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CONTENTS

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVIII, Nos. 2,3, April-June 1996 avril-juin

Probabilités et Analyse Harmonique/Probability Theory and Harmonic Analysis

Théorèmes local et de renouvellement pour une convolution généralisée sur la demi droite

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Presented by G.F.D. Duff, F.R.S.C.

Résumé. Soit $*$ la convolution sur $M(\mathbb{R}_+)$ associée à un opérateur différentiel singulier L. Pour une probabilité μ sur $\mathbb{R}_+$ avec des hypothèses de moments adéquates, on étudie comment normaliser les mesures $\{\mu^{*n}; n \in \mathbb{N}\}$ (resp. $\epsilon_x * \sum_{n=0}^{\infty} \mu^{*n}; x > 0$) pour qu'elles convergent vaguement si $n \to +\infty$ (resp. $x \to +\infty$). Les résultats dépendent de L et d'une étude fine de ses fonctions propres.

Local and renewal theorems for a generalized convolution on the half line

Abstract. Let $*$ be the convolution on $M(\mathbb{R}_+)$ associated to a singular differential operator L. If μ is a probability measure on $\mathbb{R}_+$ with suitable moment conditions. we study how to normalize the measures $\{\mu^{*n}; n \in \mathbb{N}\}$ (resp. $\{\epsilon_x * \sum_{n=0}^{\infty} \mu^{*n}\}$) in order to get weak convergence if $n \to +\infty$ (resp. $x \to +\infty$). The results depend on the type of L and on a precise study of its eigenfunctions.

1) Généralités. On considère sur $]0, +\infty[$ un opérateur différentiel $L = \partial_x^2 + A'(x)/A(x)\partial_x$ (∂_x^n désigne la dérivée nième en x). On suppose $A(x) = x^{2\alpha+1}B(x), \alpha > -1/2, B > 0$ est une fonction C^∞ paire sur $\mathbb{R}$ avec $B(0) = 1$. De plus A est croissante. $\lim_{+\infty} A = +\infty$, A'/A est décroissante. $\lim_{+\infty} A'/A = 2\rho \geq 0$ et il existe $\delta > 0, x_0 > 0$ tels que pour $x \geq x_0$ on ait

$$(1) \qquad (B'/B)(x) = 2\rho - (2\alpha+1)x^{-1} + e^{-\delta x}D(x) \text{ si } \rho > 0 \text{ ou } e^{-\delta x}D(x) \text{ si } \rho = 0.$$

où D est une fonction C^∞ sur $[x_0, +\infty[$, bornée avec toutes ses dérivées. Les opérateurs de Bessel et de Jacobi sont des exemples types avec respectivement $A(x) = x^{2\alpha+1}$ (et $\rho = 0$) et $A(x) = 2^{2\rho}(shx)^{2\alpha+1}(chx)^{2\beta+1}$($\alpha \geq \beta > -1/2$ et $\rho = \alpha + \beta + 1 > 0$) ([6]).

D'après [4] l'équation aux valeurs propres $Lu = -(\lambda^2 + \rho^2)u$ a pour tout $\lambda \in \mathbb{C}$ une unique solution φ_λ telle que $\varphi_\lambda(0) = 1$ et $\varphi'_\lambda(0) = 0$ et deux autres solutions linéairement indépendantes $\Phi_{\pm\lambda}$ telles que $\sqrt{A(x)}\Phi_{\pm\lambda}(x) \sim e^{i(\pm\lambda)x}(x \to +\infty)$ ([5]). La fonction $c : \mathbb{C} \to \mathbb{C}$ telle que $\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x)$ est la fonction d'Harish Chandra. Pour tout $\lambda \in \mathbb{C}$ et $x > 0$, on a

$$(2) \qquad \varphi_\lambda(x) = \int_0^x K(x.u)cos\,(\lambda u)\,du.$$

où $K(x,.) \geq 0$ est de classe C^∞ sur $]-x,x[$, paire et à support dans $[-x,x]$ ([8]). De plus pour tous $x,y \in \mathbb{R}_+$, il existe une probabilité notée $\epsilon_x * \epsilon_y$ (ϵ pour mesure de Dirac) à support dans $[|x-y|, x+y]$ telle que $\varphi_\lambda(x)\varphi_\lambda(y) = \langle \epsilon_x * \epsilon_y, \varphi_\lambda \rangle$. La convolée de μ et $\nu \in M(\mathbb{R}_+)$ est alors définie par $\langle \mu * \nu, f \rangle = \langle \mu \otimes \nu, u \rangle$ si f est une fonction continue et à support compact dans $\mathbb{R}_+$ et $u(x,y) = \langle \epsilon_x * \epsilon_y, f \rangle$. On munit ainsi $\mathbb{R}_+$ d'une structure d'hypergroupe commutatif de mesure de Haar $m(dx) = A(x)dx$ et de mesure de Plancherel $|c(\lambda)|^{-2}d\lambda$ ([2], [4], [8]). Le comportement de $|c(\lambda)|^{-2}$ à l'infini a été établi en [5] ; pour le voisinage de zéro nous avons utilisé [9] (th. 3.72) et obtenu le résultat suivant :

Proposition 1. Il existe des constantes positives k, k_1, k_2 telles que

1) si $\rho \geq 0$ et $\alpha > -1/2$; $\forall \lambda \in \mathbb{C}, |\lambda| > k \Rightarrow k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}$
2) si $\rho > 0$ et $\alpha > -1/2$; $\forall \lambda \in \mathbb{C}, |\lambda| \leq k \Rightarrow k_1|\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^2$
si $\rho = 0$ et $\alpha > 0$; $\forall \lambda \in \mathbb{C}, |\lambda| \leq k \Rightarrow k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}$

Sous les conditions précédentes, posant $\gamma = 1/2$ si $\rho > 0, \gamma = \alpha$ si $\rho = 0$ et $c_1(\lambda) = \lambda^{\gamma+1/2}c(\lambda)$, on a $\lim_{\lambda \to 0} c_1(\lambda) = c_1(0) \neq 0$.

2) Propriétés asymptotiques de φ_λ et Φ_λ. Soit la fonction $G = 4^{-1}(A'/A) + 2^{-1}(A'/A)' - \rho^2$. On écrira $G(x) = (\alpha^2 - 1/4)x^{-2} + \chi(x)$. D'après (1), il existe $\delta > 0, x_0 > 0$ et des fonctions $D_i (i = 1,2)$ de classe C^∞ sur $[x_0, +\infty[$ bornées avec leurs dérivées et telles que $G(x) = e^{-\delta x}D_1(x)$ si $\rho > 0$ et $\chi(x) = e^{-\delta x}D_2(x)$ si $\rho = 0$.

Théorème 1. Pour tout entier $n > 0$, tout $\lambda \in \mathbb{C}^*$ et $x > 0$, on a

$$(3) \qquad \sqrt{A(x)}\Phi_\lambda(x) = e^{i\lambda x}\sum_{s=0}^{n-1} a_s(x)(i\lambda)^{-s} + R_n(\lambda, x),$$

où les fonctions réelles a_s s'obtiennent par les relations

$$a_0(x) \equiv 1, \; 2a'_{s+1}(x) = -a''_s(x) + G(x)a_s(x) \; (s \geq 0)$$

et

$$(4) \qquad |R_n(\lambda, x)| \leq 2|\lambda|^{-n}\left(\int_x^{+\infty} |a'_n(t)|dt\right) exp\left(|\lambda|^{-1}\int_x^{+\infty} |G(t)|dt\right).$$

De plus, $a_s(x) = O(e^{-\delta x})$ si $\rho > 0$ et $a_s(x) = C_s x^{-s} + O(e^{-\delta x})$ si $\rho = 0$, où C_s est une constante.

Démonstration (idée) La fonction $\sqrt{A}\Phi_\lambda$ est solution de l'équation différentielle $v'' = ((i\lambda)^2 + G)v$; on en déduit alors une équation intégrale pour le terme d'erreur R_n qu'on résout par la méthode de variation de la constante.

Soit $W_\alpha(u) = (\pi/2)^{1/2}e^{i(1+2\alpha)\pi/4}u^{1/2}H_\alpha^{(1)}(u)$ avec $H_\alpha^{(1)}$ la fonction de Hankel de première espèce d'indice α.

Théorème 2. Si $\rho \geq 0$. il existe une constante $C > 0$. telle que pour tous $\lambda \in \mathbb{R}$ et $x > 0$

(5) $$|\sqrt{A(x)}\Phi_\lambda(x) - W_\alpha(\lambda x)| \leq$$
$$\begin{cases} C\left(\frac{|\lambda x|}{1+|\lambda|x}\right)^{1/2-\alpha}\left[exp\left(C\int_x^\infty \frac{t}{1+|\lambda|t}|\chi(t)|dt\right) - 1\right] & \text{si } \alpha > 1/2 \\ C\left[exp\left(C\int_x^\infty \frac{t}{1+|\lambda|t}|\chi(t)|dt\right) - 1\right] & \text{si } \alpha \in]-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

(6) $$|\sqrt{A(x)}\varphi_\lambda(x)| \leq C\, x^{1/2+\alpha}(1+|\lambda|x)^{-\alpha-1/2} exp\left(\int_0^x \frac{t}{1+|\lambda|t}|\chi(t)|dt\right).$$

Démonstration (idée) La fonction $\sqrt{A}\Phi_\lambda$ est solution de l'équation $w''(x) + ((1/4-\alpha^2)x^{-2} + \lambda^2)\, w(x) = \chi(x)w(x)$. Une résolution par approximations successives de donnée initiale $W_\alpha(\lambda x)$ conduit à (5). Pour (6) on applique une méthode analogue à la fonction $\sqrt{A}\varphi_\lambda$ qui satisfait la même équation mais avec une donnée initiale équivalente à $x^{1/2+\alpha}$ si $x \to 0$.

Corollaire 1(formules de Mehler-Heine). Si $\rho = 0$ ou $\rho > 0$ et $\alpha = 1/2$ on a pour $\lambda \in \mathbb{R}$

(7) $$\lim_{x\to+\infty} \sqrt{A(x)}\Phi_{\lambda/x}(x) = W_\alpha(\lambda)$$

(8) $$\lim_{x\to+\infty} \sqrt{B(x)}\varphi_{\lambda/x}(x) = \sqrt{\pi}2^{1/2-\alpha}|c_1(0)|(\Gamma(\alpha+1))^{-1}j_\alpha(\lambda),$$

où $j_\alpha(\lambda) = 2^\alpha\Gamma(\alpha+1)\lambda^{-\alpha}J_\alpha(\lambda)$ et J_α est la fonction de Bessel de première espèce.

Démonstration (idée) : (7) résulte du Théorème 2. Pour (8) on utilise (7). l'expression φ_λ en fonction de $\Phi_{\pm\lambda}$ et une méthode de calcul par inversion de Fourier qui sera détaillée ailleurs.

3) Le théorème local. La transformée de Fourier d'une probabilité $\mu \in M_1(\mathbb{R})$ est définie par $\hat{\mu}(\lambda) = \langle\mu, \varphi_\lambda\rangle$ pour $|Im\lambda| \leq \rho$. Soit alors $b(x) = -\partial^2_\lambda\varphi_\lambda(x)|_{\lambda=0}$. On voit facilement à partir de (2) que $b \geq 0$ est bornée sur $\mathbb{R}_+$ si $\rho > 0$ et que $b(x) \leq x^2$ si $\rho = 0$. Ainsi $b(\mu) = \langle\mu, b\rangle < +\infty$ pour toute $\mu \in M_1(\mathbb{R}_+)$ si $\rho > 0$ et pour μ ayant un moment d'ordre 2 si $\rho = 0$. Dans ce cas on a immédiatement

(9) $$\hat{\mu}(\lambda) = \hat{\mu}(0) - b(\mu)\lambda^2/2 + o(\lambda^2) \text{ si } \lambda \to 0.$$

Théorème 3. Soit $\mu \in M_1(\mathbb{R}_+), \mu \neq \epsilon_0$ et $b(\mu) < +\infty$. On suppose $\rho > 0$ ou $\rho = 0$ et $\alpha > 0$ et soit γ la constante de la Proposition 1. Alors la suite des mesures $n^{\gamma+1}(\hat{\mu}(0))^{-(n+\gamma+1)}\mu^{*n}$ converge vaguement vers $C_\mu\varphi_0 m (= C_\mu\varphi_0(x).A(x)dx)$ où $C_\mu = (2\pi)^{-1}|c_1(0)|^{-2}2^\gamma\Gamma(\gamma+1)b(\mu)^{-(\gamma+1)} > 0$.

Démonstration (esquissée) : pour $f \in \mathcal{D}_*(\mathbb{R})$ (espace des fonctions paires de classe C^∞ sur $\mathbb{R}$ et à support compact) et pour tout $\eta > 0$, on a par inversion de Fourier

$$n^{\gamma+1}(\hat{\mu}(0))^{-n}\int_0^{+\infty} f(x)\mu^{*n}(dx) = \left(\int_0^\eta + \int_\eta^{+\infty}\right) n^{\gamma+1}\hat{f}(\lambda)(\hat{\mu}(\lambda)\hat{\mu}(0)^{-1})^n|c(\lambda)|^{-2}\frac{d\lambda}{2\pi}.$$

On peut alors passer à la limite quand $n \to +\infty$ suivant la démarche classique grâce à (9), à la Proposition 1 et aux lemmes suivants :

Lemme 1. Soient m et $(m_i)_{i\in I}$ (avec $I = \mathbb{N}$ ou $\mathbb{R}_+$) des mesures de Radon positives sur $\mathbb{R}_+$. Si pour tout $f \in \mathcal{D}_*(\mathbb{R})$, $\lim_{i\to+\infty}\langle m_i, f\rangle = \langle m, f\rangle$ alors m_i converge vaguement vers m quand $i \to +\infty$.

Lemme 2. Soit $\mu \in M_1(\mathbb{R}_+)$. S'il existe $\alpha > 0$ tel que $[0.\alpha] \subset supp\mu$. alors pour tout $\eta > 0$, il existe $a = a(\eta) < 1$ tel que $|\hat{\mu}(\lambda)| \leq a\hat{\mu}(0)$ pour tout $\lambda \geq \eta$.

Lemme 3. Soit $\mu \in M_1(\mathbb{R}_+)$ et $x_0 \in supp\mu$ avec $x_0 \neq 0$. Alors $[0, 2x_0] \subset supp\mu^{*2}$.

Remarque : On notera que $\hat{\mu}(0) = 1$ si $\rho = 0$ et $0 < \hat{\mu}(0) < 1$ si $\rho > 0$. On remarquera aussi que pour tout $x \in \mathbb{R}_+, \varphi_0(x) > 0$ et que $\varphi_0(x) \equiv 1$ si $\rho = 0$.

4) Théorèmes de renouvellement. Pour n entier > 0. soit la fonction $d_n(x) = \partial_\lambda^n\varphi_{\lambda+i\rho}(x)|_{\lambda=0}$ $(x \geq 0)$. A l'aide de (2), on montre qu'on a toujours $|d_n(x)| \leq x^n$. Si $\rho = 0$ on a $d_n(x) \equiv 0$ si n est impair et $|d_\ell(x)| \leq 1 + |d_n(x)|$ pour tout $\ell \leq n$ si n est pair. Pour $\mu \in M_1(\mathbb{R}_+)$, on pose alors $d_n(\mu) = \langle\mu, |d_n|\rangle$. Si $\rho = 0$. on notera que $d_2(\mu) = b(\mu)$ (cf. §3). Si $\rho = 0$ et μ admet un moment d'ordre n (i.e. $\int_0^\infty x^n\mu(dx) < +\infty$), $\hat{\mu}$ est de classe C^n sur $\mathbb{R}$. De plus

Proposition 2 : Si $\rho = 0$, si n est pair et si $d_n(\mu) < +\infty$, $\hat{\mu}$ est de classe C^n sur $\mathbb{R}$.

Théorème 4. Soit $\mu \in M_1(\mathbb{R}_+), \mu \neq \epsilon_0$. Si $\rho > 0$ ou si $\rho = 0$ et $\alpha > 0$ et $d_2(\mu) < +\infty, \nu = \sum_{n=0}^{+\infty}\mu^{*n}$ est une mesure de Radon et on a

a) si $\rho > 0$ et si $d_2(\mu) < +\infty$, $\sqrt{A(x)}e^{\rho x}(\epsilon_x * \nu)$ converge vaguement si $x \to +\infty$ vers la mesure $(d_1(\mu))^{-1}m$.

b) si $\rho = 0$ et $\alpha > 0$, on suppose $d_2(\mu) < +\infty$ si $0 < \alpha < 1/2$ et pour $\alpha > 1/2$, $\alpha = n + r$ avec $n \in \mathbb{N}^*$ et $-1/2 < r < 1/2$, on suppose soit que μ a un moment d'ordre $n+2$ soit que $d_{2k+2}(\mu) < +\infty$ si $n = 2k$ ou $2k-1$. Si de plus $c_1(\lambda) = \lambda^{\alpha+1/2}c(\lambda)$ est de classe C^∞ sur $\mathbb{R}_+$ avec des dérivées polynomialement bornées, il existe alors une constante $a > 0$ telle que $x^{2\alpha}(\epsilon_x * \nu)$ converge vaguement vers am si $x \to +\infty$.

c) dans le cas de l'opérateur de Bessel, on suppose $d_2(\mu) < +\infty$ si $0 < \alpha \leq 1/2$ et pour $\alpha > 1/2, \alpha = n + r, n \in \mathbb{N}^*$ et $-1/2 < r \leq 1/2$, on suppose soit que μ a un moment d'ordre $n+2$, soit que $d_{2k+2}(\mu) < +\infty$ si $n = 2k$ ou $2k-1$. Alors $x^{2\alpha}(\epsilon_x * \nu)$ converge vaguement vers $(\alpha d_2(\mu))^{-1}m$ si $x \to +\infty$.

Démonstration (idées) a) Pour $f \in \mathcal{D}_*(\mathbb{R})$, on a $\langle \epsilon_x * \nu, f\rangle = Uf(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} \hat{f}(\lambda)(1-\hat{\mu}(\lambda))^{-1}\Phi_\lambda(x)c(-\lambda)^{-1}d\lambda$. On utilise alors l'holomorphie de la fonction $\lambda \to \Phi_\lambda(x)c(-\lambda)^{-1}$ dans le demi plan supérieur pour estimer cette intégrale sur un contour adéquat. Pour b) si $\alpha > 1/2$ on écrit $Uf(x) = (2\pi)^{-1}\int_0^{+\infty} \hat{f}(\lambda)(1-\hat{\mu}(\lambda))^{-1}(\Phi_\lambda(x)c(-\lambda)^{-1} + \Phi_{-\lambda}(x)c(\lambda)^{-1})d\lambda$, on utilise le théorème 1 en procédant à des intégrations par parties pour trouver un équivalent de chacun des termes et on utilise le théorème 2 pour évaluer le reste. Pour $0 < \alpha < 1/2$ on utilise le corollaire 1. Pour c) le calcul direct est possible car $\varphi_\lambda(x) = j_\alpha(\lambda x)$. Dans tous les cas on a besoin du lemme 1 pour conclure.

Remarques. 1) Pour des analogues classiques des théorèmes étudiés ici on peut consulter [1], [7], pour le cas de $\mathbb{R}^n$ et [3] pour le cas des espaces symétriques.

