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Homogeneous ultrametric spaces, semi-valued groups and fields

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Abstract

This note is devoted to the study of homogeneous ultrametric spaces, in particular those endowed with an appropriate group-action. We study Hahn spaces, immediate and dense extensions. Then we apply the theory developed so far to study semi-valued groups and skewfields extending the classical results on fields with Krull valuations.

1 Homogeneous Ultrametric Spaces

A. Definition of homogeneous ultrametric spaces

An ultrametric space (X, d, Γ) is said to be *homogeneous* when the following condition is satisfied: for all $\gamma \in \Gamma \setminus \{0\}$, $\alpha \in \mathrm{Max}(\mathbf{B}_\gamma)$, $D, E \in X/\mathbf{B}_\gamma$, we have $\mathrm{card}(D/\alpha) = \mathrm{card}(E/\alpha)$. It suffices that for all D, E as above, there is a bijection $\varphi = \varphi_{D,E} : D \to E$ such that $d(\varphi(x), \varphi(y)) = d(x, y)$, for all $x, y \in D$.

B. Examples

An *action* of a group G on an ultrametric space (X, d, Γ) is a homomorphism λ from G onto the group $\mathrm{Aut}(X, d, \Gamma)$ of distance preserving self-bijections of X. If there exists a faithful and transitive action of a group on (X, d, Γ), then the space is homogeneous. In this case, we say that $(G, \lambda, X, d, \Gamma)$ is a *group-space*. An important special case occurs when $X = G$ is a group and the action λ is by translation. Another important case occurs when $\mathrm{Aut}(X, d, \Gamma)$ is transitive.

The following construction yields homogeneous spaces. Let $(E_v)_{v \in V}$ be a non-empty family of sets E_v, each one with at least two elements. Let $P = \prod_{v \in V} E_v$; for $f, g \in P$, let $d(f, g) = \{v \in V \mid f(v) \neq g(v)\} \in \mathcal{P}(V)$ (Boolean algebra of subsets of V). Then $(P, d, \mathcal{P}(V))$ is an ultrametric space on which the group of automorphisms acts transitively, thus giving a group-space. This leads to interesting special cases. Taking $E_v = E$ for all $v \in V$, respectively $E_v = \{0, 1\}$ for all $v \in V$, we have the group-spaces E^V and $\mathcal{P}(V)$ (with the distances already introduced). Similar examples are obtained by taking any Boolean algebra A, with the distance $d : A \times A \to A$ given by the sum $d(a, b) = a + b = (a \wedge b') \vee (a' \wedge b)$. In the same vein, we have the space $C(V, E)$ of continuous functions from the topological space V to the discrete space E (having more than one point) and distance having values on the Boolean algebra of open and closed subsets of V.

Hahn spaces are also examples of group-spaces. We recall their definition ([10]). Let $(V, \leq)$ be a partially ordered set, let $(E_v)_{v \in V}$ be a family of sets, each one having more than one element, let $0 = (0_v)_v$ where $0_v \in E_v$ for all $v \in V$. If $f \in P = \prod_{v \in V} E_v$, let $\mathrm{supp}(f) = \{v \in V \mid f(v) \neq 0_v\}$. The Hahn space H_0 consists of all the $f \in P$ having noetherian support (i.e. the support does not contain any infinite ascending chain). Let $\mathcal{A}(V)$ be the set of all antichains of V (i.e. trivially ordered subsets of V). $\mathcal{A}(V)$ is naturally ordered as indicated in [10] and $d : H_0 \times H_0 \to \mathcal{A}(V)$ is defined by $d(f,g) = \text{Max}\{v \in V \mid f(v) \neq g(v)\}$. It may be shown that $\text{Aut}(H_0)$ acts transitively on H_0, thus giving rise to a group-space.

C. The pointed Hahn space

Let (X, d, Γ) be a homogeneous ultrametric space. In [10] we considered its Hahn space $(H_0, d, \mathcal{A}(\mathcal{V}(X)))$ associated to the skeleton $(\mathcal{E}(X), \mathcal{V}(X), \text{gr}, \sim)$ of the given space and to $0 = (0_v)_{v \in \mathcal{V}(X)}$. Given $x_0 \in X$ we define the pointed Hahn space. For each $(\alpha,\gamma) \in \mathcal{V}(X)$ let $\mathcal{E}^0(\alpha,\gamma) = \{(C, B_\gamma(x_0)) \mid C \in B_\gamma(x_0)/\alpha\}$. For each $v = (\alpha, \underset{\gamma}{=})$ let $0_v = ([x_0]_\alpha, B_\gamma(x_0))$. The associated Hahn space is denoted by $(H^0, d, \mathcal{A}(\mathcal{V}(X)))$ and it is called *the pointed Hahn space at x_0*, associated to (X, d, Γ). It is a solid spherically complete ultrametric space on which the group $\text{Aut}(H^0)$ acts transitively, thus it is a group-space. If $(H^1, d, \mathcal{A}(\mathcal{V}(X)))$ is the pointed space at $x_1 \in X$, and if (X, d, Γ) is a homogeneous space, then H^0, H^1 are isomorphic.

We compare the Hahn space H_0 with the pointed Hahn space H^0, when for every $v = (\alpha, \underset{\gamma}{=}) \in \mathcal{V}(X)$ we have $([x_0]_\alpha, B_\gamma(x_0)) \in \Omega(v)$ and thus it is identified with 0_v.

(1.1) *If (X, d, Γ) is a homogeneous ultrametric space, there is a contracting transformation from H_0 to H^0, which is the identity on H^0.*

D. Comparison of the space with its associated pointed Hahn space

We obtain the following:

(1.2) Theorem: *Let (X, d, Γ) be a homogeneous ultrametric space, let $x_0 \in X$. There exists an injective expanding transformation from (X, d, Γ) to $(H^0, d, \mathcal{A}(\mathcal{V}(X)))$.*

This result will be made more precise when $(\Gamma, \leq)$ is totally ordered.

E. Immediate extensions of homogeneous ultrametric spaces

We first obtain the following results:

(1.3) *If* $(X, d, \Gamma)\, im \prec (X', d', \Gamma)$ *and* (X', d', Γ) *is homogeneous, then so is* (X, d, Γ).

(1.4) *The relation of "immediate extension" in the class of homogeneous ultrametric spaces is inductive.*

Then, we prove the following remarkable result:

(1.5) *Let* $(X, d, \Gamma)\, im \prec (X', d', \Gamma)$. *If* (X', d', Γ) *is homogeneous, then* $\mathrm{card}(X') \leq \mathrm{card}(X^{\Gamma})$.

This leads to:

(1.6) Theorem: *Every homogeneous ultrametric space has an immediate extension which is homogeneous and maximal.*

For group-spaces, $(G, \lambda, X, d, \Gamma) \prec (G', \lambda', X', d', \Gamma')$ means that $(X, d, \Gamma) \prec (X', d', \Gamma')$, G is a subgroup of G' and for every $g \in G$, λ'_g restricted to X is equal to the automorphism λ_g of (X, d, Γ).

(1.7) *The relation of extension in the class of group-spaces is transitive and inductive.*

(1.8) *Every group-space has an immediate group-space extension which is maximal (among immediate group-space extensions).*

F. Dense extensions of homogeneous ultrametric spaces

The relation of "dense extension" in the class of homogeneous spaces is transitive but not inductive. However:

(1.9) *The relation of "immediate and dense extension" in the class of homogeneous spaces is inductive.*

Therefore:

(1.10) *Every homogeneous ultrametric space has a dense and immediate extension which is homoegeneous and maximal.*

We have also:

(1.11) *Every group-space has a dense group-space extension which is maximal.*

G. Case when $(\Gamma, \leq)$ is totally ordered

We assume now that $(\Gamma, \leq)$ is totally ordered. As a consequence we obtain more specific results.

With previous notations:

(1.12) Theorem: *If (X, d, Γ) is a homogeneous space and $(\Gamma, \leq)$ is totally ordered, there is a homogeneous space $(\hat{X}, d, \Gamma)$, which is isomorphic to (X, d, Γ) and such that the pointed Hahn space (H^0, d, Γ) associated to (X, d, Γ) is a maximal immediate extension of $(\hat{X}, d, \Gamma)$.*

Similarly:

(1.13) *With the same hypotheses and notations, (X, d, Γ) has a dense homogeneous extension which is complete and isomorphic to a subspace of (H^0, d, Γ).*

We get also the following characterization:

(1.14) *Let (X, d, Γ) be an ultrametric space with $(\Gamma, \leq)$ totally ordered. Then the following conditions are equivalent:*

a) (X, d, Γ) *is homogeneous.*

b) There exists an immediate extension of (X, d, Γ) *which is a group-space.*

2 Semi-valued and ultrametric groups

A. Definition of semi-valued and ultrametric groups

Let $(G, +)$ be a group, $(\Gamma, \leq)$ a partially ordered set with smallest element 0. A surjective mapping $v : G \to \Gamma$ is a *semi-valuation* when the following conditions are satisfied:

1. $v(x) = 0$ if and only if $x = 0$;
2. $v(-x) = v(x)$;
3. if $v(x) \leq \gamma$ and $v(y) \leq \gamma$ then $v(x + y) \leq \gamma$.

If $(\Gamma, \leq)$ is totally ordered, v is called a *valuation*. The semi-valuation v is said to be *invariant* when $v(x + y) = v(y + x)$.

An ultrametric space (G, d, Γ) where $(G, +)$ is a group and the distance is left-invariant, i.e. $d(z + x, z + y) = d(x, y)$, is called an *ultrametric group*. If moreover, $d(x + z, y + z) = d(x, y)$, we say that the distance is invariant.

Semi-valued and ultrametric groups correspond bijectively one to another, as follows: given v, let $d(x, y) = v(-x + y)$; given d, let $v(x) = d(x, 0)$.

Every ultrametric space is a group-space, where G acts on itself by translation.

B. Examples

The examples of ultrametric spaces of §1B yield ultrametric groups with invariant distance, when each E_v is a group $\neq \{0\}$. In particular, the Hahn space $(H_0, d, \mathcal{A}(V))$ is also an ultrametric group.

Another kind of examples is the following. Let G be a group, let $P(g)$ be the subgroup generated by $g \in G$. let $\Gamma = \{P(g) \mid g \in G\}$, ordered by inclusion. Defining $v : G \to \Gamma$ by $v(g) = P(g)$, then (G, v, Γ) is a semi-valued group.

C. The skeleton

Let (G, d, Γ) be an ultrametric group, and (G, v, Γ) the corresponding semi-valued group. A subgroup C of G is said to be *(v-)convex* (or *(d-)convex*), when the following condition is satisfied: if $x \in C$, $v(y) \leq v(x)$, then $y \in C$.

Let $\mathcal{C}$ denote the family of convex subgroups, ordered by inclusion. Let $\mathcal{P}$ denote the family of convex subgroups which are principal, i.e. generated by one element of G. A *jump* is a pair (C, P) when $P \in \mathcal{P}$, $C \in \mathcal{P}$ and C is maximal properly contained in P. It is easily seen that compatible equivalence relations of G correspond to convex subgroups. This gives an order-preserving bijection from $\mathcal{V}(G)$ to the set of jumps of G. Moreover, if v is invariant, then for each $(\alpha, \equiv_\gamma) \in \mathcal{V}(G)$, $\mathcal{E}(\alpha, \equiv_\gamma)$ is a group, and $\mathcal{E}^0(\alpha, \equiv_\gamma) = \{(C, B_\gamma(0)) \mid C \in B_\gamma(0)/\alpha\}$ is a subgroup called the *residue class group at* $(\alpha, \equiv_\gamma)$. Thus for an invariant semi-valuation, the local skeleton is fully determined by the jumps and the residue class groups.

D. Comparison of the ultrametric group with its canonical pointed Hahn space

Let (G, d, Γ) be an ultrametric group with invariant distance. Then $(H^0, d, \mathcal{A}(\mathcal{V}(G)))$ is also an ultrametric group with invariant distance.

For the main theorem which follows, we assume that G *is a divisible abelian group* and $d(\frac{1}{n}x, \frac{1}{n}y) = d(x, y)$ (where n is any positive integer). The following lemma, dating back to Banaschewski [1] is essential:

(2.1) *There exists a coherent family $\Sigma = (S_\beta)_{\beta \in \Xi(G)}$, where S_β is a set of representatives of G/β satisfying:*

1. *S_β is a subgroup of G;*

2. *$G = [0]_\beta \ominus S_\beta$.*

(2.2) Theorem: *There exists an injective expanding transformation $(\theta, \underline{\theta})$ from (G, d, Γ) to the pointed Hahn space with operation defined by means of Σ, such that θ is a group-homomorphism.*

E. Immediate extensions of ultrametric groups

We consider extensions and immediate extensions in the class of ultrametric groups, or equivalently, in the class of semi-valued groups.

The relation "immediate extension" in the class of ultrametric groups, resp. semi-valued groups, is transitive and inductive. Hence:

(2.3) Theorem: *Every ultrametric group, resp. semi-valued group, has an immediate extension which is maximal.*

(2.4) *If the ultrametric group is spherically complete, then it has no proper group extension.*

F. Dense extensions of ultrametric spaces

We consider dense extensions of ultrametric groups, resp. semi-valued groups. Again, the relation "dense extension" in the class of ultrametric groups, resp. semi-valued groups, is transitive and inductive.

(2.5) Theorem: *Every ultrametric group, resp. semi-valued group, has a dense extension which is maximal.*

(2.6) *If the ultrametric group is complete, then it is not dense in any proper ultrametric group extension.*

G. Case when $(\Gamma, \leq)$ is totally ordered

If $(\Gamma, \leq)$ is totally ordered, we have more specific results. With the previous notations:

(2.7) Theorem: *If (G, d, Γ) is an ultrametric group and $(\Gamma, \leq)$ is totally ordered, there is an ultrametric group $(\hat{G}, d, \Gamma)$, isomorphic to (G, d, Γ) and such that the pointed ultrametric Hahn group (H^0, d, Γ) is a maximal immediate group extension of $(\hat{G}, d, \Gamma)$.*

(2.8) Theorem: *Let (G, d, Γ) be an ultrametric group with $(\Gamma, \leq)$ totally ordered. Then:*

1. *(G, d, Γ) has a maximal immediate ultrametric group extension, which moreover is spherically complete.*

2. *(G, d, Γ) is spherically complete if and only if it does not have any proper immediate ultrametric group extension.*

(2.9) *With the same hypotheses and notations, (G, d, Γ) has a dense ultrametric group extension which is complete and isomorphic to an ultrametric subgroup of the ultrametric pointed Hahn group (H^0, d, Γ).*

3 Semi-valued and ultrametric skewfields

A. Definition of semi-valued and ultrametric skewfields

Let K be a skewfield, K_+ its additive group, $K^\bullet = K \setminus \{0\}$ the multiplicative group of non-zero elements. Let $(\Gamma, \leq)$ be a partially ordered set with smallest element 0. Assume that $\Gamma^\bullet = \Gamma \setminus \{0\}$ is an ordered monoid with neutral element ε.

A *semi-valuation* is a surjective map $v : K \to \Gamma$ satisfying the following conditions:

1. v is a semi-valuation of K_+;

2. $v(1) = \varepsilon$;

3. $v(xy) \leq v(x)v(y)$ for all x, $y \in K$.

If $v(xy) = v(x)v(y)$ for all x, y, then the semi-valuation is said to be *strict*. Then, it follows that $\Gamma^{\bullet}$ is a group. If $(\Gamma, \leq)$ is totally ordered, a strict semi-valuation is called a *valuation*.

If K, Γ are as above, the ultrametric space (K, d, Γ) is an *ultrametric skewfield* if (K_+, d, Γ) is an ultrametric group, $d(1,0) = \varepsilon$, and $d(tx, ty) \leq d(t,0) \cdot d(x,y)$, $d(xt, yt) \leq d(x,y) \cdot d(t,0)$. In the case of equalities in the above relations, we have a *strict* ultrametric skewfield. Then $\Gamma^{\bullet}$ is a group.

As in the case of groups, semi-valued skewfields correspond canonically to ultrametric skewfields and strictly semi-valued skewfields correspond to strictly ultrametric skewfields.

B. Examples

The classical examples are fields endowed with a Krull valuation and skewfields endowed with a Schilling valuation (see [7], [14]). Jaffard, Gilmer and others studied strictly semi-valued fields, as a generalization of Krull valued fields (see [5], [3]).

More examples are given by skewfields of power series with coefficients in a field and exponents in a totally ordered group (considered by Hahn [4], Krull [7], Neumann [8]) and also by fields of generalized power series (Elliott & Ribenboim [1]).

For this last example, let F be a field, $(S, +, \leq)$ a (partially) ordered abelian group, let K be the ring of generalized power series: all $f : S \to F$ having noetherian and narrow support.

It was shown in [2] that K is a field if and only if the group S is torsion-free and the order $\leq$ is subtotal (that is, if s, $t \in S$ there exists an integer $k \geq 1$ such that $ks \leq kt$ or $kt \leq ks$). Let Γ be the set of finite antichains of S ordered as follows: if γ, $\delta \in \Gamma$, $\gamma \leq \delta$ when for every $s \in \gamma$ there exists $t \in \delta$ such that $s \leq t$.

It was shown in [13] that it is possible to define a natural multiplication on $\Gamma \setminus \{\emptyset\}$, which makes it an ordered monoid. Let $v(f) = \operatorname{Max}\operatorname{supp}(f)$ for every $f \in K$. Then (K, v, Γ) is a semi-valued field. It is strict if and only if the order on S is total. With the associated distance, we obtain the ultrametric field (K, d, Γ). We have indicated in [11] the necessary and sufficient condition for the ultrametric field to be spherically complete.

C. The skeleton

We supplement the results obtained in §2, when (K, v, Γ) is a semi-valued skewfield — or for the corresponding ultrametric skewfield.

(3.1)

1. $[0]_{\underset{\varepsilon}{\equiv}} = B_\varepsilon(0)$ *is a ring. If* $\alpha \in \equiv(K)$, *then* $[0]_\alpha$ *is a right and left* $[0]_{\underset{\varepsilon}{\equiv}}$*-module.*

2. *If the distance is strict, then* $[0]_{\underset{\varepsilon}{\equiv}}$ *is invariant under inner automorphisms, every left (or right)* $[0]_{\underset{\varepsilon}{\equiv}}$*-submodule of* K *is* v*-convex, so equal to some* $[0]_\alpha$, *with* $\alpha \in \equiv(K)$.

Thus, if (K, v, Γ) is a strictly semi-valued skewfield, the following sets coincide:

(a) the set of all v-convex subgroups of K_+;

(b) the set of all two-sided $[0]_{\underset{\varepsilon}{\equiv}}$-submodules of K_+;

(c) the set of all left $[0]_{\underset{\varepsilon}{\equiv}}$-submodules of K_+;

(d) the set of all right $[0]_{\underset{\epsilon}{\equiv}}$-submodules of K_+.

