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ON GOLOD SUBALGEBRAS.

LAKHDAR HAMMOUDI.

Presented by C.K. Gupta, F.R.S.C.

ABSTRACT. - We investigate GOLOD subalgebras of a GOLOD algebra. We proved that the GOLOD algebras constructed in [2] contain an infinity of proper GOLOD subalgebras, and by the method used, we can construct a GOLOD d-algebra ($d \geq 2$) which has GOLOD d'-subalgebras ($d' \geq d$) for every d' in a finite set of integers. This generalizes TIMOFEENKO's result ([5]). Here, we establish analogous results for LIE algebras and p-groups.

From now on we call an s-algebra, an algebra with s generators.

For any field K and for every integer $d \geq 2$, GOLOD has constructed a non nilpotent d-algebra A such that every $(d-1)$-subalgebra is nilpotent. These algebras are called GOLOD-algebras. The analogous results for the LIE algebra $A\mathcal{L}$ which is generated by the generators $\bar{X}_1, ..., \bar{X}_d$ of A; and in the case, when the field K is of characteristic $p > 0$, for the subgroup $G =< 1 + \bar{X}_1, ..., 1 + \bar{X}_d >$ of the p-group $1 + A$, where 1 is the unit of K, hold ([1]). $A\mathcal{L}$ and G are respectively called GOLOD-LIE algebra and GOLOD group.

The problem of the existence of proper GOLOD d'-subalgebra in a GOLOD algebra ($d' \geq d$) was first formulated in [5], where a GOLOD d-algebra was constructed which contains a proper GOLOD d'-subalgebra. By [4], we remark once again, that proper GOLOD subalgebras of a GOLOD algebra exist.

In our paper [2] we have studied, like SOZUTOV, a certain class of GOLOD algebras. So, we focussed on proper GOLOD subalgebras of the algebras described in [2]. We prove mainly that these algebras contain an infinity of proper GOLOD subalgebras, and for any fixed finite set M of integers greater or equal to 2, we found among them some that have a proper GOLOD d'-subalgebra, for every $d' \in M$. This generalizes TIMOFEENKO's result. The cases of the associated GOLOD-LIE algebra and the GOLOD group will follow.

We point out that by [3] there is no GOLOD algebra which contains an isomorphic copy of every GOLOD algebra.

Let K be any field and let $F^{(1)}$ be the free associative algebra of polynomials without constant terms in the non-commuting indeterminates $X_1, ..., X_d$ over K. Denote by F_k the subspace of homogeneous polynomials of degree $k \geq 1$ and put $F^{(k)} = \bigoplus_{i \geq k} F_i$. It is well known that if I is a homogeneous ideal of $F^{(1)}$, then the algebra $F^{(1)}/I$ will inherit the graduation of $F^{(1)}$.

Golod has proved ([1]) that if an ideal I of $F^{(1)}$ is generated by a family of homogeneous polynomials $f_1, f_2, ...$ of increasing degrees greater or equal to 2 verifying :

For a fixed real ϵ, $0 < \epsilon < 1/2$, the number r_i of polynomials of degree i in the sequence $f_1, f_2, ...$ is such that $r_i \leq \epsilon^2(d - 2\epsilon)^{i-2}$, for every $i \geq 2$, then $F^{(1)}/I$ is an infinite-dimensional algebra. Our proofs depend significantly on the above properties, which are called Golod properties.

Let α be a real number, $1 \leq \alpha < d - 2\epsilon$ and put $\beta_k = [\alpha^k]$, where $[\alpha^k]$ means the integer part of α^k.

By means of Golod algebras, we have recently established in [2] the following theorem, which generalizes Sozutov's result.

Theorem A ([2], **Theorem 1**). - For any field, There exists a non nilpotent d-algebra $A = F^{(1)}/I$ (I is a homogeneous ideal) such that any β_t ($t \geq 1$) elements of $A^{(t)} = (F^{(t)} + I)/I$ generate a nilpotent subalgebra of a degree of nilpotency bounded by a function of the minimal and the maximal degrees of these elements.

Of course, we have the same results for the Golod-Lie algebra $A\mathcal{L}$ and the Golod group G.

Theorem 1. - For any field and choice of α, there exits a Golod d-algebra A, which has a proper Golod d'-subalgebra; where $\beta_t + 1 \leq d' \leq d^t$, $t \in N^*$.

Proof. - Let A be a d-algebra exhibited in Theorem A. Since any β_t elements of $A^{(t)}$, $t \in N^*$, generate a nilpotent subalgebra, and $A^{(t)}$ is an such that any $(d' - 1)$ elements of $A^{(t)}$ generate a nilpotent subalgebra, and there exists an infinite-dimensional d'-subalgebra of $A^{(t)}$.

Corollary 1. - For any field, there exists a Golod-Lie algebra which contains an infinity of Golod-Lie d'-subalgebras, $\beta_t + 1 \leq d' \leq d^t$, $t \in N^*$.

Corollary 2. - There exists a d-generator ($d \geq 2$) Golod p-group which contains an infinity of d'-generator Golod subgroups, where $\beta_t + 1 \leq d' \leq d^t$, $t \in N^*$.

THEOREM 2. - For any finite set M of integers greater or equal to d and for any field, there exists a GOLOD d-algebra which has GOLOD d'-subalgebras, for every $d' \in M$.

PROOF. - Let ϵ be a real, $0 < \epsilon < 1/2$. Let $d'_1 \leq d'_2 \leq \dots \leq d'_n$ be the elelments of the set M. To each d'_i associate an integer j_i stisfying $[ln(d'_i - 1)/ln(d - 2\epsilon)] + 1 \leq j_i < j_{i+1}$; $i = 1, \dots, n-1$.

According to [2] and [5], it is sufficient to construct a GOLOD d-algebra $(d \leq d'_1)$ $A = F^{(1)}/I$ such that any $(d'_i - 1)$ elements of minimum degree j_i generate a nilpotent subalgebra, and where I is an ideal generated by a family of homogeneous polynomials $f_1, f_2, \dots$ of increasing degrees greater or equal to 2, verifying the following conditions :

Let j be any multiple of $j_1, \dots, j_n$; $j = s_i j_i$, $i = 1, \dots, n$. The degrees $jN_1, jN_2, \dots, jN_k, jN_k + 1, \dots, kjN_k, \dots$ of $f_1, f_2, \dots$ are chosen such that the number of polynomials of degrees $jN_k, jN_k + 1, \dots, kjN_k$, $k = 1, 2, \dots$

does not exceed $(1/N_0)\, \epsilon^2 (d - 2\epsilon)^{N_k - 3}$, where $N_0 \geq 1$ is a constant. (1)

$$N_1 \geq 3, \qquad kN_k < N_{k+1} - 1; \qquad k = 2, 3, \dots \tag{2}$$

$$(d'_i - 1)^{s_i N_k} (s_i N_k + q_i - 1)^{q_i - 1} \leq \epsilon^2 (d'_i - 2\epsilon)^{s_i N_k - 3} - \epsilon^2 (d - 2\epsilon)^{N_k - 3} \tag{3}$$

$$q_i = (d'_i - 1)(d + d^2 + \dots + d^{J_i + s}); \qquad i = 1, \dots, n; \qquad j_i + s = k.$$

Note that these inequalities are the analogues of those in [5].

Let $V_{d'_i}$ be the free associtive algebra of polynomials without constant terms in d'_i non-commuting indeterminates. Choose the generators of $V_{d'_i}$ to be free monomials in $F^{(1)}$ of degree j_i such that for all monomials $Z_1, Z_2 \notin V_{d'_i}$, and for every monomial $T \in V_{d'_i}$, $Z_1 T Z_2 \in V_{d'_i}$ ([5], Remark 1).

Now, decompose every generator f of I, $f = f^* + u_1 \tilde{f}_1 v_1 + \dots + u_l \tilde{f}_l v_l$ such that $\tilde{f}_1, \dots, \tilde{f}_l$ are in $V_{d'_i}$ and for every monomial T summand of f^*, and every choice of $Z_1, Z_2 \in F^{(1)} \setminus \{0\}$, one has $Z_1 T Z_2 \notin V_{d'_i}$ ([5], construction of the set U). So, we obtain a new ideal $\hat{I}$ generated by $f_1^*, \tilde{f}_{1,1}, \dots, \tilde{f}_{1,l_1}, f_2^*, \tilde{f}_{2,1}, \dots, \tilde{f}_{2,l_2}, \dots$. It is obvious that $I \subseteq \hat{I}$, and so, $F^{(1)}/\hat{I}$ is a d-nilalgebra. By lemmas 4, 5 ([5]), the property (1) and, the inequality (2), $\hat{I}$ verifies the GOLOD properties. Apply now, lemma 6 ([5]), and the inequality (3) to assert that $V_{d'_i} \cap \hat{I}$ verifies the GOLOD properties too. At present, observe that $V_{d'_i} \cap I \subseteq V_{d'_i} \cap \hat{I}$, and recall that $(V_{d'_i} + I)/I \cong V_{d'_i}/(V_{d'_i} \cap I)$ is a d'_i-nilalgebra ([2], proof of THEOREM 1). So, $(V_{d'_i} + I)/I$ is a GOLOD d'_i-subalgebra of A.

Let us start the constuction of the ideal I. We will do this by induction. Suppose that we have constructed the segment $f_1, f_2, \dots, f_{n_{(t-1,t'-1)}}$ with all the required properties, and that we have used general polynomials of minimum degrees $j_1, \dots, j_k$, $k \leq n$, such that, if we denote by $I_{n_{(t-1,t'-1)}}$ the ideal generated by $f_1, f_2, \dots, f_{n_{(t-1,t'-1)}}$ then,

- Any $(d-1)$ elements of $F^{(1)}/I_{n_{(t-1,t'-1)}}$ of degrees at most $(t'-1)$ will generate a nilpotent subalgebra.

- Any $(d'_i - 1)$ elements of $F^{(1)}/I_{n_{(t-1,t'-1)}}$ of minimum degree j_i, and of maximum degree $(t'-1)$ will generate a nilpotent subalgebra, for every $i = 1, ..., k$.

Consider a GOLOD system constituted by $(d'_{k'} - 1)$ general polynomials of minimum degree $j_{k'}$, and of maximum degree $j_{k'} + \bar{s} = t'$ (k' is equal to k if $t' < j_{k+1}$ and equal to $k+1$ otherwise).

$$g_m = c^{(1)}_{1,m}\bar{X}_1^{j_{k'}} + ... + c^{(1)}_{d^{j_{k'}},m}\bar{X}_d^{j_{k'}} + c^{(2)}_{1,m}\bar{X}_1^{j_{k'}+1} + ... + c^{(2)}_{d^{j_{k'}+1},m}\bar{X}_d^{j_{k'}+1} + ... + c^{j_{k'}+s}_{d^{j_{k'}+s},m}\bar{X}_d^{j_{k'}+s}.$$

$m = 1, ..., (d'_{k'} - 1)$.

Consider all the products of length N, $g_{m_1} ... g_{m_N}$ which we decompose in homogeneous polynomials. These homogeneous polynomials will be added to the segment $f_1, f_2, ..., f_{n_{(t-1,t'-1)}}$ to constitute the system $f_1, f_2, ..., f_{n_{(t-1,t'-1)}}, ..., f_{n_{(t,t')}}$. The number of homogeneous polynomials of degrees greater or equal to $j_{k'}N$ in the new system does not exceed $(d'_{k'} - 1)^N (N + q_{k'} - 1)^{q_{k'}-1}$, where $q_{k'} = (d'_{k'} - 1)(d + d^2 + ... + d^{j_{k'}+s})$.

For a large number N, we have :

$(d'_{k'} - 1)^N (N + q_{k'} - 1)^{q_{k'}-1} \leq (1/N_0)\epsilon^2 (d - 2\epsilon)^{j_{k'}N-3}$.

We can choose N such that the inequalities (2) and (3) are also satisfied. For the homogeneous polynomials of degrees less than $j_{k'}N$, (1), (2) and (3) are also verified. The union of all these segments will give a sequence $f_1, f_2, ...$. If we denote by I the ideal generated by this sequence of polynomials, then, $A = F^{(1)}/I$ will be the desired algebra.

COROLLARY 3. - Let M be a finite set of integers greater or equal to d. For any field, there exists a GOLD-LIE d-algebra which contains GOLOD-LIE d'-subalgebras, for every $d' \in M$.

COROLLARY 4. - For every finite set M of integers greater or equal to d, there exists a d-generator GOLOD p-group which contains d'-generator GOLOD p-subgroups, for every $d' \in M$.

REMARK 1. - In the course of the proof of theorem 2, we have constructed two GOLOD algebras : $F^{(1)}/I$ and $F^{(1)}/\hat{I}$. The GOLOD d'_i-subalgebras of $F^{(1)}/\hat{I}$ verify the GOLOD properties, however, we do not know whether this is so in the GOLOD d'_i-subalgebras of $F^{(1)}/I$.

REMARK 2. - We observe that certain GOLOD subalgebras of the GOLOD algebras of theorem 2 are of infinite codimension, and thus, by [6], the corresponding GOLOD subgroups are of infinite index. On the other hand, there exist GOLOD subalgebras of finite codimension. If A is a GOLOD d-algebra such that for some $k \geq 2$, $(d^k - 1)$ elements of $A^{(k)}$ generate a nilpotent subalgebra then, $A^{(k)}$ is a GOLOD $d^{l'}$-subalgebra of finite codimension ([2], [4]). The associated GOLOD subgroup will be of finite index.

In [2], we have also established the analogue of THEOREM A for ENGEL-LIE algebras. An argument similar to that proving THEOREMS 1 and 2 will prove :

THEOREM 3. - For any field, there exists a non-nil ENGEL-LIE d-algebra which has proper non-nil ENGEL-LIE d'-subalgebras; where $\beta_t + 1 \leq d' \leq d^t$, $t \in N^*$.

THEOREM 4. - For any field, and for every finite set M of integers greater or equal to d, there exists a non-nil ENGEL-LIE d-algebra which contains non-nil ENGEL-LIE d'-subalgebras, for every $d' \in M$.

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LAKHDAR HAMMOUDI. Université de Haute-Alsace. F. S. T.
Laboratoire de Mathématiques. 4, Rue des Frères Lumières.
68093 Mulhouse Cedex. France.
E-mail address : hammoudi@math.u-strasbg.fr

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SPECTRAL PROPERTIES OF MULTIPLIERS ON TOPOLOGICAL ALGEBRAS

Noor Mohammad[1]
International Centre for Theoretical Physics, Trieste, Italy.

Presented by B. Banaschewski, F.R.S.C.

ABSTRACT

We investigate the spectral properties of a multiplier T on a Hausdorff topological algebra A possessing an orthogonal basis. We show that each multiplier T on A is determined by a sequence of complex numbers which are precisely the eigenvalues of T. It is further proved here that the point spectrum of T coincides with this sequence and the residual spectrum of T is empty.

1 Introduction and Preliminaries

Before proceeding to our results we briefly recall some relevant definitions and theorems. Regarding the general theory of multipliers we refer to [5] and [6]. Throughout this paper we assume, without mentioning explicitly, that the algebras under consideration are over the field $\mathbb{C}$ of complex numbers, while the topological spaces involved are always Hausdorff.

An algebra A is said to be *proper* if for any $x \in A$, $xA = Ax = \{0\}$ implies $x = 0$. If A has an identity, then A is proper. A mapping $T : A \rightarrow A$ is called a *multiplier* if $(Tx)\cdot y = x \cdot T(y)$ for each $x, y \in A$. Note that if A is proper, then each multiplier is linear (see for instance, [5]), Proposition 1.2, Chapter VII). We denote the set of all multipliers of A by $M(A)$. It is well-known that $M(A)$ is an algebra with the identity map of A as its identity.

In what follows, unless specified otherwise, A denotes a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. Then A is commutative ([5], Corollary 1.4, Chapter III), proper ([5], Proposition 1.6, Chapter III) and semisimple ([5], Corollary 2.5, Chapter III). Also, each coordinate functional λ_i determined by the basis $\{x_i\}$ via $x = \sum_{i=1}^{\infty} \lambda_i(x)x_i$, is continuous i.e., $\{x_i\}$ is a Schauder basis ([5], Theorem 1.12, Chapter III). Further, each λ_i is a multiplicative linear functional ([3], Lemma 1.1). We denote the set of all nonzero continuous multiplicative linear functionals on A by $\Delta(A)$. Thus $\Delta(A)$ is a subset of the dual space A' of A. We endow $\Delta(A)$ with the Gelfand topology i.e., the induced ω^*-topology from A'. Assuming that $\Delta(A)$ is non-empty and point-separating, for each $x \in A$ we put $\hat{x}(f) = f(x)$, $f \in \Delta(A)$, the Gelfand transform of x.