2) Les idées utilisées ici permettent aussi d'obtenir des théorèmes locaux et de renouvellement pour d'autres structures de convolution comme certains hypergroupes polynomiaux. On trouvera tous les détails dans une prochaine publication.

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SOME REMARKS ON COINCIDENCES AND FIXED POINTS

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Presented by E. Bierstone, F.R.S.C.

Abstract. The purpose of this note is to provide a substantial improvement of recent results of Pathak [5].

In a well-written paper, Pathak [5] obtained some interesting results improving several coincidence and fixed point theorems. In the present note we provide a substantial improvement of his main results([5,Theorems 2 and 4]) by removing the assumptions of continuity, and replacing the completeness of the space by a set of weaker conditions. We also drop the weak compatibility requirement from his Theorem 2.

We generally follow the notations and definitions used in [5]. Let (X,d) be a metric space. Let (CB(X),H) and (CL(X),H) denote respectively the hyperspaces (cf. Nadler [4]) of nonempty closed bounded subsets of X and nonempty closed subsets of X, where H is the Hausdorff metric induced by d.

The following is the main result of Pathak[5,Theorem 2].

THEOREM 1. Let (X,d) be a complete metric space, $f : X \to X$, and $T : X \to CB(X)$ be f-weak compatible continuous mappings such that $T(X) \subseteq f(X)$ and

(1) $H(Tx,Ty) \leq h \max\{d(fx,fy),d(fx,Tx),d(fy,Ty),[d(fx,Ty)+d(fy,Tx)]/2\}$

is satisfied for all $x,y \in X$ and $0 \leq h < 1$. Then there exists a point t such that $ft \in Tt$.

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The following two results improve the above theorem substantially.

THEOREM 2. Let Y ($\neq \phi$) be an arbitrary set, (X,d) a metric space, $f : Y \to X$ and $T : Y \to CL(X)$ such that $T(Y) \subseteq f(Y)$ and (1) is satisfied for all $x, y \in Y$, $0 \leq h < 1$. If one of $T(Y)$ or $f(Y)$ is a complete subspace of X, then there exists a point $t \in Y$ such that $ft \in Tt$.

Proof. Let $x_0 \in Y$ be arbitrary, and choose $x_1 \in Y$ such that $fx_1 \in Tx_0$. Such a choice is permissible since $Tx_0 \subseteq f(Y)$. Since we may assume $0 < h < 1$, we choose a point $x_2 \in Y$ such that $d(fx_2,fx_1) \leq k\, H(Tx_1,Tx_0)$, where $k = 1/\sqrt{h}$. In fact, following the constructive proof technique of Pathak [5, p. 72], we construct a sequence $\{x_n\} \subseteq Y$ such that $fx_{n+1} \in Tx_n$ and $d(fx_{n+1},fx_n) \leq k\, H(Tx_n,Tx_{n-1})$, $n = 1, 2, \ldots$. Now setting $x = x_{n+1}$ and $y = x_n$ in (1), it can be easily verified that $\{fx_n\}$ is a Cauchy sequence.

If $f(Y)$ is a complete subspace of X, then $\{fx_n\}$ being contained in $f(Y)$ has a limit in it. Call it u. Let $t \in f^{-1}u$. Then $ft = u$. By (1),

$$\begin{aligned} d(fx_{n+1},Tt) &\leq H(Tx_n,Tt) \\ &\leq h \max\{d(fx_n,ft),d(fx_n,Tx_n),d(ft,Tt), [d(fx_n,Tt)+d(ft,Tx_n)]/2\} \\ &\leq h \max\{d(fx_n,ft),d(fx_n,fx_{n+1}),d(ft,Tt),[d(fx_n,Tt)+d(ft,fx_{n+1})]/2\}. \end{aligned}$$

Making $n \to \infty$ we have $d(ft,Tt) \leq h\, d(ft,Tt)$, i.e. $ft \in Tt$. The other case, when $T(Y)$ is a complete subspace of X, essentially pertains to the previous case as $T(Y) \subseteq f(Y)$. This completes the proof.

Now we derive a hybrid fixed point theorem from Theorem 2. Recall that a point $z \in X$ is a hybrid fixed point of $f : X \to X$ and $T : X \to CL(X)$ if $fz \in Tfz$.

COROLLARY 1. Let (X,d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$ such that $T(X) \subseteq f(X)$ and (1) is satisfied for all $x, y \in X$, $0 \leq h < 1$. If $T(X)$ or $f(X)$ is a complete subspace of X, then there exists a point $t \in X$ such that $ft \in Tt$. Further, if T is f–weakly compatible and $f(ft) = ft$, then f and T have a common fixed point, indeed, $ft \in Tft$.

Proof. It comes from Theorem 2 (when $Y = X$) that there exists a point $t \in X$ such that $ft \in Tt$. If T is f – weakly compatible, then by the lemma of Pathak [5, p. 74], $fTt = Tft$. Hence $ft \in Tt$ implies $f(ft) \in f(Tt) = Tft$. Consequently ft is a fixed point of T if $f(ft) = ft$. This completes the proof.

Corley [1, Theorem 1] has shown that a hybrid fixed point is a maximal point in certain Pareto maximization problems (see [1, p. 529]). Therefore Corollary 1 has a big potential for applications to Pareto type of maximization problems.

The following is an extension of the main result of Das and Naik [2].

THEOREM 3([5,Theorem 4]). Let (X,d) be a complete metric space, and let $f, T : X \to X$ be two f – weak compatible maps such that $T(X) \subseteq f(X)$ and

(2) $\quad d(Tx,Ty) \leq h \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}$

for all $x, y \in X$ and $0 \leq h < 1$. If one of f or T is continuous, then there exists a unique common fixed point of f and T.

We improve the above theorem as follows.

THEOREM 4. Let (X,d) be a metric space, and let $f, T : X \to X$ be such that $T(X) \subseteq f(X)$ and (2) is satisfied for all $x, y \in X$ and $0 \leq h < 1$. If one of $T(X)$ or $f(X)$ is a complete subspace of X, then there exists a point $t \in X$ such that $ft = Tt$. Further,

(i) if there exist $v, w \in X$ such that $fv = Tv$ and $fw = Tw$, then $fv = fw$;

(ii) if T is f – weak compatible, then f and T have a unique common fixed point.

Proof. Pick $x_0 \in X$. Following [2] and [5], we find a Cauchy sequence $\{fx_n\} \subseteq f(X)$, where $Tx_n = fx_{n+1}$, $n = 0, 1, 2, \ldots$. If $f(X)$ is complete then, $\{fx_n\}$ has a limit in $f(X)$, say u. Let $t \in f^{-1}u$ so that $ft = u$. We note that $\{Tx_n\}$ also converges to ft. Setting $x = x_n$

and $y = t$ in (2) and passing over to the limit, we find that $d(ft,Tt) \leq h\, d(ft,Tt)$. So $ft = Tt$. Now let $v, w \in X$ be such that $fv = Tv$ and $fw = Tw$, then by (2), $d(fv,fw) = d(Tv,Tw) \leq k\, d(fv,fw)$. This proves (i). Finally, if T is f – weak compatible, then appealing to the lemma of Pathak [5], $fTt = Tft$. Thus $ft = Tt$ yields $fft = fTt = Tft$, and hence ft is a coincidence point of f and T. Therefore by (i), $ft = f(ft)$, proving ft is a common fixed point of f and T. The uniqueness of the common fixed point can be easily verified.

The following exmples provide an insight into our results, and establish clear and applicable superiority over those of [5]. The sequence $\{fx_n\}$ (respectively $\{Tx_n\}$) constructed in the proof of Theorem 4 may be called the orbit of f with respect to T (respectively the orbit of T with respect to f).

EXAMPLE 1. Let $X = \{ x : 0 \leq x \leq 2$ and x is rational $\}$ be endowed with the usual metric. Let $Tx = \{0, 2\}$, $fx = 2 - x$, $x \in X$. Then $T(X) = \{0, 2\} \subset f(X) = X$. It is easily seen that all the hypotheses of Theorem 2 (with $Y = X$ and $T(X)$ complete) are satisfied, and $ft \in Tt$, $t = 0, 2$. However, Theorem 1 can not be applied as X is not complete.

The doubling function D (cf. below) finds significance in chaotic dynamical theory (see, for instance, Devaney [3, p. 24]). We introduce an auxiliary function T so that our Theorem 4 may make the orbit of D with respect to T behave nicely.

EXAMPLE 2. Let $X = [0,1]$ be endowed with the usual metric. Let $D, T : X \to X$ be such that $Dx = 2x \bmod 1$ and $Tx = x/2$, $0 \leq x < 1/2$, $Tx = x/2 - 1/4$, $1/2 \leq x \leq 1$. Then $T(X) = [0,1/4] \subset D(X) = X$, and $|Tx - Ty| \leq (1/4)|Dx - Dy|$ for all $x, y \in X$. Theorem 3 can not be applied to T and $f = D$, since both the maps are discontinuous. However, Theorem 4 applies, and 0 is the unique common fixed point. More importantly, for any $x_0 \in X$, the sequence $\{Dx_n : Dx_n = Tx_{n-1}, n = 1, 2, \ldots\}$ converges to 0. In particular, if $0 \leq x_0 < 1/2$, then $Dx_n = 2x_0/4^n$, $n = 1, 2, \ldots$.

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UN CRITERE GENERALISE DE LE PAGE ET COMMUTATIVITE

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Presented by G.A. Elliott, F.R.S.C.

Abstract: Considering a generalized criterion of Le Page type, we give a theorem from which follow, without further calculations, all results originated by the condition $\|xy\| \leq \alpha\|yx\|$ of Le Page.

Résumé: Considérant un critère généralisé de type Le Page, nous donnons un théorème duquel découle, sans autres calculs, tous les résultats de commutativité issus de la condition $\|xy\| \leq \alpha\|yx\|$ de Le Page.

Dans toute la suite, A sera une algèbre de Banach complexe pour une norme $\|.\|$ et $C(A)$ le centre de A. Pour $x \in A$, on désignera aussi par $\rho(x)$ le rayon spectral de x.

C. Le Page a considéré ([5]) la condition $\|xy\| \leq \alpha\|yx\|$ dans une algèbre de Banach unitaire et a montré qu'elle entraîne la commutativité. Divers auteurs ont examiné le même problème dans le cas non unitaire. D'autres ont introduit des fonctions dans l'inégalité soit à gauche soit à droite, et reprennent à chaque fois l'idée de Le Page qui consiste à utiliser la fonction exponentielle. Nous donnons ici un énoncé qui recouvre tous les cas.

Théorème: Soit $(A,\|.\|)$ une algèbre de Banach, (E,p) un espace normé; $S:A \to E$ une application linéaire continue. Soit de plus F un espace vectoriel, $q:F \to R^+$ et $T:A \to F$ deux applications quelconques. S'il existe $\alpha > 0$ tel que:

$$p(S(uv)) \leq \alpha q(T(vu)) \tag{1}$$

pour tous $u,v \in A$; alors on a:

(i) $S[(xy)z] = S[z(xy)]$ pour tous $x,y,z \in A$. Si de plus A admet une unité approchée à gauche ou à droite, alors $S(xy) = S(yx)$ pour tous $x,y \in A$.

(ii) Si **(1)** est vérifiée pour tout $u \in A^1 = A \oplus C$ et tout $v \in A$, alors $S(xy) = S(yx)$ pour tous $x,y \in A$.

Preuve: Si A est non unitaire, on considère $A^1 = A \oplus C$.

(i) Nous reprenons ici l'argument classique de Le Page. Pour x,y,z dans A, soit l'application f définie, sur C et à valeurs dans E, par

$$f(\lambda) = S\left[e^{\lambda x} x y e^{-\lambda x}\right].$$

Elle est holomorphe. De plus elle est bornée car $p(f(\lambda)) \leq \alpha q(T(yx))$ pour tout λ. Elle est donc constante. On obtient alors la première conclusion à partir de $f'(0) = 0$. La deuxième résulte de la première en remplaçant x ou y (selon le cas) par e_i, où $(e_i)_i$ est une unité approchée à gauche ou à droite.

(ii) Pour x, y dans A, on considère la fonction g définie, sur C et à valeurs dans E par:

$$g(\lambda) = S\left(e^{-\lambda x} y e^{\lambda x}\right).$$

Elle est holomorphe. De plus elle est bornée car $p(g(\lambda)) \leq \alpha q(T(y))$. Elle est donc constante. On obtient alors la conclusion à partir de $g'(0) = 0$.

Comme conséquence du théorème précédent, on retrouve des résultats classiques sur la commutativité.

1°) Si on prend $E = F = A$, $S = T = Id_A$ et $p = q = \|\cdot\|$, (1) devient:

$$\|uv\| \leq \alpha \|vu\|. \qquad (2)$$

La première conclusion de l'assertion (i) dit que $A^2 \subset C(A)$. C'est (i) du théorème II.1 de [8]. Si l'algèbre admet une unité approchée à gauche ou à droite, la deuxième conclusion de (i) implique la commutativité, d'où le résultat de Baker et Pym (où il s'agit d'une unité approchée bornée ([1])) et le résultat de C. Le Page ([5]) dans le cas unitaire.

Remarque: On sait que la condition (2) n'implique pas la commutativité ([2]). Si on suppose qu'elle est vérifiée pour u dans A^1 et v dans A, alors A est commutative par (ii).

Remarque: Si on prend $E = A/_{RadA}$ et $S: A \to E$ la surjection canonique, $F = A$ et $T = Id_A$, (1) devient:

$$\|S(uv)\| \leq \alpha \|vu\|. \qquad (3)$$

Alors, d'après (i), A est commutative modulo son radical dès qu'elle admet une unité approchée (à gauche ou à droite). Cette condition ne nous semble pas avoir été considérée auparavant.

2°) Si on prend $E = F = A$, $S = T = Id_A$, $p = \|\cdot\|$ et $q = \rho$, (1) devient:

$$\|uv\| \leq \alpha \rho(vu) \qquad (4)$$

et on est ramené à (2) par $\rho \leq \|\cdot\|$.

3°) Si on prend $E = F = A$, $S = T = Id_A$, p une norme quelconque d' espace vectoriel et $q = \rho$, (1) devient:

$$p(uv) \leq \alpha\rho(vu). \tag{5}$$

Dans ([6]), Mocanu considère la condition:

$$p(x) \leq \alpha\rho(x), \ \forall x \in A. \tag{6}$$

Soient $u = x + \lambda e$ dans A^1 et v dans A. Le produit $uv = (x + \lambda e)v = xv + \lambda v$ est dans A; et l'on a:

$$p(uv) \leq \alpha\rho(uv) = \alpha\rho(vu)$$

c'est à dire (5). Alors (ii) assure que l'algèbre A est commutative. Comme cas particulier nous avons la condition:

$$v(x) \leq \alpha\rho(x) \tag{7}$$

où v est l'image numérique ([9]) et la condition:

$$\|x\| \leq \alpha\rho(x) \tag{8}$$

considérée par C. Le Page dans le cas unitaire ([5]) et par R. A. Hirschfeld, W. Zelazko dans le cas non unitaire ([3])

4°) Si on considère $F = A$, $T = Id_A$, (E, p) un espace normé quelconque et la condition:

$$p(S(uv)) \leq \alpha\|vu\|, \ u \in A^1, v \in A. \tag{9}$$

La condition (9) est équivalente à la condition:

$$p(S(xy + y)) \leq \alpha\|yx + y\| \tag{10}$$

posée dans [7]. On retrouve ainsi le résultat principal de G.Niestegge.

5°) Si on prend $E = A$, $S = Id_A$, (1) devient:

$$\|xy\| \leq \alpha q(T(yx)) \tag{11}$$

C'est la condition considérée dans [4] dont on retrouve les résultats. Par la première conclusion de (i), $A^2 \subset C(A)$ et par la deuxième, A est commutative dès qu'elle est à unité approchée (à gauche ou à droite).

Remarque: Par (ii), on sait maintenant que l'algèbre est commutative si on a (11) avec x dans A^1 et y dans A.

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ASYMPTOTIC BEHAVIOUR OF THE HALF DISC POLYNOMIALS AND RANDOM WALKS ON A DISCRETE CONE

Maher MILI

Presented by P.C. Greiner, F.R.S.C.

Abstract

In this work we give a Mehler-Heine formula for the half disc polynomials and we study the random walk on the hypergroup $(T,*)$ *defined by the set* $T = \left\{(p,q) \in \mathbb{N}^2 | p-q \in 2\mathbb{N}\right\}$ *and the convolution* $*$ *generated by these polynomials.(See [1] page 145).*

The half disc polynomials $\{Q^{(n)}_{p,q}\ ; p,q \in \mathbb{N}, p-q \in 2\mathbb{N}\}$ are defined on $\mathcal{D}_+ = \{(x,y) \in \mathbb{R}^2 |\ x^2+y^2 \leq 1 \text{ and } y \geq 0\}$ by

$$Q^{(n)}_{p,q}(x,y) = \mathcal{R}_q^{(\frac{1}{2},\frac{1}{2})}\left(x(x^2+y^2)^{-\frac{1}{2}}\right)(x^2+y^2)^{\frac{q}{2}}\mathcal{R}^{(2n-3,q+1)}_{\frac{p-q}{2}}\left(2(x^2+y^2)-1\right),$$

where $\mathcal{R}_m^{(\alpha,\beta)}$ is the Jacobi polynomial normalized by $\mathcal{R}_m^{(\alpha,\beta)}(1) = 1$.
Introducing polar coordinates, the polynomials $Q^{(n)}_{p,q}$ can be written as

$$Q^{(n)}_{p,q}(\lambda,\theta) = \mathcal{R}_q^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)(\cos\lambda)^q\mathcal{R}^{(2n-3,q+1)}_{\frac{p-q}{2}}(\cos 2\lambda) = \mathcal{R}_q^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)U^{(n)}_{p,q}(\cos\lambda),$$

where $\qquad U^{(n)}_{p,q}(\cos\lambda) = (\cos\lambda)^q\mathcal{R}^{(2n-3,q+1)}_{\frac{p-q}{2}}(\cos 2\lambda)$

with $\lambda \in [0,\frac{\pi}{2}]$ and $\theta \in [0,\pi]$. These polynomials are orthogonal with respect to the measure $dm_n(\lambda,\theta) = (\sin\lambda)^{4n-5}\cos^3\lambda\sin^2\theta d\lambda d\theta$ and we have

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi} Q^{(n)}_{p,q}(\lambda,\theta)Q^{(n)}_{p',q'}(\lambda,\theta)(\sin\lambda)^{4n-5}\cos^3\lambda\sin^2\theta d\lambda d\theta = (h^n_{p,q})^{-1}\delta_{p,p'}\delta_{q,q'},$$

where

$$h^n_{p,q} = \frac{(2n-1)(2n-2)\Gamma\left(\frac{p-q}{2}+1\right)\Gamma\left(\frac{p+q}{2}+2\right)\left(\Gamma(2n-2)\right)^2}{(q+1)^2\left(\frac{p-q}{2}+2n-2\right)^2(p+2n-1)\Gamma\left(\frac{p-q}{2}+2n-2\right)\Gamma\left(\frac{p+q}{2}+2n-1\right)}.$$

We denote by $T = \left\{(p,q) \in \mathbb{N}^2 \mid p-q \in 2\mathbb{N}\right\}$. We have for all $(p,q) \in T$,

$$\sup_{(\lambda,\theta)\in[0,\frac{\pi}{2}]\times[0,\pi]}\left|Q^{(n)}_{(p,q)}(\lambda,\theta)\right| \leq 1.$$

1. Mehler-Heine Formula .

For (p,q) $\in T$, let $\qquad \beta^2_{pq} = (p+2n-1)^2-(q+1)^2.$

Theorem 1-1: *Let $C > 0$ be an arbitrary constant. Then there exist real numbers $N_C > 0$, $C_1 > 0$ and $C_2 > 0$ such that for every $(p,q) \in T$ satisfying $\beta_{pq} \geq N_C$, and for every real number $\lambda \in \left[0, \frac{C}{\beta_{pq}}\right]$, we have*

$$\left(\frac{\sin\lambda}{\lambda}\right)^{2n-\frac{5}{2}}(\cos\lambda)^{\frac{3}{2}}U_{p,q}^{(n)}(\cos\lambda) = 2^{(2n-3)}(2n-3)!\frac{J_{2n-3}(\beta_{p,q}\lambda)}{(\beta_{p,q}\lambda)^{2n-3}} + R(p,q,\lambda),$$

where J_{2n-3} is the Bessel function of first kind and order 2n-3 and the function R satisfies the following inequality

$$|r(p,q,\lambda)| \leq C_1\lambda^2 + C_2(q+1)^2\lambda^4.$$

Corollary 1-1.(Mehler- Heine Formula): *We have*

$$\lim_{\substack{p\to+\infty, q\to+\infty \\ (p,q)\in T}} Q_{p,q}^{(n)}\left(\cos\frac{x}{\beta_{pq}}\right) = (2n-3)!\left(\frac{x}{2}\right)^{-(2n-3)} J_{2n-3}(x).$$

This limit holds uniformly in every bounded interval.

