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A New Invariant of Local Rings

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Presented by P. Ribenboim, F.R.S.C.

Abstract

We introduce the following invariant of local ring A :

$$n(A) = \sup\{\ell_A(A/\mathfrak{q})/e(\mathfrak{q}; A)\},$$

where the supremum is taken over all primary ideals \mathfrak{q} belonging to the maximal ideal of A . We describe $n(A)$ in two cases: A is generalized Cohen-Macaulay and A is the coordinate ring of a nondegenerate projective variety with $\deg(X) \leq \text{codim}(X) + 2$.

1 Introduction

The aim of this note is to study the problems stated in [11]. In 1965, D.A. Buchsbaum posed the problem to describe the difference between the length and multiplicity of ideals generated by systems of parameters by an invariant (see, e.g., [9]). An extended problem of [11] considers the same question for the quotient. More precisely, let A be a Noetherian local ring with maximal ideal \mathfrak{m} and \mathfrak{q} an \mathfrak{m} -primary ideal. We define an invariant of A as follows:

$$n(A) = \sup\{\ell_A(A/\mathfrak{q})/e(\mathfrak{q}; A)\},$$

where the supremum is taken over all \mathfrak{m} -primary \mathfrak{q} , and $\ell_A(A/\mathfrak{q})$ and $e(\mathfrak{q}; A)$ denotes the length of A -module A/\mathfrak{q} and the multiplicity of \mathfrak{q} in A , respectively (see, e.g., [7]). We note that $n(A) \geq 1$, and if $n(A) < \infty$, then we get a new invariant of the local ring A .

The important class of Cohen-Macaulay rings (see, e.g., [3]) is given in case $n(A) = 1$. The following theorem sheds some light on the importance of this invariant by describing $n(A)$ in two cases.

Theorem. *Let k be an algebraically closed field. Let X be a pure-dimensional projective subscheme of \mathbb{P}_k^N . Let A be the local ring of the cone over X at the vertex. Then we have*

- (i) *If X is locally Cohen-Macaulay, then $n(A) \leq 1 + I(A)/\text{degree}(X)$, and we have equality provided X is arithmetically Buchsbaum.*
- (ii) *If X is a nondegenerate subvariety with $\text{degree}(X) \leq \text{codimension}(X) + 2$, then $n(A) = 1 + (\dim(A) - \text{depth}(A))/\text{degree}(X)$.*

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We note that there exists nondegenerate projective variety X with $\deg(X) = \text{codim}(X) + 2$ which are not locally Cohen-Macaulay. Therefore we conclude our note with some examples.

2 Notation and preliminary results

Before proving our theorem we need to introduce some notation: Let k be an algebraically closed field. By a graded k -algebra we mean a graded Noetherian ring $A = \bigoplus_{n \geq 0} A_n$ such that $A_0 = k$ and A is finitely generated over k by A_1 . From the point of algebraic geometry we want to consider such graded k -algebras although it works mostly for finitely generated modules over local rings. Let $H_m^i(A)$ be the local cohomology groups for $i \geq 0$, where \mathfrak{m} is the irrelevant ideal of A . We set $d = \dim(A)$, and

$$I(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(H_m^i(A)).$$

If $I(A) < \infty$, then A is said to be generalized Cohen-Macaulay, and $\ell_A(A/\mathfrak{q}) - e(\mathfrak{q}; A) \leq I(A)$ for all \mathfrak{m} -primary ideals \mathfrak{q} (see, e.g., [10]).

We need to introduce strongly reducing system of parameters: A system of parameters a_1, \dots, a_d of A is said to be a strongly reducing system of parameters if a_j is not in any prime $\mathfrak{p} \neq \mathfrak{m}$ belonging to (a_1, \dots, a_{j-1}) for any $j = 1, \dots, d$. Such systems of parameters are generalizations of reducing systems of parameters in the sense of [1].

Lemma 1. $n(A) = \sup\{\ell_A(A/\mathfrak{q})/e(\mathfrak{q}; A)\}$, where \mathfrak{q} runs through all \mathfrak{m} -primary ideals generated by strongly reducing system of parameters.

