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SUPERSINGULAR K3 SURFACES WITH ARTIN INVARIANT 10

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Abstract

We compute the Artin invariant of the minimal resolutions of supersingular weighted Delsarte K3 surfaces. Consequently, we construct K3 surfaces with Artin invariant 10. It is one of the two values for the invariant that were not realized in [10].

Let k be an algebraically closed field of characteristic p (> 0). Let X_k be a K3 surface over k. It is known that $\mathrm{NS}(X_k)$, the Néron-Severi group of X_k, has rank at most 22. Put $\rho(X_k) = \mathrm{rank}_{\mathbf{Z}}\mathrm{NS}(X_k)$. As in [10], we call X_k a supersingular K3 surface if $\rho(X_k) = 22$. On a supersingular K3 surface, M. Artin [1] proved that

$$\det \mathrm{NS}(X_k) = -p^{2\sigma_0}$$

for some integer $\sigma_0 = \sigma_0(X_k)$ satisfying $1 \leq \sigma_0 \leq 10$, where $\det \mathrm{NS}(X_k)$ is the discriminant of the intersection matrix of $\mathrm{NS}(X_k)$. The integer σ_0 may be called the Artin invariant of X_k. Let W be the ring of Witt vectors over k. Denote by $H^2_{cris}(X_k/W)$ the second crystalline cohomology of X_k. There is a Chern class map:

$$c_1 : \ \mathrm{NS}(X_k) \otimes W \longrightarrow H^2_{cris}(X_k/W).$$

Write F for the endomorphism of $H^2_{cris}(X_k/W)$ induced from the Frobenius automorphism of X_k. The image of c_1 is the largest sub-F-crystal, M, such that $F(M) \subset pM$. By the Poincaré duality, the W-length of the cokernel of c_1 is equal to σ_0 (cf. [5], [7], [10]).

In [9], Shioda showed that σ_0 takes all 10 possible values; further, in [10], he used Ekedahl's algorithm on Delsarte surfaces in $\mathbb{P}^3_k$ and gave examples of K3 surfaces for all σ_0 except for $\sigma_0 = 7, 10$. In this paper [1], we refine Shioda's method and apply it to weighted Delsarte surfaces to construct supersingular K3 surfaces with Artin invariant 10.

Let $Q = (q_0, q_1, q_2, q_3)$ be a quadruplet of positive integers such that $p \not| q_i$ $(0 \leq i \leq 3)$. Let $\mathbb{P}^3_k(Q) := \mathrm{Proj}\, k[x_0, x_1, x_2, x_3]$ be the weighted projective 3-space over k of type Q graded by the condition $\deg x_i = q_i$ for $0 \leq i \leq 3$ (cf. [2], [4]). Choose $m \in \mathbf{Z}_+$ such that $p \not| m$. Let $A = (a_{ij})$ be a 4×4 matrix of integer entries satisfying

$$\begin{cases} \text{(i)} & a_{ij} > 0 \text{ and } p \not| a_{ij} \text{ for every } (i,j) \\ \text{(ii)} & p \not| \det A \\ \text{(iii)} & \sum_{j=0}^{3} q_j a_{ij} = m \text{ for } 0 \leq i \leq 3 \\ \text{(iv)} & \text{given } j,\ a_{ij} = 0 \text{ for some } i. \end{cases}$$

[1] A more detailed account of this paper has been submitted for publication elsewhere.

We define the weighted Delsarte surface in $\mathbb{P}^3_k(Q)$ of degree m with matrix A (cf. [3], [10]) to be the surface:

$$X_A: \sum_{i=0}^{3} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} = 0 \quad \subset \mathbb{P}^3_k(Q). \tag{1}$$

Weighted Delsarte surfaces are, in general, singular surfaces; let $\widetilde{X}_A$ denote the minimal resolution of X_A. If $[0,0,0,0]$ is the only simultaneous solution to the system of 4 partial derivatives of (1), then X_A is quasi-smooth. In this case, X_A has only cyclic quotient singularities of type $A_{n,\alpha}$; further, $\widetilde{X}_A$ is K3 if and only if $m = q_0 + q_1 + q_2 + q_3$ (cf. [4]).

Every weighted Delsarte surface and hence its minimal resolution are birational to a quotient of a Fermat surface. In fact, let Y_k be the Fermat surface in $\mathbb{P}^3_k$ of degree $d := \det A$.

$$Y_k: \ y_0^d + y_1^d + y_2^d + y_3^d = 0 \quad \subset \mathbb{P}^3_k.$$

Denote by μ_d the group of d-th roots of unity in $k^\times$. Put $\Gamma = \oplus_{i=0}^3 \mu_d/(\text{diagonal elements})$. Let Γ_A be a subgroup of Γ defined by

$$\Gamma_A = \left\{\gamma = \left(\prod_{j=0}^3 \lambda_j^{a_{0j}}, \prod_{j=0}^3 \lambda_j^{a_{1j}}, \prod_{j=0}^3 \lambda_j^{a_{2j}}, \prod_{j=0}^3 \lambda_j^{a_{3j}}\right) \in \Gamma \,\middle|\, (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \Gamma\right\}.$$

Then Γ_A acts on Y_k coordinate-wise and Y_k/Γ_A is birational to X_A.

For $\gamma \in \Gamma$, let γ^* be the endomorphism of $H^2_{cris}(Y_k/W)$ induced from γ. Define

$$\mathfrak{A}(Y_k) = \left\{\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \,\middle|\, \alpha_i \in \mathbf{Z}/d\mathbf{Z},\ \alpha_i \neq 0\,(0 \le i \le 3),\ \textstyle\sum_{i=0}^3 \alpha_i = 0\right\}$$
$$V(\alpha) = \{v \in H^2_{prim}(Y_k/W) \mid \gamma^*(v) = \gamma_0^{\alpha_0}\gamma_1^{\alpha_1}\gamma_2^{\alpha_2}\gamma_3^{\alpha_3} \cdot v \ \text{ for all } \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in \Gamma\}$$

where $H^2_{prim}(Y_k/W)$ denotes the primitive part of $H^2_{cris}(Y_k/W)$ and $\mathrm{rank}_W V(\alpha) = 1$ for $\alpha \in \mathfrak{A}(Y_k) \cup \{0\}$. Then we have

$$H^2_{cris}(Y_k/W) \cong V(0) \oplus \bigoplus_{\alpha \in \mathfrak{A}(Y_k)} V(\alpha)$$

(cf. [6], [8]). The Γ_A-invariant submodule is given as follows.

Proposition. *Let X_A be the weighted Delsarte surface in $\mathbb{P}^3_k(Q)$ of degree m with matrix A. Let Y_k be the Fermat surface in $\mathbb{P}^3_k$ of degree $d = \det A$. Put*

$$\mathfrak{A}(X_A) = \left\{\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathfrak{A}(Y_k) \,\middle|\, \textstyle\sum_{i=0}^3 a_{ij}\alpha_i \equiv 0 \pmod{d} \ \textit{for}\ 0 \le j \le 3\right\}$$

Then

$$H^2_{cris}(Y_k/W)^{\Gamma_A} \cong V(0) \oplus \bigoplus_{\alpha \in \mathfrak{A}(X_A)} V(\alpha).$$

Note that $H^2_{cris}(\widetilde{X}_A/W)$ and $H^2_{cris}(Y_k/W)^{\Gamma_A}$ differ only by classes of exceptional cycles.

Assume that the minimal resolution $\widetilde{X}_A$ of X_A is K3. Then there exist unique α_0 and $\alpha_{ss} \in \mathfrak{A}(X_A)$ such that $V(\alpha_0)$ and $V(\alpha_{ss})$ are of type $(0,2)$ and $(2,0)$, respectively. Here

$V((\alpha_0,\alpha_1,\alpha_2,\alpha_3))$ is of type $(2-q,q)$ if $\alpha_0+\alpha_1+\alpha_2+\alpha_3=d(q+1)$ (cf. [8]). Given $\boldsymbol{\alpha}_{ss}=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$, we define

$$e_A=\frac{d}{\gcd(\alpha_0,\alpha_1,\alpha_2,\alpha_3)}.$$

Lemma. *Let X_A be a weighted Delsarte surface with matrix A. Assume that $\widetilde{X}_A$ is K3. Then $\widetilde{X}_A$ is supersingular if and only if $p^\mu\equiv -1 \pmod{e_A}$ for some integer $\mu\geq 1$.*

Let r be the smallest positive integer such that $F^r(V(\boldsymbol{\alpha}_0))=V(\boldsymbol{\alpha}_{ss})$. Put $I_0=\{p^i\boldsymbol{\alpha}_0 \mid 1\leq i\leq r\}$. Then the image of the map $c_1:\mathrm{NS}(\widetilde{X}_A)\otimes W\longrightarrow H^2_{cris}(\widetilde{X}_A/W)$ is

$$c_1(\mathrm{NS}(\widetilde{X}_A)\otimes W)\cong V(0)\oplus\bigoplus_{\boldsymbol{\alpha}\in I_0}pV(\boldsymbol{\alpha})\oplus\bigoplus_{\boldsymbol{\alpha}\in I\setminus I_0}V(\boldsymbol{\alpha})\oplus E$$

where E denotes the classes of exceptional cycles arising from the desingularization Y_k/Γ_A (cf. [5], [10]). Hence the W-length of the image of c_1 (and so $\sigma_0(\widetilde{X}_A)$) is equal to r.

Theorem. *Let X_A be a quasi-smooth weighted Delsarte surface in $\mathbb{P}^3_k((q_0,q_1,q_2,q_3))$ of degree m with matrix A. Write $\widetilde{X}_A$ for the minimal resolution of X_A. Assume that there exists a positive integer μ such that $p^\mu\equiv -1 \pmod{e_A}$; let μ_0 be the smallest positive integer among such μ's. Assume also that $m=q_0+q_1+q_2+q_3$. Then $\widetilde{X}_A$ is a supersingular K3 surface and the Artin invariant of $\widetilde{X}_A$ is equal to μ_0.*

Example 1. Let X_A be a weighted Delsarte surface in $\mathbb{P}^3_k((1,1,1,3))$ defined by the equation:

$$x_0^5x_1+x_1^5x_2+x_2^6+x_3^2=0$$

(here $p\neq 2,3,5$). X_A is quasi-smooth; in fact, X_A is smooth ($\widetilde{X}_A=X_A$) since $\mathbb{P}^3_k((1,1,1,3))$ has singularity only at $[0,0,0,1]$ and this point is not on X_A. As $m=q_0+q_1+q_2+q_3$, X_A is K3. We have $d=2^2\cdot 3\cdot 5^2$ and $\boldsymbol{\alpha}_{ss}=(90,48,42,150)$. Hence $e_A=2\cdot 5^2$. Therefore

$$\rho(X_A)=\begin{cases}2 & \text{if } p\equiv 1,11,21,31,41 \pmod{50}\\ 22 & \text{otherwise.}\end{cases}$$

When X_A is supersingular, we obtain

$$\sigma_0=\begin{cases}10 & \text{if } p\equiv \pm3,\pm27,\pm33,\pm37 \pmod{50}\\ 5 & \text{if } p\equiv \pm9,\pm29 \pmod{50}\\ 2 & \text{if } p\equiv \pm43 \pmod{50}\\ 1 & \text{if } p\equiv -1 \pmod{50}\end{cases}$$

Example 2. Let X_A be a weighted Delsarte surface in $\mathbb{P}^3_k((1,1,1,3))$ defined by the equation:

$$x_0^5x_1+x_1^5x_2+x_2^3x_3+x_3^2=0$$

(here $p \neq 2,3,5$). This X_A is also smooth ($\widetilde{X}_A = X_A$) and K3. We find $d = 2 \cdot 3 \cdot 5^2$ and $\alpha_{ss} = (30, 24, 42, 54)$. Hence $e_A = 5^2$. Therefore

$$\rho(X_A) = \begin{cases} 2 & \text{if } p \equiv 1, 6, 11, 16, 21 \pmod{25} \\ 22 & \text{otherwise.} \end{cases}$$

When X_A is supersingular, we obtain

$$\sigma_0 = \begin{cases} 10 & \text{if } p \equiv \pm 2, \pm 3, \pm 8, \pm 12 \pmod{25} \\ 5 & \text{if } p \equiv 4, 9, 14, 19 \pmod{25} \\ 2 & \text{if } p \equiv \pm 7 \pmod{25} \\ 1 & \text{if } p \equiv -1 \pmod{25} \end{cases}$$

Remark. We must modify our method to realize the Artin invariant 7 since there is no integer d such that the maximal order of the units in $(\mathbb{Z}/d\mathbb{Z})^\times$ is equal to 14.

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ADAPTED NUMERICAL SCHEME FOR VANISHING-LAG DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. This note presents a numerical method for solving vanishing-lag delay differential equations. For asymptotically vanishing-lag as $t \to \infty$, once the lag is sufficiently small, the solution can be continued by solving an ordinary differential equation which approximates the original delay equation. Numerical tests, by means of a Fortran program called SYSDEL, which is available from the authors, show that the theoretical results are valid in practice.

RÉSUMÉ. On présente une méthode numérique pour résoudre des équations différentielles avec un retard qui peut s'annuler. Si le retard s'annule asymptotiquement, dès qu'il est suffisamment petit, on continue la solution en résolvant une équation différentielle approchée. Les résultats numériques obtenus au moyen du programme SYSDEL en Fortran, disponible sur demande, montrent que les résultats théoriques sont valides dans la pratique.

Subject-classification: AMS(MOS): 65L06, 65L05
Keywords: Delay differential equations, vanishing-lag

1. Introduction. Delay differential equations (DDEs) play a central role in the mathematical modeling of important real-life problems. Thus there is continued interest in the numerical treatment of DDEs. A large class of DDEs can formulated as follows:

$$y'(t) = f\Big(t, y(t), y\big(\alpha(t, y(t))\big)\Big), \quad t \in [a, b], \quad \text{and} \quad y(t) = \phi(t), \quad t \in [\bar{a}, a], \tag{1.1}$$

where the vector functions y, f, ϕ and α have domains and ranges given by

$$y : [\bar{a}, b] \to \mathbb{R}^n, \quad f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad \phi : [\bar{a}, a] \to \mathbb{R}^n, \quad \alpha : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n,$$

respectively, and the number $\bar{a} = \min\limits_{\substack{a \le t \le b \\ i=1,\dots,n}} \big\{\alpha_i\big(t, y(t)\big)\big\}$ denotes the minimum of the delays. The functions ϕ and α are assumed to be sufficiently smooth. To simplify vector notation for systems, here and below, we have set

$$y\big(\alpha(t, y)\big) := \Big[y_1\big(\alpha_1(t, y)\big), \dots, y_n\big(\alpha_n(t, y)\big)\Big]^T. \tag{1.2}$$

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The above DDEs can be divided into two families, those with nonvanishing-lag and those with vanishing- or asymptotically vanishing-lag. A lag is said to be *nonvanishing* if the delay functions satisfy inequalities of the form

$$\alpha_i(t, y(t)) < t - \epsilon, \qquad \text{for some } \epsilon > 0 \text{ and all } t > a.$$

A lag is said to be *vanishing* at t_0 if $\alpha_i(t_0, y(t_0)) = t_0$ and there exists $\eta > 0$ such that

$$\alpha_i(t, y(t)) < t - \epsilon, \qquad \text{for all } t \text{ satisfying } |t - t_0| > \eta.$$

Finally, a lag is said to be *asymptotically vanishing* if

$$\alpha_i(t, y(t)) \to t, \qquad \text{as } t \to +\infty.$$

In this note, we present two methods based on an adapted ODE scheme to solve vanishing-lag and asymptotically vanishing-lag DDEs. Both methods are implemented in a Fortran program called SYSDEL, available from the authors.

2. Method 1 for solving vanishing-lag problems. The numerical scheme proposed in [3] for solving nonvanishing-lag state-dependent DDEs, can be turned into a vanishing-lag problem solver by using an extrapolation polynomial of the appropriate degree to approximate the solution when the delay time falls beyond the history queue.

2.1. Localization of the derivative jump discontinuities. For notational simplicity, we shall consider only single-lag scalar DDEs. It is known [5] that any high-order numerical scheme for solving DDEs needs to locate the derivative jump discontinuities to a specific accuracy in order to include them in the set of mesh points. These discontinuities are characterized as the set of zeros of the nonlinear switching function $g(t) = \alpha(t, y(t)) - Z$ where Z is a previous jump point. These zeros are efficiently located by the method embodied in the following theorem which is proved in [3].