2. Law of Large Numbers .

It is proved by T.H.Koornwinder in [5] that the polynomials $Q_{p,q}^{(n)}$ satisfy the following linearization formula

$$Q_{p,q}^{(n)}(\lambda,\theta)Q_{p',q'}^{(n)}(\lambda,\theta) = \sum_{(p",q")\in T} C(p,q,p',q',p",q")Q_{p",q"}^{(n)}(\lambda,\theta),$$

where there are finitely nonzero terms and the coefficients $C(p,q,p',q',p",q")$ are nonnegative. Then we can define a convolution structure on $M(T)$, the space of bounded measures on T, by taking

$$\delta_{(p,q)} * \delta_{(p',q')} = \sum_{(p",q")\in T} C(p,q,p',q',p",q")\delta_{(p",q")}$$

for two Dirac measures and, more generally, for all μ and ν in $M(T)$ we have

$$\mu * \nu = \sum_{(i,j)\in T}\sum_{(k,l)\in T} \mu(i,j)\nu(k,l)\delta_{(i,j)} * \delta_{(k,l)}.$$

Proposition 2-1: *With the convolution $*$, the involution $(p,q)^- = (p,q)$ and the unit element $(0,0)$, $(T,*)$ is a commutative hypergroup with the Haar measure $\sum_{(p,q)\in T} h_{pq}^n\delta_{(p,q)}$ and dual $\mathcal{D}_+$ with Plancherel measure $dm_n(\lambda,\theta)$.*

Remark 2-1: *This hypergroup has been cited in [1] page 145.*

Definition 2-1: *i) By a random walk of law μ on $(T,*)$ we mean every Markov chain on T with markovian kernel*

$$P\Big((i,j),A\Big) = (\delta_{(i,j)} * \mu)(A), \quad A \subset T,\ (i,j) \in T.$$

The iterated kernels of P are given by

$$P^{(n)}\Big((i,j),A\Big) = (\delta_{(i,j)} * \mu^n)(A),$$

where $\mu^n = \mu * \mu * \cdots * \mu$, the n^{th} power of convolution of μ.

ii) We say that the probability μ on T is adapted if the hypergroup generated by mean of the convolution structure $*$ by the support of μ is equal to T.(See [4]).

Taking $r = cos\lambda$, the polynomial $Q^{(n)}_{p,q}$ can be written as

$$Q^{(n)}_{p,q}(\lambda,\theta) := Q^{(n)}_{p,q}(r,\theta) = \frac{1}{(q+1)}U_q(cos\theta)r^q\mathcal{R}^{(2n-3,q+1)}_{\frac{p-q}{2}}(2r^2-1),$$

where U_q is the Tchebycheff polynomial of the second kind of degree q.

We denote by $M_1(T)$ the set of probability measures on T. The Fourier transform of a probability μ in $M_1(T)$ is the function given by

$$\hat{\mu}(\lambda,\theta) = \sum_{(p,q)\in T} \mu(p,q)Q^{(n)}_{p,q}(\lambda,\theta).$$

For a random vector (X,Y) of law μ on T, we have

$$\hat{\mu}(\lambda,\theta) = \mathbb{E}\Big(Q^{(n)}_{X,Y}(\lambda,\theta)\Big)$$

and for μ and ν in $M_1(T)$, we have the relation

$$\widehat{\mu*\nu}(\lambda,\theta) = \hat{\mu}(\lambda,\theta)\hat{\nu}(\lambda,\theta).$$

<u>**Definition**</u> **2-2:** *For a probability μ on T, we define the numbers $V_1(\mu)$ and $V_2(\mu)$, if they exist, by*

$$V_1(\mu) = \frac{\partial}{\partial r}\hat{\mu}(r,\theta)\Big|_{\substack{r=1\\ \theta=0}} = \sum_{(p,q)\in T} \mu(p,q)\frac{\partial}{\partial r}Q^{(n)}_{p,q}(r,\theta)\Big|_{\substack{r=1\\ \theta=0}},$$

$$V_2(\mu) = \frac{\partial^2}{\partial\theta^2}\hat{\mu}(r,\theta)\Big|_{\substack{r=1\\ \theta=0}} = -\sum_{(p,q)\in T} \mu(p,q)\frac{\partial^2}{\partial\theta^2}Q^{(n)}_{p,q}(r,\theta)\Big|_{\substack{r=1\\ \theta=0}}.$$

<u>**Proposition**</u> **2-2:** *For $(p,q)\in T$, we have*

$$V_1(\delta_{(p,q)}) = \frac{1}{8(n-1)}\Big(p^2 - q^2 + (4n-2)p + (4n-6)q\Big)$$

$$V_2(\delta_{(p,q)}) = \begin{cases} \frac{1}{3}(q^2+q-2) & , \ for\ all\ (p,q)\in T\ and\ q>1 \\ 1 & , \ for\ all\ (p,1)\in T \\ 0 & , \ for\ all\ (p,0)\in T\ . \end{cases}$$

<u>**Proposition**</u> **2-3:** *For every μ and ν in $M_1(T)$, we have*

$$V_i(\mu*\nu) = V_i(\mu) + V_i(\nu), \qquad i = 1,2.$$

<u>**Notations**</u>

i) In the following, we denote $V_i(\delta_{(p,q)}) = V_i(p,q)$, $\quad i = 1,2$. Then $V_i(\mu)$ can be considered as the integral $< V_i, \mu >$ of the function V_i with respect to the measure μ.

ii) We mean by $S_k = (X_k, Y_k)$, the position at the instant k of a random walk of law μ in $(T,*)$ starting from $(0,0)$.

Lemma 2-1: *The processes*

$$8(n-1)V_1(S_k) = X_k^2 - Y_k^2 + (4n-2)X_k + (4n-6)Y_k \ ,$$

$$3V_2(S_k) = \begin{cases} Y_k^2 + Y_k - 2 & , \ if \ Y_k \notin \{0,1\} \\ 1 & , \ if \ Y_k = 1 \\ 0 & , \ if \ Y_k = 0 \end{cases}$$

are two positive submartingales.

Lemma 2-2: *Let $V(S_k) = T_k$ be a positive submartingale. Then the sequence of random variables $k^{-2}T_k$ converges to zero almost surely.*

Corollary 2-1: *The sequences of random variables $(k^{-2}X_k)_{k\in\mathbb{N}}$ and $(k^{-2}Y_k)_{k\in\mathbb{N}}$ converge to zero almost surely.*

Corollary 2-2.(Law of Large Numbers): *We suppose that μ has a moment of order 2. Then* $\lim_{k\to+\infty}\left(\frac{X_k}{k}, \frac{Y_k}{k}\right) = (0,0)$, *a.s.*

3. The Limit Central Theorem .

Proposition 3-1: *For every $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we have*

$$\lim_{k\to+\infty} \mathbb{E}\left(\frac{\sin(\frac{Y_k+1}{\sqrt{k}})\theta}{(Y_k+1)\sin\frac{\theta}{\sqrt{k}}} U^{(n)}_{X_k,Y_k}(\cos\frac{\lambda}{\sqrt{k}})\right) = e^{-a\frac{\theta^2}{2}} e^{-b\lambda^2},$$

where a and b are two positive constants depending only on μ.

Corollary 3-1: *For fixed λ, θ, we have*

$$\lim_{k\to+\infty} \mathbb{E}\left(\frac{\sin(\frac{Y_k}{\sqrt{k}})\theta}{\frac{Y_k}{\sqrt{k}}\theta} \Lambda_{2n-3}(\beta_{X_k,Y_k}\frac{\lambda}{\sqrt{k}})\right) = e^{-a\frac{\theta^2}{2}} e^{-b\lambda^2}.$$

Remark 3-1: *Let $\mu \in M_1(\mathbb{R}_+ \times \mathbb{R}_+)$, the set of probability measures on $\mathbb{R}_+ \times \mathbb{R}_+$. The Hankel-mixed transform of μ is defined by*

$$\mathcal{H}_{\frac{1}{2},2n-3}(\mu)(\lambda,\theta) = \frac{\sqrt{\pi}}{2}\int_0^{+\infty}\int_0^{+\infty} \Lambda_{\frac{1}{2}}(\theta x)\Lambda_{2n-3}(\lambda y)\mu(dxdy).$$

If we consider the density of probability $\mu_1(x,y)$ with respect to the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$ given by

$$\mu_1(x,y) = 2\sqrt{\frac{2}{\pi}}a^{-\frac{3}{2}}(4b)^{-(2n-2)}x^2y^{4n-5}e^{-\frac{x^2}{2a}}e^{-\frac{y^2}{4b}},$$

then : $\quad \mathcal{H}_{\frac{1}{2},2n-3}(\mu_1)(\lambda,\theta) = e^{-a\frac{\theta^2}{2}}e^{-b\lambda^2}.$

The Paul Lévy continuity theorem is available for the transform $\mathcal{H}_{\frac{1}{2},2n-3}$, (see [3]), so we have

Thorem 3-1.(Limit Central Theorem): *Let $\mu \in M_1(T)$ be such that*

- *μ is adapted,*
- *the random walk of law μ on T has a moment of order 2.*

Then the sequence of random vectors $\left(k^{\frac{-1}{2}}X_k, k^{\frac{-1}{2}}Y_k\right)$ *converges in law when* k *tends to infinity to the random vector whose probability density is given by*

$$2\sqrt{\frac{2}{\pi}}a^{-\frac{3}{2}}(4b)^{-(2n-2)}xy^2(x^2-y^2)^{2n-3}e^{-\frac{(x^2-y^2)}{2b}}e^{-\frac{y^2}{2a}} \quad x \geq 0, y \geq 0, x \geq y.$$

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INVERSION OF THE WEYL INTEGRAL TRANSFORM AND THE RADON TRANSFORM ON $\mathbb{R}^n$ USING GENERALIZED WAVELETS

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Presented by J. Arthur, F.R.S.C.

Abstract: We use the generalized continuous wavelet transform associated with the Bessel operator on $]0,+\infty[$, to derive inversion formulas for the Weyl integral transform and the Radon transform on $\mathbb{R}^n$, $n \geq 2$.

I. The Weyl integral transform.

Notations : We denote by

- $L^p(x^{2\alpha+1}dx)$, $p \in [1,+\infty[$, $\alpha > -1/2$, the space of measurable functions f on $[0,+\infty[$ such that $\int_0^\infty |f(x)|^p x^{2\alpha+1}\,dx < +\infty$.
- $L^1([0,+\infty[, dx)$ the space of integrable functions on $[0,+\infty[$ with respect to the Lebesgue measure
- $L^\infty([0,+\infty[, dx)$ the space of essentially bounded functions on $[0,+\infty[$ with respect to the Lebesgue measure.

1) Fourier-Bessel transform—Generalized convolution product.

The Fourier-Bessel transform $\mathcal{F}(f)$ of a function f in $L^1(x^{2\alpha+1}dx)$ is defined by

$$\forall \lambda \in \mathbb{R},\ \mathcal{F}(f)(\lambda) = \int_0^\infty f(x)\, j_\alpha(\lambda x) x^{2\alpha+1} dx,$$

where $j_\alpha(s) = 2^\alpha \Gamma(\alpha+1)\, s^{-\alpha} J_\alpha(s)$, with J_α the Bessel function of the first kind and order α.

Remark: For each $\lambda \in \mathbb{C}$, the function $x \longrightarrow j_\alpha(\lambda x)$ is the unique solution of the equation.

$$L_\alpha u = -\lambda^2 u,\ u(0) = 1,\ u'(0) = 0,$$

where $L_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}$ is the Bessel operator on $]0,+\infty[$.

The generalized translation operators T_x, $x \geq 0$, are defined for smooth functions on $[0,+\infty[$ by

$$T_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2+y^2+2xy\cos\theta})(\sin\theta)^{2\alpha} d\theta.$$

The generalized convolution product of two smooth functions f and g is defined by

$$\forall x \geq 0,\quad f \# g(x) = \int_0^\infty T_x f(y) g(y)\, y^{2\alpha+1} dy.$$

For the properties of the Fourier-Bessel transform and the generalized convolution product, we can see [3] and [4].

2) The Weyl integral transform.

The Weyl integral transform W_α is defined in [4] for suitable functions on $[0,+\infty[$ by

$$\forall\ y \geq 0,\ W_\alpha(f)(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_y^\infty (x^2-y^2)^{\alpha-1/2} f(x)\, x\, dx.$$

For f in $L^1(x^{2\alpha+1}dx)$ the function $W_\alpha(f)$ belongs to $L^1([0,+\infty[, dx)$, and we have the relation: $\mathcal{F}(f) = \mathcal{F}_0 \circ W_\alpha(f)$, where $\mathcal{F}_0$ is the Fourier-cosine transform defined on $L^1([0,+\infty[, dx)$ by

$$\forall \lambda \in \mathbb{R},\quad \mathcal{F}_0(f)(\lambda) = \int_0^\infty f(x)\cos(\lambda x)dx.$$

The Riemann-Liouville integral transform B_α is defined on $L^\infty([0,+\infty[, dx)$ by

$$B_\alpha(f)(x) = \frac{2\,\Gamma(\alpha+1)\,x^{-2\alpha}}{\sqrt{\pi}\,\Gamma(\alpha+1/2))}\int_0^x f(y)(x^2-y^2)^{\alpha-1/2}\,dy.$$

<u>Remarks</u>:(i) For all $\lambda \in \mathbb{R}$ and $x \geq 0$ we have : $j_\alpha(\lambda x) = B_\alpha(\cos(\lambda.))(x)$.

(ii) For all $f \in L^1(x^{2\alpha+1}dx)$ and $g \in L^\infty([0,+\infty[, dx)$ we have

$$\int_0^\infty W_\alpha(f)(y)\,g(y)\,dy = \int_0^\infty f(x)\,B_\alpha(g)(x)\,x^{2\alpha+1}dx.$$

II. The Radon transform on $\mathbb{R}^n$.

<u>Notation</u> : We denote by $L^p(\mathbb{R}^n)$, $p \in [1, +\infty]$, the space of mesurables functions on $\mathbb{R}^n$ such that for $p \in [1, +\infty[$, $\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{1/p} < +\infty$,and $\|f\|_\infty = \underset{x \in \mathbb{R}^n}{ess\ sup}|f(x)| < +\infty$.

Let f be a function on $\mathbb{R}^n$, integrable on each hyperplane in $\mathbb{R}^n$. We define the Radon transform Rf of f by

$$Rf(\theta,s) = R_\theta f(s) = \int_{<x,\theta>=s} f(x)\,dm(x),$$

where $<,>$ denotes the usual inner product in R^n, θ in the unit sphere S^{n-1} of $\mathbb{R}^n$, $s \in \mathbb{R}^n$ and dm is the Lebesgue measure on the hyperplane $\{x \in \mathbb{R}^n : <x,\theta> = s\}$(see [1]).

For $f \in L^1(\mathbb{R}^n)$ and $\theta \in S^{n-1}$ the function $R_\theta(f)$ belongs to $L^1(\mathbb{R})$, and we have the identity : $\hat{f}(s\theta) = (R_\theta f)^\wedge(s)$,where ^ denotes the usual Fourier transform defined on $L^1(\mathbb{R}^n)$ by

$$\hat{h}(\lambda) = \int_{\mathbb{R}^n} h(x)\,e^{-i<\lambda,x>}dx.$$

<u>Remark</u> : Let $f(x) = f_0(\|x\|)$ be a radial function in $L^1(\mathbb{R}^n)$. Then Rf is independent of $\theta \in S^{n-1}$ and we have the relation

(II.1) $$R_\theta f = \frac{\pi^{n/2}}{\Gamma(n/2)}W_{(n-2)/2}(f_0)$$

<u>Notation</u>: We denote by $L^\infty(S^{n-1} \times \mathbb{R})$ the space of essentially bounded functions on $S^{n-1} \times \mathbb{R}$, with respect to the measure $d\theta dt$,where $d\theta$ is the surface measure on S^{n-1}.

The dual Radon transform ${}^tR$ is defined on $L^\infty(S^{n-1} \times \mathbb{R})$ by

$${}^tRg(x) = \int_{S^{n-1}} g(\theta,<x,\theta>)d\theta.$$

<u>Remark:</u>(i) For all $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(S^{n-1} \times \mathbb{R})$ we have

$$\int_{S^{n-1}}\int_{\mathbb{R}} Rf(\theta,s)\,g(\theta,s)dsd\theta = \int_{\mathbb{R}^n} f(x)\,{}^tRg(x)dx.$$

(ii) If $g \in L^\infty(\mathbb{R})$ then ${}^tRg$ is radial and we have the relation

$${}^tRg(x) = \frac{2\pi^{n/2}}{\Gamma(n/2)}\,B_{(n-2)/2}\,g(\|x\|).$$

III. Generalized wavelets associated with the Bessel operator.

1) Classical continuous wavelet transform on $\mathbb{R}^n$.

The classical continuous wavelet transform on $\mathbb{R}^n$,is defined for regular functions by

$$S_g^{(n)}(f)(a,b) = \int_{\mathbb{R}^n} f(x)\overline{g_{a,b}^0(x)}dx, \ a>0, b \in \mathbb{R}^n,$$

where $g_{a,b}^0(x) = a^{-n/2} g((x-b)/a)$ and g is a classical wavelet on $\mathbb{R}^n$, i.e.a function in $L^2(\mathbb{R}^n)$ satisfying the admissibility condition: There exists a constant $0 < C_g^0 < +\infty$, such that $C_g^0 = \int_0^\infty |\hat{g}(a\lambda)|^2 \frac{da}{a}$, for almost every $\lambda \in \mathbb{R}^n$.

This transform is studied in [2]. In particular we have the following inversion formula.

(III.1)
$$f(x) = \frac{1}{C_g^0} \int_0^\infty \left(\int_{\mathbb{R}^n} S_g^{(n)}(f)(a,b) g_{a,b}^0(x)dx\right) \frac{da}{a^{n+1}}, \text{ a.e,}$$

where both f and $\hat{f}$ are in $L^1(\mathbb{R}^n)$.

