(M)$, where $\mathfrak{S}$ denotes cyclic sum. Hence, a Kähler manifold is necessarily an almost Kähler manifold, and an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. A non-Kähler almost Kähler manifold is called a *strictly almost Kähler* manifold. Examples of strictly almost Kähler manifold have been constructed by many authors ([1], [2], [8], [16] and so on). Concerning integrability of almost Kähler manifolds, the following conjecture of Goldberg is still open ([5]): The almost complex structure of a compact almost Kähler-Einstein manifold is integrable. Some progress has been made by several authors under some additional curvature conditions ([3], [4], [10], [13], [14], [15] and so on).

Taking account of the fact that an irreducible symmetric space is an Einstein space, it is worthwhile to study the integrability of almost Kähler structures on a locally symmetric space. Murakoshi, Sekigawa, and the author proved that a four-dimensional compact almost Kähler locally symmetric space is a Kähler manifold ([9]). Moreover, Sekigawa and the author constructed the first example of a strictly almost Kähler structure on a symmetric space, which is the one on $\mathbb{H}^3 \times \mathbb{R}$, the Riemannian product manifold of the three-dimensional hyperbolic space and the real line ([11]). However, to my knowledge, there are no known examples of strictly almost Kähler symmetric spaces of dimension ≥ 6.

In the present paper, we study almost Kähler structures on $\mathbb{H}^m \times \mathbb{R}^{2n-m}$, the Riemannian product of the m-dimensional hyperbolic space and the $(2n - m)$-dimensional Euclidean space, and construct uncountably many strictly almost Kähler structures on $\mathbb{H}^3 \times \mathbb{R}^{2n-3}$ $(2n \geq 6)$.

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Gromov showed that on a non-compact, connected manifold M, any almost complex structure is realized by a symplectic form (with arbitrary choice of the element in the 2-dimensional de Rham cohomology group) ([7]). However, for a specified Riemannian metric on M, it is not always true that M admits an almost complex structure J such that (J, g) is an almost Kähler structure. In fact, in Section 3, we shall prove that $\mathbb{H}^m \times \mathbb{R}^{2n-m}$ admits an almost Kähler structure only when $m = 3$.

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2. Preliminaries

We denote by $\mathbb{H}^m$ and $\mathbb{R}^{2n-m}$ the m-dimensional hyperbolic space of constant sectional curvature -1 and the $(2n-m)$-dimensional Euclidean space, respectively. We regard the Riemannian product manifold $\mathbb{H}^m \times \mathbb{R}^{2n-m}$ as the Riemannian manifold $M_m = (\mathbb{R}^{2n}_+, g)$, where

$$\mathbb{R}^{2n}_+ = \{ (x_1, \dots, x_m, x_{m+1}, \dots, x_{2n}) \in \mathbb{R}^{2n} \mid x_1 > 0 \},$$

$$g = \frac{1}{x_1^2} \sum_{i=1}^{m} dx_i \otimes dx_i + \sum_{i=m+1}^{2n} dx_i \otimes dx_i.$$

We put $X_i = x_1(\partial/\partial x_i)$ $(i = 1, \dots, m)$ and $X_i = \partial/\partial x_i$ $(i = m+1, \dots, 2n)$. Then $\{X_1, \dots, X_{2n}\}$ forms a global orthonormal frame field on M_m. Let ∇ be the Levi-Civita connection of the Riemannian metric g and put $\Gamma_{ijk} = g(\nabla_{X_i} X_j, X_k)$. Then

$$\Gamma_{ii1} = -\Gamma_{i1i} = 1 \tag{2.1}$$

for $i = 2, \dots, m$ and are otherwise zero. We denote by R the Riemannian curvature tensor, where we assume that R is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for $X, Y \in \mathfrak{X}(M_m)$. The components $R_{ijkl} = g(R(X_i, X_j)X_k, X_l)$ of R are given by

$$R_{ijkl} = -(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \tag{2.2}$$

for $1 \leq i, j, k, l \leq m$ and are otherwise zero.

Now, let (J, g) be an almost Kähler structure on M_m. We put $J_{ij} = g(JX_i, X_j)$ and $\nabla_i J_{jk} = g((\nabla_{X_i} J)X_j, X_k)$ for $1 \leq i, j, k \leq 2n$. Then, it is obvious that

$$J_{ij} = -J_{ji}, \qquad \sum_{k=1}^{2n} J_{ik} J_{jk} = \delta_{ij} \tag{2.3}$$

for $1 \leq i, j \leq 2n$. Furthermore, from (2.1), we see that the almost Kähler condition $\mathfrak{S}_{i,j,k} \nabla_i J_{jk} = 0$ is equivalent to the following first order partial differential equations:

$$\begin{aligned}
&X_1 J_{jk} + X_j J_{k1} + X_k J_{1j} - 2J_{jk} = 0, && \text{if} \quad 1 < j < k \leq m, \\
&X_1 J_{jk} + X_j J_{k1} + X_k J_{1j} - J_{jk} = 0, && \text{if} \quad 1 < j \leq m < k \leq 2n, \\
&X_1 J_{jk} + X_j J_{k1} + X_k J_{1j} = 0, && \text{if} \quad m < j < k \leq 2n, \\
&X_i J_{jk} + X_j J_{ki} + X_k J_{ij} = 0, && \text{if} \quad 1 < i < j < k \leq 2n.
\end{aligned} \tag{2.4}$$

3. Examples

We assume that (M_m, g) admits an almost Kähler structure (J, g). Since our aim is to extend the example in [11], in the rest of the present paper, we assume that the smooth functions J_{ij} on M are functions on $\mathbb{R}^{2n-m}$, namely $J_{ij} = J_{ij}(x_{m+1}, \dots, x_{2n})$. Then, the system (2.4) reduces to the following:

$$
\begin{array}{lll}
 & J_{jk} = 0, & \text{if } \ 1 < j < k \le m, \\
 & X_k J_{1j} = J_{jk}, & \text{if } \ 1 < j \le m < k \le 2n, \\
 & X_j J_{k1} + X_k J_{1j} = 0, & \text{if } \ m < j < k \le 2n, \\
(3.1) & X_i J_{jk} + X_j J_{ki} + X_k J_{ij} = 0, & \text{if } \ m < i < j < k \le 2n, \\
 & X_j J_{ki} + X_k J_{ij} = 0, & \text{if } \ 1 < i \le m < j < k \le 2n, \\
 & X_k J_{ij} = 0, & \text{if } \ 1 < i < j \le m < k \le 2n.
\end{array}
$$

First of all, we shall prove the following.

Proposition. *If the Riemannian manifold (M_m, g) admits an almost Kähler structure (J, g), then the dimension m of the hyperbolic space in M_m must be three.*

Proof. We recall the curvature identity due to Gray [5]:

$$R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} = 2\sum_{a=1}^{2n} (\nabla_a J_{ij}) \nabla_a J_{kl}$$

for $1 \le i, j, k, l \le 2n$, where $R_{ij\bar{k}\bar{l}} = g(R(X_i, X_j)JX_k, JX_l)$ and so on. In particular,

$$R_{ijij} - 2R_{ij\bar{i}\bar{j}} + R_{\bar{i}\bar{j}\bar{i}\bar{j}} + R_{\bar{i}j\bar{i}j} + 2R_{\bar{i}ji\bar{j}} + R_{i\bar{j}i\bar{j}} = 2\sum_{a=1}^{2n} (\nabla_a J_{ij})^2. \tag{3.2}$$

From (2.1)~(2.3) and (3.2),

$$
\begin{aligned}
1 - 4J_{ij}^2 &+ \sum_{a=1}^{m} J_{ia}^2 + \sum_{a=1}^{m} J_{ja}^2 + \sum_{a,b=1}^{m} J_{ia}^2 J_{jb}^2 - \sum_{a,b=1}^{m} J_{ia} J_{ib} J_{ja} J_{jb} \\
&= 2\left\{ (X_1 J_{ij})^2 + \sum_{a=2}^{m} (X_a J_{ij} - \delta_{ai} J_{1j} - \delta_{aj} J_{i1})^2 + \sum_{a=m+1}^{2n} (X_a J_{ij})^2 \right\}
\end{aligned}
$$

for $2 \le i < j \le m$. Hence, from (3.1), we have $J_{1i}^2 + J_{1j}^2 = 1$ for all i, j with $2 \le i < j \le m$. From (2.3), $\sum_{k=1}^{2n} J_{1k}^2 = 1$. Therefore, we conclude $m = 3$. □

Now, we consider $\mathbb{H}^3 \times \mathbb{R}^3$ and write down strictly almost Kähler structures on it.

Example. Let J be the (1.1)-tensor field defined by

$$
(J_{ij}) = \begin{pmatrix}
0 & \cos\theta & \sin\theta & 0 & 0 & 0 \\
-\cos\theta & 0 & 0 & -\frac{\partial\theta}{\partial x_4}\sin\theta & -\frac{\partial\theta}{\partial x_5}\sin\theta & -\frac{\partial\theta}{\partial x_6}\sin\theta \\
-\sin\theta & 0 & 0 & \frac{\partial\theta}{\partial x_4}\cos\theta & \frac{\partial\theta}{\partial x_5}\cos\theta & \frac{\partial\theta}{\partial x_6}\cos\theta \\
0 & \frac{\partial\theta}{\partial x_4}\sin\theta & -\frac{\partial\theta}{\partial x_4}\cos\theta & 0 & \frac{\partial\theta}{\partial x_6} & -\frac{\partial\theta}{\partial x_5} \\
0 & \frac{\partial\theta}{\partial x_5}\sin\theta & -\frac{\partial\theta}{\partial x_5}\cos\theta & -\frac{\partial\theta}{\partial x_6} & 0 & \frac{\partial\theta}{\partial x_4} \\
0 & \frac{\partial\theta}{\partial x_6}\sin\theta & -\frac{\partial\theta}{\partial x_6}\cos\theta & \frac{\partial\theta}{\partial x_5} & -\frac{\partial\theta}{\partial x_4} & 0
\end{pmatrix},
$$

where $\theta = \theta(x_4, x_5, x_6)$ is a smooth function on $\mathbb{R}^3$ satisfying

$$(3.3) \qquad \left(\frac{\partial\theta}{\partial x_4}\right)^2 + \left(\frac{\partial\theta}{\partial x_5}\right)^2 + \left(\frac{\partial\theta}{\partial x_6}\right)^2 = 1, \qquad \frac{\partial^2\theta}{\partial x_4^2} + \frac{\partial^2\theta}{\partial x_5^2} + \frac{\partial^2\theta}{\partial x_6^2} = 0.$$

It is easy to verify that $(\mathbb{H}^3 \times R^3, J, g)$ is a strictly almost Kähler manifold.

Remark 1. Let $(c_4, c_5, c_6) \in \mathbb{R}^3$ with $c_4^2 + c_5^2 + c_6^2 = 1$, and define $\theta(x_4, x_5, x_6) = c_4x_4 + c_5x_5 + c_6x_6$. Then, the function θ satisfies (3.3), and hence, the corresponding almost complex structure J determines a strictly almost Kähler structure on $\mathbb{H}^3 \times \mathbb{R}^3$. This means that $\mathbb{H}^3 \times \mathbb{R}^3$ admits uncountably many strictly almost Kähler structures.

Remark 2. In the higher dimensional case, it is difficult to write down general solutions satisfying (2.3) and (3.1). However, we may observe that $\mathbb{H}^3 \times \mathbb{R}^n$ ($n > 3$ and odd) admits uncountably many strictly almost Kähler structures.

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Real Prime-Producing Quadratics

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Dedicated to the memory of Daniel Shanks

Presented by J.B. Friedlander, F.R.S.C.

Abstract

The purpose of this paper is to announce results about quadratic polynomials of positive discriminant that generate maximal numbers of consecutive, distinct primes for an initial range of input values. This provides the real analogue of the recent result that we provided for the complex side in [2], which generalized the well-known Rabinowitsch criterion for class number one in complex quadratic fields. This generalization linked the prime-production of quadratic polynomials with negative discriminant to complex quadratic fields with class groups of exponent one or two. In this paper we link the real quadratic prime-producers to class groups of real quadratic fields with exponent one or two.