Theorem 1. *Consider the DDE*

$$y'(t) = f\Big(t, y(t), y\big(\alpha(t, y(t))\big)\Big), \quad t \in [a, b], \quad \textit{and} \quad y(t) = \phi(t), \quad t \in [\bar{a}, a]. \qquad (2.1)$$

Consider also the continuous and discrete switching functions,

$$g(t) = \alpha(t, y(t)) - Y, \qquad g_h(t_i) = \alpha(t_i, y_i) - Y_h,$$

where y_i is a numerical approximation to the solution $y(t_i)$ of the DDE, and Y and Y_h are the exact and the approximate derivative jump discontinuities of $y(t)$ and y_i, respectively. Assume that the function $\alpha(t, y)$ is Lipschitzian with respect to y with Lipschitz constant

M_α and Z is a zero of $g(t)$ in (t_j, t_{j+1}). Then, by extrapolating Newton's backward interpolation polynomial $P_{g_h}(t)$ for $g_h(t_i)$, one can approximate Z by the zero Z_P of $P_{g_h}(t)$ to order $O\left(h^{\min\{p/r, p/s\}}\right)$ provided

(1) *the degree of the interpolation polynomial is at least $(p-1)$,*
(2) $|Y - Y_h| = O(h^p)$,
(3) *the global integration method is of order at least p,*
(4) *the zeros, Z_P^* and Z_P, nearest to Z, of P_g and P_{g_h} are of multiplicity r and s, respectively, where P_g is Newton's backward interpolation polynomial interpolating g at the points $\{t_{j-p+1}, \dots, t_j\}$,*
(5) *the divided difference $g[t_j, \dots, t_{j-p+1}, Z]$ is bounded.*

2.2. Adapted Runge–Kutta methods for vanishing-lag problems. When using an explicit Runge–Kutta formula to solve an n-dimensional vanishing- or asymptotically vanishing-lag problem of the form (1.1), we approximate the solution at the delay time $\alpha(t, y(t))$ by a vector-valued polynomial, $Q_q^h(\alpha(t_i, y(t_i)))$, of appropriate degree as done in [7]. The components of Q_q^h are taken to be a q-point Hermite polynomial interpolating or extrapolating the solution as the delay time falls in, or beyond, the history queue, respectively.

An *adapted* r-stage Runge–Kutta method for solving vanishing- and asymptotically vanishing-lag problems can be formulated as follows:

$$y_{n+1} = y_n + h\Psi\Big(t_n, y_n, Q_q^h(\alpha(t_n, y_n)), h\Big), \tag{2.2}$$

where the increment vector function is

$$\Psi\Big(t_n, y_n, Q_q^h(\alpha(t_n, y_n)), h\Big) = \sum_{i=1}^{r} c_i \boldsymbol{k}_i,$$

and, for $l = 1, \dots, n$, the lth component of $\boldsymbol{k}_i$ is

$$k_{il} = f_l\left(t_n + \lambda_i h, y_n + h\sum_{j=1}^{i-1} \beta_{ij} \boldsymbol{k}_j, Q_q^h\left(\alpha\left(t_n + \lambda_i h, y_n + h\sum_{j=1}^{i-1} \beta_{ij}\boldsymbol{k}_j\right)\right)\right). \tag{2.3}$$

In [3], a set of conditions has been given which ensure the convergence of the present scheme. If the degree of Q_q^h is at least p and $q \geq p$, then the local truncation error is of order $p+1$, and hence, by using Theorem 2 proved in [3], one easily concludes that the global truncation error of our method is still of order p.

3. Method 2 for solving asymptotically vanishing-lag problems. For notational simplicity, we restrict ourselves to the scalar case with a state-independent delay function $\alpha(t)$ satisfying $\alpha(t) \to t$ as $t \to +\infty$. However, the results of this section are still valid for a system of equations with asymptotically vanishing state-dependent delays.

Since $\alpha(t) \to t$ as $t \to +\infty$, then for all $\epsilon > 0$, there exists t_ϵ such that, for all $t > t_\epsilon$ we have $|\alpha(t) - t| < \epsilon$.

Hence, if we assume that the solution $y(t)$ is twice differentiable for $t \geq a$, and $|y''(t)|\,|\alpha(t)-t|^\eta \leq M$ for some $\eta \in [0,2)$ and a positive constant M, then it is reasonable that, for $t \geq t_\epsilon$, one can approximate $y(\alpha(t))$ to a required accuracy by its first degree Taylor polynomial. Consequently, the problem is no longer to solve a delay problem.

If $y(t)$ is not twice continuously differentiable, it is not advised to use Method 2; however Method 1 will give a good approximate solution.

The *modified problem* has the following form: find a solution, $\bar{y}(t)$, of

$$\bar{y}'(t) = f\big(t, \bar{y}(t), \bar{y}(t) + [\alpha(t) - t]\bar{y}'(t)\big), \quad t \geq t_\epsilon, \quad \text{and} \quad \bar{y}(t_\epsilon) = y(t_\epsilon), \tag{3.1}$$

where $y(t)$ is the numerical solution of the delay problem found by Method 1.

For $t > t_\epsilon$, the solution of the modified problem can be efficiently and simply obtained by means of a non-delay ODE solver. In the following theorem, proved in [4], it is shown that under some specific conditions, the solution of the modified problem can approximate the true solution to any order of accuracy.

Theorem 2. *Consider the delay differential equation:*

$$y'(t) = f\big(t, y(t), y(\alpha(t))\big), \quad t \geq a, \quad \textit{and} \quad y(t) = \phi(t), \quad t \in [\bar{a}, a]. \tag{3.2}$$

Assume that

(1) $\exists\, t_0 \in \mathbb{R}$ *and a positive constant* c_0 *such that* $\forall t \geq t_0$, y *and* z *in* $\mathbb{R}$, $f_{zz}(t, y, z)$ *is continuous and* $|f_{zz}(t, y, z)| \leq c_0$;

(2) $|f_y(t, y, y)| + |f_z(t, y, y)| \leq c(t)$ *for some* $c(t) \in L^1(\mathbb{R})$ *and* $\forall y \in \mathbb{R}$ *and*[1] $f_z(t, y, y) \to 0$ *as* $t \to +\infty$;

(3) *the solution* $y(t)$ *of* (3.2) *is of class* C^2;

(4) $\exists$ *a positive constant* M *and* $\eta \in [0,2)$ *such that* $|\alpha(t) - t|^{2-\eta} \in L^1(\mathbb{R})$ *and* $|y''(t)|\,|\alpha(t) - t|^\eta \leq M \quad \forall t \geq t_0$.

Then, $\forall \epsilon > 0$, $\exists\, t_\epsilon \geq t_0$ *such that the solution* $\bar{y}(t)$ *of the differential equation*

$$\bar{y}'(t) = f\big(t, \bar{y}(t), \bar{y}(t) + [\alpha(t) - t]\bar{y}'(t)\big), \quad t \geq t_\epsilon, \quad \textit{and} \quad \bar{y}(t_\epsilon) = y(t_\epsilon) + \delta, \tag{3.3}$$

satisfies $|\bar{y}(t) - y(t)| \leq K(\epsilon + \delta)$, *where* K *depends only on* $\|c\|_1$.

Remark 1. In general, it is not practically possible to find the exact solution $y(t)$ to problem (1.1); nevertheless an approximation of order $O(h^p)$ is possible. Hence, the δ in (3.3) is generally of order $O(h^p)$, where h is the step size used by the numerical integration and p is the order of the global integration method. If ϵ in the above theorem is also of order $O(h^p)$, it is easily seen that the global truncation error of the new numerical scheme is still of order p.

[1]The need of this assumption, which is missing in [4], was pointed out to the authors by the Editor of the *Math. Rep.*

4. Numerical results. The following abbreviations and notation will be used:

TOL: tolerance for the maximum norm of the error estimate,

NFE: number of function evaluations,

t_ϵ: transition time from Method 1 to Method 2,

t_f: final integration point,

MRE: maximum relative error in the components of the solution at time t_f.

4.1. Solution of vanishing-lag problems by Method 1. A Runge–Kutta formula pair of orders $(5,6)$ is used as the global integration method. The solution at the delay time is approximated by a 3-point Hermite polynomial used as an interpolant when the delay time falls in the history queue, and otherwise as an extrapolant in a neighbourhood of a vanishing-lag point. The derivative jump discontinuities are located by using the algorithm given in [3].

The step size control policy bounds the local truncation error per unit step of the Runge–Kutta formula pair. At each integration step, an estimate of the local truncation error is given by EST $= \|\hat{y}_i - y_i\|/h$, where $\hat{y}_i$ and y_i are the numerical solutions given by the 6th- and 5th-order formulas, respectively, and h is the previous step size. If EST > Tolerance, the step is reduced, otherwise it may be increased, as described in [3].

The accuracy and cost of Method 1 is illustrated by the following example.

Example 1. Consider the state dependent delay system [2], with vanishing lag at t=1,

$$2y_1'(t) = y_2(t), \qquad y_2'(t) = -y_2\left(\exp(1-y_2(t)\right)\left[y_2(t)\right]^2\left(\exp(1-y_2(t)\right), \qquad t \in [0.5, 5],$$

$$y_1(t) = \ln t, \qquad y_2(t) = \frac{1}{t}, \qquad t \in [0, 0.1].$$

The exact solution is $y_1(t) = \ln t$, $y_2(t) = 1/t$. The numerical results at $t_f = 5$ for different values of the error tolerance are listed in Table 1.

TABLE 1. Numerical results for Example 1 at $t_f = 5$.

TOL	NFE	MRE
10^{-6}	553	1.85E−06
10^{-8}	959	5.05E−09
10^{-10}	1946	2.85E−11
10^{-12}	4214	1.11E−13

4.2. Comparison of Methods 1 and 2 for asymptotically vanishing-lag problems. The numerical solution of an asymptotically vanishing-lag problem is started by Method 1. Then from a point t_ϵ on, to be fixed by the user, Method 2 solves an ODE which approximates the given delay equation.

The results of the two methods are compared in the following example.

Example 2. Consider the asymptotically vanishing-lag DDE, as $t \to \infty$,

$$2y'(t) = \frac{y\left(t - t^{-2}\right)}{(1 + t^2)\arctan\left(t - t^{-2}\right)}, \quad t \in [1.5, 50], \quad \text{and} \quad y(t) = \arctan(t), \quad t \in [0, 1.5],$$

whose exact solution, $y(t) = \arctan(t)$, is infinitely differentiable. The results obtained by Method 1 and Method 2 with $t_\epsilon = 15$, are listed in Table 2 at $t_f = 50$.

TABLE 2. Numerical results for Example 2 at $t_f = 50$.

	Method 1		Method 2, $t_\epsilon = 15$	
TOL	NFE	MRE	NFE	MRE
10^{-6}	770	1.08E−09	525	1.14E−09
10^{-8}	833	7.36E−10	553	7.75E−10
10^{-10}	1680	8.51E−12	805	2.87E−11
10^{-12}	3528	4.50E−14	1344	2.05E−11

5. Conclusion. Numerical results indicate that vanishing-lag problems can be solved accurately by Method 1. In the case of an asymptotically vanishing-lag, under appropriate smoothness conditions, the solution can be started by Method 1 up to a time t_ϵ, to be fixed by the user, and then continued by Method 2 applied to an approximate ODE whose solution is an accurate approximation to the exact solution and is obtained with a relatively small number of function evaluations. It is seen from the numerical results that Method 2 is comparable with other known methods [2,6].

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ROUND-OFF STABILITY OF ITERATIONS FOR MULTIVALUED OPERATORS

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Presented by M.A. Akcoglu, F.R.S.C.

ABSTRACT: A.M. Ostrowski's classical theorem for stability of single-valued operators is extended to multivalued operators.

KEYWORDS: Stable iteration, Banach contraction, multivalued contraction, fixed point.

Mathematics Subject Classifications(1991): 65D15, 41A25, 47H10, 54C60, 54H25.

1. INTRODUCTION. Let (X, d) be a metric space and $T: X \to X$. The concept of a fixed point iteration procedure given by $x_{n+1} = f(T, x_n)$ being T-stable or stable with respect to T has been (formally) defined by Harder and Hicks [3]. Ostrowski's first stabillity result [7] (cf. Corollary 4 below) for Banach contractions has recently been extended to various classes of (Banach type) single-valued operators by Harder-Hicks [3], Rhoades [7]-[8] and Singh et al. [10]. The purpose of this paper is to extend Ostrowski's theorem [op. cit.] to multivalued contractions.

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2. MULTIVALUED CONTRACTIONS. Consistent with [6], p.620, we will use the following notation where (X, d) is a metric space:

$CB(X) = \{ A: A$ is nonempty closed bounded subset of $X\}$,

$CL(X) = \{ A: A$ is nonempty closed subset of $X\}$.

For $A, B \in CL(X)$ and $\epsilon > 0$,

$N(\epsilon, A) = \{ x \in X : d(x, a) < \epsilon \text{ for some } a \in A \}$,

$E_{A,B} = \{ \epsilon > 0: A \subseteq N(\epsilon, B), B \subseteq N(\epsilon, A) \}$,

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \phi, \\ +\infty & \text{if } E_{A,B} = \phi. \end{cases}$$

H is called the *generalized Hausdorff metric* (resp. *Hausdorff metric*) for $CL(X)$ (resp. $CB(X)$) induced by d.

An *orbit* of a multivalued map T at a point x_0 is a sequence $\{ x_n: x_n \in Tx_{n-1}, n = 1, 2, \ldots\}$. For a single-valued operator T, this orbit is $\{ x_n: x_n = Tx_{n-1}, n = 1, 2, \cdots\}$.

The following is Nadler's (now classic) fixed point theorem for multivalued contractions (see [1],[4]-[6] and [11]).

THEOREM 1. *Let X be a complete metric space and let $T: X \to CL(X)$ be a multivalued contraction, that is,*

(1.1) $$H(Tx, Ty) \leq q\, d(x, y)$$

for all x, y in X, where $q < 1$ is a positive number.

Then:

(i) for every $x_0 \in X$, there exists an orbit $\{ x_n \}$ of T at x_0 and $p \in X$ such that $\lim_n x_n = p$;

(ii) the point p is a fixed point of T, i.e, $p \in Tp$; and

(iii) $d(x_n, p) \leq [(q^{1-\lambda})^n/(1 - q^{1-\lambda})]\, d(x_0, x_1)$, where $\lambda < 1$ is a positive number.

Indeed, Nadler [5] proved (i)-(ii) of Theorem 1 for $T: X \to CB(X)$, and the last result (iii) is essentially due to Ćirić [1]. If T is single-valued, i.e., if $T: X \to X$, then (1.1) becomes $d(Tx, Ty) \leq q\, d(x, y)$, which is the well-known *Banach contraction* condition.

The iteration procedure (i) is used very often in numerical praxis. However, in actual computation, because of rounding-off or discretization of the function, an approximate sequence $\{y_n\}$ is used in place of sequence $\{x_n\}$. Refer Harder-Hicks [3] for an excellent exegesis on this aspect.

3. STABILITY OF MULTIVALUED OPERATORS. Let X be a metric space and $T: X \to CL(X)$. For a point $x_0 \in X$, let

(*) $$x_{n+1} \in f(T, x_n)$$

denote some iteration procedure. Let the sequence $\{x_n\}$ be convergent to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and set

$$\epsilon_n = H(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, \dots.$$

If $\lim_n \epsilon_n = 0$ implies that $\lim_n y_n = p$ then the iteration process defined in (*) is said to be *T-stable* or *stable* with respect to T. Recall that this definition for a single-valued operator is due to Harder-Hicks [2]-[3], (see also [8]-[10]).

Ostrowski's stability theorem (cf. Corollary 4) says that Picard iterative procedure for (single-valued) Banach contractions is stable. Now we extend it to multivalued contractions.