We prove also another inversion formula.

Theorem III.1: Let $g \in L^2(\mathbb{R}^n)$ be a classical wavelet such that $\hat{g} \in L^\infty(\mathbb{R}^n)$. Then for $f \in L^2(\mathbb{R}^n)$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{C_g^0} \int_\varepsilon^\delta \int_{\mathbb{R}^n} S_g^n(f)(a,b) g_{a,b}^0(x)db) \frac{da}{a^{n+1}}$$

belongs to $L^2(\mathbb{R}^n)$ and satisfies: $\lim_{\varepsilon\to 0, \delta\to+\infty} \|f^{\varepsilon,\delta} - f\|_2 = 0$.

2) Generalized wavelets.

A generalized wavelet is an even function g in $L^2(x^{2\alpha+1}dx)$ satisfying the admissibility condition

(III.2)
$$0 < C_g = \int_0^\infty |\mathcal{F}(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

Proposition III.1: Let $0 \neq g \in L^2(x^{2\alpha+1}dx)$ such that $\mathcal{F}(g)$ is right continuous in 0 and satisfying : $\exists\, \gamma > 0$ such that $\mathcal{F}(g)(\lambda) - \mathcal{F}(g)(0) = O(\lambda^\gamma)$, as $\lambda \to 0^+$.Then the condition (III.2) is equivalent to $\mathcal{F}(g)(0) = 0$.

Remark: Let $g(x) = g_0(\|x\|)$ be a radial function on $\mathbb{R}^n$. Then g is a classical wavelet on $\mathbb{R}^n$ if and only if g_0 is a generalized wavelet associated with the Bessel operator $L_{(n-2)/2}$.

Let $g \in L(x^{2\alpha+1}dx)$ be a generalized wavelet. We define for smooth functions on $[0,+\infty[$, the generalized continuous wavelet transform by

$$\Phi_g(f)(a,b) = \int_0^\infty f(x)\overline{g_{a,b}(x)}x^{2\alpha+1}dx, \ a>0, b \geq 0,$$

where $g_{a,b}(x) = \frac{1}{a^{2\alpha+1}} T_a g_a(x)$, with $g_a(x) = \frac{1}{\sqrt{a}} g(x/a)$, and T_b, $b \geq 0$, are the generalized translation operators associated with Bessel operator L_α.

Theorem III.2: (Plancherel formula) Let $g \in L^2(x^{2\alpha+1}dx)$ be a generalized wavelet. Then for all f in $L^2(x^{2\alpha+1}dx)$ we have

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1}dx = \frac{1}{C_g}\int_0^\infty\int_0^\infty |\Phi_g(f)(a,b)|^2 b^{2\alpha+1}db\frac{da}{a^2}.$$

Theorem III.3: (Inversion formula) Let $g \in L^2(x^{2\alpha+1}dx)$ be a generalized wavelet. For f in $L^1(x^{2\alpha+1}dx)$ such that $\mathfrak{F}(f)$ belongs to $L^1(x^{2\alpha+1}dx)$ we have

$$f(x) = \frac{1}{C_g}\int_0^\infty \left(\int_0^\infty \Phi_g(f)(a,b) g_{a,b}(x) b^{2\alpha+1} db\right) \frac{da}{a^2}, \text{ a.e.}$$

Another inversion formula is as follows.

Theorem III.4: Let $g \in L^2(x^{2\alpha+1}dx)$ be a generalized wavelet such that $\mathfrak{F}(g) \in L^\infty([0,+\infty[,dx)$. Then for f in $L^2(x^{2\alpha+1}dx)$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{C_g}\int_\varepsilon^\delta \int_0^\infty \Phi_g(f)(a,b) g_{a,b}(x) b^{2\alpha+1} db \frac{da}{a^2}$$

belongs to $L^2(x^{2\alpha+1}dx)$ and we have: $\lim_{\varepsilon \to 0, \delta \to +\infty} \|f^{\varepsilon,\delta} - f\|_{2,\alpha} = 0$.

IV.I. Inversion of the Weyl integral transform and the Radon transform on $\mathbb{R}^n$, using generalized wavelets.

1) Inversion of the Weyl integral transform.

Proposition IV.1: Let $g \in (L^1 \cap L^2)(\mathbb{R})$ be an even classical wavelet such that $\hat{g} \in L^1(\mathbb{R})$ and satisfying

(IV.1) $\exists\ \gamma > 2\alpha + 1$ such that $\hat{g}(\lambda) = O(\lambda^\gamma)$, as $\lambda \longrightarrow 0^+$.

Then

(i) The function $B_\alpha g$ is a generalized wavelet in $L^2(x^{2\alpha+1}dx)$ such that $\mathfrak{F}(B_\alpha g)$ belongs to $L^\infty([0,+\infty[,dx)$.

(ii) For all f in $L^1(x^{2\alpha+1}dx)$ we have the relation

$$\Phi_{B_\alpha g}(f)(a,b) = \frac{1}{2a^{2\alpha+1}} B_\alpha[S_g^1(W_\alpha f)(a,.)](b).$$

From this proposition and Theorems III.3, III.4, we deduce the following inversion formulas for the Weyl integral transform.

Theorem IV.1: Let $g \in (L^1 \cap L^2)(\mathbb{R})$ be an even classical wavelet such that $\hat{g} \in L^1(\mathbb{R})$ and satisfying (IV.1). Then

(i) For all $f \in L^1(x^{2\alpha+1}dx)$ such that $\mathfrak{F}(f) \in L^1(x^{2\alpha+1}dx)$ we have

$$f(x) = \frac{1}{2C_{B_\alpha g}}\int_0^\infty \left(\int_0^\infty B_\alpha[S_g^1(W_\alpha f)(a,.)](b)\,(B_\alpha g)_{a,b}(x)\, b^{2\alpha+1} db\right) \frac{da}{a^{2\alpha+3}}, \text{ a.e.}$$

(ii) For all f in $(L^1 \cap L^2)(x^{2\alpha+1}dx)$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{2C_{B_\alpha g}}\int_\varepsilon^\delta \int_0^\infty B_\alpha[S_g^1(W_\alpha f)(a,.)](b)(B_\alpha g)_{a,b}(x)\, b^{2\alpha+1} db \frac{da}{a^{2\alpha+3}}$$

satisfies: $\lim_{\varepsilon \to 0, \delta \to +\infty} \|f^{\varepsilon,\delta} - f\|_{2,\alpha} = 0$.

2) Inversion of the Radon transform on $\mathbb{R}^n$.

Proposition IV.2: Let $g \in (L^1 \cap L^2)(\mathbb{R})$ be an even classical wavelet such that $\hat{g} \in L^1(\mathbb{R})$ and satisfying

(IV.2) $\exists\ \gamma > n - 1$ such that $\hat{g}(\lambda) = O(\lambda^\gamma)$, as $\lambda \to 0^+$.

Then

(i) The function ${}^tRg$ is a radial classical wavelet on $\mathbb{R}^n$ such that $({}^tRg)^\wedge \in L^\infty(\mathbb{R}^n)$.

(ii) For all f in $L^1(\mathbb{R}^n)$ we have

$$S^{(n)}_{{}^tRg}(f)(a,b) = a^{(1-n)/2}\int_{S^{n-1}} S^{(1)}_g(R_\theta f)(a, <b,\theta>)d\theta.$$

To inverte Rf for radial functions f on $\mathbb{R}^n$, we use Relation (II.1) and Theorem IV.1. For general functions f on $\mathbb{R}^n$ we deduce from Proposition IV.2, Relation (III.1) and Theorem III.1, the following inversion formulas for the transform R.

<u>Theorem</u> IV.2: Let $g\in (L^1 \cap L^2)(\mathbb{R})$ be an even classical wavelet such that $\hat{g} \in L^1(\mathbb{R})$ and satisfying (IV.2).

(i) For all f in $L^1(\mathbb{R}^n)$ such that $\hat{f}$ belongs to $L^1(\mathbb{R}^n)$, we have

$$f(x) = \frac{1}{C^0_{{}^tRg}}\int_0^\infty \left[\int_{\mathbb{R}^n}\left(\int_{S^{n-1}}\left(S^{(1)}_g(R_\theta f)(a,<b,\theta>)d\theta\right)({}^tRg)^0_{a,b}(x)db\right]\frac{da}{a^{(3n+1)/2}}, \text{ a.e.}$$

(ii) For all in $(L^1 \cap L^2)(\mathbb{R}^n)$ and $0<\varepsilon<\delta<\infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{C^0_{{}^tRg}}\int_\varepsilon^\delta \int_{\mathbb{R}^n}\left(\int_{S^{n-1}} S^{(1)}_g(R_\theta f)(a, <b,\theta>)d\theta\right)({}^tRg)^0_{a,b}(x)db\frac{da}{a^{(3n+1)/2}}$$

satisfies: $\lim_{\varepsilon\to 0,\delta\to+\infty} \|f^{\varepsilon,\delta}-f\|_2=0.$

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A note on rings with projective socle

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Presented by I. Halperin, F.R.S.C.

Abstract

We present an example of a right hereditary ring for which the left socle is not projective, in response to a remark of Xue Weimin.

1 Introduction

Throughout this note all rings have an identity. In [5] Nicholson and Watters call a ring R a *left PS*-ring if its left socle $\mathrm{Soc}(_R R)$ is projective as a left R-module. Right PS-rings are defined similarly and in [5] an example is given of a left PS-ring which is not a right PS-ring. A further instance of this asymmetry is given by Xue Weimin in [7] with an example of a left semihereditary ring that is not right PS. Xue notes that his example is not left hereditary and raises the question of whether a one-sided hereditary ring must be PS on both sides. We respond here with an example of a right hereditary ring R which is not a left PS-ring.

2 The example

Let S be the ring of all eventually constant rational sequences. Thus S is obtained by adjoining a unit to $\mathbb{Q}^{(\mathbb{N})}$, the direct sum of countably infinitely many copies of the field of rationals $\mathbb{Q}$. Then S is a commutative countable von Neumann regular ring and so, by [3, Theorem 1], is hereditary.

If $(a_1, a_2, \ldots, a_n. a, a, a, \ldots)$ is an element of S which is eventually constant at a and if $b \in Q$ then the multiplication

$$(a_1, a_2, \ldots, a_n, a, a, a, \ldots)b = ab$$

ensures $\mathbb{Q}$ as a left S-module. With the obvious right $\mathbb{Q}$-module multiplication, $\mathbb{Q}$ is then an S-$\mathbb{Q}$-bimodule. Thus we may form the generalized triangular matrix ring

$$R = \begin{bmatrix} S & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}.$$

Since $\mathbb{Q}$ is a field and S is a von Neumann regular hereditary ring, it follows from the upper triangular analogue of [2, Theorem 4.7] (a special case of [6, Theorem 5]) that R is right hereditary.

We now show that R is not a left PS-ring. To do this we employ a part of [5, Theorem 2.4] which shows that R is left PS if and only if for each maximal left ideal L of R either $L = Re$ for some idempotent e in R or the right annihilator $\mathrm{r}(L)$ of L is 0.

It is easy to see that

$$L = \begin{bmatrix} \mathbb{Q}^{(\mathbb{N})} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$$

is a maximal left ideal of R which is not generated by an idempotent. On the other hand,

$$\begin{bmatrix} \mathbb{Q}^{(\mathbb{N})} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

so that $\mathrm{r}(L) \neq 0$. It now follows from above that R is not left PS, as claimed. (As an alternative argument, it is straightforward to see that, for the minimal left ideal K generated by

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

if we define $f : R \to K$ to be right multiplication by x then f is an epimorphism which does not split and so K is not projective.)

Finally we note that the ring R has featured elsewhere. In particular, it appears in [4] as an example, attributed to L. W. Small, of a right hereditary ring of left global dimension 2 which is finitely generated over its centre. Its opposite ring also appears as Example 2.3 of [1] (as an example of a left hereditary ring module-finite over its centre in which there are principal right ideals that are not projective).

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ONDELETTES SUR L'INTERVALLE ET OPERATEURS D'EXTENSION

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Résumé

L'objet de ce travail est de construire des opérateurs d'extension sur les espaces de Sobolev $H^k(]-\infty,0])$ et $H^k_0([0,+\infty[)$, $k \geq 0$, liés à des analyses multirésolutions sur chaque demi-droite. Ces extensions permettent d'obtenir des bases d'ondelettes biorthogonales à support compact de $L^2(\mathbb{R})$ adaptées à l'étude de l'espace de Sobolev $H^k(\mathbb{R})$. Nous décrivons auparavant les analyses multirésolutions orthogonales et biorthogonales sur l'intervalle.

L'idée générale consiste à diviser un intervalle I en deux intervalles I_1 et I_2 (décomposition du domaine [1]), de construire deux analyses multirésolutions biorthogonales respectivement dans $L^2(I_1)$ et $L^2(I_2)$ (espaces des fonctions de carré intégrables sur I_1 et I_2 par rapport à la mesure de Lebesque), et ensuite, d'appliquer certains opérateurs d'extension "naturels" pour obtenir une analyse multirésolution biorthogonale de $L^2(I)$. Plus précisément, ces opérateurs d'extension conservent la compacité des supports des fonctions de base, leur régularité ainsi que la localisation en temps-fréquence.

Pour réaliser une telle construction, il faut savoir auparavant définir la notion d'analyse multirésolution sur un intervalle [6]. Pour cela nous partons de l'analyse orthogonale $V_j(\mathbb{R})$ de I. Daubechies [2] à fonctions d'échelle φ et ondelette associée ψ, et on note :

$$V_j([0,1]) = \text{Vect}\{\varphi_{j,k/_{[0,1]}}\ ,\ \varphi_{j,k} \in V_j(\mathbb{R})\}$$

et

$$\mathcal{V}_j([0,1]) = \text{Vect}\{\varphi_{j,k}\ ,\ \text{supp}\ \varphi_{j,k} \in [0,1]\}$$

Définition 1 *: Une suite* $\{V_j\}_{j \geq j}$ *de sous-espaces fermés de* $L^2([0,1])$ *est dite une analyse multirésolution sur* $L^2([0,1])$ *associée à* $V_j(\mathbb{R})$ *si elle vérifie :*

i) $\forall j \geq j_o\ ,\ \mathcal{V}_j([0,1]) \subset V_j \subset V_j([0,1])$

ii) $\forall j \geq j_o\ ,\ V_j \subset V_{j+1}$

La proposition suivante décrit une base du supplémentaire (non orthogonal) de V_j dans V_{j+1}.

Proposition 1 : Soit j_o le plus petit entier vérifiant $2^j \geq 4N-4$. Pour $j \geq j_o$ on note :
$X_j = Vect\{\psi_{j,k}, 0 \leq k \leq 2^j - 2N+1; \varphi_{j+1,2k}, 2^j - 2N + 2 \leq k \leq 2^j - N ; \varphi_{j+1,2k+1}, 0 \leq k \leq N-2\}$.
Alors

i) $\dim X_j = 2^j$

ii) il existe J tel que $\forall j \geq J$, $V_{j+1} = V_j \oplus X_j$

Pour obtenir des bases orthonormées d'ondelettes sur l'intervalle $[0,1]$, il suffit d'orthonormaliser par Gram-Schmidt ([3] , [4]) . La différence entre ces bases est dans la construction de V_{j_o}. Nous allons introduire maintenant la notion d'analyse multirésolution biorthogonale sur l'intervalle.

Définition 2 : Une analyse multirésolution biorthogonale sur $L^2([0,1])$ associée à une analyse multirésolution biorthogonale $(V_j(\mathbb{R}), V^*_j(\mathbb{R}))$ de $L^2(\mathbb{R})$ est une suite de couples (V_j, V^*_j) de sous-espaces fermés de $L^2([0,1])$ telle que :

i) $\mathcal{V}_j([0,1]) \subset V_j \subset V_j([0,1])$ et $\mathcal{V}^*_j([0,1]) \subset V^*_j \subset V^*_j([0,1])$.

ii) $V_j \subset V_{j+1}$ et $V^*_j \subset V^*_{j+1}$

iii) $L^2([0,1]) = V_j \oplus (V^*_j)^{\perp}$

Soit $(V_j(R) , V^*_j(R))$ une analyse multirésolution biorthogonale de $L^2(\mathbb{R})$ à fonctions d'échelle conjuguées (g , g^*). On suppose que supp $g = [N_1, N_2]$ et on note :

$$P^{\alpha}_i(x) = \sum_{k \leq -N_1 - 1} k^i g(x-k) \; ; \; P^{\beta}_i(x) = \sum_{k \geq -N_2 + 1} k^i g(x-k)$$

Notre construction est décrite par le théorème suviant :

Théorème 1 : Soit $(V_j(\mathbb{R}) , V^*_j(\mathbb{R}))$ une analyse multirésolution biorthogonale de $L^2(\mathbb{R})$ à fonctions d'échelle conjuguées (g , g^*) à support compact, et (V_j , V^*_j) celle de $L^2([0,1])$ associée à $(V_j(\mathbb{R}) , V^*_j(\mathbb{R}))$. On suppose que :

(i) g est dérivable de dérivée $g'(x) = \tilde{g}(x) - \tilde{g}(x-1)$

(ii) V_j contient les demi-constantes :

$$P^{\alpha}_{o,j} = P^{\alpha}_o(2^j x)_{/[0,1]} \text{ et } P^{\beta}_{o,j} = P^{\beta}_o(2^j x - 2^j)_{/[0,1]}.$$

Si on pose :

$$\tilde{V}_j = \{f \in L^2([0,1]) \,/\, \exists g \in V_j, f = g'\} \quad et \quad \tilde{V}_j^* = \{f \in L^2([0,1]) \,/\, f \in V_j \; et \; f(0) = f(1) = 0\},$$

alors $(\tilde{V}_j, \tilde{V}_j^*)$ *est une analyse multirésolution biorthogonale de* $L^2([0,1])$. *De plus, si* P_j *(resp* $\tilde{P}_j$*) désigne le projecteur oblique de* $L^2([0,1])$ *sur* V_j *(resp* $\tilde{V}_j$*) parallèlement à* $(V_j^*)^\perp$ *(resp* $(\tilde{V}_j^*)^\perp$ *) alors on a la formule de commutation suivante :*

$$\frac{d}{dx} o P_j = \tilde{P}_j o \frac{d}{dx}.$$

Un exemple fondamental est obtenu par d dérivations et intégrations à partir de l'analyse de I. Daubechies. En plus l'analyse multirésolution biorthogonale obtenue est adaptée à l'étude des fonctions régulières des espaces de Sobolev $H^k([0,1])$ et $H_0^{-k}([0,1])$ $(k \in \mathbb{Z})$.

Nous décrivons maintenant les analyses multirésolutions biorthogonales segmentées (sur chaque demi-droite) et les opérateurs d'extension associés. Pour cela, on divise $\mathbb{R}$ (l'axe des temps) en $]-\infty, 0]$ (le passé) et $[0,+\infty[$ (le futur). Le temps $t = 0$ correspond au présent.