1 Notation and Preliminaries

In [2], we gave a complete explanation of why polynomials of negative discriminant, such as Euler's polynomial $x^2 + x + 41$, produce primes for an initial range of input values. The reasons are intimately linked to the exponent of the class groups of the underlying complex quadratic fields. In this paper, we look at the real side, and provide explanations for the prime-production of quadratic polynomials of positive discriminant.

The polynomial, which is the template for the study in [2], has a real analogue as follows. First recall that a *fundamental discriminant* or *field discriminant* is an integer Δ such that $\Delta = \frac{4}{\sigma^2}D$, where D, called the *radicand associated with* Δ, is square-free, $\sigma = 2$ if $D \equiv 1 \pmod 4$, and $\sigma = 1$

otherwise. In this paper we will only be concerned with *positive* fundamental discriminants.

Definition 1.1 *Let $\Delta > 0$ be a fundamental discriminant having $N+1$ distinct prime factors, with p being the largest. Suppose that q denotes the product of the N distinct prime factors of Δ* excluding p, *and $q = 1$ if $N = 0$. Then, the following is called the q^{th}* Euler-Rabinowitsch polynomial:

$$F_{\Delta,q}(x) = qx^2 + (\alpha - 1)qx + ((\alpha - 1)q^2 - \Delta)/(4q),$$

where $\alpha = 1$ if $4q|\Delta$, and $\alpha = 2$ otherwise.

It is the prime-production of $F_{\Delta,q}(x)$ that we will study herein and link to the exponent of the class group of the underlying real quadratic field. Recall that the exponent of the class group of $\mathbb{Q}(\sqrt{D})$, denoted $\mathcal{C}_\Delta$, is the smallest positive integer e_Δ such that $\{I\}^{e_\Delta} \sim 1$, where $\{I\}$ denotes the equivalence class of the ideal I in $\mathcal{C}_\Delta$, and $\sim$ denotes equivalence in the class group with 1 representing the principal class.

2 Results

We maintain the notation set down in section one throughout the balance of the paper. We provide the following results. However, the proofs, far too lengthy to reproduce here, will appear elsewhere.

The first result tells us what must necessarily follow from the q^{th} Euler-Rabinowitsch polynomial producing consecutive, distinct initial values up to a Minkowski bound $\sqrt{\Delta}/2$.

Theorem 2.1 *In what follows,* (1) $\Rightarrow$ (2)*:*

(1) $|F_{\Delta,q}(x)|$ *is 1 or prime for any integer x with $0 \le x < (\sqrt{\Delta}/2 - \alpha + 1)/\alpha$.*

(2) $e_\Delta \le 2$ *and* $\sqrt{\Delta}/2 < p$.

The following result shows that when $\Delta \equiv 1 \pmod 8$ and (1) of Theorem 2.1 hold, then we are reduced to a single value. This is the analogue of the result for complex quadratics given in [2, p. 20].

Corollary 2.1 *If* $\Delta \equiv 1 \pmod 8$ *and* (1) *of Theorem 2.1 holds, then* $\Delta = 33$.

We now turn to the case where $\Delta \equiv 0 \pmod 8$.

Theorem 2.2 *Let* $\Delta \equiv 0 \pmod 8$. *If* $|F_{\Delta,q}(x)|$ *is* 1 *or prime for all non-negative integers* $x < \sqrt{D}$, *then one of the following holds:*

(1) $D = r^2 + 1 = 2s$ *with either* $D = 2$ *or else* $s \equiv 1 \pmod 4$ *is prime,* $h_\Delta = 2$, *and* r *is the only split prime less than* $\sqrt{D}$.

(2) $D = \lfloor\sqrt{D}\rfloor^2 + 2 = 2s$ *where* $s \equiv 3 \pmod 4$ *is prime,* $h_\Delta = 1$ *and there are no split primes less than* $\sqrt{D}$.

(3) $D = q^2\lfloor\sqrt{D}/q\rfloor^2 + q$, $h_\Delta = 2^{N-1}$, *and there are no split primes less than* $\sqrt{D}$.

(4) $D = [(r+3)/2]^2 - 2 = 2s$ *with* $s = 2[(r+3)/4]^2 - 1$ *prime,* $h_\Delta = 1$, $r > \sqrt{D}$ *is prime and there are no split primes less than* $\sqrt{D}$.

The following table illustrates Theorem 2.2. In fact, it follows from [1, Theorem 6.2.2, p. 202] that the list is complete with one possible exceptional value remaining, whose existence would be a counterexample to the generalized Riemann hypothesis (GRH). As a result of this last fact, this is the only illustration that we are able to provide herein, since the known values for which the remaining results are known to hold under GRH, are far too numerous to tabulate.

Table 2.1. $D \equiv 0 \pmod 2$ with $|F_{\Delta,q}(x)| = 1$ or prime for all $x \in T = \{0, 1, 2, \ldots, \lfloor\sqrt{D}\rfloor\}$.

(a) $D = r^2 + 1 > 2$, $r < \sqrt{D}$, $h_\Delta = 2$.

D	r	$F_{\Delta,q}(x)$	$\lvert F_{\Delta,q}(x)\rvert$ for all $x \in T$
10	3	$2x^2 - 5$	5, 3, 3, 13
26	5	$2x^2 - 13$	13, 11, 5, 5, 19, 37
122	11	$2x^2 - 61$	61, 59, 53, 43, 29, 11, 11, 37, 67, 101, 139, 181
362	19	$2x^2 - 181$	181, 179, 173, 163, 149, 131, 109, 83, 53, 19
			19, 61, 107, 157, 211, 269, 331, 397, 467, 541

(b) $D = \lfloor\sqrt{D}\rfloor^2 + 2$, $h_\Delta = 1$.

D	$F_{\Delta,q}(x)$	$\lvert F_{\Delta,q}(x)\rvert$ for all $x \in T$
6	$2x^2 - 3$	3, 1, 5
38	$2x^2 - 19$	19, 17, 11, 1, 13, 31, 53

(c) $D = [(r+3)/2]^2 - 2$, $r > \sqrt{D}$, $h_\Delta = 1$.

D	r	$F_{\Delta,q}(x)$	$\lvert F_{\Delta,q}(x)\rvert$ for all $x \in T$
14	5	$2x^2 - 7$	7,5,1,11
62	13	$2x^2 - 31$	31,29,23,13,1,19,41,67
398	37	$2x^2 - 199$	199,197,191,181,167,149,127,101,71,37, 1,43,89,139,193,251,313,379,449,523

We now turn to $\Delta \equiv 4 \pmod 8$.

Theorem 2.3 *Let $\Delta \equiv 4 \pmod 8$. If $|F_{\Delta,q}(x)|$ is 1 or prime for all non-negative integers $x < (\sqrt{D} - 1)/2$, then $e_\Delta \le 2$, and one of the following holds:*

(1) *$D = \lfloor\sqrt{D}\rfloor^2 + q$ with $h_\Delta = 2^{N-1}$, and there are no split primes $p < \sqrt{D}$.*

(2) *$D = (\lfloor\sqrt{D}\rfloor + 1)^2 - 2$, with $h_\Delta = 1$ and $r = 2\lfloor\sqrt{D}\rfloor - 1$, is the only split prime less than $\sqrt{\Delta}$.*

(3) *$D = (\lfloor\sqrt{D}\rfloor + 1)^2 - s = st$ with $s < t$ primes, $h_\Delta = 2$ and $r = \lfloor\sqrt{D}\rfloor + 1 - (s+1)/2$, is the only split prime less than $\sqrt{D}$.*

(4) *$D = (\lfloor\sqrt{D}\rfloor + 1)^2 - s$, where s is the second largest prime dividing Δ, and $h_\Delta = 2^{N-1}$.*

(5) *$D = \lfloor\sqrt{D}\rfloor^2 + s$ where s is the second largest prime dividing Δ, and $h_\Delta = 2^{N-1}$.*

(6) *$D = (\lfloor\sqrt{D}\rfloor + 1)^2 - q$, with $h_\Delta = 2^{N-1}$, and $r = 2\lfloor\sqrt{D}\rfloor - q + 1$, is the only split prime less than $\sqrt{\Delta}$.*

The remaining case where $\Delta \equiv 1 \pmod 4$, reduces to the case $D = \Delta \equiv 5 \pmod 8$ by virtue of Corollary 2.1. The case where $h_\Delta = 1$ is special and the only result we have is the following, which remains unresolved.

Conjecture 2.1 *If $\Delta = q_1q_2 \equiv 5 \pmod 8$ with $q_1 \equiv q_2 \equiv 3 \pmod 4$ primes with $q_1 < q_2$, then the following are equivalent:*

(1) $|F_{\Delta,q_1}(x)|$ *is 1 or prime for all non-negative integers $x < (\sqrt{D} - 2)/4$.*

(2) $D = q_1^2s^2 \pm 4q_1$ *or* $4q_1^2s^2 - q_1$, *and* $h_\Delta = 1$.

We do however have a result in the remaining case where $h_\Delta > 1$.

Theorem 2.4 *Let $\Delta \equiv 5 \pmod 8$ with $h_\Delta > 1$. If $|F_{\Delta,q}(x)|$ is 1 or prime for all non-negative integers x with $x < (\sqrt{D} - 2)/4$, then $e_\Delta \leq 2$, and one of the following must hold (where s is the second largest prime dividing D):*

(1) $D = \lfloor\sqrt{D}\rfloor^2 + 4q$, $h_\Delta = 2^{N-1}$, *and there are no split primes less than* $\sqrt{D}/2$.

(2) $D = (3r + s + 1)^2 - 4q$ *with* $h_\Delta = 2 = 2^{N-1}$, $s \equiv 1 \pmod 3$, *and r is prime.*

(3) $D = l^2 - 4s$, *with* $h_\Delta = 2^{N-1}$.

(4) $D = l^2 + 4s$, *with* $h_\Delta = 2^{N-1}$.

(5) $D = l^2 \pm 4$, *with* $h_\Delta = 2$.

(6) $D = l^2 - 4q$, *with* $h_\Delta = 2^{N-1}$.

(7) $D = l^2 + 4q$, *with* $h_\Delta = 2^{N-1}$.

Therefore, the above represents the completion of the project designed to find all Euler-Rabinowitsch polynomials producing initial consecutive distinct primes. Complete lists of them are given in [1], and the lists are known to be complete with one possible exception, whose existence would be a counterexample to the GRH.

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On the Global Existence of Mild Solutions of Initial Value Problems

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Abstract. *In this note we give a global existence result for mild solutions of semilinear initial value problems in abstract spaces where the linear part is the infinitesimal generator of a compact C_0-semigroup.*

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Let A be the infinitesimal generator of a compact C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, $\{B(t)\}_{t\geq 0}$ be a continuous family of bounded linear operators on X and $f : [0,\infty) \times X \to X$ a continuous function. We consider the following semiliniar initial value problem:

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + B(t)x(t) + f(t, x(t)), \quad t \geq 0. \\ x(0) = x_0, \quad x_0 \in X. \end{cases} \tag{1}$$

If (1) has a classical solution then this solution x satisfies the integral equation:

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\Big(B(s)x(s) + f(s, x(s))\Big)\, ds \tag{2}$$

A continuous solution x of the integral equation (2) will be called a *mild solution* of the initial value problem (1). We recall [3, Th. 6.2.2]:

Let A be the infinitesimal generator of a compact semigrup $T(t)$, $t \geq 0$ on X. If $f : [0,\infty) \times X \to X$ is continuous and maps bounded sets in $[0,\infty) \times X$

into bounded sets in X then for every $x_0 \in X$ the initial value problem (1) has a mild solution x on a maximal interval of existence $[0, t_{\max})$. If $t_{\max} < \infty$ then $\lim_{t \uparrow t_{\max}} ||x(t)|| = \infty$.

Using this result we will prove the following:

Theorem. *(i) Let $f : [0, \infty) \times X \to X$ be continuous, satisfying the inequality*

$$||f(t, x)|| \leq \gamma(t) w(||x||) \tag{3}$$

where $w : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function, $w(t) > 0$ for $t > 0$ and $\int_1^\infty \frac{ds}{w(s)} = \infty$ and $\gamma : [0, \infty) \to [0, \infty)$ is continuous. Then the maximal mild solutions of (1) are defined on $[0, \infty)$.

(ii) In addition to (i), assume that $\int_0^\infty \big(\gamma(s) + ||B(s)||\big)\, ds < \infty$. If $||T(t)|| \to 0$ as $t \to \infty$, then $||x(t)|| \to 0$ as $t \to \infty$.