Proof. Applying [12], (VIII.22) we have an \mathfrak{m} -primary ideal $\mathfrak{q}' \subseteq \mathfrak{q}$ generated by a system of parameters such that $e(\mathfrak{q}'; A) = e(\mathfrak{q}; A)$. Using Proposition 2 of [2], we see that \mathfrak{q}' can be generated by a strongly reducing system of parameters. This proves Lemma 1.

Moreover, we need the following lemma.

Lemma 2. Let X be a pure-dimensional projective subscheme of \mathbb{P}_k^N of dimension $(d-1) \geq 1$. Let A be the coordinate ring of X and a_1, \dots, a_d a system of parameters of A . Then we have

$$e((a_1, \dots, a_d)A; A) = \deg(X) \cdot \left(\prod_{i=1}^d \deg(a_i) \right).$$

Proof. The elements a_1, \dots, a_d of A give raise to homogeneous polynomials f_1, \dots, f_d of $k[X_0, \dots, X_N]$. The ideal $I(Y) = (f_1, \dots, f_d)$ defines a complete intersection Y of \mathbb{P}_k^N . Since a_1, \dots, a_d is a system of parameters of A , we have $X \cap Y = \emptyset$. By taking the cones over X and Y , denoted by $c(X)$ and $c(Y)$ resp., we get that the intersection $c(X) \cap c(Y)$ is the vertex P . The reduction theorem of P. Samuel [8], (II.5.7.b) yields that $e((a_1, \dots, a_d)A; A) = i(X, Y; P)$, where $i(X, Y; P)$ is A. Weil's intersection number of X and Y at P . Bezout's theorem provides $i(X, Y; P) = \deg(X) \cdot \deg(Y) = \deg(X) \cdot \left(\prod_{i=1}^d \deg(f_i) \right)$. This proves Lemma 2.

3 Proof of the theorem

It is enough to prove our theorem for the graded k -algebra $A = k[X_0, \dots, X_N]/I(X)$, where $I(X)$ is the defining ideal of X . We set the irrelevant ideal $\mathfrak{m} = (X_0, \dots, X_N)$ of A . We set $d = \dim(A) \geq 2$ and $r = \text{depth}(A) \geq 1$.

(i) Let \mathfrak{q} be any \mathfrak{m} -primary ideal of A . Since X is locally Cohen-Macaulay, we have $\ell_A(A/\mathfrak{q}) - e_A(\mathfrak{q}; A) \leq I(A)$. Thus we see $\ell_A(A/\mathfrak{q})/e_A(\mathfrak{q}; A) \leq 1 + I(A)/e_A(\mathfrak{q}; A) \leq 1 + I(A)/\text{deg}(X)$ by Lemma 2. In Buchsbaum case we get the equality. Hence the assertion is proved.

(ii) In case $\text{deg}(X) = \text{codim}(X) + 1$, we know that X is a variety of minimal degree. Thus we see A is a Cohen-Macaulay ring, that is, $n(A) = 1$. Now let us assume that $\text{deg}(X) = \text{codim}(X) + 2$ and A is not Cohen-Macaulay. By [6], Theorem B, we see $H_{\mathfrak{m}}^i(A) = 0$ for all $i \neq r, d$ and $H_{\mathfrak{m}}^r(A) \cong k[y_1, \dots, y_{r-1}]^{\vee(r-2)}$, where y_1, \dots, y_{r-1} are algebraically independent elements of degree 1 of A . Firstly we want to show that $\ell_A(A/\mathfrak{q})/e_A(\mathfrak{q}; A) \leq 1 + (d-r)/\text{deg}(X)$ for any \mathfrak{m} -primary ideal \mathfrak{q} . By Lemma 1 and the proof of Proposition 2 of [2], we may assume $\mathfrak{q} = (f_1, \dots, f_d)$ such that f_1, \dots, f_d is a strongly reducing system of parameters for A and f_1, \dots, f_{r-1} is a system of parameters for $k[y_1, \dots, y_{r-1}]$. Put $d_i = \text{deg}(f_i)$ for all i . We set $B = A/(f_1, \dots, f_{r-1})A$ and $\bar{\mathfrak{q}} = \mathfrak{q}B$. Since f_i is a non-zero-divisor of $A/(f_1, \dots, f_{i-1})A$ for $i = 1, \dots, r-1$, we get $e_A(\mathfrak{q}; A) = e_B(\bar{\mathfrak{q}}; B)$, see, e.g., [7], (14.11). Thus we have $\ell_A(A/\mathfrak{q})/e_A(\mathfrak{q}; A) = \ell_B(B/\bar{\mathfrak{q}})/e_B(\bar{\mathfrak{q}}; B) \leq 1 + I(B)/e_B(\bar{\mathfrak{q}}; B) = 1 + I(B)/(\text{deg}(X) \cdot \prod_{i=1}^{r-1} d_i)$ by using Lemma 2. In order to show that $I(B) = (d-r) \cdot \prod_{i=1}^{r-1} d_i$, we need the following claim.