Soit $V_j(\mathbb{R})$ l'analyse multirésolution de I. Daubechies. On définit les opérateurs d'extension "basse fréquence" E_j et E'_j par :

$$E_j : V_j(]-\infty, 0]) \longrightarrow V_j(\mathbb{R}) \; ; \; E'_j : V_j([0,+\infty[) \longrightarrow V_j(R)$$

$$\varphi_{j,k/]-\infty,0]} \longrightarrow \varphi_{j,k} \qquad \varphi_{j,k/[0,+\infty[} \longrightarrow \varphi_{j,k}$$

et également les deux opérateurs E_o^* et $E_o^{*'}$ d'extension par 0 :

$$E_o^* : L^2(]-\infty, 0]) \longrightarrow L^2(\mathbb{R}) \quad ; \quad E_o^{*'} : L^2([0,+\infty[) \longrightarrow L^2(\mathbb{R})$$

$$f \longrightarrow \bar{f} : \bar{f}_{/]-\infty,0]} = f, \; \bar{f}_{/(\mathbb{R})\backslash]0,\infty[} = 0 \qquad f \longrightarrow \bar{f} : \bar{f}_{/[0,+\infty[} = f, \; \bar{f}_{/\mathbb{R}\backslash[0,+\infty[} = 0.$$

On a alors le résultat fondamental suivant :

<u>*Théorème 2*</u>

i) Il existe une analyse multirésolution biorthogonale $(V_j(]-\infty, 0]), V_j^*(]-\infty, 0]))$ *de* $L^2(]-\infty, 0])$ *et une analyse multirésolution biorthogonale* $(V_j([0,+\infty[), V_j^*([0,+\infty[))$ *de* $L^2([0,+\infty[)$ *telles que si on pose* $V_j = E_j(V_j(]-\infty, 0])) \oplus E'_j(V_j([0,+\infty[))$ *et* $V_j^* = E_o^*(V_j^*(]-\infty,0])) \oplus E_o^{*'}(V_j^*([0,+\infty[))$ *alors* (V_j, V_j^*) *est une analyse multirésolution biorthogonale de* $L^2(\mathbb{R})$.

ii) On considère les espaces $W_j(]-\infty, 0]) = V_{j+1}(]-\infty, 0]) \cap (V_j^*]-\infty, 0))^\perp$ *et* $W_j^*(]-\infty, 0]) = V_{j+1}^*(]-\infty, 0]) \cap (V_j(]-\infty, 0]))^\perp$. *On définit de même les espaces* $W_j([0,+\infty[)$ *et* $W_j^*([0,+\infty[)$.

Si on pose :

$W_j = E_j(W_j(]-\infty, 0])) \oplus E'_j(W_j([0, +\infty[))$ *et* $W_j^* = E_0^*(W_j^*(]-\infty,0])) \oplus E_0^{*\prime}(W_j^*([0,+\infty[))$,

alors W_j *et* W_j^* *sont en dualité pour le produit scalaire de* $L^2(\mathbb{R})$ *et on a :*

$$V_{j+1} = V_j \oplus W_j \text{ et } V_{j+1}^* = V_j^* \oplus W_j^*.$$

iii) Si on pose :

$$V_j^{(1)}(]-\infty,0]) = \{f \in L^2(]-\infty,0]) \,/\, \exists g \in V_j(]-\infty,0]), f = g'\}$$

et

$$V_j^{(-1)}(]-\infty,0]) = \{f \in L^2(]-\infty,0]) \,/\, f' \in V_j^*(]-\infty,0]) \text{ et } f(0) = 0\}$$

alors $(V_j^{(1)}(]-\infty,0]), V_j^{(-1)}(]-\infty,0]))$ *est une analyse multirésolution biorthogonale de* $L^2(]-\infty, 0])$. *Il existe aussi une analyse multirésolution biorthogonale de* $L^2([0,+\infty[)$, *notée* $(V_j^{(1)}([0, +\infty[) , V_j^{(-1)}[0, +\infty[)$.

iv) Si on définit $V_j^{(1)}$ et $V_j^{(-1)}$ *de la même manière que (i) en remplaçant* $V_j(]-\infty, 0])$ *par* $V_j^{(1)}(]-\infty,0])$ *et* $V_j^*(]-\infty,0])$ *par* $V_j^{(-1)}(]-\infty,0])$ *(de même pour les espaces définis sur* $[0,+\infty[$*) alors* $(V_j^{(1)}, V_j^{(-1)})$ *est une analyse multirésolution biorthogonale de* $L^2(\mathbb{R})$. *De plus, si* P_j *(resp.* $P_j^{(1)}$*) désigne le projecteur oblique de* $L^2(\mathbb{R})$ *sur* V_j *(resp.* $V_j^{(1)}$ *parallèlement à* $(V_j^*)^{\perp}$ *(resp.* $(V_j^{(-1)})^{\perp}$*), alors on a :*

$$\frac{d}{dx} o P_j = P_j^{(1)} o \frac{d}{dx}.$$

Conclusion

Les opérateurs d'extension E_j et E'_j sont fournis par construction et conservent la régularité et la localisation des fonctions de base. Si on suppose que la fonction d'échelle φ est de classe $C^{p+\varepsilon}$ et on itère le processus décrit ci-dessus, alors l'analyse multirésolution $(V_j^{(d)}(]-\infty,0]), V_j^{(-d)}(]-\infty,0]))$ est adaptée à l'étude des espaces de Sobolev $H^k(]-\infty, 0])$ et $H_0^{-k}(]-\infty,0])$ pour $0 \le k \le p-d$ (ou $H^{-k}(]-\infty,0])$ et $H_0^k(]-\infty,0])$ pour $0 \le k \le d$). On a la même propriété pour l'analyse biorthogonale $(V_j^{(d)}([0,+\infty[), V_j^{(-d)}([0,+\infty[))$ de $L^2([0,+\infty[)$.

Par suite, l'analyse multirésolution biorthogonale $(V_j^{(d)}, V_j^{(-d)})$ de $L^2(\mathbb{R})$ est adaptée à l'étude des espaces de Sobolev $H^k(\mathbb{R})$ pour $0 \le k \le p-d$, et $H^{-k}(\mathbb{R})$ pour $0 \le k \le d$.

Nous signalons enfin qu'une deuxième construction d'analyse multirésolution biorthogonale segmentée est explicitée dans $L^2([-1,1])$ en partant d'une analyse multirésolution symétrique de $L^2(\mathbb{R})$ ([5]).

Nous avons montré dans ce travail qu'on peut construire des analyses multirésolutions orthogonales (ou biorthogonales) de $L^2([0,1])$ engendrées par translation et dilatation à partir d'un nombre fini de fonctions de base. Le processus de "dérivation et intégration" permet d'engendrer des nouvelles analyses multirésolutions adaptées à l'étude des fonctions régulières sur [0, 1].

Les analyses multrésolutions segmentées montrent qu'on peut analyser un signal ou une fonction à l'aide d'une information dans le passé, d'une relaxation du passé dans un futur proche, et d'une information dans le futur propre.

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STARLIKENESS OF CERTAIN INTEGRAL OPERATORS

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Abstract

The object of this paper is to investigate the starlikeness of a certain class of integral operators which are defined for analytic functions. The result obtained here extends and sharpens some recent results due to Kim *et al.* [1], Owa *et al.* [6], and Miller and Mocanu [3].

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are *analytic* in the *open* unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be *starlike of order* α if it satisfies the inequality:

$$\Re\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha \qquad (0 \leq \alpha < 1;\ z \in \mathcal{U}). \tag{1}$$

We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the condition (1). We note that $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) = \mathcal{S}^* \quad (0 \leq \alpha < 1)$, where $\mathcal{S}^*$ is the familiar class of starlike functions in $\mathcal{U}$.

Miller and Mocanu [3] developed a general family of integral operators which map subsets of $\mathcal{A}$ into $\mathcal{S}^*$, and which includes many of the previously considered integral operators. Their results provide an extension and improvement of numerous earlier results (see also Kim *et al.* [2]). By applying a result of Miller and Mocanu [3], Kim *et al.* [1] proved

Theorem 1 (Kim *et al.* [1]). *Let* $f(z) \in \mathcal{S}^*(\beta)$ *and* $g(z) \in \mathcal{S}^*(\beta)$. *Suppose also that*

$$F(z) := \left\{\frac{\gamma+\alpha+1}{z^\gamma}\int_0^z \{f(t)\}^\alpha g(t)\, t^{\gamma-1}\, dt\right\}^{1/(\alpha+1)} \qquad (\alpha > 0;\ \gamma > 0). \tag{2}$$

Then $F(z) \in \mathcal{S}^*(\beta)$.

More recently, Owa *et al.* [6] proved a generalization of Theorem 1, which is recalled here as

Theorem 2 (Owa *et al.* [6]). *Let* $f(z) \in \mathcal{S}^*(\eta_1)$ *and* $g(z) \in \mathcal{S}^*(\eta_2)$. *Then the function* $F(z)$ *defined by*

$$F(z) := \left\{ \frac{\gamma+\alpha+1-\eta}{z^{\gamma}} \int_0^z \{f(t)\}^{\alpha} g(t) t^{\gamma-1-\eta} \, dt \right\}^{1/(\alpha+1-\eta)} \tag{3}$$

$$(\alpha \geq 0;\ \gamma > 0;\ 0 \leq \eta \leq \eta_2;\ \alpha\eta_1 + \eta_2 - \eta \leq 1) \tag{4}$$

belongs to the class

$$\mathcal{S}^* \left(\frac{\alpha\eta_1 + \eta_2 - \eta}{\alpha + 1 - \eta} \right).$$

Clearly, by setting $\eta_1 = \eta_2$ and $\eta = 0$ in Theorem 2, we arrive at Theorem 1 of Kim *et al.* [1]. In this note, we make use of a result of Mocanu *et al.* [5] in order to establish a theorem which will weaken the hypothesis of Theorem 2 while sharpening the conclusion. As an application of our main result (Theorem 3 below), we also sharpen some results of Miller and Mocanu [3].

2. A Preliminary Lemma

We shall need the following Lemma to prove our main result.

Lemma (Mocanu *et al.* [5]). *Let $\beta > 0$ and $\beta + \gamma > 0$, and consider the integral operator $I_{\beta,\gamma}(h)$*

$$I_{\beta,\gamma}(h)(z) = \left\{ \frac{\beta+\gamma}{z^{\gamma}} \int_0^z \{h(t)\}^{\beta} t^{\gamma-1} \, dt \right\}^{1/\beta} \qquad (h(z) \in \mathcal{A}). \tag{5}$$

If $\rho \in [-\gamma/\beta, 1)$, $h(z) \in \mathcal{A}$, and

$$\Re \left\{ \frac{zh'(z)}{h(z)} \right\} \geq \rho,$$

then the function

$$G(z) := I_{\beta,\gamma}(h)(z)$$

is analytic in $\mathcal{U}$ and satisfies the inequality:

$$\Re \left\{ \frac{z\,G'(z)}{G(z)} \right\} \geq W(\rho;\beta,\gamma) := \inf_{|z|<1} \Re\{H(z)\}, \tag{6}$$

where

$$H(z) := \frac{(1-z)^{2(\rho-1)\beta}}{\beta \int\limits_0^1 t^{\beta+\gamma-1}(1-zt)^{2(\rho-1)\beta} \, dt} - \frac{\gamma}{\beta}.$$

This result is sharp and $W(\rho;\beta,\gamma) \geq \rho$. The extremal function is given by

$$G_0(z) := I_{\beta,\gamma}(k)(z) \qquad \left(k(z) := z(1-z)^{2(\rho-1)} \right).$$

Remark 1. More general forms of the Lemma may be found in the work of Miller and Mocanu [4].

3. The Main Result and Its Consequences

By appealing to the Lemma, we shall prove our main result contained in

Theorem 3. *Let α, β, γ, δ, and σ be real numbers satisfying*

$$\alpha \geq 0,\ \beta > 0,\ \sigma \geq 0,\ \text{and } \beta + \gamma = \alpha + \delta > 0.$$

Suppose that the function $\Phi(z)$ is analytic in $\mathcal{U}$ and satisfies the conditions:

$$\Phi(0) = 1 \quad \text{and} \quad \Phi(z) \neq 0 \qquad (z \in \mathcal{U}).$$

If $f(z) \in \mathcal{S}^*(\eta_1)$ and $g(z) \in \mathcal{S}^*(\eta_2)$, then the function $F(z)$ defined by

$$F(z) := J(f,g)(z) = \left\{ \frac{\beta + \gamma}{z^\gamma \Phi(z)} \int_0^z \{f(t)\}^\alpha \{g(t)\}^\sigma t^{\delta - \sigma - 1}\, dt \right\}^{1/\beta}, \tag{7}$$

with

$$\delta + \alpha\eta_1 + (\eta_2 - 1)\sigma \geq 0, \tag{8}$$

satisfies the inequality:

$$\Re\left\{ \frac{z F'(z)}{F(z)} + \frac{1}{\beta} \cdot \frac{z \Phi'(z)}{\Phi(z)} \right\} \geq W(\rho; \beta, \gamma) \qquad (z \in \mathcal{U}), \tag{9}$$

where $\rho = (\delta + \alpha\eta_1 + \sigma\eta_2 - \sigma - \gamma)/\beta$ and $W(\rho;\beta,\gamma)$ is given by (6). *This result is sharp, the extremal function being given by*

$$F_0(z) := J(k_1, k_2)(z)$$

$$\left(k_1(z) := z(1-z)^{2(\eta_1 - 1)}; \quad k_2(z) := z(1-z)^{2(\eta_2 - 1)}\right).$$

Proof. In terms of the functions $f(z) \in \mathcal{S}^*(\eta_1)$ and $g(z) \in \mathcal{S}^*(\eta_2)$, we define the function $h(z)$ by

$$h(z) = f(z) \left\{ \frac{f(z)}{z} \right\}^{(\gamma - \delta)/\beta} \left\{ \frac{g(z)}{z} \right\}^{\sigma/\beta} \tag{10}$$

Then $h(z) \in \mathcal{A}$ and

$$\begin{aligned} \Re\left\{ \frac{z h'(z)}{h(z)} \right\} &= \Re\left\{ \frac{\beta + \gamma - \delta}{\beta} \cdot \frac{z f'(z)}{f(z)} + \frac{\sigma}{\beta} \cdot \frac{z g'(z)}{g(z)} + \frac{\delta - \gamma - \sigma}{\beta} \right\} \\ &\geq \frac{\delta + \alpha\eta_1 + \sigma\eta_2 - \sigma}{\beta} - \frac{\gamma}{\beta} \qquad (z \in \mathcal{U}). \end{aligned} \tag{11}$$

Setting $\rho = (\delta + \alpha\eta_1 + \sigma\eta_2 - \sigma - \gamma)/\beta$, we see from (8) that $\rho \in [-\gamma/\beta, 1)$. Hence the function $h(z)$ satisfies all the conditions of the Lemma. A simple calculation shows that $F(z)$, as given by (7), can be written as

$$F(z)=\frac{I_{\beta,\gamma}(h)(z)}{\{\Phi(z)\}^{1/\beta}}=\frac{G(z)}{\{\Phi(z)\}^{1/\beta}}, \tag{12}$$

where $I_{\beta,\gamma}$ is given by (5). Consequently, applying the Lemma, we conclude that

$$\Re\left\{\frac{zF'(z)}{F(z)}+\frac{1}{\beta}\cdot\frac{z\Phi'(z)}{\Phi(z)}\right\}=\Re\left\{\frac{zG'(z)}{G(z)}\right\}\geq W(\rho;\beta,\gamma) \qquad (z\in\mathcal{U}).$$

For the functions

$$k_1(z)=z(1-z)^{2(\eta_1-1)}\in\mathcal{S}^*(\eta_1)$$

and

$$k_2(z)=z(1-z)^{2(\eta_2-1)}\in\mathcal{S}^*(\eta_2),$$

we see that

$$k_1(z)\left\{\frac{k_1(z)}{z}\right\}^{(\gamma-\delta)/\beta}\left\{\frac{k_2(z)}{z}\right\}^{\sigma/\beta}=k(z)=z(1-z)^{2(\rho-1)}$$

and

$$J(k_1,k_2)(z)=\frac{I_{\beta,\gamma}(k)(z)}{\{\Phi(z)\}^{1/\beta}}. \tag{13}$$

Hence the sharpness of the assertion of Theorem 3 follows from the Lemma. This evidently completes the proof of Theorem 3.

Remark 2. In Theorem 3, ρ and $W(\rho;\beta,\gamma)$ may both be negative. If $\rho\geq 0$, then we see from the Lemma that $W(\rho;\beta,\gamma)\geq 0$. In the case when

$$\max\left\{\frac{\beta-\gamma-1}{2\beta},-\frac{\gamma}{\beta}\right\}=\rho_0\leq\rho<1,$$

the value of $W(\rho;\beta,\gamma)$ given by (6) can be replaced by

$$W(\rho;\beta,\gamma)=H(-1)=\frac{1}{\beta}\left\{\frac{(\beta+\gamma)\,2^{-2\beta(1-\rho)}}{{}_2F_1\left[2\beta(1-\rho),\beta+\gamma;\beta+\gamma+1;-1\right]}-\gamma\right\}, \tag{14}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function.

Remark 3. Integral operators of the type (7), involved in Theorem 3, are becoming increasingly useful in the study of such subclasses of analytic and univalent functions as the class $\mathcal{S}^*(\alpha)$ (*cf.*, *e.g.*, [3] and [6]). Indeed, by assigning appropriate special values to the various parameters involved in Theorem 3, we can derive several consequences of Theorem 3. Only one corollary and three examples will be listed here.

If we set $\Phi(z)\equiv 1$, $\sigma=1$, $\beta=\alpha+1-\eta$, and $\delta=\gamma+1-\eta$ in Theorem 3, we obtain the following

Corollary. *Let* $f(z)\in\mathcal{S}^*(\eta_1)$ *and* $g(z)\in\mathcal{S}^*(\eta_2)$. *Then the function* $F(z)$ *defined by* (3), *with*

$$\alpha \geq 0, \ \ \gamma \geq 0, \ \ and \ \eta \leq \alpha\eta_1 + \eta_2, \tag{15}$$

belongs to the class

$$\mathcal{S}^*(W(\rho; 1+\alpha-\eta, \gamma)) \qquad \left(\rho := \frac{\alpha\eta_1 + \eta_2 - \eta}{1+\alpha-\eta}\right),$$

where $W(\rho; 1+\alpha-\eta, \gamma)$ *is given by* (6) *with* $\beta = 1+\alpha-\eta$. *This result is sharp.*

This Corollary extends and improves both Theorem 1 and Theorem 2. The following three examples would improve the corresponding results of Miller and Mocanu [3]. In Example 3, in particular, we have chosen $\Phi(z) \not\equiv 1$.