The set $[0]_{\underset{\epsilon}{\equiv}} = B_\epsilon(0)$ is called the *semi-valuation ring* or the *unit ball* of (K, v, Γ); also denoted by B.

If the distance or semi-valuation is strict, then $\mathrm{Max}(\underset{\epsilon}{\equiv})$ corresponds to the set $\Omega(K)$ of all maximal two-sided ideals of the semi-valuation ring. For each $\alpha \in \mathrm{Max}(\underset{\epsilon}{\equiv})$ the skewfield $B_\epsilon(0)/[0]_\alpha$ is called the *residue skewfield* of (K, v, Γ) at α.

If the distance is strict, there is a natural bijection between the sets $\mathrm{Max}(\underset{\gamma}{\equiv})$, $\mathrm{Max}(\underset{\gamma'}{\equiv})$ (for any γ, $\gamma' \in \Gamma \setminus \{0\}$), as easily seen. This implies:

(3.2) *If (K, d, Γ) is a strictly ultrametric skewfield, for each $(\alpha, \underset{\gamma}{\equiv}) \in \mathcal{V}(K)$ there exists $M \in \Omega(K)$ such that $\mathcal{E}(\alpha, \underset{\gamma}{\equiv}) \cong K/M$ (as additive groups), and $\mathcal{E}^0(\alpha, \underset{\gamma}{\equiv}) \cong B/M$ (as rings); in particular $\mathcal{E}^0(\alpha, \underset{\gamma}{\equiv})$ is a skewfield.*

Thus, the local skeleton is fully determined by Γ and the residue skewfields.

D. Immediate extensions of ultrametric skewfields

We consider extensions and immediate extensions in the classes of semi-valued skewfields and of ultrametric skewfields. We have the following characterization:

(3.3) *Let (K', v', Γ) be a strictly semi-valued skewfield, which is an extension of (K, v, Γ) (as semi-valued skewfields). The extension is immediate if and only if the following conditions are satisfied:*

1. *The mapping $M' \mapsto M' \cap K$ from $\Omega(K')$ to $\Omega(K)$ is a bijection.*

2. *$B' = B + M'$, for every $M' \in \Omega(K')$.*

From this characterization, it follows that the relation of "immediate extension" holds for two strictly semi-valued skewfields if and only if it also holds for the associated ultrametric skewfields.

The relation "immediate extension" in the classes of semi-valued skewfields and of ultrametric skewfields is transitive and inductive.

Also:

(3.4) *If an ultrametric skewfield is spherically complete, then it does not have any proper immediate ultrametric skewfield extension.*

E. Dense extensions of ultrametric spaces

We consider dense extensions of semi-valued skewfields and of ultrametric skewfields and we show:

(3.5) *Let (K, v, Γ) be a semi-valued skewfield, let (K', v', Γ) be a strictly semi-valued skewfield which is an extension of (K, v, Γ). This extension is dense if and only if the following conditions are satisfied:*

1. *For every $x' \in K'$ there exists $x \in K \cap (x' + B')$.*

2. *For every $M' \in \Omega(K')$, $B' = M' + (K \cap B')$.*

The relation "dense extension" in the classes of semi-valued skewfields and of ultrametric skewfields is transitive and inductive.

(3.6) Theorem: *Every semi-valued (resp. ultrametric) skewfield has a dense extension which is maximal.*

(3.7) *If the ultrametric skewfield is complete, then it does not have any proper dense ultrametric skewfield extension.*

F. Case when $(\Gamma, \leq)$ is totally ordered

Now we indicate the more specific results obtained when $(\Gamma, \leq)$ is totally ordered. A strictly valued field is just a field endowed with a Krull valuation and the theory is classical. We show that in this case, we can deduce the following (known) results for valued fields from our results on ultrametric groups ((2.7), (2.8), (2.9)). For the proof, we need Kaplansky's theorems [6] about the adjunction of pseudolimits to valued fields.

(3.8) Theorem: *The valued field (K, v, Γ) is without any proper immediate field extension if and only if (K, v, Γ) is spherically complete. Every valued field has an immediate field extension which is spherically complete.*

(3.9) Theorem: *Let (K, v, Γ) be a valued field. Assume that its residue field has characteristic 0. Assume furthermore that the value group Γ^* is root-closed and that the valuation ring B contains a subfield F with $B = F \oplus M$, where M denotes the valuation ideal. Then with respect to a suitably chosen set of representatives, the pointed Hahn group H^0 of (2.7) becomes a field of formal power series, into which (K, v, Γ) is mapped by a value- and field-operations preserving monomorphism. Moreover H^0 is a maximal immediate extension of the image of (K, v, Γ) under this monomorphism.*

(3.10) Theorem: *Let (K, v, Γ) and its pointed Hahn group (H^0, v, Γ) be as in (3.9). Then (K, v, Γ) has (with respect to the associated ultrametric d) a dense field extension which is complete and value-isomorphic to a valued subfield of (H^0, v, Γ).*

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A BURNSIDE RING THEORETIC PROOF OF T.Y. LAM'S RESULTS CONCERNING THE ARTIN EXPONENT OF FINITE GROUPS.

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Presented by P. Ribenboim, F.R.S.C.

Abstract

For a finite group G of order a power of a prime p say p^α, it is well known (see [2]) that the Artin exponent $A(G)$, depends on whether G is cyclic or not. $A(G) = 1$ if and only if G is cyclic and $A(G) = p^{\alpha-1}$ when G is noncyclic excluding the cases for which G is one of the special class of 2-groups considered here in Section 4. In this paper we demonstrate a Burnside ring theoretic proof of these results.

1 Introduction

Recall that for a finite group G that the set of isomorphism classes of finite left G-sets form a commutative semi-ring with respect to disjoint union and cartesian products. Let $B^+(G)$ denote this semi-ring. The universal ring associated to $B^+(G)$ is called the *Burnside ring* $B(G)$ of G. For every subgroup U of G there exists a canonical homomorphism, $\chi_U : B(G) \rightarrow Z$, (the integers) which maps every finite G-set S onto the cardinality, $\chi_U(S) := \#S^U$, of its subset $S^U := \{s \in S \mid us = s \text{ for all } u \in U\}$, of U invariant elements. So we can consider $B(G)$ in a canonical way as a subring of the *ghost ring*, $\tilde{B}(G) := \prod_{U \leq G} Z$, of G, consisting of all maps from all subgroups of G into Z which are constant on each conjugacy class of subgroups, by identifying each $x \in B(G)$ with the associated map $U \rightarrow \chi_U(x)$, (also denoted by x) from the set of all subgroups of G into Z. It is well known (see[6] chapter 1) that there is an exact sequence $0 \rightarrow B(G) \rightarrow \tilde{B}(G) \rightarrow \prod_{(U)} (Z/(N_G(U) : U))$, where (U) runs over a set of G-conjugacy class of subgroups. For more details on the Burnside ring see [6].

Now let $\mathcal{G}(QG)$ denote the Grothendieck ring of all rational representations of G and $\mathcal{G}_C(QG)$ the ideal in $\mathcal{G}(QG)$ generated by rational representations of G induced from cyclic subgroups. The Artin exponent of G has been computed by finding the characteristic of the quotient ring $\mathcal{G}(QG)/\mathcal{G}_C(QG)$ (see[2]). In this paper, we prove a reformulation in Burnside ring theoretic terms of some results concerning the Artin exponent of finite p-groups which arises in the context of the definition in section 2.

There are various reasons why the Artin exponents are studied. As can be observed, these ideas are natural generalizations of number theoretic notions which in themselves have aesthetic values. Moreover there are applications in other branches of mathematics. For example, a full knowledge of these invariants may sometimes yield nontrivial information about the induction theorems for various functor K on the category of finite groups (see[3]). Finally and above all the procedure used here affirms the utility of the Burnside ring in the representation theory of finite groups.

2 Definition of the Artin exponent in terms of Burnside rings

One can also describe the elements in $\tilde{B}(G)$ which are in $B(G)$ as follows:

Definition and Theorem 2.1 *Let G be a finite group. For $\mathcal{U}$ a family of subgroups of G closed with respect to conjugation, define $e_{\mathcal{U}} \in \tilde{B}(G)$ by,*

$$e_{\mathcal{U}}(U) := \begin{cases} 1, & \text{if } U \in \mathcal{U} \\ 0, & \text{if } U \notin \mathcal{U} . \end{cases}$$

Then $|G| \cdot e_{\mathcal{U}} \in B(G)$ for any $\mathcal{U}$. Furthermore define,
$A(G\ ,\ \mathcal{U}) := min(n \in N \mid n \cdot e_{\mathcal{U}} \in B(G))$. The integer $A(G\ ,\ \mathcal{U})$ is said to be an Artin exponent of G. We have that $|G| \geq A(G\ ,\ \mathcal{U})$. More precisely, $A(G\ ,\ \mathcal{U})$ divides $|G|$ (see[4]).

Theorem 2.2 *Let $x \in \tilde{B}(G)$. Then $x \in B(G)$ if and only if for every $U \trianglelefteq V \leq G$ with $(V : U)$ a prime power one has that, $\sum_{vU \in V/U} x(< v\ ,\ U >) \equiv 0(mod\ (V : U))$.*

Proof: see [1]

Furthermore one has,

Corollary 2.3 *$n \cdot e_{\mathcal{U}} \in B(G)$ if and only if for every $U \trianglelefteq V \leq G$ with $(V : U)$ a prime power, $\sum_{vU \in V/U} n \cdot e_{\mathcal{U}}(< v\ ,\ U >) \equiv 0(mod\ (V : U))$. where*
$e_{\mathcal{U}}(< v\ ,\ U >) := \#\{vU \in V/U \mid < v\ ,\ U > \in \mathcal{U}\}$.

It then implies that $n \cdot e_{\mathcal{U}} \in B(G)$ if and only if $(V : U)$ divides

$$n \cdot \#\{vU \in V/U \mid < v\ ,\ U > \in \mathcal{U}\}.$$

In the important special case where $\mathcal{U} := \{\ C \leq G \mid C \text{ cyclic}\ \}$ we obtain a reformulation of T.Y. Lam's definition and call (putting $A(G) := A(G\ ,\ \mathcal{U})$ in this case), $A(G)$ the Artin exponent of G.

3 Some useful results concerning p-groups

For the proofs of the results in this Section, see [4]. For the reader's convenience we list the notation used here; For G a finite group, we denote by: $\{1_G\}$, the trivial subgroup of G; p, a prime number; $Z(G)$, the center of G; $Z_G(N)$, the centralizers of N in G; $< T >$, subgroup of G generated by T; $< T >_N$, the normal subgroup of G generated by T; g_h, the g conjugate of h; $[g\ ,\ h]$, the commutator of g and h, in particular $[S\ ,\ T] := < [s\ ,\ t] \mid s \in S,\ t \in T >$.

Lemma 3.1 *Let G be a finite p-group and let U and H be subgroups of G such that U is a cyclic normal subgroup with $U \subseteq H$. Let $C(H)$ denote the set of all cyclic subgroups V of H with $U \leq V$ and $(V : U) = p$, and let $C'(H) = C'_U(H)$ denote the subset of $C(H)$, consisting of those $V \in C(H)$ which are normal in H. We have:*
(a.) $|C(H)| \equiv |C'(H)|$ *(mod p).*
(b.) $H' = H'_U := \{h \in H \mid [h\ ,\ H] \subseteq U\}$, *is a normal subgroup of H.*
(c.) $C'(H) = C(H')$
(d.) If $G' \subseteq H \trianglelefteq G$, *then* $|C(H)| = |C(G')|$*(mod p).*

Lemma 3.2 *If G is abelian, then the following are equivalent:*
(i) $|C(G)| = 1$
(ii) $|C(G)| \not\equiv 0 (mod\ \ p)$
(iii) G is cyclic

Lemma 3.3 *With the definitions and assumptions of (3.1), if G is a 2-group, one has,*
(a.) $|C(H)| \equiv 1(2) \Rightarrow Z(G)$ *is cyclic.*
(b.) If $[G\ ,\ G] \subseteq U$ *and* $|C(H)|$ *is odd, then G is cyclic or nonabelian of order* 8.
(c.) If $H \leq G$ *with* $[H\ ,\ H] \subseteq U \subseteq H$ *and* $|\{V \in C(H) \mid V \subseteq H\}| \equiv 1(mod\ 2)$, *then H is cyclic or nonabelian of order* 8.
(d.) If $|C(H)| \equiv 1(mod\ 2)$, *then either H' is cyclic or G is nonabelian of order* 8.
(e.) If $|C(H)| \equiv 1(mod\ 2)$, *then G contains a cyclic normal subgroup of index* 2.

Lemma 3.4 *Let G be a noncyclic abelian p-group* $(p \neq 2)$ *and* $U \trianglelefteq G$, $|U| = p$. *Then there exists a normal subgroup* $V \trianglelefteq G$ *of G of order* p^2, *containing U and isomorphic to* $Z_p \times Z_p$.

4 The Artin Exponent

We shall be concerned with results dealing with p-groups. The global discussion is established similarly - more precisely by observing that the condition that G is a p group can be removed. This is discussed in a separate paper (see [5]). By a standard argument (see[4]), we can show that to calculate the Artin exponent one only needs to find for each prime p the minimum of all $n \in N$ such that $(V\ :\ U)$ divides $n \cdot c(U\ ,\ V)$ holds for all $U \trianglelefteq V \leq G$ with U cyclic, $U \leq Z(V)$ and V a p-group, where $c(U\ ,\ V) := \#\{vU \in V/U \mid < v\ ,\ U > \text{ is cyclic }\}$. This is precisely what we shall do here to get $A(G)$. We shall need the following result (see[7]);

Lemma 4.1 *Let G be a p-group of order p^n. If there exists a $g \in G$ of order p^{n-1} then G has one of the following presentations for some h in G.*

(A) *If G abelian:*

(i) $n \geq 1$, Z_{p^n}: $h^{p^n} = 1$. *or*

(ii) $n \geq 2$, $g^{p^{n-1}} = 1$, $h^p = 1$, $gh = hg$.

(B) *If G is non-abelian and p odd, $n \geq 3$.*

(iii) $g^{p^{n-1}} = 1$, $h^p = 1$, $hgh^{-1} = g^{1+p^{n-2}}$.

(C) *If G nonabelian and $p = 2$, $n \geq 3$.*

(iv) $g^{2^{n-1}} = 1$, $h^2 = g^{2^{n-2}}$, $hgh^{-1} = g^{-1}$, *i.e. the generalized quaternion group:***(Q)**.

(v) $g^{2^{n-1}} = 1$, $h^2 = 1$, $hgh^{-1} = g^{-1}$, *i.e. the dihedral group:***(D)**.

(D) *If G is non-abelian and $p = 2$, $n \geq 4$.*

(vi) $g^{2^{n-1}} = 1$, $h^2 = 1$, $hgh^{-1} = g^{1+2^{n-2}}$.

(vii) $g^{2^{n-1}} = 1$, $h^2 = 1$, $hgh^{-1} = g^{-1+2^{n-2}}$. *i.e. the semi-dihedral group:***(SD)**.

Also we observe (see [4]),

Theorem 4.2 *Let G be a noncyclic p-group ($p \neq 2$) of order p^α. Let $\mathcal{U}$ denote the set of subgroups of G closed with respect to conjugation. Define for $2 \leq \beta \leq \alpha$, $x_\beta \in \tilde{B}(G)$ by*

$$x_\beta := \begin{cases} p^{\alpha-\beta}, & \text{if } |U| < p^\beta \\ p^{\alpha-\beta}, & \text{if } |U| = p^\beta \text{ and } U \text{ is noncyclic} \\ 0, & \text{otherwise} . \end{cases}$$

Then x_β is contained in $B(G)$.

Proposition 4.3 *Let G be a p-group of order p^α and consider $\mathcal{U}$, the family of all cyclic subgroups of G. For $n \in N$ and a subgroup $U \leq G$, let $x_n \in \tilde{B}(G)$ be defined by*

$$x_n = \begin{cases} n, & \text{if } U \text{ is cyclic} \\ 0, & \text{otherwise} \end{cases}$$

Then, $A(G) = 1$ if and only if, G is cyclic.

Proof: Assume $U \trianglelefteq V \leq G$, U cyclic, $|V| = p^\delta$, (p a prime). Then $x_n \in B(G)$ implies

$$\sum_{vU \in V/U} x(<v\,,\,U>) = n \cdot \#\{\, vU \in V/U \mid <v\,,\,U> \text{ cyclic} \,\} \equiv 0 (mod\,(V:U)).$$

Since n is arbitrary, we get $A(G)$ by making the following assumptions (see[4]): Set $|U| = p$ and $V = G$. These assumptions remain valid for the proof of (4.4) and (4.5). Then if $n = 1$, one has that $\#\{gU \in G/U \mid <g\,,\,U> \text{ is cyclic}\} \equiv 0(mod\,(G:U))$. That is, $|G| \mid \#\{g \in G \mid <g\,,\,U> \text{ is cyclic}\}$. So for all $g \in G$, $<g\,,\,U>$ is cyclic. This implies that G is cyclic since if G is not cyclic then from (3.4) there exists a normal subgroup $V \trianglelefteq G$ such that V contains U and $V \cong Z_p \times Z_p \cong U \times Z_p = U \times <g>$ for some $g \in G$. That is $<g\,,\,U> = V$ for some $g \in G$. Therefore we must have that G is cyclic.