Claim. For $j = 1, \dots, r$, we have $H_{\mathfrak{m}}^i(A/(f_1, \dots, f_{j-1})A) = 0$ for $i \neq r-j+1, d-j+1$ and $H_{\mathfrak{m}}^{r-j+1}(A/(f_1, \dots, f_{j-1})A) \cong (k[y_1, \dots, y_{r-1}]/(f_1, \dots, f_{j-1})k[y_1, \dots, y_{r-1}])^{\vee(r-d_1-\dots-d_{j-1}-2)}$.

Proof of Claim. We use induction. The case $j = 1$ follows from [6]. Let us consider the case $j \geq 2$. Let us put $R = A/(f_1, \dots, f_{j-2})A$. By the short exact sequence

$$0 \rightarrow R(-d_{j-1}) \rightarrow R \rightarrow R/f_{j-1}R \rightarrow 0$$

and the hypothesis of induction, we get $H_{\mathfrak{m}}^i(R/f_{j-1}R) = 0$ for $i \neq r-j+1, r-j+2, d-j+1$ and the exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^{r-j+1}(R/f_{j-1}R) \rightarrow H_{\mathfrak{m}}^{r-j+2}(R)(-d_{j-1}) \rightarrow H_{\mathfrak{m}}^{r-j+2}(R) \rightarrow H_{\mathfrak{m}}^{r-j+2}(R/f_{j-1}R).$$

Since $H_{\mathfrak{m}}^{r-j+2}(R) \cong (k[y_1, \dots, y_{r-1}]/(f_1, \dots, f_{j-2})k[y_1, \dots, y_{r-1}])^{\vee(r-d_1-\dots-d_{j-2}-2)}$ by induction and f_{j-1} is a non-zero-divisor of $k[y_1, \dots, y_{r-1}]/(f_1, \dots, f_{j-2})k[y_1, \dots, y_{r-1}]$, we have $H_{\mathfrak{m}}^{r-j+2}(R/f_{j-1}R) = 0$ provided $r < d-1$, and $H_{\mathfrak{m}}^{r-j+1}(R/f_{j-1}R) \cong (k[y_1, \dots, y_{r-1}]/(f_1, \dots, f_{j-1})k[y_1, \dots, y_{r-1}])^{\vee(r-d_1-\dots-d_{j-1}-2)}$. Hence the claim is proved.

By the claim we have

$$I(B) = \sum_{i=0}^{d-r} \binom{d-r}{i} \ell_B(H_{\mathfrak{m}}^i(B))$$

$$\begin{aligned}
&= (d-r) \cdot \ell_B(k[y_1, \dots, y_{r-1}]/(f_1, \dots, f_{r-1})k[y_1, \dots, y_{r-1}]) \\
&= (d-r) \cdot \prod_{i=1}^{r-1} d_i.
\end{aligned}$$

Thus we see $\ell_A(A/q)/e_A(q; A) \leq 1 + I(B)/(d \deg(X) \prod_{i=1}^{r-1} d_i) = 1 + (d-r)/\deg(X)$. Hence we proved that $n(A) \leq 1 + (d-r)/\deg(X)$.