The following result determines the continuity of a multiplier T on A.

Theorem 1.1 ([4], Theorem (2.2)). *Let A be a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. Then each $T \in M(A)$ has closed graph.*

Corollary 1.2 ([4], Corollary (2.3)). *Each multiplier on a proper complete metrizable (in particular, B_0 or Frechet or Banach) algebra is continuous.*

We shall need the following result.

Theorem 1.3 ([4], Theorem 2.5 and Corollary (2.8)). *Let A be a topological algebra with an orthogonal basis $\{x_i\}$. Then to each $T \in M(A)$ there corresponds a continuous function $\mu^T : \Delta(A) \to \mathbb{C}$ defined by $\mu^T(f) = foT(x)$, where $x \in A$ is chosen such that $f(x) = 1$, satisfying the relation*

$$(Tx)^\wedge(f) = \mu^T(f)\hat{x}(f)$$

for all $x \in A$ and all $f \in \Delta(A)$.

If in addition, A is a Banach algebra then

$$\|\mu^T\|_\infty = \sup|\mu^T(f)| \leq \|T\|, \ f \in \Delta(A)$$

i.e., μ^T is a bounded function on $\Delta(A)$.

2 Spectral properties of multipliers

We wish to study the spectral properties of a multiplier T on A. Throughout, A denotes a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. As we observed earlier, such an algebra A is commutative, proper, semisimple and its basis is a Schauder basis.

Theorem 2.1 *Let A be a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. Then each $T \in M(A)$ is determined by a sequence $\{\mu_i^T\}$ of complex numbers, where $\mu_i^T = \mu^T(\lambda_i)$ for all $i \geq 1$. Furthermore, for all $x \in A$ we have*

$$Tx = \sum_{i=1}^{\infty} \lambda_i(x)\mu_i^T x_i \, . \tag{1}$$

Proof. See [5, p. 225].

Theorem 2.2 *Let A be a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. Then for all $T \in M(A)$ and for all $j \geq 1$, x_j is an eigenvector for T with an eigenvalue μ_j^T.*

Proof. By setting $x = x_j$, $(j \geq 1)$ in equation (1) we get

$$Tx_j = \sum_{i=1}^{\infty} \lambda_i(x_j)\mu_i^T x_i \,, \quad (j \geq 1) \,.$$

Since $\lambda_i(x_j) = \delta_{ij}$ for all $i, j \geq 1$, we have

$$Tx_j = \mu_j^T x_j \,. \tag{2}$$

From (2) it follows that x_j is an eigenvector for T with eigenvalue μ_j^T.

We next show that the point spectrum of T, denoted by $\sigma_P(T)$, coincides with the set $\{\mu_i^T : i \geq 1\}$ and the residual spectrum $\sigma_r(T)$ of T is empty.

Theorem 2.3 *For each $T \in M(A)$ we have*

a) $\sigma_P(T) = \{\mu_i^T : i \geq 1\}$

b) $\sigma_r(T)$ is empty.

Proof.

a) By virtue of Theorem (2.2) we have $\mu^T(\Delta(A)) = \{\mu_i^T : i \geq 1\} \subset \sigma_P(T)$, where $\mu^T(\Delta(A))$ denotes the range of μ^T.

To prove the reverse inclusion, let μ be an eigenvalue of T. Then there exists a non-zero element x of A such that $(\mu I - T)x = 0$. Therefore, $((\mu I - T)x)^\wedge = \{\mu - \mu^T\}\hat{x} = \hat{0}$. Since $\hat{x} \neq \hat{0}$ as A is semisimple ([5], corollary (2.5), Chapter III), there exists an element $f \in \Delta(A)$ such that $\hat{x}(f) \neq 0$. Hence it follows from above that $(\mu - \mu^T)(f) = 0$, i.e., $\mu^T(f) = \mu$. Thus $\mu \in \mu^T(\Delta(A))$. Hence $\sigma_P(T) \subset \mu^T(\Delta(A))$ and consequently $\sigma_P(T) = \{\mu_i^T : i \geq 1\}$.

b) Let us assume to the contrary, i.e., $\sigma_r(T)$ is non-empty, and let $\mu \in \sigma_r(T)$. Then, by part (a) above, we have $\mu \notin \sigma_P(T)$, i.e. $\mu \neq \mu_i^T$ for each $i \geq 1$. Setting $z_i = (\mu - \mu_i^T)^{-1}x_i$, we see that $(\mu I - T)z_i = \{\mu - \mu_i^T)^{-1} \cdot (\mu I - T)x_i$ for each $i \geq 1$. By equation (2), $Tx_i = \mu_i^T x_i$. it follows that $(\mu I - T)z_i = x_i$. In other words, for each $i \geq 1$ the element x_i belongs to the space $(\mu I - T)(A)$.

Now let K be the linear space spanned by the sequence $\{x_i\}$. Then clearly K is dense (with respect to topology of A) in A and as we have seen before, $\bar{K} = \overline{(\mu I - T)(A)} = A$.

This implies that $\mu \notin \sigma_r(T)$; an obvious contradiction.

Remark. The sequence $\{\mu_i^T\}$ of complex numbers associated to each $T \in M(A)$ may not be bounded in general. However, if $M(A)$ is a Q-algebra (i.e. the set of all quasi-regular elements is open), then $\{\mu_i^T\}$ is a bounded sequence since each $\mu_i^T \in \sigma_P(T)$ for $i \geq 1$, and every element of a Q-algebra has a compact spectrum ([7], Lemma E.3). We also have

Theorem 2.4 *Let A be a Banach algebra with an orthogonal basis $\{x_i\}$. Then the sequence $\{\mu_i^T\}$ associated to each $T \in M(A)$ is bounded. Moreover, the map $\psi : M(A) \to \ell_\infty$ defined by $\psi(T) = \{\mu_i^T\}$ is a continuous isomorphism which is also norm-decreasing.*

Proof. See [5, p. 225].

As an immediate consequence of Theorem (2.3), we derive the following result due to P. Aiena [1].

Corollary 2.1 *Let A be a Banach algebra with an orthogonal basis $\{x_i\}$. Then for each $T \in M(A)$ we have*

i) $\sigma_P(T) = \{\mu_i^T : i \geq 1\}$

ii) $\sigma_r(T)$ is empty.

Application. Finally we present few applications of our results.

Example 1. Let $H(D)$ denote the algebra of all holomorphic functions defined on the open disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with point-wise addition and scalar multiplication. With the Cauchy-Hadamard product and th compact-open topology, $H(D)$ is a commutative B_0-algebra (i.e., complete metrizable locally convex) possessing an orthogonal basis $\{e_n : n \geq 0\}$, where $e_n(z) = z^n$ for $z \in D$. The element $e(z) = \sum_{n=0}^{\infty} z^n$ is the identity of $H(D)$ (see [5], Chapter III, p. 97). Therefore, each multiplier T on $H(D)$ is given by a multiplication operator $T_g(f) = g * f$, for some $g \in H(D)$ ([5], Proposition 1.4, Chapter VII)). It can be easily verified that $\sigma_P(T) = \{\mu_n^T : n \geq 0\}$ where $\{\mu_n^T\}$ is a sequence of eigenvalues of T corresponding to the orthogonal basis $\{e_n : n \geq 0\}$ as in Theorem 2.2. Moreover, $\sigma_r(T)$ is empty.

Example 2. Consider the set s of all complex sequences with coordinate-wise operations. Then s is a commutative Fréchet algebra with identity and possessing an orthogonal basis $\{e_n : n \geq 1\}$ ([5], Chapter II, Example 3.4). One can see, in this case, that for any multiplier $T \in M(s)$ the point spectrum $\sigma_P(T)$ is precisely the set of all eigenvalues $\{\mu_n^T : n \geq 1\}$, moreover $\sigma_r(T) = \phi$.

For more examples of such algebras we refer to [5].

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Department of Mathematics,
Quaid-i-Azam University,
Islamabad, Pakistan.

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ALGEBRES DE BANACH PRESQUE COMMUTATIVES[1]

O.H. CHEIKH
Presented by J. Lambek, F.R.S.C.

Abstract: *We give a condition implying the commutativity of a Banach algebra modulo its Jacobson radical. It concerns the set of invertible elements. Other equivalent conditions involving the spectral radius and the norm are also given.*

Les algèbres qu'on considère ici sont complexes. On entend par algèbre presque commutative une algèbre E telle que E/Rad E soit commutative où Rad E désigne le radical de Jacobson de E. Si E est une algèbre de Banach unitaire, on désigne par G l'ensemble des éléments inversibles de E. Il est bien connu que pour tout $x \in G$, la boule ouverte de centre x et de rayon $\left[\|x^{-1}\|\right]^{-1}$ est contenue dans G.

On montre dans cette note que si pour tout $x \in G$, la boule ouverte de centre x et de rayon $\left[\rho(x^{-1})\right]^{-1}$ est contenue dans G, alors E est presque commutative (ρ est le rayon spectral).

Dans [2], Hirschfeld et Zelazko ont montré que dans une algèbre de Banach sans éléments quasi-nilpotents, la sous-additivité et la sous-multiplicativité du rayon spectral entraînent la

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Mots clés: Algèbre de Banach commutative, Radical de Jacobson, rayon spectral.

commutativité. Par la suite, Aupetit [1] et Zemanek [4] ont montré que dans les algèbres de Banach ces notions sont équivalentes à la commutativité modulo le radical.

On considère ici des conditions plus faibles faisant intervenir à la fois le rayon spectral et la norme et on montre que celles-ci sont, elles aussi, équivalentes à la presque commutativité.

Théorème 1: *Soit E une algèbre de Banach unitaire. Les propriétés suivantes sont équivalentes:*

1. $\rho(xy) \leq \rho(x).\|y\|$ *pour tout x et tout y dans E.*
2. *Pour tout $x \in G$, la boule ouverte de centre x et de rayon $\left[\rho(x^{-1})\right]^{-1}$ est contenue dans G.*
3. $\rho(x+y) \leq \rho(x)+\|y\|$ *pour tout x et tout y dans E.*
4. *E est presque commutative.*

Preuve: 1.$\Rightarrow$2. Soit $y \in E$ tel que $\|y\| \prec \left[\rho(x^{-1})\right]^{-1}$. Alors $x+y=x(1+x^{-1}y)$ et $\rho(x^{-1}y) \leq \rho(x^{-1}).\|y\|$ d'après 1.. Il en résulte que $x+y \in G$.

2.$\Rightarrow$3. Soit λ un nombre complexe tel que $|\lambda| \succ \rho(x)+\|y\|$. Alors $(\lambda-x)$ est inversible. De plus $\lambda-(x+y)=(\lambda-x)\left[1-(\lambda-x)^{-1}y\right]$ et $\rho\left[(\lambda-x)^{-1}\right].\|y\| \leq \dfrac{\|y\|}{|y|-\rho(x)} \prec 1$. Par suite λ - $(x+y)$ est inversible. Il en résulte que $\rho(x+y) \leq \rho(x)+\|y\|$.

3.$\Rightarrow$ 4. On reprend le raisonnement de B. Aupetit [1, p. 49]:

Si $x,y \in E$, on pose $g(\lambda)=\frac{1}{\lambda}\left[e^{\lambda x}.y.e^{-\lambda x}-y\right]; \lambda \in \mathbb{C}^{*}$. Alors $\rho(g(\lambda))$ est sous-harmonique.

Comme $\rho(g(\lambda)) \leq \dfrac{\rho(y)+\|y\|}{|\lambda|}$ le théorème de Liouville pour les fonctions sous-harmoniques donne $\rho(g(\lambda)) \equiv 0$. On a donc $\rho(xy\text{-}yx)=0$. Un résultat de Le Page [3] montre alors que

$xy - yx \in \mathrm{Rad}\, E$. On en conclut que $E/\mathrm{Rad}\, E$ est commutative.

4.$\Rightarrow$1. Évident.

Les propriétés précédentes sont également équivalentes à la sous-additivité, à la sous-multiplicativité et à l'uniforme continuité du rayon spectral:

Théorème 2: *Dans une algèbre de Banach unitaire E, les propriétés suivantes sont équivalentes:*

i: $\rho(xy) \leq \rho(x).\rho(y)$ *pour tout x et tout y dans E*

ii: $\rho(x+y) \leq \rho(x)+\rho(y)$ *pour tout x et tout* $y \in E$

iii: $|\rho(x)-\rho(y)| \leq \|x-y\|$ *pour tout x et tout* $y \in E$.

4: E est presque commutative.

Preuve: compte tenu des équivalences déjà établies dans le théorème 1, il suffit de montrer que 4. $\Rightarrow$ ii. $\Rightarrow$ 3. $\Rightarrow$ iii. $\Rightarrow$ 4. et 4. $\Rightarrow$ i. $\Rightarrow$ 1. L'implication iii. $\Rightarrow$ 4. se fait exactement comme 3. $\Rightarrow$ 4. dans le théorème précédent. Toutes les autres implications sont évidentes.

Pour les algèbres de Banach non nécessairement unitaires, on a le résultat suivant:

Théorème 3: *Soit E une algèbre de Banach. Les propriétés suivantes sont équivalentes:*

a: $\rho(xy) \leq \rho(x).\|y\|$ *pour tout x et tout y dans E.*

b: Pour tout x quasi-inversible, de quasi-inverse x°*, la boule ouverte de centre x et de rayon* $\left[1+\rho(x^{\circ})\right]^{-1}$ *est contenue dans l'ensemble des éléments quasi-inversibles de E.*

c: $\rho(x+y) \leq \rho(x)+\|y\|$ *pour tout x et tout y de E.*

d: $|\rho(x)-\rho(y)| \leq \|x-y\|$ *pour tout x et tout y de E.*

e: *E est presque commutative.*

f: $\rho(xy) \leq \rho(x).\rho(y)$ *pour tout x et tout y de E.*

g: $\rho(x+y) \leq \rho(x)+\rho(y)$ *pour tout x et tout y de E.*

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Adresse de l'auteur:

Département de mathématique et d'informatique.

Institut Supérieur Scientifique.

B.P: 5026 Nouakchott - MAURITANIE

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F-INJECTIVITY IN NEGATIVE DEGREE AND TIGHT CLOSURE IN GRADED COMPLETE INTERSECTION RINGS

Nobuo HARA

Presented by J. Lambek, F.R.S.C.

Abstract. Let R be a commutative Noetherian $\mathbf{N}$-graded ring over a perfect field of characteristic $p > 0$. We say that R is F-injective in negative degree if the Frobenius action on the negatively graded pieces of the highest local cohomology of R is injective. We will show that a graded complete intersection ring with isolated singularity has this property if the the characteristic p is sufficiently large, and apply this result to computation of tight closure in such rings.

The notion of tight closure of an ideal in a commutative ring of characteristic $p > 0$ introduced by Hochster and Huneke [HH1] has been successfully applied to various problems in commutative algebra. Among others, the Briançon-Skoda theorem implies the deep connection between F-rational or F-regular rings, which are defined in terms of tight closure, and rational singularities in characteristic 0.

In spite of its importance, not so many attempts had been done for concrete computation of tight closure. In [S], Smith gives a new method of computing the tight closure of a homogeneous s.o.p. (system of parameters) ideal in a graded ring, which is especially useful in two-dimensional case. But in higher dimensional cases, one cannot always determine the tight closure even in hypersurface rings.

In this short note, we will compute the tight closure of a homogeneous s.o.p. ideal in a graded complete intersection ring with isolated singularity, by combining Smith's method with the observation by Fedder and Watanabe [FW] on F-injectivity — the injectivity of the Frobenius map on the local cohomology module.

In [FW], it is proved that F-injectivity of a graded ring R, together with negativity of the a-invariant $a(R)$, implies F-rationality. But if we look at their proof carefully, even when $a(R) \geq 0$, we find that the injectivity of the Frobenius map on the negatively graded pieces of the local cohomology $H^{\dim R}_{\mathfrak{m}}(R)$ (Watanabe called this property "F-injectivity in negative degree" [HW]) plays an important role in determining the tight closure of an s.o.p. ideal.

