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POLYNOMIAL OR POWER SERIES EXTENSIONS WITH LINEARLY ORDERED INTERMEDIATE RINGS

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Presented by P. Ribenboim, F.R.S.C.

Abstract. It is proved that if $R \subset T$ are commutative rings and X is an indeterminate, then the set of rings between $R[X]$ and $T[X]$ is linearly ordered by inclusion (if and) only if $T = R$.

All the rings in this note are assumed commutative, with 1. Given rings $D \subset E$, we also assume that the 1 of the subring D is the same as the 1 of the ring E. Consider rings $R \subset T$. For many properties $\mathcal{P}$, it follows that T is algebraic over R - or nearly so - if one requires that $\mathcal{P}$ be satisfied by all the ring extensions between R and T. For instance, if R is a field and T is an integral domain such that $A \subset B$ satisfies LO for all rings $R \subset A \subset B \subset T$, then $\text{tr. deg}_R(T) \leq 1$ (see [4, Proposition 3.7]). (Following [7, p. 28], we use LO, GU, and INC to denote the lying-over, going-up, and incomparable properties, respectively.) Similarly, by [3, Corollary 4], $A \subset B$ satisfies INC for all rings $R \subset A \subset B \subset T$ if and only if each element of T satisfies a polynomial over R with unit content.

Our focus in this note is on extensions of polynomial rings. Once again, consider rings $R \subset T$, and let X be an indeterminate over

T. For motivation, recall from [4, Corollaries 3.2 and 3.6] that $A \subset B$ satisfies LO (or GU) for all rings $R[X] \subset A \subset B \subset T[X]$ if and only if T is integral over R. (Cf. [2, Lemma, p. 160].) There is no lack of examples of INC-behavior in this context either, as $A \subset B$ satisfies INC for all rings $R[X] \subset A \subset B \subset T[X]$ if T is integral over R. We next sharpen the focus to a particular kind of INC-extension.

Recall from [6] that a ring extension $D \subset E$ is called a Δ-extension if $A + B$ is a ring for all rings A and B contained between D and E. If $D \subset E$ is a Δ-extension, then it follows by combining [6, Lemma 3] with [3, Corollary 4] that $D \subset E$ satisfies INC, indeed that $A \subset B$ satisfies INC for all rings $D \subset A \subset B \subset E$. It now seems natural to ask the following question: which ring extensions $R \subset T$ are such that $A \subset B$ is a Δ-extension for all rings $R[X] \subset A \subset B \subset T[X]$?

One case of the above question is easy to settle. Recall from [1] (cf. also [5] and [8]) that an extension $D \subset E$ of distinct rings is said to be adjacent if there is no ring contained properly between D and E. It is trivial that any adjacent extension of rings is a Δ-extension. Thus, a special case of the above question would be the following: which ring extensions $R \subset T$ are such that $A \subset B$ is an adjacent extension for all rings $R[X] \subset A \subset B \subset T[X]$? Unfortunately, the answer is easy for this special case: $T = R$. Indeed, if R is a proper subring of T, then for each non-negative integer n, $R + XR + \cdots + X^nR + X^{n+1}T[X]$ is a ring contained strictly between $R[X]$ and $T[X]$.

Matters become more interesting if we focus on the following class intermediate between the adjacent extensions and the Δ-extensions. We say that a ring extension $D \subset E$ is a λ-extension ("λ" for "linear") in case the set of rings contained between D and E is linearly ordered by inclusion. (The second-named author is preparing an extensive manuscript on λ-extensions.) It is evident that any adjacent extension is a λ-extension and that every λ-extension is a Δ-extension; examples show that neither of these assertions has a valid converse. We are thus led to the following special case of the above question concerning Δ-extensions: which ring extensions $R \subset T$ are such that $A \subset B$ is a λ-extension for all rings $R[X] \subset A \subset B \subset T[X]$? The answer is given in our main result, which is stated next and which indicates that, for these purposes, λ-extensions behave like adjacent extensions.

THEOREM. Let $R \subset T$ be rings and X an indeterminate over T. Then $R[X] \subset T[X]$ is a λ-extension if and only if $T = R$.

Before proving the Theorem, we give two preliminary results. It is interesting to note, by [6, Lemma 4], that the condition on $D \subset E$ in the following Proposition implies that $D \subset E$ is a Δ-extension; this condition also evidently implies integrality (and thus, LO, GU, and INC for all the intermediate extensions).

PROPOSITION. Let $D \subset E$ be distinct rings such that $E = D + De$ for some $e \in E$. Consider the conductor $I = (D: E) := \{f \in E : fE \subset D\}$. Then there is an inclusion - preserving and inclusion - reflecting bijection between the set of rings contained between D and E and the set of ideals of D/I.

Proof. The D-module homomorphism $f: D \to E/D$, $d \mapsto de + D$, has $\ker(f) = (D:E) = I$ and $\operatorname{im}(f) = (De + D)/D = E/D$. Hence, $D/I \cong E/D$ as D-modules. We thus have an order-isomorphism between the set of ideals of D/I and the set of D-modules between D and E. It now suffices to show that if M is a D-module contained between D and E, then M is a ring. For this, consider $\alpha, \beta \in M$. We need only show that $\alpha\beta \in M$. Write $\alpha = d_1 + d_2e$ and $\beta = d_3 + d_4e$, with all $d_i \in D$. Observe that $d_2e = \alpha - d_1 \in M$; similarly, $d_4e \in M$. As $e^2 = a + be$ for some $a, b \in D$,

$$\alpha\beta = d_1d_3 + d_1(d_4e) + d_3(d_2e) + d_2d_4a + d_2b(d_4e)$$

is a sum of elements of M, whence $\alpha\beta \in M$, to complete the proof.

LEMMA. (a) If $D \subset E$ are rings, then $(D[X]:E[X]) = (D:E)[X]$.

(b) If D is a nonzero ring, then the ideals of $D[X]$ are not linearly ordered by inclusion.

Proof. (a) may be proved by straightforward calculations. As for (b), observe that $(X + 1)D[X]$ and $XD[X]$ are incomparable ideals.

Proof of Theorem. The "if" assertion is trivial. Suppose that the converse fails. Then $R[X] \subset T[X]$ is a λ-extension and $T \neq R$. First, we claim that the set of R-modules contained between R and T is linearly ordered by inclusion. Indeed, if I and J are

R-modules contained between R and T, then

$$A := R + IX + X^2T[X] \text{ and } B := R + JX + X^2T[X]$$

are rings contained between $R[X]$ and $T[X]$. As $R[X] \subset T[X]$ is a λ-extension, either $A \subset B$ or $B \subset A$. Hence, by comparing coefficients of X, we have that either $I \subset J$ or $J \subset I$, thus proving the above claim.

Next, we claim that if I is an R-module contained between R and T, then either $I = R$ or $T = I^2 \; (:= \{\Sigma c_jd_j : c_j, d_j \in I\})$. Indeed,

$$C := R + RX + X^2T[X] \text{ and } D := R + IX + I^2X^2 + X^3T[X]$$

are rings contained between $R[X]$ and $T[X]$. As $R[X] \subset T[X]$ is a λ-extension, either $C \subset D$ or $D \subset C$. Hence, by comparing coefficients of X^2 if $C \subset D$ and coefficients of X if $D \subset C$, we have that either $T = I^2$ or $I = R$, thus proving the second claim.

Our final claim is that $T = R + Rv$ for some $v \in T$. To see this, choose $u \in T \setminus R$, and consider $E := R + Ru$. Without loss of generality, $T \neq E$. As $E \neq R$, it follows by applying the result of the preceding paragraph to $I = E$ that $T = E^2 = R + Ru + Ru^2$. Now, since $T \neq E = R + Ru$, we see that u^2 does not belong to E. In particular, $R + Ru^2$ is not contained in E. Thus, by the claim established in the first paragraph of the proof, $E \subset R + Ru^2$. Hence, $T = E + (R + Ru^2) = R + Ru^2$; so, by taking $v = u^2$, we have proved the final claim.

We may now complete the proof. Let v be as in the preceding paragraph. Applying the Proposition to the extension $R[X] \subset T[X] = R[X] + R[X]v$, we see that the ideals of $R[X]/(R[X] : T[X])$ are linearly ordered by inclusion, since $R[X] \subset T[X]$ is a λ-extension. However, by part (a) of the Lemma,

$$R[X]/(R[X] : T[X]) = R[X]/(R : T)[X] \cong (R/(R : T))[X];$$

and, by part (b) of the Lemma, the ideals of $(R/(R:T))[X]$ are not linearly ordered by inclusion, since $R/(R:T)$ is a nonzero ring. This (desired) contradiction completes the proof.

REMARK. (a) The above Theorem may be generalized by replacing X with any nonempty set $\{X_\alpha\}$ of indeterminates. This is a consequence of the one-variable case (i.e., the Theorem) and the fact that if $A \subset B$ are rings such that $A[\{X_\alpha\}] \subset B[\{X_\alpha\}]$ is a λ-extension, then $A \subset B$ is a λ-extension.

(b) The results for power series are more complicated only because the "power series" analogue of part (b) of the Lemma is false. The above methods may be modified to prove the following. Let R be a proper subring of a ring T. Then $R[[X]] \subset T[[X]]$ is a λ-extension if and only if $T/(R:T)$ is a two-dimensional algebra over the field $R/(R:T)$; but if $|\{X_\alpha\}| > 1$, then $R[[\{X_\alpha\}]] \subset T[[\{X_\alpha\}]]$ is not a λ-extension, essentially since $(R/(R:T))[[\{X_\alpha\}]]$ is not chained.

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ASYMPTOTIC EXPANSION AND GENERALIZED SCHLÄFLI INTEGRAL REPRESENTATION FOR THE EIGENFUNCTION OF A SINGULAR SECOND ORDER DIFFERENTIAL OPERATOR.

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Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. In this work we consider the eigenfunction $V(\lambda,t)$ satisfying a condition at infinity of a singular second order differential operator on $(0,+\infty)$. We give an asymptotic expansion of this solution with respect to the variable λ as $|\lambda| \to +\infty$, which permits to establish a generalized Schläfli integral representation for the function $V(\lambda,t)$. Next we give some applications of these results.

I. Asymptotic expansion and generalized Schläfli integral representation of the function $V(\lambda,t)$.

We consider the following class of singular second order differential equations of the form

(1) $$u''(t) - [\tfrac{\alpha^2-1/4}{t^2} - \lambda^2 - \chi(t)]u(t) = 0,\ \alpha > -1/2,\ t > 0,\ \lambda \in \mathbb{C},$$

where χ is a C^∞-function on $]0,+\infty[$ of the form $\chi(t) = \frac{a}{t^2} + e^{-\delta t}F(t)$, $\delta > 0$, $a \in \mathbb{R}$, here F and its derivatives are bounded on all interval $[t_0,+\infty[$, $t_0 > 0$.

This class contains (after some transformation) a wide class of singular second order differential equations of the form

(2) $$\Delta u(t) + \lambda^2 u(t) = 0,\ t > 0,\ \lambda \in \mathbb{C},$$

with

(3) $$\Delta u(t) = \tfrac{1}{A(t)}[A(t)u'(t)]' + q(t)u(t),$$

where $A(t) = t^{2\alpha+1}C(t)$, $\alpha > -1/2$, and C is an even positive C^∞-function on $\mathbb{R}$ such that $C(0) = 1$.

The Bessel and Jacobi operators whose functions A and q are given respectively by $A(t) = t^{2\alpha+1}$, $q(t) = 0$, and $A(t) = 2^{2(\alpha+\beta+1)}\sinh(t)^{2\alpha+1}\cosh(t)^{2\beta+1}$, $q(t) = (\alpha+\beta+1)^2$, $\alpha > -1/2$, $\beta \in \mathbb{R}$, are of the type (3). Also the radial part of the Laplace-Beltrami operator on a symmetric riemannian space of rank one is of type (3).

We denote by $V(\lambda,t)$ the unique solution of the equation (1) which is C^∞ on $]0,+\infty[$ with respect to the variable t and satisfying $V(\lambda,t) = (i\lambda)^{-(\alpha+1/2)}e^{-i\lambda t}U(\lambda,t)$ with $U(\lambda,t) \to 1$ as $t \to +\infty$. This function is analytic on $\{\lambda \in \mathbb{C} \ :\ Im(\lambda) < 0\}$ with respect to the variable λ. (See [1]).

Using Olver's method (see [3] page 219) we prove the following asymptotic expansion of the function $V(\lambda,t)$ with respect to the variable λ as $|\lambda| \to +\infty$, which intervens the Mac-Donald function K_ν (see [5] p 78).

Theorem I.1. *For all* $m \in \mathbb{N}$, *we have*

i) $\forall t \in]0,+\infty[$, $\forall \lambda \in \mathbb{C}$, $Im(\lambda) < 0$,

$$V(\lambda,t)=\sqrt{\frac{2t}{\pi}}\sum_{p=0}^{m}a_p(t)\frac{K_{\alpha+p}(i\lambda t)}{(i\lambda)^{\alpha+p}}+\mathcal{R}_m(\lambda,t),$$

where a_p, $p\in\mathbb{N}$, are C^∞-functions on $]0,+\infty[$ given by the recursive relations

$$\begin{cases} a_0(t)=1,\\ a_{p+1}(t)=-\frac{1}{2}\int_t^{+\infty}[a_p''(s)+\frac{1-2(\alpha+p)}{s}a_p'(s)+(\frac{p(2\alpha+p)}{s^2}+\chi(s))a_p(s)]ds, \end{cases}$$

here χ is the function given in the equation (1).

ii) The remainder $\mathcal{R}_m(\lambda,t)$ satisfies, for $t>0$ and $\lambda\in\mathbb{C}$, $Im(\lambda)<0$:

$$|\mathcal{R}_m(\lambda,t)|\leq\frac{k_{\lambda,m}(t)e^{-|Im(\lambda)|t}}{|\lambda|^{\alpha+m+3/2}}\left(\int_t^{+\infty}|a_{m+1}'(s)|ds\right)\,exp\{\frac{1}{|\lambda|}\int_t^{+\infty}|\frac{\alpha^2-1/4}{s^2}-\chi(s)|ds\}$$

and

$$|\frac{d}{dt}\mathcal{R}_m(\lambda,t)|\leq\frac{k_{\lambda,m}(t)e^{-|Im(\lambda)|t}}{|\lambda|^{\alpha+m+1/2}}\left(\int_t^{+\infty}|a_{m+1}'(s)|ds\right)\,exp\{\frac{1}{|\lambda|}\int_t^{+\infty}|\frac{\alpha^2-1/4}{s^2}-\chi(s)|ds\},$$

where

$$k_{\lambda,m}(t)=\begin{cases} c(m,\alpha)\frac{1+(|\lambda|t)^{\alpha+m-1/2}}{(|\lambda|t)^{\alpha+m-1/2}} & , \quad if \quad m\geq 1,\\ c(\alpha)\frac{1+(|\lambda|t)^{\alpha-1/2}}{(|\lambda|t)^{\alpha-1/2}} & , \quad if \quad m=0 \quad and \quad \alpha\geq 1/2,\\ c(\alpha) & , \quad if \quad m=0 \quad and \quad 0\leq\alpha<1/2,\\ c(-\alpha) & , \quad if \quad m=0 \quad and \quad -1/2\leq\alpha<0, \end{cases}$$

and $c(m,\alpha)$, $c(\alpha)$ and $c(-\alpha)$ *are positive constants.*

The following propositions gives properties of the remainder $\mathcal{R}_m(\lambda,t)$ and its derivatives.

Proposition I.1. *If $\alpha=k+r$, $k\in\mathbb{N}$ and $r\in]-1/2,1/2]$, then for all m, p, in $\mathbb{N}$ with $0\leq p\leq m+k$, and $\sigma>0$, there exists a C^∞-function $C_{m,p}$ on $]0,+\infty[$, bounded on every interval $[t_0,+\infty[$, $t_0>0$, such that for all $t>0$, $\lambda\in\mathbb{C}$, $Im(\lambda)<-\sigma$, we have*

$$|\frac{d^p}{dt^p}\mathcal{R}_m(\lambda,t)|\leq C_{m,p}(t)\frac{e^{-|Im(\lambda)|t}}{|\lambda|^{\alpha+m-p+3/2}}.$$

Proposition I.2. *Let $\alpha=k+r$, $k\in\mathbb{N}$, $r\in]-1/2,1/2]$ and $m\in\mathbb{N}$. The function $\mathcal{R}_m(\lambda,t)$ given in the theorem I.1, possesses the following integral representation*

$$\mathcal{R}_m(\lambda,t)=\int_t^{\infty}\mathcal{K}_m(t,s)e^{-i\lambda s}ds,\ t>0,\ \lambda\in\mathbb{C},\ Im(\lambda)<0.$$

Here the kernel $\mathcal{K}_m$ satisfies

i) For all $t>0$, $s\rightarrow\mathcal{K}_m(t,s)$ is a C^{m+k}-function on $]0,+\infty[$ with support in $[t,+\infty[$.

ii) For all $s>0$, $t\rightarrow\mathcal{K}_m(t,s)$ is a C^{m+k}-function on $]0,+\infty[$ with support in $[0,s]$.

Proposition I.3. *i) For all $m\in\mathbb{N}$ and $(t,s)\in]0,+\infty[\times]0,+\infty[$, $0<t<s$, we have*

$$\mathcal{K}_m(t,s)=\frac{1}{2^{\alpha-1/2}\Gamma(\alpha+1/2)}(\frac{s^2-t^2}{t})^{\alpha-1/2}\widetilde{\mathcal{K}}_m(t,s),$$

where

- for all $t>0$, $s\rightarrow\widetilde{\mathcal{K}}_m(t,s)$ is a C^m-function on $]0,+\infty[$ with support in $[t,+\infty[$,

- for all $s>0$, $t\rightarrow\widetilde{\mathcal{K}}_m(t,s)$ is a C^m-function on $]0,+\infty[$ with support in $[0,s]$.

ii) We have the recursive relation

$$\widetilde{\mathcal{K}}_m(t,s) = \frac{a_{m+1}(t)}{2^{m+1}(\alpha+1/2)_{m+1}}\left(\frac{s^2-t^2}{t}\right)^{m+1} + \widetilde{\mathcal{K}}_{m+1}(t,s)$$

where $(\alpha+1/2)_{m+1} = \dfrac{\Gamma(\alpha+m+3/2)}{\Gamma(\alpha+1/2)}$.