Next we will show $n(A) \geq 1 + (d-r)/\deg(X)$. In order to prove this, we take a linear strongly reducing system of parameters z_1, \dots, z_d for A such that z_1, \dots, z_{r-1} is a system of parameters for $k[y_1, \dots, y_{r-1}]$. We set $B = A/(z_1, \dots, z_{r-1})A$, $\mathfrak{q} = (z_1, \dots, z_d)$ and $\bar{\mathfrak{q}} = \mathfrak{q}B$. Note that B is Buchsbaum by our claim. Hence $\ell_B(B/\bar{\mathfrak{q}}) - e_B(\bar{\mathfrak{q}}; B) = I(B)$. Thus we have $\ell_A(A/q)/e_A(q; A) = \ell_B(B/\bar{\mathfrak{q}})/e_B(\bar{\mathfrak{q}}; B) = 1 + I(B)/e_B(\bar{\mathfrak{q}}; B) = 1 + (d-r)/\deg(X)$, where the last equality follows from our claim and Lemma 2. Hence we see $n(A) \geq 1 + (d-r)/\deg(X)$.

Therefore we obtain that $n(A) = 1 + (d-r)/\deg(X)$. This completes the proof of our theorem.

4 Examples

We discuss in conclusion some examples. Following our introduction we want to study a subvariety X with $\deg(X) = \text{codim}(X) + 2$ which is not locally Cohen-Macaulay.

Example 1. Let X be the surface given parametrically by

$$\{s^3, s^2t, stu, su(s-u), u^2(s-u)\}.$$

Let A be the coordinate ring of X . Then we have the following facts: X is a irreducible and reduced surface of degree 4. It follows from [9], (V.5.2) that X is not locally Cohen-Macaulay. Our theorem shows that $\ell_A(A/q) < 2e(q; A)$ for all \mathfrak{m} -primary ideals \mathfrak{q} .

This example gives a subvariety with $n(A) \leq 2$. However, there are subvarieties with $n(A) \geq 2$ which are even arithmetically Buchsbaum of codimension 2. Example 2 describes such varieties which show that Problem 1 of [11] is not true in general.

Example 2. Let k be an infinite field, and $S = k[X_0, \dots, X_n], n \geq 4$. Let E_i be the i -th syzygy S -module of k for $i = 3, \dots, n-1$. By the well-known theorem of Evans and Griffith (see, e.g., [4]) there are prime ideals \mathfrak{p}_i of S of height 2 such that S/\mathfrak{p}_i is a normal domain and we have the following short exact sequence:

$$0 \rightarrow F \rightarrow E_i \rightarrow \mathfrak{p}_i \rightarrow 0$$

with some free S -module F . This exact sequence provides

$$H_m^q(S/\mathfrak{p}_i) \cong \begin{cases} 0 & \text{for } q \neq i-1, n-1 \\ k & \text{for } q = i-1, \end{cases}$$

where $\mathfrak{m} = (X_0, \dots, X_n)$. Hence S/\mathfrak{p}_i is a Buchsbaum ring. We can take a subring $R = k[y_0, \dots, y_{n-2}]$ of S such that S/\mathfrak{p}_i is a maximal Buchsbaum R -module. We set $\mathfrak{n} = (y_0, \dots, y_{n-2})$ in R . By Goto's structure theorem [5], S/\mathfrak{p}_i is isomorphic to the

$(i - 1)$ -th syzygy module of R , say F_{i-1} , as R -module. Since $I_S(S/\mathfrak{p}_i) = I_R(S/\mathfrak{p}_i)$ and $\ell_S((S/\mathfrak{p}_i)/n(S/\mathfrak{p}_i)) = \ell_R((S/\mathfrak{p}_i)/n(S/\mathfrak{p}_i))$, we have only to consider the category of R -module. Now let us calculate $\ell(F_{i-1}/nF_{i-1})$ and $I(F_{i-1})$. By the construction of the syzygy module, $F_{i-1} \otimes R/n$ is an R/n -vector space of rank $(n - 1)!/((i - 1)!(n - i)!)$. Hence we see

$$\ell_R((S/\mathfrak{p}_i)/n(S/\mathfrak{p}_i)) = \ell_R(F_{i-1}/nF_{i-1}) = \binom{n-1}{i-1}.$$

On the other hand

$$I(S/\mathfrak{p}_i) = I(F_{i-1}) = \sum_{j=0}^{n-2} \binom{n-2}{j} \ell_R(H_n^j(F_{j-1})) = \binom{n-2}{i-1}.$$