We will show that a graded complete intersection ring with isolated singularity is F-injective in negative degree if the characteristic p is "sufficiently large" (Theorem 4) by modifying Fedder's method [F1] [F2], and compute the tight closure in such rings for some examples.

Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian N-graded ring defined over a perfect field $k = R_0$ of characteristic $p > 0$, and $\mathfrak{m} = \bigoplus_{n>0} R_n$ be the irrelevant maximal ideal of R. Assume furthermore that R is Cohen-Macaulay. We will keep these assumptions throughout this paper for simplicity, although they are necessary only partly for the definitions and the argument. We always use the letter q for a power p^e of p.

Definition. The *tight closure* I^* of an ideal $I \subset R$ is the ideal defined by $x \in I^*$ iff there exists a non-zero-divisor $c \in R$ such that $cx^q \in I^{[q]}$ for $q = p^e \gg 0$, where $I^{[q]}$ is the ideal generated by the q-th powers of the elements of I.

Set $d = \dim R$ and consider the highest local cohomology module $H^d_{\mathfrak{m}}(R)$ of R. Since R is a graded ring, $H^d_{\mathfrak{m}}(R)$ carries the structure of a graded R-module. We will write its n-th graded piece as $[H^d_{\mathfrak{m}}(R)]_n$ for an integer n.

Definition. The Frobenius ring homomorphism $r \mapsto r^p$ of R induces the Frobenius map

$$F_n : [H^d_{\mathfrak{m}}(R)]_n \longrightarrow [H^d_{\mathfrak{m}}(R)]_{pn}$$

on each graded piece of $H^d_{\mathfrak{m}}(R)$. We say that R is *F-injective* (resp. *F-injective in negative degree*) if F_n is injective for every integer n (resp. for $n < 0$). We define *F-injectivity in degree* $\leq n_0$ for a fixed integer n_0, etc. in the same way.

The following characterization of tight closure in terms of F-injectivity in negative degree, to which we will give a brief proof, is essentially due to Fedder and Watanabe:

Proposition 1 [FW]. *Let $x_1, \dots, x_d$ be a homogeneous s.o.p. of R. Assume that R is F-injective in degree $\leq -n_0$ for a natural number n_0 and strongly F-regular in codimension $d-1$ (the second condition is satisfied if R is regular in codimension $d-1$). If a homogeneous element $z \in R$ with $\deg(z) \leq \sum_{i=1}^d \deg(x_i) - n_0$ is in the tight closure $(x_1, \dots, x_d)^*$ of the ideal $(x_1, \dots, x_d)$, then z lies in $(x_1, \dots, x_d)$ itself.*

Proof. First note that $H^d_{\mathfrak{m}}(R) \cong \lim R/(x_1^t, \dots, x_d^t)$ and the direct limit map $R/(x_1^t, \dots, x_d^t) \longrightarrow H^d_{\mathfrak{m}}(R)$ is injective since R is Cohen-Macaulay. Note also that the degree of the element $\zeta \in H^d_{\mathfrak{m}}(R)$ represented by $z \bmod (x_1, \dots, x_d) \in R/(x_1, \dots, x_d)$ is equal to $\deg(z) - \sum_{i=1}^d \deg(x_i) \leq -n_0$.

Now suppose that $z \notin (x_1, \dots, x_d)$. Then ζ is not zero, and its e-times Frobenius image $\zeta^q \in H^d_{\mathfrak{m}}(R)$ is not zero for any $q = p^e$ since R is F-injective in degree $\leq -n_0$. As $H^d_{\mathfrak{m}}(R)$ is an Artinian R-module, there exists $c_e \in R$ such that $c_e\zeta^q$ is not zero but lies in the socle of $H^d_{\mathfrak{m}}(R)$. The socle is finitely generated, so that there exists an integer n_1 such that $\deg(c_e\zeta^q) \geq n_1$ for every $q = p^e$. Thus we see $\deg(c_e) \to +\infty$ as $e \to +\infty$. Hence by [HH2] c_e is a test element for $e \gg 0$, which means, by definition, $c_e z^q \in (x_1^q, \dots, x_d^q)$ as $z \in (x_1, \dots, x_d)^*$. But this contradicts $c_e\zeta^q \neq 0$, and we are done. ∎

On the other hand, we have the following easy

LEMMA 2. *Let $x_1, \dots, x_d$ be a homogeneous s.o.p. of R, and assume that $d = \dim R \geq 1$. If a homogeneous element z satisfies $\deg(z) \geq \sum_{i=1}^d \deg(x_i)$, then $z \in (x_1, \dots, x_d)^*$.*

Proof. Let $\zeta \in H^d_{\mathfrak{m}}(R)$ be taken as in the proof of Proposition 1. Then $\deg(\zeta) \geq 0$ by the assumption. Since $H^d_{\mathfrak{m}}(R)$ is Artinian, if we pick a non-zero-divisor $c \in R$ of sufficiently large degree, then $c\zeta^q = 0$ for all $q = p^e$. Hence $\zeta \in (0)^*$ in $H^d_{\mathfrak{m}}(R)$, which implies $z \in (x_1, \dots, x_d)^*$. ∎

Taking into account the homogeneity of the tight closure of a homogeneous ideal [HH3], Proposition 1 and Lemma 2 show:

COROLLARY 3. *Let R satisfy the assumption in Proposition 1, and assume that $d = \dim R \geq 1$. For a homogeneous s.o.p. $x_1, \dots, x_d$ of R we set $N = \sum_{i=1}^d \deg(x_i)$. Then*

$$(x_1, \dots, x_d) + \bigoplus_{n \geq N} R_n \subseteq (x_1, \dots, x_d)^* \subseteq (x_1, \dots, x_d) + \bigoplus_{n > N - n_0} R_n.$$

In particular, in the case that R is F-injective in negative degree we have

$$(x_1, \dots, x_d)^* = (x_1, \dots, x_d) + \bigoplus_{n \geq N} R_n.$$

Thus being F-injective in negative degree is the key point to determine the tight closure of a homogeneous s.o.p. ideal. We will show that a graded complete intersection ring with only isolated singularity has this property for "sufficiently large p".

Let $S = k[X_1, \dots, X_s]$ be a polynomial ring over a field k, and $f_1, \dots, f_t$ be a regular sequence in S. We define the Jacobian ideal $J(f_1, \dots, f_t)$ associated with $f_1, \dots, f_t$ to be the ideal of S generated by the maximal minors of the Jacobian matrix $(\partial f_i / \partial X_j)_{1 \leq i \leq t, 1 \leq j \leq s}$.

THEOREM 4. *Let $S = k[X_1, \dots, X_s]$ be a graded polynomial ring over a perfect field k of characteristic $p > 0$ with $\deg(X_i) = e_i > 0$. Let $f_1, \dots, f_t$ be a homogeneous regular sequence in S such that $(X_1^{k_1}, \dots, X_s^{k_s}) \subseteq J(f_1, \dots, f_t)$ for some natural numbers $k_1, \dots, k_s$. For a natural number n, if the inequality $pn \geq \sum_{i=1}^s k_i e_i$ holds, then the graded complete intersection ring $R = S/(f_1, \dots, f_t)$ is F-injective in degree $\leq -n$. In particular, if $p \geq \sum_{i=1}^s k_i e_i$, then R is F-injective in negative degree.*

Proof. Set $f = \prod_{i=1}^t f_i$ and $R_i = S/(f_1, \dots, f_i)$. For $i = 1, \dots, t$ we consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{i-1} & \xrightarrow{f_i} & R_{i-1} & \longrightarrow & R_i & \longrightarrow & 0 \\ & & {\scriptstyle f_i^{p-1}F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & \\ 0 & \longrightarrow & R_{i-1} & \xrightarrow{f_i} & R_{i-1} & \longrightarrow & R_i & \longrightarrow & 0 \end{array}$$

with exact rows, where F denotes the Frobenius maps. If we take the induced exact sequences of local cohomologies associated with the maximal ideal $\mathfrak{n} = (X_1, \dots, X_s)$ of S and compose the connected homomorphisms between the highest local cohomologies, we obtain the following commutative diagram:

$$\begin{array}{ccc} H^{s-t}_{\mathfrak{n}R}(R) & \longrightarrow & H^s_{\mathfrak{n}}(S) \\ F\downarrow & & f^{p-1}F\downarrow \\ H^{s-t}_{\mathfrak{n}R}(R) & \longrightarrow & H^s_{\mathfrak{n}}(S). \end{array}$$

Here F denotes the induced Frobenius maps, and the horizontal maps $H^{s-t}_{\mathfrak{n}R}(R) \to H^s_{\mathfrak{n}}(S)$ are injective graded homomorphisms of degree $-\deg(f)$. Note also that $H^s_{\mathfrak{n}}(S)$ can be identified with the naturally graded module $E_S = (X_1 \cdots X_s)^{-1}k[X_1^{-1}, \dots, X_s^{-1}]$ of inverse polynomials ([GW], see also [Ha]) whose S-module structure is defined by

$$g.\varphi = (\text{the negative } \mathbf{Z}^s\text{-graded part of the product } g\varphi)$$

for $g \in S$ and $\varphi \in E_S$. Thus to see the F-injectivity of R in degree $\leq -n$, it is sufficient to show:

Claim. The map $f^{p-1}F : E_S \longrightarrow E_S$ which sends $\varphi \in E_S$ to $f^{p-1}.\varphi^p \in E_S$ is injective in degree $\leq -n - \deg(f)$.

We will prove this by induction on the embedding codimension t of R under the hypothesis in our theorem. In the case $t = 0$, the above map is nothing but the Frobenius map $F : E_S \longrightarrow E_S$, which is obviously injective. Suppose that $t > 0$. First note that the assumption $(X_1^{k_1}, \dots, X_s^{k_s}) \subseteq J(f_1, \dots, f_t)$ for $R = S/(f_1, \dots, f_t)$ implies the corresponding condition $(X_1^{k_1}, \dots, X_s^{k_s}) \subseteq J(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_t)$ for a graded complete intersection ring $S/(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_t)$ of embedding codimension $t-1$ for $i = 1, \dots, t$ since $J(f_1, \dots, f_t) \subseteq J(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_t)$. Hence, by induction, we may assume that the map $(f/f_i)^{p-1}F$: $E_S \longrightarrow E_S$ is injective in degree $\leq -n - \deg(f/f_i)$, and so is in degree $< -n - \deg(f)$.

Now to prove the claim, set $I = (X_1^{k_1}, \dots, X_s^{k_s})$, and suppose on the contrary that there exists a non-zero homogeneous inverse polynomial $\varphi \in E_S$ of degree $\leq -n - \deg(f)$ such that $f^{p-1}.\varphi^p = 0$. If we take a natural number m and a homogeneous polynomial $\tilde{\varphi} \in S$ such that $\varphi = \tilde{\varphi}.(X_1 \cdots X_s)^{-m}$, then the condition $f^{p-1}.\varphi^p = f^{p-1}\tilde{\varphi}^p.(X_1 \cdots X_s)^{-pm} = 0$ is equivarent to the condition $f^{p-1}\tilde{\varphi}^p \in I^{[p]} = (X_1^{pm}, \dots, X_s^{pm})$. On the other hand, we have $(f/f_i)^{p-1}.\varphi^p \neq 0$, so that $(f/f_i)^{p-1}\tilde{\varphi}^p \notin I^{[p]}$ for each i. Hence there exist positive integers $r_1, \dots, r_t \leq p-1$ such that $f_1^{r_1} \cdots f_t^{r_t}\tilde{\varphi}^p \in I^{[p]}$ and $f_1^{r_1} \cdots f_i^{r_i-1} \cdots f_t^{r_t}\tilde{\varphi}^p \notin I^{[p]}$. If we take the partial derivatives of the former, we get

$$\sum_{i=1}^{t} r_i f_1^{r_1} \cdots f_i^{r_i-1} \cdots f_t^{r_t}\tilde{\varphi}^p \partial f_i/\partial X_j \in I^{[p]}.$$

for $j = 1, \dots, s$. If we set $\tilde{\varphi}_i = f_1^{r_1} \cdots f_i^{r_i-1} \cdots f_t^{r_t}\tilde{\varphi}^p$, then the matrix equation

$$\begin{bmatrix} \partial f_1/\partial X_{i_1} & \dots & \partial f_t/\partial X_{i_1} \\ \dots & \dots & \dots \\ \partial f_1/\partial X_{i_t} & \dots & \partial f_t/\partial X_{i_t} \end{bmatrix} \begin{bmatrix} r_1\tilde{\varphi}_1 \\ \vdots \\ r_t\tilde{\varphi}_t \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \pmod{I^{[p]}}$$

holds for $1 \le i_1 < \dots < i_t \le s$. Denoting the determinant of the matrix in the left hand side by $\Delta_{i_1 \dots i_t}$, we have $r_i \Delta_{i_1 \dots i_t} \bar{\varphi}_i \in I^{[p]}$ for each i. Since r_i are not divisible by p, and the Jacobian ideal $J = J(f_1, \dots, f_t)$ is generated by $\Delta_{i_1 \dots i_t}$'s, $\bar{\varphi}_i$ lies in the ideal $I^{[p]} : J \subseteq I^{[p]} : (X_1^{k_1}, \dots, X_s^{k_s}) = (X_1^{pm-k_1} \cdots X_s^{pm-k_s})S + I^{[p]}$. As the homogeneous polynomial $\bar{\varphi}_i = f_1^{r_1} \cdots f_i^{r_i - 1} \cdots f_t^{r_t} \bar{\varphi}^p$ does not belong to $I^{[p]}$, it must have a term $aX_1^{\nu_1} \cdots X_s^{\nu_s}$ with each $\nu_j \ge pm - k_j$ and $0 \ne a \in k$. Therefore

$$\sum_{j=1}^{s} (pm - k_j)e_j \le \sum_{j=1}^{s} \nu_j e_j = \deg(\bar{\varphi}_i) \le (p-1)\deg(f) - \deg(f_i) + p\deg(\bar{\varphi}).$$

Summing up these inequalities for $i = 1, \dots, t$, we obtain

$$t\sum_{j=1}^{s} (pm - k_j)e_j \le (pt - t - 1)\deg(f) + pt\deg(\bar{\varphi}).$$

This inequality, together with $\deg(\varphi) = \deg(\bar{\varphi}) - m\sum_{j=1}^{s} e_j \le -n$, implies $ptn \le t\sum_{j=1}^{s} k_j e_j - (t+1)\deg(f)$, which contradicts the assumption of our theorem. Thus the claim is proven. ∎

Remarks. (i) Fedder has a similar result in the case $a(R) < 0$ [F2]. Moreover his result in [F3] implies that a 2-dimensional Cohen-Macaulay graded ring is F-injective in negative degree for "sufficiently large p" (see also [HW]).

(ii) By Theorem 4 it follows that any graded complete intersection ring with isolated singularity is F-injective in sufficiently small degrees.

(iii) If R is a hypersurface ring (i.e., $t = 1$), then it is immediate from the proof of the theorem that the lower bound "$pn \ge \sum_{i=1}^{s} k_i e_i$" of p can be improved to "$pn > \sum_{i=1}^{s} k_i e_i - 2\deg(f)$". We use this fact in the following examples.

EXAMPLES.

1. Let $R = k[X_1, \dots, X_n]/(X_1^d + \dots + X_n^d)$, and let x_i denote the image of X_i in R. If p does not divide d and $p > n(d-1) - 2d$, then R is F-injective in negative degree, and

$$(x_1, \dots, x_{n-1})^* = (x_1, \dots, x_{n-1}) + R_{\ge n-1} = (x_1, \dots, x_{n-1}, x_n^{n-1}).$$

2. $R = k[X, Y, Z, W]/(X^2 + Y^3 + Z^7 + W^{42})$ is F-injective in negative degree if $p > 42$. In this case the tight closures of homogeneous s.o.p. ideals are:

$$(y, z, w)^* = (x, y, z, w),$$
$$(x, z, w)^* = (x, y^2, z, w),$$
$$(x, y, w)^* = (x, y, z^6, w), \text{ and}$$
$$(x, y, z)^* = (x, y, z, w^{41}).$$

It is possible to compute the tight closures for lower p. For example, we can verify by direct manipulation of Frobenius action on the local cohomology module that the tight closure $(x, y, z)^*$ coincides with (x, y, z, w^{41}) if $p = 37, 31, 29, 23$; (x, y, z, w^{40}) if $p = 41, 19, 17$; (x, y, z, w^{39}) if $p = 13, 11$; and (x, y, z, w^{34}) if $p = 5$.