If we let $\Phi(z) \equiv 1$ and $\beta + \gamma = \delta + \alpha = 1$ in Theorem 3, we obtain

Example 1. Let $0 \leq \alpha < 1$, $\beta \geq 1$, and $0 \leq \sigma \leq 2(1-\alpha)$. If $f \in \mathcal{S}^*$ and $g \in \mathcal{S}^*(\frac{1}{2})$, then

$$\left\{z^{\beta-1}\int_0^z \left\{\frac{f(t)}{t}\right\}^{\alpha}\left\{\frac{g(t)}{t}\right\}^{\sigma} dt\right\}^{1/\beta} \in \mathcal{S}^*(W(\rho;\beta,1-\beta)) \tag{16}$$

$$\left(\rho := \frac{\beta-\alpha-\frac{1}{2}\sigma}{\beta} \geq 0\right).$$

For $\beta = 1$, we find from (16) that

$$\int_0^z \left\{\frac{f(t)}{t}\right\}^{\alpha}\left\{\frac{g(t)}{t}\right\}^{\sigma} dt \in \mathcal{S}^*(W(\rho;1,0)) \tag{17}$$

where $\rho = 1-\alpha-\frac{1}{2}\sigma$ and

$$W(\rho;1,0) = \begin{cases} \frac{2\rho-1}{2-2^{2(1-\rho)}} & (\rho \neq \frac{1}{2}) \\ \frac{1}{2\,\ell n 2} & (\rho = \frac{1}{2}). \end{cases} \tag{18}$$

If the parameter σ is constrained further by $0 \leq \sigma \leq \frac{2}{3}$, then

$$\int_0^z \left\{\frac{f(t)}{t}\right\}^{\sigma}\left\{\frac{g(t)}{t}\right\}^{\sigma} dt \in \mathcal{S}^*(W(1-3\sigma/2;1,0)).$$

If we let $\Phi(z) \equiv 1$, $\beta = 1$, and $\delta = 1+\gamma-\alpha$ in Theorem 3, we obtain

Example 2. Let $\alpha \geq 0$, $\gamma \geq 0$, and $0 \leq \sigma \leq 2-\alpha$. If $f, g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, then

$$\frac{1+\gamma}{z^{\gamma}}\int_0^z \left\{\frac{f(t)}{t}\right\}^{\alpha}\left\{\frac{g(t)}{t}\right\}^{\sigma} t^{\gamma}\, dt \in \mathcal{S}^*(W(\rho;1,\gamma)) \tag{19}$$

$$\left(\rho := 1-\frac{1}{2}\alpha-\frac{1}{2}\sigma\right).$$

For $\alpha = \gamma = 1$, we find from (19) that

$$\frac{2}{z}\int_0^z f(t)\left\{\frac{g(t)}{t}\right\}^{\sigma} dt \in \mathcal{S}^*\left(W\left(\frac{1-\sigma}{2};1,1\right)\right) \qquad (0 \leq \sigma \leq 1),$$

where

$$W\left(\frac{1-\sigma}{2};1,1\right) = \begin{cases} \frac{1}{2(2\ell n2-1)} - 1 & (\sigma = 1) \\ \frac{1}{2(1-\ell n2)} - 1 & (\sigma = 0) \\ \frac{\sigma(\sigma-1)}{2(2^{\sigma}-\sigma-1)} - 1 & (\sigma \neq 1;\ \sigma \neq 0). \end{cases} \tag{20}$$

If we set $\Phi(z) = e^z$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, and $\delta = \sigma = 0$ in Theorem 3, we obtain

Example 3. If $f \in \mathcal{S}^*(\eta_1)$, then

$$\left\{z\,e^{-z}\int_0^z t^{-1} f(t)\,dt\right\}^{\frac{1}{2}} \in \mathcal{S}^*\left(\tfrac{1}{2}W(\eta_1;1,0)\right), \tag{21}$$

where $W(\eta_1;1,0)$ is given by (18) with, of course, $\rho = \eta_1$.

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MULTINOMIALS IN MODIFIED CLIFFORD ANALYSIS

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Abstract

A generalized multinomial theorem and geometric series expansion for paravector variables in Clifford algebras is presented as a consequence of a modified approach to Clifford analysis.

AMS subject classification: 30G35, *Keywords*: Clifford Analysis, multinomial expansion

1. Clifford Algebras. Let (V,q) be a quadratic space over the field K (characteristic $\neq 2$) with dimension $d < \infty$, that means V is a d-dimensional vector space over K endowed with a quadratic form $q: V \to K$ ([9]). Further let $e_1, \ldots, e_d$ be an orthogonal basis for V (with respect to q). Then the (universal) *Clifford algebra* C_d over (V,q) is the associative algebra of vector space dimension 2^d generated by the products $e_{i_1} \cdot \ldots \cdot e_{i_r}$, $1 \leq i_1 < \ldots < i_r \leq d$, $r \in \{1, \ldots, d\}$, and $1 \in K$ with the relations $e_i e_j = -e_j e_i$ for $i \neq j$ and $e_i^2 = -q(e_i)1 =: -q(e_i)$. Every element x of C_d can be written as $x = x_0 + \sum_{\nu=1}^{d} x_\nu e_\nu + \sum_{\nu<\mu} x_{\nu\mu} e_\nu e_\mu + \ldots + x_{1\ldots d} e_1 \cdot \ldots \cdot e_d$ with unique $x_\nu \in K$. Special cases for $K = \mathbb{R}$ and the quadratic form $q(x) = \sum_{\nu=1}^{d} x_\nu^2$ are $C_1 \cong \mathbb{C}$ and $C_2 \cong \mathbb{H}$, the skew field of the quaternions.

In the following we identify so-called *paravectors* [8] $x = x_0 + \sum_{\nu=1}^{d} x_\nu e_\nu$ with elements of K^{d+1}. We remark that for such x the n-th power of x, x^n, is again a paravector [6]. Of course paravectors with vanishing first coordinate ($x_0 = 0$) can be identified with vectors, the elements of V resp. K^d; if the last coordinate vanishes ($x_d = 0$) the paravectors will be called *shortened*.

2. Modified Clifford Analysis. Now we specialize to $K := \mathbb{R}$ and, to get simpler formulas, $q(x) = \sum_{\nu=1}^{n} x_\nu^2$, i.e. $e_j^2 = -1$ for $j = 1, \ldots, n$. Clifford analysis extends complex analysis to Clifford algebras using a generalization of the CAUCHY-RIEMANN equations; solutions of this new system of partial differential equations are called *Clifford analytic* [4] or *regular* [3] or *monogenic* [1] depending what generalization is meant. In classical Clifford analysis simple functions such as polynomials (and even the identity map) are not Clifford analytic. However, LEUTWILER [6] used a "hyperbolic modification" making $f(x) = x^n$, $x = x_0 + \sum_{\nu=1}^{d} x_\nu e_\nu$, and the derivatives $\frac{\partial f}{\partial x_\nu}$ $(\nu = 0, \ldots, d-1)$ solutions of his modified system of differential equations, namely

$$x_n\left(\frac{\partial f_0}{\partial x_0} - \sum_{\nu=1}^{d} \frac{\partial f_\nu}{\partial x_\nu}\right) + (d-1) f_d = 0,$$

$$\frac{\partial f_\nu}{\partial x_\mu} = \frac{\partial f_\mu}{\partial x_\nu}, \quad \frac{\partial f_\nu}{\partial x_0} = -\frac{\partial f_0}{\partial x_\nu} \quad (\nu, \mu = 1, \ldots, d).$$

Thereby f is a C^1 vector valued function $f = f_0 + \sum_{\nu=1}^{d} f_\nu e_\nu$ defined on an open subset of $\mathbb{R}^{d+1}$, as proposed by CNOPS [2], we call solutions of this system *hyperbolic monogenic*. "Hyperbolic" means thereby that Leutwilers modification is based on the hyperbolic metric in the upper half space $\mathbb{R}_+^{d+1}$ instead of the euclidean one. As a consequence, power series in $x \in \mathbb{R}^{d+1}$ with *real* coefficients, become hyperbolic monogenic.

It is natural to have a closer look at the derivatives of the n-th power x^n when interested in homogeneous polynomials which represent hyperbolic monogenic functions. To get polynomials of degree n we study two classes of derivatives:

3. Polynomials I — and a generalized multinomial theorem. For $x \in \mathbb{R}^{d+1}$ and any multiindex $k = (k_1, \ldots, k_{d-1}) \in \mathbb{N}_0^{d-1}$, $|k| := \sum_{\nu=1}^{d-1} k_\nu$, $k! := \prod_{\nu=1}^{d-1} k_\nu!$, we define the polynomials

$$L_n^k(x) := \frac{1}{k!} \cdot \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdot \ldots \cdot \partial x_{d-1}^{k_{d-1}}} x^{n+|k|}$$

of degree n. Obviously $L_n^k(x)$ is hyperbolic monogenic. For example $L_n^0(x) = x^n$, $L_1^k(x) \in \{x, e_\nu x + x e_\nu, e_\nu x e_\nu - 2x, \frac{1}{2}(e_\mu x e_\nu + e_\nu x e_\mu), \ldots\}$ for $k \in \{(0,\ldots,0), (0,\ldots,1,\ldots,0), (0,\ldots,2,\ldots,0), (0,\ldots,1,\ldots,1,\ldots,0), \ldots\}$, respectively. The multiindex k counts the appearance of e_ν in the products $e_{\nu_1} x e_{\nu_2} x e_{\nu_3} x \ldots$

An easy consequence of this definition together with the chain rule is the following *recursion formula*:

$$L_n^k(x) = x L_{n-1}^k(x) + \sum_{\nu=1}^{d-1} e_\nu L_n^{k-1_\nu}(x), \quad k - 1_\nu := (k_1, \ldots, k_\nu - 1, \ldots, k_{d-1}),$$

provided $L_n^k(x) = 0$ if any component of k is negative. This recursion could also be used as a definition for our polynomials. Another useful observation are formulas for the *derivatives*:

$$\tfrac{\partial}{\partial x_0} L_n^k(x) = (n + |k|) L_{n-1}^k(x), \quad \tfrac{\partial}{\partial x_\nu} L_n^k(x) = (k_\nu + 1) L_{n-1}^{k+1_\nu}(x) \quad (\nu \neq d).$$

These look even more pretty when we introduce a "normalized" version of the L_n^k:

$$\tilde{L}_n^k(x) := \frac{1}{\binom{n+|k|}{k}} L_n^k(x), \quad \text{where} \quad \binom{n+|k|}{k} := \frac{(n+|k|)!}{k_1! \cdot \ldots \cdot k_{d-1}! n!} = \frac{(n+|k|)!}{k! n!};$$

in this case the above reads simply as

$$\tfrac{\partial}{\partial x_0} \tilde{L}_n^k(x) = n \tilde{L}_{n-1}^k(x), \quad \tfrac{\partial}{\partial x_\nu} \tilde{L}_n^k(x) = n \tilde{L}_{n-1}^{k+1_\nu}(x) \quad (\nu \neq d).$$

This is just what could be expected from regarding the complex case where $\frac{\partial z^n}{\partial x} = n z^{n-1}$, $\frac{\partial z^n}{\partial y} = n i z^{n-1}$ for $z = x + iy \in \mathbb{C}$.

The following lemma is the key for the proof of the multinomial theorem:

Lemma 3.1: *For all $x \in \mathbb{R}^{d+1}$ and $a \in \mathbb{R}$ the expansions*

$$L_n^k(x + a) = \sum_{p=0}^{n} \binom{n+|k|}{p} a^p L_{n-p}^k(x) \quad \textit{and} \quad L_n^k(x + a e_\nu) = \sum_{p=0}^{n} \binom{k_\nu + p}{k_\nu} a^p L_{n-p}^{k+p1_\nu}(x)$$

with $k + p1_\nu := (k_1, \ldots, k_\nu + p, \ldots, k_{d-1})$ are valid for $k = (k_1, \ldots, k_{d-1}) \in \mathbb{N}_0^{d-1}$ and $n \in \mathbb{N}_0$.

The proofs can be done by induction using the recursion formula.

For given $n \in \mathbb{N}_0$, $k \in \mathbb{N}_0^{d-1}$ denote by $\sum_{p=0}^{n}$ the iterated sum

$$\sum_{p=0}^{n} := \sum_{p_0=0}^{n} \sum_{p_1=0}^{n-p_0} \sum_{p_2=0}^{n-p_0-p_1} \cdots \sum_{p_{d-1}=0}^{n-p_0-\ldots-p_{d-2}}.$$

Combining the two expansions from lemma 3.1 we obtain

Theorem 3.2: *Let $n \in \mathbb{N}_0$, $k \in \mathbb{N}_0^{d-1}$. Then for all $x \in \mathbb{R}^{d+1}$ and all shortened paravectors $y = y_0 + \sum_{\nu=1}^{d-1} y_\nu e_\nu \in \mathbb{R}^d$ the expansion*

$$L_n^k(x+y) = \sum_{p=0}^{n} \binom{n+|k|}{p_0} y_0^{p_0} \prod_{\nu=1}^{d-1} \binom{k_\nu + p_\nu}{k_\nu} y_\nu^{p_\nu} L_{n-|p|}^{k+p}(x)$$

holds, where $k + p := (k_1 + p_1, \dots, k_{d-1} + p_{d-1})$.

Switching now to the normalized polynomials $\tilde{L}_n^k$ introduced above and evaluating 3.2 at $x := x_d e_d$ and $y := x_0 + \sum_{\nu=1}^{d-1} x_\nu e_\nu$ yields — notation as in 3.2 —: $\tilde{L}_n^k(x) = \sum_{p=0}^{n} \binom{n}{p} \prod_{\nu=0}^{d-1} x_\nu^{p_\nu} x_d^{n-|p|} \tilde{L}_{n-|p|}^{k+p}(e_d)$. A more symmetric reformulation of this expansion is possible by setting $q := (p, n-|p|) =: (q_0, \dots, q_d)$:

Theorem 3.3 (Generalized Multinomial Theorem).
$\tilde{L}_n^k(x) = \sum_{|q|=n} \frac{n!}{q!} \ell_{k,n,q} \prod_{\nu=0}^{d} x_\nu^{q_\nu}$ with coefficients $\ell_{k,n,q} := \tilde{L}_{q_d}^{k+q_1 1_1 + \dots + q_{d-1} 1_{d-1}}(e_d) = \tilde{L}_{n-|p|}^{k+p}(e_d)$.

The coefficients $\tilde{L}_{n-|p|}^{k+p}(e_d)$ can be determined explicitly; in the special case $k = 0$ we get

$$\tilde{L}_n^0(x) = x^n = (x_0 + \sum_{\nu=1}^{d} x_\nu e_\nu)^n = \sum_{|q|=n} \frac{n!}{q!} \prod_{\nu=0}^{d} x_\nu^{q_\nu} L_{n-|p|}^{p}(e_d)$$

which is the paravector equivalent of the well-known multinomial theorem

$$(x_0 + \sum_{\nu=1}^{d} x_\nu)^n = \sum_{|q|=n} \frac{n!}{q!} \prod_{\nu=0}^{d} x_\nu^{q_\nu}.$$

4. Polynomials II — and a generalized geometric series expansion. Another way to gain homogeneous hyperbolic monogenic polynomials of degree n is to differentiate not with respect to x but with respect to a new shortened variable $y \in \mathbb{R}^d$, $y \neq 0$: for n, k, x as above with the limitation $|k| \leq n+1$, and $y = y_0 + \sum_{\nu=1}^{d-1} y_\nu e_\nu$ define

$$E_n^k(x) := \frac{1}{k!(n+1-|k|)!} \cdot \frac{\partial^{n+1}}{\partial y_0^{n+1-|k|} y_1^{k_1} \cdots y_{d-1}^{k_{d-1}}} [(yx)^n y].$$

Then again chain rule gives a *recursion formula*: $E_n^k(x) = x E_{n-1}^k(x) + \sum_{\nu=1}^{d-1} e_\nu x E_{n-1}^{k-1_\nu}(x)$, but derivative formulas are not so easy to see: for instance

$$\frac{\partial}{\partial x_0} E_n^k(x) = (n+|k|) E_{n-1}^k(x) - (n-|k|+2) \sum_{\nu=1}^{d-1} E_{n-1}^{k-2\cdot 1_\nu}(x).$$

Some examples are $E_n^0(x) = x^n$, $E_1^k(x) \in \{x,\ e_\nu x + x e_\nu,\ e_\nu x e_\nu,\ e_\mu x e_\nu + e_\nu x e_\mu, \dots\}$.
We note that the generating polynomial $(yx)^n y$ can be reproduced by summing up the E_n^k for fixed n in the following way:

Lemma 4.1: $\sum_{k=0}^{n+1} y_0^{n+1-|k|} \prod_{\nu=1}^{d-1} y_\nu^{k_\nu} E_n^k(x) = (yx)^n y.$

(Here $\sum_{k=0}^{n+1}$ has to be understood as in 3.2.) The proof is again by induction using the recursion formula. Applying now geometric series expansion to this result gives

Theorem 4.2: $\sum_{n=0}^{\infty}\sum_{k=0}^{n+1} y_0^{n+1-|k|}\prod_{\nu=1}^{d-1} y_\nu^{k_\nu} E_n^k(x) = (y^{-1}-x)^{-1}$ for all $x \in \mathbb{R}^{d+1}$ with $|x| < \frac{1}{|y|}$.

In the case $y = 1$ this remains the standard geometric series expansion provided we have set $0^0 = 1$.

5. Connection. The two definitions given in 3 and 4 are not independent. Another inductive argument using all the recursion and derivative formulas above shows for example that

$$E_n^k(x) = \sum_{p_1=0}^{[\frac{k_1}{2}]} \cdots \sum_{p_{d-1}=0}^{[\frac{k_{d-1}}{2}]} \binom{n+1}{p} L_n^{k-2p}(x)$$

and there is a similar formula for producing $L_n^k(x)$ out of the $E_n^k(x)$. One consequence of this linear dependency is that the E_n^k are hyperbolic monogenic, which could also be proved directly.

6. Remark. The results in the case $d = 2$ (quaternionic paravectors) have been proved already in [7]; moreover in this case for fixed n the polynomials E_n^k resp. L_n^k, $|k| \le n+1$ form a basis for the vector space of all homogeneous polynomials of degree n representing hyperbolic monogenic functions. This is wrong in the case $d = 3$ and all higher dimensions ([5], [6]).

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MONGE-AMPERE EQUATIONS ON CERTAIN MANIFOLDS WITH POSITIVE FIRST CHERN CLASS

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Presented by T. Bloom, F.R.S.C.

I-Introduction.

Let P_m be the complex projective space of complex dimension m endowed with the Fubini-Study Kähler metric. In a local chart its components are given by $(m+1)\,\partial_{\lambda\bar\mu}\,\text{Log}(1+|z_1|^2+...+|z_m|^2)$, where $z_i = \frac{Z_i}{Z_o}$ $(1 \le i \le m)$ and $[Z_o,Z_1,...,Z_m]$ are the homogeneous coordinates of P_m. This metric is Einstein.

Let X be the blow-up of P_m at the point [0,1,0,...,0]. It is the submanifold of $P_m \times P_{m-1}$ defined by the family of equations $Z_iY_j = Z_jY_i$ for i,j = 0, 2, ..., m, where $[Y_o,Y_2,...,Y_m]$ are the homogeneous coordinates of P_{m-1}.

Let us denote π_o (resp. π_1) the projection of X on P_m (resp. P_{m-1}), and by ω_p the Fubini-Study metric of P_p $(p\ge 2)$, then $g = 2\pi_o^*\,\omega_m + (m-1)\,\pi_1^*\,\omega_{m-1}$ is a Kähler metric which belongs to the first Chern class of X.

There exists an open dense set U of X isomorphic to $\mathbb{C}^m$ on which the components of g are given by

$$g_{\lambda\bar\mu} = 2\,\partial_{\lambda\bar\mu}\,\text{Log}(1+|z_1|^2+...+|z_m|^2)+(m-1)\,\partial_{\lambda\bar\mu}\,\text{Log}(1+|z_2|^2+...+|z_m|^2)\,.$$

In this paper we prove (see corollary 1 below) that for every $\lambda < \frac{1}{2m}$, the Monge-Ampère equation

$$\text{Log}\,M(\varphi) = -\lambda\varphi + f$$

has a solution, where φ is C^∞ g-admissible on X (i.e. $g'_{\lambda\bar\mu} = g_{\lambda\bar\mu} + \partial_{\lambda\bar\mu}\varphi > 0$),

$$M(\varphi) = \det((g'g^{-1})) = \det((\delta^\lambda_{\ \mu} + \nabla^\lambda_{\ \mu}\varphi))$$

and f is the C^∞ function on X given by the geometric data: $R_{\lambda\bar\mu} = g_{\lambda\bar\mu} + \partial_{\lambda\bar\mu} f$, $R_{\lambda\bar\mu}$ being the components of the Ricci tensor relative to the metric g.