The converse is clear. □

Proposition 4.4 *Keeping the assumptions of (4.3), one has that if G is a noncyclic p-group of order p^{α} and $p \neq 2$ then $A(G) = p^{\alpha-1}$.*

Proof: From the assumptions and proof of (4.3) one has that because G is noncyclic that $|G|$ does not divide $x(<g\, ,\, U>) \;=\; \#\{gU \in G/U \;\mid\; <g\, ,\, U> \text{ cyclic }\}$. But,

$$\begin{aligned} x(<g\, ,\, U>) &= \#\{gU \in G/U \mid <g\, ,\, U> \text{ is cyclic }\} \\ &= \sum_{\beta \geq 1}(p^{\beta-1} - p^{\beta-2}) \cdot \#\{W \leq G \mid W \text{ is cyclic of order } p^{\beta},\ W \supseteq U\} \\ &= 1 + p - 1 \cdot \#\{W \leq G \mid W \text{ is cyclic of order } p^{2},\ W \supseteq U\} \\ &\quad + \sum_{\beta \geq 3}(p^{\beta-1} - p^{\beta-2}) \cdot \#\{W \leq G \mid W \text{ is cyclic of order } p^{\beta},\ W \supseteq U\}. \end{aligned}$$

It suffices to look at the number $\#\{W \leq G \mid W \text{ is cyclic of order } p^{2},\ W \supseteq U\}$. Recall that from (3.2) and (3.1), one has that, $\#\{W \leq G \mid W \text{ is cyclic of order } p^{2},\ W \supseteq U\} \equiv 0 (mod\ p)$. This implies that $A(G) = p^{\alpha-1}$. □

Proposition 4.5 *With the assumptions of (4.3) one has that if G is noncyclic 2-group of order 2^{α}, then $A(G\, ,\, \mathcal{U}) = 2^{\alpha-1}$, unless* **G = Q, D** *or* **SD** *(as defined in Lemma 4.1), in which cases we have $A(G) = 2$.*

Proof: Following the proof of (4.3) one has that if,
$\#\{W \leq G \mid W \text{ is cyclic of order } 4,\ W \supseteq U\} \equiv 0(mod\ 2)$, then $A(G\, ,\, \mathcal{U}) = 2^{\alpha-1}$. On the other hand if, $\#\{W \leq G \mid W \text{ is cyclic of order } 4,\ W \supseteq U\} \not\equiv 0(mod\ 2)$ then by (3.3) G is cyclic or nonabelian group of order 8. Since G is noncyclic it must be nonabelian of order 8 and so must correspond to one the groups **Q, D** or **SD** (of order 8) as presented in (4.1) and it is easy to see by direct computation that in any of these cases that $A(G) = 2$. □

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MEHLER INTEGRAL TRANSFORMS ASSOCIATED WITH JACOBI FUNCTIONS WITH RESPECT TO THE DUAL VARIABLE.

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Abstract. We prove a Mehler representation for Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the dual variable λ. We exploit this representation to define a pair of dual integral transforms $\chi_{\alpha,\beta}$ and its transposed ${}^t\chi_{\alpha,\beta}$. We give some spaces of functions on which $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are isomorphisms and we establish inversion formulas for these transforms.

1. Dual Mehler representation of the Jacobi function.

For $\alpha \geq \beta \geq -1/2$, $\lambda \in \mathbb{C}$, $t \geq 0$, the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ is defined by $\varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda), \alpha+1, -\operatorname{sh}^2 t)$, where $\rho = \alpha+\beta+1$ and ${}_2F_1$ is the Gauss hypergeometric function. The function $t \dashrightarrow \varphi_\lambda^{(\alpha,\beta)}(t)$ is the unique solution of the following differential equation $\Delta_{\alpha,\beta}\, \varphi_\lambda^{(\alpha,\beta)}(t) = -(\lambda^2 + \rho^2)\, \varphi_\lambda^{(\alpha,\beta)}(t)$, satisfying $\varphi_\lambda^{(\alpha,\beta)}(0) = 1$, $\frac{d}{dt}\varphi_\lambda^{(\alpha,\beta)}(0) = 0$, where $\Delta_{\alpha,\beta} = \frac{1}{A_{\alpha,\beta}(t)}\frac{d}{dt}\left[A_{\alpha,\beta}(t)\frac{d}{dt}\right]$ is the Jacobi differential operator, with $A_{\alpha,\beta}(t) = 2^{2\rho}(\operatorname{sh} t)^{2\alpha+1}(\operatorname{ch} t)^{2\beta+1}$.

For special values of α and β, the Jacobi functions are interpreted as spherical functions on non compact riemannian symmetric spaces of rank one, for more details see the very interesting survey of T.H. Koornwinder [2].

We give the following integral representation for the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the dual variable λ.

Theorem 1.1: Let $\alpha \geq \beta \geq -\frac{1}{2}$, $(\alpha, \beta) \neq (-\frac{1}{2}, -\frac{1}{2})$. Then the function $\varphi_{\lambda_o}^{(\alpha,\beta)}(t)$ possesses the following dual Mehler representation

$$\forall t \geq 0,\ \forall \lambda_o \in \mathbb{R},\quad \varphi_{\lambda_o}^{(\alpha,\beta)}(t) = \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_o,\lambda)\cos\lambda t\, d\lambda.$$

where $b_{\alpha,\beta}(\lambda_o, \lambda)$ is a nonnegative kernel.

Example. For $\alpha = 1/2$ and $\beta = -1/2$, a computation gives the kernel $b_{1/2,-1/2}(\lambda_o,\lambda)$. We have

$$\forall \lambda_0, \lambda \in \mathbb{R},\qquad b_{1/2,-1/2}(\lambda_o,\lambda) = \frac{\pi \operatorname{sh}\pi\lambda_o}{2\lambda_o}\left[\frac{\operatorname{ch}\pi\lambda + \operatorname{ch}\pi\lambda_o}{(\operatorname{ch}\pi(\lambda_o+\lambda)+1)(\operatorname{ch}\pi(\lambda_o-\lambda)+1)}\right].$$

In the following we give the main properties of the kernel $b_{\alpha,\beta}$.

Theorem 1.2.: The kernel $b_{\alpha,\beta}$ satisfies the following properties.

i) For all $\lambda_o, \lambda \in \mathbb{R}$, $b_{\alpha,\beta}$ is nonnegative and we have $\frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_o,\lambda)d\lambda = 1$.

ii) The kernel $b_{\alpha,\beta}(\lambda_o, \lambda)$ can be extended to an even analytic function on the set $\{(\lambda_o, \lambda)\in\mathbb{C}^2 / |\mathrm{Im}\,\lambda_o| + |\mathrm{Im}\,\lambda| < \rho\}$.

iii) For all $\lambda \in \mathbb{R}$, the function $\lambda_o \longrightarrow b_{\alpha,\beta}(\lambda_o, \lambda)$ is even and analytic on the strip $\{(\lambda_o\in \mathbb{C} / |\mathrm{Im}\,\lambda_o| < \rho\}$.

iv) There exists a constant $c > 0$, such that for all $\lambda \in \mathbb{R}$ and $\lambda_o \in \mathbb{C}$, $|\mathrm{Im}\,\lambda_o| < \rho$, we have

$$|b_{\alpha,\beta}(\lambda_o, \lambda)| \leq c\left(\frac{1}{\rho-|\mathrm{Im}\,\lambda_o|} + \frac{1}{(\rho-|\mathrm{Im}\,\lambda_o|)^2}\right).$$

v) For all $\lambda_o\in\mathbb{C}$, $|\mathrm{Im}\,\lambda_o| < \rho$, the function $\lambda \longrightarrow b_{\alpha,\beta}(\lambda_o, \lambda)$ is an even C^∞- functions on $\mathbb{R}$, rapidly decreasing..

<u>Remark:</u> It is well known (see [2]) that the function $t \longrightarrow \varphi_\lambda^{(\alpha,\beta)}(t)$ possesses the following Mehler representation:

$$\forall t > 0,\ \forall\lambda \in\mathbb{C},\ \varphi_\lambda^{(\alpha,\beta)}(t) = \int_0^t K_{\alpha,\beta}(t,u)\cos\lambda u\, du,$$

where $K_{\alpha,\beta}(t, .)$ is a nonnegative even function continuous on $]-t,t[$ and with support in $[-t,t]$.
We observe that the kernels $K_{\alpha,\beta}(t,u)$ and $b_{\alpha,\beta}(\lambda_o, \lambda)$ have not the same properties.

2. The Fourier -Jacobi transform and its inverse.

Let us define the following functions spaces

- $S_*(\mathbb{R})$ the space of even C^∞- functions on $\mathbb{R}$, rapidly decreasing.
- $S_*^r(\mathbb{R}) = \{(\mathrm{ch}\,t)^{-2\rho/r} S_*(\mathbb{R})\}$, for $0 < r \leq 2$.

Clearly $S_*^r(\mathbb{R})$ is invariant under $\Delta_{\alpha,\beta}$. Give $S_*^r(\mathbb{R})$ the topology defined by the semi-norms

$$Q_{n,m}(f) = \sup_{t\geq 0} |(\mathrm{ch}\,t)^{2\rho/r}(1+t)^n \Delta_{\alpha,\beta}^m f(t)|$$

-$D_r = \{\lambda\in\mathbb{C} / |\mathrm{Im}\,\lambda| < (2/r-1)\rho\}$.

- H_*^r the space of even holomorphic functions ψ in D_r, C^∞ in $\overline{D}_r$ and rapidly decreasing that is :

$$\forall\, m,\ n \in \mathbb{N},\ P_{n,m}(\psi) = \sup_{\lambda\in\overline{D}_r} |(1+\lambda^n)\frac{d^m}{d\lambda^m}\psi(\lambda)| < +\infty .$$

Notice that $H_*^2 = S_*(\mathbb{R})$ and if $r\leq s$ then $H_*^r \subset H_*^s$. The space H_*^r is equipped with the topology defined by the semi-norms $P_{n,m}$.

<u>Notations:</u> we denote by

- $d\mu_{\alpha,\beta}(t) = (2\pi)^{-1/2}\, 2^{2\rho}\, (\mathrm{sh}\,t)^{2\alpha+1}(\mathrm{ch}\,t)^{\beta+1}\,dt$.
- $d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-1/2} |C_{\alpha,\beta}(\lambda)|^{-2}\, d\lambda$ where $C_{\alpha,\beta} = \dfrac{2^{\rho-i\lambda}\,\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma(\frac{\alpha+\beta+1+i\lambda}{2})\Gamma(\frac{\alpha-\beta+1+i\lambda}{2})}$.

<u>Definition 2.1</u>: (i) The Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$ of $f \in L^1([0,+\infty[, d\mu_{\alpha,\beta})$ (the space of integrable functions on $[0,+\infty[$ with respect to the measure $\mu_{\alpha,\beta}$) is defined by

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)d\mu_{\alpha,\beta}(t).$$

(ii) The inverse Fourier-Jacobi transform $\mathcal{F}^{-1}_{\alpha,\beta}$ of $h \in L^1([0,+\infty[, dv_{\alpha,\beta})$ (the space of integrable functions on $[0,+\infty[$ with respect to the measure $v_{\alpha,\beta}$) is defined by

$$\mathcal{F}^{-1}_{\alpha,\beta}(h)(t) = \int_0^\infty h(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)dv_{\alpha,\beta}(\lambda).$$

We have the following Paley-Wiener type theorem for the Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$.

Theorem 2.1 : The Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$ is an isomorphism from $S^{\prime}_{*}(\mathbb{R})$ onto $H^{\prime}_{*}$.

3 - Dual Mehler transforms associated with the Jacobi functions.

Notations: we denote by

. $C^\infty_*(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$.

. $C_{*,b}(\mathbb{R})$ the space of even, continuous and bounded functions on $\mathbb{R}$.

. For $f \in C_{*,b}(\mathbb{R})$, $\|f\|_\infty = \sup_{x\in\mathbb{R}}|f(x)|$.

Definition 3.1 : The dual Mehler transform denoted by $\chi_{\alpha,\beta}$ is defined for a suitable function f by

$$\forall \lambda_0 \in \mathbb{R}, \quad \chi_{\alpha,\beta}(f)(\lambda_0) = \frac{2}{\pi}\int_0^\infty b_{\alpha,\beta}(\lambda_0,\lambda)f(\lambda)d\lambda.$$

Remark: We have for all $\lambda_0 \in \mathbb{R}$, and $u \geq 0$, $\varphi^{(\alpha,\beta)}_{\lambda_0}(u) = \chi_{\alpha,\beta}(\cos(.u))(\lambda_0)$.

Proposition 3.1 : i) For $f \in C_{*,b}(\mathbb{R})$, $\chi_{\alpha,\beta}(f) \in C_{*,b}(\mathbb{R})$ and we have $\|\chi_{\alpha,\beta}(f)\|_\infty \leq \|f\|_\infty$.

ii) If $f \in L^1([0,+\infty[, d\lambda)$, (the space of integrable functions on $[0,+\infty[$ with respect to the Lebesgue measure $d\lambda$), then the function $\lambda_0 \dashrightarrow \chi_{\alpha,\beta}(f)(\lambda_0)$ is well defined on the strip $|\operatorname{Im}\lambda_0| < \rho$ and it is analytic on this strip.

iii) The transform $\chi_{\alpha,\beta}$ is defined on $H^{!}_{*}$ and it is injective on this space.

Definition 3.2 : The transposed of $\chi_{\alpha,\beta}$ denoted by ${}^t\chi_{\alpha,\beta}$ is defined for $g \in L^1([0,+\infty[, dv_{\alpha,\beta})$ by

$$\forall \lambda \in \mathbb{R}, \quad {}^t\chi_{\alpha,\beta}(g)(\lambda) = \int_0^\infty b_{\alpha,\beta}(\lambda_0,\lambda)g(\lambda_0)dv_{\alpha,\beta}(\lambda_0).$$

We have the following properties for the transform ${}^t\chi_{\alpha,\beta}$.

i) For $g \in L^1([0,+\infty[, dv_{\alpha,\beta})$, the function ${}^t\chi_{\alpha,\beta}(g)$ is analytic on the strip $|\operatorname{Im}\lambda| < \rho$.

ii) For $f \in L^1([0,+\infty[, d\lambda)$ and $g \in L^1([0,+\infty[, dv_{\alpha,\beta})$, we have the following duality relation

$$\int_0^\infty \chi_{\alpha,\beta}(f)(\lambda_0)g(\lambda_0)dv_{\alpha,\beta}(\lambda_0) = \frac{2}{\pi}\int_0^\infty f(\lambda)\,{}^t\chi_{\alpha,\beta}(g)(\lambda)d\lambda.$$

iii) For all $g \in S_*(\mathbb{R})$, we have the following formula $\mathcal{F}^{-1}_{\alpha,\beta}(g) = \mathcal{F}^{-1}_0 \circ {}^t\chi_{\alpha,\beta}(g)$ where $\mathcal{F}^{-1}_{\alpha,\beta}$ denotes the inverse Fourier-Jacobi transform and $\mathcal{F}_0$ the Fourier-cosine transform.

From the previous results we deduce the following theorem.

Theorem 3.1 : The transform ${}^t\chi_{\alpha,\beta}$ is a topological isomorphism from $S_*(\mathbb{R})$ onto $H^{!}_{*}$.

Remark: It is well known that the Mehler representation of the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ with respect to the variable t permits to define the Abel transform and its dual which are permutations operators between the differential operator $\Delta_{\alpha,\beta}$ and $\frac{d^2}{dt^2}$ (see [3]).

A natural question arises, whether or not the operators $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$ are permutations operators with some differential operators or distributions to define?

(a) The first example, $\alpha = 1/2$ and $\beta = -1/2$: We have $L\chi_{1/2,-1/2} = \chi_{1/2,-1/2} D^2$ and ${}^t\chi_{1/2,-1/2} L = D^2\, {}^t\chi_{1/2,-1/2}$ where $L = D^2 + \frac{2}{\lambda} D$ and $D = \frac{d}{d\lambda}$.

(b) The second example $\alpha = 3/2$ and $\beta = 1/2$: We have $P\,\chi_{3/2,1/2} = \chi_{3/2,1/2}\, Q$ and ${}^t\chi_{3/2,1/2}\, P = Q\, {}^t\chi_{3/2,1/2}$ where $P = D^2 + (\frac{4\lambda}{\lambda^2+1} + \frac{2}{\lambda})D + \frac{2}{\lambda^2+1}$ and

$$Q = D^2 - \frac{1}{\lambda^2+4}\sum_{n>o}(-1)^n(a_n+b_n)D^{2n} + \frac{2\lambda}{4+\lambda^2}D - \frac{3}{\lambda^2+1}\sum_{n\geq o}(-1)^n b_n \delta^{(2n)},$$

here $\delta^{(k)}$ denotes the k-derivative of the Dirac distribution at 0 and (a_n) and (b_n) are the coefficients given in the following expansions

$$\frac{4t}{sh2t}ch2t = \sum_{n\geq o} a_n t^{2n} \quad\text{and}\quad \frac{4t}{sh\,2t} = \sum_{n\geq o} b_n t^{2n}.$$

We observe that in the first example there exists a differential operator which gives the permutation relations with D^2. In the second example there exists a distribution Q which gives the permutation relations with the differential operator P.

4 - Inversion formulas for the operators $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$.

In this section we shall define some functions spaces on which we can inverte the operators $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$.

Notations: We denote by

. $S_{*,o}(\mathbb{R})$ the subspace of $S_*(\mathbb{R})$ consisting of functions f satisfying : $\forall n\in\mathbb{N}$, $\frac{d^{2n}}{d\lambda^{2n}}f(0) = 0$.

. $S^\perp_{*,o}(\mathbb{R})$ the subspace of $S_*(\mathbb{R})$ consisting of functions f satisfying :

$$\forall n\in\mathbb{N},\ \int_o^\infty f(\lambda)\lambda^{2n}d\lambda = 0.$$

. $H^r_{*,o}$ the subspace of H^r_* consisting of functions h satisfying : $\forall n\in\mathbb{N}$, $\int_o^\infty h(\lambda)\lambda^{2n}d\lambda = 0$.

. $H^r_{*,\perp}$ the subspace of H^r_* consisting of functions h satisfying :

$$\forall n\in\mathbb{N},\ \int_o^\infty h(\lambda)c_n(\lambda)d\nu_{\alpha,\beta}(\lambda) = 0,$$

where $c_n(\lambda) = \int_o^\infty b_{\alpha,\beta}(\lambda,s)s^{2n}ds$.

Theorem 4.1 : i) The Fourier-cosine transform $\mathcal{F}_o$ is a topological isomorphism from

- $S^\perp_{*,o}(\mathbb{R})$ onto $S_{*,o}(\mathbb{R})$.

$-(\text{ch } t)^{-2\rho} S_{*,o}(\mathbb{R})$ onto $H_{*,0}^{2/3}$.

ii) The Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$ is a topological isomorphism from $(\text{ch } t)^{-2\rho/r} S_{*,o}(\mathbb{R})$ onto $H_{*,\perp}^{r}$.

<u>Proposition</u> 4.1 : i) The operator $\mathcal{M}_o$ defined by $\mathcal{M}_o(f) = (2\pi)^{-1/2}\mathcal{F}_o[A_{\alpha,\beta}\, \mathcal{F}_o^{-1}(f)]$ is a topological isomorphism from $H_{*,0}^{2/3}$ onto $S_{*,o}(\mathbb{R})$.

ii) The operator $\mathcal{M}$ defined by $\mathcal{M}(f) = (2\pi)^{-1/2}\mathcal{F}_{\alpha,\beta}[A_{\alpha,\beta}.\, \mathcal{F}_{\alpha,\beta}^{-1}(f)]$ is a topological isomorphism from $H_{*,\perp}^{r}$ onto $H_{*,\perp}^{r/(1-r)}$, $0 < r < 1$.