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Department of Mathematics, Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169 Japan
E-mail: 62m508@cfi.waseda.ac.jp

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A Cohomology for Dynamical Systems

Oleg I. Bogoyavlenskij*

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Abstract: A cohomology is introduced for dynamical systems on smooth manifolds. It is proved that the cohomologies $H^2(V, M^{2k})$ and $H^4(V, M^{2k})$ are infinite-dimensional for an arbitrary C-integrable and non-degenerate Hamiltonian system V on a symplectic manifold M^{2k}.

I. Let $V(x)$ be a smooth vector field on a manifold M^n and

$$\dot{x}^i = V^i(x^1, \cdots, x^n) \tag{1}$$

be the corresponding dynamical system. We denote Λ_V^m the space of differential m-forms ω_m on M^n which are invariant with respect to system (1).

Let us consider the complex of V-invariant differential forms on M^n

$$0 \rightarrow \Lambda_V^0 \xrightarrow{d} \Lambda_V^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_V^{n-1} \xrightarrow{d} \Lambda_V^n \rightarrow 0. \tag{2}$$

Definition 1 *The quotient space*

$$H^*(V, M^n) = \mathrm{Ker}\, d / \mathrm{Im}\, d \tag{3}$$

is called the cohomology of the dynamical system V (1).. The wedge product of differential forms induces a ring structure in $H^*(V, M^n)$.

We have the ring homomorphism

$$\alpha: \quad H^*(V, M^n) \rightarrow H^*(M^n) \tag{4}$$

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that transforms a cohomology class of the invariant closed q-forms into the corresponding de Rham's cohomology class [4] of the general closed q-forms. For any constant $c \neq 0$, we have the isomorphism $H^*(cV, M^n) = H^*(V, M^n)$. For $c = 0$ the cohomology $H^*(0, M^n)$ is isomorphic to the de Rham cohomology [4] $H^*(M^n)$.

Remark 1. Using Duff's results [5] it is possible to generalize the constructions of this paper for dynamical systems on a manifold M^n with boundary.

II. The homomorphism α (4) has an inverse and therefore is an isomorphism for the following dynamical systems:

Assume that all trajectories of the dynamical system (1) are closed curves and have the same period T. Let φ_τ

$$\varphi_\tau : \quad M^n \to M^n, \quad \varphi_T = \mathrm{id} \tag{5}$$

be the corresponding action of the circle S^1. For any closed q-form ω_q we construct the q-form

$$\alpha^{-1}\omega_q = \frac{1}{T}\oint \varphi_\tau^*\omega_q \mathrm{d}\tau. \tag{6}$$

Obviously, the q-form $\alpha^{-1}\omega_q$ is closed and invariant with respect to all diffeomorphisms (5). The q-form $\alpha^{-1}\omega_q$ belongs to the same de Rham's cohomology class in $H^q(M^n)$ as the closed q-form ω_q because the q-forms $\varphi_\tau^*\omega_q$ are homotopically equivalent to ω_q for all τ. Therefore $\alpha \circ \alpha^{-1} = \mathrm{id}$ in $H^q(M^n)$ and hence the map α is an isomorphism.

III. Let dynamical system (1) be a generic non-integrable Hamiltonian system. Then $V^i = P_1^{ij} H_{,j}$ where P_1 is a non-degenerate Poisson structure on M^{2k}. The corresponding cohomology is isomorphic to the sum

$$H^*(V, M^{2k}) = \mathbb{R}[u]/u^{k+1}\mathbb{R}[u] + H^{2k}(V, M^{2k}) \tag{7}$$

of the quotient-ring of polynomials of a single variable u and the infinite-dimensional group $H^{2k}(V, M^{2k})$ that has a trivial law of multiplication. The generator $u \in H^2(V, M^{2k})$ corresponds to the invariant symplectic structure $\omega_1 = P_1^{-1}$. The linear independent elements of the infinite-dimensional group $H^{2k}(V, M^{2k})$ are represented by the invariant closed $2k$-forms

$$\omega_F = F(H)\omega_1 \wedge \cdots \wedge \omega_1. \tag{8}$$

There are k factors ω_1 in the wedge product (8), $F(H)$ is an arbitrary smooth function of the single variable and $H(x)$ is the Hamiltonian function.

IV. Let P_1^{ij} be a non-degenerate Poisson structure on a manifold M^{2k}. Let us consider a completely integrable in Liouville's sense Hamiltonian system

$$\dot{x}^i = P_1^{i\alpha} H_{,\alpha}, \qquad H_{,\alpha} = \partial H/\partial x^\alpha. \tag{9}$$

Definition 2 *Hamiltonian system (9) is called C-integrable if it is integrable in Liouville's sense and all its invariant submanifolds are compact.*

These invariant submanifolds are tori $\mathbb{T}^k$:

$$\mathbb{T}^k: \quad I_1 = c_1, \cdots, \quad I_k = c_k, \quad 0 \le \varphi_i \le 2\pi. \tag{10}$$

The Liouville Theorem [1,2,6] implies that almost all points of the manifold M^n (excluding a set $S \subset M^n$, $\dim S \le n-1$) are covered by a system of open toroidal domains $\mathcal{O}_m \subset M^{2k}$ with the action-angle coordinates $I_1, \cdots, I_k, \varphi_1, \cdots, \varphi_k$. In these coordinates the completely integrable system (9) has the form

$$\dot{I}_\ell = 0, \quad \dot{\varphi}_\ell = \frac{\partial H(I)}{\partial I_\ell}. \tag{11}$$

The completely integrable Hamiltonian system (9) is called non-degenerate if the condition for the Hessian matrix

$$\det \left\| \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta} \right\| \neq 0 \tag{12}$$

is met almost everywhere in the action-angle coordinates (11).

Definition 3 *A (p,q) tensor T on the manifold M^n is called C-invariant if it is invariant with respect to a C-integrable non-degenerate Hamiltonian system (9).*

We proved in [3] the following Theorem.

Theorem 1 *1) In the toroidal domain $O = B_r \times \mathbb{T}^k \subset M^n$ a closed 2-form ω_c is C-invariant if and only if it has the form ($\ell = 1, \cdots, k$)*

$$\omega_c = \mathrm{d}\Big(\frac{\partial B(J)}{\partial J_\ell}\Big) \wedge \mathrm{d}\varphi_\ell + \mathrm{d}f_\ell(I) \wedge \mathrm{d}I_\ell, \quad J_\ell(I) = \frac{\partial H(I)}{\partial I_\ell} \tag{13}$$

where $B(J_1, \cdots, J_n)$ and $f_\ell(I_1, \cdots, I_k)$ are arbitrary smooth functions.

The Hamiltonian system (11) has the form

$$\dot{J}_\ell = 0, \quad \dot{\varphi}_\ell = J_\ell \tag{14}$$

in the toroidal coordinates J_ℓ (13) and φ_ℓ. Let θ be a differential 1-form

$$\theta = \theta_i(J, \varphi)\mathrm{d}J_i + \theta_{i+k}(J, \varphi)\mathrm{d}\varphi_i. \tag{15}$$

Theorem 2 *1) Differential 1-form θ (15) is C-invariant if and only if*

$$\theta = \theta_i(J)\mathrm{d}J_i. \tag{16}$$

2) Any closed C-invariant 1-form θ (15) is exact.

Proof. 1) For the 1-form θ (15), the invariance equation has the form

$$(L_V\theta)_\beta = \dot{\theta}_\beta + V^\alpha_{,\beta}\theta_\alpha = 0 \tag{17}$$

where L_V is the Lie derivative. In view of (14) and (15) equations (17) imply

$$\dot{\theta}_i = -\theta_{i+k}, \quad \dot{\theta}_{i+k} = 0. \tag{18}$$

In view of (14), solutions to system (18) have the form

$$\theta_i(t) = -\tilde{\theta}_{i+k}(J)t + \tilde{\theta}_i(J), \quad \theta_{i+k}(t) = \tilde{\theta}_{i+k}(J). \tag{19}$$

Components $\theta_\alpha(J, \varphi)$ of any smooth 1-form (15) are bounded on any torus $\mathbb{T}^k$ (10). Solutions (19) are bounded for all t if and only if $\tilde{\theta}_{i+k}(J) = 0$ for $i = 1, \cdots, k$. Therefore using (19) and the fact that general trajectories of the C-integrable non-degenerate Hamiltonian system (11), (14) are dense everywhere on the tori $\mathbb{T}^k$ we obtain that the 1-form θ is invariant if and only if it has the form (16).

2) If the C-invariant 1-form θ (16) is closed then applying Poincare's Lemma we obtain $\theta = \mathrm{d}F(I)$.

Proposition 1 *Any C-invariant differential 3-form ω_3 has the form*

$$\omega_3 = b_{i\ell m}(J)\mathrm{d}J_i \wedge \mathrm{d}J_\ell \wedge \mathrm{d}\varphi_m + c_{i\ell m}(J)\mathrm{d}J_i \wedge \mathrm{d}J_\ell \wedge \mathrm{d}J_m \tag{20}$$

where coefficients $c_{i\ell m}(J)$ are alternating and $b_{i\ell m}(J)$ satisfy the equations

$$b_{i\ell m}(J) + b_{\ell m i}(J) + b_{m i \ell}(J) = 0, \quad b_{i\ell m}(J) = -b_{\ell i m}(J). \tag{21}$$

Theorem 3 *1) A closed differential 3-form ω_3 is invariant with respect to the C-integrable non-degenerate Hamiltonian system (11) if and only if it has the form*

$$\omega_3 = \mathrm{d}\left(\frac{\partial B_i(J)}{\partial J_m} + b_{im}(J)\right) \wedge \mathrm{d}J_i \wedge \mathrm{d}\varphi_m + \mathrm{d}(a_{i\ell}(J)\mathrm{d}J_i \wedge \mathrm{d}J_\ell) \tag{22}$$

in the toroidal coordinates J_i, φ_i. Here $B_i(J)$ are arbitrary smooth functions of $J_1, \cdots, J_k$, and coefficients $a_{i\ell}(J)$ and $b_{im}(J)$ satisfy the equations $a_{i\ell}(J) = -a_{\ell i}(J)$, $b_{im}(J) = b_{mi}(J)$.

2) Any closed C-invariant differential 3-form ω_3 is exact. The equation $\omega_3 = d\tilde{\omega}_2$ holds where the C-invariant 2-form $\tilde{\omega}_2$ has the form

$$\tilde{\omega}_2 = \left(\frac{\partial B_i(J)}{\partial J_m} + \frac{\partial B_m(J)}{\partial J_i} + b_{im}(J)\right)\mathrm{d}J_i \wedge \mathrm{d}\varphi_m + a_{i\ell}(J)\mathrm{d}J_i \wedge \mathrm{d}J_\ell. \tag{23}$$

The proof of Proposition 1 and Theorem 3 is based on the same ideas as proofs of Theorems 1 and 2 and will be published elsewhere.

Theorem 4 *Assume that a C-integrable non-degenerate Hamiltonian system (11) is defined in an open toroidal domain $\mathcal{O} = B_r \times \mathbb{T}^k$. Then the first five cohomology groups have the form*

$$H^0(V,\mathcal{O}) = \mathbb{R}^1, \quad H^1(V,\mathcal{O}) = 0, \quad H^2(V,\mathcal{O}) = \mathbb{R}^\infty, \tag{24}$$

$$H^3(V,\mathcal{O}) = 0, \quad H^4(V,\mathcal{O}) = \mathbb{R}^\infty.$$

Proof. Theorem 2 implies that each C-invariant closed 1-form is the exterior derivative of some first integral. That means $H^1(V,\mathcal{O}) = 0$. Theorem 3 implies that each C-invariant closed 3-form is the exterior derivative of some

C-invariant 2-form. That means $H^3(V, \mathcal{O}) = 0$. Theorem 1 and Theorem 2 imply that $H^2(V, \mathcal{O}) = \mathbb{R}^\infty$. The Proposition 1 implies that the wedge product $\omega_1 \wedge \omega_2$ of two generic C-invariant closed 2-forms ω_1 and ω_2 (13) is not the exterior derivative of any C-invariant 3-form (20). Hence the cohomology $H^4(V, \mathcal{O})$ is infinite-dimensional.

V. The "toroidal surgeries" method presented in [3] provides a smooth extension of any invariant 2-form ω_c (13) on the whole manifold M^{2k}. Therefore, the second and the fourth cohomologies $H^2(V, M^{2k})$ and $H^4(V, M^{2k})$ are infinite-dimensional. Hence we obtain the following consequence.

Corollary 1 *The infinite-dimensionality of the cohomologies $H^2(V, M^{2k})$ and $H^4(V, M^{2k})$ is the necessary condition for the non-degenerate integrability of the dynamical system V on the manifold M^{2k}.*

Concluding Remark. We have introduced in this paper the cohomology $H^*(V, M^n)$ that is an invariant characterizing the global properties of dynamical systems on smooth manifolds.

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Department of Mathematics and Statistics
Queen's University, Kingston, Canada, K7L 3N6

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Martingale Problem Characterization for Superprocesses

Guillaume Leduc

Presented by D.A. Dawson, F.R.S.C.

Abstract

The martingale problem for superprocesses with parameters (ξ, Φ, k) is studied where $k(ds)$ may not be absolutely continuous with respect to the Lebesgue measure. We show that for any process X which *partially* solves a certain martingale problem, an extended form of the liftings defined in [2] can be found, which permits the full martingale problem to be well defined. A sequence of (ξ, Φ, k^n)-superprocesses approximating the (ξ, Φ, k)-superprocess can be found, where $k^n(ds)$ has the form $\phi(s, \xi_s)ds$. Using an argument in [4], applied to the (ξ, Φ, k^n)-superprocesses, we derive that the full martingale problem is well posed.

1 Martingale problem: statement and existence

Let us define once and for all the following objects: let $T > 0$ be a constant, we consider only our processes during the time interval $[0, T]$. Let $\xi = (\xi_t, \mathcal{F}, \pi_x)$ be a (time homogeneous) Hunt process on a metrizable Luzin space $(E, \mathcal{E})$. It is always the case that π_x induces a positive contraction semigroup $S_t(f)(x)$ on $b\mathcal{E}$ defined by $S_t(f)(x) := \pi_x f(\xi_t)$. Let L be the set of bounded measurable functions f such that $S_t(f)(x)$ is *strongly* continuous, that is $\|S_t(f)(.) - S_{t+h}(f)(.)\|_\infty \to 0$ as $h \to 0$. Let A be the (strong infinitesimal) generator of S_t and $\mathcal{D}(A)$ its domain.[1] We assume that L is an algebra. Let ρ be the metric of the Ray topology (c.f. [5, II.17] for instance). The Hunt property of ξ is preserved by a passage to the topology generated by metric $\Delta(x,y) = d(x,y) + \rho(x,y)$ (see [5, Th. V.47.5] for instance). For now on, it is the only metric on E we will consider. Let $k(ds)$ be a continuous non negative additive functional of ξ satisfying the condition

$$h_t^r(x) := \pi_x k(r,t) \to 0 \text{ uniformly in } x \text{ as } t - r \to 0. \tag{1}$$

(Note that this is equivalent to the so called "admissibility condition" in [2] according to [1, Lemma 3.3.1]). We assume that $h_t^r(.) \in L$ for every r, t. Let also $b(x)$ and $\ell(x, d\mu)$ be respectively a measurable function and a kernel satisfying the following conditions:

$$0 \leq b(x) \leq 1, \; 0 \leq \int_0^\infty u^2 \ell(x, du) \leq 1. \tag{2}$$

Throughout this paper we pose

$$\Phi(x, f) = b(x) f^2(x) + \int_0^\infty \mathcal{E}(uf)\, \ell(x, du) \tag{3}$$

where $\mathcal{E}(z) = e^{-z} + z - 1$. In the same spirit as [3], we assume that for every $\phi \in \mathcal{D}(A)$, $\Phi(x, \phi) \in L$. We will state and prove the martingale problem for the superprocess with parameters (ξ, Φ, k). In this paper, every measure valued process considered will be a canonical càdlàg process and the triple $X = (X_t, \Im, P_{r,\mu})$ will always denote such processes. The filtrations $\{\Im_t^r\}_{t \in [r,\infty)}$ are assumed to satisfy the usual hypothesis and $P_{r,\mu}(X_r = \mu) = 1$.