Proof. (i) We observe that f maps bounded sets in $[0, \infty) \times X$ into bounded sets in X. Therefore there exists a maximal mild solution of (1) on $[0, t_{\max})$.

Suppose that $t_{\max} < \infty$. Then $\lim_{t \uparrow t_{\max}} ||x(t)|| = \infty$. From (2) it follows that

$$||x(t)|| \leq ||T(t)x_0|| + \int_0^t ||T(t-s)|| \Big(||B(s)||\, ||x(s)|| + ||f(s, x(s))||\Big)\, ds$$

$$\leq K_2 + K_1 \int_0^t \big(\gamma(s) + ||B(s)||\big) \big(w(||x(s)||) + ||x||\big)\, ds, \quad t \in [0, t_{\max}),$$

where $K_1 = \sup_{t \in [0, t_{\max})} ||T(t)|| < \infty$ and $K_2 = K_1 ||x_0||$.

Applying the Bihari inequality (see [1]) we obtain

$$||x(t)|| \leq G^{-1}(K_1 \int_0^t \big(\gamma(s) + ||B(s)||\big)\, ds), \quad t \in [0, t_{\max}), \tag{4}$$

where $G(u) = \int_{K_2}^u \frac{ds}{w(s)+s}\, ds$, $u > 0$. Evidently G is a increasing continuous function, $G(K_2) = 0$ and $G(\infty) = \infty$ since $\int_1^\infty \frac{ds}{s + w(s)} = \infty$ in our hypotheses (see [2]). Therefore $G(K_2) + K_1 \int_0^t \big(\gamma(s) + ||B(s)||\big)\, ds \in \mathrm{Dom}(G^{-1})$ for which (4) makes sense.

From (4) we obtain

$$\limsup_{t\uparrow t_{max}} \|x(t)\| \le G^{-1}(K_1 \int_0^{t_{max}} \Big(\gamma(s) + \|B(s)\|\Big)\, ds) < \infty$$

witch is a contradiction with our assumption. Therefore $t_{max} = \infty$.

(ii) First we remark that the maximal mild solution of (1) are global by (i). From the principle of uniform boundedness it results that there exists a constant $M > 0$ such that $\|T(t)\| \le M,\ \ t \ge 0$. Therefore, like in (i), we obtain

$$\|x(t)\| \le G^{-1}(G(M\|x_0\|) + M \int_0^t \Big(\gamma(s) + \|B(s)\|\Big)\, ds)$$
$$\le G^{-1}(G(M\|x_0\|) + Mk) = L < \infty$$

where $k = \int_0^\infty \Big(\gamma(s) + \|B(s)\|\Big)\, ds$.

Given any $\varepsilon > 0$ we choose $t_1 \ge 0$ so that

$$M(w(L) + L) \int_{t_1}^\infty \Big(\gamma(s) + \|B(s)\|\Big)\, ds < \varepsilon.$$

Observe that

$$x(t) = T(t)x_0 + T(t - t_1) \int_0^{t_1} T(t_1 - s)\Big(B(s)x(s) + f(s, x(s))\Big)\, ds +$$
$$+ \int_{t_1}^t T(t - s)\Big(B(s)x(s) + f(s, x(s))\Big)\, ds, \quad t \ge t_1,$$

and therefore

$$\|x(t)\| \le \|T(t)\|\, \|x_0\| + \|T(t - t_1)\|\, \| \int_0^{t_1} T(t_1 - s)\Big(B(s)x(s) + f(s, x(s))\Big)\, ds\| +$$
$$+M\ (w(L) + L) \int_{t_1}^t \Big(\gamma(s) + \|B(s)\|\Big)\, ds < 2\varepsilon \quad \text{for all large } t.$$

Thus $x(t) \to 0$ as $t \to \infty$.□

Remark. *As the proof shows, it is enough to asume that γ is locally integrable in (i). Therefore, taking $w(t) = t,\ t \ge 0$, the classical global existence result from [3, Section 6.2] is a particular case of our theorem.*

For $f(t, x) = \frac{1}{t^2+1} x \ln(\|x\| + 1),\ t \ge 0,\ x \in X$, and $B(t) = 0, t \ge 0$, we can apply our theorem whereas the result in [3] is not applicable.

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INTEGRO-DIFFERENTIAL EQUATIONS ASSOCIATED WITH THE BESSEL OPERATOR ON THE COMPLEX DOMAIN

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Presented by G.F.D. Duff, F.R.S.C.

Abstract. In this work we consider the Bessel differential operator on $\mathbb{C}\setminus\{0\}$ defined by

$$\Delta = \frac{d^2}{dz^2} + \frac{2\alpha+1}{z}\frac{d}{dz} \ ; \ \alpha \in \mathbb{C} \setminus \{-1,-2,-3,\ldots\},\ \mathrm{Re}(\alpha) > -1/2 .$$

We define a generalized convolution product associated with Δ denoted # and we study the integro-differential equations of the type $\mu \# f = \sum_{n=0}^{\infty} a_n \Delta^n f$, where (a_n) is a sequence of complex numbers and μ is a measure over the real line. We characterize the solutions of these equations as restrictions on $\mathbb{R}$ of even entire functions of exponential type .

0. Introduction.

We consider the Bessel differential operator defined on $\mathbb{C}\setminus\{0\}$ by

$$\Delta = \frac{d^2}{dz^2} + \frac{2\alpha+1}{z}\frac{d}{dz} \ ; \ \alpha \in \mathbb{C} \setminus \{-1,-2,-3,\ldots\},\ \mathrm{Re}(\alpha) > -1/2.$$

By using the method of generalized Taylor series in the sense of Delsarte (see [2]), we associate with Δ generalized translation operators which permit to define a generalized convolution product of a convenient measure μ on the real line and an even analytic function f on $\mathbb{C}$, denoted $\mu \# f$.

In this work, we are interesting in the study of the following integro-differential equations

$$\mu \# f(z) = \sum_{n\geq 0}^{\infty} a_n \Delta^n f(z) . \tag{1}$$

where $(a_n)_{n\geq 0}$ is a sequence of complex numbers.

These equations characterize a class of even and entire functions of exponential type. In fact this study shows that when the measure μ satisfies $\int_{\mathbb{R}} e^{\sigma|x|} d|\mu| < +\infty$, where σ is a positive number, then every even entire function of exponential type less than σ, is a solution of such equations and conversely if f is an even C^∞-function on $\mathbb{R}$ satisfying the equation (1) and if $\sum_{n\geq 0} a_n z^{2n}$ is analytic inside the disk $|z| < a$, $a \leq \sigma$, then f is the restriction on $\mathbb{R}$ of an even entire function of exponential type at most a.

Next, we assume that $\alpha \in \mathbb{R}$, $\alpha > -1/2$. In this case, the restriction on $\mathbb{R}$ of the generalized translation operators associated with Δ possess an integral representation which is available for even continuous functions on $\mathbb{R}$, so that we can consider equations of the type $\mu \# f = \Delta^n f$ when f is an even C^{2n}-function on $\mathbb{R}$ and μ is a convenient measure. We establish, under some assuptions, that every even C^{2n}-function on $\mathbb{R}$ satisfying $\mu \# f = \Delta^n f$ is the restiction on $\mathbb{R}$ of an even entire function of exponential type.

1 - Harmonic Analysis Associated with the Bessel operator Δ.

We denote by $j_\alpha(\lambda z)$ the unique solution of the following system

$$\begin{cases}\Delta u = -\lambda^2 u \ , \ \lambda \in \mathbb{C} \ , \\ u(0) = 1 \ , u'(0) = 0 \ , \end{cases}$$

The function $j_\alpha(\lambda z)$ is given by

$$j_\alpha(\lambda z) = \begin{cases} 2^\alpha \Gamma(\alpha+1) \dfrac{J_\alpha(\lambda z)}{(\lambda z)^\alpha} \ , & \text{if } \lambda z \neq 0 \ , \\ 1 & , \text{ if } \lambda z = 0 \ , \end{cases}$$

where J_α is the Bessel function of first kind and order α , (see [4]).

The function j_α has the following properties :

i) There exists a constant $k > 0$ such that : $\forall z \in \mathbb{C}$, $|j_\alpha(z)| \leq k\, e^{|\mathrm{Im}\, z|}$

ii) The function j_α can be expanded in entire series as follows

$$j_\alpha(\lambda z) = \sum_{n=0}^{\infty} (-1)^n \lambda^{2n} b_n(z)$$

where the functions b_n , $n \in \mathbb{N}$, are defined by $b_n(z) = \dfrac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha+n+1)} (\dfrac{z}{2})^{2n}$.

We remark that for all $z \in \mathbb{C}$ and for all $n \in \mathbb{N}$, we have $|b_n(z)| \leq \dfrac{|z|^{2n}}{(2n)!}$ and $\Delta b_{n+1} = b_n$.

<u>Notations</u> : We denote by

- $\mathcal{H}_*(\mathbb{C})$ the space of even entire functions on $\mathbb{C}$, provided with the topology of uniform convergence on every compact of $\mathbb{C}$.
- $\mathcal{H}'_*(\mathbb{C})$ the topological dual of $\mathcal{H}_*(\mathbb{C})$.
- $\mathrm{Exp}_*(\mathbb{C})$ the space of even entire functions of exponential type ; we have

$$\mathrm{Exp}_*(\mathbb{C}) = \bigcup_{a>0} \mathrm{Exp}_{*,a}(\mathbb{C})$$

where $\mathrm{Exp}_{*,a}(\mathbb{C}) = \{f \in \mathcal{H}_*(\mathbb{C}) \,/\, \sup_{\lambda \in \mathbb{C}} |f(\lambda)\, e^{-a|\lambda|}| < \infty\}$.

The space $\mathrm{Exp}_{*,a}(\mathbb{C})$ provided with the norm $N_a(f) = \sup_{\lambda \in \mathbb{C}} |f(\lambda) e^{-a|\lambda|}| < \infty$, is a Banach space.

We provide the space $\mathrm{Exp}_*(\mathbb{C})$ with the inductive limit topology.

<u>Definition 1.1</u> : The generalized translation operators associated with Δ, denoted by T_z , $z \in \mathbb{C}$, are defined on $\mathcal{H}_*(\mathbb{C})$ by : $\forall \omega \in \mathbb{C}$, $T_z f(\omega) = \sum_{n=0}^{\infty} b_n(\omega) \Delta^n f(z)$.

We give the following properties of the operators T_z , (see [3], for more details).

<u>Proposition 1.1</u> : The operators T_z satisfy the following properties :

i) For every $z \in \mathbb{C}$, the operator T_z is a linear and continuous map from $\mathcal{H}_*(\mathbb{C})$ into itself.

ii) For all function f in $\mathcal{H}_*(\mathbb{C})$ and for every $z \in \mathbb{C}$ we have

$$T_z f(\omega) = T_\omega f(z) \ , \ T_0 f(z) = f(z) \ , \ \Delta T_z f = T_z \Delta f.$$

iii) For all $z \in \mathbb{C}$, $\omega \in \mathbb{C}$ and $\lambda \in \mathbb{C}$, we have $T_z j_\alpha(\lambda \omega) = j_\alpha(\lambda\omega) j_\alpha(\lambda z)$.

iv) If $\alpha \in \mathbb{R}, \alpha > -1/2$, we have for an even continuous function on $\mathbb{R}$:

$$\forall x, y \in \mathbb{R} \ , \ T_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} \int_0^\pi f(\sqrt{x^2 + y^2 + 2xy\cos\theta})(\sin\theta)^{2\alpha} d\theta \ .$$

Definition 1.2 : The Fourier-Bessel transform $\mathcal{F}_B$ of an analytic functional $S \in \mathcal{H}'_*(\mathbb{C})$ is defined by :

$$\forall \lambda \in \mathbb{C}, \mathcal{F}_B(S)(\lambda) = \langle S, j_\alpha(\lambda .)\rangle.$$

K. Trimèche has established in [3] the following Paley-Wiener type theorem.

Theorem 1.1 : The Fourier-Bessel transform $\mathcal{F}_B$ is a topological isomorphism from $\mathcal{H}'_*(\mathbb{C})$ onto $Exp_*(\mathbb{C})$.

With the help of Theorem 1.1 we prove the following Polya representation type theorem for an even entire function of exponential type.