THEOREM 2. *Let X be a complete metric space and $T: X \to CL(X)$ such that (1.1) holds for all $x, y \in X$. Let x_0 be an arbitrary point in X and $\{x_n\}_{n=1}^{\infty}$ an orbit for T at x_0 such that $\{x_n\}_{n=1}^{\infty}$ is convergent to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X, and set*

$$\epsilon_n = H(y_{n+1}, Ty_n), \quad n = 0, 1, 2, \dots.$$

Then

(I) $$d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j}\epsilon_j.$$

Further, if Tp is singleton then

(II) $\lim_n y_n = p$ *if and only if* $\lim_n \epsilon_n = 0$.

PROOF. Let n be a nonnegative integer. Then, since T satisfies (1.1),

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\le H(Tx_n, y_{n+1}) \le H(Tx_n, Ty_n) + H(Ty_n, y_{n+1}) \\ &\le q\, d(x_n, y_n) + \epsilon_n \le q[\, q\, d(x_{n-1}, y_{n-1}) + \epsilon_{n-1}] + \epsilon_n \\ &\le q^2 d(x_{n-1}, y_{n-1}) + q\,\epsilon_{n-1} + \epsilon_n. \end{aligned}$$

Inductively, $d(x_{n+1}, y_{n+1}) \le q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j}\epsilon_j$.

Now the relation (I) follows immediately from

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + d(x_{n+1}, y_{n+1}).$$

To prove (II), first assume $y_n \to p$ as $n \to \infty$. Note that $H(p, Tp) = 0$ since, by hypothesis, $Tp = \{p\}$. Then for any nonnegative integer n,

$$\epsilon_n = H(y_{n+1}, Ty_n) \le d(y_{n+1}, p) + H(p, Tp) + H(Tp, Ty_n)$$

$$\le d(y_{n+1}, p) + q\, d(p, y_n).$$

Therefore $\lim_n y_n = p$ implies $\lim_n \epsilon_n = 0$.

Now, suppose $\epsilon_n \to o$ as $n \to \infty$. Since $0 < q < 1$ and $x_n \to p$ as $n \to \infty$, the first two terms on the right hand side of (I) vanish in the limit. Consequently

$$\lim_n d(p, y_{n+1}) \le \lim_n \Big(\sum_{j=0}^{n} q^{n-j}\epsilon_j \Big).$$

Let A denote the lower triangular matrix with entries $a_{nj} = q^{n-j}$. Then $\lim_n a_{nj} = 0$ for each j and $\lim_n \big(\sum_{j=0}^{n} a_{nj}\big) = \lim_n \big(\frac{1-q^{n+1}}{1-q}\big) = \frac{1}{1-q}$. Therefore A is multiplicative (i.e., for any convergent sequence $\{ s_n \}$, $\lim_n A(s_n) = \frac{1}{1-q} \lim_n s_n$ (cf. [9], p. 692)). Since $\lim_n \epsilon_n = 0$, $\lim_n \big(\sum_{j=0}^{n} q^{n-j}\epsilon_j\big) = 0$, proving $\lim_n y_n = p$. This completes the proof of the theorem.

We remark that, in (II) of the above theorem, $p \in X$ is not required to be the unique fixed point of T. The related condition emphasizes that Tp contains just one point.

The following, due to an idea of [11, p. 226], is another extension of Ostrowski's stability theorem for Banach contractions.

<u>THEOREM 3</u>. *Let all the hypotheses of Theorem 2 hold, wherein the definition of* ϵ_n *is replaced by the following*

$$\epsilon_n = d(y_{n+1}, p_n), \quad p_n \in Ty_n, \quad n = 0, 1, 2, \cdots.$$

Then

(III) $$d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j}(H_j + \epsilon_j),$$

where $H_j = H(x_{j+1}, Tx_j)$. *Further, if* Tp *is singleton, then*

(IV) $\lim_n y_n = p$ *if and only if* $\lim_n \epsilon_n = 0$.

PROOF. For any nonnegative integer n,

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\le d(x_{n+1}, p_n) + d(p_n, y_{n+1}) \le H(x_{n+1}, Ty_n) + \epsilon_n \\ &\le H(x_{n+1}, Tx_n) + H(Tx_n, Ty_n) + \epsilon_n \\ &\le H_n + q\, d(x_n, y_n) + \epsilon_n \\ &\le H_n + q\, [H_{n-1} + q\, d(x_{n-1}, y_{n-1}) + \epsilon_{n-1}] + \epsilon_n \\ &\le q^2\, d(x_{n-1}, y_{n-1}) + q\, (H_{n-1} + \epsilon_{n-1}) + (H_n + \epsilon_n) . \end{aligned}$$

Inductively, $d(x_{n+1}, y_{n+1}) \le q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} (H_j + \epsilon_j)$, and the relation (III) follows as in the proof of (I).

To prove (IV), first assume $y_n \to p$ as $n \to \infty$.

Then $\epsilon_n = d(y_{n+1}, p_n) \le H(y_{n+1}, Ty_n)$.

This, as in the proof of Theorem 2, gives $\lim_n \epsilon_n = 0$.

Now assume that $\lim_n \epsilon_n = 0$. From (III),

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} t_j,$$

where $t_j = H_j + \epsilon_j$. In view of the (corresponding part of the) proof of Theorem 2, it is sufficient to show that the sequence $\{ t_j \}$ is convergent to 0. Since, by assumption, the sequence $\{ \epsilon_j \}$ is convergent to zero, it is enough to show that $\{ H_n \}$ is also convergent to 0. Since T being contraction is continuous,

$$\lim_n H_n = \lim_n H(x_{n+1}, Tx_n) = H(p, Tp) = 0.$$

This completes the proof.

REMARK 3. Relations (II) and (IV) say that the Picard sequence of iterates for multivalued contractions is stable at a fixed point p provided Tp is singleton. Further, in view of (iii) of Theorem 1, relation (I) (resp. (III)) gives an upper bound for error while estimating $d(y_n, p)$.

COROLLARY 4. **(Ostrowski's stability theorem** [7], see also [2], [4, p.101], [8], [10]). *Let (X, d) be a complete metric space and $T: X \to X$ such that T is a Banach contraction (with contraction constant q). Let p be the fixed point of T. Let x_0 be an arbitrary point in X, and put $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Let $\{ y_n \}$ be a sequence in X, and $\epsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \cdots$. Then, for $n = 0, 1, 2, \cdots$,*

$$(1)\quad d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} \epsilon_j.$$

Also
(2) $\lim_n y_n = p$ *if and only if* $\lim_n \epsilon_n = 0$.

PROOF. It is exactly derivable from Theorem 2 simply by noting that $\epsilon_n = H(y_{n+1}, Ty_n) = d(y_{n+1}, Ty_n)$ when T is single-valued. As regards its derivation from Theorem 3, one may note that $H_j = H(x_{j+1}, Tx_j) = d(x_{j+1}, x_{j+1}) = 0$ where T is single-valued, and (III) becomes (1). Thus Theorems 2-3 are appropriate extensions of this corollary.

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SPECTRAL SYNTHESIS AND REFLEXIVE OPERATORS

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Presented by P.A. Fillmore, F.R.S.C.

Abstract

We announce results which connect some aspects of basis theory with results about reflexivity of operators. These include an example of a non-reflexive compact operator which allows spectral synthesis. The selection problem for strong M-bases is shown to have a negative solution in Hilbert space. The results show that a number of statements in the literature are incorrect.

Recall that an operator A on H is said to allow *spectral synthesis* (see for example [M]) if every invariant subspace M of A is the span of root vectors corresponding to non-zero eigenvalues of A. One of the questions motivating our work was the following:

Question 1. *If K is a compact operator on Hilbert space such that all its root vectors are eigenvectors and if K allows spectral synthesis, is K reflexive?*

By an obvious application of Sarason's lemma [RR], Theorem 7.1 it follows that, to prove reflexivity, it is sufficient to prove that for any operator K satisfying the hypotheses, its inflation $K \oplus K$ on $H \oplus H$ also allows spectral synthesis. This attack on the problem was followed in [F] which claimed a positive answer to Question 1, but our results show that the answer is, in fact, negative.

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The hypotheses of this question were related to some basis-theoretical conditions by Markus [M]. A sequence $(f_n)_{n=1}^{\infty}$ of vectors of a Hilbert space H is called an *M-basis* (see e.g. [ALL]) if $\bigvee_{n=1}^{\infty} f_n = H$ and there exists a sequence $(f_n^*)_{n=1}^{\infty}$ biorthogonal to it, $\langle f_m, f_n^* \rangle = \delta_{mn}$, such that $\bigvee_{n=1}^{\infty} f_n^* = H$. An M-basis $(f_n)_{n=1}^{\infty}$ is called *strong*, if additionally $x \in \bigvee\{f_n : \langle x, f_n^* \rangle \neq 0\}$ for every vector x of H. We also need a subspace version of a strong M-basis: suppose $(N_j)_{j=1}^{\infty}$ is a sequence of non-zero subspaces of H such that $\bigvee_{j=1}^{\infty} N_j = H$. We call the sequence *separated* if for each $j \in \mathbb{N}$, we have $N_j \oplus N^j = H$, where $N^j = \bigvee\{N_k : k \neq j\}$. If P_j denotes the projection on N_j along N^j, we say that $(N_j)_{j=1}^{\infty}$ is *strongly complete* if in addition $x \in \bigvee\{P_j x : j = 1, 2, ...\}$ for each $x \in H$. It is shown in [M] that if A is compact and all the root vectors of A are eigenvectors then A allows spectral synthesis if and only if its eigenspaces form a strongly complete sequence.

The following result is proved in [ELS] Theorem 2.

Theorem. *Let $(N_j)_{j=1}^{\infty}$ be a separated sequence of subspaces H with $\bigvee N_j = H$. Then $(N_j)_{j=1}^{\infty}$ is strongly complete if and only if for every collection of strong M-bases $\{f_k^j : k = 1, 2, \cdots\}$ of N_j, a strong M-basis of H is given by the union $\{f_k^j : j = 1, 2, \cdots, k = 1, 2, \cdots\}$.*

Suppose now that K satisfies the hypotheses of our Question 1. Then, from above, its sequence $(N_j)_{j=1}^{\infty}$ of (finite-dimensional) eigenspaces is strongly complete. Choosing a basis $\{f_k^j : k = 1, 2, \cdots n_j\}$ of each subspace N_j, from our Theorem above, $\{f_k^j : j = 1, 2, \cdots n_j, k = 1, 2, \cdots\}$ is a strong M-basis of H. The eigenspaces of $K \oplus K$ are $(N_j \oplus N_j)_{j=1}^{\infty}$ and $\{(f_k^j, 0) : j = 1, \cdots n_j, k = 1, 2, \cdots\} \cup \{(0, f_k^j) : j = 1, \cdots n_j, k = 1, 2, \cdots\}$ forms a strong M-basis of $H \oplus H$.

Lemma 2 of [F] claims that, at least for finite-dimensional subspaces $(N_j)_{j=1}^{\infty}$, a stronger version of our Theorem holds, namely that the phrase "for every collection of strong M-bases" may be replaced by "for **some** collection of strong M-bases". This would be sufficient to establish that the eigenspaces of $K \oplus K$ form a strongly complete sequence and so prove the reflexivity of K. Unfortunately this claimed result is not valid. In [ELS], we give explicit examples (Examples 2 and 3) of sequences of 2-dimensional subspaces $(N_j)_{j=1}^{\infty}$ which are **not** strongly complete and a selection of a basis of each N_j such that the union of these bases forms a strong M-basis of H. Our work also provides an example of a non-reflexive operator which satisfies all the hypotheses of Question 1.

The above examples arise from our solution in [ELS] of the so-called selection problem of strong M-bases in Hilbert space. This is expressed using the concept of block sequence which we now define: given an M-basis $(f_n)_{n=1}^{\infty}$, and a sequence $0 = n_0 < n_1 < \dots$ of integers, we call a *block* sequence of $(f_n)_{n=1}^{\infty}$ any sequence $(g_k)_{k=1}^{\infty}$ of non-zero vectors with

$$g_k \in \bigvee_{i=n_{k-1}+1}^{n_k} f_i \qquad (k = 1, 2, \dots)$$

The selection problem asks whether given a strong M-basis $(f_n)_{n=1}^{\infty}$, every block sequence of it is also a strong M-basis on the space it spans. Terenzi [T] gave the first (negative) solution to this problem in a specially constructed Banach space. The counterexample in [ELS] is based on results in [KLP]. It is the first example in Hilbert space and also improves on the one in [T] by having 2-dimensional blocks as against blocks of rapidly increasing dimension.

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A note on the diophantine equation $x^4-y^4=z^p$

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Presented by J. Friedlander, F.R.S.C.

Abstract

Let p be an odd prime. In this note we prove that the equation $x^4-y^4=z^p$ has no positive integer solutions (x,y,z) satisfy $\gcd(x,y)=1$, $2|z$ and $p\nmid z$.

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of integers and positive integers respectively. Let p be a prime. More than three hundred years ago, Fermat proved that if $p=2$, then the equation

(1) $$x^4-y^4=z^p, \quad x,y,z\in\mathbb{N},$$

has no solutions (x,y,z) (see [2, page 2]). In this note we deal with the case that p is an odd prime. Let a,b be positive integers with $a>b$, and let $c=a^4-b^4$. Further let m,n be positive integers satisfy

$$m\equiv\begin{cases}1 \pmod 4, & \text{if } p\equiv 1 \pmod 4,\\ 3 \pmod 4, & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

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and $n=(mp-1)/4$. Then (1) has a trivial infinity of solutions $(x,y,z)=(ac^n, bc^n, c^m)$ with $\gcd(x,y)>1$. Recently , under the assumption that the Taniyama - Shimura conjecture is true , Darmon [1] proved that if $p\geq 11$, then (1) has no solutions (x,y,z) with $\gcd(x,y)=1$. In this note , with some elementary methods , we give an unconditional proof of the following result :

Theorem . If p is an odd prime , then (1) has no solutions (x,y,z) satisfy $\gcd(x,y)=1$, $2|z$ and $p\nmid z$.

Proof . Let (x,y,z) be a solution of (1) satisfies $\gcd(x,y)=1$, $2|z$ and $p\nmid z$. Since x and y are both odd , and since y appears only in the term y^4 in (1) , we can change y to $-y$ if necessary so as to force $x\equiv y \pmod 4$. Then we have

(2) $$x^2+y^2=2z_1^p , \quad x+y=2z_2^p , \quad x-y=2^{p-2}z_3^p ,$$

where z_1, z_2, z_3 are positive integers satisfying

(3) $$z=2z_1z_2z_3 .$$

From (2) , we get

(4) $$(x^2+y^2)/2=z_2^{2p}+2^{2p-6}z_3^{2p}=z_1^p .$$

Since $x\equiv y \pmod 4$ and p are all odd , by (2) , we have $2z_1\equiv 2z_1^p=x^2+y^2\equiv 2 \pmod 8$ and $2z_2\equiv 2z_2^p=x+y\equiv 2x \pmod 4$. This implies that $z_1\equiv 1 \pmod 4$ and $2\nmid z_2$. From (4) , we get

(5) $$z_1^p-z_2^{2p}=(z_1-z_2^2)\left(\frac{z_1^p-z_2^{2p}}{z_1-z_2^2}\right)=2^{2p-6}z_3^{2p} .$$

Since $p\nmid z$ and $(z_1^p-z_2^{2p})/(z_1-z_2^2)=z_1^{p-1}+z_1^{p-2}z_2^2+\cdots+z_2^{2(p-1)}\equiv p \pmod 4$, we see from (3) and (5) that $p\nmid z_3$, $z_1-z_2^2=2^{2p-6}z_3^{2p}$ and

(6) $$\frac{z_1^p - z_2^{2p}}{z_1 - z_2^2} = z_{32}^{2p} ,$$

where z_{31}, z_{32} are positive integers satisfying $z_{31} z_{32} = z_3$ and $2 \nmid z_{32}$.