Moreover we have $\mathcal{M} = ({}^t\chi_{\alpha,\beta})^{-1} \circ \mathcal{M}_o \circ {}^t\chi_{\alpha,\beta}$.

<u>Theorem</u> 4.2 : i) The transform $\chi_{\alpha,\beta}$ is a topological isomorphism from $S_{*,o}^{\perp}(\mathbb{R})$ onto $H_{*,\perp}^{1}$.

ii) The transform ${}^t\chi_{\alpha,\beta}$ is a topological isomorphism from $H_{*,\perp}^{1}$ onto $H_{*,0}^{2/3}$.

<u>Theorem</u> 4.3:We have the following inversion formulas for the operators $\chi_{\alpha,\beta}$ and ${}^t\chi_{\alpha,\beta}$:

i) for all $f \in H_{*,\perp}^{1}$, $f = \chi_{\alpha,\beta} \circ \mathcal{M}_o \circ {}^t\chi_{\alpha,\beta}(f)$,

ii) for all $f \in H_{*,\perp}^{1/2}$, $f = \chi_{\alpha,\beta} \circ {}^t\chi_{\alpha,\beta} \circ \mathcal{M}(f)$,

iii) for all $f \in H_{*,0}^{2/3}$, $f = {}^t\chi_{\alpha,\beta} \circ \chi_{\alpha,\beta} \circ \mathcal{M}_o(f)$,

iv) for all $f \in H_{*,0}^{1/2}$, $f = {}^t\chi_{\alpha,\beta} \circ \mathcal{M} \circ \chi_{\alpha,\beta}(f)$.

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CLOSED MODEL STRUCTURES FOR ALGEBRAIC MODELS OF r-CONNECTED SPACES

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ABSTRACT. In this Note we give, for any $r \geq 0$, a Quillen's simplicial model structure to the category **Simp(Gp)** of simplicial groups where the weak equivalences are those morphisms $f_\bullet$ such that $\pi_q(f_\bullet)$ is an isomorphism for $q \geq r+1$. We then make explicit typical constructions for the associated homotopy theory as the cylinder and path objects and the loop and suspension functors, and we show that this homotopy theory is equivalent to the homotopy theory in the category of the r-connected spaces.

1. PRELIMINARIES.- In what follows, **Simp(Gp)** will denote the category of simplicial groups and if $G_\bullet \in$ **Simp(Gp)**, $N_\cdot(G_\bullet)$ and $\pi_\cdot(G_\bullet)$ will denote the normalized complex and homotopy groups of $G_\bullet$ in the sense of Moore. The n-th-simplicial kernel of a $(n-1)$ truncated simplicial group $G_{\bullet tr}$, denoted $\Delta^n(G_{\bullet tr})$, is the subgroup of $(G_{n-1})^{n+1}$ whose elements are those $(x_0, ..., x_n)$ such that $d_i x_j = d_{j-1} x_i$ for $i < j$. Using this simplicial kernel construction, one has a functor, $cosk^{n-1}$, from the category of $(n-1)$-truncated simplicial groups to **Simp(Gp)**, which is right adjoint to the $(n-1)$-truncating functor, tr^{n-1}, which truncates the simplicial group at dimension $n-1$. We will denote $Cosk^{n-1} = cosk^{n-1} tr^{n-1}$ and $\Delta^n(G_\bullet) = \Delta^n(tr^{n-1}(G_\bullet))$.

Given a $(n-1)$-truncated simplicial group $G_{\bullet tr}$, we will denote $\Lambda^n_k(G_{\bullet tr})$, $0 \leq k \leq n$, the group of open k-horns of $G_{\bullet tr}$, which is the subgroup of $(G_{n-1})^n$ whose elements are those $(x_0, \ldots, \hat{x}_k, \ldots, x_n)$ such that $d_i x_j = d_{j-1} x_i$, $i < j$, $i, j \neq k$. For $G_\bullet$ a simplicial group, we will denote $\Lambda^n_k(G_\bullet) = \Lambda^n_k(tr^{n-1}(G_\bullet))$.

The category **Simp(Gp)** is a simplicial category (see [6]) where the function complex $\underline{Hom}_{\mathbf{Simp(Gp)}}(G_\bullet, H_\bullet)$ has as n-simplices the simplicial maps $F_\bullet : G_\bullet \times \Delta[n] \to H_\bullet$ such that $F_t : G_q \to H_q$, given by $F_t(x) = F_q(x, t)$, is a group morphism. Also, given a simplicial set $K_\bullet$ and simplicial groups $G_\bullet$ and $H_\bullet$, there are simplicial groups $G_\bullet \otimes K_\bullet$ and $H_\bullet^{K_\bullet}$ where $(G_\bullet \otimes K_\bullet)_n = \coprod_{k \in K_n} (G_n)_k$ with $(G_n)_k = G_n$, and $(H_\bullet^{K_\bullet})_n = Hom_{\mathbf{Simp(Set)}}(K_\bullet \times \Delta[n], G_\bullet)$. In particular, the simplicial group $H_\bullet^{\Delta[1]}$ can be identified with the simplicial group whose n-simplices are

$$\{(x_0, \ldots, x_n) \in H^{n+1}_{n+1} \ / \ d_i x_i = d_i x_{i-1} \ 1 \leq i \leq n\}$$

and the face and degeneracy operators are given by:

$$d_i(x_0, \ldots, x_n) = (d_{i+1}x_0, \ldots d_{i+1}x_{i-1}, d_i x_{i+1}, \ldots, d_i x_n), \ 0 \leq i \leq n$$

$$s_j(x_0, \ldots, x_n) = (s_{j+1}x_0, \ldots s_{j+1}x_j, s_j x_j, \ldots, s_j x_n), \ 0 \leq j \leq n$$

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If one considers the simplicial subset $\overset{\bullet}{\Delta}[n]$ (respec. $\Delta[n,k]$) of $\Delta[n]$ (see [2]), it is easy to see that to give an element of $\Delta^n(G_\bullet)$ (respec. $\Lambda^n_k(G_\bullet)$) is equivalent to give a simplicial morphism $F\overset{\bullet}{\Delta}[n] \to G_\bullet$ (respec. $F\Delta[n,k] \to G_\bullet$) where F is the free group functor.

Proposition 1.1. Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups. Then,

i) The natural morphism $G_n \to H_n \times_{\Delta^n(H_\bullet)} \Delta^n(G_\bullet)$ is surjective if and only if $f_\bullet$ has the RLP (right lifting property) with respect to the simplicial morphism $F\overset{\bullet}{\Delta}[n] \to F\Delta[n]$ induced by the canonical inclusion $\overset{\bullet}{\Delta}[n] \hookrightarrow \Delta[n]$.

ii) The natural morphism $G_n \to H_n \times_{\Lambda^n_k(H_\bullet)} \Lambda^n_k(G_\bullet)$, $0 \le k \le n$, is surjective if and only if $f_\bullet$ has the RLP with respect to the simplicial morphism $F\Delta[n,k] \to F\Delta[n]$, $0 \le k \le n$, induced by the canonical inclusion $\Delta[n,k] \hookrightarrow \Delta[n]$.

A morphism satisfying the condition *ii)* will be called a Kan fibration in dimension n.

Lemma 1.2. Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in **Simp(Gp)**. Then $f_\bullet$ is a Kan fibration in dimension n if and only if $N_n(f_\bullet)$ is surjective.

The following lemma is used in the characterizations of the (trivial) fibrations given in propositions 2.3 and 2.4.

Lemma 1.3. Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups.

i) If $f_\bullet$ verifies the RLP with respect to $F\Delta[n+1,n+1] \to F\Delta[n+1]$, $\pi_n(f_\bullet)$ is surjective and $\pi_{n-1}(f_\bullet)$ is injective, then $f_\bullet$ verifies the RLP with respect to $F\overset{\bullet}{\Delta}[n] \to F\Delta[n]$.

ii) If $f_\bullet$ verifies the RLP with respect to $F\overset{\bullet}{\Delta}[n] \to F\Delta[n]$, then $\pi_n(f_\bullet)$ is surjective and $\pi_{n-1}(f_\bullet)$ is injective.

iii) If $f_\bullet$ verifies the RLP with respect to both morphisms $F\overset{\bullet}{\Delta}[n+1] \to F\Delta[n+1]$ and $F\overset{\bullet}{\Delta}[n] \to F\Delta[n]$, then $f_\bullet$ is a Kan fibration in dimension $n+1$.

For $r > 0$, $\mathbf{Simp(Gp)}_r$ will denote the full subcategory of **Simp(Gp)** consisting of the r-reduced simplicial groups, i.e., reduced to the identity in dimensions $< r$. The inclusion functor $I : \mathbf{Simp(Gp)}_r \to \mathbf{Simp(Gp)}$ has a right adjoint E_r, given by $E_r(G_\bullet) = Ker(G_\bullet \to Cosk^{r-1}G_\bullet)$ and also, it has a left adjoint L_r which can be defined in a dual way to E_r. The category $\mathbf{Simp(Gp)}_r$ supports a closed model structure (see [7]) with cofibrations and weak equivalences defined to be those morphisms in $\mathbf{Simp(Gp)}_r$ which are cofibrations and weak equivalences respectively in **Simp(Gp)** and with fibrations defined by the RLP with respect to the trivial cofibrations (the dual process to this one has been studied in [1]).

If $\mathcal{T}_r$ denotes the category of the $(r-1)$-connected topological spaces, then one has

Theorem 1.4. ([7], th 3.1 and th 6.1).- There is an equivalence of the homotopy theory in $\mathcal{T}_r$ with the homotopy theory of the closed model category $\mathbf{Simp(Gp)}_{r-1}$.

2. THE $\bar{r}$-STRUCTURE IN Simp(Gp).- We will consider the distinguished classes of morphisms given in the following:

Definition 2.1.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups and $r \geq 0$.

i) $f_\bullet$ is said to be an $\bar{r}$-fibration if it is a Kan fibration in dimensions $\geq r+1$.

ii) $f_\bullet$ is said to be a weak $\bar{r}$-equivalence if $\pi_q(f_\bullet)$ is an isomorphism for $q \geq r$.

iii) $f_\bullet$ is said to be an $\bar{r}$-cofibration if it has the LLP with respect to the $\bar{r}$-trivial fibrations.

This kind of structure has been considered recently in the categories of topological spaces and simplicial sets (see [4]). Note that, for $r = 0$, the above concepts give the well known model structure in Simp(Gp), [6], and that this structure can be generalized, in a complementary way to the $\bar{r}$-structure, by considering weak equivalences as those morphisms inducing isomorphisms in the homotopy groups until dimension r (see [3] for simplicial sets or topological spaces and [5] for simplicial groups).

Let us note also that all simplicial group is $\bar{r}$–fibrant and that one has the following sequences of inclusions

$$(fibs.) \subseteq (\bar{1} - fibs.) \subseteq \ldots \subseteq (\bar{r} - fibs.) \subseteq (\overline{r+1} - fibs.) \subseteq \ldots$$
$$(triv.\ fibs.) \subseteq (\bar{1} - triv.\ fibs.) \subseteq \ldots \subseteq (\bar{r} - triv.\ fibs.) \subseteq (\overline{r+1} - triv.\ fibs.) \subseteq \ldots$$

Proposition 2.2.- If $f_\bullet : G_\bullet \to H_\bullet$ is an $\bar{r}$-fibration, I is the image of $N_\bullet(f_\bullet)$ and $H_\bullet(I)$ denotes the homology of the complex I, there is a long exact sequence

$$\longrightarrow \pi_{r+1}(Ker f_\bullet) \longrightarrow \pi_{r+1}(G_\bullet) \longrightarrow \pi_{r+1}(H_\bullet) \longrightarrow \pi_r(Ker f_\bullet) \longrightarrow \pi_r(G_\bullet) \longrightarrow H_r(I)$$

and a factorization of the morphism $\pi_r(f_\bullet)$, $\pi_r(G_\bullet) \longrightarrow H_r(I) \xrightarrow{\gamma} \pi_r(H_\bullet)$, where γ is injective.

Fibrations and trivial fibrations can be characterized in terms of lifting properties (RLP) with respect to families of morphisms which have sequentally small domains, and also, by using the Moore complex functor.

Proposition 2.3.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups. Then, the following are equivalent:

i) $f_\bullet$ is an $\bar{r}$-fibration.

ii) $f_\bullet$ has the RLP with respect to the families of morphisms $F\Delta[p,k] \to F\Delta[p]$, $p \geq r+1, 0 \leq k \leq p$.

iii) $N_p(f_\bullet)$ is surjective for $p \geq r+1$.

Proposition 2.4.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups. Then, the following are equivalent:

i) $f_\bullet$ is an $\bar{r}$-trivial fibration.

ii) $f_\bullet$ has the RLP with respect to the families of morphisms $F\dot{\Delta}[p] \to F\Delta[p]$, $p \geq r+1$ and $F\Delta[r+1,k] \to F\dot{\Delta}[r+1]$, $0 \leq k \leq r+1$.

iii) $N_{q+1}(f_\bullet)$ is surjective, $\pi_q(Ker\ f_\bullet) = 0$, $q \geq r$ and $\pi_r(f_\bullet)$ is surjective.

Using the characterizations given in propositions 2.3 and 2.4, and the so called "small object argument", we can show the factorization axiom and the lifting axiom required in the proof of the following:

Theorem 2.5.- With the definitions of $\overline{r}$-fibration, $\overline{r}$-cofibration and weak $\overline{r}$-equivalence, $r \geq 0$, the category **Simp (Gp)** is a closed simplicial model category.

In the following we will characterize the $\overline{r}-$ cofibrations.

Proposition 2.6.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups. Then $f_\bullet$ is an $\overline{r}$-cofibration if and only if the following conditions are satisfied:

i) $f_\bullet$ is a cofibration,

ii) $tr^{r-1}(f_\bullet)$ is an isomorphism,

iii) For any $y \in H_r$ there is $x_0, \dots, x_r \in G_r$ such that $(f_r(x_0), \dots, f_r(x_r), y) \in \Delta^{r+1}(H_\bullet)$.

Corollary 2.7.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a cofibration of simplicial groups. If $tr^r(f_\bullet)$ is an isomorphism, then $f_\bullet$ is an $\overline{r}-$cofibration.

Note that $f_\bullet : F\overset{\bullet}{\Delta}[r] \to F\Delta[r]$ is a cofibration of simplicial groups with $tr^{r-1}(f_\bullet)$ an isomorphism which is not an $\overline{r}$-cofibration. On the other hand, the morphism $f_\bullet : F\Delta[r+1,k] \to F\overset{\bullet}{\Delta}[r+1]$ is a cofibration of simplicial groups with $tr^{r-1}(f_\bullet)$ an isomorphism (but $tr^r(f_\bullet)$ not) which is an $\overline{r}$-cofibration.

The characterization of the $\overline{r}$-cofibrations applied to the morphism $0 \to G_\bullet$ gives the following:

Corollary 2.8.- A simplicial group is $\overline{r}-$cofibrant if and only if it is free and $r-$reduced.

Using Proposition 2.6 we can prove that the simplicial structure in **Simp(Gp)** is compatible with the $\overline{r}$-model structure (see [6]) and so we have the following:

Teorema 2.9.- The category **Simp(Gp)**, with the $\overline{r}$-structure, is a closed simplicial model category.

3. THE HOMOTOPY THEORY ASSOCIATED TO THE $\overline{r}$-STRUCTURE

.- We use the adjoint situation $I \vdash E_r$ to compare the homotopy theory associated to **Simp(Gp)** with the $\overline{r}$-structure and the homotopy theory of the closed model category $\mathbf{Simp(Gp)}_r$.

Proposition 3.1.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in $\mathbf{Simp(Gp)}_r$. Then:

i) $f_\bullet$ is a cofibration if and only if $I(f_\bullet)$ is an $\overline{r}-$cofibration.
ii) $f_\bullet$ is a weak equivalence if and only if $I(f_\bullet)$ is a weak $\overline{r}$-equivalence.
iii) $f_\bullet$ is a fibration if and only if $I(f_\bullet)$ is an $\overline{r}-$fibration.
iv) $I(f_\bullet)$ is a fibration implies that $f_\bullet$ is a fibration.

Proposition 3.2.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in **Simp(Gp)**. Then:

i) $f_\bullet$ is an $\overline{r}$-fibration if and only if $E_r f_\bullet$ is a fibration.
ii) $f_\bullet$ is a weak $\overline{r}$-equivalence if and only if $E_r f_\bullet$ is a weak equivalence.
iii) $f_\bullet$ is a fibration (weak equivalence) implies that $E_r f_\bullet$ is a fibration (weak equivalence).

The counit in the above adjunction is a weak $\overline{r}-$ equivalence and so we have (see th. 3, chapter I,4 , [6]) the following:

Theorem 3.3.- The above adjunction induces, for $r \geq 0$, an equivalence of categories

$$Ho_{\overline{r}}(\mathbf{Simp(Gp)}) \cong Ho(\mathbf{Simp(Gp)}_r).$$

and an equivalence between the homotopy theory in **Simp(Gp)** associated to the $\overline{r}-$structure and the homotopy theory in $\mathbf{Simp(Gp)}_r$.

Remarks:

1. Using Theorem 2.9 we have that, if $G_\bullet$ is an $\overline{r}$-cofibrant simplicial group, $G_\bullet \otimes \Delta[1]$ is a cylinder object for the $\overline{r}$-structure, i.e., we have a factorization of the codiagonal morphism

$$G_\bullet \coprod G_\bullet \xrightarrow{i_0+i_1} G_\bullet \otimes \Delta[1] \xrightarrow{\sigma} G_\bullet$$

where σ is a weak $\overline{r}$-equivalence and $i_0 + i_1$ is an $\overline{r}$-cofibration.
Also, for any simplicial group $H_\bullet$, (which is $\overline{r}$-fibrant), we have

$$Hom_{Ho_{\overline{r}}(\mathbf{Simp(Gp)})}(G_\bullet, H_\bullet) \cong [G_\bullet, H_\bullet]$$

where the member on the right denotes the set of simplicial homotopy classes of simplicial morphisms from $G_\bullet$ to $H_\bullet$. On the other hand, by using theorem 3.3, one has that

$$Hom_{Ho_{\overline{r}}(\mathbf{Simp(Gp)})}(G_\bullet, H_\bullet) \cong Hom_{Ho(\mathbf{Simp(Gp)}_r)}(G_\bullet, E_r H_\bullet)$$

and this last set is (see [7], prop. 3.6) the set $[G_\bullet, E_r H_\bullet]$ of simplicial homotopy classes of maps from $G_\bullet$ to $E_r H_\bullet$, or equivalently, the set $[G_\bullet, H_\bullet]$, since $G_\bullet \otimes \Delta[1]$ is a cylinder for the $\overline{r}$-structure.