Theorem 1.2 : Let f be an even entire function of exponentiel type a , a > 0. Then f has the following integral representation

$$f(z) = \int_{|\omega|=a+\varepsilon} j_\alpha(z\omega)F(\omega)d\omega ,$$

where $\varepsilon > 0$ and F is an analytic function outside the disk centered at the origin and with radius a.

From Theorem 1.2 and the properties of the function j_α , we deduce the following.

Proposition 1.1 : Let f be an even entire function of exponential type a , a > 0. Then for every $n \in \mathbb{N}$, the function $\Delta^n f$ is even, entire and of exponential type a.

Proposition 1.2 : Let f be an even entire function of exponential type a , a > 0. Then we have for every $z \in \mathbb{C}$, $\omega \in \mathbb{C}$ and $\varepsilon > 0$

$$|T_z f(\omega)| \le k_\varepsilon e^{(a+\varepsilon)(|z|+|\omega|)} ,$$

where k_ε is a positive constant.

Notation : Let $\sigma > 0$, we denote by $M_\sigma(\mathbb{R})$ the space of Radon measures on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} e^{\sigma|x|}d|\mu| < +\infty$

Definition 1.3 : Let f be an even entire function of exponenetial type a, a > 0, and $\mu \in M_\sigma(\mathbb{R})$ with $\sigma > a$. The convolution product associated with Δ of the function f and the measure μ is the function denoted $\mu \# f$, defined by

$$\forall z \in \mathbb{C}, \mu \# f(z) = \int_{\mathbb{R}} T_z f(y) d\mu(y).$$

Proposition 1.3 : Let f be an even entire function of exponential type a > 0, and $\mu \in M_\sigma(\mathbb{R})$ with $\sigma > a$. Then the function $\mu \# f$ is even, entire and of exponential type a.

Definition 1.4 : Let μ be a measure in $M_\sigma(\mathbb{R})$. The Fourier-Bessel of the measure μ is the function denoted $\hat{\mu}$, defined by : $\forall z \in \mathbb{C}, \hat{\mu}(z) = \int_{\mathbb{R}} j_\alpha(zy)d\mu(y)$

We remark that for $\mu \in M_\sigma(\mathbb{R})$, the function $\hat{\mu}$ is even and analytic in the strip $|\text{Im } z| < \sigma$.

2 - Integro-differential equations associated with the Bessel operator Δ on the space of entire functions of exponential type.

Theorem 2.1 :Let $\mu \in M_\sigma(\mathbb{R})$. Then every even entire function of exponential type a, $0< a < \sigma$, satisfies the following equation

$$\forall z \in \mathbb{C}\ ,\ \mu \# f(z) = \sum_{n=0}^{\infty} a_n \Delta^n f(z)\ , \tag{2.1}$$

where

$$a_n = \frac{(-1)^n}{(2n)!} \frac{d^{2n}}{dz^{2n}} \hat{\mu}(z)|_{z=0}. \tag{2.2}$$

Proposition 2.1 : Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers such that $\sum_{n\geq 0} a_n z^{2n}$ is analytic in the disk centered at the origin and with radius $\sigma > 0$, and $\mu \in M_\sigma(\mathbb{R})$. If the equation (2.1) is satisfied by any function f in $Exp._a(\mathbb{C})$, with $0< a < \sigma$. Then a_n is given by the formula (2.2).

Theorem 2.2: Let $\mu \in M_\sigma(\mathbb{R})$ and f an even C^∞-function on $\mathbb{R}$ satisfying

$$\forall x \in \mathbb{R}\ ,\ \mu \# f(x) = \sum_{n\geq 0} a_n \Delta^n f(x)\ , \tag{2.3}$$

where $(a_n)_{n\geq 0}$ is a sequence of complex numbers such that the series $\sum_{n\geq 0} a_n z^{2n}$ is analytic inside the disk $|z| < a$, $0 < a < \sigma$. Then f is the restriction on $\mathbb{R}$ of an even and entire function of exponential type at most a.

Example.

Let μ be the measure defined by $d\mu(y) = e^{-y} y^{2\alpha+1} 1_{[0,+\infty[}(y)dy$, $\alpha > -1/2$. Consider the equation

$$\mu \# f = \sum_{n=0}^{+\infty} a_n \Delta^n f. \tag{2.4}$$

where a_n is given by $a_n = \dfrac{\Gamma(\alpha+n+\frac{3}{2})}{n!\,\Gamma(\alpha+\frac{3}{2})}$.

By computation, we have, for $|z| < 1$, $\hat{\mu}(z) = \dfrac{2^{2\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+3/2)}{\sqrt{\pi}(1+z^2)^{\alpha+3/2}}$. Then Theorem 2.1 shows that (2.4) is satisfied by every even entire function of exponential type less than 1.

Proposition 2.2: Let $\mu \in M_\sigma(\mathbb{R})$ and u_0 a zero of multiplicity N of the function

$$g(z) = \hat{\mu}(z) - \sum_{n=0}^{k} (-1)^n c_n z^{2n},$$

where $(c_n)_{0\leq n\leq k}$ is a finite sequence in $\mathbb{C}$. The function defined by $f(x) = \sum_{m=0}^{N-1} a_m h_{m,u_0}(x)$ is a solution of the equation

$$\mu \# f(x) = \sum_{n=0}^{k} c_n \Delta^n f(x).,$$

where $h_{m,u_0}(x) = x^m j_\alpha^{(m)}(u_0 x)$ and $(a_m)_{0\leq m\leq N-1}$ is a finite sequence in $\mathbb{C}$.

The previous result can be extended to the infinite case as follows .

Proposition 2.3 : Let $\mu \in M_\sigma(\mathbb{R})$ and u_0 a zero of multiplicity N of the function

$$g(z) = \hat{\mu}(z) - \sum_{n=0}^{k}(-1)^n c_n z^{2n}.$$

Suppose that the series $\sum_{n\geq 0} c_n z^{2n}$ is analytic in the disk $|z| \leq a < \sigma$ and $|u_0| < a$. Then every function of the form $f(x) = \sum_{m=0}^{N-1} a_m h_{m,u_0}(x)$, , is a solution of the integro-differential equation

$$\mu \# f(x) = \sum_{n=0}^{+\infty} c_n \Delta^n f(x).$$

Example.

For $\alpha > -1/2$, let $d\mu(t) = \begin{cases} t^{2\alpha+1}\,dt & , \text{ if } 0<t<1, \\ 0 & , \text{ otherwise.} \end{cases}$

To construct solutions of the equation

$$\mu \# f = \Delta f \tag{2.5}$$

we look at the roots of the function $g(z) = \hat{\mu}(z) + z^2 = \frac{1}{2(\alpha+1)} j_{\alpha+1}(z) + z^2.$

Since there are an infinite number of zeros of this function, then there are an infinite number of solutions of (2.5).

Proposition 2.4 : Let f be an even entire function of exponential type a, $a > 0$, $\sum_{n\geq 0} a_n z^{2n}$ an analytic function in the disk $|z| \leq b$, which contains the conjugate indicator diagram of f and $\mu \in M_\sigma(\mathbb{R})$, with $\sigma \geq b$. If moreover f satisfies the equation

$$\mu \# f = \sum_{n=0}^{+\infty} a_n \Delta^n$$

Then we have

$$f(z) = \sum_{k=0}^{M} \sum_{n=0}^{m_k - 1} \beta_n z^n j_\alpha^{(n)}(\lambda_k z)$$

where (λ_k), $0 \leq k \leq M$, is the set of zeros of multiplicity m_k of the function

$$g(z) = \hat{\mu}(z) - \sum_{n=0}^{\infty}(-1)^n a_n z^{2n}$$

which are contained in the conjugate indicator diagram of f. and β_n are constants depending on f.

3 - Integro-differential equations associated with the Bessel operator Δ on the space of even C^{2n}-functions on $\mathbb{R}$.

In the following we suppose that $\alpha \in \mathbb{R}$, $\alpha > -1/2$.

Notations : We denote by

- $C_*^k(\mathbb{R})$, $k \in \mathbb{N}$, the space of even C^k-functions on $\mathbb{R}$.
- $C_*^\infty(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$.

Definition 3.1 : Let f be an even continuous function on $\mathbb{R}$. We say that the nonnegative and even function ψ is a bounding function of f, if we have

i) $\forall x \in \mathbb{R}$, $|f(x)| \leq \psi(x)$

ii) There exists a constant $A(\psi)$ such that

$$\forall x,y \in \mathbb{R}\ , T_x\ \varphi(y) \le A(\varphi)\ \varphi(x)\ \varphi(y).$$

The smallest constant satisfying the latter inequality will be called the supporting constant.

We have the following results .

Proposition 3.1: i)Let f be such that $\Delta^m f \in C_*^k(\mathbb{R})$. Then $f \in C_*^{m+k}(\mathbb{R})$.

ii)Let $f \in C_*^2(\mathbb{R})$ and μ a measure on $\mathbb{R}$. If φ is a bounding function of f, satisfying $\int_{\mathbb{R}} \varphi(y) d|\mu|(y) < +\infty$, then $\mu \# f \in C_*^2(\mathbb{R})$ and we have $\Delta(\mu \# f) = \mu\ \#\Delta f$.

iii)Let f be in $C_*^\infty(\mathbb{R})$. Suppose that there exist a positive constant B and an even, nonnegative and continuous function φ on $\mathbb{R}$ such that

$$\forall x \in \mathbb{R}\ , |\Delta^n f(x)| \le B^{2n}\ \varphi(x)\ \ ,\ \ \forall n \in \mathbb{N}.$$

Then f is the restriction on $\mathbb{R}$ of an even entire function of exponential type B.

Proposition 3.2: Let $f \in C_*^2(\mathbb{R})$, φ a bounding function of f and μ a measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \varphi(x) d|\mu|(x) = V < +\infty.$$

If f is a solution of the equation $\mu \# f = \Delta f$, then f is the restriction on $\mathbb{R}$ of an even entire function of exponential type at most $(AV)^{1/2}$, where A is the supporting constant.

Theorem 3.1: Let f be in $C_*^{2n}(\mathbb{R})$, $n \ge 1$, satisfying

i) $|f(x)| \le M e^{\tau|x|}$,

ii) $\mu \# f = \Delta^n f$,

where μ is a measure on $\mathbb{R}$ such that $\tilde{V} = \int e^{(\tau+1)|x|} d|\mu|(x) < +\infty$. Then f is the restriction on $\mathbb{R}$ of an even entire function of exponential type at most $\tilde{V}^{\,1/2n}$.

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LINEAR STABILITY OF COUETTE–POISEUILLE FLOW BETWEEN ROTATING PERMEABLE CYLINDERS

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Presented by K.B. Ranger, F.R.S.C.

ABSTRACT. The convective instability boundary of a circular Couette flow in the annular region bounded by two co- or counter-rotating coaxial porous cylinders with axial and outward or inward radial flows is studied. Bifurcation diagrams are obtained for axisymmetric and non-axisymmetric disturbances and computational results are compared with experimental data.

RÉSUMÉ. On étudie la frontière d'instabilité convective d'un courant de Couette circulaire dans la région annulaire bornée par deux cylindres poreux coaxiaux rotatifs en sens semblable ou opposés, avec flot axial et flot radial entrant ou sortant. On obtient des diagrammes de bifurcation pour des perturbations axisymétriques et non-axisymétriques et l'on compare les résultats numériques et expérimentaux.

1. Introduction. A rotating filter usually consists of an inner rotating porous cylinder and an outer stationary impermeable one. As a suspension moves axially between the cylinders, the filtrate passes radially through the wall of the inner cylinder and the concentrate is collected at the exit end of the annulus. Thus, in a rotating filter, a circular Couette flow, an axial flow caused by an axial pressure gradient, and a radial flow through the porous wall of the inner cylinder are superposed.

Theoretical and experimental studies of convective instability boundary of a viscous flow in an annulus between two rotating coaxial cylinders has been surveyed, for instance, in [1]. In [2], it is found that a Couette flow with a radial flow and axisymmetric perturbations is stabilized by an inward radial flow and destabilized by a weak outward radial flow. A strong outward flow has a stabilizing effect on the onset of Taylor vortices.