Since $\gcd(x^2+y^2, x+y) = 2$ if $\gcd(x, y) = 1$ and $2 \nmid xy$, we see from (2) that $\gcd(z_1, z_2^2) = 1$. For any positive integer n, let

$$E(n) = \frac{z_1^n - (z_2^2)^n}{z_1 - z_2^2} .$$

Then, (6) can be written as

(7) $$E(p) = z_{32}^{2p} .$$

Since $z_1 \equiv z_2^2 \equiv 1 \pmod 4$ and $2 \nmid z_{32}$, we see from (7) that $p \equiv 1 \pmod 4$. Then there exists a square nonresidue a modulo p with $2 \nmid a$. Further, by [3, Lemma], we have

(8) $$\left(\frac{E(p)}{E(a)}\right) = \left(\frac{p}{a}\right) ,$$

where $(\cdot / \cdot)$ is Jacobi's symbol. Since $p \equiv 1 \pmod 4$, we get from (7) and (8) that

$$\left(\frac{E(p)}{E(a)}\right) = \left(\frac{(z_{32}^p)^2}{E(a)}\right) = 1 , \quad \left(\frac{E(p)}{E(a)}\right) = \left(\frac{p}{a}\right) = \left(\frac{a}{p}\right) = -1 ,$$

a contradiction. Thus, the theorem is proved.

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FINITE CONJUGACY AND NILPOTENCY IN LOOPS OF UNITS

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Presented by V. Dlab, F.R.S.C.

ABSTRACT. Let $\mathcal{U}(RL)$ denote the Moufang loop of units in an alternative loop ring RL. In this paper, we give necessary and sufficient conditions for $\mathcal{U}(RL)$ to be nilpotent or to have the finite conjugacy property when R is the ring of rational integers or a field.

1. Introduction. An *alternative* ring is a ring which satisfies the left and right alternative laws, $x(xy) = x^2y$ and $(yx)x = yx^2$. Any associative ring is alternative, but in this paper we are concerned primarily with alternative rings which are not associative. The Cayley numbers is undoubtedly the best known example of such a ring.

A *Moufang* loop is a loop in which $x(y \cdot xz) = (xy \cdot x)z$ is an identity. Any loop of units (invertible elements) contained in an alternative ring is a Moufang loop. For example, the standard basis elements of the Cayley numbers, together with their negatives, form a Moufang loop of order 16, and one which is *Hamiltonian* (all its subloops are normal). We refer the reader to [17] and [13] for information about alternative rings and Moufang loops, respectively.

Generalizing the terminology of group theory, we say that a Moufang loop L is FC, or has the *finite conjugacy property*, if, for all $\ell \in L$, the set $\{x^{-1}\ell x \mid x \in L\}$ is finite. The concept of nilpotency in loop theory, like that for groups, is a measure of the deviation of a loop from an abelian group, so it involves associators as well as commutators. If a, b and c are elements of a loop L, the *commutator* of a and b and the *associator* of a, b and c are the elements (a,b) and (a,b,c) of L, respectively, defined by

$$ab = ba(a,b) \quad \text{and} \quad (ab)c = (a \cdot bc)(a,b,c).$$

If X, Y, Z are subsets of L, we write (X,Y) for the set of all commutators (x,y), $x \in X$, $y \in Y$, (X,Y,Z) for the set of all associators (x,y,z), $x \in X$, $y \in Y$, $z \in Z$, and $\langle X\rangle$

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for the subloop of L generated by X. Let $\gamma_0(L) = L$, $\gamma_1(L) = \langle (L,L),(L,L,L)\rangle$ and, for $i \geq 1$,

$$\gamma_{i+1}(L) = \langle (L,\gamma_i(L)),(\gamma_i(L),L),(L,L,\gamma_i(L)),(L,\gamma_i(L),L),(\gamma_i(L),L,L)\rangle.$$

The subloop $\gamma_1(L)$ is also denoted L' and called the *commutator/associator subloop* of L. The loop L is *nilpotent* (Bruck uses the term "centrally nilpotent" in Chapter VI of his well-known treatise [1]) if $\gamma_n(L) = \{1\}$ for some positive integer n, which is then called the *nilpotency class* of L.

Let L be a loop and suppose that the loop ring RL is alternative, but not associative, for any commutative and associative ring R with unity. Then the loop L (which, as we have observed, is necessarily Moufang) has many special properties, including nilpotence and finite conjugacy [2]. In fact, L is nilpotent of class 2 and, for any $\ell \in L$, the set $\{x^{-1}\ell x \mid x \in L\}$ has cardinality at most 2.

The complete set $\mathcal{U}(RL)$ of units in RL is a Moufang loop containing L and it is natural to wonder if $\mathcal{U}(RL)$ inherits any of the properties of L. In this connection, and for various rings R, we have recently explored the possibility that $\mathcal{U}(RL)$ is nilpotent or has the finite conjugacy property and it is our purpose here to report our findings.

2. Integral Alternative Loop Rings.

Over the ring Z of rational integers, nilpotency and finite conjugacy in $\mathcal{U}(\mathrm{Z}L)$ are equivalent. In fact, we have established the following theorem [9].

Theorem 2.1. *Suppose* $\mathrm{Z}L$ *is an alternative, but not associative, ring. Then the following are equivalent:*

1. $\mathcal{U}(\mathrm{Z}L)$ *is FC;*
2. $\mathcal{U}(\mathrm{Z}L)$ *is nilpotent;*
3. $\mathcal{U}(\mathrm{Z}L)$ *is nilpotent of class* 2*;*
4. *The set* T *of torsion elements of* L *form an abelian group or a Moufang Hamiltonian 2-loop such that for any* $t \in T$ *and any* $x \in L$*, we have* $x^{-1}tx = t^{\pm 1}$*. Moreover, if* T *is an abelian group and* $x \in L$ *is any element which does not centralize* T*, then* $x^{-1}tx = t^{-1}$ *for all* $t \in T$*.*

A *torsion* element in a loop is an element of finite order. As a consequence of Theorem 2.1, if $\mathcal{U}(\mathbb{Z}L)$ is nilpotent or FC, then the only torsion elements of $\mathbb{Z}L$ are *trivial*; that is, of the form $\pm\ell$, $\ell \in L$ [10]. In particular, the torsion units of $\mathbb{Z}L$ form a subloop.

3. **Alternative Loop Algebras over Fields.** More recently, we have examined nilpotency and finite conjugacy in alternative loop algebras over fields and found the situation to be quite different from the case of loop rings over $\mathbb{Z}$. It is interesting to contrast our results for the cases that L is or is not a torsion loop.

Theorem 3.1. *Let L be a torsion loop and F a field such that FL is alternative. Then*

1. *$\mathcal{U}(FL)$ is an FC loop if and only if both F and L are finite* [6].
2. *$\mathcal{U}(FL)$ is nilpotent if and only if F has characteristic* 2 [7].

Thus we see, for example, that if L is a finite loop, it is the field which alone determines whether or not $\mathcal{U}(FL)$ is nilpotent or FC. We do not know if nilpotency or finite conjugacy of a Moufang loop implies that the torsion units form a subloop, but, as with loop rings over the integers, such is the case for unit loops in the alternative loop algebras of torsion loops.

Theorem 3.2. [8] *Let L be a torsion loop and F a field such that FL is alternative. Then the torsion units of FL form a subloop if and only if F has positive characteristic p and either $p = 2$ or F is algebraic over its prime field.*

Turning to the case that L is not a torsion loop, we use T to denote the set of torsion units in L and note that, for any loop considered in this paper, T is always a subloop [9, Lemma 2.1]. We consider finite conjugacy and nilpotency of the unit loop $\mathcal{U}(FL)$ separately.

Theorem 3.3. [6] *Let L be a loop with torsion subloop $T \neq L$. Let F be a field such that FL is an alternative algebra.*

1. *If the characteristic of F is 0, $\mathcal{U}(FL)$ is FC if and only if T is central in L and, if it is also infinite, then $T = \mathbb{Z}(2^\infty) \times B$ where B is a finite group, and there exists an integer k such that F does not contain roots of unity of order 2^k.*

2. *If the characteristic of F is $p > 0$ and L contains an element of order p, then $\mathcal{U}(FL)$ is FC if and only if $p = 2$ and $T = L' \times A$, where A is a finite abelian group of odd order.*
3. *If the characteristic of F is $p > 0$ and L does not contain an element of order p, then $\mathcal{U}(FL)$ is FC if and only if T is an abelian group and one of the following occurs:*
   (i) *FT is finite and, for all $t \in T$ and all $x \in L$, we have $xtx^{-1} = t^{p^r}$ for some integer $r \geq 0$, a multiple of $[F\colon \mathcal{P}]$, where $\mathcal{P}$ denotes the prime field of F.*
   (ii) *T is finite and central.*
   (iii) *T is central and of the form $\mathbb{Z}(2^\infty) \times B$ with B finite, and there exists an integer k such that F does not contain roots of unity of order 2^k.*

Theorem 3.4. [7] *Let L be a loop with torsion subloop $T \neq L$. Let F be a field such that FL is an alternative algebra.*

1. *If the characteristic of F is 0, or if $\operatorname{char} F = p > 0$ and L contains no element of order p, then $\mathcal{U}(FL)$ is nilpotent if and only if either T is central or $|F| = p = 2^\beta - 1$ for some positive integer β, T is an abelian group of exponent $2(p-1)$ and, for all $x \in L$ and all $t \in T$, we have $x^{-1}tx = t$ or t^p.*
2. *If the characteristic of F is $p > 0$ and L contains an element of order p, then $\mathcal{U}(FL)$ is nilpotent if and only if $p = 2$.*

Once again, nilpotency or finite conjugacy of $\mathcal{U}(FL)$ implies that the torsion units of $\mathcal{U}(FL)$ form a subloop, for we have

Theorem 3.5. [8] *Let L be a loop with torsion subloop $T \neq L$ and F a field such that FL is alternative. Then*

1. *If the characteristic of F is 0, then the product of torsion units in FL is a torsion unit if and only if T is an abelian group, for each $t \in T$ and $x \in L$, we have $xtx^{-1} = t^i$ for some i and, for each noncentral element $t \in T$, F contains no root of unity whose order is the order of t.*

2. *If the characteristic of F is $p > 0$, then the product of torsion units in FL is a torsion unit if and only if $p = 2$ or T is an abelian group and, if it is not central, then $\overline{\mathcal{P}}$, the algebraic closure in F of the prime field of F, is finite and, for all $x \in L$ and all $t \in T$ of order relatively prime to p, we have $xtx^{-1} = t^{p^r}$ for some positive integer r, a multiple of $[\overline{\mathcal{P}} : \mathcal{P}]$.*

4. **Conclusion.** It is appropriate to observe that the questions we have considered in this paper have all previously been settled in the case of group rings. In fact, the literature is rather extensive. For finite conjugacy of unit groups over fields or the integers, we refer the reader to [4] and [15] respectively. Nilpotence in group rings is the subject of [5] and [16]. The interested reader should also consult [14, Chapter VI]. Both finite conjugacy and nilpotency of the unit group are related to the property that the torsion units in a group ring form a subgroup; see [12], [11] and [3].

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COMPLEX REPRESENTATIONS OF GL(2, q)

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ABSTRACT. A direct construction of the irreducible matrix representations over $\mathbb{C}$ of GL(2, q) is given. The crucial ingredient is a recent result on the existence of certain primitive elements in the quadratic extension of a finite field.

1. OVERVIEW

The character table of the groups GL(2, q) is well-known, either through Green's treatment of the characters of the groups GL(n, q), or by more ad-hoc methods. The actual representations, i.e. homomorphisms from GL(2, q) to GL($\cdot, \mathbb{C}$), have been described using indirect methods by several authors (see [5, 7, 11]).

The problematic case is that of the so-called cuspidal representations of degree $q-1$. These arise by inflation of their restrictions to the Borel subgroup, but computation is complicated by the nonnaturality of the embedding of $\mathbb{Z}_{q^2-1}$ into GL(2, q). We use the following recent result to choose a suitable system of generators for GL(2, q) in terms of which the conjugacy class structure can be expressed explicitly. We use this to give a new construction of the cuspidal representations.

Theorem 1 ([1, 10]). *For any prime power q, there is a primitive element ζ of* GF(q^2) *over* GF(q) *(i.e. an element of multiplicative order q^2-1) with trace* $\operatorname{Tr}\zeta = \zeta + \zeta^q = 1$.

Apart from this result, our construction uses only elementary techniques at the level of a first graduate course in groups and representations. This allows for a shorter and perhaps more easily motivated approach than previously. The underlying argument in §5 comes from [3], where it is used, without number-theoretic complications, to construct the degree $(p \pm 1)/2$ representations of SL(2, p). The author wishes to thank John Dixon for discussions on this topic.

We take q to be a prime power greater than 2 throughout.

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2. Structure of GL(2, q)

Let $\mathbb{F} = \mathrm{GF}(q)$ and $\mathbb{E}$ be its quadratic extension. We denote by G the group $\mathrm{GL}(2,\mathbb{F}) = \mathrm{GL}(2,q)$ of order $(q-1)^2q(q+1)$. For any generator ϵ of the multiplicative group $\mathbb{F}^*$ (we choose one below), G is generated by the elements

$$\begin{array}{cccc} & z & x & t \\ & \begin{pmatrix} 1 & \\ & \epsilon \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} & \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\ \text{order} & q-1 & q & 4 \end{array}$$

Let $y = t^{-1}zt = \begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}$ and $w = yz = \begin{pmatrix} \epsilon & \\ & \epsilon \end{pmatrix}$. The elements x, y, and z generate the Borel (triangular) subgroup T of index $q+1$.

The conjugacy classes of G are classified by the way in which the minimal polynomial of their elements factors over $\mathbb{E}$, as follows:

(1) The minimal polynomial is linear. This gives $q-1$ classes denoted by $[\epsilon^c]$, each consisting of the single element w^c $(0 \le c < q-1)$.
(2) The minimal polynomial is a square. This gives $q-1$ classes $[\epsilon^c]^2$ with representatives w^cx. The stabilizer of w^cx under conjugation is $H = \langle w, x\rangle$ of order $q(q-1)$, so each class has size q^2-1.
(3) The minimal polynomial has two distinct roots in $\mathbb{F}$. This gives $(q-1)(q-2)/2$ classes $[\epsilon^a, \epsilon^b]$ $(0 \le a < b < q-1)$ with representatives y^az^b (note that y^az^b is conjugate to y^bz^a in G). The stabilizer subgroup is $\langle y, z\rangle$ so each class has size $q(q+1)$.
(4) The minimal polynomial has roots ξ and ξ^q for some $\xi \in \mathbb{E} \setminus \mathbb{F}$. The corresponding classes $[\xi, \xi^q]$ arise from some fixed embedding $\mathbb{E}^*$ in G. The stabilizer subgroup is $\mathbb{E}^*$, of order q^2-1, so each class has size $q(q-1)$.

The characteristic polynomial of tz^rx^s is $X^2 - sX + \epsilon^r$. By Theorem 1 we can choose a primitive element ζ in $\mathbb{E}$ with trace 1, and then define ϵ to be the norm ζ^{q+1} of ζ. With this choice of ϵ, tzx is an element of order q^2-1 and $\zeta \mapsto tzx$ fixes an embedding of $\mathbb{E}^*$ in G, completely determining the conjugacy classes in (4).

Define $\mathcal{K}$ to consist of the elements of $\mathbb{Z}/(q^2-1)\mathbb{Z}$ of the form $\alpha q + \beta$ for $0 \le \alpha < \beta < q$. Then $\mathcal{K}$ has $q(q-1)/2$ elements and $\mathbb{Z}/(q^2-1)\mathbb{Z}$ can be partitioned into three subsets: $\mathcal{K}$, $q\cdot\mathcal{K}$, and $\mathcal{R} = \{m(q+1) \mid m = 0, \ldots, q-2\}$. The conjugacy classes in (4) can be indexed as $[\zeta^k, \zeta^{kq}]$ for $k \in \mathcal{K}$.

Finally, it is easy to see that the conjugacy classes of the Borel subgroup T are derived from those of G as follows. The singleton classes $[\epsilon^c]$ remain unchanged. The classes $[\epsilon^c]^2$ also remain, but now have size $q-1$. The classes $[\zeta^k, \zeta^{qk}]$ are not present. The elements y^az^b and y^bz^a are no longer conjugate, so there are distinct classes $[\epsilon^a, \epsilon^b]$ and $[\epsilon^b, \epsilon^a]$, each of size q. In particular, for c fixed, the elements w^cx^d are conjugate in T for $d \not\equiv 0 \pmod q$. Similarly, for fixed $a \not\equiv b \pmod{q-1}$, the elements $y^az^bx^d$ are conjugate in T for all d.