2. For any $H_\bullet \in$ **Simp(Gp)** the simplicial group $H_\bullet^{\Delta[1]}$ is a path space object in **Simp(Gp)** for the $\overline{r}$-structure, since one has a factorization of the diagonal morphism

$$H_\bullet \xrightarrow{\rho} H_\bullet^{\Delta[1]} \xrightarrow{(\partial_0,\partial_1)} H_\bullet \times H_\bullet$$

where $\rho_n(x) = (s_0x, \ldots, s_nx)$ is a weak $\overline{r}$-equivalence and (∂_0, ∂_1), induced by the morphisms $(\partial_0)_n(x_0, \ldots, x_n) = d_0x_0$ and $(\partial_1)_n(x_0, \ldots, x_n) = d_{n+1}x_n$, is an $\overline{r}$-fibration.

3. Given $H_\bullet \in$ **Simp(Gp)**, the fibre of the morphism (∂_0, ∂_1), $\Omega H_\bullet$, verifies that $\pi_n(H_\bullet) \cong \pi_{n-1}(\Omega H_\bullet)$, $n \geq 1$, and it is weak equivalent to the loop complex associated to $H_\bullet$ defined in [2]. Then, both constructions determine the same functor $\Omega : Ho(\mathbf{Simp(Gp)}) \to Ho(\mathbf{Simp(Gp)})$ and then Ω determines the loop functor $\Omega^{\overline{r}}$ in $Ho_{\overline{r}}(\mathbf{Simp(Gp)})$ and there exist natural equivalences at the level of the homotopy categories $\Omega^{\overline{r}} \cong \Omega E_{r+1} \cong E_r\Omega$.

4. Given $G_\bullet \in$ **Simp(Gp)**, the cofibre $\Sigma G_\bullet$ of the morphism i_0+i_1, which can be identified as $(\Sigma G_\bullet)_n = \coprod_{i=1}^{n}(G_n)_i$, $(G_n)_i = G_n$, determines a functor $\Sigma :$ **Simp(Gp)** $\to$ **Simp(Gp)** which is left adjoint to Ω. Now, by remark 1, Σ determines the suspension functor $\Sigma^{\bar{r}}$ in $Ho_{\bar{r}}$ (**Simp(Gp)**) which is naturally equivalent to $L_r\Sigma$.

5. The functor Ω : **Simp(Gp)** $\to$ **Simp(Gp)** carries weak $\bar{r}$-equivalences into weak $(\overline{r-1})$-equivalences, and so it induces a functor

$$\Omega : Ho_{\bar{r}}(\mathbf{Simp(Gp)}) \to Ho_{\overline{r-1}}(\mathbf{Simp(Gp)})$$

On the other hand, the suspension functor Σ induces a functor

$$\Sigma^{\overline{r-1}} : Ho_{\overline{r-1}}(\mathbf{Simp(Gp)}) \to Ho_{\bar{r}}(\mathbf{Simp(Gp)})$$

and the identity functor induces another functor

$$Ho_{\overline{r-1}}(\mathbf{Simp(Gp)}) \to Ho_{\bar{r}}(\mathbf{Simp(Gp)})$$

The composition will be denoted by $\Sigma : Ho_{\overline{r-1}}(\mathbf{Simp(Gp)}) \to Ho_{\bar{r}}(\mathbf{Simp(Gp)})$ and this functor is left adjoint to $\Omega : Ho_{\bar{r}}(\mathbf{Simp(Gp)}) \to Ho_{\overline{r-1}}(\mathbf{Simp(Gp)})$ since, given $G_\bullet$, $H_\bullet \in$ **Simp(Gp)**, we have

$$Hom_{Ho_{\bar{r}}(\mathbf{Simp(Gp)})}(\Sigma G_\bullet, H_\bullet) \cong Hom_{Ho(\mathbf{Simp(Gp)})}(\Sigma G_\bullet, E_r H_\bullet) \cong$$
$$\cong Hom_{Ho(\mathbf{Simp(Gp)})}(G_\bullet, \Omega E_r H_\bullet) \cong Hom_{Ho(\mathbf{Simp(Gp)})}(G_\bullet, E_{r-1}\Omega H_\bullet) \cong$$
$$\cong Hom_{Ho_{\overline{r-1}}(\mathbf{Simp(Gp)})}(G_\bullet, \Omega H_\bullet)$$

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CLOSED MODEL STRUCTURES FOR n-TYPES OF SIMPLICIAL GROUPS

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ABSTRACT. We consider in the category **Simp(Gp)** of simplicial groups weak n-equivalences, for any $n \geq 0$, as those morphisms $f_\bullet$ such that $\pi_q(f_\bullet)$ is an isomorphism for $0 \leq q \leq n$. With suitable definitions of n-fibration and n-cofibration we have a closed model structure in **Simp(Gp)** whose homotopy theory is equivalent to that one in the category of the connected and $(n+1)$-coconnected CW-complexes. Cylinder and path objects for this n-structure are then given and we show that the associated homotopy relation is obtained by truncating the simplicial homotopy relation.

1.PRELIMINARIES.- From now on, **Simp(Gp)** will denote the category of simplicial groups and if $G_\bullet$ is a simplicial group, $N_\bullet(G_\bullet)$ and $\pi_\bullet(G_\bullet)$ will denote the normalized complex and homotopy groups of $G_\bullet$ in the sense of Moore.

Using the simplicial kernel construction, one has a functor, $cosk^{n-1}$, from the category of $(n-1)$-truncated simplicial groups to **Simp(Gp)**. This functor is right adjoint to the $(n-1)$-truncating functor, tr^{n-1}, which truncates a simplicial group at dimension $n-1$. We will denote $Cosk^{n-1} = cosk^{n-1}tr^{n-1}$. The truncating functor tr^{n-1} has also a left adjoint, sk^{n-1}, called the $(n-1)$-skeleton functor, which can be described by iterating the notion of n-simplicial cokernel (see [2]). We will write $Sk^{n-1} = sk^{n-1}tr^{n-1}$ and then, this functor is left adjoint to $Cosk^{n-1}$.

The category **Simp(Gp)** is a closed model category in the sense of Quillen, [6], where the cofibrations were characterized as the retracts of the free simplicial group maps. Next we recall some constructions in the homotopy theory associated to this closed model structure.

For any simplicial group $G_\bullet$ and $I = \Delta[1]$, $G_\bullet \otimes I$ is the simplicial group with $(G_\bullet \otimes I)_n = \coprod_{i=0}^{n+1} (G_n)_i$, $(G_n)_i = G_n \ \forall i$ and natural simplicial operations. If $G_\bullet$ is cofibrant, this construction gives a cylinder object for $G_\bullet$ in **Simp(Gp)** since one has a factorization of the codiagonal morphism, $G_\bullet \coprod G_\bullet \xrightarrow{i_0+i_1} G_\bullet \otimes I \xrightarrow{\sigma_\bullet} G_\bullet$, where $\sigma_\bullet$, which is the morphism induced by the identities, is a weak equivalence and $i_0 + i_1$, which is the induced morphism by the first and last inclusions respectively, is a cofibration.

Given $H_\bullet \in$ **Simp(Gp)** recall that $H_\bullet^I = \underline{Hom}_{\mathbf{Simp(Sets)}}(I, H_\bullet)$ is the simplicial group whose n-simplices are $(H_\bullet^I)_n = Hom_{\mathbf{Simp(Sets)}}(I \times \Delta[n], H_\bullet) \cong \{(x_0, ..., x_n) \in H_{n+1}^{n+1} \ / \ d_i x_i = d_i x_{i-1}, 1 \leq i \leq n\}$. $H_\bullet^I$ is a path space object for $H_\bullet$ in **Simp(Gp)** since one has a factorization of the diagonal morphism, $H_\bullet \xrightarrow{s_\bullet} H_\bullet^I \xrightarrow{(\partial_0, \partial_1)} H_\bullet \times H_\bullet$, where $s_\bullet$ is a weak equivalence and (∂_0, ∂_1) is a fibration.

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These cylinder and path constructions determine adjoint functors

$$-\otimes I \vdash (-)^I : \mathbf{Simp(Gp)} \to \mathbf{Simp(Gp)}$$

and also they determine adjoint functors (suspension and loop functors)

$$\Sigma \vdash \Omega : \mathbf{Simp(Gp)} \to \mathbf{Simp(Gp)}$$

where $\Sigma G_\bullet$, which is the cofibre of the morphism $i_0 + i_1$, is the simplicial group with $(\Sigma G_\bullet)_n = \coprod_{i=1}^{n} (G_n)_i$, $(G_n)_i = G_n \; \forall i$, and $\Omega H_\bullet$, which is the fibre of (∂_0, ∂_1)), is the simplicial group with $(\Omega H_\bullet)_n = \{(x_0, \ldots, x_n) \in H_{n+1}^{n+1} \,/\, d_i x_i = d_i x_{i-1} \; 1 \le i \le n;\; d_0 x_0 = d_{n+1} x_n = 0\}$.

The category **Simp(Gp)** is a closed simplicial model category, [6], where the function complex $\underline{Hom}_{\mathbf{Simp(Gp)}}(G_\bullet, H_\bullet)$ has as n-simplices the simplicial maps $F_\bullet : G_\bullet \times \Delta[n] \to H_\bullet$ such that $F_t : G_q \to H_q$, given by $F_t(x) = F_q(x,t)$, is a group morphism. Given morphisms $f_\bullet, g_\bullet : G_\bullet \to H_\bullet$, $f_\bullet$ is simplicially homotopic to $g_\bullet$ if $f_\bullet$ is homotopic to $g_\bullet$ when $f_\bullet$ and $g_\bullet$ are regarded as 0-simplices of $\underline{Hom}_{\mathbf{Simp(Gp)}}(G_\bullet, H_\bullet)$. A simplicial homotopy from $f_\bullet$ to $g_\bullet$ is then a simplicial map $F_\bullet : G_\bullet \times I \to H_\bullet$ such that $F_t : G_q \to H_q$ is a group morphism and such that $d_0 F_\bullet = f_\bullet$ and $d_1 F_\bullet = g_\bullet$. This is equivalent to give either a simplicial morphism $\kappa_\bullet : G_\bullet \otimes I \to H_\bullet$ verifying $\kappa_\bullet i_0 = f_\bullet$ and $\kappa_\bullet i_1 = g_\bullet$ or a simplicial morphism $\kappa'_\bullet : G_\bullet \to H^I_\bullet$ verifying $\partial_0 \kappa'_\bullet = g_\bullet$ and $\partial_1 \kappa'_\bullet = f_\bullet$.

Note that to give such a morphism $\kappa_\bullet$ is equivalent to give a system of morphisms $(k^n_j : G_n \to H_n,\; 1 \le j \le n)$ satisfying the following identities:

$d_0 k^n_1 = f_{n-1} d_0$	$d_i k^n_j = k^{n-1}_{j-1} d_i$	$for\ i < j$	$s_i k^n_j = k^{n+1}_j s_i$	$for\ i \ge j$
$d_n k^n_n = g_{n-1} d_n$	$d_i k^n_j = k^{n-1}_j d_i$	$for\ i \ge j$	$s_i k^n_j = k^{n+1}_{j+1} s_i$	$for\ i < j$

Note also that to give such a morphism $\kappa'_\bullet$ is equivalent to give a system of morphisms $(h^n_j : G_n \to H_{n+1}, 0 \le j \le n)$ satisfying the following homotopy identities with respect to the face and degeneracy operators:

$d_0 h^n_0 = f_n$	$d_{n+1} h^n_n = g_n$	$d_i h^n_j = h^{n-1}_{j-1} d_i$	$for\ i < j$
$s_i h^{n-1}_j = h^n_{j+1} s_i$	$for\ i \le j$	$d_{j+1} h^n_{j+1} = d_{j+1} h^n_j$	
$s_i h^{n-1}_j = h^n_j s_{i-1}$	$for\ i > j$	$d_i h^n_j = h^{n-1}_j d_{i-1}$	$for\ i > j+1$

The equivalence between both systems of morphisms is given by the rules $k^n_i = d_i h^n_{i-1}$ and $h^n_i = k^{n+1}_{i+1} s_i$.

Let us consider now the full subcategory of **Simp(Gp)** whose objects are those simplicial groups with trivial Moore complex in dimensions $> n$. This category, denoted $\mathbf{n-Hypgd(Gp)}$ because it is the category of n-hypergroupoids of groups in the sense of Duskin-Glen, [4], was showed as a closed model category in [1]. The fibrations and weak equivalences were defined as those morphisms which are fibrations and weak equivalences respectively in **Simp(Gp)**, and the cofibrations were defined by the LLP with respect to the trivial fibrations. This category is a reflexive subcategory of **Simp(Gp)** and if $\mathcal{P}$ denotes the reflector functor, the adjunction $\mathcal{P} \vdash J$ induces an equivalence of categories (see [1])

$$Ho(\mathbf{n-Hygpd(Gp)}) \cong Ho(\mathbf{Simp(Gp)}|n-coconnected)$$

where the member on the right denotes the full subcategory of the category $Ho(\mathbf{Simp(Gp)})$ whose objects are those simplicial groups n-coconnected, i.e., with $\pi_i = 0$ for $i \ge n+1$.

Thus one has an equivalence between $Ho(\mathbf{n}-\mathbf{Hypgd(Gp)})$ and the homotopy category of the $(n+1)$-coconnected CW complexes.

2.THE n-STRUCTURE IN **Simp(Gp)**.- Let us consider, for any $n \geq 0$, the pair of adjoint functors in **Simp(Gp)**, $Sk^{n+1} \vdash Cosk^{n+1}$. We then propose the following structure in this category:

Definition 2.1.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in **Simp(Gp)** and $n \geq 0$.

i) $f_\bullet$ is said to be an n-fibration if $Cosk^{n+1}(f_\bullet)$ is a fibration in **Simp(Gp)**.

ii) $f_\bullet$ is said to be a weak n-equivalence if $Cosk^{n+1}(f_\bullet)$ is a weak equivalence in **Simp(Gp)**.

iii) $f_\bullet$ is said to be an n-cofibration if it has the *LLP* with respect to the trivial n-fibrations.

It is clear with this definition that all simplicial group is n-fibrant for all $n \geq 0$.

Note that a similar structure has been proposed in [5] for crossed complexes and in [3] for the categories of topological spaces and simplicial sets.

Proposition 2.2.- If $f_\bullet : G_\bullet \to H_\bullet$ is an n-fibration, I is the image of $N_\bullet(f_\bullet)$ and $H_\bullet(I)$ denotes the homology of the complex I, there is a long exact sequence

$$\cdots \to \pi_{n+1}(G_\bullet) \to H_{n+1}(I) \to \pi_n(Ker f_\bullet) \to \pi_n(G_\bullet) \to \pi_n(H_\bullet) \to \cdots$$

and a factorization of the morphism $\pi_{n+1}(f_\bullet)$,

$$\pi_{n+1}(G_\bullet) \xrightarrow{\epsilon} H_{n+1}(I) \xrightarrow{\gamma} \pi_{n+1}(H_\bullet)$$

where ϵ and γ are surjective.

Next, we characterize the (trivial) n-fibrations.

Proposition 2.3.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups.

i) $f_\bullet$ is an n-fibration if and only if $N_q(f_\bullet)$ is surjective for $1 \leq q \leq n+1$ and $Ker(N_{n+1}(G_\bullet) \to N_n(G_\bullet)) \to Ker(N_{n+1}(H_\bullet) \to N_n(H_\bullet))$ is surjective.

ii) $f_\bullet$ is a weak n-equivalence if and only if $\pi_q(f_\bullet)$ is an isomorphism for $0 \leq q \leq n$.

iii) $f_\bullet$ is a trivial n-fibration if and only if $N_q(f_\bullet)$ is surjective for $0 \leq q \leq n+1$ and $\pi_q(Ker(f_\bullet)) = 0$ for $0 \leq q \leq n$.

Proposition 2.4.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism of simplicial groups.

i) $f_\bullet$ is an n-fibration if and only if $f_\bullet$ has the RLP with respect to $F\Delta[q,k] \to F\Delta[q], 1 \leq q \leq n+1, 0 \leq k \leq q$, and with respect to $F\Delta[n+2,k] \to F\dot{\Delta}[n+2], 0 \leq k \leq n+2$.

ii) $f_\bullet$ is a trivial n-fibration if and only if $f_\bullet$ has the RLP with respect to $F\dot{\Delta}[q] \to F\Delta[q], 0 \leq q \leq n+1$.

Note that one has then the following inclusions of classes of morphisms:

$$\{triv.fibs.\} \subseteq \ldots \subseteq \{(n+1) - triv.fibs.\} \subseteq \{n - triv.fibs.\} \subseteq \ldots$$
$$\ldots \subseteq \{n - cofibs.\} \subseteq \{(n+1) - cofibs.\} \subseteq \ldots \{cofibs.\}$$

We can now prove the following:

Theorem 2.5.- **Simp(Gp)** is a closed model category, for any $n \geq 0$, with the structure given in definition 2.1 (this structure will be called the "n-structure").

The cofibrant objects and the cofibrations in **Simp(Gp)** with respect to the n-structure are characterized as follows:

Proposition 2.6.- Let $G_\bullet$ be a simplicial group. Then $G_\bullet$ is n-cofibrant, $n \geq 0$, if and only if $G_\bullet$ is cofibrant (i.e., a free simplicial group) and $G_\bullet = Sk^{n+1}(G_\bullet)$.

Proposition 2.7.- Let $f_\bullet : G_\bullet \to G_\bullet \coprod FU_\bullet$ be a cofibration of simplicial groups. Then $f_\bullet$ is an n-cofibration, $n \geq 0$, if and only if every element of U_q is a degenerated simplex for $q \geq n+2$.

3.THE HOMOTOPY THEORY ASSOCIATED TO THE n-STRUCTURE.- Next we will use the adjoint situation $\mathcal{P} \vdash J$ to compare the homotopy theory associated to **Simp(Gp)** with the n-structure and the homotopy theory associated to the closed model category $\mathbf{n} - \mathbf{Hypgd(Gp)}$ recalled in Section 1.

Proposition 3.1.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in $\mathbf{n} - \mathbf{Hypgd(Gp)}$. Then $f_\bullet$ is a fibration (resp. weak equivalence) if and only if $I(f_\bullet)$ is an n-fibration (resp. weak n-equivalence).

Proposition 3.2.- Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in **Simp(Gp)**.

i)If $f_\bullet$ is an n-cofibration $\mathcal{P}(f_\bullet)$ is a cofibration.

ii) $f_\bullet$ is a weak n-equivalence if and only if $\mathcal{P}(f_\bullet)$ is a weak equivalence.

iii) If $f_\bullet$ is an n-fibration then $\mathcal{P}(f_\bullet)$ is a fibration.