In 1871 Schläfli has proved the following integral representation

$$K_\nu(z) = \frac{\Gamma(1/2)(z/2)^\nu}{\Gamma(\nu+1/2)}\int_1^\infty e^{-zs}(s^2-1)^{\nu-1/2}ds,$$

where $Re(z) > 0$ and $Re(\nu) > -1/2$ (see [5] p 172).

From this relation we deduce the following formula

$$\sqrt{\frac{2t}{\pi}}\frac{K_\alpha(i\lambda t)}{(i\lambda)^\alpha} = \frac{t^{-\alpha+1/2}}{2^{\alpha-1/2}\Gamma(\alpha+1/2)}\int_t^{+\infty}(s^2-t^2)^{\alpha-1/2}e^{-i\lambda s}ds.$$

Using this relation and the proposition I.1 we establish the following result.

Theorem I.2. *The function $V(\lambda,t)$ admits the generalized Schläfli integral representation*

$$V(\lambda,t) = \int_t^\infty \mathcal{K}(t,s)e^{-i\lambda s}ds, \text{ for all } t>0,\ \lambda\in\mathbb{C},\ Im(\lambda)<0,$$

where $\mathcal{K}(t,s) = \dfrac{t^{-\alpha+1/2}}{2^{\alpha-1/2}\Gamma(\alpha+1/2)}(s^2-t^2)^{\alpha-1/2}\,\mathbb{I}_{]t,+\infty[}(s) + \mathcal{K}_0(t,s)$, *and* $\mathcal{K}_0(t,s)$ *is the function given in the proposition I.2.*

II. Applications.

In the following we give some applications of the previous study.

II.1. The operator Δ.

We consider the singular second order differential operator Δ given by the relation (3).

We assume that the functions A and q satisfy also the following conditions : $\lim\limits_{t\to+\infty}\dfrac{A'(t)}{A(t)} = 2\rho \geq 0$, and there exist two C^∞-functions F_1 and F_2 bounded together with their derivatives on all interval $[t_0,+\infty[$, $t_0>0$, such that

$$\begin{cases} \dfrac{C'(t)}{C(t)} = 2\rho - \dfrac{2\alpha+1}{t} + e^{-\delta t}F_1(t), if\ \rho>0, \\ \dfrac{C'(t)}{C(t)} = e^{-\delta t}F_1(t), if\ \rho = 0, \\ q(t) = -\rho^2 + \dfrac{\beta}{t^2} + e^{-\delta t}F_2(t), \end{cases}$$

where $\delta>0$, $\beta\in\mathbb{R}$.

It is known (see [1]) that there exists a unique function $\Phi_{-\lambda}(t)$ which is C^∞ on $]0,+\infty[$ with respect to the variable t, satisfying the differential equation

$$\begin{cases} \Delta\Phi_{-\lambda}(t) = -\lambda^2\Phi_{-\lambda}(t), \\ \Phi_{-\lambda}(t) \cong e^{-(i\lambda+\rho)t},\ (t\to+\infty). \end{cases}$$

This function is analytic on $\{\lambda \in \mathbb{C} \ : Im(\lambda) < 0\}$ with respect to the variable λ.

The function $t^{\alpha+1/2}\sqrt{C(t)}\dfrac{\Phi_{-\lambda}(t)}{(i\lambda)^{\alpha+1/2}}$ admits the asymptotic expansion with respect to the variable λ as $|\lambda \to +\infty$ given by the theorem I.1, and using the theorem I.2, we deduce the following result.

Theorem II.1. *The fonction* $\dfrac{\Phi_{-\lambda}(t)}{(i\lambda)^{\alpha+1/2}}$ *possesses the generalised Schläfli integral representation*

$$\frac{\Phi_{-\lambda}(t)}{(i\lambda)^{\alpha+1/2}} = \int_t^{\infty} \mathcal{H}(t,s)e^{-i\lambda s}ds, \text{ for all } t > 0,\ \lambda \in \mathbb{C},\ Im(\lambda) < 0,$$

where $\mathcal{H}(t,s) = \dfrac{1}{t^{\alpha+1/2}\sqrt{C(t)}}\mathcal{K}(t,s)$, *and* $\mathcal{K}(t,s)$ *is the kernel given by the theorem I.2.*

II.2. The Jacobi operator.

In this case the function $\Phi_{-\lambda} = \Phi_{-\lambda}^{(\alpha,\beta)}$, $\alpha > -1/2$, $\beta \in \mathbb{R}$, is the Jacobi function of second kind, (see [2]), given by

$$\Phi_{-\lambda}^{(\alpha,\beta)}(t) = (e^t - e^{-t})^{-i\lambda-\rho}\,{}_2F_1\Big(\frac{i\lambda-\alpha+\beta+1}{2}, \frac{i\lambda+\alpha+\beta+1}{2}; 1+i\lambda; -\sinh^{-2}(t)\Big)$$

where ${}_2F_1$ is the Gauss hypergeometric function.

In the following we assume that $\alpha > -1/2$ and $\beta = -1/2$. In this case the kernel $\mathcal{H}(t,s)$ given by the theorem II.1 can be written explicitely.

Theorem II.2. *If* $\alpha > -1/2$ *and* $\beta = -1/2$ *then the kernel* $\mathcal{H}(t,s)$ *has the form :*

$$\mathcal{H}(t,s) = \frac{(\sinh(t))^{-2\alpha}}{2^{\alpha+1/2}\Gamma(\alpha+1/2)}\Big[\mathcal{H}_1(t,s) + \int_t^s \mathcal{H}_1(t,v)P(s-v)dv\Big]$$

with

$$\mathcal{H}_1(t,s) = \Big[(\cosh(s)-\cosh(t))^{(\alpha-1/2)} + \int_t^s (\cosh(v)-\cosh(t))^{(\alpha-1/2)} f'(s-v)dv\Big]\ \mathbb{I}_{]t,+\infty[}(s),$$

P and f are given by

i) if $\alpha = 1/2$, *then* $P(s) = f(s) = 0$,

ii) if $\alpha \in]-1/2, 1/2[$, *then* $P(s) = 0$ *and*

$$f(s) = \frac{1}{\Gamma(1/2-\alpha)\Gamma(1/2+\alpha)}\int_0^s (s-v)^{\alpha-1/2}(1-e^{-v})^{-\alpha-1/2}dv$$

iii) if $\alpha = k + 1/2$, $k \in \mathbb{N}^*$, *then* $f(s) = 0$ *and*

$$P(s) = \sum_{p=0}^{k-1} \frac{s^{k-p-1}}{p!(k-p-1)!}\Big(\frac{d}{dv}\Big)^p\Big[\prod_{j=1}^{k}(v-j)\Big]_{v=0}$$

iv) if $\alpha = k + r$, $k \in \mathbb{N}^*$ *and* $r \in]-1/2, 1/2[$, *then*

$$P(s) = \sum_{p=0}^{k-1} \frac{s^{k-p-1}}{p!(k-p-1)!}\Big(\frac{d}{dv}\Big)^p\Big[\prod_{j=1}^{k}(v+1/2-j-r)\Big]_{v=0}$$

and

$$f(s) = \frac{1}{\Gamma(1/2-r)\Gamma(1/2+r)}\int_0^s (s-v)^{r-1/2}(1-e^{-v})^{-r-1/2}dv.$$

II.3. The Whittaker operator.

We denote by $\mathcal{M}_{\beta,i\lambda}$ the Whittaker function, (see [3] p 260), given by

$$\mathcal{M}_{\beta,i\lambda}(t) = t^{i\lambda+1/2}F(i\lambda-\beta-1/2;2i\lambda+1;t)$$

where $F(a;c;t)$ is the confluent hypergeometric function of first kind (see [3] p 255). The function $e^{t/2}\mathcal{M}_{\beta,i\lambda}(e^{-t})$ satisfies the following differential equation $u''(t)-[\frac{1}{4}e^{-2t}-\beta e^{-t}-\lambda^2]u(t)=0$, with $e^{t/2}\mathcal{M}_{\beta,i\lambda}(e^{-t}) \cong \frac{e^{-i\lambda t}}{i\lambda}$ $(t \to +\infty)$.

The function $e^{t/2}\mathcal{M}_{\beta,i\lambda}(e^{-t})$ admits the asymptotic expansion with respect to the variable λ as $|\lambda \to +\infty$ given by the theorem I.1, and applying the theorem I.2 we obtain:

Theorem II.3. *The function* $\frac{\mathcal{M}_{\beta,i\lambda}(e^{-t})}{i\lambda}$ *admits the Generalized Schläfli integral representation*

$$\frac{\mathcal{M}_{\beta,i\lambda}(e^{-t})}{i\lambda} = \int_t^{\infty} \mathcal{K}(s,t)e^{-i\lambda s}ds, \text{ for all } t>0,\ \lambda\in\mathbb{C},\ Im(\lambda)<0,$$

where $\mathcal{K}(t,.)$ *is continuous on* $]t,+\infty[$ *with support in* $[t,+\infty[$. *Moreover we have* $\mathcal{K}(t,s) = e^{-t/2}\, \mathbb{I}_{]t,+\infty[}(s) + e^{-t/2}\mathcal{K}_0(t,s)$, *and* $\mathcal{K}_0(t,s)$ *is the function given in the proposition I.2.*

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Intégrales Supérieures de Fubini

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Presenté par P. Ribenboim, F.R.S.C.

Résumé - On établit une version du théorème de Fubini dans un cadre très général. Les résultats obtenus sont pour une fonctionnelle sublinèaire positive, une approche qui généralise les théorèmes antérieurs de type Fubini aussi bien dans le cas dénombrablement additif que dans le cas finiment additif.

1. INTÉGRATION ESSENTIELLE. - On utilise la notation ensembliste habituelle. Sur la droite réelle achevée $\bar{\mathbb{R}}$, nous adoptons les conventions suivantes: $a + b := 0$ et $a \dot{+} b := \infty$, si $a = -b \in \{-\infty, \infty\}$. On note $a \wedge b := \min\{a, b\}, a \vee b := \max\{a, b\}, a^{\pm} := \pm a \vee 0$ et $a \cap t := (a \wedge t) \vee (-t)$, si $a, b, t \in \bar{\mathbb{R}}, t \geq 0$.

Pour tout sous ensemble $A \subset \bar{\mathbb{R}}$ on a $\inf A, \sup A \in \bar{\mathbb{R}}$, avec les conventions usuelles $\inf \emptyset := \infty$ et $\sup \emptyset := -\infty$.

Étant donné un ensemble non vide X, $\bar{\mathbb{R}}^X$ designera l'ensemble des fonctions de X dans $\bar{\mathbb{R}}$.

Les opérations et les relations entre les fonctions sont définies ponctuellement. Pour un sous ensemble $M \subset \bar{\mathbb{R}}^X$, nous posons $+M = \{f \in M; f \geq 0\}$.

Supposons que M soit un treillis vectoriel, i.e., un espace vectoriel de fonctions telles que si $f \in M$ implique $|f| \in M$; alors $f \wedge g$ et $f \vee g \in M$ pour toute $f, g \in M$.

Une fonctionnelle $q : \bar{\mathbb{R}}^X \rightarrow \bar{\mathbb{R}}$ est dite intégrale supérieure si

$$q(0) = 0,\ q\left(f \dot{+} g\right) \leq q(f) \dot{+} q(g) \text{ et } q(f) \leq q(h), \text{ si } f, g, h \in \bar{\mathbb{R}}^X, f \leq h.$$

L'intégrale supérieure q est dite reguliére en M si $q(h) = q_*(h) := -q(-h)$, pour toute $h \in M$, et elle est dite determinée par M si $q(f) = \inf\{q(g): f \leq g \in M\}$ pour toute $f \in \bar{\mathbb{R}}^X$.

Dans tout ce qui suit, on suppose que q definie sur $\bar{\mathbb{R}}^X$, est une intégrale supérieure reguliére determinée par un treillis vectoriel M.

Une fonction $f \in \bar{\mathbb{R}}^X$ est dite $q - M$-intégrable si elle est dans la fermeture de M dans $\bar{\mathbb{R}}^X$, relativement a la seminorme intègrale $q(|\cdot|)$, ou d'une mainiére équivalente á $q(f) = q_*(f) \in \mathbb{R}$.

Étant donnée $f \in \bar{\mathbb{R}}^X$, l'intégrale supérieure localisée de q est definie par:

$$q_\ell(f) := \sup\{q(f \wedge h)\,;\, h \in +M\}$$

(c'est une versión simplifiée de la definitión de [12, p. 120]).

Il est clair que q_ℓ est, á nouveau, une intégrale supérieure. $q_\ell \leq q$ dans $\bar{I\!R}^X$ et $q_\ell(f) = q(f)$ s'il existe $h \in M$ tel que $f \leq h$.

THÉORÈME 1.- Pour une fonction $f \in \bar{I\!R}^X$, les affirmations suivantes sont équivalentes:

1. f appartient á la fermeture de M dans $\bar{I\!R}^X$ relativement a la semi-norme $q_\ell(|\cdot|)$.
2. $q(f) = q_*(f) \in I\!R$.
3. Pour toute $h \in +M, f \wedge h$ est $q - M$-intégrable et $q_\ell(|f|) < +\infty$.
4. Quelque soient $h, k \in +M, (f \wedge h) \vee (-k)$ est $q - M$-intégrable et

   $$\lim_{h,k \in +M} q[(f \wedge h) \vee (-k)]$$

   existe en $I\!R$. (La limite est prise relativement a $(+M) \times (+M)$ ordonnée par $\leq$.)
5. $q^\bullet(f) = q_\bullet(f) \in I\!R$, oú

   $$q^\bullet(f) := \inf_{k \in +M}\left(\sup_{h \in +M} q[(f \wedge h) \vee (-k)]\right)$$

   et $q_\bullet(f) := -q^\bullet(-f)$.

Une fonction $f \in \bar{I\!R}^X$ est dite $q - M$-essentiellement intégrable si elle vérifie les conditions équivalentes 1-5 anterieures. L'ensemble de toutes les fonctions $q - M$-essentiellement intégrables est designé par M^{q_ℓ}.

2. THÉORÈME DE TYPE FUBINI. - On etudie des systèmes produit par rapport a deux intégrales superieures q_1 et q_2 données.

Les notations et définitions sur les systèmes produit, utilisées dans la suite, sont les mêmes que dans [9], [11] et [1].

Supposons que pour $i = 1, 2$, X_i est un ensemble arbitraire non vide, $M_i \subset \bar{I\!R}^{X_i}$ est un treillis vectoriel et q_i est une intégrale supérieure reguliére et déterminée par M_i.

Soint $X_3 := X_1 \times X_2$ et $f \in \bar{\mathbb{R}}^{X_3}$. Pour chaque $x \in X_1$, on pose $f_x(y) := f(x,y)$, pour tout $y \in X_2$ et $(q_2 f)(x) := q_2(f_x)$.

Soient $M_3 \subset \bar{\mathbb{R}}^{X_3}$ un treillis vectoriel et q_3 une intégrale superiéure en $\bar{\mathbb{R}}^{X_3}$ déterminéé par M_3. Par $q_{i,\ell}, i = 1,2,3$, nous notons la localisation intégrale correspondante.

On appelle (X_3, M_3, q_3) système integrale produit si les conditions suivantes sont vérifiées:

1. $f_x \in M_2$, pour tout $x \in X_1$.
2. $q_2 f \in M_1$.
3. $q_3(f) = q_1(q_2 f)$.

Pour obtenir un théorème de type Fubini satisfaisant, les conditions suivantes apparaissent d'une maniére naturelle (voir [9], [6] et [1]):

(α) Pour $f \in +\bar{\mathbb{R}}^{X_3}$ et $x \in X_1$ il existe $g \in M_2^{q_{2,\ell}}$ tel que $f_x \leq g$.

(β) Pour $h \in +M_1$ et $g \in M_2$ il existe $k \in M_3$ tel que $g(y) \leq k(x,y)$, si $h(x) > 0$, pour $x \in X_1$ et $y \in X_2$.

(γ) Si $h \in +M_1$, alors $1 \wedge h \in M_1$ (Condition de Stone) et $q_1(h \wedge \varepsilon) \to 0$, si $\varepsilon \to 0$ (continuité en 0).

LEMME 2. - Si les conditions (β) et (γ) sont vérifiées et $f \in +\bar{\mathbb{R}}^{X_3}$ vérifie (α), alors

$$q_{1,\ell} \circ q_{2,\ell}(f) \leq q_{3,\ell}(f).$$

Étant donneé une intégrale supérieure arbitraire q sur $\bar{\mathbb{R}}$, un ensemble $A \subset X$ est dit $q-$nul si $q(\chi_A) = 0$.

THÉORÈME 3. (Théorème de type Fubini) - Supposons que les conditions (β) et (γ) soient vérifeés. Si $f \in M_3^{q_{3,\ell}}$ verifie (α), alors on a les affirmations suivantes:

1. Il existe une suite (A_n) d'ensembles $q_{1,\ell}-$nuls dans X_1 telle que

$$\left\{x \in X_1; f_x \notin M_2^{q_{2,\ell}}\right\} \subset \bigcup_{n=1}^{\infty} A_n.$$

2. $q_{2,\ell}f, (q_{2,\ell}f)_* \in M_1^{q_{1,\ell}}$.

3. $q_{3,\ell}(f) = q_{1,\ell} \circ q_{2,\ell}(f)$.

Esquisse de la démonstration: Lemma 2 nous donne

$$q_{3,\ell}(f) \geq q_{1,\ell} \circ q_{2,\ell}(f) \geq \left\{ \begin{matrix} (q_{1,\ell})_*(q_{2,\ell}f) \\ q_{1,\ell}((q_{2,\ell}f)_*) \end{matrix} \right\} \geq q_{1,\ell*} \circ q_{2,\ell*}(f) \geq (q_{3,\ell})_*(f).$$

Posons $h(x) := q_{2,\ell}(f_x) - (q_{2,\ell})_*(f_x)$, pour tout $x \in X_1$ et

$$A_n := \left\{ x \in X_1; h(x) > \tfrac{1}{n} \right\}$$

avec $n \in I\!N$. Alors, si $x \in X_1 - \bigcup_{n=1}^{\infty} A_n$, on a $h(x) = 0$.