3. Character Table of GL(2, q)

The character table of G is as follows. Here ω is a primitive (q^2-1)th root of 1, and $\Omega = \omega^{q+1}$. Both superscripts and subscripts indicate exponentiation (so

$\Omega^a_C = \Omega^{aC}$); the lower position is used to distinguish between representations of the same type.

	$[\epsilon^c]$	$[\epsilon^c]^2$	$[\epsilon^a, \epsilon^b]$	$[\zeta^k, \zeta^{qk}]$
ϕ_C	Ω^{2c}_C	Ω^{2c}_C	$\Omega^a_C \Omega^b_C$	Ω^k_C
σ_C	$q\Omega^{2c}_C$	0	$\Omega^a_C \Omega^b_C$	$-\Omega^k_C$
$\psi_{A,B}$	$(q+1)\Omega^c_A \Omega^c_B$	$\Omega^c_A \Omega^c_B$	$\Omega^a_A \Omega^b_B + \Omega^b_A \Omega^a_B$	0
ρ_K	$(q-1)\Omega^c_K$	$-\Omega^c_K$	0	$-(\omega^k_K + \omega^{qk}_K)$

The degree 1 representations ϕ_C arise from the characters of $G/G' \cong \mathbb{F}^*$ and are given by $\varphi_C(g) = \chi_C(\det g)$. The representation σ_0 is given by $\phi_0 \oplus \sigma_0 = \pi$, where π is the permutation representation of G acting on $\mathbf{P}^1(\mathbb{F})$ in the standard manner. The other σ_C are given by tensoring with the ϕ_C. The representations $\psi_{A,B}$ are induced from the degree 1 representations of T given by $\tilde{\psi}_{A,B}\begin{pmatrix} \alpha & * \\ & \beta \end{pmatrix} = \chi_A(\alpha)\chi_B(\beta)$ for distinct characters χ_A and χ_B of $\mathbb{F}^*$.

The characters of the remaining so-called *cuspidal* representations ρ_K can be obtained by judicious fiddling with the character table (see [4, Section 5.2]). Alternatively, the character tables of all the groups $\mathrm{GL}(n, q)$ can be determined by a method of J. A. Green (see [6] or [9, Chapter IV]). Here again the parabolic subgroups (the Borel subgroup in our case) play an important role, and the cuspidal characters arise from a change of basis for a system of symmetric polynomials. Green's approach fixes a one-to-one correspondence between irreducible characters and conjugacy classes, shown in the table above by equating corresponding lowercase and uppercase variables. Since the representations ρ_K correspond to the classes $[\zeta^k, \zeta^{qk}]$, it is not surprising that giving the representations ρ_K explicitly is as sensitive a problem as fixing the conjugacy class structure of G precisely in terms of a fixed system of generators.

4. Conjugacy classes of $tz^r x^s$

We construct a look-up table specifying the conjugacy classes of the elements $tz^r x^s$. Define the sequence $\{T_k\}$ for $k \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ by the equations

(1) $T_0 = T_1 = 1$.
(2) $T_{qk} = T_k$.
(3) $T_k = 1 - \sum_{1 \le m \le k/2} \binom{k}{m} \epsilon^m T_{k-2m} \pmod q$.

Proposition 2. *Let r vary from 0 to $q-2$ and s vary from 0 to $q-1$. There is exactly one element $tz^r x^s$ in each conjugacy class of G except for the classes $[\epsilon^c]$, which contain no such elements.*

In particular, $tx^2 \in [1,1]$; $tz^r x^s \in [\zeta^k, \zeta^{qk}]$ if $r \equiv k \pmod{q-1}$ and $s \equiv T_k \pmod q$ for $k \in \mathcal{K}$; and $tz^r x^s \in [\epsilon^{k/(q+1)}]^2$ if $r \equiv k \pmod{q-1}$ and $s \equiv T_k \pmod q$ for $k \in \mathcal{R}, k \neq 0$. Otherwise, $tx^r x^s$ has distinct eigenvalues in $\mathbb{F}$.

Proof. We first observe $tz^r x^s$ is never diagonal, and so cannot be in $[\epsilon^c]$.

Now, let $\xi \in \mathbb{E}$ have norm $N = \xi^{q+1}$. The norm of ξ^k is N^k. Expanding $(\xi + \xi^q)^k$ by the binomial theorem and grouping the terms for m and $m-k$, we see that the trace $T_k = \xi^k + \xi^{qk}$ of ξ^k for $k > 0$ satisfies

$$T_1^k = \sum_{0 \le m \le k/2} \binom{k}{m} N^m T_{k-2m} \pmod q.$$

We set $T_0 = 1$, so the case $m = k/2$ works out.

Setting $\xi = \zeta$, our trace 1 primitive element, we obtain equation (3). The field structure gives (2). This shows that the conjugacy classes of $tz^r x^s$ are as given. The special treatment of tx^2 is necessary since $T_0 \neq \operatorname{Tr}\zeta^0 = 2$.

Since $tz^{ab}x^{\epsilon^a+\epsilon^b}$ belongs to $[\epsilon^a, \epsilon^b]$, there is an element $tz^r x^s$ in each nondiagonal class. Uniqueness follows by counting the number of elements $tz^r x^s$ and the number of classes. □

5. Constructing the degree $q-1$ representations

Fix $K \in \mathcal{K}$. Calculation of the inner product shows that the restriction $\operatorname{Res}_T \rho_K$ is irreducible, and in fact, $\operatorname{Res}_T \rho_K = \operatorname{Ind}_H^T L_K$, where $H = \langle w, x\rangle$ and L_K is the degree 1 representation of H given by

$$L_K(w) = \Omega_K, \text{ and } L_K(x) = \eta,\ \eta \text{ a primitive } q\text{th root of } 1.$$

Indeed, T is metabelian so any irreducible representation must be monomial.

Doing the induction using the basis $z^0, z^1, \ldots, z^{q-2}$ for $\mathbb{C}T$ over $\mathbb{C}H$, we obtain

$$\begin{aligned} \rho_K(w) &= \Omega_K I \\ \rho_K(x) &= \operatorname{Diag}(\eta, \eta^{\epsilon}, \eta^{\epsilon^2}, \ldots, \eta^{\epsilon^{q-2}}) \\ \rho_K(z) &= \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}. \end{aligned} \tag{1}$$

Now, the relation $ztz = wt$ is preserved by ρ_K. This implies that $\rho_K(t)$ has the form

$$\rho_K(t) = \begin{pmatrix} \alpha_0 & \Omega_K\alpha_1 & \Omega_K^2\alpha_2 & \cdots & \Omega_K^{q-2}\alpha_{q-2} \\ \alpha_1 & \Omega_K\alpha_2 & \Omega_K^2\alpha_3 & \cdots & \Omega_K^{q-2}\alpha_0 \\ \alpha_2 & \Omega_K\alpha_3 & \Omega_K^2\alpha_4 & \cdots & \Omega_K^{q-2}\alpha_1 \\ & & \vdots & & \\ \alpha_{q-2} & \Omega_K\alpha_0 & \Omega_K^2\alpha_1 & \cdots & \Omega_K^{q-2}\alpha_{q-3} \end{pmatrix} \tag{2}$$

Expanding from (1) and (2) now forces

$$\chi_{\rho_K}(tz^r x^s) = \operatorname{Tr}\rho_K(tz^r x^s) = \sum_{n=0}^{q-2} \alpha_{r+2n}\Omega_K^{r+n}\eta^{s\epsilon^n} \tag{3}$$

Here the indices on α are taken $\pmod{q-1}$.

We sum (3) over s with weighting η^{-s}. The terms on the right involve the sum $\sum_{s=0}^{q-1}\eta^{s(\epsilon^n-1)}$, which sums to q if $n=0$ and vanishes otherwise. Hence the α_r are given by

$$\alpha_r q\Omega_K^r = \sum_{s=0}^{q-1}\eta^{-s}\chi_{\rho_K}(tz^r x^s) \tag{4}$$

Since the character values on the right can be identified using Proposition 2, this fully determines ρ_K.

6. More on Theorem 1

The known proofs of Theorem 1 ([1, 10]) involve rather delicate arguments using character sums to show the result holds except possibly for a finite list of q's. These remaining cases (roughly 200 in [10]) are checked by computer.

For our purposes, we require a method to find a specific $\epsilon \in \mathbf{F}$ such that $X^2-X+\epsilon$ is the minimal polynomial of a primitive element of $\mathbf{E}$. This can be done using the following result.

Theorem 3 (Alanen-Knuth, [8, Theorem 3.18]**).** *A monic quadratic $f \in \mathbf{F}[X]$ with constant term ϵ is the minimal polynomial of a primitive element of $\mathbf{E}$ iff ϵ is a generator of $\mathbf{F}^\bullet$ and the least integer m for which X^m is congruent to an element of $\mathbf{F}$ (mod $f(X)$) is $m=q+1$. In this case, X^m is congruent to ϵ.* □

This procedure for finding a suitable ϵ, together with the calculation of the T_k, seems particularly suited to computer methods.

Theorem 1 has been generalized in [2] to show that for any $n \geq 2$, there exists a primitive element of GF(q^n) over GF(q) of arbitrary trace T, except for the cases $T=0, n=2$ and $T=0, n=3, q=4$.

7. Example, $q = 3$

We conclude by computing ρ_K for the simplest case, that of $q=3$. Let $\eta = \exp 2\pi i/3$ and $\omega = \exp 2\pi i/8$, so $\Omega_K = (-1)^K$. The only generator of $\mathbf{F}^\bullet$ is $\epsilon = 2$. We choose $\mathcal{K} = \{1,2,5\}$. For $K \in \mathcal{K}$, we see from (1), (2), and (4) that

$$\rho_K(x) = \begin{pmatrix}\omega & \\ & \omega^2\end{pmatrix},\ \rho_K(z) = \begin{pmatrix} & 1\\ 1 & \end{pmatrix},\ \text{and } \rho_K(t) = \begin{pmatrix}\alpha_0 & (-1)^K\alpha_1\\ \alpha_1 & (-1)^K\alpha_0\end{pmatrix}.$$

where

$$3\alpha_0 = \chi_{\rho_K}(t) + \eta^2\chi_{\rho_K}(tx) + \eta^1\chi_{\rho_K}(tx^2), \text{ and}$$
$$3(-1)^K\alpha_1 = \chi_{\rho_K}(tz) + \eta^2\chi_{\rho_K}(tzx) + \eta^1\chi_{\rho_K}(tzx^2).$$

From Proposition 2, we see that $T_0 = T_1 = T_3 = 1$, $T_2 = T_6 = 0$, $T_4 = 1$, and $T_5 (= T_{15}) = T_7 = 2$, and so $tx^2 \in [1,1]$, $tzx \in [\zeta,\zeta^3]$, $t \in [\zeta^2,\zeta^6]$, $tx \in [2,2]$, $tzx^2 \in [\zeta^5,\zeta^7]$, and tz has distinct eigenvalues in $\mathbf{F}$. Hence

$$3\alpha_0 = -(\omega_K^2+\omega_K^6) - \eta^2(-1)^K - \eta, \text{ and}$$
$$3(-1)^K\alpha_1 = -\eta^2(\omega_K+\omega_K^3) - \eta(\omega_K^5+\omega_K^7).$$

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Free Bicompletion of Enriched Categories

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This is the second of a series of comptes rendus on free bicomplete categories. Here we extend the results of the first to enriched categories. In the third we shall introduce a game theoretic semantic for bicomplete categories as in [Blass].

§ 1. Bicomplete $\mathcal{V}$-categories

For the basic concepts of enriched category theory see [Kelly]. To fix the notation and terminology we shall review briefly a few concepts. For any monoidal category $\mathcal{V}$ (sometime called a *tensor category*), we shall denote $\mathcal{V}^r$ the monoidal category obtained by reversing the tensor product of $\mathcal{V}$: $A \otimes^r B = B \otimes A$. Recall that $\mathcal{V}$ is *closed* if for every $A \in \mathcal{V}$ the functors $X \mapsto A \otimes X$ and $X \mapsto X \otimes A$ have right adjoints, denoted respectively by $X \mapsto A\backslash X$ and $X \mapsto X/A$. A category $\mathcal{C}$ *enriched over* $\mathcal{V}$, also called a *$\mathcal{V}$-category*, is defined by a map $[-,-] : Ob\mathcal{C} \times Ob\mathcal{C} \to Ob\mathcal{V}$ together with associative composition laws

$$[A,B] \otimes [B,C] \to [A,C]$$

and units $1_A : I \to [A,A]$. The map $[-,-]$ is sometime denoted by $\mathcal{C}[-,-]$. Observe that a $\mathcal{V}^r$-category is defined by composition laws

$$[B,C] \otimes [A,B] \to [A,C]$$

An *arrow* $A \to B$ of $\mathcal{C}$ is an arrow $I \to \mathcal{C}[A,B]$ in $\mathcal{V}$. The category $\mathcal{V}$ is itself a $\mathcal{V}$-category with $\mathcal{V}[A,B] = A\backslash B$, and a $\mathcal{V}^r$-category with $\mathcal{V}^r[A,B] = B/A$. The *opposite* $\mathcal{C}^{op}$ of a $\mathcal{V}$-category is a $\mathcal{V}^r$-category with $\mathcal{C}^{op}[A^o, B^o] = \mathcal{C}[B,A]$ where the map $A \mapsto A^o$ is a formal bijection $\mathcal{C} \simeq \mathcal{C}^{op}$. In the absence of indication to the contrary, we shall suppose that all functors and natural transformations are $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. A functor F is an *embedding* if the arrows $\mathcal{C}[A,B] \to [FA,FB]$ are invertible.

In this paper $\mathcal{V}$ denotes an arbitrary but fixed closed monoidal category that admits small limits and colimits. When $\mathcal{C}$ is small the category $[\mathcal{C}^{op}, \mathcal{V}]$ of $\mathcal{V}^r$-functors $\mathcal{C}^{op} \to \mathcal{V}$ is enriched over $\mathcal{V}$.

We shall denote by $X \otimes A$ the *tensor* and AX the *cotensor* of an object $X \in \mathcal{C}$ with an object $A \in \mathcal{V}$, when they exist. A $\mathcal{V}$-category $\mathcal{C}$ is *cocomplete* (resp. complete) if it admits tensors and cotensors, and limits and colimits of small diagrams.

We shall use upper and lower integrals. Recall that a *$\mathcal{V}$-graph* is a set S with a map $[-,-] : S \times S \to Ob\mathcal{V}$. A *weight* on S is a map $w : S \to Ob\mathcal{V}$ together with a family of arrows $[i,j] \otimes w(j) \to w(i)$. A *diagram* $D : S \to \mathcal{C}$ is a map $D : S \to Ob\mathcal{C}$ together with a family of arrows $[i,j] \to [D(i), D(j)]$. In a cocomplete category the *lower integral*

$$\int_{i\in S} D(i)\otimes w(i) = D\otimes w$$

can be defined as the colimit of the diagram of canonical maps

$$D(i)\otimes w(i) \longleftarrow D(i)\otimes [i,j]\otimes w(j) \longrightarrow D(j)\otimes w(j),$$

where i and j run through S. A functor $F : \mathcal{C} \to \mathcal{D}$ between cocomplete categories is *cocontinuous* iff the canonical arrow $F(D)\otimes w \to F(D\otimes w)$ is invertible for any (small) weighted diagram (D, w).

Dually, a *co-weight* on a $\mathcal{V}$-graph S is a diagram $w : S \to \mathcal{V}$. The *upper integral*

$$\int^{i\in S} w(i)D(i) = wD$$

is the limit of the diagram of canonical maps

$$w(i)D(i) \longrightarrow w(i)\otimes [i,j]D(j) \longleftarrow w(j)D(j)$$

where i and j run through S. A functor F is *continuous* iff the canonical arrow $F(wD) \to wF(D)$ is invertible for any (small) coweighted diagram (D, w).

Definition 1: An object $A \in \mathcal{C}$ of a cocomplete (resp. complete) $\mathcal{V}$-category is *σ-atomic* (resp. *π-atomic*) if the functor $[A, -] : \mathcal{C} \to \mathcal{V}$ (resp. $[-, A] : \mathcal{C}^{op} \to \mathcal{V}$) is cocontinuous. An object is *atomic* if it is both $\sigma-$ and $\pi-$atomic.