Definition 1 (partial-martingale problem for A) *The process $X = (X_t, \Im, P_{r,\mu})$ will be said to be a solution to the* ***partial martingale problem for*** *A if for every $\phi \in \mathcal{D}(A)$*

$$< X_t, \phi > - < X_r, \phi > - \int_r^t < X_s, A\phi > ds \tag{4}$$

is a $P_{r,\mu}$-martingale for $s \in [\, r, t]$.

[1] Note that it can be proved that $\mathcal{D}(A)$ is dense in $b\mathcal{E}$, when $b\mathcal{E}$ is furnished with the topology of bounded pointwise convergence.

The full martingale problem requires for its statement the notion of a lifting of an additive functional:

Definition 2 (Lifting) *Let $X = (X_t, \Im, P_{r,\mu})$ be a (canonical càdlàg M_f-valued) process and let $a(ds)$ be an additive functional of ξ. A natural[2] right continuous additive functional $A(ds)$ of X will be called a **lifting** of $a(ds)$ if for every $t \geq r$ $A(r,s] + < X_s, \pi_. a(s,t] >$ is a $P_{r,\mu}$-martingale for $s \in [\, r, t]$.*

The following guaranties the existence and uniqueness of liftings for every solution $X = (X_t, \Im, P_{r,\mu})$ to the partial martingale problem.

Theorem 1 *(existence of liftings and uniqueness) Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for A. Then for every additive functional $a(ds)$ of ξ satisfying (1), there exists a unique lifting $A(ds)$ of X.*

Notation: Let f be a progressively measurable bounded function. Then the additive functional $\Phi(\xi_s, f(s,\xi_s))k(ds)$ satisfies (1), and we will denote by $K^{\Phi(f)dk}(ds)$ the lifting of $\Phi(\xi_s, f(s,\xi_s))k(ds)$.

The next Theorem characterizes the $(\xi, \Phi, k)-$superprocess in terms of a martingale problem. Theorem 1 above, is needed for the statement of this martingale problem to be well defined. This theorem is the central result of this paper.

Theorem 2 *(full martingale problem) Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem, and let $P_{r,\mu}^{(\xi,\Phi,k)}$ be the distribution of the $(\xi, \Phi, k)-$superprocess. If*

$$\begin{array}{l} \exp(- < X_t, \phi >) + \int_r^t \exp(- < X_s, \phi >) < X_s, A\phi > ds \\ - \int_r^t \exp(- < X_s, \phi >) K^{\Phi(\phi)dk}(ds) \end{array} \tag{5}$$

is a $P_{r,\mu}$ -martingale for every $\phi \in \mathcal{D}(A)$, then $P_{r,\mu} = P_{r,\mu}^{(\xi,\Phi,k)}$. Conversely, the $(\xi, \Phi, k)-$superprocess is solution to the partial martingale problem and (5) is a $P_{r,\mu}^{(\xi,\Phi,k)}$-martingale.

[2] *An additive functional $A(ds)$ is said to be natural if the processes $t \mapsto A(r,t]$ are predictable.*

A solution $X = (X_t, \Im, P_{r,\mu})$ to the partial martingale for A which is such that (5) is a $P_{r,\mu}$-martingale will be called a solution to the **full martingale problem** for (A, Φ, k). Essentially using Ito's formula, it can be proved that the (ξ, Φ, k)-superprocess is a solution to the full martingale problem. We will concentrate our attention, in this paper, to the proof of uniqueness of the solution. This requires to establish the existence of a "smooth approximation" for the (ξ, Φ, k)-superprocess, which is done in the next section.

2 Smooth approximation for superprocesses

Definition 3 *We say that a mapping $\psi(s,x)$ is smooth for the strong generator $(A, \mathcal{D}(A))$ of ξ, or simply that $\psi(s,x)$ is* **smooth for** A*, if 1°) $\psi(s,.)$ belongs to $\mathcal{D}(A)$ for every s, 2°) $\frac{\partial}{\partial s}\psi$ (s,x) exists and $\left\| \frac{\psi(s+h,.)-\psi(s,.)}{h} - \frac{\partial}{\partial s}\psi(s,.) \right\|_\infty \to 0$ 3°) $\psi, \frac{\partial}{\partial s}\psi$ and $A\psi$ are bounded and strongly continuous.*

Definition 4 *We will say that superprocesses X^n with parameters (ξ, Φ, k^n) are an* **A-smooth approximating sequence** *for the superprocess X with parameters (X, Φ, k), if: 1°) $k^n(ds)$ converges to $k(ds)$ in $L^1(\pi_x)$ for every $x \in E$, 2°) $\sup_{x\in E} \|\pi_x k^n(r,t)\|_\infty < \infty$ for every $r, t \geq 0$, 3°) the log-Laplace functional v^n of X^n converges to the log-Laplace functional v of X. 4°) v^n is smooth for A, 5°) $Av^n_{s,T} + \frac{\partial}{\partial s}v^n_{s,T} = \Phi(.,v^n_{s,T})\lambda^n(s,.)$.*

Theorem 3 *There exists an A-smooth approximating sequence of the form $(\xi, \Phi, \lambda^n(s,\xi_s)ds)$ for the superprocess with parameters (ξ, Φ, k).*

3 The partial martingale problem

Several consequences can be derived from the knowledge that a process $X = (X_t, \Im, P_{r,\mu})$ is a solution to the partial martingale problem. In this section we investigate some of these properties.

Theorem 4 ***(continuity of liftings)*** *Every lifting $A(ds)$ of an additive functional $a(ds)$ satisfying (1) is a continuous additive functional.*

Theorem 5 ***(induced convergence for liftings)*** *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for A, and let $\psi^n_s(x)$ be a* *sequence of measurable functions such that* $\sup_{s,x,n} |\psi_s^n(x)| < \infty$ *and* $\psi_s^n(x) \to \psi_s(x)$ *for every* $x \in E$ *and every* $s \geq r$. *Suppose that* $\lambda^n(s,\xi_s)ds$ *converges to* $k(ds)$ *in* $L^1(\pi_x)$ *for every* $x \in E$, *and that* $\sup_{x\in E} \left\| \pi_x \int_r^t \lambda^n(s,\xi_s)ds \right\|_\infty < \infty$ *for every* $r,t \geq 0$ *Let* $K_i^n(ds), i = 1,2$, *be given by*

$$\begin{aligned} K_1^n(ds) &= < X_s, \Phi(.,\psi_s^n(.))\lambda^n(s,.) > ds \\ K_2^n(ds) &= K^{\Phi(\psi^n)dk}(ds). \end{aligned}$$

Then $K_i^n(ds)$ *converges in* $L^1(P_{r,\mu})$ *to the lifting* $K^{\Phi(\psi)dk}(ds)$ *of* $\Phi(\xi_s,\psi_s(\xi_s))k(ds)$.

4 Martingale problem: uniqueness

With the use of the previously established results, we can show that the solution to the full martingale problem for (A,Φ,k) is unique, as stated in Theorem 2. In fact, the first step consists in establishing the following Theorem:

Theorem 6 *Let* X_t *be a solution to the full martingale problem for* (A,Φ,k) *and let* ψ *be smooth for* A. *Then*

$$\begin{aligned} &\exp(- < X_t,\psi_t >) + \textstyle\int_r^t \exp(- < X_s,\psi_s >) < X_s, A\psi_s + \frac{\partial}{\partial_s}\psi_s > ds \\ &- \textstyle\int_r^t \exp(- < X_s,\psi_s >) K^{\Phi(\psi)dk}(ds) \end{aligned}$$

is a $P_{r,\mu}-$*martingale, where* $K^{\Phi(\psi)dk}(ds)$ *is the lifting of* $\Phi(\xi_s,\psi_s)k(ds)$.

Let now the $(\xi,\Phi,\lambda^n(s,\xi_s)ds)$-superprocesses form an A smooth approximating sequence for the (ξ,Φ,k)-superprocess. Let v^n be the coresponding log-Laplace functionals. From Theorem (6) we get that

$$\begin{aligned} &\exp\left(- < X_t, v_{t,T}^n >\right) + \textstyle\int_r^t \exp\left(- < X_s, v_{s,T}^n >\right) < X_s, Av_{s,T}^n + \frac{\partial}{\partial_s}v_{s,T}^n > ds \\ &- \textstyle\int_r^t \exp\left(- < X_s, v_{s,T}^n >\right) K^{\Phi(\xi_s,v_{s,T}^n)dk}(ds) \end{aligned}$$

is a martingale. Putting $x_t^n =< X_t, v_{t,T}^n >$, the equality $Av_{s,T}^n(.) + \frac{\partial}{\partial_s}v_{s,T}^n(.) = \Phi(.,v_{s,T}^n)\lambda^n(s,.)$ gives

$$e^{-x_t^n} = M_t^n(\phi) + \textstyle\int_r^t e^{-x_t^n} K_1^n(ds) - \int_r^t e^{-x_t^n} K_2^n(ds) \qquad (6)$$

where

$$\begin{aligned} K_1^n(ds) &:= < X_s, \Phi(., v_{s,T}^n)\lambda^n(s,.) > ds \\ K_2^n(ds) &:= K^{\Phi(v_{.,T}^n)dk}(ds). \end{aligned}$$

Clearly $e^{-x_t^n} \rightarrow e^{-x_t}$ pointwise and in $L^1(P_{r,\mu})$ where $x_t :=< X_t, v_{t,T} >$ and v is the log-Laplace functional of the superprocess (ξ, Φ, k).

According to Theorem 5, $K_i^n(ds)$ in $L^1(P_{r,\mu})$ to $K(ds) := K^{\Phi(v_{.,T})dk}(ds)$ for $i = 1, 2$. But this implies that

$$\int_0^t e^{-x_t^n} K_1^n(ds) - \int_0^t e^{-x_t^n} K_2^n(ds) \rightarrow 0 \tag{7}$$

where the convergence holds in $L^1(P_{r,\mu})$.

That forces $M_t^n(\phi)$ to converge in $L^1(P_\mu)$ to a limit $M_t(\phi)$ which has to be a martingale, and we get

$$P_{r,\mu}\left(e^{-x_T}\right) = P_{r,\mu}\left(e^{-x_r}\right), \text{ that is,}$$

$$P_{r,\mu}\left(\exp\left(- < X_T, \phi >\right)\right) = \exp\left(- < \mu, v_{r,T} >\right).$$

In other words, X_t is the superprocess with parameters (ξ, Φ, k).

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SUR LES ALGEBRES DE BANACH VERIFIANT $x^{n(x)} A x^{n(x)} = x A x$ POUR TOUT $x \in A$

E. ILLOUSSAMEN (*)

Presented by J. Lambek, F.R.S.C.

Résumé: Nous montrons ici que si A est une algèbre de Banach complexe vérifiant la propriété : Pour tout $x \in A$ il existe un entier $n = n(x)$ tel que $x^n A x^n = x A x$, alors A est commutative modulo son radical (de Jacobson). De plus son radical premier coincide avec l'ensemble de ses éléments quasi-nilpotents. Nous donnons également un résultat dans le cas particulier ou A vérifie la condition $x^2 A x^2 = x A x$ pour tout $x \in A$.

1-Introduction. Dans [4], les auteurs ont donné un théorème de structure pour les algèbres de Banach (complexe ou réelles) semi-premières vérifiant $x^2 A x^2 = xAx$ pour tout $x \in A$. Plus précisement, ils ont montré le résultat suivant:

Théorème 1 [4]. Soit A une algèbre de Banach (complexe ou réelle) semi-première. Si A vérifie $x^2 A x^2 = xAx$ pour tout $x \in A$, alors A est somme direct d'idéaux $M_1, \dots, M_n$ (n étant un entier naturel) tels que chaque M_i soit isomorphe soit à $\mathbb{R}$, soit à $\mathbb{C}$ soit au corps des quaternions. Si A est complexe, alors tous les M_i sont isomorphes à $\mathbb{C}$ et A sera commutative.

D'autre part, certains autres auteurs se sont intéressés aux algèbres de Banach complexes A vérifiants $Ax^2 = Ax$ pour tout $x \in A$ (voir par exemple [2],[3],[5],[6],...). La description complète de ces algèbres a été établi dans [3] où l'on donne le théorème suivant:

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Théorème 2 [3]. Soit A une algèbre de Banach complexe. Alors $Ax^2 = Ax$ pour tout $x \in A$ si et seulement si $A = B \oplus R$ (R étant le radical de Jacobson de A) tels que $AR = \{0\}$ et B est isomorphe à $\mathbb{C}^n$ pour un certain $n \geqslant 0$.

Remarquons que si A est une algèbre de Banach vérifiant $Ax^2 = Ax$ pour tout $x \in A$, alors le radical premier de A coincide avec son radical (de Jacobson) noté R . Cela découle du fait que $R^2 = \{0\}$ (cf. lemme 3.1. [3]). Donc, d'après les théorèmes 1 et 2 ci-dessus la classe des algèbres de Banach semi-premières vérifiant $Ax^2 = Ax$ $(x \in A)$ coincide avec la classe des algèbres de Banach semi-premières vérifiant $x^2Ax^2 = xAx$ pour tout $x \in A$. C'est la classe des algèbres de Banach isomorphes à $\mathbb{C}^n$ ($n \in N$).

Afin d'alléger l'écriture, nous utiliserons les notations suivantes: Soit A une algèbre de Banach

1-On dira que A vérifie la condition (α) si $Ax^2 = Ax$ pour tout $x \in A$.

2--On dira que A vérifie la conditon (β) si $x^2Ax^2 = xAx$ pour tout $x \in A$.

3- On dira que A vérifie la condition (β') si pour tout $x \in A$ il existe $n = n(x)$ tel que $x^nAx^n = xAx$.

Il est evidement clair que la condition (β) implique la condition (β').

Dans ce papier, nous donnons quelques propriétés des algèbres de Banach complexes vérifiant (β') et qui ne sont pas nécessairement semi-premières.

Nous donnons ensuite un théorème de structure pour les algèbres de Banach vérifiant à la fois la condition (β) et la condition supplémentaire $xRx = \{0\}$ pour tout $x \in A$. A la fin, nous donnons un exemple d'algèbre de Banach qui vérifie (β) et qui ne vérifie pas (α).

Dans toute la suite, les algèbres considérées seront sur $\mathbb{C}$. Si A est une algèbre, R désignera son radical (de Jacobson), N l'ensemble de ses éléments nilpotents et Q celui de de ses éléments quasi-nilpotents.

2-Quelques propriétés des algèbres de Banach vérifiant la condition (β').

Voici un résultat qui regroupe certaines propriétés de ce type d'algèbres.

Proposition 2.1. Soit A une algèbre de Banach vérifiant (β'). Alors on a:
(i) $N = R = Q$
(ii) $xAx = \{0\}$ pour tout $x \in R$.
(iii) l'algèbre quotient A / R est commutative.

Preuve. Signalons tout d'abord que si $x \in A$ et si $n = n(x)$ alors on a :

$$(*) \qquad xAx = x^n A x^n = x^{2n-1} A x^{2n-1} = x^{3n-2} A x^{3n-2} = ...$$

(i) Montrons que $N \subseteq R$. Soit $x \in A$ tel que $x^p = 0$ pour un certain entier p. Soit $n = n(x)$, il existe k, entier, tel que $kn-(k-1) \geq p$. D'après (*), $xAx = \{0\}$, or $(ax)^n \in axAx = \{0\}$, d'où $r(ax) = 0$ (r désigne le rayon spectral) pour tout $a \in A$ et d'après [1,p.126] $x \in R$; ce qui montre que $N \subseteq R$. D'autre part, on a toujours $R \subseteq Q$. Considérons maintenant $x \in Q$, et soit $n = n(x)$, comme $x^{3n-2}Ax^{3n-2} = xAx$ et puisque $3n-2 \geq 4$, il existe $b \in A$ tel que $bx^4 = x^3$. Donc $b^k x^{3+k} = x^3$ pour entier $k \geq 1$. Par suite :

$$||x^3|| \leq ||b||^k \, ||x^3|| \, ||x^k||$$

ce qui donne

$$||x^3||^{1/k} \leq ||b|| \, ||x^3||^{1/k} \, ||x^k||^{1/k}$$

or $r(x) = 0$, donc nécéssairement $x^3 = 0$ sinon on aurait une contradiction. Donc $Q \subseteq N$, ce qui finit la preuve de (i).

(ii) D'après ce qui précéde, tout élément de R est nilpotent. Le résultat est alors une conséquence immédiate de (*).