3. APPLICATIONS. - (Voir [3] et [1].) Étant donné un treillis vectoriel $M \subset \bar{I\!R}^X$ et une fonctionnelle non négative I dans M, avec

$$q(f) = I^-(f) := \inf\{I(g); f \leq g \in +M\}$$

on obtient la classe M^{q_ℓ} des fonctions intégrables abstraites de Riemann dans [3], classe qui contient la "one sided-completion" de Loomis [10, p.178].

Si en plus on suppose la condition de continuité de Daniell (i.e., $I(h_n) \to 0$, avec $h_n \in +M, h_n(x) \searrow 0$ pour chaque $x \in X$) avec

$$q(f) := \inf\left\{ \sum_{n=1}^{+\infty} I(h_n); f \leq \sum_{n=1}^{+\infty} h_n, h_n \in +M \right\}.$$

pour toute $f \in \bar{I\!R}^X$, on obtient la classe M^{q_ℓ} des fonctions I-intégrables au sens classique de Daniell. Dans ce cas les hypothèses du lemme 2 sont vérifiées (voir [3, p.424]).

Le théorème 3 donne un résultat de type Fubini, même sans aucune hypothèse de continuité monotone pour l'intégrale èlémentaire, en differant du processus d'intégration de Daniell.

Considérons maintenant M et I induits par une fonction d'ensemble finiment additive μ sur un semi anneau Ω, i.e., $M = M_\Omega :=$ classe des fonctions étageés á valeurs réelles définies sur Ω, $I = I_\mu := \int \cdot d\mu$. (Les conditions (β) et (γ) sont vérifiées d'une maniére automatique.)

Avec $q := I_\mu^-$ comme auparavant, on obtient la classe $M_\Omega^{q_\ell}$ des fonctions μ-intégrables abstraites Riemann de Günzler [8, A.146], classe laquelle contient l'espace $L^1(X, \Omega, \mu, I\!R)$ des fonctions μ-intégrables au sens de Dunford-Schwartz.

Si Ω est un σ-anneau et $q = I_\mu^-$ avec μ dénombrablement additive, alors $R_1\left(\mu, \bar{I\!R}\right) = L_1\left(\mu, \bar{I\!R}\right)$ = la classe des fonctions μ-intégrables au sens de Lebesgue.

Le théorème 3 etablit, avec une preuve simplifiée , un théorème de type Fubini pour l'intégration par rapport aux mesures finiment additives, lequel généralise les resultats correspondants de [9] et [12]. (Voir aussi [1].)

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Inversion Formulas for The Generalized Radon Transform and Its Dual Associated with Singular Partial Differential Operators

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Presented by P.G. Rooney, F.R.S.C.

Abstract. We consider the generalized Radon transform ${}^tR_{\alpha,\beta}$ and its dual $R_{\alpha,\beta}$; $\alpha, \beta \geq 0$, associated with two singular partial differential operators D_1 and D_2. We give some results about the harmonic analysis associated with the operators D_1 and D_2. We deduce inversion formulas for the operators $R_{\alpha,\beta}$ and ${}^tR_{\alpha,\beta}$ and a Plancherel theorem for the operator ${}^tR_{\alpha,\beta}$. Next we prove that the operators $R_{\alpha,\beta}$ and ${}^tR_{\alpha,\beta}$ are transmutation operators on some spaces of functions.

I. Generalized Radon transform and its dual associated with the operators D_1 and D_2

Notations. We denote by

- $\mathcal{E}_*(\mathbb{R})$ the space of even $\mathcal{C}^\infty$ functions on $\mathbb{R}$.
- $S_*(\mathbb{R})$ the space of even $\mathcal{C}^\infty$ functions on $\mathbb{R}$, rapidly decreasing together with their derivatives.
- $\mathcal{D}_*(\mathbb{R})$ the space of even $\mathcal{C}^\infty$ functions on $\mathbb{R}$ with compact support.

For $\alpha, \beta \geq 0$, such that $\beta-\alpha>0$, the Sonine transform $S_{\beta-1/2,\ \alpha-1/2}$ is defined on $\mathcal{E}_*(\mathbb{R})$ by

$S_{\beta-1/2,\ \alpha-1/2}(f)(x) = \frac{2\Gamma(\beta+1/2)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1/2)} \int_0^1 (1-u^2)^{\beta-\alpha-1} f(x\,u)\, u^{2\alpha}\, du$, and its dual on $\mathcal{D}_*(\mathbb{R})$ by

$${}^tS_{\beta-1/2,\ \alpha-1/2}(g)(y) = \frac{2\Gamma(\beta+1/2)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1/2)} \int_y^\infty (x^2-y^2)^{\beta-\alpha-1} g(x)\, x\, dx\ .$$

Let $\gamma>0$, the Bessel operator $L_{\gamma-1/2}$ on $]0, +\infty[$ is defined by $L_{\gamma-1/2} = \frac{d^2}{dr^2} + \frac{2\gamma}{r}\frac{d}{dr}$.

i) The Riemann-Liouville transform $R_{\gamma-1/2}$ is defined on $\mathcal{E}_*(\mathbb{R})$ by

$$R_{\gamma-1/2}(f)(x) = \frac{2\Gamma(\gamma+1/2)}{\sqrt{\pi}\,\Gamma(\gamma)} \int_0^1 (1-y^2)^{\gamma-1} f(x\,y)dy \quad \text{.(see [4])}$$

ii) The Weyl transform is defined on $\mathcal{D}_*(\mathbb{R})$ by

$$W_{\gamma-1/2}(f)(x) = \frac{2\Gamma(\gamma+1/2)}{\sqrt{\pi}\,\Gamma(\gamma)} \int_x^\infty (y^2-x^2)^{\gamma-1} f(y)\, y\, dy$$

iii) The generalized translation operator $T_x^{\gamma-1/2}$ is defined on $\mathcal{E}_*(\mathbb{R})$ by

$$T_x^{\gamma-1/2}(f)(y) = \frac{\Gamma(\gamma+1/2)}{\sqrt{\pi}\,\Gamma(\gamma)}\int_0^\pi f\left(\sqrt{x^2+y^2+2xy\cos\theta}\right)(\sin\theta)^{2\gamma-1}d\theta$$

Let D_1 and D_2 be the singular partial differential operators given by

$$D_1 = \frac{\partial^2}{\partial x^2} + \frac{2\beta}{x}\frac{\partial}{\partial x}; \qquad D_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - D_1;\ \alpha \geq 0,\ \beta \geq 0$$

Theorem I.1. The system of partial differential equations

$$\begin{cases} D_1u(r,x) = -\lambda^2u(r,x)\,,\ \lambda \in \mathbb{C} \\ D_2u(r,x) = -\mu^2u(r,x)\,,\ \mu \in \mathbb{C} \\ u(0,0) = 1,\ \dfrac{\partial u}{\partial x}(r,0) = 0,\ \dfrac{\partial u}{\partial r}(0,x) = 0 \end{cases}$$

admits a unique solution given by $\varphi_{\mu,\lambda}^{(\alpha,\beta)}(r,x) = j_\alpha(r\sqrt{\mu^2+\lambda^2})j_{\beta-1/2}(\lambda x)$,where $j_\gamma(s) = (2^\gamma\,\Gamma(\gamma+1)J_\gamma(x))/s^\gamma$ is the normalized Bessel function. J_γ being the Bessel function of first kind and order γ ([5]) .

Notations. We denote by

- $\mathcal{E}_*(\mathbb{R}^2)$ the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^2$, even with respect to each variable.
- $S_*(\mathbb{R}^2)$ the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^2$, even with respect to each variable and rapidly decreasing together with their derivatives.
- $\mathcal{D}_*(\mathbb{R}^2)$ the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^2$, even with respect to each variable and with compact support.

Definition I.1. The generalized dual Radon transform $R_{\alpha,\beta}$ associated with the operators D_1 and D_2 is defined on $\mathcal{E}_*(\mathbb{R}^2)$ by

i) If $\beta \geq 0$ and $\alpha-\beta>0$, $R_{\alpha,\beta}(f)(r,x) = a_\alpha\int_0^1(1-t^2)^{\alpha-1/2}S_{\alpha-1/2,\beta-1/2}\circ T_x^{\beta-1/2}\circ R_{\beta-1/2}(f(rt,.))(r\sqrt{1-t^2})dt$

ii) If $\alpha \geq 0$ and $\beta-\alpha>0$, $R_{\alpha,\beta}(f)(r,x) = a_\alpha\int_0^1(1-t^2)^{\alpha-1/2}S_{\beta-1/2,\alpha-1/2}\circ T_{r\sqrt{1-t^2}}^{\beta-1/2}\circ R_{\alpha-1/2}(f(rt,.))(x)dt$

iii) If $\alpha=\beta\geq 0$, $R_{\alpha,\alpha}(f)(r,x) = a_\alpha\int_0^1(1-t^2)^{\alpha-1/2}T_x^{\alpha-1/2}\circ R_{\alpha-1/2}(f(rt,.))(r\sqrt{1-t^2})\,dt$

where $a_\alpha = 2\Gamma(\alpha+1)/\sqrt{\pi}\,\Gamma(\alpha+1/2)$

Remark. For $(\mu,\lambda) \in \mathbb{C}^2$, we have $\varphi_{\mu,\lambda}^{(\alpha,\beta)}(r,x) = R_{\alpha,\beta}(\cos(\mu.)\cos(\lambda.))(r,x)$.

Definition I.2. The generalized Radon transform ${}^{t}R_{\alpha,\beta}$ associated with the operators D_1 and D_2 is defined on $\mathcal{D}_*(\mathbb{R}^2)$ by

i) If $\beta \geq 0$ and $\alpha-\beta>0$, ${}^{t}R_{\alpha,\beta}(g)(t,x) = a_\alpha \int_t^\infty (r^2-t^2)^{\alpha-1/2} S_{\alpha-1/2,\beta-1/2} \circ W_{\beta-1/2} \circ T_x^{\beta-1/2}(g(r,.))(\sqrt{r^2-t^2})\, r\,dr$

ii) If $\alpha \geq 0$ and $\beta-\alpha>0$, ${}^{t}R_{\alpha,\beta}(g)(t,x) = a_\alpha \int_t^\infty (r^2-t^2)^{\alpha-1/2} W_{\alpha-1/2} \circ T_{\sqrt{r^2-t^2}}^{\alpha-1/2} \circ {}^{t}S_{\beta-1/2,\alpha-1/2}(g(r,.))(x)\, r\,dr$

iii) If $\alpha=\beta \geq 0$, ${}^{t}R_{\alpha,\alpha}(g)(t,x) = a_\alpha \int_t^\infty (r^2-t^2)^{\alpha-1/2} W_{\alpha-1/2} \circ T_x^{\alpha-1/2}(g(r,.))(\sqrt{r^2-t^2})\, r\,dr$

II. Paley-Wiener and Plancherel theorems for the generalized Fourier transform associated with the operators D_1 and D_2.

Notations. We denote by

- $\Gamma = \mathbb{R}^2 \cup \{(i\mu,\lambda) ; (\mu,\lambda) \in \mathbb{R}^2 \text{ and } |\mu| \leq |\lambda|\}$
- $L^2(r^{2\alpha+1}x^{2\beta}drdx)$ the space of square integrable functions on $[0,+\infty[\times[0,+\infty[$ with respect to the measure $r^{2\alpha+1}x^{2\beta}drdx$.
- $L^2(d\gamma)$ the space of square integrable functions on Γ with respect to the measure $d\gamma$ given by

$$\int_\Gamma f(z,\lambda)d\gamma(z,\lambda) = \frac{2^{1-2\alpha-2\beta}}{\Gamma^2(\alpha+1)\Gamma^2(\beta+1/2)} \left\{ \int_0^\infty \int_0^\infty f(t,x)(t^2+x^2)^\alpha x^{2\beta} t\,dt\,dx + \int_0^\infty \int_0^x f(it,x)(x^2-t^2)^\alpha x^{2\beta} t\,dt\,dx \right\}$$

- $S_*(\Gamma)$ the space of functions $g : \Gamma \longrightarrow \mathbb{C}$, infinitely differentiable, even with respect to each variable and rapidly decreasing together with their derivatives.
- $\mathbb{H}_{*,0}(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$, even with respect to each variable rapidly decreasing of exponential type and such that for all $k \in \mathbb{N}$:

$$\sup\{(1-\mu^2+2\lambda^2)^k |f(i\mu,\lambda)|;\ (\mu,\lambda) \in \mathbb{R}^2,\ |\mu| \leq |\lambda|\} < +\infty$$

- $\mathcal{E}'_*(\mathbb{R}^2)$ the space of distributions on $\mathbb{R}^2$, even with respect to each variable, with compact support.
- $\mathcal{H}_{*,0}(\mathbb{C}^2)$ the space of entire functions $g : \mathbb{C}^2 \longrightarrow \mathbb{C}$, even with respect to each variable, slowly increasing of exponential type and such that ther exists $k \in \mathbb{N}$:

$$\sup\{(1-\mu^2+2\lambda^2)^{-k} |g(i\mu,\lambda)|,\ (\mu,\lambda) \in \mathbb{R}^2,\ |\mu| \leq |\lambda|\} < +\infty$$

Definition II.1. The generalized Fourier transform $\mathcal{F}_{\alpha,\beta}$ associated with the operators D_1 and D_2 is defined on $S_*(\mathbb{R}^2)$ by $\mathcal{F}_{\alpha,\beta}(f)(\mu,\lambda) = \int_0^\infty \int_0^\infty f(r,x)\varphi_{\mu,\lambda}^{(\alpha,\beta)}(r,x)\, r^{2\alpha+1} x^{2\beta}\, drdx$; $(\mu,\lambda) \in \Gamma$, and on $\mathcal{E}'_*(\mathbb{R}^2)$ by $\mathcal{F}_{\alpha,\beta}(T)(\mu,\lambda) = \langle T, \varphi_{\mu,\lambda}^{(\alpha,\beta)} \rangle$; $(\mu,\lambda) \in \mathbb{C}^2$.

Using analogous proofs as in [3], we obtain the following results

Theorem II.1. i) The generalized Fourier transform $\mathcal{F}_{\alpha,\beta}$ is an isomorphism from $S_*(\mathbb{R}^2)$ onto $S_*(\Gamma)$. The inverse transform is given by $\mathcal{F}^{-1}_{\alpha,\beta}(f)(r,x) = \int_\Gamma f(z,\lambda)\, \varphi^{(\alpha,\beta)}_{z,\lambda}(r,x)\, d\gamma(z,\lambda)$.

ii) For all $f \in S_*(\mathbb{R}^2)$, we have the Plancherel formula for $\mathcal{F}_{\alpha,\beta}$:

$$\int_0^\infty \int_0^\infty |f(r,x)|^2 r^{2\alpha+1} x^{2\beta} dr dx = \int_\Gamma |\mathcal{F}_{\alpha,\beta}(f)(z,\lambda)|^2 d\gamma(z,\lambda)$$

iii) The generalized Fourier transform $\mathcal{F}_{\alpha,\beta}$ can be extended to an isometric isomorphism from $L^2(r^{2\alpha+1} x^{2\beta} dr dx)$ onto $L^2(d\gamma)$.

Theorem II.2.(of Paley-Wiener) The generalized Fourier transform $\mathcal{F}_{\alpha,\beta}$ is an isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ (respectively $\mathcal{E}'_*(\mathbb{R}^2)$)onto $\mathbb{H}_{*,0}(\mathbb{C}^2)$(respectively $\mathcal{H}_{*,0}(\mathbb{C}^2)$).

III. Fractional powers of the Laplacian and of the Bessel operator .

Notations . We denote by

- $S'_*(\mathbb{R})$ the space of even tempered distributions on $\mathbb{R}$.
- $S'_*(\mathbb{R}^2)$ the space of tempered distributions on $\mathbb{R}^2$, even with respect to each variable.

a) Fractional powers of the Laplacian on $\mathbb{R}^2$.([1] , [2])

For $\gamma \in \mathbb{C} - \mathbb{Z}$, the fractional power $(-\Delta)^\gamma$ of the laplacian $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2}$ on $\mathbb{R}^2$ is defined on $S_*(\mathbb{R}^2)$ by $(-\Delta)^\gamma f(r,x) = (2^{2\gamma}\Gamma(1+\gamma)/\pi\Gamma(-\gamma))\big(T_{(s^2+y^2)^{-\gamma-1}} * f\big)(r,x)$, where $*$ is the classical convolution product on $\mathbb{R}^2$ and $(2^{2\gamma}\Gamma(1+\gamma)/\pi\Gamma(-\gamma))T_{(s^2+y^2)^{-\gamma-1}}$ is the distribution of $S'_*(\mathbb{R})$ given by the function $(2^{2\gamma}\Gamma(1+\gamma)/\pi\Gamma(-\gamma))(s^2+y^2)^{-\gamma-1}$.

Again, for $\gamma \in \mathbb{C}$, $\gamma \notin \mathbb{N} \cup (-(1/2) - \mathbb{N})$, we define the operators $(-J_1)^\gamma$ and $(-J_2)^\gamma$ on $S_*(\mathbb{R}^2)$ by $(-J_1)^\gamma f(r,x) = b_\gamma \int_{\mathbb{R}} \frac{f(s,x)}{|r-s|^{2\gamma+1}} ds$, and $(-J_2)^\gamma f(r,x) = b_\gamma \int_{\mathbb{R}} \frac{f(r,y)}{|x-y|^{2\gamma+1}} dy$, where $b_\gamma = \frac{2^{2\gamma}\Gamma(1+\gamma)}{\sqrt{\pi}\,\Gamma(-\gamma)}$.

b) Fractional powers of the Bessel operator on $]0,+\infty[$.