The stability problem of a Couette flow with an axial flow due to an axial pressure gradient has been solved [3, 4] for different values of the Reynolds number, Re, defined in terms of the mean axial velocity. In the presence of non-axisymmetric disturbances [5, 6, 7], the critical Taylor number, Ta_c, does not increase monotonically with Re. In [6] numerical results indicate that there exist regions of stabilization and destabilization in the (Re, Ta)-plane; moreover, theoretical and experimental data are in good agreement for $\mathrm{Re} \leq 40$, but for $\mathrm{Re} \geq 40$ the discrepancy increases slightly with Re.

Numerous applications of rotating filter separators have stimulated interest in the study of the stability of superposed axial, radial and circular Couette flows, and recently experiments [8] have uncovered different flow regimes.

In this note we study the combined influence of axial and radial flows on the convective stability of a circular Couette flow. The linear stability problem of both axisymmetric

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(toroidal) and non-axisymmetric modes is solved for different values of the axial and radial Reynolds numbers. Numerical results are given for the cases of co-rotating and counter-rotating cylinders, and theoretical results are compared with experimental data [8].

2. Formulation of the problem. Consider two infinitely long coaxial porous cylinders, of radii $R_1 < R_2$, which are rotating with constant angular velocities ω_1 and ω_2, respectively. Let the measure of length, time, velocity, and pressure be given by $h = (R_2 - R_1)/2$, $t = h^2/\nu$, $v = \omega_1 h$, and $p = \rho\nu\omega_1$, respectively. We introduce a system of cylindrical coordinates (r, θ, z) with the z-axis coinciding with the common vertical axis of the cylinders and where r and z are dimensionless variables. The annular region between the cylinders is filled with a viscous incompressible fluid. We assume that a Couette flow coexists together with an inward or outward radial flow through the cylinder walls and an axial flow caused by a constant axial pressure gradient. If the velocity vector is a function of r only, the dimensionless Navier–Stokes equations have a steady axisymmetric solution of the form

$$v_r = U_0(r), \quad v_\theta = V_0(r), \quad v_z = W_0(r), \quad p = p_0(r, z) =: p_{01}(r) + p_{02}(z), \tag{1}$$

with

$$U_0(r) = \frac{\alpha}{Tr}, \qquad V_0(r) = Ar^{\alpha+1} + \frac{B}{r}, \tag{2}$$

$$W_0(r) = \frac{dp_{02}}{dz}\left[\frac{r_2^2 - r_1^2}{2(2-\alpha)(r_2^\alpha - r_1^\alpha)} r^\alpha + \frac{r_1^2 r_2^\alpha - r_2^2 r_1^\alpha}{2(2-\alpha)(r_2^\alpha - r_1^\alpha)} - \frac{r^2}{2(2-\alpha)}\right], \tag{3}$$

where

$$A = \frac{\mu r_2^2 - r_1^2}{r_2^{\alpha+2} - r_1^{\alpha+2}}, \qquad B = \frac{r_1^2 r_2^{\alpha+2} - \mu r_2^2 r_1^{\alpha+2}}{r_2^{\alpha+2} - r_1^{\alpha+2}},$$

$$r_1 = \frac{2\eta}{1-\eta}, \quad r_2 = \frac{2}{1-\eta}, \quad \mu = \frac{\omega_2}{\omega_1}, \quad \eta = \frac{R_1}{R_2}, \quad T = \frac{\omega_1 (R_2 - R_1)^2}{4\nu}.$$

$\alpha = u_1 R_1/\nu$ is the radial Reynolds number, and u_1 is the radial velocity at $r = r_1$. The radial flow is inward if $\alpha < 0$ and outward if $\alpha > 0$. It is assumed in (1)–(3) that $dp_{02}(z)/dz = \text{const}$ and $\alpha \neq -2$. If $\alpha = -2$, the solution contains a logarithm; but this case is excluded from our analysis.

It is convenient to define the axial Reynolds number as $\mathrm{Re} = 2W_{av}h/\nu$, where

$$W_{av} = \frac{1}{\pi(r_2^2 - r_1^2)} \int_0^{2\pi} d\theta \int_{r_1}^{r_2} rW_0(r)\,dr \tag{4}$$

is the average velocity. We seek a perturbed solution to the Navier–Stokes equations in the form

$$\begin{bmatrix} v_r \\ v_\theta \\ v_z \\ p \end{bmatrix} = \begin{bmatrix} U_0(r) \\ V_0(r) \\ W_0(r) \\ p_0(r,z) \end{bmatrix} + \begin{bmatrix} u(r) \\ v(r) \\ w(r) \\ q(r) \end{bmatrix} e^{-\lambda t + ikz + in\theta}, \tag{5}$$

where $u(r)$, $v(r)$, $w(r)$ and $q(r)$ are the complex amplitudes of the normal perturbations, k is the wave number, $\lambda = \alpha + i\beta$ is a complex eigenvalue, and n is the azimuthal mode number. The numbers $n = 0$ and $n = 1, 2, \ldots$ correspond to axisymmetric (toroidal) and non-axisymmetric perturbations, respectively.

Substituting (5) into the Navier–Stokes equations and using a standard linearization procedure, we obtain the following system of ordinary differential equations for $u(r)$, $v(r)$, $w(r)$ and $q(r)$,

$$-\lambda u + T\left(U_0 \frac{du}{dr} + \frac{dU_0}{dr} u + \frac{V_0}{r} i n u + W_0 i k u - \frac{2}{r} v V_0\right) = -\frac{dq}{dr} + Lu - \frac{u}{r^2} - i n \frac{2}{r^2} v,$$

$$-\lambda v + T\left(u \frac{dV_0}{dr} + U_0 \frac{dv}{dr} + \frac{V_0}{r} i n v + W_0 i k v + \frac{u}{r} V_0 + \frac{v}{r} U_0\right) = -\frac{inq}{r} + Lv - \frac{v}{r^2} + i n \frac{2}{r^2} u,$$

$$-\lambda w + T\left(u \frac{dW_0}{dr} + U_0 \frac{dw}{dr} + \frac{V_0}{r} i n w + W_0 i k w\right) = -i k q + Lw,$$

$$\frac{du}{dr} + \frac{u}{r} + \frac{inv}{r} + i k w = 0,$$

where

$$L := \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} - k^2.$$

The boundary conditions for u, v and w are zero at $r = r_i$ for $i = 1, 2$. This problem is transformed into an eigenvalue problem by means of a pseudospectral collocation method based on Chebyshev polynomials. The eigenvalues, $\lambda_s = \alpha_s + i\beta_s$, $s = 1, 2, \ldots$, determine the stability of the flow (2)–(3), which is said to be stable if $\alpha_s > 0$ for all s, and convectively unstable if $\alpha_s < 0$ for at least one value of s. The imaginary part, β_s, of the eigenvalue λ_s determines the axial velocity of the translation of the vortices. The IMSL routine GVLCG is used to solve this problem as explained, for instance, in [9].

We shall restrict attention to the determination of the convective instability curves in the (k, T)-plane, that is, the absolute minimum (k_c, T_c) of each such curve, where k_c is the critical wave number and $T_c = \min_k T$. Our results are given in terms of the critical Taylor number

$$\mathrm{Ta}_c = \frac{\omega_1 R_1 (R_2 - R_1)}{\nu} = \frac{4\eta}{1-\eta} T_c. \tag{6}$$

3. Numerical results.

3.1. *Radial flow without axial flow:* $\alpha \neq 0$ *and* $\mathrm{Re} = 0$. This case generalizes the analysis of axisymmetric perturbations ($n = 0$), found in [2], to non-axisymmetric perturbations ($n \neq 0$) since the latter perturbations are the most unstable ones for some negative values of μ.

In Fig. 1, Ta_c is plotted as a function of α for $\mathrm{Re} = 0$, $\eta = 0.8$, and $\mu = -1$, -0.5 and 0. First, we discuss results for outward radial flow ($\alpha > 0$). It is seen that Ta_c decreases for small but increasing α. As α becomes sufficiently large, the outward radial flow begins to stabilize the Couette flow; but as α grows further, Ta_c increases.

For $\mu = 0$ and $\mu = -0.5$, only axisymmetric perturbations lead to instability whereas for $\mu = -1$ there is a sequence of transitions from the most unstable mode ($n = 2$) at $\alpha = 0$ to the axisymmetric mode as α increases. For $\mu = -1$, segments AB, BC and CD and DE correspond to modes $n = 1, 2, 3, 4$, respectively. For $\mu = -0.5$, the corresponding transition points are marked with primes (A', etc.). The part of the curves on the right of the points A and A' correspond to axisymmetric mode with $n = 0$.

Second, we consider inward radial flow ($\alpha < 0$). A stabilization of the Couette flow is observed in Fig. 1 as the intensity of the inward radial flow increases, that is, as α decreases.

We see that stabilization is stronger for negative α. For instance, with $\mu = 0$, $\mathrm{Ta}_c = 119.27$ and $\mathrm{Ta}_c = 214.21$ for $\alpha = 20$ and $\alpha = -20$, respectively. If $\mu = 0$, the flow is unstable with respect to the sole axisymmetric perturbations. If $\mu = -0.5$, there is a sequence of transitions from the axisymmetric mode (at $\alpha = 0$) to the first, second and third ($n = 1, 2, 3$) non-axisymmetric modes at $\alpha = -2.47$, -4.4 and -12.5, respectively. If $\mu = -1$, the stability curve also goes through a sequence of bifurcations from the second non-axisymmetric mode (at $\alpha = 0$) to the third and fourth ($n = 4$) non-axisymmetric modes.

As shown in [10], the critical Taylor number for non-axisymmetric disturbances without radial flow is only slightly greater than for axisymmetric disturbances. Therefore, the appearance of non-axisymmetric modes for $\mu < 0$ and some α can be associated with the influence of these parameters on the mutual location of the stability curves for the non-axisymmetric and symmetric cases.

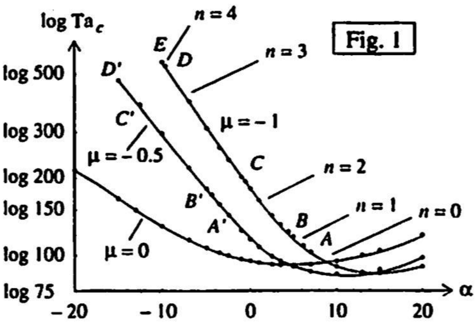
3.2. *Radial flow with axial flow: $\alpha \neq 0$ and* $\mathrm{Re} \neq 0$. All computations for this case have been made with $\eta = 0.5$ and $\mu = -0.5$, 0, and 0.2. The results for the case $\alpha = -2.5$ are shown in Figs. 2 and 3. Only axisymmetric disturbances correspond to instability if $\mu = 0$. However, for large Re the stability curve in the (k, Ta)-plane has two minima (see Fig. 3 in [9]); this explains the existence of a corner at the point $(\mathrm{Re}, \mathrm{Ta}_c) = (39.5, 279.3)$ on the stability curve for $\mu = 0$ (see Fig. 2), where a transition from $k_c = 1.31$ to $k_c = 3.02$ takes place. Note that such nonmonotonic behavior of the convective instability curves in the (k, Ta)-plane has been observed [4] in the absence of radial flow, but for larger values of Re.

The case $\mu = 0.2$ is characterized by a sequence of bifurcations from the axisymmetric mode to the non-axisymmetric mode with $n = 6$ as Re grows from 0 to 50. A slight discontinuity in the axial size of the cells can be observed at each bifurcation point. The non-axisymmetric mode with $n = 1$ changes to the axisymmetric one at $\mathrm{Re} = 7.8$ for $\mu = -0.5$; then Ta_c increases with Re, but instability remains axisymmetric, at least up to $\mathrm{Re} = 50$.

It is known that even a classical Taylor–Couette flow (without radial and axial flows) may give rise to a rich variety of flow patterns. For example, fifteen different flow regimes were found in [11]. The variety of possible instability phenomena certainly increases when both centrifugal and shear instability mechanisms are present. The present study has focussed on the linear stability analysis and the determination of the convective stability boundary. Further work (both experimental and theoretical) is needed for a better understanding of the physics of flow (for example, the appearance of non-axisymmetric vortices).

4. **Comparison with experimental results.** Circular Couette flow with axial flow and no radial flow has been studied experimentally in [6, 12, 13, 14].