(iii) Pour montrer que A / R est commutative, nous exploitons la technique utilisée dans la preuve du lemme 3.1. de [3]. Soit alors S une représentation irréductible de A sur un A-module de Banach X.D'après le théorème de densité de Jacobson[1, p.123] il existe $a \in A$ tel que $S(a)x_1 = x_2$ et $S(x_2) = 0$. Il existe également $b \in A$ tel que $S(b)x_2 = x_1$. Nous avons alors $S(aba)x_1 = x_2$. Soit maintenant c un élément quelconque de A et soit un entier $n \geq 2$, on a $S(ca^n)x_1 = 0$ et donc $S(a^n c a^n)x_1 = 0$, ce qui montre que $aba \in a^n A a^n$ pour tout $n \geq 2$, ce qui est absurde car $aba \in aAa$ et A vérifie (*) . D'où $\dim X = 1$. Donc $S(A)$ est isomorphe à $\mathbb{C}$. Il en résulte que $S(u)S(v) = S(v)S(u)$, c'est à dire que $uv-vu \in \operatorname{Ker} S$ pour tous $u,v \in A$ et pour toute représentation irréductible S de A. Donc $uv-vu \in R$.

Remarques:

1- Si A est une algèbre de Banach vérifiant (β') , alors le radical premier de A coincide avec R. Donc A est semi-première si et seulement si elle est semi-simple. Par suite si A est semi-première alors elle est commutative.

2- Les propriétés (i),(ii) et (iii) de la proposition précédente restent évidemment vraies dans toute algèbre de Banach vérifiant (β).

3- Soit A une algèbre de Banach vérifiant (β), on vérifie facilement que A / R vérifie également (β). Donc d'après le théorème 1, A / R est isomorphe à $\mathbb{C}^n$ pour un certain entier positif n.

3- Un théorème de structure pour certaines algèbres de Banach vérifiant (β).

En exploitant la technique utilisée dans [3], nous obtenons le résultat suivant :

Théorème 3.1. Soit A une algèbre de Banach vérifiant la condition (β). Supposons de plus que $xRx = \{0\}$ pour tout $x \in A$. Alors $A = B \oplus R$, où B est isomorphe à $\mathbb{C}^n$ (n étant un entier positif). De plus $xAx = \{0\}$ pour tout $x \in R$.

Preuve. Dans la proposition 2.1, on a montré que $xAx = \{0\}$ pour tout $x \in A$. Considérons l'algèbre A / R, cette algèbre vérifie bien la condition (β). Donc d'après [4], A / R est isomorphe à $\mathbb{C}^n$ pour un certain entier $n \geqslant 0$. Il existe alors une suite $F_1, \dots,$ F_n d'éléments de A / R tels que $F_i^2 = F_i$, $F_i F_j = 0$ ($i \neq j$) et $A/R = \mathbb{C}F_1 \oplus \dots \oplus \mathbb{C}F_n$. Soit $s : A \longrightarrow A/R$ la surjection canonique et soient $f_1, \dots, f_n$ éléments de A tels que $s(f_i) = F_i$, $i=1,\dots,n$.On a $s(f_i^2 - f_i) = 0$, ce qui donne $f_i^2 - f_i \in R$. Donc $f_i(f_i^2-f_i)f_i = \{0\}$. De même $f_if_i(f_i^2-f_i)f_i = 0$ et $f_if_i(f_i^2 - f_i)f_if_i = 0$. Nous obtenons alors $f_i^3 = f_i^6$. Soit $u_i = f_i^3$, u_i est idempotent, de plus $s(u_i) = F_i$, $i = 1, \dots, n$. Maintenant pour $i \neq j$, on a $s(u_iu_j) = F_iF_j = 0$, donc $u_iu_j \in R$, ce qui donne $u_iu_ju_i = u_iu_iu_ju_i = 0$. On a aussi $(u_i+u_j)u_iu_j(u_i+u_j) = 0$. En effectuant les calculs, nous obtenons $u_iu_j = 0$. Notons B le sous espace engêndré par $\{u_1,\dots,u_n\}$. B est alors isomorphe à $\mathbb{C}^n$. Soit maintenant $x \in A$, $s(x) = \mu_1 F_1 + \dots + \mu_n F_n$. Cela donne $x - \mu_1 u_1 - \dots - \mu_n u_n \in R$. Donc $A = B + R$. Montrons maintenant que $B \cap R = \{0\}$. Soit $x \in B \cap R$, x est combinaison linéaire de $u_1,\dots,u_n$. D'après la proposition 2.1.i , $xu_ix = 0$ pour tout i, ce qui montre que les coefficiants de x dans la base $\{u_1,\dots,u_n\}$ sont tous nuls. Par suite $x = 0$ et donc $A = B \oplus R$.

Remarque. La condition $xRx = \{0\}$, mentionnée dans le théorème précédent, est évidemment vérifiée si $R = \{0\}$. C'est le cas encore si $A^3 = \{0\}$.

Comme cela a été mentionné dans l'introduction, les algèbres de Banach vérifiant (β) ont été complétement caractérisées dans [3]. Il est donc naturelle de se demander s'il existe un exemple d'algèbre de Banach vérifiant (β) et qui ne vérifie pas (α). Une telle algèbre existe. En voici un exemple.

Exemple. Soient e_1, e_2, e_3 trois symboles. Posons $e_ie_je_k = 0$ pour tous $i,j,k \in \{1,2,3\}$. Considérons l'algèbre, notée A, engendrée par e_1, e_2, e_3 .C'est une algèbre de dimension finie. Munissons cette algèbre de la norme canonique. A est alors une algèbre de Banach non commutative radicale (A coincide avec son radical R)et il est facile de voir que cette algèbre vérifie la condition (β), puisque $xAx = \{0\}$ pour tout $x \in A$, mais ne véri- fie pas la condition (α) car $e_2e_1 \in Ae_1$ mais $Ae_1{}^2 = \{0\}$. On constate aussi que $A^3 = \{0\}$.

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Ecole Normale Supérieure Takaddoum
Département de Mathématiques
Avenue Oued Akreuch
B. P. 5118 RABAT
MAROC

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PALEY-WIENER THEOREMS FOR THE SCHRODINGER OPERATOR ON $\mathbb{R}$.

M.N. LAZHARI

Presented by G.F.D. Duff, F.R.S.C.

Abstract.
In this paper we define and study generalized Fourier transforms associated with some class of Schrodinger operators on $\mathbb{R}$. Next, we establish Paley-Wiener type theorem which characterize some functionnal spaces by their generalized Fourier transforms.

1. Introduction

We consider the symmetric differential operator $(L, D_0(L))$ defined by:

$$D_0(L) = \mathcal{D}(\mathbb{R}) \text{ and } Lu(x) = -\frac{d^2u}{dx^2}(x) + q(x)u(x),\ u \in \mathcal{D}(\mathbb{R}),$$

where $\mathcal{D}(\mathbb{R})$ is the space of C^∞-functions on $\mathbb{R}$, with compact support and q is a measurable function satisfying

$$\int_{-\infty}^{+\infty} (1+ \mid x \mid) \mid q(x) \mid dx < +\infty.$$

The operator (L, D_0) has a unique self-adjoint extension (L, D_L), where (see [2])

$$D_L = \{f \in L^2(\mathbb{R}) \ :\ f, f' \text{ are absolutely continuous and } \mathrm{L}(f) \in L^2(\mathbb{R})\}.$$

On the other hand, for $\mu \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} \ :\ (\mathcal{I}m(\lambda) > 0) \text{ or } (\mathcal{I}m(\lambda) = 0 \text{ and } \mathcal{R}e(\lambda) \geq 0)\}$, the differential equation $Lu = \mu^2 u$ possesses two linear independent solutions $E_\pm(.,\mu)$ satisfying $\lim\limits_{x\to\pm\infty} e^{\mp i\mu x}E_\pm(x,\mu) = 1$, which are called generalized eigenfunctions.

We associate with the spectral decomposition of the self adjoint operator (L, D_L) two generalized Fourier transforms defined by

$$\mathcal{F}_\pm(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\ E_\pm(x, \mp\mu)dx,\ \mu \in \mathbb{R},\ f \in \mathcal{D}(\mathbb{R}).$$

__Properties__: *For all μ in $\mathbb{C}_+$ and t in $\mathbb{R}$, we have*

$$E_+(t,\mu) = e^{i\mu t} + \int_t^{+\infty} \frac{\sin\mu(s-t)}{\mu} q(s)E_+(s,\mu)ds$$

and

$$E_-(t,\mu) = e^{-i\mu t} + \int_{-\infty}^t \frac{\sin\mu(s-t)}{\mu} q(s)E_-(s,\mu)ds.$$

In particular, there exist constants $C_\pm$, independent of μ, such that

$$\mid E_\pm(t,\mu) \mid \leq C_\pm e^{\mp\mathcal{I}m(\mu)t}.$$

Theorem 1.1: (See [1] and [2]): *There exists kernels* $K_\pm(t,s)$ *with support respectively in* $\{(t,s)\in\mathbb{R}^2 \ : \ t\le s\}$ *and* $\{(t,s)\in\mathbb{R}^2 \ : \ t\ge s\}$ *such that*

$$E_+(t,\mu)=e^{i\mu t}+\int_t^{+\infty}K_+(t,s)e^{i\mu s}ds$$

and

$$E_-(t,\mu)=e^{-i\mu t}+\int_{-\infty}^{t}K_-(t,s)e^{-i\mu s}ds.$$

Furthermore these kernels are respectively the unique solution of the following integral equations:

$$K_+(t,s)=\frac{1}{2}\int_{\frac{t+s}{2}}^{+\infty}q(u)du-\int_{\frac{t+s}{2}}^{+\infty}[\int_0^{\frac{s-t}{2}}q(x-y)K_+(x-y,x+y)dy]dx$$

and

$$K_-(t,s)=\frac{1}{2}\int_{-\infty}^{\frac{t+s}{2}}q(u)du-\int_{-\infty}^{\frac{t+s}{2}}[\int_{\frac{s-t}{2}}^{0}q(x-y)K_-(x-y,x+y)dy]dx.$$

Corollary 1.2:

$$K_\pm(t,s)\le\frac{1}{2}\sigma_\pm\Big(\frac{t+s}{2}\Big)exp(\epsilon_\pm(t)).$$

Corollary 1.3: *If* q *is a* C^n*-functions,* $n\in\mathbb{N}$*, (respectively* C^∞*-function) on* $\mathbb{R}$*, then the kernels* $K_\pm$ *are* C^n*-function (respectively* C^∞*-function) on* $\mathbb{R}^2$.

Corollary 1.4: *Let* a *be in* $\mathbb{R}$*. We have*

1) if the support of q *is in* $]-\infty,a]$*, then*

$$\frac{t+s}{2}\ge a\Longrightarrow K_+(t,s)=0,$$

2) if the support of q *is in* $[a,+\infty[$*, then*

$$\frac{t+s}{2}\le a\Longrightarrow K_-(t,s)=0.$$

Corollary 1.5:

1) if the support of q *is in* $]-\infty,a]$*,* $a\in\mathbb{R}$*, (respectively in* $[a,+\infty[$ *), then for all* t *in* $\mathbb{R}$*, the solution* $\mu\longrightarrow E_+(t,\mu)$ *(respectively* $\mu\longrightarrow E_-(t,\mu)$ *) can be extended to an analytic function on* $\mathbb{C}$*.*

2) if the support of q *is compact then for all* t *in* $\mathbb{R}$*, the solutions* $\mu\longrightarrow E_\pm(t,\mu)$ *are analytic on* $\mathbb{C}$*.*

The generalized Fourier transforms $\mathcal{F}_\pm$ are injective (see [3] and [4]) and are related to the ordinary Fourier transform $\mathcal{F}_0$ on $\mathbb{R}$ by the relation

$$\mathcal{F}_\pm(f) = \mathcal{F}_0 \circ (I +^t K_\pm)(f),\ f \in \mathcal{D}(\mathbb{R})$$

where

$$\mathcal{F}_0(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\mu x} dx,\ \mu \in \mathbb{R},\ f \in \mathcal{D}(\mathbb{R})$$

and ${}^tK_\pm$ are the operators defined respectively by

$${}^tK_+(f)(x) = \int_{-\infty}^{x} K_+(u,x) f(u) du,\ f \in \mathcal{D}(\mathbb{R})$$

and

$${}^tK_-(f)(x) = \int_{x}^{\infty} K_-(u,x) f(u) du,\ f \in \mathcal{D}(\mathbb{R}).$$

2. The study of the operators ${}^tK_\pm$.

In the following we state theorems which characterize some functionnal spaces on which the operators $I +^t K_\pm$ are bijective.

Let a be in $\mathbb{R}$, n in $\mathbb{N}$ and $R > 0$. We denote by
- $C^n_{R,a}$ the space of C^n-functions on $\mathbb{R}$, with support in $[-R+a, R+a]$,
- $\mathcal{D}_{R,a}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$, with support in $[-R+a, R+a]$.

Proposition 2.1: *We suppose that the support of the function q is in $]-\infty, a]$. Then we have*
i) $(I +^t K_+)(C^1_{R,a}) \subset C^1_{R,a}$.
ii) If q is C^n on $\mathbb{R}$ then $(I +^t K_+)(C^{n+1}_{R,a}) \subset C^1_{R,a}$.
iii) If q is C^∞ on $\mathbb{R}$, then $(I +^t K_+)(\mathcal{D}_{R,a}(\mathbb{R})) \subset \mathcal{D}_{R,a}(\mathbb{R})$.

Notation:
We put

$$N^R_+(s,u) = \begin{cases} K_+(u,s) & if \quad -R+a \le u \le s \le +\infty, \\ 0 & elsewhere \end{cases}$$

We consider the following integral equations:

$$h(s) = f(s) + \int_{-\infty}^{s} K_+(u,s) f(u) du, \tag{1}$$

and

$$h(s) = f(s) + \int_{-\infty}^{s} N^R_+(u,s) f(u) du, \tag{2}$$

where h is a given function and f is an unknown function.

__Proposition__ **2.2:** *We suppose that the function h is in $C^1_{R,a}$. Then*
i) The support of every solution f of (2) is in $[-R+a, R+a]$.
ii) Let f be a function with support in $[-R+a, R+a]$, then f is a solution of (1) if and only if f is a solution of (2).

__Proposition__ **2.3:** *Let h be a function in $C^1_{R,a}$. Then the integral equation (2) possesses a unique solution f in $C^1_{R,a}$.*

The proof of the following theorem is a consequence of the previous propositions.

__Theorem__ **2.4:** *We suppose that the support of the function q is in $]-\infty, a]$. Then, the operator $I +^t K_+$ is bijective*
i) from $C^1_{R,a}$ onto itself,
ii) from $C^{n+1}_{R,a}$ onto itself, if q is C^n on $\mathbb{R}$,
iii) from $\mathcal{D}_{R,a}(\mathbb{R})$ onto itself, if q is C^∞ on $\mathbb{R}$.

In the same way we prove the following theorem .

__Theorem__ **2.5:** *We suppose that the support of the function q is in $[a, +\infty[$. Then, the operator $I +^t K_-$ is bijective*
i) from $C^1_{R,a}$ onto itself,
ii) from $C^{n+1}_{R,a}$ onto itself, if q is C^n on $\mathbb{R}$,
iii) from $\mathcal{D}_{R,a}(\mathbb{R})$ onto itself, if q is C^∞ on $\mathbb{R}$.

3. __Paley-Wiener type theorems.__

For all n in $\mathbb{N}$ and $R > 0$, we denote by
-$\mathcal{H}^{n+1}_R$ the space of analytic functions ψ on $\mathbb{C}$ such that

$$\begin{cases} \forall m \in \{0, 1, \cdots, n+1\},\ \exists c_m > 0\ s.t. \\ \forall \mu \in \mathbb{C},\ |\psi(\mu)| \leq c_m(1+|\mu|)^{-m} e^{|\mathcal{I}m(\mu)|R}, \end{cases}$$

-$\mathcal{H}_R$ the space of functions in $\mathcal{H}^{n+1}_R$, for all n in $\mathbb{N}$.