For $a \in \mathbb{C}$ and $\alpha > -1/2$, we denote by $|x|^a$ the even tempered distribution on $\mathbb{R}$ defined by

$$<|x|^a, \varphi> = \int_0^\infty \varphi(x)\, x^{a+2\alpha+1} dx, = <x_+^{a+2\alpha+1}, \varphi>, \quad \varphi \in S_*(\mathbb{R})$$

The mapping a —> $|x|^a$ can be extended to an holomorphic distribution valued function on $\mathbb{C} - \{a \,/\, a = -2(\alpha+k),\ k \in \mathbb{N}^*\}$

For $a \in \mathbb{C} - \mathbb{N} \cup (-\alpha - \mathbb{N}^*)$, the fractional power $(-L_\alpha)^a$ of the Bessel operator $L_\alpha = \frac{d^2}{dr^2} + \frac{2\alpha+1}{r}\frac{d}{dr}$ is defined on $S_*(\mathbb{R})$ by $(-L_\alpha)^a f(x) = \frac{2^{2a}\Gamma(a+\alpha+1)}{\Gamma(\alpha+1)\Gamma(-a)} |y|^{-2a-2\alpha-2} \#_\alpha f(x)$, where $\#_\alpha$ is the Bessel convolution product defined by $T \#_\alpha f(x) = <T, T^\alpha_x f>$; $T \in S'_*(\mathbb{R})$, $f \in S_*(\mathbb{R})$. T^α_x being the generalized translation operator associated with the Bessel operator.

IV. Inversion formulas for the operators $R_{\alpha,\beta}$ and ${}^tR_{\alpha,\beta}$.

Notations. We denote by

- $S_{*,0}(\mathbb{R}^2)$ the subspace of $S_*(\mathbb{R}^2)$ consisting of functions f such that for all one variable polynomials P and Q we have $\int_0^\infty P(r)f(r,x)\,dr = 0$; $x \geq 0$, and $\int_0^\infty Q(x)f(r,x)\,dx = 0$; $r \geq 0$
- $S_*^0(\mathbb{R}^2)$ the susbspace of $S_*(\mathbb{R}^2)$ consisting of functions f such that the function

(IV.1) $$\tilde{\mathfrak{F}}_{\alpha,\beta}(f)(\mu,\lambda) = \int_0^\infty\int_0^\infty f(r,x)\, j_\alpha(r\mu)\, j_{\beta-1/2}(\lambda x)\, r^{2\alpha+1} x^{2\beta} dr dx; \qquad (\mu,\lambda) \in \mathbb{R}^2$$

is supported in $\{(t,x) \in \mathbb{R}^2, |t| > |x| > 0\}$.

- S the distribution of $S_*'(\mathbb{R}^2)$ defined by $< S, \varphi > = \int_0^\infty \varphi(z,z)\, z^{2\beta}\, dz$; $\varphi \in S_*(\mathbb{R}^2)$.
- $\overset{r}{L}_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}$, $\overset{x}{L}_{\beta-1/2} = \frac{\partial^2}{\partial x^2} + \frac{2\beta}{x}\frac{\partial}{\partial x}$, $\Delta_{\alpha,\beta} = (\overset{x}{L}_{\beta-1/2} - \overset{r}{L}_\alpha)(-\overset{r}{L}_\alpha)^{2\alpha}(-\overset{x}{L}_{\beta-1/2})^\beta$
- For $\alpha, \beta \geq 0$ and $\beta-\alpha>0$, we define the Sonine transform $S_{\beta,\alpha}$ on $S_*'(\mathbb{R}^2)$by

$$<S_{\beta,\alpha}(T), \varphi> = < T_{(r,x)}, {}^tS_{\beta,\alpha}(\varphi(.,x))(r)>, \qquad \varphi \in S_*(\mathbb{R}^2)$$

- $T_{\alpha,\beta}$ the distribution of $S_*'(\mathbb{R}^2)$ defined by

i) If $\alpha = \beta$, $T_{\alpha,\beta} = (\pi^{5/2} 2^{1-6\alpha})/(\Gamma^3(\alpha+1)\Gamma^3(\alpha+1/2))\Delta_{\alpha,\alpha}(S)$

ii) If $\alpha > \beta$, $T_{\alpha,\beta} = (\pi^{5/2}\Gamma(\beta+1)) \ / \ (2^{4\alpha+2\beta+1}\Gamma^4(\alpha+1)\Gamma^3(\beta+1/2))\Delta_{\alpha,\beta}\, S_{\alpha,\beta}(S)$

iii) If $\beta > \alpha$, $T_{\alpha,\beta} = (\pi^{5/2} 2^{1-2\alpha-4\beta})/(\Gamma^2(\alpha+1)\Gamma^4(\beta+1/2)\Gamma(\beta+1))\Delta_{\alpha,\beta}\, S_{\beta,\alpha}(S)$

- $\mathcal{H}$ is the Hilbert transform defined by $\mathcal{H}(f)(t,x) = \frac{-i}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \mathrm{sgn}(r)\, f(s,x)\, e^{-i(t-s)r}\, ds\right)dr$.

- The generalized convolution product is defined for $f \in S_*(\mathbb{R}^2)$ and $T \in S_*'(\mathbb{R}^2)$ by $(T \# f)(r,x) = <T, \mathfrak{T}_{(r,x)}f>$, where $\mathfrak{T}_{(r,x)}$ is the generalized translation operator associated with the operators D_1 and D_2, given by

$$\mathfrak{T}_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1/2)}{\pi\,\Gamma(\alpha+1/2)\Gamma(\beta)}\int_0^\infty\int_0^\infty f(\sqrt{r^2+s^2+2rs\cos\theta}, \sqrt{x^2+y^2+2xy\cos\varphi})(\sin\theta)^{2\alpha}(\sin\varphi^{2\beta-1} d\theta d\varphi; \ s,y \geq 0$$

- $c_{\alpha,\beta} = (\pi^2)/(2^{2\alpha+2\beta+1}\Gamma^2(\alpha+1)\Gamma^2(\beta+1/2))$

Theorem IV.1 i) The operator $K^1_{\alpha,\beta} = c_{\alpha,\beta}\mathcal{H}\frac{\partial}{\partial r}(-\Delta)^\alpha(-J_2)^\beta$ is an isomorphism from $S_{*,0}(\mathbb{R}^2)$ onto itself.

ii) The operator $K^2_{\alpha,\beta}(g) = T_{\alpha,\beta}\# g$ is an isomorphism from $S_*^0(\mathbb{R}^2)$ onto itself.

iii)The operator $K^3_{\alpha,\beta} = c_{\alpha,\beta}(-J_1)^{1/4}(-J_2)^{\beta/2}(-\Delta)^{\alpha/2}$ is an isomorphism from $S_{*,0}(\mathbb{R}^2)$ onto itself.

Theorem IV.2. i) For $f \in S_{*,0}(\mathbb{R}^2)$ and $g \in S_*^0(\mathbb{R}^2)$, we have the inversion formulas for $R_{\alpha,\beta}$

$$g = R_{\alpha,\beta}\, K^1_{\alpha,\beta}\, {}^tR_{\alpha,\beta}(g); \quad f = K^1_{\alpha,\beta}\, {}^tR_{\alpha,\beta}\, R_{\alpha,\beta}(f)$$

ii) For $f \in S_{*,0}(\mathbb{R}^2)$ and $g \in S_*^0(\mathbb{R}^2)$, we have the inversion formulas for ${}^tR_{\alpha,\beta}$

$$f = {}^tR_{\alpha,\beta}\, K^2_{\alpha,\beta}\, R_{\alpha,\beta}(f); \quad g = K^2_{\alpha,\beta}\, R_{\alpha,\beta}\, {}^tR_{\alpha,\beta}(g)$$

V. Plancherel theorem for the operator ${}^tR_{\alpha,\beta}$.

Notations. We denote by

- $L_0^2(r^{2\alpha+1}x^{2\beta}drdx)$ the space of square integrable functions f on $[0,+\infty[\times[0,+\infty[$ with respect to the measure $r^{2\alpha+1}x^{2\beta}drdx$ such that the function $\tilde{\mathcal{F}}_{\alpha,\beta}(f)$ defined by the relation (IV.1) is supported in $\{(t,x)\in\mathbb{R}^2, |t|>|x|>0\}$.
- $L^2(dr\,dx)$ the space of square integrable functions on $[0,+\infty[\times[0,+\infty[$ with respect to the measure $drdx$.

Proposition V.1. Let g be in $S_*^0(\mathbb{R}^2)$, we have the Plancherel formula for ${}^tR_{\alpha,\beta}$

$$\int_0^{+\infty}\int_0^{+\infty}|g(r,x)|^2\, r^{2\alpha+1}x^{2\beta}\,dr\,dx = \int_0^{+\infty}\int_0^{+\infty}|K^3_{\alpha,\beta}\,{}^tR_{\alpha,\beta}(g)(r,x)|^2\,dr\,dx$$

Theorem V.1. (of Plancherel for ${}^tR_{\alpha,\beta}$) The operator $K^3_{\alpha,\beta}\,{}^tR_{\alpha,\beta}$ can be extended to an isometric isomorphism from $L_0^2(r^{2\alpha+1}x^{2\beta}drdx)$ onto $L^2(dr\,dx)$.

VI. Transmutation operators.

Theorem VI.1. i) The generalized Radon transform ${}^tR_{\alpha,\beta}$ is bijective from $S_*^0(\mathbb{R}^2)$ onto $S_{*,0}(\mathbb{R}^2)$ and satisfies the following permutation relations

$${}^tR_{\alpha,\beta}(D_2(f))(r,x) = \frac{\partial^2}{\partial r^2}\,{}^tR_{\alpha,\beta}(f)(r,x); \quad {}^tR_{\alpha,\beta}(D_1(f))(r,x) = \frac{\partial^2}{\partial x^2}\,{}^tR_{\alpha,\beta}(f)(r,x)$$

ii) The generalized dual Radon transform $R_{\alpha,\beta}$ is bijective from $S_{*,0}(\mathbb{R}^2)$ onto $S_*^0(\mathbb{R}^2)$ and satisfies the following permutation relations

$$D_2(R_{\alpha,\beta}(g))(r,x) = R_{\alpha,\beta}(\frac{\partial^2}{\partial r^2}g)(r,x); \quad D_1(R_{\alpha,\beta}(g))(r,x) = R_{\alpha,\beta}(\frac{\partial^2}{\partial x^2}g)(r,x)$$

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AN EXACT SOLUTION OF REYNOLDS VISCOUS FLOW EQUATIONS FOR THE TIME DEPENDENT MOTION OF A SPHERE IN AN INFINITE LIQUID.

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Abstract. The Reynolds equations for viscous, incompressible flow contain an additional term to the Navier-Stokes equations which renders the equations truly non-linear in the highest order derivatives. An exact solution is found for the velocity field from Reynolds equations describing the time dependent exponentially decaying motion of a solid sphere in an infinite liquid. Also considered is the two dimensional analogue of an infinitely long circular cylinder moving perpendicular to its axis.

Introduction. In 1883 [1] Reynolds published a paper where he presented a set of hydrodynamical equations describing the motion of a viscous fluid and are in fact a generalization of the Navier-Stokes equations. At the time his paper was written Reynolds considered the Navier-Stokes equations were not totally consistent with the no slip boundary condition. Another author in the same era Crudeli [2] came to a similar conclusion using a different argument.

The Reynolds equations contain an additional term multiplied by the square of a length scale which when equated to zero reduces to the Navier-Stokes equations. In time dependent parallel flow the authors of [3] claim the additional term influences the rate of decay of vortex motion. The Reynolds equations are more non linear than the Navier-Stokes and are no longer quasi-linear, or linear to the highest order derivatives.

It is not the intent of the present note to contest the correctness, or otherwise, of the Reynolds equations but merely to compare some physical results with the corresponding flow as depicted by the Navier-Stokes equations. The specific motion considered here is the motion of a sphere translating and rotating with exponential time decay in a viscous incompressible liquid relative to a uniform stream which also decays exponentially with time. It is possible to construct an exact analytic velocity field in a concise finite form from the governing flow equations, which reduces to an exact Navier-Stokes solution when the length scale is set equal to zero. A compact expression is found for the force on the sphere.

The two-dimensional analogue where the sphere is replaced by a circular cylinder is briefly considered using a similar analysis.

The flow equations

The Reynolds equations for the motion of a viscous, incompressible liquid are described by

$$\frac{d}{dt}\underline{q} - l^2\nabla^2\frac{d\underline{q}}{dt} = -\underline{\nabla}\frac{p}{\rho_0} + \nu\nabla^2\underline{q}, \tag{1}$$

$$(\underline{\nabla}\cdot\underline{q}) = 0, \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\underline{q}\cdot\underline{\nabla}), \tag{2}$$

where the symbols have their usual meaning except for l which is a length scale and will be discussed later in connection with the boundary conditions. In this paper it is assumed that the fluid velocity $\underline{q}$ can be written as

$$\underline{q} = \left(\underline{Q} + V_0\hat{k}\right)e^{-\alpha^2\nu t}, \tag{3}$$

with V_0 a constant speed, and

$$\underline{Q} \equiv \underline{Q}\left(x, y, x + \frac{V_0 e^{-\alpha^2\nu t}}{\alpha^2\nu}\right), \tag{4}$$

where α is a constant which has the dimension $(\text{length})^{-1}$. Since

$$\begin{aligned}\frac{d}{dt}\underline{q} = {} & e^{-2\alpha^2\nu t}\left[e^{\alpha^2\nu t}\underline{Q}_t + V_0\underline{Q}_z\right] - \alpha^2\nu\left(\underline{Q} + V_0\hat{k}\right)e^{-\alpha^2\nu t} \\ & + \underline{\nabla}\left\{\frac{1}{2}|\underline{Q}|^2 e^{-2\alpha^2\nu t}\right\} - \left[\underline{Q}\times\operatorname{curl}\underline{Q}\right]e^{-2\alpha^2\nu t},\end{aligned} \tag{5}$$

and

$$\begin{aligned}\nabla^2\frac{d}{dt}\underline{q} = {} & e^{-2\alpha^2\nu t}\nabla^2\left(e^{\alpha^2\nu t}\underline{Q}_t + V_0\underline{Q}_z\right) - \alpha^2\nu e^{-\alpha^2\nu t}\nabla^2\underline{Q} \\ & + \underline{\nabla}\nabla^2\left\{\frac{1}{2}|\underline{Q}|^2 e^{-2\alpha^2\nu t}\right\} - \nabla^2\left\{\left[\underline{Q}\times\operatorname{curl}\underline{Q}\right]e^{-2\alpha^2\nu t}\right\},\end{aligned} \tag{6}$$

it follows that the equations of motion are satisfied exactly if $\underline{Q}$ satisfies the Beltrami equation

$$\operatorname{curl}\underline{Q} = \left[\underline{\nabla}\times\underline{Q}\right] = \beta\underline{Q}, \qquad \beta^2 = \frac{\alpha^2}{(1+\alpha^2 l^2)} \tag{7}$$

and the pressure field satisfies

$$\frac{p}{\rho_0} + \frac{1}{2}|\underline{Q}|^2 e^{-2\alpha^2\nu t} - l^2\nabla^2\left\{\frac{1}{2}|\underline{Q}|^2 e^{-2\alpha^2\nu t}\right\} - V_0 z\alpha^2\nu e^{-\alpha^2\nu t} = \text{constant}. \tag{8}$$

The specific flow to be discussed here is the translating motion of a sphere with velocity $V_0\hat{k}e^{-\alpha^2\nu t}$, and rotation with angular velocity $\omega e^{-\alpha^2\nu t}$, *relative* to a uniform stream with velocity $V_0\hat{k}e^{-\alpha^2\nu t}$. If $O(x, y, z)$ is a fixed frame of reference then the velocity of the sphere is $2V_0e^{-\alpha^2\nu t}$, and its angular velocity is $\omega e^{-\alpha^2\nu t}$. At infinity $x^2+y^2+z^2\to\infty$, the velocity of the fluid is $\underline{q}\to V_0e^{-\alpha^2\nu t}\hat{k}$. If $O'(x,y,z')$, where $z'=z+\frac{V_0e^{-\alpha^2\nu t}}{\alpha^2\nu}$ is a frame of reference moving parallel to the z-axis with velocity $V_0e^{-\alpha^2\nu t}\hat{k}$, then the relevant boundary conditions for $\underline{Q}$ are expressed by

$$\underline{Q}=V_0\hat{k} \qquad \text{on} \quad S, \qquad \underline{Q}\to 0, \qquad \text{at infinity.} \tag{9}$$

The equation of the solid sphere S referred to the fixed frame $O(x,y,z)$ is

$$x^2+y^2+\left[z+\frac{V_0e^{-\alpha^2\nu t}}{\alpha^2\nu}\right]^2=a^2, \tag{10}$$

or if (r,θ,ϕ) are spherical polar coordinates in the moving frame $O'(x,y,z')$ defined by

$$x=r\sin\theta\cos\phi, \quad y=r\sin\theta\cos\phi, \quad z'=r\cos\theta, \tag{11}$$

the boundary is more simply represented by $r=a$. Since the Beltrami force-free field equation is invariant under the transformation $(x,y,z)\to\left(x,y,z+\frac{V_0e^{-\alpha^2\nu t}}{\alpha^2\nu}\right)$ an axisymmetric velocity field $\underline{Q}$ see [4] with swirl is given by

$$\underline{Q}=\text{curl}\left\{\frac{\chi}{r\sin\theta}\hat{\phi}\right\}+\frac{\beta\chi\hat{\phi}}{r\sin\theta}, \tag{12}$$

where $\hat{\phi}=-\sin o\hat{i}+\cos\phi\hat{j}$, and $\chi=\chi(r,\theta)$, satisfies

$$\left(L_{-1}+\beta^2\right)\chi=0, \qquad L_{-1}\equiv\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{13}$$

The boundary conditions in terms of χ are defined by

$$\chi_\theta=V_0a^2\sin\theta\cos\theta, \qquad \chi_r=V_0a\sin^2\theta \tag{14}$$

$$\beta\chi=a^2\omega\sin^2\theta, \qquad \text{at} \quad r=a, \tag{15}$$

and

$$\chi/r\sin\theta\to 0 \qquad \text{as} \quad r\to\infty. \tag{16}$$

The solution of the boundary value problem for χ is straightforward see [5], and is given by $\chi=G(r)\sin^2\theta$. where

$$\begin{aligned}G(r)=\frac{V_0}{2\beta^3a}\bigg\{&\beta\left[\beta^2a^2-3+\frac{3a}{r}\right]\cos\beta(r-a)\\&+\left[3\beta^2a+\frac{(3-\beta^2a^2)}{r}\right]\sin\beta(r-a)\bigg\},\end{aligned} \tag{17}$$

provided that $2\omega = V_0\beta$. It is noted that for a given V_0, ω and α, the length scale l, if not known, is expressed by

$$l = \frac{\left[\alpha^2 V_0^2 - 4\omega^2\right]^{\frac{1}{2}}}{2\omega\alpha}, \qquad 4\omega^2 < \alpha^2 V_0^2. \tag{18}$$

In the case of the Navier-Stokes equations $l = 0$, and ω, α, V_0 are not all independent since $2\omega = \alpha V_0$.