In Fig. 4, our results for $\eta = 0.85$ are compared with results in [8], which is devoted to the study of processes in rotating filter separators. Experimental points corresponding to different flow regimes are indicated by the symbols ⊙ for Couette–Poiseuille flow, ⊡ for laminar vortices, and △ for helical vortices. Theoretical data are represented by a solid line, where segments AB and BC correspond to axisymmetric and non-axisymmetric instability, respectively. The agreement is satisfactory for small values of Re. However, as Re grows, the discrepancy between experimental and theoretical data slightly increases. Note that the



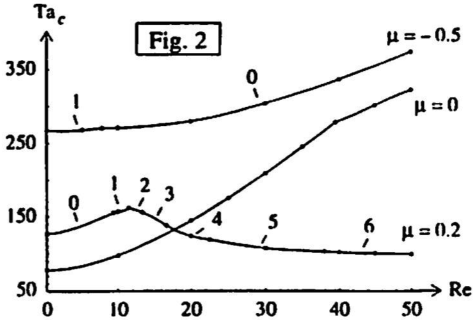
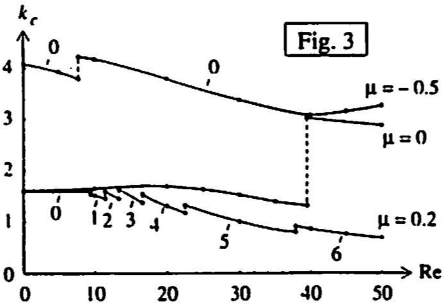


FIGURE 1. Convective instability curves for $\mu = -1$, -0.5 and 0, at $\mathrm{Re} = 0$, $\eta = 0.8$ for outward ($\alpha > 0$) and inward ($\alpha < 0$) radial flows. Segments AB, BC, CD, and DE of the curve for $\mu = -1$ correspond to the modes $n = 1, 2, 3, 4$, respectively. Segments $A'B'$, $B'C'$, and $C'D'$ of the curve for $\mu = -0.5$ correspond to the modes $n = 1, 2, 3$, respectively.

FIGURE 2. Convective instability curves for $\mu = -0.5$, 0 and 0.2, at $\alpha = -2.5$, $\eta = 0.5$. The numbers identifying the segments of the curves correspond to instabilities with respect to toroidal ($n = 0$) or non-axisymmetric ($n > 0$) perturbations.



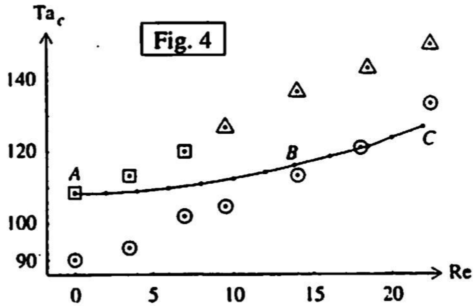


FIGURE 3. Critical wave number k_c versus axial Reynolds number Re for $\mu = -0.5$, 0 and 0.2. at $\alpha = -2.5$, $\eta = 0.5$. The numbers identifying the segments of the curves correspond to instabilities with respect to toroidal ($n = 0$) or non-axisymmetric ($n > 0$) perturbations.

FIGURE 4. Comparison between theoretical and experimental [8] data at $\alpha = 0$ and $\eta = 0.85$. The symbols are: ⊙ for Couette-Poiseuille flow, ⊡ for laminar vortices, and △ for helical vortices. The solid line represents the theoretical stability curve, where segments AB and BC correspond to laminar ($n = 0$) and helical ($n = 1$) vortices, respectively.

same phenomenon has been observed in [6, 15]. The existence of a vortex development length is suggested in [6] as a possible explanation for this discrepancy.

This discrepancy may be explained by two important factors which cannot be taken into account in the framework of our model. First, in experiments [8] and in rotating filter separators the outer cylinder is impermeable, while in our model it is permeable. Unfortunately, a simple flow with structure (1) does not exist if one cylinder is permeable and the other one is impermeable. In fact, if we assume as in (1) that $v_r(r) = U_0(r)$, then the function $U_0(r)$ satisfies a first-order differential equation and one cannot impose two boundary conditions on this function, one at $r = r_1$ and the other at $r = r_2$. Secondly, the axial flow in [8] is not uniform. Instead, it changes with the axial coordinate, whereas we assume that $v_z = lV_0(r)$ (see (1)). It is also found [8] that helical vortices disappear if $\alpha \neq 0$. However, our computations show that helical vortices disappear and instability becomes axisymmetric for values of $|\alpha|$ significantly larger than those reported in [8].

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Decay estimates for a strongly damped nonlinear wave equation

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Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. We prove the uniform stabilization of solutions to the strongly damped wave equation $u_{tt} = \alpha u_{xxt} + (1 + \beta \int_0^1 u_y^2\, dy)u_{xx}$ in $(0,1) \times R_+$, $u(0,t) = u(1,t) = 0, t > 0$, and $u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,1)$.

1. Introduction

Consider the nonlocal partial differential equation

$$\begin{cases} u_{tt} - \alpha u_{xxt} - (1 + \beta \int_0^1 u_y^2\, dy)u_{xx} = 0 & x \in (0,1), t > 0, \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in (0,1). \end{cases} \qquad (P)$$

where α, β are two positive constants with no specific assumptions on their relative magnitudes.Problem (P) is the approximation of the following problem (P_ε) which describes the transverse deflection of an extensible beam

$$\begin{cases} u_{tt} + \varepsilon u_{xxxx} - \alpha u_{xxt} - (1 + \beta \int_0^1 u_y^2\, dy)u_{xx} = 0 & x \in (0,1), t > 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in (0,1). \end{cases} \qquad (P_\varepsilon)$$

The question of wellposedness for (P_ε) is well in hand.the reader is referred to Ball[1].Holmes and Marsden[7].Dickey[3].Taboada and You[10] and Fitzgibbon[5].Global wellposedness of (P) is more difficult.Recently, by using the Galerkin arguments (as in Böhm and Taboada[2].Pohozaev[9] and Dix and Torrejón[4]).Fitzgibbon and Parrott[6] have been successful in proving it.More precisely they proved that for all $(u_0, u_1) \in D(A^1) \times D(A^2)$ and $T > 0$,we have

$$\lim_{\varepsilon \to 0^+} \Big(\sup_{t \in [0,T]} \|u_\varepsilon(.,t) - u(.,t)\|_\infty \Big) = 0$$

where u_ε and u are strong solutions to (P_ε) and (P) respectively,and A is the operator defined by

$$A : D(A) \subset L^2(\Omega) \to L^2(\Omega) \quad Au(x) = -u''(x) \quad \text{with} \quad D(A) = \{u, u \in H^2(0,1) \cap H_0^1(0,1)\}.$$

To the best of my knowledge,there is no result concerning the asymptotic behavior of the solution $u(.,t)$ of (P) as t goes to infinity.In this paper .we shall prove that the energy of the solution to (P) decays to zero uniformly.

2.Statement and proof of the main theorem

It should be evident that the problem (P) may be written abstractly as

$$u_{tt}(t)+\alpha Au_t(t)+Au(t)+\beta\|A^{\frac{1}{2}}u(t)\|^2Au(t)=0,\quad t>0. \tag{2.1}$$

$$u(0)=u_0,\quad u_t(0)=u_1. \tag{2.2}$$

where A is the operator defined by

$$A:D(A)\subset L^2(\Omega)\rightarrow L^2(\Omega)\quad Au(x)=-u''(x)\quad \text{with}\quad D(A)=\{u,u\in H^2(0,1)\cap H_0^1(0,1)\}.$$

$\|.\|$ is the usual norm of $L^2(\Omega)$,and $H_0^1(0,1)$, $H^2(0,1)$ are the usual Hilbert Sobolev spaces.

We define the energy of the solution u of (P) by the formula

$$E(t):=E(u(t))=\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|A^{\frac{1}{2}}u(t)\|^2+\frac{\beta}{4}\|A^{\frac{1}{2}}u(t)\|^4.$$

A simple computation gives

$$E'(t)=-\alpha\|A^{\frac{1}{2}}u_t(t)\|^2\leq 0.$$

that is

$$E(0)-E(t)=\int_0^t\int_0^1\alpha|A^{\frac{1}{2}}u_t|^2\,dxds. \tag{2.3}$$

the energy is non-increasing.Our main result is the following

Main theorem
We have

$$E(t)\leq cE(0)e^{-\omega t}\quad \forall t\geq 0.$$

where c and ω are two positive constants independents of the initial data $(u_0,u_1)\in D(A^4)\times D(A^2)$.

Before giving the proof of the main theorem,we recall the following lemma.
Lemma 2.1(Komornik [8],th.8.1)

Let $E : R_+ \rightarrow R_+$ be a non-increasing function and assume that there is a constant $T > 0$ such that

$$\int_t^{+\infty} E(s)\,ds \le TE(0)E(t), \quad \forall t \in R_+.$$

then we have

$$E(t) \le E(0)e^{1-\frac{t}{T}}, \quad \forall t \ge 0.$$

Proof of the main theorem

We multiply the equation (2.1) with u and we integrate over $(0,1) \times (0,T)$,we obtain that

$$0 = \int_0^T \int_0^1 (u_{tt}u + \alpha Au_t u + Auu + \beta|A^{\frac{1}{2}}u|^2 Auu)\,dxdt.$$

after integration by parts,we conclude that

$$0 = \left[\int_0^1 uu_t\,dx\right]_0^T - \int_0^T \int_0^1 u_t^2 + \int_0^T \int_0^1 \alpha Au_t u + \int_0^T \int_0^1 |A^{\frac{1}{2}}u|^2 + \int_0^T \int_0^1 \beta|A^{\frac{1}{2}}u|^4.$$

that is

$$0 = \left[\int_0^1 uu_t\,dx\right]_0^T - 2\int_0^T \int_0^1 u_t^2 + \left(\int_0^T \int_0^1 (u_t^2 + |A^{\frac{1}{2}}u|^2 + \beta|A^{\frac{1}{2}}u|^4)\,dxdt\right) + \int_0^T \int_0^1 \alpha Au_t u.$$

whence

$$2\int_0^T E(t)\,dt \le -\left[\int_0^1 u_t u\,dx\right]_0^T + 2\int_0^T \int_0^1 u_t^2\,dxdt - \int_0^T \int_0^1 \alpha Au_t u\,dxdt$$

$$\le c_1 E(0) - \alpha \int_0^T \int_0^1 Au_t u\,dxdt \quad (c_1 \text{ positive constant}).$$

by the Cauchy-Schwarz inequality,the non-increasing property of E,the Sobolev imbedding $H^1 \subset L^2$ and the relation (2.3).

On the other hand, it is easy to verify that

$$-\alpha \int_0^T \int_0^1 Au_t u\,dxdt \le c_2 E(0) \quad (c_2 \text{ positive constant}).$$

hence

$$\int_0^T E(t)\,dt \le c_3 E(0) \quad (c_3 \text{positive constant}).$$

and by lemma 2.1,we deduce the desired decay estimate.

Remark

The method of proof of the main theorem remains valid when the equation (2.1) is replaced by the equation $u_{tt} - f(\int_0^1 u_y^2\,dy)u_{xx} - \alpha u_{xxt} = 0$ with $f \in C^1([0,+\infty))$, $f(x) \geq m_0 > 0$ and/or the strong damping term u_{xxt} is replaced by a weak damping of the form δu_t, $\delta > 0$.In these cases,the global wellposedness may be obtained by the same arguments as in [6].

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Solution to a Problem Posed by H. S. M. Coxeter

by

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Presented by H.S.M. Coxeter, F.R.S.C.

1. INTRODUCTION

In [2], Prof. Coxeter posed the problem to show the identity

$$\int_1^6 \frac{\operatorname{arcsec} t}{(t+2)\sqrt{t+1}}\left[\frac{1}{\sqrt{t+3}}+2\right] \mathrm{d}t = \frac{2\pi^2}{15} \tag{1}$$

"without appealing to geometry or the computer". He established eq. (1) in [4, p. 71] by trisecting a spherical orthoscheme of known volume into three smaller ones, and in [3, p. 20] by computation.