If Ho_n(**Simp(Gp)**) denotes the homotopy category of **Simp(Gp)** with the n-structure and we use the criterion for an equivalence of homotopy theories given by Quillen in [6], we have:

Theorem 3.3.- The adjunction $\mathcal{P} \vdash J$ induces, for $n \geq 0$, an equivalence of categories

$$Ho_n(\mathbf{Simp(Gp)}) \cong Ho(\mathbf{n} - \mathbf{Hypgd(Gp)})$$

and an equivalence between the homotopy theory in **Simp(Gp)** associated to the n-structure and the homotopy theory in $\mathbf{n} - \mathbf{Hypgd(Gp)}$.

Remarks:

1. Let $G_\bullet$ be a simplicial group and let $G_\bullet \otimes_n I$ be the simplicial group defined by the following pushout diagram (where $I_0 = \overset{\bullet}{\Delta}[1]$)

$$\begin{array}{ccc} Sk^{n+1}(G_\bullet \otimes I_0) & \xrightarrow{\alpha_\bullet} & G_\bullet \otimes I_0 \\ {\scriptstyle Sk^{n+1}(i_0+i_1)}\downarrow & & \downarrow{\scriptstyle \bar{i}_0+\bar{i}_1} \\ Sk^{n+1}(G_\bullet \otimes I) & \xrightarrow{\beta_\bullet} & G_\bullet \otimes_n I \end{array}$$

If $G_\bullet$ is n-cofibrant we have a factorization of the codiagonal morphism

$$G_\bullet \coprod G_\bullet = G_\bullet \otimes I_0 \xrightarrow{\bar{i}_0+\bar{i}_1} G_\bullet \otimes_n I \xrightarrow{\bar{\sigma}_\bullet} G_\bullet$$

where $\bar{\sigma}_\bullet$ is a weak n-equivalence and $\bar{i}_0 + \bar{i}_1$ is an n-cofibration and so $G_\bullet \otimes_n I$ is a cylinder object for $G_\bullet$ in **Simp(Gp)** with the n-structure.

Note that $Coker(Sk^{n+1}(i_0 + i_1)) \cong Coker(\bar{i}_0 + \bar{i}_1) = {}^n\Sigma(G_\bullet)$, the suspension functor in **Simp(Gp)** for the n-structure. On the other hand we have that

$$Coker(Sk^{n+1}(i_0 + i_1)) \cong Sk^{n+1}(Coker(i_0 + i_1)) = Sk^{n+1}(\Sigma(G_\bullet))$$

and therefore ${}^n\Sigma(G_\bullet) \cong Sk^{n+1}(\Sigma(G_\bullet))$.

It is not very difficult to prove that the following pushout diagram also gives a cylinder object for $G_\bullet$ with respect to the n-structure

$$\begin{array}{ccc} Sk^n G_\bullet \otimes I_0 & \longrightarrow & G_\bullet \otimes I_0 \\ \downarrow & & \downarrow \\ Sk^n G_\bullet \otimes I & \longrightarrow & G_\bullet \otimes'_n I \end{array}$$

and then, one deduces easily that ${}^n\Sigma G_\bullet$ and $\Sigma(Sk^n G_\bullet)$ are weak n-equivalent.

2. Given $H_\bullet \in$ **Simp(Gp)** consider the simplicial group ${}_nH^I_\bullet$ given by the following pullback diagram

$$\begin{array}{ccc} {}_nH^I_\bullet & \longrightarrow & (Cosk^{n+1}H_\bullet)^I \\ {\scriptstyle (\bar{\partial}_0,\bar{\partial}_1)}\downarrow & & \downarrow{\scriptstyle (\partial_0,\partial_1)} \\ H^{I_0}_\bullet & \longrightarrow & (Cosk^{n+1}H_\bullet)^{I_0} \end{array}$$

This construction gives a functor which is clearly right adjoint to the first cylinder functor defined in 1) and we have factorization of the diagonal morphism

$$H_\bullet \xrightarrow{\bar{s}_\bullet} H^I_\bullet \xrightarrow{(\bar{\partial}_0,\bar{\partial}_1)} H_\bullet \times H_\bullet = H^{I_0}_\bullet$$

where $\bar{s}_\bullet$ is a weak n-equivalence and $(\bar{\partial}_0, \bar{\partial}_1)$ is an n-fibration. Thus, ${}_nH^I_\bullet$ gives a path object for $H_\bullet$ in **Simp(Gp)** for the n-structure.

Note that ${}^n\Omega H_\bullet = Ker(\bar{\partial}_0, \bar{\partial}_1) \cong Ker(\partial_0, \partial_1) = \Omega(Cosk^{n+1}H_\bullet)$. This functor ${}^n\Omega$ is right adjoint to the suspension functor ${}^n\Sigma$.

3. Given $f_\bullet, g_\bullet : G_\bullet \to H_\bullet$, we can truncate the concept of what a homotopy from $f_\bullet$ to $g_\bullet$ is, in two different ways.

On the one hand, we say that $f_\bullet$ is n-homotopic to $g_\bullet$ if there is a system of morphisms $\{k_i^p : G_p \to H_p, 1 \le p \le n+1, 1 \le i \le p\}$ satisfying the identities

$d_0 k_1^p = f_{p-1} d_0$	$d_i k_j^p = k_{j-1}^{p-1} d_i \quad for\ i < j$	$s_i k_j^p = k_j^{p+1} s_i \quad for\ i \ge j$
$d_p k_p^p = g_{p-1} d_p$	$d_i k_j^p = k_j^{p-1} d_i \quad for\ i \ge j$	$s_i k_j^p = k_{j+1}^{p+1} s_i \quad for\ i < j$

On the other hand, we can say that $f_\bullet$ is n-homotopic to $g_\bullet$ if there is a system of morphisms $(h_j^q : G_q \to H_{q+1}, 0 \le q \le n, 0 \le j \le q)$ satisfying the following homotopy identities with respect to the face and degeneracy operators:

$d_0 h_0^q = f_q$	$d_{q+1} h_q^q = g_q$	$d_i h_j^q = h_{j-1}^{q-1} d_i$	$for\ i < j$
$s_i h_j^{q-1} = h_{j+1}^q s_i$	$for\ i \le j$	$d_{j+1} h_{j+1}^q = d_{j+1} h_j^q$	
$s_i h_j^{q-1} = h_j^q s_{i-1}$	$for\ i > j$	$d_i h_j^q = h_j^{q-1} d_{i-1}$	$for\ i > j+1$

The first concept of n-homotopy is equivalent to the concept of homotopy which is derived from the first cylinder object defined in 1) (or equivalently from the path object defined in 2)), i.e., $f_\bullet$ is n-homotopic to $g_\bullet$ if and only if there is a simplicial morphism $\kappa_\bullet : G_\bullet \otimes_n I \to H_\bullet$ such that $\kappa_\bullet(\bar{i}_0 + \bar{i}_1) = f_\bullet + g_\bullet$.

The second concept of n-homotopy is equivalent to the concept of homotopy which is derived from the second cylinder object in 1), i.e., $f_\bullet$ is n-homotopic to $g_\bullet$ if there is a simplicial morphism $\kappa'_\bullet : G_\bullet \otimes'_n I \to H_\bullet$.

It is not difficult to see that the morphisms $\bar{i}_0$, $\bar{i}_1 : G_\bullet \to G_\bullet \otimes_n I$ are n-homotopic in the second sense, so that there exists a morphism $G_\bullet \otimes'_n I \to G_\bullet \otimes_n I$. This gives that the first concept of n-homotopy implies the second one. Now, if $G_\bullet$ is n-cofibrant (since $H_\bullet$ is always n-fibrant), both concepts of n-homotopy coincide, because, they are just the left homotopy relation derived from the n-structure.

Note that for any abelian group the morphisms $0, Id : K(A, n+1) \to K(A, n+1)$ are n-homotopic but obviously they are not simplicially homotopic.

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A NOTE ON THE $12(g-1)$ BOUND

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ABSTRACT. In these notes we consider a particular family of groups of conformal automorphisms on closed Riemann surfaces of genus at least two. We see that the order of such groups are bounded by the classical $12(g-1)$. As a consequence, we obtain the results in [5] and [4]. We also obtain the following: (1) Let H be a group of conformal automorphisms on a stable Riemann surface R of genus at least two, with at least one node, with the property that: if for some $h \in H$ we have that h^k keeps invariant a component R_i of R minus its nodes and it acts as the identity there, then h^k is the identity. Then H has order at most $12(g-1)$. (2) Let K be a function group, different from a Fuchsian one, with invariant component Δ. If K contains a group G such that Δ/G is a closed Riemann surface of genus $g \geq 2$, then the index $[K:G]$ is bounded by $12(g-1)$.

In many articles appear the bound $12(g-1)$ for some type of groups of conformal automorphisms on closed Riemann surfaces. Examples of groups with this bound property are the following:

(1) groups of conformal automorphisms of a closed Riemann surface of genus greater than one which can be lifted to some Schottky covering [5],
(2) groups of conformal automorphisms of a closed real Riemann surface of genus at least two, with real points, that commute with the real structure [4].

In these notes we observe that a certain family of groups of conformal automorphisms of closed Riemann surfaces of genus at least two, say g, have the property

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that their orders are bounded by $12(g-1)$. The examples above are particular cases of such a category.

First of all, we start with the following easy fact due to the Riemann-Hurwitz formula. If H is a group of conformal automorphisms of a closed Riemann surface S of genus $g \geq 2$, and S/H is different from an sphere with exactly three singular points, then $|H| \leq 12(g-1)$. In fact, the Riemann-Hurwitz formula asserts that $\text{Area}(S) = |H|\text{Area}(S/H)$. But the smaller area for the quotient S/H, under the assumption that it is not a sphere with exactly three singular points, is $\pi/3$ (the area of the sphere with four singular points of orders $2,2,2,3$). Then we have the inequality $\text{Area}(S) = |H|\text{Area}(S/H) \leq |H|\pi/3$. Since $\text{Area}(S) = 4\pi(g-1)$, the inequality $|H| \leq 12(g-1)$ follows.

The category of groups of conformal automorphisms we are interested in these notes are the following. Let S be a closed Riemann surface of genus $g \geq 2$ and $\Sigma \neq \emptyset$ a family of pairwise disjoint homotopically independent simple loops on S. Denote by $N(\Sigma)$ the normalizer of Σ in the free homotopy group of S. We set $Aut(\Sigma)$ the group of conformal automorphisms of S that preserves, up to free homotopy, the group $N(\Sigma)$.

We have that the groups considered by Zimmermann and May have the property that they leave invariant a family Σ as above. In particular, they leave invariant $N(\Sigma)$. The main result is the following:

Theorem. *The quotient $S/Aut(\Sigma)$ is not a sphere with exactly three singular points. In particular, $|Aut(\Sigma)| \leq 12(g-1)$.*

Proof. Let us consider a regular covering $\pi : \Delta \to S$ defined by $N(\Sigma)$, that is, a smaller regular covering for which the loops in $N(\Sigma)$ lift to loops. This is a planar covering [3].

The group $Aut(\Sigma)$ can be lifted to a group K of conformal automorphisms of Δ via π. Set G the Deck group of the regular covering $\pi : \Delta \to S$ defined by $N(\Sigma)$. In particular, G is a normal subgroup of K of index $|Aut(\Sigma)|$.

The planarity theorem of B. Maskit in [3] asserts that we may assume Δ to be a connected open subset of the Riemann sphere and K a group of Möbius transformations. Since the group G has finite index in K, we have that K is a Kleinian group with Δ as invariant set inside its region of discontinuity. It follows that K is a function group with an invariant component Δ_0 containing Δ. Since $\Delta/G \subset \Delta_0/G$ and Δ/G is a closed Riemann surface, we must have that $\Delta = \Delta_0$.

If $S/Aut(\Sigma) = \Delta/K$ is a sphere with exactly three singular points, then it follows from a result of I. Kra [2] that K is a Fuchsian group and, in particular, that G is a Fuchsian group. We obtain a contradiction to the fact that Σ is a non-empty family. □

Another family of conformal groups that belongs to the above category is given by the following. Let R be a stable Riemann surface of genus at least two. Assume that R has at least one node. Set H the group of conformal automorphisms of R with the following property: if for some $h \in H$ we have that h^k keeps invariant a component R_i of R minus its nodes and it acts as the identity there, then h^k is the identity. One may now produce, after fattening the nodes of R, a closed orientable surface S of the same genus as R, a group J of orientation preserving homeomorphisms of S, an isomorphism $\theta : J \to H$ and a natural deformation $d : S \to R$ satisfying $dj = \theta(j)d$ for all $j \in J$.

The Nielsen realization theorem [1] asserts that we may assume S a Riemann surface and J a group of conformal automorphisms. The family Σ in this case is given by the loops obtained from the fattening process of the nodes of R. It follows from the above theorem that the order of H is bounded by $12(g-1)$.

Another result, direct consequence of the proof of the theorem and the results of [3], is the following. If F is a function group with invariant component Δ, different from a Fuchsian group, Δ/F is a closed Riemann surface of genus $g \geq 2$, and K a Kleinian group containig F as a normal subgroup (in particular, a function group with the same invariant component) , then the index of F in K is at most $12(g-1)$.

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PAIRES DE GUELFAND GENERALISEES ET LE GROUPE D'HEISENBERG

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Presented by P.C. Greiner, F.R.S.C.

Résumé

La paire $(U_{p,q} \wedge H_{p,q}, U_{p,q})$ est une paire de Guelfand généralisée. Nous déterminons les distributions sphériques de type positif qui interviennent dans la formule de Plancherel, et nous décrivons les représentations unitaires irréductibles de $U_{p,q} \wedge H_{p,q}$ qui leurs correspondent.

Abstract

$(U_{p,q} \wedge H_{p,q}, U_{p,q})$ is a generalised Gelfand pair. We determine the spherical distributions of positive type wich appear in the Plancherel formula, and we describe the correspondent irreducible unitary representations of $U_{p,q} \wedge H_{p,q}$.

La notion de paires de Guelfand généralisées a été introduite par Thomas [3]. Dans ce travail on considère le groupe d'Heisenberg $H_{p,q} = \mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^q$ muni du produit: $(t, z_1, z_2)(t', z'_1, z'_2) = (t + t' + Im({}^t z_1 \overline{z'}_1 -{}^t z_2 \overline{z'}_2), z_1 + z'_1, z_2 + z'_2)$ $U_{p,q}$ et $U_p \times U_q$ opèrent sur $H_{p,q}$ par automorphismes de la façon suivante: $g(t,z) = (t, g(z))$ de sorte qu'on a les produits semi-directs: $G = U_{p,q} \wedge H_{p,q}$ et $G' = U_p \times U_q \wedge H_{p,q}$. $(G', U_p \times U_q)$ est une paire de Guelfand et $(G, U_{p,q})$ est une paire de Guelfand généralisée. Van Dijk a remarqué dans [4] que cette paire est de Guelfand en utilisant d'autres arguments.

Si T est une distribution sphérique pour $(G, U_{p,q})$ alors c'est une distribution de type positif sur $H_{p,q}$ invariante par $U_p \times U_q$, une telle distribution se décompose d'une façon unique suivant les fonctions sphériques de $(G', U_p \times U_q)$. L'idée de ce travail est de déterminer cette décomposition et d'utiliser l'unicité de la mesure de Plancherel pour déduire la formule de Plancherel pour $(G, U_{p,q})$ à partir de celle de $(G', U_p \times U_q)$.

Aux constantes de normalisation près les fonctions sphériques de type positif qui interviennent dans la formule de Plancherel de $(G', U_p \times U_q)$ sont données pour tout $\lambda \in \mathbb{R}^*, m, n \in \mathbb{N}$ par:

$$\Phi_{\lambda,m,n}(t,z) = e^{-i\lambda t} e^{\frac{-|\lambda|}{2}(|z_1|^2+|z_2|^2)} L_m^{p-1}(|\lambda||z_1|^2) L_n^{q-1}(|\lambda||z_2|^2)$$

$L_m^{p-1} et L_n^{q-1}$ sont des polynômes de Laguerre. La formule de Plancherel s'écrit:

$$\delta_0 = \frac{1}{\pi^{p+q-1}} \int_{\mathbb{R}^*} \sum_{m,n \in N} \Phi_{\lambda,m,n} |\lambda|^{p+q} d\lambda$$

δ_0 étant la mesure de Dirac à l'origine de $H_{p,q}$, elle correspond au noyau de $L^2(H_{p,q})$. Pour ces résultats voir Mokni [1].

1. Représentations admettant un vecteur distribution $U_{p,q}$-invariant

Les représentations unitaires irréductibles de $H_{p,q}$ qui ont une restriction non triviale sur le centre se réalisent dans les espaces de Fock-Bergman définis pour $\lambda \in \mathbb{R}^*$ par $\mathcal{H}_\lambda$ l'espace des fonctions f holomorphes sur $\mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q$ telles que:

$$\int_{\mathbb{C}^{p+q}} |f(\zeta_1, \zeta_2)|^2 e^{-|\lambda||(\zeta_1,\zeta_2)|^2} d\zeta_1 d\zeta_2 < \infty$$

La représentation π_λ associée est définie par:

$$[\pi_\lambda(t, z_1, z_2)f](\zeta_1, \zeta_2') = e^{-i\lambda t} e^{|\lambda|\langle\zeta_1, z_1\rangle} e^{|\lambda|\langle\zeta_2, \overline{z_2}\rangle} e^{-\frac{|\lambda|}{2}(|z_1|^2+|z_2|^2)} f(\zeta_1 - z_1, \zeta_2 - \overline{z_2})$$

Wolf a prouvé dans [4] que la classe de la représentation π_λ est $U_{p,q}$-stable, et qu'elle se prolonge en une représentation $\tilde{\pi}_\lambda$ de $U_{p,q} \wedge H_{p,q}$ dans $\mathcal{H}_\lambda$. $\tilde{\pi}_\lambda$ est décrite par Sternberg et Wolf dans [2]. Les représentations unitaires irréductibles de G relatives aux π_λ sont de la forme $\tilde{\pi}_\lambda \otimes \gamma$ où $\gamma \in \hat{U}_{p,q}$.

Pour simplifier les notations, on va décrire les résultats pour $\lambda = 1$, les choses sont tout à fait analogues pour λ quelconque. On pose $\pi = \tilde{\pi}_1$ et $\mathcal{H} = \mathcal{H}_1$. Pour $d \in \mathbb{Z}$ soit $\mathcal{H}_d = \{f \in \mathcal{H}; f(r\zeta_1, \frac{1}{r}\zeta_2) = r^d f((\zeta_1, \zeta_2)\}$ c'est l'adhérence du sous espace de $\mathcal{H}$ engendré par les fonctions polynômes homogènes de degré n_1 en ζ_1 et homogènes de degré n_2 en ζ_2 avec $n_1 - n_2 = d$. On note ν_d la restriction de $\pi_{|U_{p,q}}$ à $\mathcal{H}_d$ et par $\overline{\nu}_d^*$ sa contragrédiente.
Sternberg et Wolf [2] ont montré que ν_d est irréductible, et on a le théorème suivant.