The force on the sphere

The sphere experiences a force which is described by the formula

$$\underline{F} = \int_0^{2\pi}\int_0^{\pi} \underline{R}_r \Big|_{r=a} a^2 \sin\theta \, d\theta d\phi, \tag{19}$$

where the stress vector is expressed by

$$r\underline{R}_r = -p\underline{r} + \rho_0\nu\left[r\frac{\partial \underline{q}}{\partial r} - \underline{q} + \underline{\nabla}\left(\underline{q}\cdot\underline{r}\right)\right]. \tag{20}$$

On calculation it is found that the force is given by the formula

$$\begin{aligned}\underline{F} &= \frac{8\pi\rho_0\nu a}{3} e^{-\alpha^2\nu t}\hat{k}\left[G''(a) - \frac{2G(a)}{a^2}\right] - \frac{4\pi}{3}V_0\rho_0\nu\alpha^2 a^3 \hat{k} e^{-\alpha^2\nu t} \\ &= -\frac{4\pi}{3}V_0\rho_0\nu a^3(\alpha^2 + \beta^2)\hat{k}e^{-\alpha^2\nu t}.\end{aligned} \tag{21}$$

The torque on the sphere is defined by

$$\underline{G} = \int_0^{2\pi}\int_0^{\pi} \left[\underline{r} \times \underline{R}_r\right] a^2 \sin\theta \, d\theta d\phi, \tag{22}$$

On calculation it is found that

$$\underline{G} = \frac{8\pi\rho_0\nu a^3}{3}\left[a q_3'(a) - q_3(a)\right]\hat{k}e^{-\alpha^2\nu t}, \tag{23}$$

where $q_3(r) = \beta G(r)/r$, and since $G(a) = \frac{1}{2}Va^2$, $G'(a) = Va$, it follows that $\underline{G} = 0$, on $r = a$. This result would appear to be a consequence of the time dependent character of the motion.

Two-dimensional flow

In the case of two-dimensional flow there is a stream function $\psi \equiv \psi(x, y, t)$, and the velocity field is given by

$$\underline{q} = \mathrm{curl}\,(-\psi\hat{k}) = -\psi_y\hat{i} + \psi_x\hat{j}, \tag{24}$$

where ψ satisfies

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} - l^2\nabla^2\left\{\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}\right\} = \nu\nabla^4\psi, \tag{25}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. There is a solution given by

$$\psi = (\chi - V_0 y)e^{-\alpha^2\nu t}, \qquad \chi \equiv \chi\left(x + \frac{V_0 e^{-\alpha^2\nu t}}{\alpha^2\nu}, y\right) \tag{26}$$

and

$$(\nabla^2 + \beta^2)\chi = 0, \qquad \beta^2 = \frac{\alpha^2}{1+\alpha^2 l^2}. \tag{27}$$

For the translating motion of a cylinder with radius a moving with velocity $V_0\hat{i}e^{-\alpha^2\nu t}$, relative to a uniform stream with velocity $V_0\hat{i}e^{-\alpha^2\nu t}$ of the fluid as a whole, the boundary conditions are defined by

$$\begin{aligned} &-\frac{1}{r}\chi_\theta = V_0\cos\theta, \chi_r = -V_0\sin\theta, \qquad \text{at} \quad = a \\ &\underline{\nabla}\chi \to 0, \qquad \text{as} \quad r \to \infty. \end{aligned} \tag{29}$$

The polar coordinates (r,θ) are measured in the moving frame with velocity $V_0\hat{i}e^{-\alpha^2\nu t}$, and defined by

$$x' = x + \frac{V_0 e^{-\alpha^2\nu t}}{\alpha^2\nu} = r\cos\theta, \qquad y' = y = r\sin\theta. \tag{30}$$

The solution of the boundary value problem is expressed by

$$\chi = \frac{1}{2}\pi a V_0\left\{\left[Y_1(\beta a) - \beta a Y_1'(\beta a)\right]J_1(\beta r) + \left[\beta a J_1'(\beta a) - J_1(\beta a)\right]Y_1(\beta r)\right\}\sin\theta. \tag{31}$$

Where $J_1(\beta r)$. $Y_1(\beta r)$ are the Bessel functions of the first and second kind respectively, and order 1. As before, the force per unit thickness of cylinder is represented by the formal expression

$$\underline{F} = \int_0^{2\pi}\left\{-p\underline{r} + \rho_0\nu\left[\frac{\partial\underline{q}}{\partial r} - \underline{q} + \underline{\nabla}\left(\underline{q}\cdot\underline{r}\right)\right]\right\} a\,d\theta, \tag{32}$$

in which the explicit calculation is somewhat more complicated than in (21). Needless to say, the force decays with the same exponential time dependence produced by the forced flow, and reduces to the Navier-Stokes expression in the case $l = 0$, see [5].

Conclusions

The velocity fields which are exact solutions of boundary value problems for Reynolds equations have been constructed in connection with the time dependent motion of a sphere, and a circular cylinder, in an infinite viscous liquid. In the case of the sphere the force exerted by the fluid on the sphere is always less than the corresponding quantity for the Navier-Stokes equations, due to the presence of the length scale in the governing equations. The length scale if not known, can be determined in terms of the prescribed translatory velocity V_0 of the sphere and its angular velocity ω. The other fact of interest is that the decay of vortex motion with respect to time and the spatial coordinates is essentially the same as for the Navier-Stokes equations.

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CHARACTERIZATIONS OF SOME CLASSES OF QUASI-HEREDITARY ALGEBRAS

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Abstract. Inspired by the work of Mirollo and Vilonen [MV] describing the categories of perverse sheaves as module categories over certain finite dimensional algebras, Dlab and Ringel introduced [DR2] an explicit recursive construction of these algebras in terms of the algebras $A(\gamma)$. In particular, they characterized the quasi-hereditary algebras of Cline-Parshall-Scott [PS] and constructed them in this way. The present paper provides a characterization of lean, shallow and replete algebras in terms of this recursive process. A detailed presentation of the results will appear elsewhere.

Let D be a division K–algebra, C a basic K-algebra, ${}_DS_C$ and ${}_CT_D$ finite-dimensional bimodules with K acting centrally. Let $\gamma: {}_CT_D \otimes_D S_C \rightarrow {}_CC_C$, be a bimodule homomorphism whose image lies in $\operatorname{rad} C$.Let $B = D \ltimes ({}_DS_C \otimes {}_CT_D)$ be the "split" K–algebra with the coordinate-wise addition and multiplication given by

$$(d_1, s_1 \otimes t_1)(d_2, s_2 \otimes t_2) = \big(d_1d_2, d_1s_2 \otimes t_2 + s_1 \otimes t_1d_2 + s_1\gamma(t_1 \otimes s_2) \otimes t_2\big).$$

Clearly, B is a local K–algebra with $\operatorname{rad} B = S_C \otimes_C T$. It follows that S has the structure of a B-C-bimodule by $(d, s \otimes t) \cdot s' = ds' + s\gamma(t \otimes s')$ and T the structure of a C-B-bimodule by $t' \cdot (d, s \otimes t) = t'd + \gamma(t' \otimes s)t$. In [DR2], the 2×2 matrix $A = \begin{pmatrix} B & S \\ T & C \end{pmatrix}$ with multiplication given by

$$\begin{pmatrix} b & s \\ t & c \end{pmatrix} \begin{pmatrix} b' & s' \\ t' & c' \end{pmatrix} = \begin{pmatrix} bb' + (0, s \otimes t') & b \cdot s' + sc' \\ t \cdot b' + ct' & \gamma(t \otimes s') + cc' \end{pmatrix}$$

is shown to be a $A(\gamma)$ ring, viz. the quotient of the tensor algebra over the $(C \times D) - (C \times D)$–bimodule $T \otimes S$ by the ideal generated by the elements $t \otimes s - \gamma(t \otimes s)$.

Let $e_1 = \begin{pmatrix} (1,0) & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{e}_C = (e_2, e_3, \dots, e_n)$ be a complete sequence of primitive orthogonal idempotents of C so that $\sum_{i=1}^n e_i = 1: \ A_A = \bigoplus_{i=1}^{n} e_iA$. Write $\mathbf{e} = \mathbf{e}_A = (e_1, e_2, \dots, e_n)$, $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$ for $1 \leq i \leq n$ and $\varepsilon_{n+1} = 0$.

Let $\Delta_C(i)$ and $\Delta^o_C(i)$ be the right and left standard modules of $(C, \mathbf{e}_C)$, respectively (see [D1] for basic definitions and notation). Dlab and Ringel have

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shown in [DR2] that (A, e) is quasi-hereditary if and only if (C, e_C) is quasi-hereditary and S_C and ${}_CT$ have Δ_C-filtration and Δ_C^o- filtration, respectively; in fact, they have shown that all basic quasi-hereditary algebras over a perfect field K can be obtained by iterating this construction, starting with a division K-algebra C. In the present note, we are going to characterize lean algebras, as well as shallow and replete quasi-hereditary algebras A in terms of properties of C, ${}_DS_C$, ${}_CT_D$ and the homomorphism γ.

Recall that (A, e) is *lean* (see [ADL]) if

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j \text{ forall } 1 \le i, j \le n \text{ and } m = \min\{i, j\}.$$

Equivalently, (A, e) is lean if and only if (C, e_C) is lean and for $i, j \ge 2$, the products $e_iAe_1Ae_j$ $(\subseteq e_i(\operatorname{rad} A)^2 e_j)$ belong to $(\operatorname{rad} C)^2$. Since these products generate the image of γ, we have the following statement.

PROPOSITION 1. *The algebra* (A, e) *is lean if and only if* (C, e_C) *is lean and* $\operatorname{Im} \gamma \subseteq (\operatorname{rad} C)^2$.

Denoting the *standard* right and left modules of A by $\Delta(i) = \Delta_A(i)$ and $\Delta^o(i) = \Delta_A^o(i)$, respectively, (A, e) said to be *quasi-hereditary* if

$$\dim_K A = \sum_{i=1}^{n} (1/d_i) \dim_K \Delta(i) \dim_K \Delta^o(i), \tag{$*$}$$

where $d_i = \dim_K \operatorname{End} S(i)$; here, and in what follows, $S(i)$, $P(i)$ and $V(i)$ denote the simple right A–module, $P(i) \simeq e_iA$ its projective cover and $V(i)$ the kernel of the canonical epimorphism $P(i) \to \Delta(i)$. The equality $(*)$ is equivalent to the fact that $\operatorname{End} \Delta(i) \simeq \operatorname{End} S(i)$ for all $1 \le i \le n$ and that the regular representation A_A has a Δ-filtration (which has been the original definition of Cline-Parshall-Scott; see also [DR1]). Indeed, this follows from the following series of statements (A)–(C) (cf. [D2]):

(A) For every A-module X, $[X : S(i)] = (1/d_i) \dim_K Xe_i$; thus

$$[A_A : S(n)] = (1/d_n) \dim_K Ae_n = (1/d_n) \dim_K \Delta^o(n).$$

(B) Always, $\dim_K Ae_nA \le (1/d_n) \dim_K \Delta(n) \dim_K \Delta^o(n)$.

(C) The equality $\dim_K Ae_nA = (1/d_n) \dim_K \Delta(n) \dim_K \Delta^o(n)$ holds if and only if $\operatorname{End}_A \Delta(n) = \operatorname{End}_A S(n)$ and $Ae_nA \simeq \bigoplus_{\text{finite}} \Delta(n)$.

The quasi-hereditary algebra (A, e) is called *shallow* if all $\operatorname{rad} \Delta(i)$ and $\operatorname{rad} \Delta^o(i)$ $(1 \le i \le n)$ are semi-simple. This is equivalent to the fact (see [ADL]) that

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j \text{ for all } 1 \le i, j \le n \text{ and } M = \max\{i, j\}.$$

As a consequence, both Δ_C-filtration of S_C and Δ_C^o-filtration of ${}_CT$ are in this case *top filtrations* (see [ADL]), and (C, e_C) is a shallow quasi-hereditary algebra. The following example shows that these properties are not sufficient for (A, e) to be shallow.

EXAMPLE . Let $(A, \mathbf{e})$ be the path algebra of the quiver $2 \longrightarrow 1 \longrightarrow 3$; then $(A, \mathbf{e})$ is quasi-hereditary (in fact, hereditary). Since $e_2(\operatorname{rad} A)^2 e_3 \neq 0$ and $e_2(\operatorname{rad} A)e_3(\operatorname{rad} A)e_3 = 0$, $(A, \mathbf{e})$ is not shallow. However, $C = \varepsilon_2 A \varepsilon_2$ is shallow quasi-hereditary algebra and both S_C and ${}_CT$ are simple C-modules. The key missing property is leanness.

PROPOSITION 2. *The algebra $(A, \mathbf{e})$ is shallow if and only if $(C, \mathbf{e}_C)$ is a shallow quasi-hereditary algebra, S_C has a top Δ_C -filtration, ${}_CT$ has a top Δ^o_C-filtration and* $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.

Proof (of sufficiency). We need only to show that, under the conditions for C, S_C, ${}_CT$ and γ,

$$e_i(\operatorname{rad} A^2)e_j \subset e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j \text{ for all } 1 \leq i, j \leq n \text{ and } M = \max\{i, j\}.$$

First, let $i, j \geq 2$. Then, in view of the fact that $(C, \mathbf{e}_C)$ is shallow,

$$e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j = e_i(\operatorname{rad} C)\varepsilon_M(\operatorname{rad} C)e_j = e_i(\operatorname{rad} C)^2 e_j,$$

which, in turn equals to $e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)e_j$. Since, by Proposition 1, $(A, \mathbf{e})$ is lean, we get

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j \subset e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)e_j,$$

as required.

If $i = 1, j \geq 2$, the inclusion $e_1(\operatorname{rad} A)^2 e_j \subset e_1(\operatorname{rad} A)\varepsilon_j(\operatorname{rad} A)e_j$ follows from the fact that the Δ-filtration of $\operatorname{rad} P(1)$, induced by the top Δ_C-filtration of S_C (which exists by [DR2]), is a top filtration. A similar argument can be applied to $i \geq 2, j = 1$.

Recall that the quasi-hereditary algebra $(A, \mathbf{e})$ is said to be *replete* if all $V(i) = e_i A \varepsilon_{i+1} A$ are projective, top submodules of $\operatorname{rad} P(i) = e_i(\operatorname{rad} A)$. and all $V^o(i) = A\varepsilon_{i+1} A e_i$ are projective, top submodules of $\operatorname{rad} P^o(i) = (\operatorname{rad} A)e_i$ (see [ADL]). If $(A, \mathbf{e})$ is a replete quasi-hereditary algebra. then $(C, \mathbf{e}_C)$ is a replete quasi-hereditary algebra and both S_C and ${}_CT$ are projective C-modules. Again, these conditions alone do not imply that $(A, \mathbf{e})$ is replete. In the above Example, $(C, \mathbf{e}_C)$ is replete and both S_C and ${}_CT$ are simple (projective) C-modules. but $(A, \mathbf{e})$ is not replete. The missing property is leanness again.

PROPOSITION 3. *The algebra $(A, \mathbf{e})$ is a replete quasi-hereditary algebra if and only if $(C, \mathbf{e}_C)$ is a replete quasi-hereditary algebra S_C and ${}_CT$ are projective $C-$modules and* $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.

Proof (of sufficiency) We want to show that $V(i)$ is a projective, top submodule of $\operatorname{rad} P(i)$. Consider first $i \geq 2$. Since $V_C(i) = e_i(\operatorname{rad} A)\varepsilon_{i+1}A\varepsilon_2$ is projective, top C-submodule of the C-module $\operatorname{rad} P_C(i) = e_i(\operatorname{rad} A)\varepsilon_2$, $V(i) = V_C(i) \otimes_C \varepsilon_2 A$ is a projective A-module.

Moreover, since A is lean by Proposition 1, we have the equality

$$e_i(\operatorname{rad} A)^2 e_{i+1} = e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)\varepsilon_{i+1},$$

and thus can identify the top of $V(i)$ in the top of $\operatorname{rad} P(i)$ with the top of $V_C(i)$. This yields a top embedding of $V(i)$ in $\operatorname{rad} P(i)$. Furthermore, since S_C is projective C-module, $V(1) = \operatorname{rad} P(1) = S_C \otimes_C \varepsilon_2 A$ is projective A-module (trivially embedded in $\operatorname{rad} P(1)$ as a top submodule).

One can use similar arguments to deal with the left A-modules $V^{\circ}(i)$.

Let us conclude our brief note with the remark that similar characterizations of the left and right medial algebras (for a definition, see [ADL]) can be made combining the conditions of Propositions 2 and 3.

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ABSTRACT. A new class of infinite-dimesional Lie algebras, called extended affine Lie algebras, has recently been introduced by physicists. In their description, the classification of so-called tori is an important step. The class of associative tori is known as quantum tori. Alternative tori have also been classified. In this announcement, we will describe the classification of Jordan tori. An important tool in our work is Zelmanov's structure theorem for prime Jordan algebras. Jordan tori can be used to coodinatize extended affine Lie algebras of type G_2 and A_1, while alternative tori can be used for type F_4 and A_2.