Whereas the area of a spherical or of a hyperbolic *triangle* is given by $|\pi - \alpha - \beta - \gamma|$, α, β, γ being its angles. there is—to my knowledge—no such formula expressing the volume V of a spherical or of a hyperbolic *tetrahedron* T in a symmetric way through its six dihedral angles (cf. the problem posed in [6. Rem. 2. p. 108]). As a stopgap. one can represent V as the sum of the volumes of six "orthoschemes" arising by dissecting T (see [1. I.2, p. 19]). (The volume of such an orthoscheme is given by a famous formula of Lobachevsky, cf. [5, § 30. (160), p. 97].) There are different ways to dissect T and the resulting formulae are different. One way to verify analytically identities resulting in this way is differentiation with respect to a parameter controlling the size of T. In the last analysis. this method relies on Schläfli's formula (cf. [7, Satz, p. 235]. [3, p. 14], [1, (2.1), p. 26], [8, (9), p. 19])

$$\mathrm{d}V = \pm\frac{1}{n-1}\sum_{\alpha} V_\alpha \,\mathrm{d}\alpha,$$

which expresses the differential of the volume of an n-dimensional $\left\{\begin{matrix}\text{spherical}\\ \text{hyperbolic}\end{matrix}\right\}$ simplex by the volumes V_α of the $(n-2)$-dimensional subsimplices associated with the dihedral angles α.

We shall apply this procedure, and shall prove, by elementary means, the following, more general identity, which is valid for $\arccos(\sqrt{2/3}) \le \alpha \le \frac{\pi}{2}$:

$$\int_{\frac{1}{2}\cot^2\alpha}^{1} \frac{\arccos x}{(2x+1)\sqrt{x+1}}\left[\frac{\sqrt{2}\cos\alpha}{\sqrt{2x\sin^2\alpha - \cos 2\alpha}} + \frac{2}{\sqrt{x}}\right] dx$$
$$= \frac{2}{5}\int_{\alpha_0}^{\alpha} \arccos\left(\frac{\cos 2\alpha(\sin^2 t + \cos 2\alpha)}{\sin^2 t - \cos^2 2\alpha}\right) dt$$
$$- \frac{2}{5}\int_{\arccos(\sqrt{2/3})}^{\alpha} \arccos\left(\frac{\cos 2t}{1 - 2\cos 2t}\right) dt + \frac{8\pi}{5}\left(\alpha - \frac{\pi}{4}\right)_+, \quad (2)$$

where $\alpha_0 := \arccos(\cot\alpha\sqrt{1 - 2\cos 2\alpha})$ and

$$\left(\alpha - \frac{\pi}{4}\right)_+ := \begin{cases} \alpha - \frac{\pi}{4} & : \alpha \ge \frac{\pi}{4}, \\ 0 & : \alpha \le \frac{\pi}{4}. \end{cases}$$

We note that, for $\alpha = \frac{\pi}{3}$, $\alpha_0 = \arccos(\sqrt{2/3})$ and

$$\frac{\cos 2\alpha(\sin^2 t + \cos 2\alpha)}{\sin^2 t - \cos^2 2\alpha} = \frac{\cos 2t}{1 - 2\cos 2t}$$

and hence the two integrals on the right-hand side of (2) cancel each other. Therefore, for $\alpha = \frac{\pi}{3}$, (2) yields

$$\int_{1/6}^{1} \frac{\arccos x}{(2x+1)\sqrt{x+1}}\left[\frac{1}{\sqrt{3x+1}} + \frac{2}{\sqrt{x}}\right] dx = \frac{2\pi^2}{15}.$$

Up to the substitution $x = 1/t$, this is the equation (1).

We observe that both sides of (2) vanish for $\alpha = \arccos(\sqrt{2/3})$ since then $\cot\alpha = \sqrt{2}$ and $\alpha = \alpha_0$. Furthermore, elementary considerations show that all integrals appearing in (2) are well-defined and give continuous functions of α over the interval $\arccos(\sqrt{2/3}) \le \alpha \le \frac{\pi}{2}$. As we shall see below, these integrals are also differentiable in the open interval $\arccos(\sqrt{2/3}) < \alpha < \frac{\pi}{2}$, except for the first integral on the right-hand side of (2), which integral is not differentiable at $\alpha = \frac{\pi}{4}$. (This is caused by the non-differentiability of the function $\arccos x$ at $x = 1$.) Hence, by continuity, eq. (2) can be verified by showing that the derivatives of both sides with respect to α coincide for $\arccos(\sqrt{2/3}) < \alpha < \frac{\pi}{2}$, $\alpha \ne \frac{\pi}{4}$.

2. VERIFICATION OF (2) BY DIFFERENTIATION

(a) We first calculate the derivative of the left-hand side L in (2). To begin with, the fundamental theorem of calculus implies

$$\frac{d}{d\alpha}\int_{\frac{1}{2}\cot^2\alpha}^{1} \frac{\arccos x}{(2x+1)\sqrt{x+1}}\frac{2}{\sqrt{x}}\, dx = 4\arccos\left(\frac{\cot^2\alpha}{2}\right)\frac{\sin\alpha}{\sqrt{1+\sin^2\alpha}}. \quad (3)$$

Similarly, from

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\cos\alpha}{\sqrt{2x\sin^2\alpha-\cos 2\alpha}}\right) = -\frac{(2x+1)\sin\alpha}{(2x\sin^2\alpha-\cos 2\alpha)^{3/2}},$$

we infer that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\frac{1}{2}\cot^2\alpha}^{1}\frac{\sqrt{2}\cos\alpha\arccos x}{(2x+1)\sqrt{x+1}\sqrt{2x\sin^2\alpha-\cos 2\alpha}}\,\mathrm{d}x$$
$$= 2\arccos\left(\frac{\cot^2\alpha}{2}\right)\frac{\cos^2\alpha}{\sin\alpha\sqrt{1+\sin^2\alpha}} + I, \quad (4)$$

where

$$I := -\sqrt{2}\sin\alpha\int_{\frac{1}{2}\cot^2\alpha}^{1}\frac{\arccos x}{\sqrt{x+1}(2x\sin^2\alpha-\cos 2\alpha)^{3/2}}\,\mathrm{d}x.$$

The substitution $x = 2u^2-1$, the formula

$$2\arccos y = \begin{cases}\arccos(2y^2-1) & : y \geq 0,\\ 2\pi - \arccos(2y^2-1) & : y \leq 0,\end{cases}$$

partial integration and the subsequent substitution $v^2 = 1-u^2$ yield

$$\begin{aligned}
I &= -8\sin\alpha\int_{\frac{\sqrt{1+\sin^2\alpha}}{2\sin\alpha}}^{1}\frac{\arccos u}{(4u^2\sin^2\alpha-1)^{3/2}}\,\mathrm{d}u\\
&= 8\sin\alpha\left[\frac{u\arccos u}{\sqrt{4u^2\sin^2\alpha-1}}\right]_{u=\frac{\sqrt{1+\sin^2\alpha}}{2\sin\alpha}}^{1}\\
&\qquad + 8\sin\alpha\int_{\frac{\sqrt{1+\sin^2\alpha}}{2\sin\alpha}}^{1}\frac{u}{\sqrt{1-u^2}\sqrt{4u^2\sin^2\alpha-1}}\,\mathrm{d}u\\
&= -4\arccos\left(\frac{\sqrt{1+\sin^2\alpha}}{2\sin\alpha}\right)\frac{\sqrt{1+\sin^2\alpha}}{\sin\alpha} + \int_0^{\frac{\sqrt{3\sin^2\alpha-1}}{2\sin\alpha}}\frac{8\sin\alpha\,\mathrm{d}v}{\sqrt{4\sin^2\alpha-1-4v^2\sin^2\alpha}}\\
&= -2\arccos\left(\frac{\cot^2\alpha}{2}\right)\frac{\sqrt{1+\sin^2\alpha}}{\sin\alpha} + 4\arcsin\left(\sqrt{\frac{3\sin^2\alpha-1}{4\sin^2\alpha-1}}\right)\\
&= -2\arccos\left(\frac{\cot^2\alpha}{2}\right)\frac{\cos^2\alpha+2\sin^2\alpha}{\sin\alpha\sqrt{1+\sin^2\alpha}} + 2\arccos\left(\frac{\cos 2\alpha}{1-2\cos 2\alpha}\right).
\end{aligned} \quad (5)$$

Hence collecting the terms from (3), (4), and (5) we obtain for the derivative of the left-hand side in (2)

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 2\arccos\left(\frac{\cos 2\alpha}{1-2\cos 2\alpha}\right).$$

(b) Let us finally calculate the derivative $\mathrm{d}R/\mathrm{d}\alpha$ of the right-hand side R in (2) for $\alpha \neq \frac{\pi}{4}$. If

$$F(\alpha,t) := \arccos\left(\frac{\cos 2\alpha(\sin^2 t + \cos 2\alpha)}{\sin^2 t - \cos^2 2\alpha}\right),$$

then $F(\alpha,\alpha_0) = 0$ and $F(\alpha,\alpha) = \arccos(\frac{\cos 2\alpha}{1-2\cos 2\alpha})$ and thus

$$\frac{\mathrm{d}R}{\mathrm{d}\alpha} = \frac{8\pi}{5} Y\Big(\alpha - \frac{\pi}{4}\Big) + \frac{2}{5}\int_{\alpha_0}^{\alpha} \frac{\partial F(\alpha,t)}{\partial \alpha}\,\mathrm{d}t, \tag{6}$$

where

$$Y\Big(\alpha - \frac{\pi}{4}\Big) := \begin{cases} 1 : \alpha > \frac{\pi}{4}, \\ 0 : \alpha < \frac{\pi}{4} \end{cases}$$

(i.e., Y denotes Heaviside's function). Straightforwardly, one finds

$$\frac{\partial F(\alpha,t)}{\partial \alpha} = \frac{2\sin t(4\cos^4\alpha - \cos^2 t)}{(\sin^2 2\alpha - \cos^2 t)\sqrt{\cos^2\alpha_0 - \cos^2 t}},$$

and hence, substituting $u = \cos t$, we have

$$\begin{aligned}\int_{\alpha_0}^{\alpha} \frac{\partial F(\alpha,t)}{\partial \alpha}\,\mathrm{d}t &= 2\int_{\cos\alpha}^{\cos\alpha_0} \frac{4\cos^4\alpha - u^2}{(\sin^2 2\alpha - u^2)\sqrt{\cos^2\alpha_0 - u^2}}\,\mathrm{d}u \\ &= \int_{\cos\alpha}^{\cos\alpha_0} \frac{2\,\mathrm{d}u}{\sqrt{\cos^2\alpha_0 - u^2}} + \int_{\cos\alpha}^{\cos\alpha_0} \frac{8\cos^2\alpha\cos 2\alpha\,\mathrm{d}u}{(\sin^2 2\alpha - u^2)\sqrt{\cos^2\alpha_0 - u^2}}.\end{aligned} \tag{7}$$

The first integral in (7) yields $\arccos(\frac{\cos 2\alpha}{1-2\cos 2\alpha})$ and the second one can be deduced from the elementary definite integral

$$\int_a^b \frac{\mathrm{d}u}{(c^2 - u^2)\sqrt{b^2 - u^2}} = \frac{1}{2c\sqrt{c^2 - b^2}} \arccos\left(\frac{b^2(a^2 + c^2) - 2a^2c^2}{b^2(a^2 - c^2)}\right)$$

for $0 < a < b < c$. Here $a = \cos\alpha$, $b = \cos\alpha_0$, $c = \sin 2\alpha$, $\sqrt{c^2 - b^2} = \cot\alpha|\cos 2\alpha|$,

$$\frac{b^2(a^2 + c^2) - 2a^2c^2}{b^2(a^2 - c^2)} = \frac{2\cos^2 2\alpha}{(1 - 2\cos 2\alpha)^2} - 1,$$

which implies

$$\arccos\left(\frac{b^2(a^2 + c^2) - 2a^2c^2}{b^2(a^2 - c^2)}\right) = \begin{cases} 2\arccos\left(\frac{\cos 2\alpha}{1 - 2\cos 2\alpha}\right) & : \alpha \le \frac{\pi}{4}, \\ 2\pi - 2\arccos\left(\frac{\cos 2\alpha}{1 - 2\cos 2\alpha}\right) & : \alpha \ge \frac{\pi}{4}. \end{cases}$$

Therefore, (6) and (7) furnish

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}\alpha} &= \frac{8\pi}{5} Y\Big(\alpha - \frac{\pi}{4}\Big) + \frac{2}{5}\arccos\left(\frac{\cos 2\alpha}{1 - 2\cos 2\alpha}\right) \\ &\quad + \frac{4}{5}\operatorname{sign}(\cos 2\alpha)\left[2\pi Y\Big(\alpha - \frac{\pi}{4}\Big) + 2\operatorname{sign}(\cos 2\alpha)\arccos\left(\frac{\cos 2\alpha}{1 - 2\cos 2\alpha}\right)\right] \\ &= 2\arccos\left(\frac{\cos 2\alpha}{1 - 2\cos 2\alpha}\right).\end{aligned}$$

This completes the proof of formula (2).