A $\mathcal{V}$-category is *bicomplete* if it is complete and cocomplete. A functor is *bicontinuous* if it is continuous and cocontinuous. The concept of *free bicompletion* $i : \mathcal{C} \to \Lambda\mathcal{C}$ of a $\mathcal{V}$-category $\mathcal{C}$ is defined as in [J].

The equivalence $(\Lambda\mathcal{C})^{op} \simeq \Lambda(\mathcal{C}^{op})$ can be used to identify these two categories. We shall write $A^o \in \Lambda\mathcal{C}^{op}$ for $A \in \Lambda\mathcal{C}$.

Definition 2: A bicomplete $\mathcal{V}$-category $\mathcal{C}$ is *soft* if the following square of canonical arrows is a pushout

$$\begin{array}{ccc}
\int_{(i,j)\in I\times J} v(i)\otimes [A(i), B(j)]\otimes w(j) & \longrightarrow & \int_{j\in J} [X, B(j)]\otimes w(j) \\
\downarrow & & \downarrow \\
\int_{i\in I} v(i)\otimes [A(i), Y] & \longrightarrow & [X, Y]
\end{array}$$

for any upper and lower integrals $X = \int^{i\in I} v(i)A(i)$ and $Y = \int_{j\in J} B(j)\otimes w(j)$.

Theorem 1. *The free bicompletion $i : \mathcal{C} \to \Lambda\mathcal{C}$ of a $\mathcal{V}$-category $\mathcal{C}$ has the following properties:*

i) the category $\Lambda\mathcal{C}$ is soft ;

ii) for every $A \in \mathcal{C}$ the object $iA \in \Lambda\mathcal{C}$ is atomic;

iii) the functor i is an embedding;

iv) the category $\Lambda\mathcal{C}$ is the biclosure of $i(\mathcal{C}) \subset \Lambda\mathcal{C}$.

Moreover, these properties characterize the pair $(i, \Lambda\mathcal{C})$ up to equivalence of categories.

The *free completion* $i : \mathcal{C} \to \Pi\mathcal{C}$ and the *free cocompletion* $i : \mathcal{C} \to \Sigma\mathcal{C}$ of a $\mathcal{V}$-category are defined similarly to the free bicompletion. It is well known that when $\mathcal{C}$ is small we have $\Sigma\mathcal{C} = [\mathcal{C}^{op}, \mathcal{V}]$ with i the Yoneda functor. The so-called *Cauchy completion* $\mathcal{C}^c$ (see [Lawvere]) is the subcategory of σ-atoms of $\Sigma\mathcal{C}$. It is also equivalent to the subcategory of π-atoms of $\Pi\mathcal{C}$ (an object of $\mathcal{C}^c$ being often described by a pair of functors).

Proposition 1. *The canonical functors $\Sigma\mathcal{C} \to \Lambda\mathcal{C}$ and $\Pi\mathcal{C} \to \Lambda\mathcal{C}$ are embeddings and their images are respectively the full subcategories of π-atoms and of σ-atoms. Moreover $\mathcal{C}^c$ is equivalent to the full subcategory of atoms of $\Lambda\mathcal{C}$.*

Consider now the continuous extension $\Pi\Sigma\mathcal{C} \to \Lambda\mathcal{C}$ of the embedding $\Sigma\mathcal{C} \hookrightarrow \Lambda\mathcal{C}$; and dually, consider $\Sigma\Pi\mathcal{C} \to \Lambda\mathcal{C}$.

Corollary. *The functors $\Sigma\Pi\mathcal{C} \to \Lambda\mathcal{C}$ and $\Pi\Sigma\mathcal{C} \to \Lambda\mathcal{C}$ are embeddings.*

Recall [Benabou] that a *distributor* $M : \mathcal{C} \Rightarrow \mathcal{D}$ between two small $\mathcal{V}$-categories is defined to be a functor $M : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{V}$. This concept make sense even when $\mathcal{V}$ is non-symmetric; namely, we specify M by a family of arrows

$$[A, B] \otimes M(B, C) \otimes [C, D] \to M(A, D)$$

satisfying obvious conditions. For every $B \in \mathcal{D}$ we have a functor $M(-, B) : \mathcal{C}^{op} \to \mathcal{V}$ and the map $B \mapsto M(-, B)$ is a functor $M(-, =) : \mathcal{D} \to [\mathcal{C}^{op}, \mathcal{V}]$. If we use the equivalence $[\mathcal{C}^{op}, \mathcal{V}] \simeq \Sigma\mathcal{C}$ the functor $M(-, =)$ can be extended to a cocontinuous functor $\Sigma M : \Sigma\mathcal{D} \to \Sigma\mathcal{C}$. The functor ΣM has a bicontinuous extension $\Lambda M : \Lambda\mathcal{D} \to \Lambda\mathcal{C}$. The *transpose* of a distributor M is a distributor ${}^tM : \mathcal{D}^{op} \times \mathcal{C} \to \mathcal{V}^r$. The distributor ${}^tM$ defines a cocontinuous functor $\Sigma^tM : \Sigma\mathcal{C}^{op} \to \Sigma\mathcal{D}^{op}$ and by duality, a continuous functor $\Pi M^* = \Pi^tM^o : \Pi\mathcal{C} \to \Pi\mathcal{D}$. The functor ΠM^* has a bicontinuous extension $\Lambda M^* : \Lambda\mathcal{C} \to \Lambda\mathcal{D}$.

Proposition 2. *For any distributor $M : \mathcal{C} \Rightarrow \mathcal{D}$ the functor $\Lambda M^* : \Lambda\mathcal{C} \to \Lambda\mathcal{D}$ is left adjoint to the functor $\Lambda M : \Lambda\mathcal{D} \to \Lambda\mathcal{C}$. Moreover, any adjoint pair of bicontinuous functors between $\Lambda\mathcal{C}$ and $\Lambda\mathcal{D}$ is of this form.*

Remark: To every functor $f : \mathcal{C} \to \mathcal{D}$ we can associate two distributors $\Gamma^f : \mathcal{D} \Rightarrow \mathcal{C}$ and $\Gamma_f : \mathcal{C} \Rightarrow \mathcal{D}$, where $\Gamma^f(B, A) = hom(B, fA)$ and $\Gamma_f(A, B) = hom(fA, B)$. The functors $\Sigma\Gamma^f : \Sigma\mathcal{C} \to \Sigma\mathcal{D}$ and $\Pi\Gamma_f^* : \Pi\mathcal{C} \to \Pi\mathcal{C}$ are respectively the cocontinuous and the continuous extensions of f. It follows that $\Lambda\Gamma^f = \Lambda f = \Lambda\Gamma_f^*$ and therefore that Λf has bicontinuous left and right adjoints. When $\mathcal{D}$ is Cauchy complete this property characterizes the bicontinuous functors $\Lambda\mathcal{C} \to \Lambda\mathcal{D}$ of the form Λf.

The *relative completion* $j : \mathcal{C} \to \Lambda^{\pi}\mathcal{C}$ of a cocomplete $\mathcal{V}$-category $\mathcal{C}$ is defined as in [Joyal]. Recall that a cocomplete category is also complete when it is accessible (see [Makkai & Paré]).

Theorem 2. *Let $\mathcal{C}$ be a bicomplete $\mathcal{V}$-category. The relative completion $j : \mathcal{C} \to \Lambda^{\pi}\mathcal{C}$ exists and has the following properties:*

(i) the category $\Lambda^{\pi}\mathcal{C}$ is soft ;
(ii) for any $A \in \mathcal{C}$ the object jA is π-atomic;
(iii) the functor j has a cocontinuous right adjoint $k : \Lambda^{\pi}\mathcal{C} \to \mathcal{C}$
(iv) the functor j is full and faithful;
(v) $\Lambda^{\pi}\mathcal{C}$ is the biclosure of $j(\mathcal{C})$.

Moreover, these properties characterize the pair $(j, \Lambda^{\pi}\mathcal{C})$ up to equivalence of categories.

§2. The symmetric case

From this point on we shall suppose that $\mathcal{V}$ is symmetric. The (tensor) product $\mathcal{C} \times \mathcal{D}$ of two $\mathcal{V}$-categories is then defined (see [Kelly]). Recall that we have

$$[(A,B),(C,D)] = [A,C] \otimes [B,D]$$

for pairs of objets (A,B) and $(C,D) \in \mathcal{C} \times \mathcal{D}$.

Definition 3: Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be cocomplete $\mathcal{V}$- categories. A functor of two variables $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *soft* if the following commutative square of canonical arrows

$$\begin{array}{ccc} \int_{(i,j)\in I\times J} F(D(i),E(j)) \otimes v(i) \otimes w(j) & \longrightarrow & \int_{j\in J} F(X,E(j)) \otimes w(j) \\ \downarrow & & \downarrow \\ \int_{i\in I} F(D(i),Y) \otimes v(i) & \longrightarrow & F(X,Y) \end{array}$$

is a pushout for any pairs of lower integrals $\int_{i\in I} D(i) \otimes v(i) = X$ and $\int_{j\in J} E(j) \otimes w(j) = Y$

The concept of soft functors in $n \geq 3$ variables is defined similarly, but by using cubical diagrams instead of (pushout) squares. A soft functor of one variable is a cocontinuous functor.

A functor of $n \geq 2$ variables $F = F(-,\dots,-)$ is *responsive* if it is continuous in each variable, and if fixing at σ-atoms the values of any subset of $0 \leq k < n$ variables produce a soft functor. The functor $[-,-] : \Lambda\mathcal{C}^{op} \times \Lambda\mathcal{C} \to \mathcal{V}$ is an example of responsive functors (Theorem 1).

We have also the dual concepts of cosoft and coresponsive functors.

Theorem 3. *Let $F : \prod_{i=1}^{n} \mathcal{C}_i \to \mathcal{D}$ be a functor from a non-empty product of $\mathcal{V}$-categories and taking its values in a bicomplete $\mathcal{V}$-category $\mathcal{D}$. Then F has a responsive extension $F' : \prod_{i=1}^{n} \Lambda\mathcal{C}_i \to \mathcal{D}$. Morover this extension is unique up to a unique isomorphism.*

We have the dual concepts of *cosoft* and *coresponsive* functors. Theorem 3 has a dual which we do not state.

The *external tensor* and the *external dual tensor* products

$$\otimes \text{ and } \odot : \Lambda\mathcal{B} \times \Lambda\mathcal{C} \to \Lambda(\mathcal{B} \times \mathcal{C})$$

are defined as in [Joyal]. They are respectively the coresponsive and responsive extensions of $i : \mathcal{B} \times \mathcal{C} \to \Lambda(\mathcal{B} \times \mathcal{C})$. These operations are associatives up to coherent isomorphisms.

Proposition 3. *For any $X \in \Lambda\mathcal{B}$ the functor $X \otimes (-) : \Lambda\mathcal{C} \to \Lambda(\mathcal{B} \otimes \mathcal{C})$ has a right adjoint $R \mapsto X \backslash R$.*

Remark: When $\mathcal{C}$ is small let $\Delta \in \Lambda(\mathcal{C} \times \mathcal{C}^{op})$ be the element corresponding to $[-,-]$ via the embedding $[\mathcal{C}^{op} \times \mathcal{C}, \mathcal{V}] = \Sigma(\mathcal{C} \times \mathcal{C}^{op}) \subset \Lambda(\mathcal{C} \times \mathcal{C}^{op})$. It can be shown that for any $X \in \Lambda\mathcal{C}$ we have a canonical map $X \otimes X^o \to \Delta$ and a natural isomorphism $X^o \simeq X \backslash \Delta$.

As in Classical Linear Logic [Girard] we have obviously the duality isomorphism

$$(X \odot Y)^o \simeq X^o \otimes Y^o$$

We have also *mixed associativity* transformations

$$(A \odot B) \otimes C \to A \odot (B \otimes C) \qquad A \otimes (B \odot C) \to (A \otimes B) \odot C$$

(see [Lambek] for the history of this concept).

Certain structures on categories can be extended to their free bicompletion. Recall that a monoidal $\mathcal{V}$-category $\mathcal{D}$ is *closed* if for every $A \in \mathcal{D}$ the functors $X \mapsto A \otimes X$ and $X \mapsto X \otimes A$ have right adjoints, denoted respectively $X \mapsto A \backslash X$ and $X \mapsto X/A$ (see [Kelly]). An object $J \in \mathcal{D}$ is *dualizing* if the (contravariant) adjoint functors $A \mapsto A \backslash J = A^\bullet$ and $A \mapsto J/A = {}^\bullet A$ are inverse equivalences. A closed category $\mathcal{D}$ equipped with a dualizing object $J \in \mathcal{D}$ is said to be *$\star$-autonomous.*

Theorem 4. *Let $\mathcal{D} = (\mathcal{D}, \otimes, I)$ be a monoidal $\mathcal{V}$-category. The coresponsive extension of the tensor product on $\mathcal{D}$ defines a monoidal structure on $\Lambda\mathcal{D}$. Moreover $\Lambda\mathcal{D}$ is closed if $\mathcal{D}$ is, and $iJ \in \Lambda\mathcal{D}$ is dualizing if $J \in \mathcal{D}$ is.*

When $\mathcal{D}$ is closed the functor $(-\backslash-) : \Lambda\mathcal{D}^{op} \times \Lambda\mathcal{D} \to \Lambda\mathcal{D}$ is the responsive extension of the corresponding functor on $\mathcal{D}$.

Corollary. *The free bicomplete $\mathcal{V}$ category generated by a finite chain of arrows is $\star$-autonomous. In particular, $\Lambda 1$ the free bicomplete $\mathcal{V}$-category generated by one object is $\star$-autonomous.*

For any $\mathcal{V}$-category $\mathcal{C}$ let $[-,-] : \Lambda\mathcal{C}^{op} \times \Lambda\mathcal{C} \to \Lambda 1$ be the responsive extension of $j[-.-] : \mathcal{C}^{op} \times \mathcal{C} \to \Lambda 1$, where $j : \mathcal{V} \to \Lambda 1$ is given by $jA = 1 \otimes A$.

Proposition 4. *Let $\mathcal{C}$ a $\mathcal{V}$-category. The responsive functor $[-,-] : \Lambda\mathcal{C}^{op} \times \Lambda\mathcal{C} \to \Lambda 1$ defines an enrichement of $\Lambda\mathcal{C}$ over $\Lambda 1$. Moreover, $\Lambda\mathcal{C}$ is bicomplete as a $\Lambda 1$-category.*

We now sketch a generalization of the theory of distributors. It is based on the observation that we have

$$[\mathcal{C}^{op} \times \mathcal{D}, \mathcal{V}] = \Sigma(\mathcal{C} \times \mathcal{D}^{op}) \hookrightarrow \Lambda(\mathcal{C} \times \mathcal{D}^{op})$$

for any small $\mathcal{V}$-category $\mathcal{C}$ and $\mathcal{D}$.

Definition 4: A *λ-distributor* $M : \mathcal{C} \Rightarrow \mathcal{D}$ is an object $M \in \Lambda(\mathcal{C} \times \mathcal{D}^{op}) = \Lambda(\mathcal{C}, \mathcal{D})$.

Recall that distributors $M : \mathcal{B} \Rightarrow \mathcal{C}$ and $N : \mathcal{C} \Rightarrow \mathcal{D}$ can be composed $M \circ N : \mathcal{B} \Rightarrow \mathcal{D}$ The operation of composition of distributors is a functor of two variables

$$\Sigma(\mathcal{B} \times \mathcal{C}^{op}) \times \Sigma(\mathcal{C} \times \mathcal{D}^{op}) \to \Sigma(\mathcal{B} \times \mathcal{D}^{op}).$$

By taking its coresponsive extension we obtain an operation of composition $(M, N) \mapsto M \circ N$ on λ-distributors:

$$\Lambda(\mathcal{B}, \mathcal{C}) \times \Lambda(\mathcal{C}, \mathcal{D}) \to \Lambda(\mathcal{B}, \mathcal{D}).$$

Proposition 5. *For any $M \in \Lambda(\mathcal{B}, \mathcal{C})$ the functor $M \circ (-) : \Lambda(\mathcal{C}, \mathcal{D}) \to \Lambda(\mathcal{B}, \mathcal{D})$ has a right adjoint $Q \mapsto M \backslash Q$. And similarly for the functor $(-) \circ N : \Lambda(\mathcal{B}, \mathcal{C}) \to \Lambda(\mathcal{B}, \mathcal{D})$ for any $N \in \Lambda(\mathcal{C}, \mathcal{D})$.*

Remark: The composition of λ-distributors can be explained in term of other operations. More precisely, let us denote by 1 the $\mathcal{V}$-category with a single object and let $\epsilon : \mathcal{D} \times \mathcal{D}^{op} \Rightarrow 1$ be the distributor given by $\epsilon(X, Y, 1) = [X, Y]$. Observe that the category of small $\mathcal{V}$-categories and distributors is a compact closed bi-category. Then for $M \in \Lambda(\mathcal{B}, \mathcal{C})$ and $N \in \Lambda(\mathcal{C}, \mathcal{D})$ we have the formula

$$M \circ N = \Lambda(\mathcal{B}^{op} \times \epsilon \times \mathcal{D})(M \otimes N)$$

where $M \otimes N \in \Lambda(\mathcal{B}^{op} \times \mathcal{C} \times \mathcal{C}^{op} \times \mathcal{D})$. We shall formalise the whole situation later with a concept of *Linear Theory*.