Let F be a field of characteristic $\neq 2$ and T a (not necessarily associative) unital algebra over F. To say that T is *graded by an abelian group* A means $T = \bigoplus_{\alpha \in A} T_\alpha$ (direct sum of F-spaces) and $T_\alpha T_\beta \subset T_{\alpha+\beta}$ for all $\alpha, \beta \in A$. We define the centre $Z(T)$ of T as $Z(T) = \{x \in T : [xy] = (x, y, z) = (y, x, z) = (y, z, x) = 0 \text{ for all } y, z \in T\}$ where $[x, y] = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$.

Definition 1. A unital algebra $T = \bigoplus_{\alpha \in \mathbf{Z}^n} T_\alpha$ graded by $\mathbf{Z}^n$ is called an *n-torus* or simply a *torus* if $\dim_F T_\alpha = 1$ and $T_\alpha T_\beta = T_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbf{Z}^n$.

As a basic property of tori, we have:

Lemma 1. *A torus has no zero-divisors.*

In particular, the centre $Z = Z(T)$ of a torus T has no zero-divisors, and is therefore an integral domain. Let $\overline{Z}$ be the field of fractions of Z. We define $\overline{T} = \overline{Z} \otimes_Z T$ and call it the *central closure of* T. Then T imbeds into $\overline{T}$ via $x \mapsto 1 \otimes x$. We identify T as a subalgebra of $\overline{T}$.

Definition 2. A torus is called an *associative torus* if it is an associative algebra, an *alternative torus* if it is an alternative algebra, and a *Jordan torus* if it is a Jordan algebra.

Example 1. (1) Let $E = F\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$ be the associative algebra of Laurent polynomials in non-commuting variables $T_1, \dots, T_n$ over F, let $q = (q_{ij})$ be an $n \times n$ matrix such that

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$q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$, and let I_q be the ideal of E generated by $\{T_jT_i - q_{ij}T_iT_j : 1 \le i,j \le n\}$. Then the *quantum torus associated to* q is defined as $F_q = E/I_q$. One can show that F_q is a torus and that every associative torus is isomorphic to some F_q. In particular, an associative torus which is commutative is isomorphic to $F_{\mathbf{1}}$ where

$$\mathbf{1} = \mathbf{1}_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \text{ (all } q_{ij} = 1).$$

This is nothing but the algebra of Laurent polynomials in n-variable, $F[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. One can also check that F_q^+ is a Jordan torus if and only if

$$(*) \qquad \prod_{i,j} q_{ij}^{\alpha_i\beta_j} \neq -1 \text{ for all } (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \mathbf{Z}^n.$$

(2) Let

$$\jmath = \jmath_n = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} \; (q_{12} = -1,\ q_{21} = -1 \text{ and } q_{ij} = 1 \text{ for the other } i,j).$$

Then one can show that the quantum torus $F_\jmath$ is a quaternion algebra over its centre $Z = Z(F_\jmath) = F[T_1^{\pm 2}, T_2^{\pm 2}, T_3^{\pm 1}, \dots, T_n^{\pm 1}]$, which we call the *quaternion torus.* The Cayley-Dickson doubling process yields an octonion algebra $\mathbf{OT} = (F_\jmath, T_3)$ over Z, taking $T_3 \in Z$ as the structure constant. One can show that the F-algebra $\mathbf{OT}$ is an alternative torus, which we call the *octonion torus* (Table 1). This was called the alternative torus in [BGKN].

(3) Let

$$\omega = \omega_n = \begin{pmatrix} 1 & \omega & 1 & \cdots & 1 \\ \omega^2 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} \; (q_{12} = \omega,\ q_{21} = \omega^2 \text{ and } q_{ij} = 1 \text{ for the other } i,j),$$

where $\omega \in F$ is a primitive 3rd root of unity. One can show that the central closure $\overline{F_\omega}$ of the quantum torus F_ω is a central (associative) division algebra of degree 3 over the field $\overline{Z} = F(T_1^3, T_2^3, T_3, \dots, T_n)$. Thus we can construct an Albert algebra $(\overline{F}_\omega, T_3)$ over $\overline{Z}$ by Tits' 1st construction [Ja], taking $T_3 \in Z \subset \overline{Z}$ as the structure constant. Let $\mathbf{AT}$ be the Jordan F-subalgebra of $(\overline{F}_\omega, T_3) = \overline{F}_\omega \oplus \overline{F}_\omega \oplus \overline{F}_\omega$ generated by $T_1^{\pm 1}, T_2^{\pm 1}, (0,1,0)^{\pm 1}, T_4^{\pm 1}, \dots, T_n^{\pm 1}$ (Note $(0,1,0)^3 = T_3$). Then one can check that $\mathbf{AT} = F_\omega \oplus F_\omega \oplus F_\omega$ is a Jordan torus (over

F) which we call the *Albert torus* (Table 2). This example appears in [AABGP] where it is called the Jordan torus. It was also found independently by the author.

Theorem 1. *An alternative torus is isomorphic to either a quantum torus or the octonion torus.*

Remark 1. This result was proven for certain base fields (e.g. F is algebraically closed.) in [BGKN]. It is in fact true for any field F of characteristic $\neq 2$.

For the classification of Jordan tori we first prove:

Lemma 2. *A Jordan torus is strongly prime.*

Because of Lemma 2, we can apply Zelmanov's structure theorem of prime Jordan algebras [M-Z]. Hence, a Jordan torus is of Clifford type, hermitian type or exceptional type.

Theorem 2. *Let J be a Jordan torus over F, and assume that $a \in F$ implies $\sqrt{a} \in F$ and that F contains a primitive 3rd root of unity. Then*

(i) *J cannot be of Clifford type,*

(ii) *J is of hermitian type if and only if $J \cong F_q^+$ with* $(*)$ (Example 1 (1)),

(iii) *J is of exceptional type if and only if $J \cong$* $\mathbf{AT}$.

Remark 2. The first assumption for F is used for the proof of (ii), while the second is used for the proof of (iii).

Corollary 1. *Let J be as above. Then J is special if and only if $J \cong F_q^+$ with $(*)$, and J is exceptional if and only if $J \cong$* $\mathbf{AT}$.

We define the *degree* of a torus T as the degree of the generic minimal polynomial of the central closure $\overline{T}$ over $\overline{Z}$ [Ja]. Then we can show:

Lemma 3. (i) *A quantum torus of degree 2 is isomorphic to the quaternion torus F_j*

(ii) *A quantum torus of degree 3 is isomorphic to F_ω.*

Remark 3. (1) For (ii), we assume that $\omega \in F$. Otherwise, there does not exist such a torus.

(2) $F_q \cong F_{q'}$ does not imply $q = q'$ after renumbering the rows and columns of q', if necessary. For example, one can check that $F_q \cong F_{q'}$ for

$$q = j_3, q' = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \text{ or } q = \omega_3, q' = \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix}.$$

Finally, we can prove:

Corollary 2. *Let A be an alternative torus and J a Jordan torus with the same assumption for F as in Theorem 1. Then:*

(i) *A is of degree 2 if and only if A is isomorphic to either the quaternion torus or the octonion torus.*

(ii) *J is of degree 3 if and only if J is isomorphic to either F_ω^+ or the Albert torus.*

Remark 4. The central closures $\overline{F_j}$, $\overline{F_\omega^+}$, $\overline{\mathbf{OT}} = (\overline{F_j}, T_3)$, and $\overline{\mathbf{AT}} = (\overline{F_\omega}, T_3)$ are all division algebras.

APPENDIX

Table 1. Multiplication table for the octonion torus:

$$\mathbf{OT} = (F_j, T_3) = F_j \oplus F_j = \bigoplus_{\alpha \in \mathbf{Z}^n} F t_\alpha$$

where, for $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$t_\alpha = \begin{cases} (T^\alpha T_3^m, 0) & \text{if } \alpha_3 = 2m \\ (0, T^\alpha T_3^m) & \text{if } \alpha_3 = 2m+1 \quad (m \in \mathbf{Z}) \end{cases}$$

and $T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} T_4^{\alpha_4} \cdots T_n^{\alpha_n}$. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{Z}^n$, the multiplication in $\mathbf{OT}$ is given by

(I) $\alpha_3 \equiv \beta_3 \equiv 0 \pmod 2$ (all $\equiv$ are mod 2 below) :

$$t_\alpha t_\beta = (-1)^{\alpha_2 \beta_1} t_{\alpha+\beta}$$

(II) $\alpha_3 \equiv 0$, $\beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ (-1)^{\alpha_2 \beta_1 + 1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(III) $\alpha_3 \equiv 1$, $\beta_3 \equiv 0$:

$$t_\alpha t_\beta = (-1)^{\alpha_1 \beta_2} t_{\alpha+\beta}$$

(IV) $\alpha_3 \equiv 1$, $\beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ (-1)^{\alpha_1 \beta_2 + 1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

The structure constants are $\{\pm 1\}$.

Table 2. Multiplication table for the Albert torus:

$$\mathbf{AT} = F_\omega \oplus F_\omega \oplus F_\omega = \langle T_1^{\pm 1}, T_2^{\pm 1}, (0,1,0)^{\pm 1}, T_4^{\pm 1}, \dots, T_n^{\pm 1}\rangle = \bigoplus_{\alpha \in \mathbf{Z}^n} F t_\alpha$$

where, for $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$t_\alpha = \begin{cases} (T^\alpha T_3^m, 0, 0) & \text{if } \alpha_3 = 3m \\ (0, T^\alpha T_3^m, 0) & \text{if } \alpha_3 = 3m+1 \\ (0, 0, T^\alpha T_3^m) & \text{if } \alpha_3 = 3m-1 \ (m \in \mathbf{Z}) \end{cases}$$

and $T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} T_4^{\alpha_4} \cdots T_n^{\alpha_n}$. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{Z}^n$, the multiplication in $\mathbf{AT}$ is given by

(I) $\alpha_3 \equiv \beta_3 \equiv 0 \pmod 3$ (all $\equiv$ are mod 3 below) :

$$t_\alpha t_\beta = \frac{1}{2}(\omega^{\alpha_1\beta_2} + \omega^{\alpha_2\beta_1}) t_{\alpha+\beta}$$

(II) $\alpha_3 \equiv 0$, $\beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_2\beta_1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(III) $\alpha_3 \equiv 0$, $\beta_3 \equiv -1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_1\beta_2} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(IV) $\alpha_3 \equiv 1$, $\beta_3 \equiv -1$:

$$t_\alpha t_\beta = \begin{cases} \omega^{\alpha_2\beta_1} t_{\alpha+\beta} & \text{if } \alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_2\beta_1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(V) $\alpha_3 \equiv \beta_3 \equiv 1$ or $\alpha_3 \equiv \beta_3 \equiv -1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv \beta_1 \equiv \beta_2 \equiv 0 \\ -\frac{1}{2} t_{\alpha+\beta} & \text{if } [\alpha_1 \equiv \alpha_2 \equiv 0 \text{ and } (\beta_1 \not\equiv 0 \text{ or } \beta_2 \not\equiv 0)] \\ & \quad \text{or } [(\alpha_1 \not\equiv 0 \text{ or } \alpha_2 \not\equiv 0) \text{ and } \beta_1 \equiv \beta_2 \equiv 0] \\ -\frac{1}{4}(\omega^{\alpha_1\beta_2} + \omega^{\alpha_2\beta_1}) t_{\alpha+\beta} & \text{if } (\alpha_1 \not\equiv 0 \text{ or } \alpha_2 \not\equiv 0) \text{ and } (\beta_1 \not\equiv 0 \text{ or } \beta_2 \not\equiv 0) \\ & \quad \text{and } \alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv 0 \\ \frac{1}{2}(\omega^{\alpha_1\beta_2} + \omega^{\alpha_2\beta_1}) t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

Since $\omega^2 + \omega + 1 = 0$, the structure constants are

$$\{1, \omega, \omega^2, -\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2}, \frac{1}{4}, \frac{\omega}{4}, \frac{\omega^2}{4}\}.$$

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FREE GROUPS OF LIE TYPE

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Abstract

Groups are constructed which afford rings of polynomial functions. Using invariant derivations of these rings along classical lines, yields Lie algebras which are completions of free Lie algebras.

Throughout this report I will be a finite set [†] and $\mathbb{K}$ a field of characteristic 0.

For each $i \in I$ we consider a multiplicative copy $E_i := \{E_i(\lambda),\ \lambda \in \mathbb{K}\}$ of $(\mathbb{K}, +)$. Thus $E_i(\lambda)E_i(\mu) = E_i(\lambda + \mu)$. Our group F is the free product $F = *_{i \in I} E_i$.

Let W be the free associative monoid (words) on I. Then for $\boldsymbol{w} \in W$, $\boldsymbol{w} \neq 1$, one has $\boldsymbol{w} = \boldsymbol{w}_1 \cdots \boldsymbol{w}_w$ with $w > 0$ and $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_w$ in I unique (so we use bold face characters for words, and the corresponding unbold characters for their lengths). There is also a unique expression, called *reduced*, of the form $\boldsymbol{w} = \overline{\boldsymbol{w}}_1^{n_1} \cdots \overline{\boldsymbol{w}}_{\overline{w}}^{n_{\overline{w}}}$ where $\overline{\boldsymbol{w}}_n \neq \overline{\boldsymbol{w}}_{n+1}$. We set $\overline{\boldsymbol{w}} := \overline{\boldsymbol{w}}_1 \cdots \overline{\boldsymbol{w}}_{\overline{w}}$ and $\overline{W} = \{\overline{\boldsymbol{w}} : \boldsymbol{w} \in W\}$ (reduced words).

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[†] This assumption is not essential. It is made in this report for ease of exposition, to rend some of the notation and concepts used more managable.

To each $w \in W$ we attach a function $\pi_w : \mathbb{K}^w \to F$ by

$$\pi_w : (\lambda_1, \ldots, \lambda_w) \mapsto E_{w_1}(\lambda_1) \cdots E_{w_w}(\lambda_w).$$

A function $f : F \to \mathbb{K}$ is *polynomial* if for all $w \in W$ the function $f_w := f \circ \pi_w$ is polynomial. The ring of all such functions we denote by $\operatorname{Pol}(F)$.

Let $\mathcal{A}$ be the free associative $\mathbb{K}$-algebra on the set of symbols $\{x_i\}_{i \in I}$. To each $w \in W$ we attach the element $x^w := \prod_{i=1}^{w} x_{w_i} \in \mathcal{A}$. Consider the completion $\overline{\mathcal{A}}$ of $\mathcal{A}$ and corresponding Magnus group

$$M := \Big\{1 + \sum_{w \in W, w \neq \emptyset} c_w x^w,\ c_w \in \mathbb{K}\Big\} \subset \overline{\mathcal{A}}.$$

For each $i \in I$ there is a group homomorphism $\varepsilon_i : \mathbb{K} \to M$ given by

$$\varepsilon_i : \lambda \mapsto \sum_{n \geq 0} \frac{\lambda^n x^n}{n!}\,.$$

By universal nonsense these yield a (unique) group homomorphism $\varepsilon : F \to M$.

For each $a \in W$ define $X^{(a)} \in \overline{\mathcal{A}}^*$ (dual space) by $\langle X^{(a)}, \sum_{b \in W} c_b x_b \rangle = \delta_{a,b} c_b$ (Kronecker δ). This yields a function

$$X^a := X^{(a)} \circ \varepsilon : F \to \mathbb{K}.$$

These functions are polynomial and allow us to describe $\operatorname{Pol}(F)$: When can a formal expression $\sum_{s \in W} c_s X^s$ be thought of as a function on F? The answer is that the set $S := \{s \in W : c_s \neq 0\}$ must have the property that the set $S_L := \{s \in S : \overline{s} \leq L\}$ be finite for all $L \in \mathbb{N}$. We call such sets *summable*.

Proposition 1. *Every polynomial function on* F *can uniquely be written in the form* $\sum_{s \in W} c_s X^s$; $c_s \in \mathbb{K}$, *where the set* $S := \{s \in W : c_s \neq 0\}$ *is summable.* □

In view of this result the multiplicative structure of $\mathrm{Pol}(F)$ is completely determined by

Proposition 2. *Let* $\boldsymbol{a}, \boldsymbol{b} \in W$. *Then*

$$X^{\boldsymbol{a}} X^{\boldsymbol{b}} = \sum_{\boldsymbol{s} \in \boldsymbol{a} \rightsquigarrow \boldsymbol{b}} X^{\boldsymbol{s}}$$

where $\rightsquigarrow$ *is the shuffle product.* □

We now begin to look at the derivations of $\mathrm{Pol}(F)$. Recall that $\mathrm{Pol}(F)$ has a natural right F-module structure via $(x \cdot f)(y) = f(yx^{-1})$. Recall also that $\partial \in \mathrm{End}\,\mathrm{Pol}(F)$ (the $\mathbb{K}$ algebra of $\mathbb{K}$-linear endomorphisms of $\mathrm{Pol}(F)$) is *right invariant* if $x \cdot (\partial(f)) = \partial(x \cdot f)$ for all $x \in F$ and $f \in \mathrm{Pol}(F)$.

Let $\partial \in \mathrm{End}\,\mathrm{Pol}(F)$ and for all $\boldsymbol{s} \in W$ write $\partial(X^{\boldsymbol{s}}) = \sum\limits_{\boldsymbol{w} \in W} \partial^{\boldsymbol{s}}_{\boldsymbol{w}} X^{\boldsymbol{w}}$. Assume that for all summable sets S we have

CONT 1: For all $\boldsymbol{w} \in W$ the set $\{\boldsymbol{s} \in S : \partial^{\boldsymbol{s}}_{\boldsymbol{w}} \neq 0\}$ is finite.

CONT 2: The set $\{\boldsymbol{w} \in W : \exists \boldsymbol{s} \in S \text{ with } \partial^{\boldsymbol{s}}_{\boldsymbol{w}} \neq 0\}$ is summable.

Then if $f = \sum c_{\boldsymbol{s}} X^{\boldsymbol{s}} \in \mathrm{Pol}(F)$ we have $\partial(f) = \sum c_{\boldsymbol{s}} \partial(X^{\boldsymbol{s}})$. Such endomorphisms we call *continuous.*

For $\boldsymbol{w} \in W$ define $d(\boldsymbol{w}) \in \mathbb{Z}^I$ by $d(\boldsymbol{w})(i) := \mathrm{Card}\,\{n : \boldsymbol{w}_n = i\}$.