__Proposition__ **3.1:** *Let n be in $\mathbb{N}$, q a C^n-function and b in $\mathbb{R}$.*

i) If the support of q is in $]-\infty, b]$, then

$$\mathcal{F}_+(\mathcal{D}_{R,b}) \subset e^{-i\mu b}\mathcal{H}^{n+1}_R$$

and

$$\mathcal{F}_+^{-1}(e^{-i\mu b}\mathcal{H}^{n+1}_R) \subset C^{n+1}_{R,b}.$$

ii) If the support of q is in $[b, +\infty[$, then

$$\mathcal{F}_-(\mathcal{D}_{R,b}) \subset e^{i\mu b}\mathcal{H}^{n+1}_R$$

and

$$\mathcal{F}_-^{-1}(e^{i\mu b}\mathcal{H}_R^{n+1}) \subset C_{R,b}^{n+1}.$$

iii) If the support of q is in $[-\mid b\mid, \mid b\mid]$, then

$$\mathcal{F}_\pm(\mathcal{D}_{R,\pm|b|}) \subset e^{\mp i\mu|b|}\mathcal{H}_R^{n+1}$$

and

$$\mathcal{F}_\pm^{-1}(e^{\mp i\mu|b|}\mathcal{H}_R^{n+1}) \subset C_{R,\pm|b|}^{n+1}.$$

Theorem 3.2: *Let q be a C^∞-function and b in $\mathbb{R}$.*

i) If the support of q is in $]-\infty, b]$, then the transform $\mathcal{F}_+$ is bijective from $\mathcal{D}_{R,b}$ onto $e^{-i\mu b}\mathcal{H}_R$.

ii) If the support of q is in $[b, +\infty[$, then the transform $\mathcal{F}_-$ is bijective from $\mathcal{D}_{R,b}$ onto $e^{+i\mu b}\mathcal{H}_R$.

iii) If the support of q is in $[-\mid b\mid, \mid b\mid]$, then the transforms $\mathcal{F}_\pm$ is bijective from $\mathcal{D}_{R,b}$ onto $e^{\mp i\mu|b|}\mathcal{H}_R$.

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Département de Mathématiques, Faculté des Sciences de Tunis
Campus Universitaire, 1060 Tunis, Tunisie

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Diagonalizability of the equations of isotachophoresis

Dimitrios Prodanos

Presented by G.F.D. Duff, F.R.S.C.

Abstract

Diagonalizability for the equations of isotachophoresis is investigated. A theorem originally by Nijenhuis and improved by Haantjes gives an algorithmic method of determining diagonalizability, provided the system is hyperbolic. The condition of diagonalizability is equivalent to the vanishing of the Haantjes tensor. It is shown that the equations of isotachophoresis are hyperbolic in a particular domain and that their Haantjes tensor vanishes identically. A particular invariant submanifold where the equations of isotachophoresis fail to be hyperbolic is also described. It is shown that there are no coordinates diagonalizing the system in a neighbourhood of this submanifold.

1 Introduction

Isotachophoresis describes the separation of an n component mixture of electrolytes by an electric field. In the diffusionless approximation, the defining mathematical model is reduced to a system of quasilinear equations [5],

$$u^i_t = \left(\frac{c_i u^i}{\bar{u}}\right)_x, \qquad i = 1, \ldots, n, \qquad \bar{u} = \sum_{j=1}^{n} u^j, \tag{1}$$

where c_i is the velocity of transfer of the i-th electrolyte and is positive for all i.

Under certain conditions, these equations are strictly hyperbolic and can be shown to possess n Riemann invariants [5]. Unfortunately, these invariants are never explicitly presented since they are the roots of a polynomial of degree $n-1$. We present an algorithmic method of determining whether or not the system possesses Riemann invariants that does not depend on solving some polynomial equation and involves no unknown quantities. Instead, the result is formulated in terms of an intrinsic geometry connected with the Nijenhuis and Haantjes tensors of an operator tensor field.

We also describe a specific submanifold where the system (1) fails to be hyperbolic and even fails to possess n Riemann invariants, a point which seems to have been missed in some of the literature [4].

We stress that this method provides no new way of actually producing the Riemann invariants for the system (1), but only determines necessary and sufficient conditions for their existence.

2 The Haantjes theorem

Riemann invariants of a non-linear, hyperbolic system of partial differential equations in one space dimension,

$$u_t^i = \sum_{j=1}^{n} A_j^i(u) u_x^j, \qquad i = 1, \ldots, n, \tag{2}$$

are coordinates $v^1(u), \ldots, v^n(u)$, in which the system (2) has the diagonal form

$$v_t^i = \lambda_i(v) v_x^i, \qquad i = 1, \ldots, n.$$

It was Riemann who first noticed that the theory of systems of the form (2) is a tensor theory [3]. Under a smooth, invertible change of coordinates, the matrix A_j^i transforms as a (1,1) tensor, and so $A(u)$ defines an operator tensor field on the Euclidean space R^n with coordinates $u^1, \ldots, u^n$.

For a general manifold M^n, let $u, v \in T_p(M^n)$ be two tangent vectors at a point $p \in M^n$, and let $\tilde{u}$ and $\tilde{v}$ be any smooth vector fields extending u and v respectively. Then the Nijenhuis tensor N [2], determined by a given operator tensor field $A : T(M^n) \to T(M^n)$, has the form

$$N(u,v) = A^2[\tilde{u},\tilde{v}] + [A\tilde{u}, A\tilde{v}] - A([A\tilde{u},\tilde{v}] + [\tilde{u}, A\tilde{v}]), \tag{3}$$

where the bracket $[\tilde{u},\tilde{v}]$ is the usual commutator of vector fields. The Haantjes tensor [1], determined from the Nijenhuis tensor, has the simple form

$$H(u,v) = A^2 N(u,v) + N(Au, Av) - A\Big(N(Au,v) + N(u,Av)\Big). \tag{4}$$

For the remainder of this paper, we choose $M^n = R^n$ and let our (1,1) tensor field A be that determined by the system (2).

Assuming the *matrix* A_j^i is diagonalizable everywhere in a neighbourhood of the point $p \in M^n$, we can ask whether or not there exist local coordinates diagonalizing the tensor field $A(u)$. This is entirely equivalent to the problem of finding Riemann invariants for the system (2), and a necessary and sufficient condition is the vanishing of the Haantjes tensor for A. Since the matrix A_j^i is always diagonalizable for a hyperbolic system, the vanishing of the Haantjes tensor remains the only condition to check. Notice that the following theorem does not require that the matrix A^i_j have *distinct* eigenvalues (in which case the system (2) is called *strictly* hyperbolic), but only that they are real and diagonalize A.

Theorem 1 *A hyperbolic system of equations* (2) *possesses n Riemann invariants if and only if the Haantjes tensor H* (4) *vanishes identically.*

The proof of this follows directly from a result by Haantjes [1], which in turn is a simpler version of an original result by Nijenhuis [2].

3 The equations of isotachophoresis

In order to apply the Haantjes theorem, we need to determine a (1,1) tensor field A. Notice that the equations of isotachophoresis (1) are already in the form of a system of conservation laws. We can rewrite system (1) in the more general form (2) with

$$A^i_j = \frac{c_i}{\bar{u}}\left(\delta^i_j - \frac{u^i}{\bar{u}}\right). \tag{5}$$

In this system of coordinates, the Nijenhuis tensor (3) for A can be calculated and found to be non-zero in general. A direct calculation also shows that the Haantjes tensor vanishes identically in this, and hence, all coordinate systems. Thus, by Theorem 1, the system (1) possesses n Riemann invariants wherever it is hyperbolic. To answer whether the system is hyperbolic, we need to consider the characteristic equation for A.

The eigenvalues λ of A (5) satisfy the characteristic equation,

$$\det(A^i_j - \lambda\delta^i_j) = \det\left(\left(\frac{c_i}{\bar{u}} - \lambda\right)\delta^i_j - \frac{c_i u^i}{\bar{u}^2}\right) = \frac{f_1\cdots f_n}{\bar{u}^n}\det\left(\frac{c_i - \bar{u}\lambda}{f_i}\delta^i_j - 1\right) = 0,$$

where $f_i(u) = c_i u^i/\bar{u}$, and we indicate our matrices by their (i,j) entries. This can be re-written as a summation,

$$\det\left(\frac{c_i - \bar{u}\lambda}{f_i}\delta^i_j - 1\right) = \left(\prod_{\alpha=1}^{n}\frac{c_\alpha - \bar{u}\lambda}{f_\alpha}\right)\left(1 - \sum_{\beta=1}^{n}\frac{f_\beta}{c_\beta - \bar{u}\lambda}\right) = 0.$$

To solve this for λ we consider the intersection points of the graph of

$$F(u,\lambda) = \sum_{\alpha=1}^{n}\frac{f_\alpha}{c_\alpha - \bar{u}\lambda}$$

with the constant function 1.

In the quadrant $u^i > 0$, the f_i are all positive and the function $F(u,\lambda)$ has simple poles at $\lambda = c_i/\bar{u}$. For each pole $\lambda_i^* = c_i/\bar{u}$, we have that $F(u,\lambda) \to +\infty$ as λ approaches λ_i^* from below and $F(u,\lambda) \to -\infty$ as λ approaches λ_i^* from above. This means that between every λ_k^* and λ_{k+1}^*, we have a root λ_k for which $F(u,\lambda_k) = 1$. Provided that $c_1 < \ldots < c_n$, this gives us $n-1$ roots. Finally, there exists an n-th root $\lambda_1 = 0$, since

$$F(u,0) = \sum_{\alpha=1}^{n} \frac{f_\alpha}{c_\alpha} = \sum_{\alpha=1}^{n} \frac{c_\alpha u^\alpha}{c_\alpha \bar{u}} = \frac{\bar{u}}{\bar{u}} = 1,$$

Thus, provided the u^i are all positive and $c_1 < \ldots < c_n$, we have n real and distinct roots, demonstrating that the system (1) is *strictly* hyperbolic.

Although the Riemann invariants for the system (1) are presented as the roots of a non-trivial polynomial equation, the invariant v^1, corresponding to the eigenvalue $\lambda_1 = 0$, can be determined directly [5]:

$$v^1 = \sum_{\alpha=1}^{n} \frac{u^\alpha}{c_\alpha}, \qquad \frac{\partial v^1}{\partial t} = 0, \qquad v^1 = v^1(x).$$

The quantity v^1 determines a family of invariant submanifolds for the system (1).

The case $v^1 = 0$ is studied by Tsarev in [4], where he claims that the system (1) possesses n Riemann invariants. He also claims that the equations of isotachophoresis possess the further property of being semi-Hamiltonian and are integrable by the generalized hodograph transform (also presented in [4]). The property of being semi-Hamiltonian is only defined for diagonal systems however, so Tsarev's claim relies on the fact that the system (1) possesses n Riemann invariants. Unfortunately, this is simply not the case on the submanifold defined by $v^1 = 0$.

If the matrix A^i_j is not diagonalizable at a point $p \in M^n$, coordinates diagonalizing $A(u)$ in a neighbourhood of p cannot possibly exist. For $v^1 = 0$, A^i_j is nowhere diagonalizable and so the system (1) fails to possess n Riemann invariants. To see this, first realize that

$$\left.\frac{\partial F(u,\lambda)}{\partial \lambda}\right|_{\lambda=0} = \left.\sum_{\alpha=1}^{n} \frac{f_\alpha \bar{u}}{(c_\alpha - \bar{u}\lambda)^2}\right|_{\lambda=0} = \sum_{\alpha=1}^{n} \frac{f_\alpha \bar{u}}{c_\alpha^2} = \sum_{\alpha=1}^{n} \frac{c_\alpha u^\alpha \bar{u}}{c_\alpha \bar{u}} = \sum_{\alpha=1}^{n} \frac{u_\alpha}{c_\alpha},$$

which vanishes identically from the fact that $v^1 = 0$. Thus, the root at $\lambda_1 = 0$ has a multiplicity of at least 2. If we solve for the eigenvector e of λ_1, we obtain the system

$$e^i = \frac{u^i}{\bar{u}} \sum_{\alpha=1}^{n} e^\alpha, \qquad i = 1, \ldots, n.$$

This system is consistent with our original equations since it reduces to the identity

$$e^i = \frac{u^i}{\bar{u}} \sum_{\alpha=1}^{n} \frac{u^\alpha}{u^i} e^i \qquad i = 1, \ldots, n,$$

which is satisfied trivially from the definition of $\bar{u}$ (1). Provided that $u^i \neq 0$ and $\bar{u} \neq 0$, we have only a single solution for this system, which proves that there exists only one independent eigenvector for $\lambda_1 = 0$. Thus, A (5) has a non-trivial Jordan block and so fails to be diagonalizable.

4 Summary

We have determined the Haantjes tensor H (4) for the equations of isotachophoresis (1) and found it to vanish identically. Hence, from theorem 1, we have that the system (1) is diagonalizable and possesses n Riemann invariants everywhere it is hyperbolic. It has been previously claimed in the literature [5] that this system is hyperbolic in a domain and this claim is verified here. It is also shown that under certain conditions, the system (1) fails to be hyperbolic and fails to possess n Riemann invariants. This is the case for the particular invariant submanifold studied in [4].

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Department of Mathematics and Statistics,
Queen's University,
Kingston, Ontario.
email: prodanos@hilda.mast.queensu.ca

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A backward stochastic differential equation with non-Lipschitz coefficients

Adrian Constantin

Presented by G.F.D. Duff, F.R.S.C.

Abstract. *We give an example of a backward stochastic differential equation with non-Lipschitz coefficients for which the pathwise uniqueness of solutions holds.*

KEY WORDS: Stochastic Differential Equation, Pathwise Uniqueness.

Let us consider the following backward stochastic differential equation:

$$x(t) + \int_t^1 f(s, x(s), y(s))ds + \int_t^1 [g(s, x(s)) + y(s)]dw(s) = X \tag{1}$$

on $0 \leq t \leq 1$, where $\{w(t) : 0 \leq t \leq 1\}$ is a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the natural filtration $\{\mathcal{F}_t : 0 \leq t \leq 1\}$ and X is a given $\mathcal{F}_1$ -measurable random variable such that $E|X|^2 < \infty$. Moreover, if M^2 is the family of real-valued processes which are $\mathcal{F}_t$-progressively measurable and square integrable on $\Omega \times [0, 1]$ with respect to $P \times \lambda$ (here λ is the Lebesgue measure on $[0, 1]$), we assume that $f : \Omega \times [0, 1] \times R^2 \to R$ is $\mathcal{P} \times \mathcal{B} \times \mathcal{B}/\mathcal{B}$-measurable and $g : \Omega \times [0, 1] \times R \to R$ is $\mathcal{P} \times \mathcal{B}/\mathcal{B}$-measurable (where $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-progressively measurable subsets of $\Omega \times [0, 1]$ and $\mathcal{B}$ is the family of Borel subsets of R) are such that $f(.,0,0) \in M^2$, $g(.,0) \in M^2$ and for each $s \in [0, 1]$, $f(s, x, y)$ and $g(s, x)$ are continuous in $(x, y) \in R^2$ and respectively in $x \in R$.

In the field of control, $y(s)$ is regarded as an adapted control and $x(s)$ as the state of the system. The reachability problem (1) consists in choosing an adapted control $y(s)$ which drives the state $x(s)$ of the system to the given target X at time $t = 1$. We are thus looking for a pair of stochastic processes $\{x(t), y(t) : 0 \leq t \leq 1\}$ with values in R^2 which is $\mathcal{F}_t$-adapted and satisfies equation (1). Such a pair is called a solution of (1).

Pardoux and Peng (1990) showed the existence and uniqueness of the solution under the condition that $f(t,x,y)$ and $g(t,x)$ are uniformly Lipschitz continuous in (x,y) and x respectively. Mao (1995) improved the result by proving the following

Theorem. *If for all $0 \le t \le 1$ and $x_1, x_2, y_1, y_2 \in R$,*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)|^2 \le k(|x_1 - x_2|^2) + c|y_1 - y_2|^2 \quad a.s.$$

$$|g(t,x_1) - g(t,x_2)|^2 \le k(|x_1 - x_2|^2) \quad a.s.$$

where $c > 0$ is a constant and k is a concave nondecreasing function from R_+ to R_+ such that $k(0) = 0$, $k(u) > 0$ for $u > 0$ and $\int_{0+}^1 \frac{du}{k(u)} = \infty$, then the backward equation (1) has a unique solution.

Although several examples of admissible functions k are given by Mao (1995), no concrete example of a backward stochastic differential equation with non-Lipschitz coefficients is considered to which the above theorem is applicable. The aim of this note is to provide such an example.