Théorème 1

Pour tout $d \in \mathbb{Z}$ la représentation $\tilde{\pi} \otimes \overline{\nu}_d^*$ admet un vecteur généralisé non nul qui est $U_{p,q}$-invariant , il est égal à:

$$\sum_{m-n=d} \frac{1}{m!}\langle\zeta_1, \alpha_1\rangle^m \frac{1}{n!}\langle\zeta_2, \alpha_2\rangle^n.$$

2. Distributions sphériques et formule de plancherel

En remarque que pour $d \in \mathbb{Z}$, $\mathcal{H}_d = \bigoplus_{m-n=d} \mathcal{H}_m(\mathbb{C}^p) \otimes \mathcal{H}_n(\mathbb{C}^q)$, $\mathcal{H}_m(\mathbb{C}^p)$ étant l'espace des polynômes sur $\mathbb{C}^p$ homogène de degré m.

Le noyau reproduisant de $\mathcal{H}_d$ est égal à $\sum_{m-n=d} \frac{1}{m!}\langle\zeta_1, \alpha_1\rangle^m \frac{1}{n!}\langle\zeta_2, \alpha_2\rangle^n$ d'après le théorème précédent, et le noyau reproduisant de $\mathcal{H}_m(\mathbb{C}^p) \otimes \mathcal{H}_n(\mathbb{C}^q)$ est $\frac{1}{m!}\langle\zeta_1, \alpha_1\rangle^m . \frac{1}{n!}\langle\zeta_2, \alpha_2\rangle^n$.
Ces propriétés nous donnent

Théorème 2

Les distributions sphériques associées aux représentations précédentes sont données pour tout $(\lambda, d) \in \mathbb{R}^* \times \mathbb{Z}$ par:

$$ {}_\lambda\mathcal{K}_d = \sum_{m-n=d} \Phi_{\lambda,m,n}. $$

La convergence a lieu dans $\mathcal{D}'(H_{p,q})$.

Ce théorème et la formule de Plancherel pour $(G', U_p \times U_q)$ donnent:

Théorème 3

La formule de Plancherel pour la paire de Guelfand généralisée $(U_{p,q} \wedge H_{p,q}, U_{p,q})$ est donnée par

$$ \delta_0 = \frac{1}{\pi^{p+q-1}} \int_{\mathbb{R}^*} \sum_{d\in\mathbb{Z}} {}_\lambda\mathcal{K}_d |\lambda|^{p+q} d\lambda $$

Démonstration: La distribution δ_0 se décompose en somme de distributions sphériques de type positif $U_p \times U_q$-invariantes de la façon suivante:

$$ \begin{aligned} \delta_0 &= \frac{1}{\pi^{p+q-1}} \int_{\mathbb{R}^*} \sum_{n,m\in\mathbb{N}} \Phi_{\lambda,m,n} |\lambda|^{p+q} d\lambda \\ &= \frac{1}{\pi^{p+q-1}} \int_{\mathbb{R}^*} \sum_{d\in\mathbb{Z}} \Big(\sum_{m-n=d} \Phi_{\lambda,m,n} \Big) |\lambda|^{p+q} d\lambda \\ &= \frac{1}{\pi^{p+q-1}} \int_{\mathbb{R}^*} \sum_{d\in\mathbb{Z}} {}_\lambda\mathcal{K}_d |\lambda|^{p+q} d\lambda \end{aligned} $$

Le fait que les ${}_\lambda\mathcal{K}_d$ soient des distributions sphériques de type positif $U_{p,q}$-invariantes et l'unicité de la mesure de Plancherel nous donnent le résultat.

3. Expression analytique des distributions sphériques

Pour ${}_\lambda\mathcal{K}_d$ on a plus explicitement:

Théorème 4

Pour $d \in \mathbb{Z} - \{1-p, 1-p+1, ..., q-1\}$ et $\lambda \in \mathbb{R}^*$, ${}_\lambda\mathcal{K}_d$ est une fonction localement intégrable de $H_{p,q} \setminus \{z \; ; \; |z_1|^2 - |z_2|^2 \neq 0\}$ et on a pour $d > q-1$:

$$ {}_\lambda\mathcal{K}_d(t,z) = \begin{cases} |d| e^{-i\lambda t} \dfrac{e^{-\frac{|\lambda|||z_1|^2 - |z_2|^2|}{2}}}{(|\lambda|||z_1|^2 - |z_2|^2|)^{p+q-1}} L^{1-p-q}_{|d|+p-1}(|\lambda|||z_1|^2 - |z_2|^2|) \text{ sur } |z_1| > |z_2| \\ 0 \text{ sur } |z_1| < |z_2| \end{cases} $$

pour $d < 1-p$:

$$ {}_\lambda\mathcal{K}_d(t,z) = \begin{cases} |d| e^{-i\lambda t} \dfrac{e^{-\frac{|\lambda|||z_1|^2 - |z_2|^2|}{2}}}{(|\lambda|||z_1|^2 - |z_2|^2|)^{p+q-1}} L^{1-p-q}_{|d|+q-1}(|\lambda|||z_1|^2 - |z_2|^2|) \text{ sur } |z_1| < |z_2| \\ 0 \text{ sur } |z_1| > |z_2| \end{cases} $$

Les égalités ont lieu dans $\mathcal{D}'(H_{p,q} \setminus \{z \; ; \; |z_1|^2 - |z_2|^2 \neq 0\})$.

Nous avons vu dans le théorème 2 un premier lien entre les distributions sphériques de la paire de Guelfand généralisée et les fonctions sphériques de la paire de Guelfand, nous allons maintenant établir un autre lien.

Pour $(t, z) \in H_{p,q}$, soit $\Omega_{(t,z)}$ l'orbite de (t, z) dans $H_{p,q}$ sous l'action de $U_{p,q}$.

A une constante multiplicative près $\Omega_{(t,z)}$ possède une seule mesure $U_{p,q}$-invariante. Pour une fonction f sur $H_{p,q}$ on notera par $\mathcal{M}_{(t,z)}(f)$ l'intégrale de f, lorsqu'elle existe, par rapport à cette mesure.

Théorème 5

Pour $m, n \in \mathbb{N}$ tels que $m - n \geq q$ ou $n - m \geq p$, pour presque tout $(z_1, z_2) \in \mathbb{C}^{p+q}$ la fonction sphérique $\Phi_{\lambda,m,n}$ est intégrable sur $\Omega_{(t,z_1,z_2)}$ et on a:

$$_{\lambda}\mathcal{K}_{m-n}(t, z_1, z_2) = c\mathcal{M}_{(t,z_1,z_2)}(\Phi_{\lambda,m,n})$$

c est une constante ne dépendant que du signe de $m - n$, et l'égalité a lieu dans $\mathcal{D}'(H_{p,q} \setminus \{z \; ; \; |z_1|^2 - |z_2|^2 \neq 0\})$.

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Chinburg's Second Conjecture For Quaternion Fields

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1 Introduction

Let N/K be a Galois extension of number fields with Galois group $G(N/K)$. In [1] T. Chinburg introduced three important elements of the projective class group $C\mathcal{L}(\mathbf{Z}[G(N/K)])$, denoted $\Omega(N/K,i)$ (for $i = 1,2,3$), which have become known as the Chinburg invariants. In the case where N/K is a tamely ramified extension, the second Chinburg invariant $\Omega(N/K,2)$ is equal to the class of the ring of integers of N (cf. [2]) and in this case M. Taylor [10] proved that $\Omega(N/K,2)$ is equal to the class $W_{N/K}$ in $C\mathcal{L}(\mathbf{Z}[G(N/K)])$ which is manufactured from the Artin root numbers of the irreducible symplectic representations of $G(N/K)$. A. Fröhlich and Ph. Cassou-Noguès extended the definition of the class $W_{N/K}$ to the case where the extension has wild ramification, and the second Chinburg conjecture asserts that for every N/K, $\Omega(N/K,2) = W_{N/K}$ in $C\mathcal{L}(\mathbf{Z}[G(N/K)])$. There is additional evidence to support such a statement. In particular, C. Greither has recently shown that this conjecture holds for every abelian Galois extension $N/\mathbf{Q}$ of number fields having odd conductor.

In [3] it is shown that $\Omega(N/K,2) - W_{N/K}$ vanishes in the class-group of a maximal order of $\mathbf{Q}[G(N/K)]$. Little further evidence is known except the results of S. Kim [4][5] which verify the conjecture for all quaternion extensions, $N/\mathbf{Q}$, for which the prime 2 is not totally ramified.

In this note, we sketch the proof of the following extension of Kim's result.

Theorem 1.1 *Let $N/\mathbf{Q}$ be any Galois extension with $G(N/\mathbf{Q}) \cong Q_8$. Then*

$$\Omega(N/\mathbf{Q},2) = W_{N/\mathbf{Q}} \in C\mathcal{L}(\mathbf{Z}[G(N/\mathbf{Q})]) \cong (\mathbf{Z}/4)^*.$$

2 Notations and Constructions

In this section we set out our notation and recall briefly the construction of the second Chinburg invariant $\Omega(N/\mathbf{Q},2)$. We follow the notation of [9, §6.1]. We begin by recalling that the quaternion group of order 8, Q_8, is defined via generators and relations as

$$Q_8 = \{x, y \mid x^2 = y^2,\ y^4 = 1 \text{ and } xyx^{-1} = y^{-1}\}.$$

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2.1 Totally Ramified Q_8-Extensions of $\mathbb{Q}_2$

Let $N/\mathbf{Q}$ be a Galois extension with $G(N/\mathbf{Q}) \cong Q_8$ in which the prime 2 is totally ramified. Then N contains a unique biquadratic extension $K/\mathbf{Q}$. We let E and L denote the completions of K and N, respectively, at the prime 2. Then it is well known that $E = \mathbf{Q}_2(\sqrt{a}, \sqrt{b})$, where either (A) $a = 3$ and $b = 10$, or (B) $a = 2$ and $b = 3$. Moreover the biquadratic extension $E/\mathbf{Q}_2$ is contained in precisely two Q_8-extensions $L/\mathbf{Q}_2$, namely $L = E(\alpha)$, where $\alpha^2 = \pm(1 + \frac{\sqrt{3}}{3} + \frac{\sqrt{10}}{10} + \frac{\sqrt{30}}{10})$ in case A and $\alpha^2 = \pm\frac{\sqrt{6}}{6}(1+\sqrt{2})(\sqrt{2}+\sqrt{3})$ in case B. From now on we shall refer in either case to the positive choice as 'case I' and the negative choice as 'case II'. In all four cases we denote by π_L an appropriate choice of uniformising element in the local ring O_L.

2.2 The map k and the module M

We begin by constructing the following diagram of $\mathbf{Z}[Q_8]$-modules and homomorphisms:

$$\begin{array}{ccccccc} Ker(d) & \rightarrow & \mathbf{Z}[Q_8]\oplus\mathbf{Z}[Q_8] & \stackrel{d}{\rightarrow} & \mathbf{Z}[Q_8] & \rightarrow & \mathbf{Z} \\ k\downarrow & & j\downarrow & & i\downarrow & & \| \\ L^* & \rightarrow & L_0^* & \stackrel{F/1}{\rightarrow} & L_0^* & \rightarrow & \mathbf{Z} \end{array}.$$

Here the bottom row is the 2-extension constructed in [7, p. 202] which represents minus the fundamental class in $H^2(G(L/\mathbf{Q}_2); L^*) \cong \mathbf{Z}/8$, while the upper exact sequence is the beginning of the standard, periodic $\mathbf{Z}[Q_8]$-resolution of $\mathbf{Z}$.

We construct the map k as follows. Let b, b' denote the free generators of $\mathbf{Z}[Q_8] \oplus \mathbf{Z}[Q_8]$. Then the map d is defined by $d(b) = x - 1$ and $d(b') = y - 1$. $Ker(d)$ is the $\mathbf{Z}[Q_8]$-submodule of $\mathbf{Z}[Q_8] \oplus \mathbf{Z}[Q_8]$ generated by $c_1 = (1+x)b - (1+y)b'$ and $c_2 = (1+xy)b + (x-1)b'$. The map i sends 1 to π_L and $j(b') = \pi_L t_1$, $j(b) = t_2$ where $t_1, t_2 \in L_0^*$ are chosen to satisfy $F(t_1)t_1^{-1} = y(\pi_L)\pi_L^{-1}$ and $F(t_2)t_2^{-1} = x(\pi_L)\pi_L^{-1}$. The restriction map k is injective (cf. §2.3) and we define $M = L^*/\overline{Im(k)}$.

2.3 Injectivity of the map k

Here we give an outline of the proof that the map k defined in §2.2 is injective. We first note that it suffices to prove that k maps $Y = (Ker(d)) \cap k^{-1}(O_L{}^*)$ injectively to $O_L{}^*$. Set $Y_\pm = \{z \in Y \mid x^2(z) = \pm z\}$ and let $\sigma = (1+x+x^2+x^3)(1+y)$, $\tau = (1+x+x^2+x^3)(1-y)$, $\lambda = (1+y+y^2+y^3)(1-x)$ and $\rho = (1+xy+(xy)^2+(xy)^3)(1-x)$ in $\mathbf{Z}[Q_8]$. Then $Y_+ \otimes \mathbf{Q} = (\sigma Y \otimes \mathbf{Q}) \oplus (\tau Y \otimes \mathbf{Q}) \oplus (\lambda Y \otimes \mathbf{Q}) \oplus (\rho Y \otimes \mathbf{Q})$, and we show that each of the four factors maps injectively under k. This shows that $k : Y_+ \longrightarrow (O_L{}^*)_+$ is injective.

Since $Y_- \otimes \mathbf{Q}$ is a module over the rational quaternions $\mathbf{Q}[Q_8]/(x^2+1)$, the map $k : Y_- \otimes \mathbf{Q} \longrightarrow (O_L{}^*)_- \otimes \mathbf{Q}$ is injective because it maps $c_1 - c_2$ nontrivially. □

2.4 Construction of the Second Chinburg Invariant

In this section we recall briefly the definition of the second Chinburg invariant $\Omega(N/\mathbf{Q}, 2)$. Details may be found in [1] or [9].

We first construct the local invariant at the prime 2, $\Omega_2 = \Omega(L/\mathbf{Q}_2, 1+X_2)$. Setting $X_2 = \mathbf{Z}_2[G(L/\mathbf{Q}_2)]\langle a_2\rangle$, where for some $t \geq 2$,

$$a_2 = 2^t(1+\sqrt{a}+\sqrt{b}+\sqrt{ab}+\alpha), \tag{2.4.1}$$

we compose the map k with the map $L^* \to L^*/(1+X_2)$ to get a map $k' : Y_L \oplus \mathbf{Z} \to L^*/(1+X_2)$. With this notation we have

$$\Omega_2 = [Coker(k')] - [Ker(k')] = [M] - [Ker(k')] \in \mathcal{CL}(\mathbf{Z}[G(L/\mathbf{Q}_2)]),$$

where M is as in §2.2. As in [9, §2.2.3] we define a locally free $\mathbf{Z}[Q_8]$-module, X, by specifying its local completions, X_p. In particular, the above X_2 is the local completion at the prime 2. By definition, the second (global) Chinburg invariant is $\Omega(N/\mathbf{Q},2) = [X] + \Omega_2$ in our case.

3 Proof of Theorem 1.1

As noted in the introduction, the results of S. Kim [4][5] reduce us to considering only those quaternion extensions $N/\mathbf{Q}$ for which the prime 2 is totally ramified. But for such extensions the results of sections 3.1–3.3 show that

$$\Omega(N/\mathbf{Q},2) = [X] + \Omega_2 = [X] + [M] - [Ker(k')] = W_{N/\mathbf{Q}} \in \mathcal{CL}(\mathbf{Z}[G(N/\mathbf{Q})]) . \qquad \square$$

3.1 Computation of $[M]$

In this section we show that $[M]$ is trivial in $\mathcal{CL}(\mathbf{Z}[Q_8]) \cong \mathbf{Z}/2$. From the exact sequence

$$0 \to \overline{k(Ker(d))} \to L^* \to M \to 0$$

we obtain an exact cohomology sequence

$$0 \to \overline{k(Ker(d))}^{Q_8} \to (L^*)^{Q_8} \to M^{Q_8} \to H^1(Q_8; k(Ker(d))) = 0.$$

Hence,

$$M^{Q_8} \cong \frac{\mathbf{Q}_2^*}{\overline{k(Ker(d))}^{Q_8}} = \frac{\mathbf{Q}_2^*}{\langle j(\sigma b), j(\sigma b')\rangle}.$$

It can be shown that $j(\sigma b) = \sigma(t_2) = 5 + 8a$ and $j(\sigma b') = \sigma(\pi t_1) = 2c$, where $a \in \mathbf{Z}_2$ and $c \in 5 + 8\mathbf{Z}_2$. It follows easily that $2 \sim 1$ in M^{Q_8} and $M^{Q_8} \cong \mathbf{Z}/2$.

As in [9, Prop. 6.2.2] we have $M_- \cong \frac{L^*/E^*}{\langle k(c_1), k(c_2)\rangle}$. Direct computation shows that for some $u \in O_L$, $k(c_1) = \frac{1}{\pi_L}(1+\pi_L+\pi_L^2+u\pi_L^4) \sim 1+\pi_L+\pi_L^2+u\pi_L^4$ in L^*/E^* and $k(c_2) \cong 1+\pi_L^3+\pi_L^4 \bmod P_L^6$. Using the actions of $1+x, 1+y, x+y$, the squaring map and the fact that in L^*/E^*, $\pi_L^2 \sim u_7$ for some $u_7 \in U_L^7 \setminus U_L^8$, we manage to generate every filtration level U_L^n/U_L^{n+1} for $n \in \mathbb{N}, n \neq 5$. Since M_- is a left $\mathbf{H_Z}$-module which is isomorphic to $\mathbf{H_Z}/(\mathbf{H_Z}(z))$ for some $z \in \mathbf{H_Z}$, we have that $|M_-| \geq 4$. This implies that

$$M_- = \langle \pi, 1+\pi^5 \mid \pi^2 = (1+\pi^5)^2 = 1 \in L^*/E^*\rangle \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2.$$

Using the methods of [9, §6.2] it can be shown that $|M_+| \geq 2^6$, and so $|M| = |M_-|\cdot|M_+| \geq 2^8$.