Proposition 3. *Let* $\partial \in \mathrm{End}\,\mathrm{Pol}(F)$. *For* $\omega \in \mathbb{Z}^I$ *and* $\boldsymbol{s} \in W$ *write* $\partial_\omega(X^{\boldsymbol{s}}) = \sum\limits_{d(\boldsymbol{w}) = \omega + d(\boldsymbol{s})} \partial^{\boldsymbol{s}}_{\boldsymbol{w}} X^{\boldsymbol{w}}$ *(a finite sum). Assume* ∂ *is continuous. Then*

(i) ∂_ω *can uniquely be extended to a continuous operator on* $\mathrm{Pol}(F)$. *Moreover* $\partial = \sum\limits_{\omega \in \mathbb{Z}^I} \partial_\omega$ *(this last sum with the obvious meaning).*

(ii) *If* ∂ *is a derivation then so is* ∂_ω.

(iii) *If* ∂ *is right invariant then so is* ∂_ω. *Furthermore* $\partial_\omega = 0$ *unless* $\omega \in \mathbb{Z}^I_-$, *(i.e.* $\omega(i) \leq 0$ *for all* $i \in I$*).* □

For each $i \in I$, and $s \in W$ with reduced expression $s = \bar{s}_1^{n_1} \cdots \bar{s}_T^{n_T}$, define

$$\partial_i(X^s) = \begin{cases} n_1 X^{\bar{s}_1^{(n_1-1)} \bar{s}_2^{n_2} \cdots \bar{s}_T^{n_T}} & \text{if} \quad i = \bar{s}_1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that ∂_i extends uniquely to a continuous operator on Pol(F) which is also a right invariant derivation of Pol(F).

For each $w \in W$ let $\partial^w := \partial_{w_1} \cdots \partial_{w_w}$. By acting on the X^s's one sees that the ∂^w's are linearly independent, and hence that the associative subalgebra A of End Pol(F) generated by the ∂_i's is free. This together with Proposition 3 and Frederich's theorem yields:

Theorem 1. *Let Lie (F) be the Lie algebra of right invariant continuous derivations of* Pol(F). *Then*

(i) *The ∂_i's generate a subalgebra L of* Lie(F) *which is free.*

(ii) *If $\partial \in$* Lie(F) *and we write $\partial = \sum_{\omega \in \mathbb{Z}^I_-} \partial_\omega$ then each ∂_ω belongs to L.*

(iii) Lie(F) *is a completion of the free Lie algebra L. More precisely*

$$\mathrm{Lie}(F) = \Big(\prod_{\omega \in \mathbb{Z}^I_-} L_\omega\Big) \cap \Big(\bigoplus_{r \in W} \Big(\prod_{\substack{w \in W \\ \bar{w} = r}} \mathbb{K}\partial^w\Big)\Big). \qquad \square$$

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A Concept of Integrability of Dynamical Systems

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Presented by G.F.D. Duff, F.R.S.C.

Abstract: A new concept of integrability of Hamiltonian systems on a symplectic manifold M^n, $n = 2k$, is introduced. The concept of integrability describes the Hamiltonian systems that have quasi-periodic dynamics on tori $\mathbb{T}^q$ or on toroidal cylinders $\mathbb{T}^m \times \mathbb{R}^{q-m}$ of an arbitrary dimension $q = n, n-1, \cdots, 2, 1$. This concept includes, as a particular case, all Hamiltonian systems that are integrable in Liouville's classical sense, for which $q \leq n/2 = k$. A concept of integrability of a dynamical system on a manifold M^n of an arbitrary dimension n is proposed.

I. According to the terminology we adopt here, a Hamiltonian system

$$\dot{x}^\tau = V^\tau = P^{\tau\mu}\theta_\mu, \quad P = \omega^{-1}, \quad d\omega = 0, \quad d\theta = 0 \tag{1}$$

on a symplectic manifold M^{2k} is called integrable in the A-sense (or just A-integrable) if Liouville's condition [8] is satisfied: The system (1) has k functionally-independent first integrals $F_1(x), \cdots, F_k(x)$ that are in involution $\{F_i, F_j\} = P^{\tau\mu}F_{i,\tau}F_{j,\mu} = 0$. The Liouville condition implies that system (1) has an abelian k-dimensional Lie algebra $\mathcal{S}_a$ of Hamiltonian symmetries $U_i^\tau = P^{\tau\mu}F_{i,\mu}$. The abelian Lie algebra of symplectic symmetries $\mathcal{S}_a$ is isomorphic with the abelian Lie algebra of first integrals $F_j(x)$ with respect to the Poisson brackets.

Our key idea about the properties of the general integrable Hamiltonian systems (1) consists of the following:

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a) The integrable Hamiltonian system must possess functionally independent first integrals $F_1(x), \cdots, F_p(x)$ but the Liouville condition of their involutiveness is not necessary and p can be arbitrary: $0 \leq p < n$,

b) The integrable Hamiltonian system must possess an abelian $(n-p)$-dimensional Lie subalgebra of symmetries $\mathcal{S}_a \subset \mathcal{S}$ that preserve the first integrals $F_j(x)$ but the symmetries must not necessarily be symplectic.

Definition 1 *The Hamiltonian system (1) on a symplectic manifold M^{2k} is said to be integrable in the B-sense (or just B-integrable) if it possesses: (a) p functionally independent first integrals $F_1(x), \cdots, F_p(x)$, where $0 \leq p < n$, (b) an abelian $(n-p)$-dimensional Lie subalgebra $\mathcal{S}_a$ of symmetries that preserve the first integrals $F_j(x)$.*

In Liouville's 1855 definition [8] the independent conditions (*a*) and (*b*) were incorporated into the single condition of the involutiveness of first integrals. It is this hidden confusion of ideas that is probably the reason why Liouville's definition has served as the most general characterization of integrability since that time.

An example. Let symplectic manifold M^4 be the cotangent bundle of a torus $\mathbb{T}^2$, $M^4 = T^*(\mathbb{T}^2)$, with coordinates $p_1, p_2, \varphi_1, \varphi_2$, and with the non-standard symplectic structure

$$\omega = \frac{p_1 \mathrm{d}p_2 - p_2 \mathrm{d}p_1}{p_1^2 + p_2^2} \wedge (a^1 \mathrm{d}\varphi_2 - a^2 \mathrm{d}\varphi_1) + \tag{2}$$

$$+ (p_1 \mathrm{d}p_1 + p_2 \mathrm{d}p_2) \wedge (a^1 \mathrm{d}\varphi_1 + a^2 \mathrm{d}\varphi_2) + a^0 \mathrm{d}p_1 \wedge \mathrm{d}p_2 - a^0 \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2.$$

We consider the Hamiltonian system $\dot{x}^r = (\omega^{-1})^{r\mu} H_{.\mu}$ with a Hamiltonian function $H(I)$, $I = p_1^2 + p_2^2$. The system has the form

$$\dot{p}_1 = -a^0 f p_2, \quad \dot{p}_2 = a^0 f p_1, \quad \dot{\varphi}_1 = a^1 f, \quad \dot{\varphi}_2 = a^2 f, \tag{3}$$

where $f = \mathrm{d}H(I)/\mathrm{d}I$. It is easy to verify that system (3) preserves the symplectic structure ω (2). The invariant submanifolds of this system are the 3-dimensional tori

$$\mathbb{T}^3 : \quad p_1^2 + p_2^2 = c, \quad 0 \leq \varphi_1 \leq 2\pi, \quad 0 \leq \varphi_2 \leq 2\pi.$$

The trajectories of system (3) have the form (i = 1,2)

$$p_1(t) = p_1^0\cos(a^0 ft + t_0),\ p_2(t) = p_2^0\sin(a^0 ft + t_0),\ \varphi_i(t) = a^i ft + \varphi_i^0. \quad (4)$$

The trajectories (4) are quasi-periodic and everywhere dense on the 3-dimensional tori $\mathbb{T}^3$ provided that the constants a^0, a^1, a^2 are rationally independent. Therefore the Hamiltonian system (3) is not integrable in the A-sense.

However, the Hamiltonian system (3) is integrable in the B-sense. Indeed, the system has one first integral $I = p_1^2 + p_2^2$ and possesses 3-dimensional Lie algebra S_a of symmetries

$$U_1 = p_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_1},\quad U_2 = \frac{\partial}{\partial \varphi_1},\quad U_3 = \frac{\partial}{\partial \varphi_2}$$

that preserve the first integral I.

A direct product of k copies of the symplectic manifold $T^*(\mathbb{T}^2)$ with the Hamiltonian systems (3) for different functions $H(I)$ and constants a^0, a^1, a^2, provides the B-integrable Hamiltonian systems on the symplectic manifolds $M^{4k} = T^*(\mathbb{T}^{2k})$. These Hamiltonian systems have invariant coisotropic tori $\mathbb{T}^{3k}$ and are $\mathbb{T}^{3k}$-dense in general. For these systems, we have $p = k$ and $q = 3k$.

II. Recall that a torus $\mathbb{T}^k$ $(q = k)$ is a Lagrangian submanifold if its tangent bundle coincides with the ω-orthogonal bundle: $T_x(\mathbb{T}^k) = T_x(\mathbb{T}^k)^\perp$. A torus $\mathbb{T}^q$ is isotropic or coisotropic if $T_x(\mathbb{T}^q) \subset T_x(\mathbb{T}^q)^\perp$ or $T_x(\mathbb{T}^q) \supset T_x(\mathbb{T}^q)^\perp$ respectively. The Lagrangian and isotropic tori appear in the A-integrable Hamiltonian systems. These systems have been studied in numerous papers and books over the 140 years since the Liouville paper [8]. The Hamiltonian systems with coisotropic invariant tori were discovered by Parasyuk [11]. These systems and their links with KAM theory were studied by Herman [7], Parasyuk [11,12] and Moser [10]. For general. integrable systems, the invariant tori $\mathbb{T}^q$ are not necessarily Lagrangian. isotropic or coisotropic.

III. **Theorem 1** *If a Hamiltonian system (1) is integrable in the B-sense, then the following properties are realized:*

1) The components of the invariant submanifolds M^q, $q = 2k - p$, $F_j(x) = c_j$, are tori $\mathbb{T}^q$ if they are compact and toroidal cylinders $\mathbb{T}^m \times \mathbb{R}^{q-m}$ if they are non-compact. In a toroidal neighbourhood $\mathcal{O} = B_a \times \mathbb{T}^m \times \mathbb{R}^{q-m}$ of any toroidal cylinder there exist local coordinates $I_1, \cdots, I_p, \varphi_1, \cdots, \varphi_m, \rho_{m+1}, \cdots, \rho_q$. The *coordinates* I_j *run over a ball* $B_a \subset \mathbb{R}^p$. *The angle coordinates* $\varphi_1, \cdots, \varphi_m$ *run over the torus* $\mathbb{T}^m$, $0 \leq \varphi_j \leq 2\pi$, *and the coordinates* $\rho_{m+1}, \cdots, \rho_q$ *run over the Euclidean space* $\mathbb{R}^{q-m}$. *In these coordinates, the Hamiltonian system (1) has the form*

$$\dot{I}_j = 0, \quad \dot{\varphi}_\alpha(I) = \omega_\alpha(I), \quad \dot{\rho}_\gamma = \omega_\gamma(I) \tag{5}$$

and therefore it is integrable in the conventional sense.

2) For each compact component $\mathbb{T}^q$ *of the invariant submanifold* M^q, *we assume that system (5)* $\dot{I}_j = 0$, $\dot{\varphi}_\alpha(I) = \omega_\alpha(I)$ *is* $\mathbb{T}^q$*-dense. This assumption does not cause any loss of generality. The invariant symplectic structure* $\omega = P^{-1}$ *is reduced to the canonical form*

$$\omega_c = \sum_{\alpha=1}^{q}\sum_{\ell=1}^{r} a_\ell^\alpha \mathrm{d}I_\ell \wedge \mathrm{d}\varphi_\alpha + \sum_{\alpha,\beta=1}^{q} c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta + \sum_{j=1}^{(p-r)/2} \mathrm{d}I_{r+j} \wedge \mathrm{d}I_{h+j} \tag{6}$$

in the new coordinates $I_1, \cdots, I_p, \varphi_1, \cdots, \varphi_q$ *in the toroidal domain* $O = B_a \times \mathbb{T}^q$. *Here* $p + q = n = 2k$, $0 \leq r \leq p$, $h = (p+r)/2$. *The vectors* $a_\ell \in \mathbb{R}^q$ *are orthonormal null-vectors of the* $q \times q$ *constant, skew-symmetric matrix* $c_{\alpha\beta}$:

$$c_{\alpha\beta}a_\ell^\beta = 0, \quad (a_\ell, a_j) = \delta_j^\ell, \quad r = q - \operatorname{rank} \parallel c_{\alpha\beta} \parallel, \quad j, \ell = 1, \cdots, r.$$

There are $k(k+1)/2$ *canonical forms (6) that are non-isomorphic to the Liouville's classical form [1,2,8]. There are* $k-1$ *canonical forms for which tori* $\mathbb{T}^q$ *are coisotropic. There are* k *canonical forms with a non-degenerate inherited symplectic structure on the tori* $\mathbb{T}^q$ *(for* $q = 2k$ *we have* $M^{2k} = \mathbb{T}^{2k}$*). For the remaining* $(k-1)(k-2)/2$ *canonical forms (6) the invariant tori* $\mathbb{T}^q$ *are not Lagrangian, isotropic, coisotropic, or non-degenerate.*

3) A $\mathbb{T}^q$*-dense Hamiltonian system that preserves the symplectic structure (6) has the canonical form*

$$\dot{I}_1 = 0, \cdots, \dot{I}_p = 0, \quad \dot{\varphi}_\alpha = \sum_{\ell=1}^{r} \frac{\partial H(I)}{\partial I_\ell} a_\ell^\alpha + b_0^\alpha. \tag{7}$$

Here $H(I)$ *is an arbitrary smooth function of the* r *variables* $I_1, \cdots, I_r$ *and vector* b_0 *is orthogonal to the vectors* a_ℓ. *System (7) has the Hamiltonian form* $\dot{x}^\tau = (\omega_c^{-1})^{\tau\mu}\theta_\mu$, $\theta = \mathrm{d}H(I) + c_{\alpha\beta}b_0^\beta \mathrm{d}\varphi_\alpha$. *For a general function*

$H(I)$, system (7) is $\mathbb{T}^q$-dense if and only if the image space $C \subset \mathbb{R}^q$ of the $q \times q$ skew-symmetric matrix $c_{\alpha\beta}$ contains no vectors $m = (m_1, \cdots, m_q)$ with integer coordinates m_α, orthogonal to vector b_0, $(m, b_0) = 0$.

4) Any Hamiltonian system (7) is integrable in the B-sense. The variables $\varphi_1, \cdots, \varphi_q$ are supposed to run over a torus $\mathbb{T}^q$ or over a toroidal cylinder $\mathbb{T}^m \times \mathbb{R}^{q-m}$ where $m = 0, 1, \cdots, q-1$.

The proof of the key Theorem 1 is split into four parts which will be published in paper [4]. The strategy of the proof is the following:

(i) The proof of the existence of coordinates $I_1, \cdots, I_p, \varphi_1, \cdots, \varphi_m, \rho_{m+1}, \cdots, \rho_q$ where the Hamiltonian system (1) has the form (5), provided that the system is integrable in the B-sense.

(ii) The derivation of a complete classification of all closed 2-forms ω_c that are invariant with respect to the $\mathbb{T}^q$-dense dynamical system (5): $\dot{I}_j = 0$, $\dot{\varphi}_\alpha(I) = \omega_\alpha(I)$. Hence we obtain the general form of the original symplectic structure $\omega = P^{-1}$ (1) in the coordinates $I_1, \cdots, I_p, \varphi_1, \cdots, \varphi_q$.

(iii) The construction of a sequence of transformations of coordinates $I_1, \cdots, I_p, \varphi_1, \cdots, \varphi_q$ that transforms the symplectic structure ω to one of the canonical forms (6). These transformations preserve the general form of the dynamical system (5) $\dot{I}_j = 0$, $\dot{\varphi}_\alpha(I) = \omega_\alpha(I)$.

(iv) The derivation of the canonical form (7) for the dynamical system (5) $\dot{I}_j = 0$, $\dot{\varphi}_\alpha(I) = \omega_\alpha(I)$ in the newly-constructed coordinates where the symplectic structure ω has the canonical form (6).

The constructed $k(k+1)/2$ canonical forms of integrable Hamiltonian systems (7) cannot be integrated through Liouville's Theorem because the maximal number of their independent involutive first integrals is equal to $r + (p-r)/2 = (p + q - \text{rank} \parallel c_{\alpha\beta} \parallel)/2 < k$. The classical Liouville canonical form [1,2,8] is the particular case of (6) and (7) for $a_\ell^\alpha = \delta_\ell^\alpha$, $b_0^\alpha = 0$, $c_{\alpha\beta} = 0$, $p = q = r = k$.

Remark 1 The integrable Hamiltonian systems (7) are invariant with respect to the action of the torus $\mathbb{T}^q$. This action preserves the symplectic structure ω_c (6) because it has constant coefficients. However, this action is not a Poisson action as defined by Souriau [13] and by Marsden & Weinstein [9] and that was studied by Atiyah [3], Cartier [5] and Flaschka [6]. The torus $\mathbb{T}^q$ action has the Hamiltonian form (1) where the closed 1-forms θ are not exact in general.

IV. The introduced concept of B-integrability of Hamiltonian systems is a particular case of the following concept of integrability of a dynamical system $\dot{x}^i = V^i(x)$ on a manifold M^n of an arbitrary dimension n.

Definition 2 *A dynamical system* $\dot{x}^i = V^i(x^1, \cdots, x^n)$ *on a manifold* M^n *is integrable in the B-sense (or just B-integrable) if it possesses: p functionally independent first integrals* $F_1(x), \cdots, F_p(x)$, *where* $0 \leq p < n$, *and an abelian* $(n-p)$*-dimensional Lie algebra* S_a *of symmetries that preserve the first integrals* $F_j(x)$.