Let us define for $\epsilon > 0$ the functions $h_\epsilon : R \to R_+$ by

$$h_\epsilon(u) = \begin{cases} 0, & u = 0, \\ |u| \sqrt{ln(1 + \frac{1}{|u|})}, & 0 < |u| \le \epsilon, \\ \epsilon \sqrt{ln(1 + \frac{1}{\epsilon})}, & |u| \ge \epsilon. \end{cases}$$

Our result is the following

Proposition. *For $\epsilon > 0$ sufficiently small, equation (1) with*

$$f(t,x,y) = h_\epsilon(x) + \min\{|y|, 1\}, \quad (t,x,y) \in [0,1] \times R^2,$$

$$g(t,x) = h_\epsilon(x), \quad (t,x) \in [0,1] \times R,$$

satisfies the conditions of the Theorem with $c = 1$ and $k : R_+ \to R_+$ defined by

$$k(u) = \begin{cases} 0, & u = 0, \\ u \, ln(1 + \frac{1}{u}), & 0 < u < 1, \\ ln(2), & u \ge 1, \end{cases}$$

and has non-Lipschitz coefficients.

Proof. Since k is nondecreasing and concave with $\int_{0+}^{1} \frac{du}{k(u)} = \infty$, it is enough to show that for $\epsilon > 0$ sufficiently small we have

$$|h_\epsilon(x) - h_\epsilon(y)|^2 \leq k(|x-y|^2), \quad x, y \in R,$$

in order for the assumptions of the Theorem to be satisfied.

As one can easily verify, there is an $\epsilon \in (0,1)$ such that h_ϵ is strictly increasing and concave on $(0, \epsilon)$, so that for $0 \leq t_2 \leq t_1 \leq \epsilon$ we have

$$h_\epsilon(t_1) - h_\epsilon(t_2) = \int_{t_2}^{t_1} h'_\epsilon(s)ds = \int_0^{t_1-t_2} h'_\epsilon(s+t_2)ds \leq \int_0^{t_1-t_2} h'_\epsilon(s)ds = h_\epsilon(t_1-t_2).$$

We deduce that

$$|h_\epsilon(x) - h_\epsilon(y)| \leq h_\epsilon(|x-y|), \quad x, y \in R,$$

and since (recall that $\epsilon \in (0,1)$)

$$h_\epsilon^2(u) \leq k(u^2), \quad u \in R,$$

we see that the conditions of the Theorem are satisfied. However, since

$$\lim_{x\to 0} \frac{|f(t,x,0)|}{|x|} = \lim_{x\to 0} \frac{|g(t,x)|}{|x|} = \lim_{x\to 0} \sqrt{ln(1+\frac{1}{|x|})} = \infty, \quad t \in [0,1],$$

the coefficients are not Lipschitz continuous.□

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Courant Institute of Mathematical Sciences
251 Mercer Street, 10012 New York
e-mail: adrian@cims.nyu.edu

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A Fast Algorithm for Lacunary Wavelet Bases related to the Solution of PDEs

JOCHEN FRÖHLICH KAI SCHNEIDER

Presented by G.F.D. Duff, F.R.S.C.

Abstract - *We describe a transform for locally refined wavelet bases which employs the cardinal Lagrange function instead of the scaling function. This construction is extended to biorthogonal vaguelettes into which a differential operator is incorporated. The approach is relevant for the solution of nonlinear parabolic PDEs by an explicit or semi-implicit time scheme when the nonlinear term is evaluated in physical space.*

1. INTRODUCTION. – The background of the present work is constituted by an evolutionary PDE where the solution at each time step is developed in an adaptively selected set of wavelet basis functions. This viewpoint is similar to the one of wavelet compression in signal analysis. In contrast to the signal processing field the difficulty when solving a PDE is to avoid the computation of all wavelet coefficients of the solution up to the finest scale before eliminating the irrelevant ones. Just the amplitudes of the relevant set (which is supposed to be known from the previous time step) are to be determined. For this task the classical Mallat algorithm is not suitable as it requires the coefficients of the scaling functions on each scale which do not exhibit the sparsity of the wavelet coefficients and require a preliminary projection. However, for general nonlinear terms as encountered e.g. in [1] the transform between physical space and coefficient space seems to be unavoidable. It is one of the central difficulties for adaptive wavelet algorithms.

The present paper starts with an adaptive wavelet transform which is suitable for a lacunary basis. It employs a collocation projection on successively coarsened grids, similar to [4]. The basic subtraction strategy is applicable to arbitrary generating sets. But in the present context the orthogonality of the wavelet basis can be used as a second ingredience for the solution of differential equations to avoid the inversion of linear systems [3]. This is accomplished by defining biorthogonal vaguelettes that incorporate the differential operator. We therefore generalize the adaptive wavelet transform to the case of positive inhomogeneous elliptic operators by constructing the appropriate bases. This improves the algorithm of [1]. Further details and numerical results can be found in [2] together with the application of the method to the solution of nonlinear parabolic PDEs.

2. PERIODIC MULTIRESOLUTION. – A periodic multiresolution (MRA) of spaces $V_j \subset L^2(\mathbb{T})$ on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be constructed through periodization from $b^{\mathbb{R}}(x) \in L^2(\mathbb{R})$ with $b^{\mathbb{R}}_{ji}(x) = 2^{j/2}\, b^{\mathbb{R}}(2^j x - i)$ generating an MRA of $L^2(\mathbb{R})$ through [5]

$$b_{ji}(x) = \sum_{n\in\mathbb{Z}} b^{\mathbb{R}}_{ji}(x+n)\,, \quad x \in \mathbb{T} \quad , \qquad \text{or} \qquad \hat{b}(k) = \hat{b}^{\mathbb{R}}(k)\,, \quad k \in \mathbb{Z} \qquad (1)$$

with

$$\hat{b}(k) = \int_0^1 b(x)\, e^{-2\pi i k x}\, dx \quad , \qquad \hat{b}^{\mathbb{R}}(\omega) = \int_{-\infty}^{\infty} b^{\mathbb{R}}(x)\, e^{-2\pi i \omega x}\, dx$$

These relations permit to deduce scaling functions ϕ, wavelets ψ and required filters from the nonperiodic case. The cardinal Lagrange functions in V_j are given by

$$\hat{S}_{ji}(k) = 2^{-j}\,\hat{b}_{ji}(k) / \sum_{n\in\mathbb{Z}} \hat{b}_{ji}(k + 2^j n) \tag{2}$$

provided that the denominator is different from zero. Observe that the latter is just the discrete Fourier transform of b_{ji} sampled at the points $n/2^j$.

3. TRANSFORM FOR A LACUNARY BASIS. – Any function $f_J \in V_J$ can be developed in the corresponding wavelet basis

$$f_J(x) = \sum_j \sum_i d_{ji}\,\psi_{ji}(x) \tag{3}$$

(we skip details such as index bounds, $\psi_{-1,0} := \phi_{00}$ etc.). The term "lacunary" is used here to indicate that not the whole set of basis functions in V_J is employed in the representation (3) but just a certain subset, adapted to a given function. Mostly, a so-called cone condition is fulfilled, but this is no prerequisite for the sequel.

The classical wavelet transform (WT) consists of a first projection step onto V_J and a subsequent decomposition in terms of scaling functions and wavelets. For the solution of PDEs a collocation projection is mostly applied in the first step for diverse reasons. Then, however, it is natural to use the cardinal function which is at the origin of this projection in every decomposition step $j = J, \ldots, 0$ instead of the scaling function, i.e.

$$f_j(x) = \sum_i f_j(\tfrac{i}{2^j})\, S_{ji}(x) = \sum_i f_{j-1}(\tfrac{i}{2^{j-1}})\, S_{j-1,i}(x) + \sum_i d_{j-1,i}\,\psi_{j-1,i}(x) \tag{4}$$

Due to the orthogonality of the functions ψ_{ji} and since $\langle S_{j-1,i}\,,\psi_{j-1,n}\rangle = 0$ the wavelet amplitudes d_{ji} are computed by the filters $D_i^j = \langle S_{ji}\,,\psi_{j-1,0}\rangle$ where $\langle .,. \rangle$ is the usual scalar product. Subsequently, the contribution of the second sum in the rhs of (4) is subtracted at the even grid points to get

$$f_{j-1}(\tfrac{i}{2^{j-1}}) = f_j(\tfrac{i}{2^{j-1}}) - \sum_n d_{j-1,n}\,\psi_{j-1,n}(\tfrac{i}{2^{j-1}}) \tag{5}$$

When working with the entire set of basis functions in V_J all operations are conveniently carried out in Fourier space by FFT. However, the WT of f_J does not seem to have any advantage for a direct (i.e. non-iterative) solution of a differential equation. In that case employing the functions ϕ_{Ji}, S_{Ji}, or b_{Ji} is much simpler and leads to the same result.

The above WT has been set up to be executed in physical space for a lacunary basis set (where FFT is inapplicable), as it works with the values at grid points in physical space. The filters D^j and the functions ψ_{ji} will generally have non-compact support. But in that case they exhibit fast decay which allows truncation in space up to a given precision. The successive coarsening of the employed grids leads to an $O(n\,M)$ operation count if the resulting filters have length M and n entries are retained in (3). The price to be payed for the finite filter length is a slight error on each level which can be controlled, however. On the other hand, the evaluation of f_J which is often costly in the PDE context is not required at all points $n/2^J$ but just at a subset defined by the selected wavelet functions. The inverse transform is analogeous and again based on (4) with x replaced by $n/2^j$. Note that for even n the first sum on the rhs contains only one entry. In summary, the cardinal function can be a convenient means to accomplish the simultaneous projections with locally varying finest grid and to relate the amplitudes to values at these grid points which is required in the PDE context.

4. OPERATOR–ADAPTED DECOMPOSITION. – Consider a linear operator L of order s with constant coefficients and positive symbol $\sigma(\xi) = \sum_{m=0}^{s} a_m (2\pi i\xi)^m > 0$ (i.e. $a_0 > 0$). For the periodic case ξ is replaced by $k \in \mathbb{Z}$. The aim now is to solve the differential equation

$$Lu(x) = f(x) \quad , \quad x \in \mathbb{T} \tag{6}$$

We set $u(x) = \sum_j \sum_i d_{ji}\, \psi_{ji}(x)$ with ψ belonging to a sufficiently smooth multiresolution and restrict $j < J$ for some large $J \in \mathbb{N}$. Then the image of V_J is $V_{L;J} = span\{Lb_{Ji}\}$. In many cases the related cardinal Lagrange functions $S_{L;ji}$ can be constructed as with eq. (2) replacing $\hat{b}_{ji}(k)$ with $\sigma(k)\hat{b}_{ji}(k)$. This allows to project the rhs of (6) onto $V_{L;J}$ by collocation

$$f_{L;J}(x) = \sum_i f(\frac{i}{2^J})\, S_{L;Ji}(x) \tag{7}$$

A Petrov-Galerkin method with test functions θ_{ji} is now used to determine the amplitudes d_{ji} of the solution. Solving (6) can thereby be made equivalent to representing the rhs as

$$f_{L;J}(x) = \sum_i \sum_j d_{ji}\, L\psi_{ji}(x) = \sum_i \sum_j \langle f_{L;Ji}\,,\, \theta_{ji}\rangle\, \mu_{ji}(x) \tag{8}$$

with the biorthogonal vaguelettes $\theta_{ji} = L^{-1*}\psi_{ji}$ and $\mu_{ji} = L\psi_{ji}$ constructed from the symbol. Several properties of these functions are reported in [6] where they are used with a different transform. In particular it can be shown that even if θ_{ji} is equivalent to the convolution of ψ_{ji} with the Greens function of the operator, it decays rapidly if the wavelets have sufficient vanishing moments.

With the above tools we now extend the WT of the previous section to the operator adapted case. The central equation is

$$f_{L;j}(x) = \sum_i f_{L;j}(\frac{i}{2^j}) S_{L;ji}(x) = \sum_i f_{L;j-1}(\frac{i}{2^{j-1}})\, S_{L;j-1,i}(x) + \sum_i d_{j-1,i}\, \mu_{j-1,i}(x) \tag{9}$$

Hence, the filters $D^j_{L;i} = \langle S_{L;ji}\,,\, \theta_{j-1,0}\rangle$ have to be employed to determine the amplitudes of the solution through

$$d_{j-1,i} = \sum_n f_{L;j}(\frac{n}{2^j})\, D^j_{L;n-2i}$$

(recall that in general $S_{L;ji} \neq L\, S_{ji}$ so that this expression does not simplify). Furthermore, ψ is replaced by μ in the subtraction analogeous to (5).

5. NUMERICAL RESULTS. – In this section we present results for Meyer wavelets ($b^{\mathbb{R}} = \phi_{Mey}$). The required computations in Fourier space are straightforward and furthermore their numerical support in physical space with low precision is even slightly smaller than the one of quintic spline wavelets [1]. The formulae for the exact filters in the operator adapted spline wavelet case ($b^{\mathbb{R}} = N_m$) are more involved and reported in [2] together with the related results.

The truncation of the filters advocated in the previous section introduces an error. It does not alter the perfect reconstruction property of the decomposition–recomposition scheme but the orthogonality which is relevant for the inversion of the differential operator. In Table I we report as an example $E = \max\{\langle\psi_{j0}\,,\,\psi_{km}\rangle_Q - \delta_{j0}\delta_{km}\}$ with different truncations ($J = 10$, double precision, full index set) for the collocation transform, its inverse, and the operator adapted decomposition. In the later case E is set up with θ and μ where $L = \lambda - \partial_{xx}$ with $\lambda = 150$. We observe that an asymmetric truncation improves the result, e.g. to 5.2E-4 for $K_Q = 40, K_S = 20$.

We furthermore solved a Helmholtz equation under the conditions of [1]. The new transform results in less than half the amount of work compared to the older method which employed the scaling function in the intermediate decomposition step.

6. CONCLUSION. The proposed operator adapted wavelet transform for a lacunary basis constitutes an appropriate framework for the solution of PDEs by adaptive wavelet-type bases. Generalization to higher dimensions is immediate. Its efficiency depends on the sizes of the resulting filters. Future work will be concerned with reducing their lengths by using MRAs with compactly supported cardinal functions and biorthogonal wavelets.

K_Q, K_S	WT	WT^{-1}	operator adapted
full grid	8.7 E-14	5.5 E-14	1.5 E-12
50, 50	2.7 E-6	3.9 E-6	1.7 E-4
30, 30	1.6 E-4	9.1 E-5	1.5 E-3
20, 20	8.6 E-4	6.3 E-4	1.1 E-2

Table I: Error in orthogonality relations when quadrature and substraction occuring in the transform (evaluation of S_{jl} and ψ_{jl} for WT^{-1}) are stopped at K_Q and K_S grid points from the wavelet center, respectively.

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Jochen Fröhlich:
Konrad-Zuse-Zentrum für Informationstechnik Berlin, Heilbronner Straße 10, 10711 Berlin, Germany

Kai Schneider:
Department of Chemistry, University of Kaiserslautern, 67663 Kaiserslautern, Germany

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Mailing Addresses

O.I. Bogoyavlenskij	Department of Mathematics & Statistics Queen's University Kingston, Ontario, K7L 3N6, Canada
O.H. Cheikh	Département de mathématiques et d'informatique Institut Supérieur Scientifique B.P. 5026, Nouakchott, Mauritanie
A. Constantin	Courant Institute of Mathematical Sciences 251 Mercer Street New York, N.Y. 10012, U.S.A.
J. Fröhlich	Konrad-Zuse-Zentrum für Informationstechnik Berlin Heilbronner Strasse 10, D-10711 Berlin, Germany
L. Hammoudi	Université de Haute-Alsace F.S.T. Laboratoire de Mathématiques 4, Rue des Fréres Lumières F-68093 Mulhouse Cedex France
N. Hara	Department of Mathematics Waseda University 3-4-1 Okubo Shinjuku-ku, Tokyo 169 Japan
E. Houssamen	Ecole Normale Supérieure Takaddoum Département de Mathématiques Avenue Oued Akreuch B.P. 5118 RABAT, MAROC
M.N. Lazhari	Département de Mathématiques Faculté des Sciences de Tunis Campus Universitaire 1060 Tunis, Tunisie
G. Leduc	Department of Mathematics Carleton University Ottawa, Ontario K1S 5B6 Canada
N. Mohammad	Department of Mathematics Quaid-i-Azam University Islamabad, Pakistan
D. Prodanos	Department of Mathematics & Statistics Queen's University Kingston, Ontario, K7L 3N6, Canada
K. Schneider	Department of Chemistry University of Kaiserslautern D-67663 Kaiserslautern, Germany

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