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A note on "Fluid velocity fields derived from the Navier-Stokes equations."

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In connection with [1] some terms in equations (6) have been omitted and should be replaced by

$$\phi_{x_1} = \frac{\left[B_{x_2} + \nu\nabla^2 Q_2 - \frac{\partial Q_2}{\partial t} + W_1\ u_3\right]}{W_3}, \tag{1}$$

$$\phi_{x_2} = \frac{\left[W_2\ u_3 - B_{x_1} - \nu\ \nabla^2 Q_1 + \frac{\partial Q_1}{\partial t}\right]}{W_3}. \tag{2}$$

Also it is not sufficiently general to set K = constant, and this function can be chosen as an arbitrary function of position and time. However this procedure is cumbersome and can be replaced by a simpler approach to determine the same result.

The analysis of [1] is substantially complete except to show that the equations

$$\text{div}\ (\underline{q} \cdot \underline{\nabla})\ \underline{q} = 0,\ \underline{q} = u_j \hat{x}_j \tag{3}$$

$$(\underline{q}\cdot\underline{\nabla})\,u_3 \;=\; -P_{x_3} \;+\; \nu\nabla^2 u_3,\ \text{div}\,\underline{q} \;=\; 0 \tag{4}$$

in which u_3 is arbitrarily prescribed and P is a three dimensional harmonic, that is $\nabla^2 P = 0$, have a mutually consistent solution for $u_1,\ u_2$. First it is possible to construct solutions from (4) by standard methods since the equations are linear-inhomogeneous. Now with $\gamma = \underline{\omega} + \lambda \underline{q},\ \underline{\omega} =$ curl $\underline{q}$ and λ arbitrary it is found that

$$\text{div}(\underline{q}\cdot\underline{\nabla})\,\underline{q} \;=\; \nabla^2\frac{1}{2}\,|\underline{q}|^2 + \text{div}\,[\underline{\omega}\times\underline{q}] \;=\; \nabla^2\frac{1}{2}\,|\underline{q}|^2 + \text{div}\,[\underline{\gamma}\times\underline{q}] \tag{5}$$

$$= \nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\gamma}) - (\underline{\gamma}\cdot\underline{\omega}) \;=\; \nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\gamma}) - |\underline{\omega}|^2 - \lambda\,(\underline{q}\cdot\underline{\omega})$$

This expression vanishes if

$$\lambda \;=\; \frac{1}{(\underline{q}\cdot\underline{\omega})}\left\{\nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\gamma}) - |\underline{\omega}|^2\right\} \tag{6}$$

where $(\underline{q}\cdot\underline{\omega}) \neq 0$, in the fluid region. In this case it follows that

$$\underline{\gamma} \;=\; \underline{\omega} + \frac{\underline{q}}{(\underline{q}\cdot\underline{\omega})}\left\{\nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\gamma}) - |\underline{\omega}|^2\right\}. \tag{7}$$

If $\underline{\gamma} \;=\; \underline{\beta} + \underline{\nabla}\chi$, then equation (7) can be written as

$$\underline{\beta} + \underline{\nabla}\chi \;=\; \underline{\omega} \;+\; \frac{\underline{q}}{(\underline{q}\cdot\underline{\omega})}\left\{\nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\beta}) - |\underline{\omega}|^2\right\} \tag{8}$$

and elimination of χ produces the equation

$$\text{curl}\,\underline{\beta} \;=\; \text{curl}\,\underline{\omega} + \text{curl}\left\{\frac{\underline{q}}{(\underline{q}\cdot\underline{\omega})}\left[\nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\beta}) - |\underline{\omega}|^2\right]\right\}. \tag{9}$$

To show that equation (9) can be satisfied without restricting $\underline{q}$ it is first convenient to set

$$\underline{R} \;=\; Rj\,\hat{x}j \;=\; \underline{\beta} - \underline{\omega} - \frac{\underline{q}}{(\underline{q}\cdot\underline{\omega})}\left\{\nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\beta}) - |\underline{\omega}|^2\right\} + \underline{\nabla}\chi \tag{10}$$

then the equations

$$\frac{\partial R_3}{\partial x_2} - \frac{\partial R_2}{\partial x_3} \;=\; (\hat{x}_1\cdot\text{curl}\,(\underline{\beta}-\underline{\omega}) + (\hat{x}_1\cdot\text{curl}\,\Big\{\frac{\underline{q}}{(\underline{q}\cdot\underline{\omega})}$$

$$\Big[\nabla^2\frac{1}{2}\,|\underline{q}|^2 + (\underline{q}\cdot\text{curl}\,\underline{\beta}) - |\underline{\omega}|^2\Big]\Big\}) \;=\; 0. \tag{11}$$

$$\frac{\partial R_1}{\partial x_3} - \frac{\partial R_3}{\partial x_1} = (\hat{x}_2 \cdot \text{curl } (\underline{\beta} - \underline{\omega})) + (\hat{x}_2 \cdot \text{curl } \{\frac{\underline{q}}{(\underline{\omega} \cdot \underline{q})}$$

$$[\nabla^2 \frac{1}{2} |\underline{q}|^2 + (\underline{q} \cdot \text{curl } \underline{\beta}) - |\underline{\omega}|^2]\}) = 0, \quad (12)$$

are linear-inhomogeneous and local solutions exist for curl $\underline{\beta}$ without restricting $\underline{q}$, apart from $(\underline{\omega} \cdot \underline{q}) \neq 0$ in the fluid. In fact if $\underline{Z} = Z_j \hat{x}_j = \text{curl } \underline{\beta}$ the system expressed by (11) and (12) can be recast as a first order system of the form

$$A_{ijk} \frac{\partial Z_i}{\partial x_j} + B_{ik} Z_i + C_k = 0, \; k = 1, \; 2, \; \frac{\partial Z_i}{\partial x_i} = 0, \quad (13)$$

and the A_{ijk}, B_{ik}, C_k depend on u_j. Even though from a constructive point of view it is a cumbersome procedure to exhibit the solutions for Z_i explicitly it is sufficient for the present purpose to be assured that such solutions exist, at least locally, and this is confirmed by standard theory [2]. It now follows from (11), (12) that $\frac{\partial R_3}{\partial x_2} - \frac{\partial R_2}{\partial x_3} = \frac{\partial R_1}{\partial x_3} - \frac{\partial R_3}{\partial x_2} = 0$, and it is always possible to choose R_j such that $\frac{\partial R_2}{\partial x_1} - \frac{\partial R_1}{\partial x_2} = 0$, in which case (9) is satisfied subject to $(\underline{q} \cdot \underline{\omega}) \neq 0$. The meaning of this result is that the solution space of div $(\underline{q} \cdot \underline{\nabla})\, \underline{q} = 0$ is sufficiently large as to encompass or include solutions of equations (4). There is then a mutually consistent solution of (3) and (4) which is identified in [1].

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NOTICE OF EDITORIAL AND OTHER CHANGES IN 1997

At his own request, the current issue is the final one to be edited by George Duff. Beginning with Vol. XIX, 1997 the Mathematical Reports are to be co-edited by Vlastimil Dlab of Carleton University and M. Ram Murty of Queen's University. Submissions of Research Announcements should, effective immediately, be sent to the new headquarters:

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Each submission can be in English or in French and should contain an abstract in the language not used in the text. For convenience of reviewers each submission which is merely an announcement of results should be accompanied by a detailed version (not for publication) of the necessary proofs. The editors encourage submission in TeX with disc to be provided after acceptance of the paper.

Accompanying the change in editorship is a change in format (7 in. by 10 in.) and cover design. The new editors also intend to solicit from time to time longer survey papers of an expository nature.

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JULY 1997 — **JUILLET 1997**

25-29 International Conference on Differential Equations with Applications to Biology (Halifax, Nova Scotia)
Shigui Ruan: ruan@cs.dal.ca

28-8 Séminaire de Mathématiques Supérieures - Séminaire Scientifique OTAN (Montréal)

AUGUST 1997 — **AOÛT 1997**

18-23 Eight International Colloquium on Differential Equations (Plovdiv, Bulgaria)
Drumi Bainov (P.O. Box 45, Sofia 1504, Bulgaria)

OCTOBER 1997 — **OCTOBRE 1997**

29-1 Sixth SIAM Conference on Applied Linear Algebra (Snowbird, Utah)
J.A. George (University of Waterloo)

DECEMBER 1997 — **DÉCEMBRE 1997**

14-16 CMS Winter Meeting / Réunion d'hiver de la SMC (Victoria, British Columbia)
Monique Bouchard, meetings@cms.math.ca

JANUARY 1998 — **JANVIER 1998**

10-13 Joint Mathematics Meetings (Baltimore, Maryland)
A.H. Daly: AMS, meet@math.ams.org

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MARCH 1997 **MARS 1997**

10-15 Workshop on Calogero-Moser-Sutherland type models and their various applications (Montréal)
Louis Pelletier: pelletl@CRM.UMontreal.CA

16-17 An International Conference on Graph Theory (Waterloo)
http://math.uwaterloo.ca/CandO_Dept/tutte/announce.html or tutte@math.uwaterloo.ca

17-21 Model Theory of Analytic Functions (Toronto, Ontario) *model@fields.utoronto.ca*

17-21 Workshop on The Bispectral Problem(Montréal)
Louis Pelletier:pelletl@CRM.UMontreal.CA

21-22 AMS Special Session on Approximation Theory (Memphis, Tennessee)
George A. Anastassiou: anastasg@mathsci.msci.memphis.edu

APRIL 1997 **AVRIL 1997**

5-13 Workshop on General Combinatorial Group Theory (Montréal)
Louis Pelletier: pelletl@UMontreal.CA

8-11 FRACTAL 97, 4th International Working Conference (Denver, CO)
http://www.kingston.ac.uk/~ap.s412

21-26 CRM Workshop on Transverbal Designs and Orthogonal Arrays (Kitchener Waterloo, ON)
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MAY 1997 **MAI 1997**

5-11 Geometry and Complexity (Toronto, Ontario)
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19-30 Workshop on Experimental Mathematics and Combinatorics (Montréal)
Louis Pelletier: pelletl@UMontreal.CA

21-23 International Conference Examines Interaction Between Computer Models and Experimental Measurements (Rhodes, Greece)
Sue Owen: WIT, wit@witcmi.ac.uk

30-1 18th Annual Meeting Canadian Applied Mathematics Society - Société Canadienne de Mathématiques Appliquées
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JUNE 1997 **JUIN 1997**

1-4 Annual Meeting of the Statistical Society of Canada (New Brunswick)
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3-6 Fifth Bi-annual Conference dedicated to Computations in Commutative Algebra and Algebraic Geometry(Herstmonceux Castle, East Sussex, England)
Anthony V. Geramita: geramita@dima.unige.it

6-8 International Linear Algebra Society (ILAS) Workshop on Fast Algorithms for Control, Signals and Image Processing (Winnipeg, Manitoba)
P.M. Shivakumar: insmath@cc.umanitoba.ca

6-8 Annual Meeting of Canadian Society for History and Philosophy of Mathematics (St. John's, Newfoundland)
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7-9 CMS Summer Meeting / Réunion d'été de la SMC (Winnipeg, Manitoba)
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9 Session on Linear Algebra for the Canadian Mathematical Society (Winnipeg, Manitoba)
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9-20 Workshop on Algebraic Combinatorics (Montréal)
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15 21 Conference in Honour of Vladimir Arnol'd (Toronto, Ontario) *arnold@fields.utoronto.ca*

25-28 Conference on Combinatorial Methods with Applications to Probability and Statistics (McMaster)
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23-27 Symplectic Geometry (Toronto, Ontario)
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JULY 1997 **JUILLET 1997**

14-18 SIAM 45th Anniversary Meeting (Stanford, CA)
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18-31 38th International Mathematical Olympiad (Mar Del Plata, Argentina)