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Free Bicomplete Categories

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This is the first of a series of comptes rendus on free bicomplete categories. A category is *bicomplete* if it admits limits and colimits. A functor is *bicontinuous* if it preserves limits and colimits. The *free bicompletion* ΛC of a category C is the bicomplete category freely generated by C. More precisely, we have a functor $C \to \Lambda C$ satisfying a universal property with respect to bicomplete categories and bicontinuous functors. One motivation for this work is to extend to categories the classical results of Phillip M. Whitman on free lattices. We introduce the concept of *soft functors* and prove that the hom functor of a free bicomplete category is soft. The result has a central role in a theorem characterizing free bicomplete categories by a few combinatorial properties. We introduce the concept of *responsive functors* and prove that any functor defined on a product of categories has a responsive extension to the product of their bicompletions. It follows that ΛC is monoidal closed, or $\star$-autonomous when C is. In particular $\Lambda 1$, the bicomplete category freely generated by a single object, is $\star$-autonomous. We also show that any ΛC is enriched over $\Lambda 1$. In the second compte rendu we shall consider free bicompletion of enriched categories. In a third we shall discuss the game-theoretic semantics of free bicompletions. This work was partly motivated by Andreas Blass's game semantic of linear logic [Blass].

§ 1. The characterization theorem

We first recall Whitman's theorem on free lattices (see [Whitman]). A lattice is a poset with binary infima and suprema. The *free lattice* generated by a poset P is a lattice $L(P)$ equipped with an order-preserving map $i : P \to L(P)$ satisfying the following universal property: for any order-preserving map $f : P \to L$ with values in a lattice L there is a unique lattice map $f' : L(P) \to L$ such that $f'i = f$.

Theorem 0. *(Whitman) Let $i : P \to L(P)$ be the free lattice generated by a poset P. Then for any a,b in P and x, y, u and v in $L(P)$ we have:*

(i) $x \wedge y \leq u \vee v$ iff $x \wedge y \leq u$ or $x \wedge y \leq v$ or $x \leq u \vee v$ or $y \leq u \vee v$;
(ii) $i(a) \leq u \vee v$ iff $i(a) \leq u$ or $i(a) \leq v$;
(iii) $x \wedge y \leq i(b)$ iff $x \leq i(b)$ or $y \leq i(b)$;
(iv) $i(a) \leq i(b)$ iff $a \leq b$;
(v) the lattice $L(P)$ is generated by $i(P)$.

Moreover, these properties characterize the free lattice generated by P.

The central thing here is the condition (i). Our first task will be that of formulating an analogous condition for categories. In the absence of any indication to the contrary, we shall suppose that all limits and colimits are finite –that is, taken over finite diagrams. But the results and proofs remain valid with κ-small limits and colimits where κ is any infinite regular cardinal, and also with all small ones.

Recall (see [Mac Lane]) that a category is complete (resp. cocomplete) if it admits limits (resp. colimits), that a functor is continuous (resp. cocontinuous) if it preserves limits (resp. colimits). A category is *bicomplete* if it is complete and cocomplete. A functor is *bicontinuous* if it is continuous and cocontinuous.

A *free bicompletion* of a category $\mathcal{C}$ is a bicomplete category $\Lambda\mathcal{C}$ equipped with a functor $i : \mathcal{C} \to \Lambda\mathcal{C}$ such that:

i) (existence) for any functor $F : \mathcal{C} \to \mathcal{B}$ with values in a bicomplete category there exist a bicontinuous functor $F' : \Lambda\mathcal{C} \to \mathcal{B}$ such that $F' \circ i = F$ (we shall say that F' is a bicontinuous extension of F) ;

ii) (uniqueness) if $F', F'' : \Lambda\mathcal{C} \to \mathcal{B}$ are two bicontinous extensions of F then there is a unique isomorphism $u : F' \to F''$ such that $u \circ i = id_F$.

Remark: It follows from this definition that the free bicompletion of a category is unique up to an equivalence of categories. Moreover, the equivalence is unique up to an isomorphism of functors; and the isomorphism itself is unique.

When $\mathcal{C}$ is small the existence of $\Lambda\mathcal{C}$ can be proved by standard categorical methods that we shall not discuss. For large $\mathcal{C}$ the existence can be proven using Theorem 1.

Definition 1: Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a functor of two variables between cocomplete categories. We shall say that F is *soft* if for any pair of finite diagrams $D : I \to \mathcal{A}$ and $E : J \to \mathcal{B}$ the following commutative square of canonical maps

$$\begin{array}{ccc} \varinjlim_{I\times J} F(D,E) & \longrightarrow & \varinjlim_{I} F(D, \varinjlim_{J} E) \\ \downarrow & & \downarrow \\ \varinjlim_{J} F(\varinjlim_{I} D, E) & \longrightarrow & F(\varinjlim_{I} D, \varinjlim_{J} E) \end{array}$$

is a pushout. A bicomplete category $\mathcal{B}$ is *soft* if the hom functor $\mathcal{B}^{op} \times \mathcal{B} \to \mathit{Sets}$ is soft .

The concept of soft functors in $n \geq 3$ variables is defined similarly, but by using cubical diagrams instead of (pushout) squares. A soft functor of one variable is just a cocontinuous functor.

Remark: Note that the functors $F(A, -)$ obtained by fixing one of the variables of a soft functor need not be cocontinuous. However, when A is an initial object, $F(A, -)$ is cocontinuous. A similar observation can be made for soft functors of $n \geq 3$ variables.

Let $\mathcal{B}$ be a soft category. Then for any pair of finite diagrams $D : I \to \mathcal{B}$ and $E : J \to \mathcal{B}$ the square

$$\begin{array}{ccc} \varinjlim_{I\times J} hom(D,E) & \longrightarrow & \varinjlim_{I} hom(D, \varinjlim_{J} E) \\ \downarrow & & \downarrow \\ \varinjlim_{J} hom(\varinjlim_{I} D, E) & \longrightarrow & hom(\varinjlim_{I} D, \varinjlim_{J} E) \end{array}$$

is a pushout. In particular, every arrow $f : \varprojlim D \to \varinjlim E$ has either a factorization $f = gp_i$, or a factorization $f = q_j h$, where $p_i : \varprojlim D \to D_i$ and $q_j : E_j \to \varinjlim E$ are canonical. This is the categorical analogue of property i) in the Whitman theorem.

Definition 2: An object $A \in C$ of a cocomplete (resp. complete) category is *σ-atomic* (resp. *π-atomic*) if the functor $hom(A,-) : C \to Sets$ (resp. $hom(-,A) : C^{op} \to Sets$) is cocontinuous. An object which is both σ- and π-atomic is said to be *atomic*.

Remark: In a soft category the σ-atoms (resp. π-atoms) are closed under limits (resp. colimits). In particular, the terminal object $\top$ (resp. initial object $\bot$) is σ-atomic (resp. π-atomic).

Theorem 1. *For any category C, the free bicompletion $i : C \to \Lambda C$ has the following properties:*

(i) the category ΛC is soft ;

(ii) for any $A \in C$ the object iA is atomic;

(iii) the functor i is full and faithful;

(iv) the category ΛC is bigenerated by $i(C)$.

Moreover, these properties characterize the pair $(i, \Lambda C)$ up to an equivalence of categories.

In this theorem, condition (iv) means that ΛC is the closure of $i(C)$ under the operations of limits and colimits.

The *free completion* ΠC and the *free cocompletion* ΣC of a category C are defined similarly to the free bicompletion. Let $[C^{op}, Sets]$ be the category of functors $C^{op} \to Sets$. It is well known that ΣC is the full subcategory of $[C^{op}, Sets]$ whose objects are the finitely presentable functors, and that i is defined by the Yoneda embedding. Recall that the *Karoubi completion* KC is obtained by splitting the idempotents of C. It is the full subcategory of σ-atoms of ΣC, and also the full subcategory of π-atoms of ΠC.

Proposition 1. *The canonical functors $\Sigma C \to \Lambda C$ and $\Pi C \to \Lambda C$ are full and faithful. Their (essential) images are respectively the full subcategories of π-atoms and of σ-atoms. Moreover, KC is equivalent the full subcategory of atoms.*

Consider now the continuous extension $\Pi\Sigma C \to \Lambda C$ of $\Sigma C \to \Lambda C$. Dually, we have a cocontinuous functor $\Sigma\Pi C \to \Lambda C$.

Corollary. *The canonical functors $\Sigma\Pi C \to \Lambda C$ and $\Pi\Sigma C \to \Lambda C$ are full and faithful.*

For the rest of this section we shall remove the condition of finiteness that we had imposed on limits and colimits, supposing only that they should be small. Recall that a *distributor* $M : C \Rightarrow \mathcal{D}$ between two small categories is a functor $M : C^{op} \times \mathcal{D} \to Sets$. For every $B \in \mathcal{D}$ we have a functor $M(-,B) : C^{op} \to Sets$ and the map $B \mapsto M(-,B)$ is a functor $M(-,=) : \mathcal{D} \to [C^{op}, Sets]$. Using the equivalence $[C^{op}, Sets] \simeq \Sigma C$ we can extend $M(-,=)$ to a cocontinuous functor $\Sigma M : \Sigma\mathcal{D} \to \Sigma C$. Furthermore, the functor ΣM has a bicontinuous extension $\Lambda M : \Lambda\mathcal{D} \to \Lambda C$. The *transpose* of M is a distributor ${}^tM : \mathcal{D} \times C^{op} \to Sets$ defining a cocontinuous functor $\Sigma^t M : \Sigma C^{op} \to \Sigma\mathcal{D}^{op}$ and, by duality, a continuous functor $\Pi M^* : \Pi C \to \Pi\mathcal{D}$. The functor ΠM^* has a bicontinuous extension $\Lambda M^* : \Lambda C \to \Lambda\mathcal{D}$.

Proposition 2. *For any distributor $M : \mathcal{C} \Rightarrow \mathcal{D}$ the functor $\Lambda M^\bullet : \Lambda\mathcal{C} \to \Lambda\mathcal{D}$ is left adjoint to the functor $\Lambda M : \Lambda\mathcal{D} \to \Lambda\mathcal{C}$. Moreover, any adjoint pair of bicontinuous functors between $\Lambda\mathcal{C}$ and $\Lambda\mathcal{D}$ is of this form.*

Remark: Following [J. Benabou] to every functor $f : \mathcal{C} \to \mathcal{D}$ we can associate two distributors $\Gamma^f : \mathcal{D} \Rightarrow \mathcal{C}$ and $\Gamma_f : \mathcal{C} \Rightarrow \mathcal{D}$, where $\Gamma^f(B, A) = hom(B, fA)$ and $\Gamma_f(A, B) = hom(fA, B)$. The functors $\Sigma\Gamma^f : \Sigma\mathcal{C} \to \Sigma\mathcal{D}$ and $\Pi\Gamma_f^\bullet : \Pi\mathcal{C} \to \Pi\mathcal{C}$ are respectively the cocontinuous and the continuous extensions of f. It follows that $\Lambda\Gamma^f = \Lambda f = \Lambda\Gamma_f^\bullet$, and therefore that Λf has bicontinuous left and right adjoints. When $\mathcal{D}$ is Karoubi complete this property characterizes the bicontinuous functors $\Lambda\mathcal{C} \to \Lambda\mathcal{D}$ of the form Λf.

§ 2. Extension theorems

We shall say that a functor of two variables $F = F(-,-)$ is *responsive* if it is continuous in each variable and soft, and if the functors $F(A,-)$ and $F(-,B)$ are cocontinuous for σ-atoms A or B. The functor $hom(-,-) : \Lambda\mathcal{C}^{op} \times \Lambda\mathcal{C} \to Sets$ is an example of a responsive functor (Theorem 1). More generally, we shall say that a functor of $n \geq 2$ variables $F = F(-,\dots,-)$ is *responsive* if it is continuous in each variable, and if fixing at σ-atoms the values of any subset of $0 \leq k < n$ variables produces a soft functor.

Theorem 2. *Let $F : \prod_{i=1}^n \mathcal{C}_i \to \mathcal{E}$ be a functor taking its values in a bicomplete category $\mathcal{E}$. Then F has a responsive extension $F' : \prod_{i=1}^n \Lambda\mathcal{C}_i \to \mathcal{E}$. Moreover this extension is unique up to a unique isomorphism.*

We have the dual concepts of *cosoft* and *coresponsive* functors. Theorem 2 has a dual which we do not state.

The *external tensor product*

$$\otimes : \Lambda\mathcal{B} \times \Lambda\mathcal{C} \to \Lambda(\mathcal{B} \times \mathcal{C})$$

is defined to be the coresponsive extension of $i : \mathcal{B} \times \mathcal{C} \to \Lambda(\mathcal{B} \times \mathcal{C})$. We have a natural associativity isomorphism $X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ in $\Lambda(\mathcal{B} \times \mathcal{C} \times \mathcal{D})$ for $X \in \Lambda\mathcal{B}$, $Y \in \Lambda\mathcal{C}$ and $Z \in \Lambda\mathcal{D}$.

Proposition 3. *For any $X \in \Lambda\mathcal{B}$ the functor $X \otimes (-) : \Lambda\mathcal{C} \to \Lambda(\mathcal{B} \odot \mathcal{C})$ has a right adjoint $R \mapsto X \backslash R$.*

The *external dual tensor product*

$$\odot : \Lambda\mathcal{B} \times \Lambda\mathcal{C} \to \Lambda(\mathcal{B} \times \mathcal{C})$$

is defined to be the responsive extension of $i : \mathcal{B} \times \mathcal{C} \to \Lambda(\mathcal{B} \times \mathcal{C})$. It is also associative and we have the duality isomorphism

$$(X \odot Y)^o \simeq X^o \otimes Y^o$$

familiar in Classical Linear Logic [Girard]. We have also *mixed associativity* transformations

$$(A \odot B) \otimes C \to A \odot (B \otimes C) \qquad A \otimes (B \odot C) \to (A \otimes B) \odot C$$

(this concept has been discovered in logic by Grishin; the categorical form independently by de Paiva and by Cockett & Seely; see [Lambek]).

Certain structures on categories can be extended to their free bicompletion. Recall that a monoidal category $\mathcal{D}$ is *closed* if for every $A \in \mathcal{D}$ the functors $X \mapsto A \otimes X$ and $X \mapsto X \otimes A$ have right adjoints, denoted respectively $X \mapsto A\backslash X$ and $X \mapsto X/A$ (see [Kelly]). An object $J \in \mathcal{D}$ is *dualizing* if the (contravariant) adjoint functors $A \mapsto A\backslash J = A^{\bullet}$ and $A \mapsto J/A = {}^{\bullet}A$ are inverse equivalences. A closed category $\mathcal{D}$ is $\star$*-autonomous* if it is equipped with a dualizing object $J \in \mathcal{D}$.