To prove the existence of solutions, we use the continuity method and we have only to prove that solutions of

$$\mathrm{Log}\, M(\varphi) = -\lambda t\varphi + f \qquad (0 < \varepsilon < t < 1)$$

are uniformly bounded, since higher estimates are given by Aubin [3] and Yau [10]. To establish the uniform bound of these solutions, we estimate the Tian's constant $\alpha(M)$ (see [8], [9] and [2]) and use Aubin's method[4].

Anlogous results are proved for the manifold Y which is the blow-up of P_2 at points [0,1,0] and [0,0,1].

We know that X and Y have positive first Chern class, that their automorphisms groups are not reductive and that their Futaki invariants does not vanish. So, the Monge-Ampère equation defined with $\lambda = 1$, which is relative to the existence of Einstein-Kähler metrics, has no solution (see [6], [7]).

<u>II-Theorem 1.</u>

Let $\varphi \in C^{\infty}(X)$ be g-admissible. Then, for all $\alpha < \frac{1}{2(m+1)}$, we have

$$\int_X \exp(-\alpha\varphi)\, dv \le C^{ste} \exp\left(\frac{-\alpha}{V_m}\int_X \varphi\, dv\right)$$

where dv is the volume element on X given by $dv = \det((g_{\lambda\bar{\mu}}))dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_m \wedge d\bar{z}_m$ and $V_m = \int_X dv$.

<u>Basic lemma</u> [4]. Let S_2 be the two-dimensional sphere, carrying a metric $\bar{g}$ with volume V. Let ψ be a real function defined on S_2, $\bar{g}$-admissible. Let us set

$$\tilde{\psi} = \psi - \frac{1}{4\pi}\int_{S_2} \psi\, dv_s,$$

where dv_s is the canonical area element of S_2.

Then, for $\beta < \frac{2\pi}{V}$, we have $\int_{S_2} e^{-\beta\tilde{\psi}}\, dv_s \le C^{ste}$ (depending only on β and V).

<u>Proof of Theorem 1.</u>

Let $\varphi(z_1)$ be the restriction of φ to the set S_2^* obtained by fixing $z_2,\ldots,z_m$ in X. We have

$$g_{1\bar{1}} = 2\partial_{1\bar{1}} \mathrm{Log}(1+|z_1|^2) + \partial_{1\bar{1}} f_1$$

where

$$f_1 = 2\,\mathrm{Log}\left(\frac{1+|z_1|^2+\ldots+|z_m|^2}{1+|z_1|^2}\right).$$

$\varphi(z_1)$ is admissible for the metric $g_{1\bar{1}}$ .Since the volume of S_2^* for the metric $g_{1\bar{1}}$ is equal to

$$V = \sqrt{-1}\int_{\mathbb{C}} g_{1\bar{1}}\; dz_1 \wedge d\bar{z}_1 = 4\pi,$$

using the basic lemma, for $\alpha < \frac{2\pi}{V} = \frac{1}{2}$, we have

$$\text{(1)} \qquad \int_{S_2^*} e^{-\alpha\varphi(z_1)}\, dv_s \le C^{ste} \exp\Big(\frac{-\alpha}{4\pi}\int_{S_2^*}\varphi(z_1)dv_s\Big).$$

Now let us set, for $2 \le \lambda,\mu \le m$

$$\tilde{\varphi}(z_2,\ldots,z_m) = \frac{\sqrt{-1}}{2\pi}\int_{S_2^*} \varphi(z_1,\ldots,z_m)\,\frac{dz_1\wedge d\bar{z}_1}{\left(1+|z_1|^2\right)^2}$$

and

$$\tilde{g}_{\lambda\bar{\mu}}(z_2,\ldots,z_m) = \frac{\sqrt{-1}}{2\pi}\int_{S_2^*} g_{\lambda\bar{\mu}}(z_1,\ldots,z_m)\,\frac{dz_1\wedge d\bar{z}_1}{\left(1+|z_1|^2\right)^2}.$$

An explicit computation of $\tilde{g}_{\lambda\bar{\mu}}(z_2,\ldots,z_m)$ gives

$$\tilde{g}_{\lambda\bar{\mu}}(z_2,\ldots,z_m) = (m+1)\,\partial_{\lambda\bar{\mu}}\,\mathrm{Log}\,(1+|z_2|^2+\ldots+|z_m|^2) + \partial_{\lambda\bar{\mu}}\,\psi,$$

where

$$\psi = 2\,\frac{\mathrm{Log}(1+|z_2|^2+\ldots+|z_m|^2)}{|z_2|^2+\ldots+|z_m|^2}.$$

Since $\tilde{\varphi}$ is $\tilde{g}$ -admissible, $\tilde{\varphi}+\psi$ is admissible for the metric $\frac{m+1}{m}\omega_{m-1}$ on P_{m-1} . In [4] Aubin proves that $\alpha(P_m) = \frac{1}{m+1}$; this yields, for $\beta < \frac{1}{m}$,

$$\int_{P_{m-1}} \exp\Big(\frac{-\beta m}{m+1}(\tilde{\varphi}+\psi)\Big)\, dv_{m-1} \le C^{ste} \exp\Big(\frac{-\beta m}{(m+1)V_{m-1}}\int_{P_{m-1}}(\tilde{\varphi}+\psi)dv_{m-1}\Big)$$

with dv_{m-1} the volume element of the metric ω_{m-1} on P_{m-1} and V_{m-1} the corresponding volume. Hence, for $\alpha < \frac{1}{m+1}$, we get

$$\int_{P_{m-1}} e^{-\alpha(\varphi+\psi)} dv_{m-1} \le C^{ste} \exp\left(\frac{-\alpha}{V_{m-1}} \int_{P_{m-1}} (\tilde{\varphi}+\psi)\, dv_{m-1}\right)$$

and, since ψ is positive and bounded,

$$\text{(2)} \qquad \int_{P_{m-1}} e^{-\alpha\tilde{\varphi}} dv_{m-1} \le C^{ste} \exp\left(\frac{-\alpha}{V_{m-1}} \int_{P_{m-1}} \tilde{\varphi}\, dv_{m-1}\right).$$

Considering φ as function of z_1 and using (1), since $\alpha < \frac{1}{m+1} < \frac{1}{2}$, we have

$$\int_{S_2^*} e^{-\alpha\varphi(z_1)} dv_s \le C^{ste} \exp\left(\frac{-\alpha}{4\pi} \int_{S_2^*} \varphi(z_1) dv_s\right) \text{ and consequently}$$

$$\int_{S_2^*} e^{-\alpha\varphi(z_1)} dv_s \le C^{ste} \exp\{-\alpha\, \tilde{\varphi}\}.$$

Integrating both sides of this inequality over P_{m-1} and using (2), we obtain

(always for $\alpha < \frac{1}{m+1}$)

$$\text{(3)} \qquad \int_{P_{m-1}} \int_{S_2^*} \exp(-\alpha\varphi(z_1,\dots,z_m))\, dv_s(z_1) dv_{m-1}(z_2,\dots,z_m)$$
$$\le C^{ste} \exp\left(\frac{-\alpha}{V_{m-1}} \int_{P_{m-1}} \tilde{\varphi}\, dv_{m-1}\right).$$

Then, comparing the measures $dv(z_1,z_2,\dots,z_m)$ and $dv_s(z_1)dv_{m-1}(z_2,\dots,z_m)$, and using Hölder's inequality, one shows that for any β and any $\varepsilon > 0$, we have

$$\text{(4)} \qquad \int_X e^{-\beta\varphi} dv \le C^{ste}(\varepsilon)\, \mathrm{Max}\left(1, \int_{P_{m-1}\times S_2^*} \exp(-(2+\varepsilon)\beta\varphi) dv_s dv_{m-1}\right).$$

Finally one shows that

$$\text{(5)} \qquad \left(- \int_{P_{m-1}} \tilde{\varphi} dv_{m-1}\right) \le C^{ste}.$$

Choosing $\beta < \frac{1}{2(m+1)}$ and ε sufficiently small in (4) the theorem follows from (4), (3) and (5). This lower bound of $\alpha(X)$ allows us to prove the following corollaries:

Corollary 1.

For all $\lambda < \frac{1}{2m}$ the Monge-Ampère equation Log $M(\varphi) = -\lambda\varphi + f$, admits at least one solution φ C^∞g-admissible on X.

Corollary 2.

For all $\lambda < \frac{1}{2m}$, there exists a Kähler metric g on X satisfying Ricci(g) $\geq \lambda g$.

III- Following an analogous pattern we obtain results concerning the manifold (Y, g^2), where the metric g^2 belongs to the first Chern class of Y and its components are given in some dense open subset by

$$g^2_{\lambda\bar{\mu}} = \partial_{\lambda\bar{\mu}} \text{Log}(1+|z_1|^2+|z_2|^2) + \partial_{\lambda\bar{\mu}} \text{Log}(1+|z_1|^2) + \partial_{\lambda\bar{\mu}} \text{Log}(1+|z_2|^2).$$

Theorem 2.

Let $\varphi \in C^\infty(Y)$ be g^2-admissible. Then for $\alpha < \frac{1}{4}$, we have

$$\int_Y e^{-\alpha\varphi} dv \leq C^{ste} \exp\left(\frac{-\alpha}{V_2}\int_Y \varphi\, dv\right)$$

where dv is the volume element of Y for the metric g^2 and $V_2 = \int_Y dv$.

As in Theorem 1, we have the following two corollaries:

Corollary 3.

For all $\lambda < \frac{3}{8}$, the Monge-Ampère equation Log $M(\varphi) = -\lambda\varphi + f$ admits at least one solution C^∞g-admissible on Y.

Corollary 4.

For all $\lambda < \frac{3}{8}$ there exists a Kähler metric g on Y satisfying Ricci(g) $\geq \lambda$ g.

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On the Hamiltonian formalism defined in the framework of the Poisson calculus

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Presented by J.E. Marsden, F.R.S.C.

Abstract

We introduce a new approach to the Hamiltonian formalism by making use of some basic concepts of the Poisson calculus. Certain interrelations between the contravariant exterior operator σ and the usual exterior differential operator d connected by a Hamiltonian system are presented. The idea also is shown to be applicable in the bi-Hamiltonian case.

1 Introduction

On a Poisson manifold (M, P), equipped with a Poisson bivector P, one can define the Poisson calculus — a natural generalization of the differential calculus of forms. It is based solely on the possibility to extend the Poisson bracket of functions to 1-forms as it was shown for the first time in [1] (see also [2]).

Denote by $\Lambda^k(M)$ and $V^k(M)$ the spaces of differential k-forms and k-vectors (contravariant skew-symmetric tensors) respectively. Then P has an associated homomorphism (isomorphism if P is not-degenerate) $\# : \Lambda^1(M) \to V^1(M)$, defined by

$$\beta(\alpha^\#) = P(\alpha, \beta), \qquad \alpha, \beta \in \Lambda^1(M).$$

This map can be generalized as $\# : \Lambda^k(M) \to V^k(M)$, such that

$$\omega^\#(\alpha_0, \ldots, \alpha_k) = (-1)^k \omega(\alpha_1^\#, \ldots, \alpha_k^\#). \qquad \omega \in \Lambda^k(M).$$

Now one can define a unique $\mathbb{R}$-bilinear operation $\{\ ,\ \} : \Lambda^1(M) \otimes \Lambda^1(M) \to \Lambda^1(M)$ given by the formula

$$\{\alpha, \beta\} = L_{\alpha^\#}\beta - L_{\beta^\#}\alpha - d\beta(\alpha^\#). \tag{1}$$

This operation provides $\Lambda^1(M)$ with a Lie algebra structure such that $\#$ is a Lie algebra homomorphism (isomorphism if P is non-degenerate),

$$\{\alpha, \beta\}^\# = [\alpha^\#, \beta^\#].$$

Furthermore, we can define now the contravariant exterior differential operator $\sigma : V^k(M) \to V^{k+1}(M)$ for $Q \in V^{k+1}(M)$ by

$$\begin{aligned}(\sigma Q)(\alpha_0, \ldots, \alpha_k) = \\ \sum_{i=0}^{k} (-1)^i \alpha_i^\#(Q(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_k)) + \\ \sum_{i<j=0}^{k} (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots \alpha_k),\end{aligned} \tag{2}$$

where $\alpha_i \in \Lambda^1(M)$, and the hat denotes missing arguments. The formula (2) for the operator σ coincides with the analogous expression for the exterior differential of forms d and so has the same algebraic properties (for more details see [4]).

Definition 1.1 *Given a Poisson manifold (M, P). Then a q-vector $Q \in V^q(M)$ is σ-closed if $\sigma Q = 0$ and σ-exact if there exists an $q-1$-vector $R \in V^{q-1}(M)$ such that $Q = \sigma R$.*

Moreover, the action of σ on any $Q \in V^k(M)$ can be expressed as follows [3]:

$$\sigma Q = -[P, Q], \tag{3}$$

where the bracket is that of Schouten [5]. This remarkable relation leads to a new formalism for the Hamiltonian theory. This is the subject of the considerations that follow.

2 The Hamiltonian formalism

Starting from Lichnerowicz's work [6] generalizing symplectic manifolds to Poisson ones, the Hamiltonian formalism has also been modified. Thus the most common representation for a Hamiltonian system is now

$$X_H = PdH, \tag{4}$$

where X_H is a Hamiltonian vector field with H as the corresponding Hamiltonian. While P denotes a Poisson bivector, i.e. a skew-symmetric 2-contravariant tensor field satisfying the condition

$$[P, P] = 0, \tag{5}$$

here the bracket is again the Schouten one. The representation (4) for a Hamiltonian system has the obvious advantage over the classical one:

$$i_{X_H}\omega = -dH. \tag{6}$$

for the former one (4) can be defined in terms of a degenerate Poisson bivector, while the representation (6) — only in terms of a non-degenerate symplectic form ω. Clearly, we always can derive (4) from (6) by using the substitution $P := \omega^{-1}$. However, inverse is not always true due to possible degeneracy of P.

If we apply (3) for $k = 0$, i.e. to a function $f \in V^0(M)$, we shall obtain the following expression

$$\sigma f = -[P, f]. \tag{7}$$

Then the right hand side of (7) in the index notations (on a local coordinate chart) is given by

$$-[P, f]^i = P^{i\alpha}\frac{\partial f}{\partial x^\alpha},$$

(we use the Einstein summation convention) which is exactly the expression for a Hamiltonian vector field (see (4)). This observation leads to the following

Definition 2.1 *Let (M, P) be a Poisson manifold and $H \in V^0(M)$ $(V^0(M) = C^\infty(M))$ a given function. Then the σ-exact vector field X_H defined by H:*

$$X_H = \sigma H \tag{8}$$

is called the Hamiltonian vector field with the energy function H — the Hamiltonian. We call the triple (M, P, X_H) a Hamiltonian system.

Proposition 2.1 *Given a Hamiltonian system (M, P, X_H), then for any $Q \in V^k(M)$ the relation holds*

$$L_{X_H}\sigma Q = \sigma L_{X_H} Q, \tag{9}$$

where σ denotes the contravariant operator associated with the Poisson bivector P.

Proposition 2.2 *For an arbitrary 1-form α defined on a Poisson manifold (M, P) the following statements are equivalent:*
1) a 1-form $\alpha \in \Lambda^1(M)$ is exact: $\alpha = df$ for some $f \in \Lambda^0(M)$,
2) the vector field $\alpha^\# \in V^1(M)$ is σ-exact (i.e., is Hamiltonian): $\alpha^\# = \sigma f$ for the same $f \in V^0(M)$

Definition 2.2 *Let (M, P) be a Poisson manifold, then two functions $f, g \in V^0(M)$ are said to be in involution with respect to P iff*

$$< \sigma f, dg >= - < df, \sigma g >= 0. \tag{10}$$

Here the bracket $< \ , \ >$ denotes the usual contraction between vectors and 1-forms.

Definition 2.3 *A vector field X on a Poisson manifold (M, P) is called locally Hamiltonian if it is σ-closed:*

$$\sigma X = 0. \tag{11}$$

In view of Definition 2.1 and the condition (3) a Poisson bivector is σ-closed:

$$\sigma P = 0, \tag{12}$$

and one can immediately notice a striking resemblance between the condition (12) and the closure of a symplectic form:

$$d\omega = 0 \tag{13}$$

In fact, they are equivalent in the framework of the Hamiltonian formalism, i.e. for a given non-degenerate Poisson bivector we have

$$\sigma P = 0 \Leftrightarrow d\omega = 0, \tag{14}$$

where $\omega^{-1} = P$ and the σ-operator is defined by P.

3 The bi-Hamiltonian case

Consider a double Poisson manifold (M, P_1, P_2) defined by two Poisson bivectors P_1 and P_2 (we assume P_2 is not-degenerate) in the general position, which means the operator $A := P_2^{-1} P_1$ has distinct eigenvalues. Accordingly, there are two σ-operators: σ_1 and σ_2 associated with P_1 and P_2 respectively.

Definition 3.1 *A vector field X_{H_1,H_2} on a double Poisson manifold (M, P_1, P_2) is called bi-Hamiltonian iff it is simultaneously σ_1 and σ_2-exact, i.e.*

$$X_{H_1,H_2} = \sigma_1 H_1 = \sigma_2 H_2, \tag{15}$$

here H_1 and H_2 are the corresponding Hamiltonians. We call the quadruple $(M, P_1, P_2, X_{H_1,H_2})$ a bi-Hamiltonian system.

Definition 3.2 *Two Poisson bivectors P_1 and P_2 are called compatible iff*

$$P_i \text{ is } \sigma_j - \text{closed}, \quad i, j = 1, 2. \tag{16}$$

The compatibility condition (16) leads to integrability in Arnol'd-Liouville's sense of the associated bi-Hamiltonian vector field. This problem has been thoroughly surfaced in works by Magri [7], Magri and Morosi [8], Gel'fand and Dorfman [9, 10] and many others. We call the bi-Hamiltonian vector fields defined by pairs of compatible Poisson bivectors the *bi-Hamiltonian in the Magri-Morosi-Gel'fand-Dorfman (MMGD) sense vector fields.*

Proposition 3.1 *Given a bi-Hamiltonian in the MMGD sense system $(M, P_1, P_2, X_{H_1,H_2})$, then for any $Q \in V^k(M)$ the relation holds*

$$\sigma_1 L_{X_{H_1,H_2}} \sigma_2 Q = \sigma_2 L_{X_{H_1,H_2}} \sigma_1 Q. \tag{17}$$

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ENHANCED EDDY CURRENT MATHEMATICAL MODELS USING THE PHASE-SHIFTED FIELDS OF TWO COILS

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Presented by K.B. Ranger, F.R.S.C.