Theorem 2 *Assume that a dynamical system* $\dot{x}^i = V^i(x)$ *on a manifold* M^n *is B-integrable. Then the general submanifolds* $M_c \subset \mathcal{D}$ *of constant level of first integrals* $F_j(x) = c_j$ *are tori* $\mathbb{T}^{n-m}$ *or toroidal cylinders* $\mathbb{T}^q \times \mathbb{R}^{n-m-q}$. *The dynamical system* $\dot{x}^i = V^i(x)$ *has the form (5) in the corresponding toroidal coordinates and therefore it is integrable in the conventional sense.*

The proof is obtained by the same methods as the proof of Theorem 1.

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ON THE STABILITY OF MULTIPLICATIVE ADDITIVE MAPPINGS

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Presented by J. Aczel, F.R.S.C.

Abstract. We consider the question of whether an approximately additive, approximately multiplicative function between normed algebras must be near an additive multiplicative function. It is shown that the answer is positive for normed algebras in which the norm is multiplicative.

This paper deals with a stability question posed by G. L. Forti in [2]. Let A and B be normed algebras, and suppose that f: A→B is both approximately additive and approximately multiplicative in the sense that there are two constants $\delta_1, \delta_2 > 0$ for which

$$(1) \qquad \|f(x+y) - f(x) - f(y)\| \le \delta_1 (\|x\| + \|y\|),$$
$$(2) \qquad \|f(xy) - f(x)f(y)\| \le \delta_2 \|x\| \|y\|$$

for all $x, y \in$ A. The question is whether there must exist a multiplicative additive map h: A→B which approximates f in the sense that

$$(3) \qquad \|f(x) - h(x)\| \le M \|x\|, \qquad x \in \mathrm{A},$$

for some constant M. It will be shown that the answer to this question is affirmative for those normed algebras in which the norm is multiplicative, i.e.

$$\|xy\| = \|x\| \|y\|.$$

This applies also for instance when A or B is a field with a real valuation $\|.\|$, such as the rationals Q with p-adic valuation. Henceforth we assume that A and B are such multiplicatively normed algebras.

Z. Gajda has shown in [3] that inequality (1) alone is not enough for stability of the additive equation $\varphi(x+y) - \varphi(x) - \varphi(y) = 0$. In particular, there exists a function f: **R**→**R** satisfying (1) for some $\delta_1 > 0$ and such that for every additive φ the quotient

$$\frac{|f(x) - \varphi(x)|}{|x|}, \qquad x \in \mathbf{R} \setminus \{0\},$$

is unbounded. This is one reason for studying the system (1), (2).

Additional motivation comes from some results of B. E. Johnson [4], [5]. He is concerned primarily with the case where f is either a bounded linear map between Banach algebras or a bounded linear functional on a Banach algebra. Johnson's results are not comparable with those presented here, for several reasons. First, he considers continuous maps on Banach algebras; we assume neither continuity of f nor completeness of either A or B. Second, his maps are assumed to be linear, so that (1) is satisfied trivially. Third, Johnson asks whether for each $M > 0$ there exists a $\delta_2 > 0$ such that (2) implies (3) for some multiplicative linear h; thus he requires a stronger type of stability than the one posed here. He finds an affirmative answer for many classes (and a negative answer for one class) of Banach algebras. Recently, P. Semrl [6] has extended some of the results of Johnson by taking f to be approximately additive in the sense of (1), rather than linear. He finds that, in the case of real Banach algebras, such an approximate homomorphism must be near a homomorphism if δ_1, δ_2 are sufficiently small.

Our main result is the following.

Theorem 1. If $f : \mathsf{A} \to \mathsf{B}$ satisfies (1) and (2), then either f is additive and multiplicative or f has the form

$$f(x) = \|x\|\, q(x), \qquad x \in \mathsf{A}, \tag{4}$$

where $q : \mathsf{A} \to \mathsf{B}$ satisfies

$$\|q(x)\| \leq \frac{1}{2}\left(1 + \sqrt{1 + 4\delta_2}\right), \qquad x \in \mathsf{A}. \tag{5}$$

Proof: Putting $x = y = 0$ in (1), we see that $f(0) = 0$. Define q: $\mathsf{A} \to \mathsf{B}$ by

$$q(x) := \begin{cases} \| x \|^{-1} f(x), & \text{if } x \neq 0 \\ 0 & , \text{ if } x = 0. \end{cases}$$

Then (4) is obvious, and by (2) we have

$$\|q(xy) - q(x)\, q(y)\| \leq \delta_2.$$

Applying a result of Baker [1], we conclude that q is either multiplicative or it is bounded as in (5). In the latter case we are finished. Now suppose that q is multiplicative.

Since q is multiplicative , we have

$$\|q(x^n)\| = \|q(x)\|^n.$$

This means that if $\|q(x_0)\| > 1$ for some x_0 in A, then q is unbounded. That is, if q is bounded then $\|q(x)\| \leq 1$ for all x in A, which means that (5) is again valid.

Now we consider the only case remaining – that q is multiplicative and unbounded. By (4), we see that f is multiplicative. It only remains to be shown that f is additive in this case. To this end, put $x = tu$ and $y = tv$ in (1). Then we obtain

$$\begin{aligned}\|f(t)\|\,\|f(u+v) - f(u) - f(v)\| &\\ &= \|f(t)\,[f(u+v) - f(u) - f(v)]\| \\ &= \|f(tu + tv) - f(tu) - f(tv)\| \\ &\leq \delta_1\,(\|tu\| + \|tv\|) \\ &= \delta_1\,\|t\|\,(\|u\| + \|v\|),\end{aligned}$$

for all $t, u, v \in \mathsf{A}$. By (4), this means that

$$\|q(t)\|\,\|f(u+v) - f(u) - f(v)\| \leq \delta_1\,(\|u\| + \|v\|)$$

for all $t, u, v \in \mathsf{A}$. Since q is unbounded, it must be the case that

$$f(u+v) - f(u) - f(v) = 0, \qquad u, v \in \mathsf{A}.$$

Therefore f is indeed additive, and the proof is finished.

Example 1. If q is constant, satisfies (5) and $\| q \| \leq \delta_1$, then (4) defines f satisfying (1), (2).

From Theorem 1, we can easily deduce the following.

Corollary 1. If f: $\mathsf{A} \rightarrow \mathsf{B}$ satisfies (1) and (2), then there exists an additive and multiplicative h: $\mathsf{A} \rightarrow \mathsf{B}$ such that

$$(6) \qquad \|f(x) - h(x)\| \leq \frac{1}{2}\left(1 + \sqrt{1 + 4\delta_2}\right) \|x\|, \qquad x \in \mathsf{A}.$$

Proof: Apply Theorem 1. If f is additive and multiplicative take $h = f$; otherwise take $h = 0$.

Theorem 1 can be generalized in the following way. Suppose now that, instead of (2), f satisfies

$$\|f(xy) - f(x)f(y)\| \leq \delta_2 \|x\|^p \|y\|^p \tag{7}$$

for some real number p. We shall prove the following.

Theorem 2. Let f: $\mathsf{A} \to \mathsf{B}$ satisfy (1) and (7) for some positive constants δ_1, δ_2, and some constant p. Then one of the following three possibilities occurs:

(i) f is additive and multiplicative, or

(ii) f has the form

$$f(t) = \|t\|^p q(t) \tag{8}$$

with q satisfying (5), or

(iii) f is multiplicative, not additive, and satisfies

$$\|f(t)\| \leq \|t\| \tag{9}$$

for all t in A.

Proof: We proceed as in the proof of Theorem 1, but define $q(x)$ by dividing $f(x)$ by $\| x \|^p$. As before, we find that q is either multiplicative and unbounded, or bounded as in (5). In the latter case, we have conclusion (ii) of the theorem. If q is multiplicative and unbounded, then either f is additive (giving (i)), or

$$\|f(t)\| \leq M \|t\|,$$

for all t in A, for some constant M. But since f is multiplicative we have here

$$\|f(t)\|^n = \|f(t)^n\| = \|f(t^n)\| \leq M \|t^n\| \leq M \|t\|^n$$

for any positive integer n. Hence $\|f(t)\| \leq M^{1/n} \|t\|$, which shows that M can be replaced by 1. This gives conclusion (iii) and completes the proof of Theorem 2.

Example 2. Let $\mathsf{A} = \mathsf{B} = \mathbf{R}$, let $p = 2$ in (7), and let $q(x) = \min\{1, 1/|x|\}$. Then the f defined by (8), namely

$$f(t) = \min\{t^2, |t|\}$$

is a solution of (1) and (7) for $\delta_1, \delta_2 \geq 1$. This solution of category (ii) fits neither category (i) nor (iii) of Theorem 2. This function f also provides a solution of (1) and (2), again if $\delta_1, \delta_2 \geq 1$, which is of the form (4) with $q(t) = \min\{ |t|, 1\}$.

From Theorem 2, we also draw the following consequence when $A = B = \mathbf{R}$.

Corollary 2. .If $A = B = \mathbf{R}$ in Theorem 2, then case (iii) contains only the function $f(x) = |x|$, and this function satisfies (1) if and only if $\delta_1 \geq 1$.

Proof: In case (iii), f being multiplicative and bounded on an interval, it follows that $f(x)$ must be of the form $|x|^c$ or $|x|^c \operatorname{sgn} x$ or constant (= 1 or 0).But because of (9) and the fact that f is not additive, we are left with only $|x|$. This function satisfies (1) only if $\delta_1 \geq 1$.

We conclude by turning to the consideration of approximate homomorphisms of multiplicatively normed algebras. (Of course, in the case of fields with real valuation, we have been considering approximate homomorphisms all along.) Let A and B be normed algebras with multiplicative norms and a common scalar field $K = \mathbf{R}$ or $\mathbf{C}$. By an *approximate homomorphism*, we mean a map f: $A \rightarrow B$ which satisfies (1), (2), and additionally

$$(10) \qquad \|f(\lambda x) - \lambda f(x)\| \leq g(|\lambda|)\, \|x\|$$

for all $\lambda \in K$ and $x \in A$, where g: $\mathbf{R}_+ \rightarrow \mathbf{R}_+$ ($\mathbf{R}_+$ the nonnegative reals) is arbitrary. We prove the following.

Theorem 3. Let f: $A \rightarrow B$ be an approximate homomorphism of multiplicatively normed algebras A and B. Then either f is a homomorphism or f has the form (4) with q bounded as in (5). In particular, there exists a homomorphism h such that (6) is fulfilled.

Proof: Applying Theorem 1, we find that either f is both additive and multiplicative or f has the form (4) with q satisfying (5). In the latter case there is nothing more to prove. Suppose that the former holds.

From the proof of Theorem 1, we see that it suffices to consider the case in which (cf. (4)) the map q is unbounded as well as multiplicative. By multiplicativity, (10) with $x = tu$ yields

$$\|[f(\lambda t) - \lambda f(t)]\, f(u)\| = \|f(\lambda tu) - \lambda f(tu)\| \leq g(|\lambda|)\, \|tu\|.$$

Thus

$$\|f(\lambda t) - \lambda f(t)\| \, \|f(u)\| \leq g(|\lambda|) \, \|t\| \, \|u\|.$$

By (4), this transforms into

$$\|f(\lambda t) - \lambda f(t)\| \, \|q(u)\| \leq g(|\lambda|) \, \|t\|, \qquad u \neq 0.$$

Since q is unbounded, this means that

$$f(\lambda t) = \lambda f(t),$$

for all $\lambda \in K$ and $t \in A$. Thus f is homogeneous and consequently a homomorphism.

The final statement of Theorem 3 follows easily, in the same manner as the Corollary to Theorem 1.

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THE SIGNATURE OF KAEHLER SURFACES IMMERSED INTO $HP^n(1)$

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Presented by T. Bloom, F.R.S.C.

Abstract. In this note we give some interesting topological restrictions for totally complex immersion of Kaehler surfaces into the quaternion projective space $HP^n(1)$.

1. Introduction

Let $HP^n(1)$ be a quaternion projective space with constant quaternion sectional curvature 1[4], and M a totally complex submanifold of $HP^n(1)$(for the definition of totally complex submanifold cf.[2]). We know that M has a natural Kaehler structure.

Recall the standard imbeddings[2]:

$$\phi_1: CP^n(1) \text{ --- } HP^n(1),$$

along with the following standard imbedding[5]:

$$\tau: CP^1(1) \times CP^1(1) \text{ --- } CP^3(1),$$

We define the imbedding of $\phi_2 = \phi_1 \cdot \tau$.

Now let $j: M \longrightarrow HP^n(1)$ be a totally complex immersion of a 2-dimensional compact Kaehler submanifold into $HP^n(1)$. We denote by sign(M) and h the signature of M and the second fundamental form of the immersion respectively. In this paper we obtain the followingtheorems.

Theorem 1. For M we have

$$(1.1) \qquad 32\pi^2 \text{sign}(M) \geq \int_M (4 - |h|^4)*1$$

Key words and phrases. signature, Kaehler surface, quaternion projective space.

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where * denotes the Hodge star operator and the equality holds if and only if M is an imbedded submanifold congruent to one of the following embeddings: ϕ_1: $CP^2(1) \longrightarrow HP^2(1)$; ϕ_2: $CP^1(1) \times CP^1(1) \longrightarrow HP^3(1)$.

Theorem 2. (i) If M has positive total scalar curvature, then the second betti number of M satisfies

$$b_2 \leqslant 2 + \frac{1}{32\pi^2}\int_M (|h|^4 - 4)*1$$

where the equality holds if and only if M is congruent to one of the following embeddings: ϕ_1: $CP^2(1) \to HP^2(1)$; ϕ_2: $CP^1(1) \times CP^1(1) \to HP^3(1)$.

(ii) If M has $\mathrm{sign}(M) \leqslant 0$, then

$$\int_M |h|^4 *1 \geqslant 4\mathrm{vol}(M)$$

where the equality holds if and only if M is congruent to the following standard imbedding: ϕ_2: $CP^1(1) \times CP^1(1) \longrightarrow HP^3(1)$.

2.Preliminaries

Let M be a 2-dimensional compact Kaehler manifold. Let w^1, w^2 be the local field of unitary coframes. Then the Keahler 2-form ϕ, the Ricci form Q and the scalar curvature r are given by

$$\phi = \frac{\sqrt{-1}}{8\pi}\sum w^a \wedge \bar{w}^a, \qquad Q = \frac{\sqrt{-1}}{4\pi}\sum \rho_{a\bar{b}} w^a \wedge \bar{w}^b, \qquad r = 2\sum \rho_{a\bar{a}}$$

where $\rho_{a\bar{b}}$ is the local components of the Ricci tensor ρ of M. It is well-known that the first chern class c_1 is represented by Q. we denote $|R|$ and $|\rho|$ the length of the curvature and Ricci tensor respectively. The signature of M can be expressed by the following formulas(cf. [1] [3])

$$(2.1) \quad 48\pi^2 \mathrm{sign}(M) = \int_M (2|\rho|^2 - 2|R|^2)*1$$

$$(2.2) \quad \mathrm{sign}(M) = \sum_{p,q=0}^{2} (-1)^q b_{p,q}$$

where the $b_{p,q}$ denotes the dimensional of the space of the harmonical forms of bidegree (p, q) on M.

3. Proof of Theorem 1

Since M is a totally complex immersion into $HP^n(1)$, as in [6], we can obtain (3.1) $r = 6 - |h|^2$

$$(3.2)\quad \frac{1}{2}\Delta|h|^2 = |\bar{\nabla}h|^2 + 2|h|^2 - 2tr(\Sigma A_u^2)^2 - \Sigma(trA_uA_v)^2$$

where Δ is the Laplacian, $\bar{\nabla}h$ the covariant derivative of h and A_u the Weingarten maps associated with orthonormal basis $\eta_1, \dots, \eta_{4n-4}$ of the normal space. From the equation of Gauss, we have

$$(3.3)\quad |\rho|^2 = 9 - 3|h|^2 + tr(\Sigma A_u^2)^2$$

$$(3.4)\quad |R|^2 = 12 - 4|h|^2 + 2\Sigma(trA_uA_v)^2$$

From [6] ,we have

$$(3.5)\quad 2\Sigma(trA_uA_v)^2 \leq |h|^4$$

Taking the integral of the both side of (3.2) and using Green's Theorem, we have

$$(3.6)\quad \int_M |\bar{\nabla}h|*1 = \int_M (2tr(\Sigma A_u^2)^2 + \Sigma(trA_uA_v)^2 - 2|h|^2)*1$$

Now combing (2.1) with (3.3), (3.4) and (3.6), we have

$$(3.7)\quad 48\pi^2 sign(M) = \int_M (|\bar{\nabla}h|^2 + 6 - 3\Sigma(trA_uA_v)^2)*1$$

From (3.7), (3.5) and (3.1) we get

$$48\pi^2 sign(M) \geq \int_M |\bar{\nabla}h|^2*1 + \frac{3}{2}\int_M (4 - |h|^4)*1 \geq \frac{3}{2}\int_M (4 - |h|^4)*1$$

From which (1.1) following. Suppose the equlity holds in (1.1), i.e.,

$$(3.8)\quad 48\pi^2 sign(M) = \frac{3}{2}\int_M (4 - |h|^4)*1$$

Then M has parallel second fundamental form, from the classification of submanifolds in $HP^n(1)$ which have parallel second fundamental form [7] and (3.8), we know that M is congruent to either ϕ_1 or ϕ_2.

4. Proof of Theorem 2

Since the totally scalar curvature $\int_M r*1$ is positive, a result of [8] implies that all plurigenera of M vanish. In particular we have

$b_{2,0} = 0$. Then using $b_{2,0} = 0$, $b_{2,2} = b_{0,0} = 1$, $b_{p,q} = b_{q,p}$ and Serre duality, from (2.2) we get
$\mathrm{sign}(M) = 2 - b_2$. Therefore (i) of the Theorem 2 follows from Theorem 1.

If M has $\mathrm{sign}(M) \leqslant 0$, from Theorem 1 we obtain $\int_M |h|^4 *1 \geqslant 4\mathrm{vol}(M)$. Moreover $\int_M |h|^4 *1 = 4\mathrm{vol}(M)$ implies $\mathrm{sign}(M) = 0$ and the equality in (1.1). Therefore M is congruent to $\phi_2\colon CP^1(1) \times CP^1(1) \longrightarrow HP^3(1)$.

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