Theorem 3. *Let $\mathcal{D} = (\mathcal{D}, \otimes, I)$ be a tensor category. The coresponsive extension of the tensor product on $\mathcal{D}$ defines a monoidal structure on $\Lambda\mathcal{D}$. Moreover $\Lambda\mathcal{D}$ is closed if $\mathcal{D}$ is, and $iJ \in \Lambda\mathcal{D}$ is dualizing if $J \in \mathcal{D}$ is.*

When $\mathcal{D}$ is closed the functor $(-\backslash-) : \Lambda\mathcal{D}^{op} \times \Lambda\mathcal{D} \to \Lambda\mathcal{D}$ is the responsive extension of the corresponding functor on $\mathcal{D}$. The theorem shows that free bicompletions of $\star$-autonomous categories are $\star$-autonomous. The posets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ are $\star$-autonomous categories on setting $x \otimes y = 0 \vee (x + y - n)$ at taking 0 for dualising object.

Corollary. *The free bicomplete category generated by a finite chain of arrows is $\star$-autonomous. In particular, $\Lambda 1$ the bicomplete category freely generated by one object is $\star$-autonomous.*

For the next proposition we remove the condition of finiteness normally imposed on limits and colimits and only suppose they are small. Let $j : Sets \to \Lambda 1$ be the coproduct-preserving functor sending a singleton to 1. For any category C let $[-,-] : \Lambda C^{op} \times \Lambda C \to \Lambda 1$ be the responsive extension of $jhom(-.-) : C^{op} \times C \to \Lambda 1$.

Proposition 4. *For any free bicomplete category ΛC, the functor $[-,-] : \Lambda C^{op} \times \Lambda C \to \Lambda 1$ defines an enrichement of ΛC over $\Lambda 1$. The category ΛC is bicomplete as a category enriched over $\Lambda 1$.*

§ 3. Relative completions

The *relative completion* of a cocomplete category C is a bicomplete category $\Lambda^{\pi} C$ equipped with a cocontinuous functor $j : C \to \Lambda^{\pi} C$ such that:

i) (existence) for any cocontinuous functor $F : C \to \mathcal{B}$ with values in a bicomplete category there exists a bicontinuous functor $F' : \Lambda^{\pi} C \to \mathcal{B}$ such that $F' \circ j = F$;
ii) (uniqueness) if $F', F'' : \Lambda^{\pi} C \to \mathcal{B}$ are two bicontinous extensions of F then there is a unique isomorphism $u : F' \to F''$ such that $u \circ j = id_F$.

The *relative cocompletion* $j : C \to \Lambda^{\sigma} C$ of a complete category is defined dually.

When C is small the existence of $\Lambda^{\pi} C$ can be proved by standard categorical methods which we shall not discuss. The construction of free bicompletion of a small category can be broken up into two steps, and this in two ways:

$$\Lambda C \simeq \Lambda^{\sigma} \Pi C \simeq \Lambda^{\pi} \Sigma C$$

Let C a small cocomplete category . The category $[C^{op}, Sets]^{\pi}$ of continuous functors $C^{op} \to Sets$ is bicomplete since it is a full reflective subcategory of the presheaf category $[C^{op}, Sets]$. For any $A \in C$ we have $yA = hom(-, A) \in [C^{op}, Sets]^{\pi}$, and the Yoneda functor $y : C \to [C^{op}, Sets]^{\pi}$ is cocontinuous.

The *edge* of a cocontinuous functor $l : C \to B$ is defined to be the functor

$$l^{\dagger} : B \to [C^{op}, Sets]^{\pi}$$

given by $l^{\dagger}(B) = hom(l(-), B)$ for any $B \in B$.

Theorem 4. *For any cocomplete small category C the relative completion $j : C \to \Lambda^{\pi}C$ has the following properties:*
- *(i) the category $\Lambda^{\pi}C$ is soft ;*
- *(ii) for any $A \in C$ the object jA is π-atomic;*
- *(iii) the edge functor $j^{\dagger} : \Lambda^{\pi}C \to [C^{op}, Sets]^{\pi}$ is cocontinuous;*
- *(iv) the functor j is full and faithful;*
- *(v) $\Lambda^{\pi}C$ is the biclosure of $j(C)$ under limits and colimits.*

Moreover, these properties characterize the pair (j, C) up to an equivalence of categories.

It follows from (iii) and (iv) that $j^{\dagger}$ is the bicontinuous extension of the Yoneda functor y. When C is bicomplete the identity functor $C \to C$ has a bicontinuous extension $k : \Lambda^{\pi}C \to C$ and (iii) can be replaced by the following condition:

(iii)′ the functor j has a cocontinuous right adjoint $k : \Lambda^{\pi}C \to C$.

The theorem remains valid if finite diagrams are replaced by κ-small diagrams. It further remains valid with all small digrams in the case where C is accessible [Makkai & Paré]; in this case C is bicomplete and *(iii)′* can be used instead of *(iii)*.

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Master symmetries and higher order invariants for finite dimensional bi-Hamiltonian systems

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Abstract

We find the Virasoro algebra of master symmetries constructed for the isotropic harmonic oscillator and the Toda lattice. For either system a suitable conformal invariance scaling basic tensor invariants is found. The corresponding master symmetries connected by the Virasoro commutator relation lead to exact hierarchies of higher order invariants. The approach is shown to be applicable in the bi-Hamiltonian case.

1 Introduction

We shall study the dynamical systems admitting the Hamiltonian description

$$X = PdH, \tag{1.1}$$

where X is a Hamiltonian vector field, P is a Poisson tensor and H is the corresponding Hamiltonian function. In the case of P being kernel-free, we can consider the symplectic form $\omega : \omega = P^{-1}$ instead. Then the equation (1.1) becomes

$$i_X\omega = dH. \tag{1.2}$$

A vector field Y commuting with the initial Hamiltonian vector field X

$$[Y, X] = 0$$

is called a *symmetry* of the Hamiltonian system (1.1).

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The notion of a *master symmetry* for the vector field X introduced in [1] we define as a vector field Z satisfying $[[Z, X], X] = 0$, provided $[Z, X] \neq 0$.

This is the case, in particular, when Z is a *conformal invariance* for X, i.e. $L_Z X = kX, k \in \mathbf{R}$.

Consider a (1, 1) tensor A the Nijenhuis tensor N_A [4] of which vanishes:

$$N_A := A^2[X,Y] + [AX, AY] - A([X, AY] + [AX, Y]) = 0, \qquad (1.3)$$

where X, Y is an arbitrary pair of vector fields. We shall call such tensor A a *recursion operator*.

In the bi-Hamiltonian case, namely when the dynamical system (1.1) admits two Hamiltonian descriptions:

$$X = P_0 dH_0 = P_1 dH_1, \qquad (1.4)$$

where P_0, P_1 are compatible Poisson tensors [6], i.e. their Schouten bracket [3] vanishes: $[P_0, P_1] = 0$. This compatibility condition garantees integrability of the system (1.4) [5–8] and can be reformulated in an alternative way. If one of the Poisson tensors P_0, P_1 is nondegenerate, for example — P_0, we can construct a (1, 1) tensor $A := P_1 P_0^{-1}$. Then the compatibility condition is equivalent to the fact that A is a recursion operator [6]. If, initially, the bi-Hamiltonian vector field X_0 preserves a symplectic form ω_1 and a Poisson tensor P, we have the following bi-Hamiltonian description

$$X_0 = P_0 dH_0 \qquad \omega_1 X_0 = dH_1, \qquad (1.5)$$

and the Hamiltonians H_0, H_1 are connected by the corresponding recursion operator $A := P_0\omega_1 : dH_1 = AdH_0$.

Define $X_n := A^n X, \omega_{n+1} := \omega_1 A^n$ and $P_n := A^n P_0$. Then by Gelfand-Dorfman-Magri-Morosi's theorem [5–8] all X_n's constitute a commutative Lie algebra of bi-Hamiltonian vector fields relative to the higher order symplectic structures ω_n and the Poisson tensors P_n. The functions $H_n := 1/nTr(A^n)$ are invariants of the vector fields X_n, in involution relative to the Poisson brackets defined by the Poisson tensors P_n.

If an appropriate conformal invariance is found, we can construct a hierarchy of higher order master symmetries [2].

Proposition 1.1 *Let X_0 be a bi-Hamiltonian vector field defined by a symplectic form ω_1 and a Poisson tensor P_0, in addition the operator $A := \omega_1 P_0$ is recursive.*

Assume there is a conformal symmetry Z_0 for X_0, ω_1, P_0:

$$L_{Z_0}X_0 = \alpha X_0, \qquad L_{Z_0}P_0 = \beta P_0, \qquad L_{Z_0}\omega_1 = \gamma\omega_1; \qquad \alpha, \beta, \gamma \in \mathbf{R}$$

Then, defining $Z_n := A^n Z_0$ we can derive for all n, m the following hierarchies of higher order invariants

$$L_{Z_n}X_m = (\alpha + m(\beta + \alpha)X_{n+m}, \qquad L_{Z_n}A = (\beta + \gamma)A^{n+1}.$$

$$L_{Z_n}\omega_m = (-\beta + (m+n)(\beta+\gamma))\omega_{n+m},$$

$$L_{Z_n}P_m = (\beta + (m-n)(\beta+\gamma))P_{n+m},$$

$$L_{Z_n}H_m = (\alpha - \beta + (m+n)(\beta+\gamma))H_{n+m},$$

$$L_{Z_n}Z_m = (m-n)(\beta+\gamma)Z_{n+m}. \tag{1.6}$$

If A is invertible, — n and m are arbitrary integers, otherwise — only positive. All Z_n's are master symmetries for the hierarchy of bi-Hamiltonian vector fields $\{X_n\}, n \in \mathbf{Z}$.

It is remarkable that the hierarchy of master symmetries $\{Z_n\}, n \in \mathbf{Z}$ with the commutator relation (1.6) form a Lie algebra isomorphic to the Virasoro algebra [9, 10]. Indeed, the latter one, denoted by Vir, is a Lie algebra over $\mathbf{C}$ with the basis $L_n (n \in \mathbf{Z})$, c and the following commutator

$$[L_m, L_n] = (m-n)L_{m-n} + \delta_{m,-n}(m^3 - m)/12c, \qquad [c, L_n] = 0. \tag{1.7}$$

Assuming c is a constant the needed isomorphism follows.

2 Applications

2.1 The harmonic oscillator

The isotropic one-dimensional oscillator is defined by the equations

$$dq/dt = p, \qquad dp/dt = -q. \tag{2.1}$$

It is a Hamiltonian system on $M = \mathbf{R}^2$ with the canonical symplectic form $\omega_0 := -dp \wedge dq$ and the corresponding Hamiltonian function $H_0 := \frac{1}{2}(q^2 + p^2)$. The vector field $X_0 = -q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}$ yields another Hamiltonian description for the symplectic form ω_1 and the Hamiltonian H_1 given by

$$\omega_1 := -(q^2 + p^2)dp \wedge dq, \qquad H_1 := \frac{1}{2}ln(2H_0).$$

The operator $A := \omega_0\omega_1^{-1}$ is found to be $A = (q^2+p^2)^{-1}(dq\odot\frac{\partial}{\partial q}+dp\otimes\frac{\partial}{\partial p}) = 1/(2H_0)(dq\otimes\frac{\partial}{\partial q}+dp\otimes\frac{\partial}{\partial p})$. A is diagonal and in the action-angle coordinates (r,φ), where $r^2 = p^2+q^2$, $\varphi = arctan(q/p)$, has the eigenvalue depending only on the action coordinate r. Therefore, by Nijenhuis' theorem [4] A is recursion.

We use the vector field $Z_0 := \frac{q}{q^2+p^2}\frac{\partial}{\partial q}+\frac{p}{q^2+p^2}\frac{\partial}{\partial p}$ as a conformal invariance for $P_0 := \omega_0^{-1}, \omega_1$ and X_0. Indeed, direct calculations show

$$L_{Z_0}X_0 = 0, \qquad L_{Z_0}P_0 = -2P_0, \qquad L_{Z_0}\omega_1 = 0.$$

In this case Z_0 is not a master symmetry, but only a symmetry of the vector field X_0.

Defining $Z_n := A^n Z_0$, $\omega_{n+1} := \omega_1 A^n$, $P_n := A^n P_0$ and applying Proposition 1.1, we derive the following higher order invariants for the isotropic harmonic oscillator

$$\begin{aligned} L_{Z_n}X_m &= -2mX_{n+m}, & L_{Z_n}Z_m &= -2(m-n)Z_{n+m}, \\ L_{Z_n}A &= -2A^{n+1}, & L_{Z_n}P_m &= -2(m-n+1)P_{n+m}, \\ L_{Z_n}\omega_m &= -2(m+n-1)\omega_{n+m}, & L_{Z_n}H_m &= -2(m+n-1)H_{n+m}. \end{aligned}$$

We note, that n and m are arbitrary integers on account of invertibility of the recursion operator A. The master symmetries $\{Z_n\}$, $n \in \mathbf{Z}$ constitute the Virasoro algebra up to isomorphism.

2.2 The Toda lattice

Consider the finite, non-periodic Toda lattice. In terms of the canonical coordinates q^i and momenta p_i, $i = 1,2,\ldots,n$ it is given by

$$\begin{aligned} dq^i/dt &= p_i, \\ dp_i/dt &= exp(-(q^i-q^{i-1})) - exp(-(q^{i+1}-q^i)), \end{aligned} \tag{2.2}$$

This system takes the Hamiltonian form (1.2) and its Hamiltonian function H_0 is defined by the formulae $H_0 := \frac{1}{2}\sum_{i=1}^n p_i^2 + \sum_{i=1}^{n-1} exp(-(q_{i+1}-q_i))$, while the corresponding symplectic form ω_0 is canonical: $\omega_0 = -\sum_{i=1}^n dq^i \wedge dp_i$. This particular case of the Toda lattice was studied in [11] from the bi-Hamiltonian point of view. There was found the second symplectic form ω_1 with the Hamiltonian H_1

$$\omega_1 = \sum_{i=1}^{n-1} e^{(q^i-q^{i+1})}dq^i \wedge dq^{i+1} + \sum_{i=1}^{n} p_i dq^i \wedge p_i + \frac{1}{2}\sum_{i<j}^{n} dp_i \wedge p_j$$

$$H_1(q,p) = \frac{1}{3}\sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n}(p_i + p_{i+1})exp(-(q^{i+1} - q^i)).$$

Furthemore, the corresponding operator $A := \omega_1\omega_0^{-1}$ given by the formulae

$$A = \sum_{i=1}^{n} p_i\frac{\partial}{\partial q^i}\otimes dq^i + \sum_{i=1}^{n-1} exp(-(q^{i+1} - q^i))(\frac{\partial}{\partial p_{i+1}}\otimes dq^i - \frac{\partial}{\partial p_i}\otimes dq^{i+1})$$

$$+\frac{1}{2}\sum_{i<j}(\frac{\partial}{\partial q^i}\otimes dp_j - \frac{\partial}{\partial q^j}\otimes dp_i) + \sum_{i=1}^{n} p_i\frac{\partial}{\partial p_i}\otimes p_i$$

was proved to be recursion [11]. This leads to integrability of the Toda lattice as a bi-Hamiltonian system of the type (1.5).

We introduce the vector field Z_0 given by

$$Z_0 := \sum_{i=1}^{n}[2(n+1-i)\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial p_i}], \qquad (2.3)$$

for which one finds

$$L_{Z_0}X_0 = -X_0, \qquad L_{Z_0}\omega_1 = 2\omega_1, \qquad L_{Z_0}P_0 = -P_0,$$

where $P_0 := \omega_0^{-1}$ and X_0 is the vector field of the system (2.2). Consequently, the vector field Z_0 is a conformal invariance for the system. Setting $Z_n := A^nZ_0$, $X_n := A^nX_0$, $\omega_{n+1} := \omega_1A^n$, $P_n := A^nP_0$ and applying Proposition 1.1, we come up with the following exact hierarchies of higher order invariants of the Toda lattice for all $n, m \in \mathbf{Z}$:

$$L_{Z_n}X_m = (-1+m)X_{n+m}, \qquad L_{Z_n}Z_m = (m-n)Z_{n+m},$$
$$L_{Z_n}A = A^{n+1}, \qquad L_{Z_n}P_m = (m-n-1)P_{n+m},$$
$$L_{Z_n}\omega_m = (m+n+1)\omega_{n+m}, \qquad L_{Z_n}H_m = (m+n)H_{n+m}.$$

Thus, we have constructed all the higher order invariants for the Toda lattice using the Virasoro algebra of master symmetries Z_n, $n \in \mathbf{Z}$.

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