ABSTRACT. Analytical solutions are obtained for the problem of the interaction of a pair of phase-shifted electromagnetic fields generated by two coaxial single-turn coils carrying currents $I_1 e^{j\omega t}$ and $I_2 e^{j(\omega t+\psi)}$, respectively, where $\psi \neq 0$. Five geometrically different media are considered. Numerical results show that ψ is the most important parameter. If the values of ψ and of the other parameters are chosen properly, then the curve representing the change in impedance can lie in any of the four quadrants of the complex plane. These results can be used for developing more sensitive and more selective eddy current testing methods.

RÉSUMÉ. On obtient une expression analytique pour l'interaction des champs electromagnétiques produits par deux bobines à un seul tour sous courants $I_1 e^{j\omega t}$ et $I_2 e^{j(\omega t+\psi)}$, où $\psi \neq 0$. Les bobines sont situées dans l'espace libre relatif à cinq media de formes géometriques différentes. Les résultats numériques montrent que ψ est le paramètre le plus important. Pour un choix judicieux des valeurs de ψ et des autres paramètres, la courbe représentant le changement d'impédance peut se trouver dans n'importe quel des quadrants du plan complexe. Ces résultats peuvent servir au développement de méthodes de détection au moyen des courants de Foucault plus sensibles et plus sélectives.

1. Introduction. In this Note, we present analytical solutions to the problem of the interaction of the phase-shifted fields of a pair of coaxial single-turn coils (Coil 1 and Coil 2 of radius R_1 and R_2) carrying currents $I_1 e^{j\omega t}$ and $I_2 e^{j(\omega t+\psi)}$, respectively. The coils are situated in free space, M_0, (a) above a conducting half-space, M_1, (b) above a two-layer medium, M_1 and M_2, (c) coaxially inside a conducting nonmagnetic tube, M_1, (d) outside a uniformly conducting nonmagnetic ball, M_1, the coil axes passing through the centre of the ball. Lastly in case (e), parallel rectangular coils are approximated by two pairs of infinitely long parallel wires (L_1, L_2 and L_3, L_4) in M_0 and parallel to a uniform nonmagnetic conducting half-space, M_1. These five cases are shown in the left parts of Figs. 1–5. The case of non-shifted fields is treated in [1] and [2].

We present formulae for the induced change in impedance, Z^{ind}, in each of the five cases. The formula of case (c) is derived. Cases (a) and (b) are treated in greater detail in [3].

The following standard notation is used. The magnetic constant is μ_0. For $i = 1, 2$, μ_i and σ_i are the relative magnetic permeability and conductivity of M_i, respectively. In the cases (c), (d) and (e), the formulae have been obtained with $\mu_1 = 1$.

Key words and phrases. Eddy current nondestructive evaluation; phase shifted fields; Hankel transform, Fourier cosine transform, Fourier–Legendre transform.

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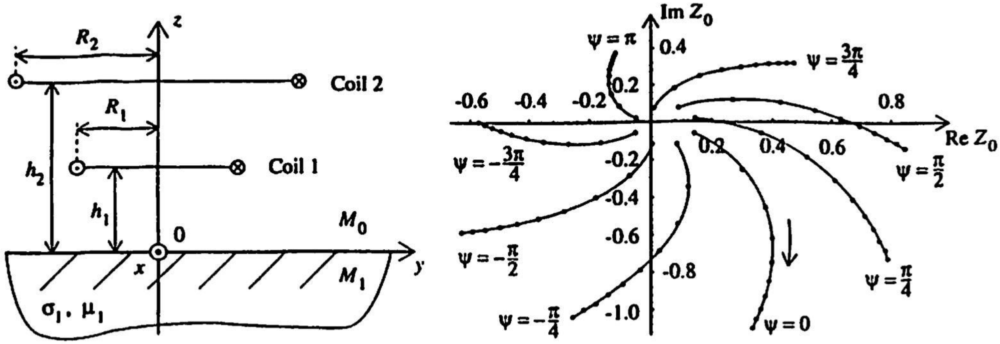

FIGURE 1. In case (a), (left) coils above uniform conducting half-space, M_1, and (right) plot of $Z_0(\beta, \psi)$ for $\alpha = 0.1$, $\mu_1 = 1$, $H = 0.9$, $\rho = 1$ and $I = 2$.

The values of the phase difference, ψ, and the other parameters of the problem have been chosen such that the curve representing the change in impedance can be situated in any quadrant of the complex plane as shown in the right parts of the five figures.

2. Case (a). Using the dimensionless variables

$$\beta = R_1\sqrt{\omega\sigma_1\mu_0\mu_1}, \qquad \alpha = \frac{h_1}{R_1}, \qquad H = \frac{h_2}{h_1}, \qquad \rho = \frac{R_2}{R_1}, \qquad I = \frac{I_2}{I_1},$$

we have the following expression,

$$Z^{\text{ind}} = \omega\mu_0\pi R_1 Z_0, \tag{1}$$

for the induced change in impedance of Coil 1, where

$$Z_0 = j\beta\int_0^\infty \frac{s\mu_1 - \sqrt{s^2+j}}{s\mu_1 + \sqrt{s^2+j}}\, J_1^2(\beta s)\, e^{-2\alpha\beta s}\, ds$$
$$+ j\beta I\rho e^{j\psi}\int_0^\infty \frac{s\mu_1 - \sqrt{s^2+j}}{s\mu_1 + \sqrt{s^2+j}}\, J_1(\beta s) J_1(\beta\rho s)\, e^{-\alpha\beta(1+H)s}\, ds. \tag{2}$$

3. Case (b). Using the following dimensionless variables

$$\alpha = \frac{h_1}{R_1}, \quad \beta = R_1\sqrt{\omega\sigma_1\mu_0\mu_1}, \quad \gamma = \frac{d_1}{R_1}, \quad \delta = \frac{\sigma_2}{\sigma_1}, \quad H = \frac{h_2}{h_1}, \quad I = \frac{I_2}{I_1}, \quad \rho = \frac{R_2}{R_1},$$

where d_1 is the thickness of layer M_1, we obtain the change in impedance of Coil 1 in the form

$$Z^{\text{ind}} = \omega\pi\mu_0 R_1 Z_0, \tag{3}$$

where, for $\mu_1 = \mu_2 = 1$,

$$Z_0 = j\beta\int_0^\infty D(s) J_1^2(\beta s)\, e^{-2\alpha\beta s}\, ds + j\beta I\rho e^{j\psi}\int_0^\infty D(s) J_1(\beta s) J_1(\beta\rho s)\, e^{-\alpha\beta(1+H)s}\, ds,$$

$$D(s) = \frac{A_+B_- + A_-\left(s + \sqrt{s^2+j\delta}\right) e^{-2\beta\gamma\sqrt{s^2+j}}}{A_+B_+ + A_-B_-\, e^{-2\beta\gamma\sqrt{s^2+j}}},$$

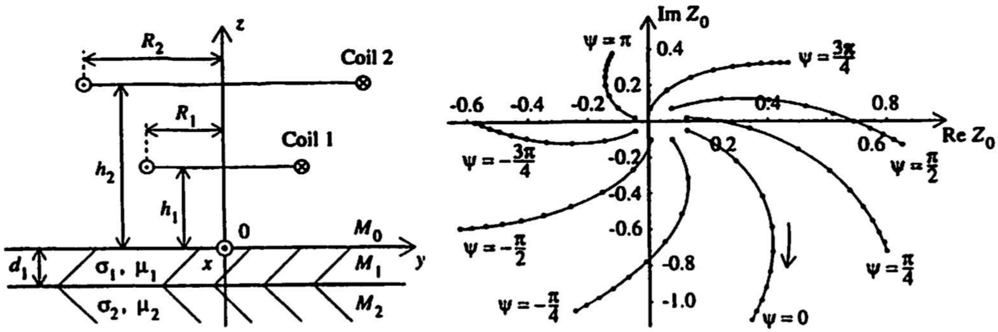

FIGURE 2. In case (b), (left) coils above conducting two-layer medium M_1 and M_2, and (right) $Z_0(\beta, \psi)$ for $\alpha = 0.1$, $\mu_1 = \mu_2 = 1$, $H = 0.9$, $\rho = 1$, $\gamma = 0.1$, $\delta = 0.8$ and $I = 2$.

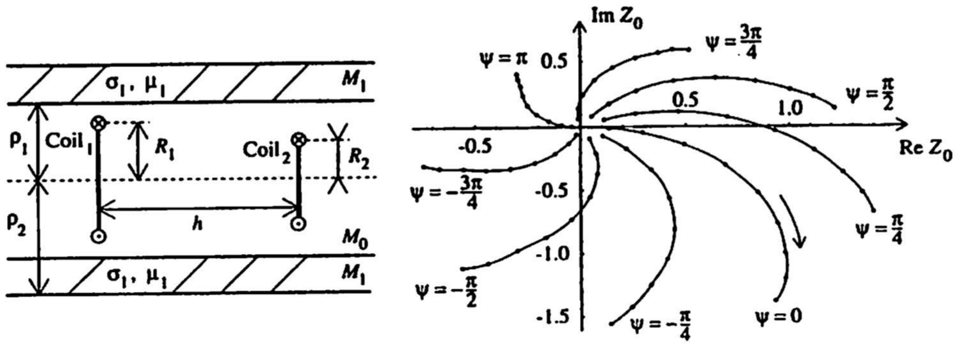

FIGURE 3. In case (c), (left) coils coaxially inside a conducting tube, and (right) plot of $Z_0(\beta, \psi)$ for $r_0 = 0.9$, $\mu_1 = 1$, $H = 0.2$, $R = 1$, $\rho = 1.2$ and $I = 2$.

with the abbreviations $A_\pm = \sqrt{s^2 + j} \pm \sqrt{s^2 + j\delta}$ and $B_\pm = s \pm \sqrt{s^2 + j}$.

4. Case (c). Suppose that Coil 1, of radius R_1, is coaxially located inside a conducting nonmagnetic tube, M_1, of inner and outer radii ρ_1 and ρ_2, respectively. The conductivity, σ_1, of the tube is constant. Coil 2, of radius R_2, is coaxially located with respect to Coil 1 at distance h from the latter, as shown in Fig. 3 (left).

If Coil 2 is absent, the induced vector potential in M_0 is given by

$$A_1(r, z) = \frac{\mu_0 I_1 \rho_1^2 r_0}{\pi} \int_0^\infty C(\lambda) I_1(\lambda r_0) I_1(\lambda r) \cos \lambda z \, d\lambda, \tag{4}$$

in terms of the cylindrical coordinates (r, φ, z) with the origin at the centre of Coil 1. Here r and z are dimensionless coordinates such that the inner radius, ρ_1, of the tube is chosen as the unit of length. The function

$$C(\lambda) = A(\lambda)/B(\lambda), \tag{5}$$

is given by

$$A(\lambda) = [qK_0(q\rho)K_1(\lambda\rho) - \lambda K_0(\lambda\rho)K_1(q\rho)]\,[qK_1(\lambda)I_0(q) + \lambda K_0(\lambda)I_1(q)] + [qI_0(q\rho)K_1(\lambda\rho) + \lambda K_0(\lambda\rho)I_1(q\rho)]\,[\lambda K_0(\lambda)K_1(q) - qK_1(\lambda)K_0(q)],$$

$$B(\lambda) = [qK_0(q\rho)K_1(\lambda\rho) - \lambda K_0(\lambda\rho)K_1(q\rho)]\,[\lambda I_0(\lambda)I_1(q) - qI_1(\lambda)I_0(q)] + [qI_0(q\rho)K_1(\lambda\rho) + \lambda K_0(\lambda\rho)I_1(q\rho)]\,[\lambda I_0(\lambda)K_1(q) + qI_1(\lambda)K_0(q)],$$

and

$$r_0 = \frac{R_1}{\rho_1}, \qquad \rho = \frac{\rho_2}{\rho_1}, \qquad q = \sqrt{\lambda^2 + j\beta^2}, \qquad \beta = \rho_1\sqrt{\omega\sigma_1\mu_0}.$$

The functions $I_\nu(s)$ and $K_\nu(s)$, for $\nu = 0, 1$, are the modified Bessel functions of the first and second kinds, respectively.

If Coil 1 is absent and Coil 2 is the only source of primary field, a similar formula can be obtained for $A_2(r, z)$ by replacing I_1, r_0 and z in (4) by $I_2 e^{j\psi}$, r_1 and $z - h$, respectively, where $r_1 = R_2/\rho_1$.

By the superposition principle, the total induced vector potential in region M_0 is the sum $A = A_1 + A_2$, that is,

$$A(r, z) = \frac{\mu_0 I_1 \rho_1^2 r_0}{\pi} \int_0^\infty C(\lambda) I_1(\lambda r_0) I_1(\lambda r) \cos\lambda z \, d\lambda + \frac{\mu_0 I_2 e^{j\psi} \rho_1^2 r_1}{\pi} \int_0^\infty C(\lambda) I_1(\lambda r_1) I_1(\lambda r) \cos\lambda(z - h)\, d\lambda, \tag{6}$$

where $C(\lambda)$ is given by (5).

Using formula (6) and computing the impedance change of Coil 1, we obtain

$$Z^{\text{ind}} = 2\mu_0 \rho_1^2 \omega r_0^2 Z_0,$$

where

$$Z_0 = j\int_0^\infty C(\lambda) I_1(\lambda r_0)[I_1(\lambda r_0) + IR e^{j\psi} I_1(\lambda r_1)\cos\lambda H]\, d\lambda, \tag{7}$$

$H = h/\rho_1$, $I = I_2/I_1$ is the ratio of the current amplitudes in the coils and $R = R_2/R_1$ is the ratio of the radii.

5. Case (d). In this case, we obtain an expression for the change in impedance of Coil 1 in the form

$$Z^{\text{ind}} = 2\pi\omega\rho_c^2 R_1^2 \mu_0 \sin\theta_1 Z_0,$$

where

$$Z_0 = j\sum_{n=1}^{\infty} \frac{1}{2n(n+1)} \left(\frac{\rho_c}{\rho_1}\right)^{2n} P_n^{(1)}(\cos\theta_1)\Bigg[\sin\theta_1 P_n^{(1)}(\cos\theta_1) + I e^{j\psi}\left(\frac{\rho_1}{\rho_2}\right)^{n-1} \sin\theta_2 P_n^{(1)}(\cos\theta_2)\Bigg] \left[\frac{(2n+1)J_{n+1/2}(k\rho_c)}{k\rho_c J_{n-1/2}(k\rho_c)} - 1\right], \tag{8}$$

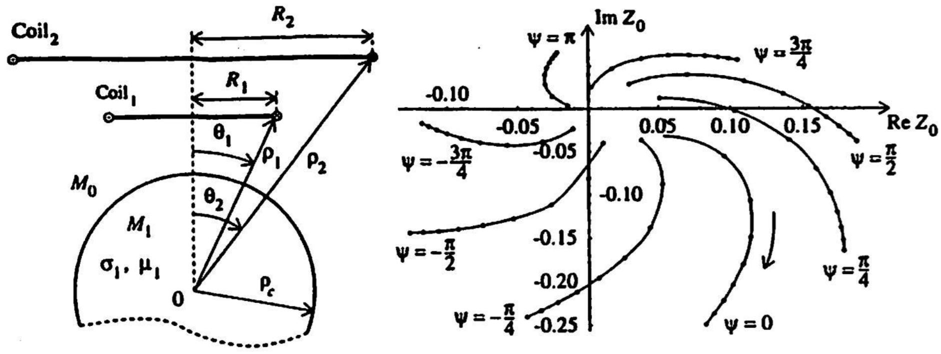

FIGURE 4. In case (d), (left) coils outside ball M_1 with coaxis going through centre of M_1, and (right) plot of $Z_0(\beta,\psi)$ for $\mu_1 = 1$, $\rho_1 = 1.2$, $\rho_2 = 1.3$, $\rho_c = 0.9$, $R = R_2/R_1 = 1$ and $I = 2$.

$k = \beta\sqrt{-j}$, $\beta = R_1\sqrt{\omega\sigma\mu_0}$ and $J_\nu(z)$ is the Bessel function of the first kind of order ν and $P_n^{(1)}(\xi)$ is the associated Legendre function of order 1. All the parameters in (8) are dimensionless and R_1 is chosen as length scale.

6. Case (e). By determining the induced impedance change per unit length, we obtain the formula

$$Z^{\text{ind}} = \frac{j\omega\mu_0}{2\pi} Z_0,$$

where

$$Z_0 = \int_0^\infty \frac{s-\sqrt{s^2+j}}{s+\sqrt{s^2+j}}\,\frac{e^{-2\alpha\beta s}}{s}\Big\{2 - 2\cos\beta s + I\,e^{j\psi}\,e^{-\beta s(\alpha_1-\alpha)}\big[\cos\beta s(y_1-y_3) - \cos\beta s(y_1-y_4) - \cos\beta s(y_2-y_3) + \cos\beta s(y_2-y_4)\big]\Big\}\,ds, \tag{9}$$

the distance, $c = y_2 - y_1$, between wires L_1 and L_2 is chosen as the unit of length,

$$\alpha = \frac{h}{c}, \qquad \alpha_1 = \frac{h_1}{c}, \qquad I = \frac{I_2}{I_1}, \qquad \beta = c\sqrt{\omega\sigma\mu_0},$$

and the variables y_1, y_2, y_3 and y_4 are measured in units of c. Without loss of generality, one can assume that $y_1 = -0.5$ and $y_2 = 0.5$.

7. Numerical results and discussion. The change in impedance, Z_0, was computed for different values of the parameters. In the figures, the ten dots on each curve cluster in the direction of increasing β as indicated by an arrow.

It was found that the most important parameter is the phase shift, ψ. This fact can be explained as follows. On the one hand, the accumulation of energy in Coil 1 depends on the phase difference, ψ, between the two currents, and affects the real part of Z_0. On the other hand, the demagnetizing action of the medium on both coils also depends on ψ but it affects the imaginary part of Z_0. The curve representing the change in impedance can be located in any quadrant of the complex plane (in contrast with the case $\psi = 0$) provided the parameters of the problem (especially ψ) are suitably chosen.

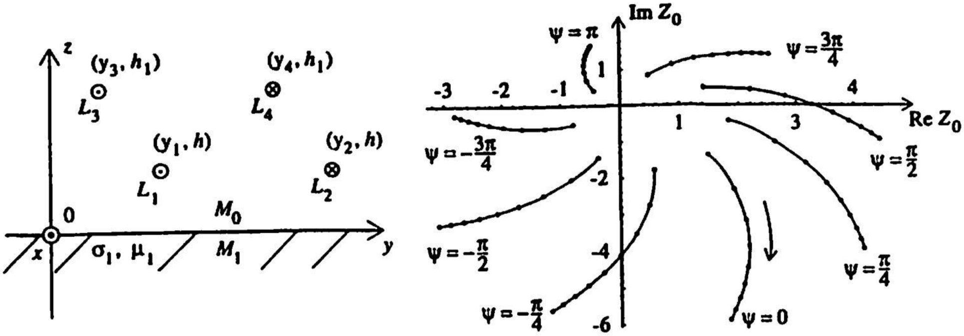

FIGURE 5. In case (e), (left) two of double lines, L_1, L_2 and L_3, L_4, and conducting half-space, M_1, and (right) plot of $Z_0(\beta, \psi)$ for $\alpha = 0.1$, $\mu_1 = 1$, $H = 0.9$, $\rho = 1$ and $I = 2$.

The parameter I is also important since the accumulated energy and the medium demagnetizing action also depend on its value.

These peculiarities of Z_0 can be used for developing eddy current testing methods with higher testing sensitivity and finer selectivity.

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