We now use the actions of $x\pm 1, y\pm 1, x^2+1, x+y$ and the squaring map on the leading terms in the expansions of $k(c_1)$ and $k(c_2)$ to compute the filtration levels generated by the $k(c_i)$'s and deduce that

$$M = \langle \pi, 1+\pi^5 \,|\, \pi^{16} = (1+\pi^5)^{16} = 1 \in L^*/\overline{k(Ker(d))}\rangle \cong \mathbf{Z}/16 \oplus \mathbf{Z}/16.$$

Lemma 3.1 *$[M]$ is trivial in $\mathcal{CL}(\mathbf{Z}[Q_8]) \cong \mathbf{Z}/2$.*

Proof: Consider the induction map $Ind_{\langle x\rangle}^{Q_8} : \mathcal{CL}(\mathbf{Z}\langle x\rangle) \to \mathcal{CL}(\mathbf{Z}[Q_8])$. Since $\langle x\rangle \cong \mathbf{Z}/4$, $\mathcal{CL}(\mathbf{Z}\langle x\rangle) = 0$, and since $Ind_{\langle x\rangle}^{Q_8}([\mathbf{Z}/16]) = [\mathbf{Z}/16 \oplus \mathbf{Z}/16]$,

$$[M] = [\mathbf{Z}/16 \oplus \mathbf{Z}/16] = 0 \in \mathcal{CL}(\mathbf{Z}[Q_8]) \qquad \square$$

3.2 Computation of $[X]$

The purpose of this section is to sketch the proof of the following result.

Proposition 3.2 *In $\mathcal{CL}(\mathbf{Z}[Q_8])$,*

$$[X] = \begin{cases} -W_{N/\mathbf{Q}} & \text{in case I} \\ W_{N/\mathbf{Q}} & \text{in case II} \end{cases}.$$

As in [9, 6.3.2] we let $1, \chi_1, \chi_2, \chi_1\chi_2$ and ν denote the irreducible complex representations of Q_8, where χ_1, χ_2 and $\chi_1\chi_2$ are one dimensional and ν is two dimensional. The Hom-description representative of $[X] - W_{N/\mathbf{Q}}$ is (cf. equation 2.4.1):

$$\chi \mapsto \begin{cases} \frac{(a_2|\chi)}{\tau(\chi)} & , \text{ if } p = 2 \\ 1 & , \text{ otherwise} \end{cases}.$$

Here $(\cdot|\cdot)$ denotes the resolvent homomorphism while $\tau(\chi)$ denotes the local Gauss sum. Define

$$h : R(Q_8) \to \mathbf{Z}_2^* \quad \text{by} \quad h(\chi)\cdot 2^{\beta(\chi)} = \frac{(a_2|\chi)}{\tau(\chi)}$$
$$\text{and} \quad \chi_+ : (\mathbf{Z}/8)^* \to \{\pm 1\} \quad \text{by} \quad \chi_+(-1) = 1 \text{ and } \chi_+(3) = -1.$$

Then, in $\mathcal{CL}(\mathbf{Z}[Q_8]) \cong (\mathbf{Z}/4)^*$

$$H = [X] - W_{N/\mathbf{Q}} \equiv \frac{(a_2|\nu)}{\tau(\nu)}\chi_+\left(\frac{(a_2|1+\chi_1+\chi_2+\chi_1\chi_2)}{\tau(1+\chi_1+\chi_2+\chi_1\chi_2)}\right) \; mod\ 4$$

Further computations show that (here $W_2 = W_{\mathbf{Q}_2}(\nu)$ denotes the local root number at 2 applied to ν)

$$(a_2|\nu) \equiv \begin{cases} 1 & \text{mod } 4, \text{ in case I} \\ -1 & \text{mod } 4, \text{ in case II} \end{cases}; \quad W_2 = \begin{cases} -1 & \text{in case A} \\ 1 & \text{in case B} \end{cases};$$

$$\text{and}\quad (a_2 \mid 1+\chi_1+\chi_2+\chi_1\chi_2) = \begin{cases} 2^\beta\cdot 15 & \text{in case A} \\ 2^\beta\cdot 3 & \text{in case B} \end{cases}.$$

Letting $f(\nu)$ denote the Artin conductor of ν, we compute that

$$f(\nu) = \prod_{\substack{p \text{ ramified in } N/\mathbf{Q} \\ p\neq 2}} p^2 \equiv 1 \bmod 4.$$

Using the fact that $W_{finite}(\nu) = \prod_{p \ finite} W_{\mathbf{Q}_p}(\nu) = \sqrt{f(\nu)}\tau(\nu)$, we deduce that

$$H \equiv \begin{cases} -\left(\frac{2}{D_0}\right)\chi_+(D_0) & \bmod 4 & \text{in case I} \\ \left(\frac{2}{D_0}\right)\chi_+(D_0) & \bmod 4 & \text{in case II} \end{cases},$$

where D_0 denotes the odd part of $\sqrt{D_{E/\mathbf{Q}}}$.

The result follows, since $\left(\frac{2}{D_0}\right)\chi_+(D_0) = 1$. □

3.3 Computation of $[Ker(k')]$

In this section, we state the results needed in the computation of $[Ker(k')]$. Following [9], we use the formula given in [11], p. 88:

$$[g] \equiv g(\nu)\,(-1)^{(1/4)\log_2(g(1+\chi_1+\chi_2+\chi_1\chi_2))}\ (\bmod 4)\,.$$

In our case, for σ,τ,ρ,λ defined as in the proof of the injectivity of the map k, we compute $\sigma(t_2) \bmod U^5_{\mathbf{Q}_2}$, $\tau(t_2) \bmod U^{11}_{\mathbf{Q}_2(\sqrt{a})}$, $\rho(t_2/\pi t_1) \bmod U^{11}_{\mathbf{Q}_2(\sqrt{ab})}$, $\lambda(\pi t_1) \bmod U^9_{\mathbf{Q}_2(\sqrt{b})}$ and $k((1-x^2)(c_1-c_2)) \bmod U^{16}_L$ to get

$$g(1)\in 1+8\mathbf{Z}_2,\quad g(\chi_1)\in \begin{cases} 3+8\mathbf{Z}_2 & \text{in case A} \\ 7+8\mathbf{Z}_2 & \text{in case B} \end{cases},$$

$$g(\chi_1\chi_2)\in 1+8\mathbf{Z}_2,\quad g(\chi_2)\in \begin{cases} 7+8\mathbf{Z}_2 & \text{in case A} \\ 3+8\mathbf{Z}_2 & \text{in case B} \end{cases},$$

$$\text{and}\ g(\nu)\in \begin{cases} 1+4\mathbf{Z}_2 & \text{in case I} \\ 3+4\mathbf{Z}_2 & \text{in case II} \end{cases}.$$

Finally, using the above formula for $[g]$, we have

$$[Ker(k')] = [g] = \begin{cases} (-1)^{(1/4)\log_2(5)} & \equiv -1 \bmod 4, & \text{in case I} \\ (-1)(-1)^{(1/4)\log_2(5)} & \equiv 1 \bmod 4, & \text{in case II} \end{cases}.$$

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On A Parametric Diesel Function

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Abstract

We consider the solution of a nonlinear differential equation related to diesel fuel spray. We investigate monotonicity properties of the logarithmic derivative of the solution. This work was motivated by an inverse problem in diesel engineering.

1. Introduction. One of the components of the parameterization method for mathematical models is the introduction of a suitable parametric representation of the characteristics under consideration. Parametric representations play a significant role in solving some technological problems. In the cases we are interested in, the appropriate parameter is a function . For example the phenomenological model of atomized liquid spray uses to introduce the necessary functional parameter through a differential equation. This model is the basis for computational problems of the diesel working process. For details and references see [3]. It also leads to purely mathematical questions whose solutions are of independent mathematical interest and may have applications to Engineering.

We shall consider the parametric representation for the case of simplified ideal diesel fuel spray. This is the case of constant injection pressure and negligible evaporation. According to the model in [4] and [2] the fuel spray length S may be represented as

$$S(\tau, V, K) = y(x)V^{1/2}/K, \quad x = KV^{1/2}\tau,$$

where the monotone increasing function $y(x)$ satisfies the nonlinear initial value problem

$$y'' = \frac{(1-y')^2}{x-y} - (y')^\alpha, \quad x > 0, \quad y(0) = 0, \; y'(0) = 1. \tag{1.1}$$

Here τ is time, V is the rate of injection, $\alpha \geq 1$ and $K > 0$ are constants (independent of τ) but depend on the physical properties of the gas and fuel involved. It is often assumed that $\alpha = 3/2$, which is the average value of α, in some sense [4]. The function y is a parameter in the system which

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includes some other equations, [4], [5], [2]. In [2] strong numerical evidence has been established to support the assertion

(1.2) $[xy'(x)/y(x)]' < 0,$ for $\alpha = 3/2$ and $x > 0.$

The property (1.2) can be used to prove the following. Given any two points on the spray curve (τ_1, s_1) and (τ_2, s_2) with $\tau_1 < \tau_2$ one can find the corresponding values of K and V and a unique value of x, $x > 0$, $K = \sqrt{x\, y(x)/(s_1\tau_1)}$ and $V = x s_1/(y(x)\tau_1)$ if and only if $(\tau_2/\tau_1)^{1/3} < s_2/s_1 < \tau_2/\tau_1$. Necessary and sufficient conditions, such as the ones mentioned above, allow us to investigate some inverse problems for diesel fuel spray. The question is to search for connections among the conditions of the injection from the knowledge of the parameters of the diesel spray [4]. It turns out that the inequality (1.2) is just one inequality among many properties of the logarithmic derivative of the function y.

Recall that a function f defined on an interval I is completely monotonic on I if it has derivatives of all orders on I and $(-1)^n \dfrac{d^n f}{dx^n} \geq 0$ for $x \in I$. When I is $(0, \infty)$ we say that f is completely monotonic. Bernstein's theorem, [10], asserts that f is completely monotonic if and only if there is a positive measure $d\mu$ supported on a subset of $[0, \infty)$ such that $f(x) = \int_{0+}^{\infty} e^{-xt} d\mu(t), x > 0$. An important concept in probability theory is the concept of infinite divisibility, [1]. The connection between infinite divisibility and complete monotonicity of measures supported on subsets of $[0, \infty)$ is the fact that $d\mu$ is infinitely divisible if and only if $\int_{0+}^{\infty} e^{-xt} d\mu(t) = e^{-h(x)}$ where $h(0) = 0$, h is infinitely differentiable on $[0, \infty)$ and h' is completely monotonic. Note that

$$h'(x) = -F'(x)/F(x), \quad \text{with} \quad F(x) = \int_{0+}^{\infty} e^{-xt} d\mu(t).$$

This raises the question of complete monotonicity of $-F'(x)/F(x)$ for certain functions F. In the cases of specific probability distributions F is usually a special function and special function methods have used to establish the complete monotonicity of several quotients of special functions and the infinite divisibility of the corresponding probability distributions, see for example [9]. Since many special functions are solutions of linear differential or difference equations with suitable boundary conditions one would like to establish criteria for the complete monotonicity of logarithmic derivatives of solutions of differential equations. This led to the interesting work of P. Hartman [6], [7], [8] on the complete mononicity of the logarithmic derivatives of solutions of linear ordinary differential equations.

In this paper we give some support to our conjecture that (1.2) is just an instance of the complete monotonicity of $xy'(x)/y(x)$ for α in an interval containing $\alpha = 3/2$. It is clear that the above conjecture implies the infinite divisibility of a probability distribution whose moment generating function is $exp\{-\int_0^x ty'(t)/y(t)\, dt\}$. The differential equation (1.1) is nonlinear and as such does not fit into Hartman's theory. This raises the question of possible extending Hartman's theory to include nonlinear differential equations such as (1.1).

2. Some Recurrence Formulas. In (1.1), let $\alpha \in [1,\infty)$ and

$$y = \sum_{n=1}^{\infty} C_n x^n, \quad C_1 = 1.$$

The next set of calculations uses only properties of formal power series. We shall prove that the differential equation (1.1) has a unique formal power series solution. In particular this will show that (1.1) has at most one solution which is analytic in a neighborhood of $x = 0$.

We shall first establish some recurrence formulas for logarithmic derivative of the function y. From the above series for y we find that (1.1) implies the following recurrence relations

$$\sum_{n=0}^{N} \left((n+2)(N+3)C_{n+2} + g_n\right) C_{N+2-n} = 0, \quad N = 0, 1, \cdots. \tag{2.1}$$

where the coefficients g_n are defined by

$$[y'(x)]^\alpha = \sum_{n=0}^{\infty} g_n x^n, \quad g_0 = 1. \tag{2.2}$$

Equation (2.1) leads to the following recursive definition of C_n

$$C_{N+2} = [(N+1)(N+6)]^{-1}[6 \sum_{n=1}^{N-1} \left((N+3)(n+2)C_{n+2} + g_n\right) C_{N+2-n} - g_N]. \tag{2.3}$$

Note that g_n depends only on $C_2, \cdots C_{n+1}$, hence the right-hand side of (2.3) contains only $C_2, \cdots C_{N+1}$.

Lemma 2.1 *Let* $\phi = \sum_{n=0}^{\infty} a_n x^n$, $a_0 = 1$, *be a formal power series and let* $\phi^\alpha = \sum_{n=0}^{\infty} P_n x^n$, $\alpha \in R$. *Then* $P_0 = 1$ *and*

$$P_N = \frac{1}{N} \sum_{n=0}^{N-1} P_n a_{N-n}(\alpha(N-n) - n), \quad N \geq 1. \tag{2.4}$$

Proof. The formal power series identity $\alpha \phi^\alpha \phi' = (\phi^\alpha)' \phi$ easily implies (2.4).

Using Lemma 2.1 and (2.2), we obtain

$$g_N = \frac{1}{N} \sum_{n=0}^{N-1} g_n (N+1-n) C_{N+1-n}(\alpha(N-n) - n), \quad N \geq 1. \tag{2.5}$$

Let

$$h(x) = xy'(x)/y(x) = \sum_{n=0}^{\infty} h_n x^n. \tag{2.6}$$

It follows that $h_0 = 1$ and

$$(2.7) \qquad h_N(\alpha) = h_N = NC_{N+1} - \sum_{n=1}^{N-1} h_n C_{N+1-n}, \quad N \geq 1.$$

The relationships (2.3), (2.5) and (2.7) define the sequence $\{h_n(\alpha)\}_0^\infty$ recursively for any $\alpha < \infty$.

Now we consider the limiting case $\alpha \to \infty$. Let $y(x) = x + g(x)$ in (1.1). Then

$$(2.8) \qquad g'' = -\frac{(g')^2}{g} - (1+g')^\alpha .$$

Again, we use only properties of formal power series. Let

$$\lim_{\alpha\to\infty} \alpha^2 g\left(\frac{x}{\alpha}\right) = f(x),$$

then

$$\lim_{\alpha\to\infty} \alpha g'\left(\frac{x}{\alpha}\right) = f'(x), \quad \textit{and} \quad \lim_{\alpha\to\infty} g''\left(\frac{x}{\alpha}\right) = f''(x).$$

We have from (2.8)

$$(2.9) \qquad f'' = -\frac{(f')^2}{f} - e^{f'} \quad \textit{where} \quad f(x) = \sum_{n=2}^{\infty} f_n x^n.$$

Equation (2.9) gives

$$(2.10) \qquad \sum_{n=0}^{N} ((n+2)(N+3)f_{n+2} + q_n) f_{N+2-n} = 0, \quad N = 0, 1, \cdots$$

where the coefficients q_n are defined by

$$(2.11) \qquad e^{f'(x)} = \sum_{n=0}^{\infty} q_n x^n, \quad q_0 = 1.$$

Lemma 2.2 *For every* $N \geq 1$

$$(2.12) \qquad q_N = \frac{1}{N} \sum_{n=0}^{N-1} q_n (N+1-n)(N-n) f_{N+1-n},$$

where coefficients f_n *and* q_n *are defined by* (2.9) *and* (2.11) *respectively.*

Proof. Equating like powers of x in the identity $f''e^{f'} = \left(e^{f'}\right)'$ gives (2.12).

Using (2.10) and Lemma 2.2, we have for $N = 0, 1, \ldots$

$$(2.13)\quad f_{N+2} = \frac{1}{(N+1)(N+6)}\left[6\sum_{n=1}^{N-1}((N+3)(n+2)f_{n+2} + q_n)f_{N+2-n} - q_N\right].$$

It is easy to see that for h defined by (2.6)

$$\lim_{\alpha\to\infty}\left(h\left(\frac{x}{\alpha}\right) - 1\right)\alpha = f'(x) - \frac{f(x)}{x}$$

We define

$$(2.14)\quad h_0(\infty) := 1, \quad h_n(\infty) := \left\{f' - \frac{f}{x}\right\}_n = nf_{n+1}, \quad n = 1, 2, \cdots.$$

Formulas (2.12), (2.13) and (2.14) define the sequence $\{h_n(\infty)\}_0^\infty$ recursively.

3. Number of Alternating Signs. Let $N(\alpha)$, $\alpha \in [1, \infty]$ be the maximum number $n \in \mathcal{N}$ such that

$$h_e(\alpha)h_{e-1}(\alpha) < 0, \quad e = 1, \ldots, n$$

where coefficients h_e are defined by (2.7) or (2.14). Since $h_0 = 1$, $h_1 = -1/6$ for any α, and

$$h_2(\alpha) = \frac{\alpha}{21} - \frac{1}{36}, \alpha < \infty \quad \textit{and} \quad h_2(\infty) = \frac{1}{21},$$

we conclude that $N(\alpha) \geq 2$ for $\alpha \in [1, \infty]$. The recurrence relations of Section 2 yield

$$N(1) = 3, \quad N(1.1) = 5, \quad N(1.2) = 19, \quad N(1.21) = 159.$$

Further calculations show that the function $N(\alpha)$ is rapidly increasing and a computer search indicates that it is very likely to be unbounded for $\alpha \in [3/2, \infty]$. For example,

$$N(1.25) > 600, \quad N(1.3) > 1,600, \quad N(1.35) > 4,000.$$

We used the function $\frac{y(\beta x)}{\beta^2}$ instead of $y(x)$ with some suitable value of $\beta > 0$ to overcome overflow in computations. The corresponding logarithmic derivative is $h(\beta x)$ with the Taylor coefficients $h_n\beta^n (n = 0, 1, \ldots)$.

It seems reasonable to expect that, in fact, $N(\alpha) = \infty$ for α in some interval $[\alpha_0, \alpha_1]$. The above-mentioned ideas and many computer calculations of function h and some of its derivatives led us to the following conjecture.

Conjecture 3.3 *The function h is completely monotonic for any value of the parameter $\alpha \in [\alpha_0, \alpha_1]$, for some $\alpha_0 \in (1.2, 1.5]$ and $\alpha_1 \in (1.5, \infty]$.*

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