Note that for Buchsbaum R -module F_{i-1}

$$e(n; F_{i-1}) = \binom{n-1}{i-1} - \binom{n-2}{i-1} = \binom{n-2}{i-2}.$$

Thus we get by Lemma 2:

$$\deg(\mathfrak{p}_i) = e(n; S/\mathfrak{p}_i) = e(n; F_{i-1}) = \binom{n-2}{i-2}.$$

By our theorem, (i), we have

$$n(S/\mathfrak{p}_i) = 1 + \frac{I(S/\mathfrak{p}_i)}{\deg(\mathfrak{p}_i)} = 1 + \frac{\binom{n-2}{i-1}}{\binom{n-2}{i-2}} = \frac{n-1}{i-1}.$$

We apply this result in order to discuss Problem 1 of [11]. This problem is the following: Let X and Y be irreducible and reduced subschemes of \mathbb{P}_k^n . Is then

$$2 \cdot \deg(X) \cdot \deg(Y) \geq \deg(X \cap Y)?$$

Take $X = X_i$ given by the prime ideal \mathfrak{p}_i of S for some i . Let Y be a complete intersection of degree 1 such that $\dim(X \cap Y) = \dim(X) + \dim(Y) - n = -1$. By Lemma 2 and the above calculations we get

$$n(S/\mathfrak{p}_i) \cdot \deg(X) \cdot \deg(Y) = \deg(X \cap Y),$$

where we set $\deg(X \cap Y) = \deg(c(X) \cap c(Y))$. But $n(S/\mathfrak{p}_i) = (n-1)/(i-1) > 2$ provided $n+1 > 2i$. Hence Problem 1 of [11] is not true in general.

References

- [1] Auslander, M. and Buchsbaum, D., Codimension and multiplicity, Ann. Math. 68(1958), 625-657.
- [2] Boda, E. and Vogel, W., On systems of parameters, local intersection multiplicity and Bezout's Theorem, Proc. Amer. Math. Soc. 78(1980), 1-7.

- [3] Bruns, W. and Herzog, J., *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press 1993.
- [4] Evans, E.G.Jr. and Griffith, P.A., *Szygies*, London Math. Soc. Lecture Note Series 106, Cambridge University Press 1985.
- [5] Goto, S., *Maximal Buchsbaum modules over regular local rings and a structure theorem for generalized Cohen-Macaulay modules*, Advanced Studies in Pure Mathematics 11, Commutative Algebra and Combinatorics, pp. 39-64, Kinokuniya-North Holland, 1987.
- [6] Hoa, L.T., Stückrad, J. and Vogel, W., *Towards a structure theory for projective varieties of degree = codimension + 2*, J. Pure Appl. Algebra 71(1991), 203-231.
- [7] Matsumura, H., *Commutative ring theory*, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press 1986.
- [8] Samuel, P., *Méthodes d'algèbre abstraite en géométrie algébrique*, Ergebnisse der Math., Springer 1955.
- [9] Stückrad, J. and Vogel, W., *Buchsbaum rings and applications*, Springer, Berlin, 1986.
- [10] Trung, N.V., *Towards a theory of generalized Cohen-Macaulay modules*, Nagoya Math. J. 102 (1986), 1-49.
- [11] Vogel, W., *New problems concerning the classical length-multiplicity*, New Zealand J. Math., 1995(to appear).
- [12] Zariski, O. and Samuel, P., *Commutative algebra*, vol. I and II, Van Nostrand, Princeton, N.J., 1958 and 1960.

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Incompatible Poisson Structures and Integrable Hamiltonian Systems

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Abstract: Supplemental invariant Poisson structures P_c which are incompatible with the original Poisson structure P_1 are discovered for an arbitrary completely integrable Hamiltonian system. For the non-degenerate case, the complete classification of the invariant Poisson structures P_c is obtained. The instability of the property of compatibility of the supplemental invariant Poisson structure P_2 with P_1 is established. The concepts of dynamical compatibility (DC) and dynamical compatibility in the strong form of pairs of Poisson structures are introduced.

I. In his 1978 paper [4] Magri proved, using the Lenard scheme, a general Theorem that states that a dynamical system or system of partial differential equations that preserves two compatible non-degenerate Poisson structures (the bi-Hamiltonian system) is completely integrable in the Liouville sense. Since then more than one hundred papers and several books were published devoted to the investigation of the diverse properties of compatible pairs of Poisson structures and bi-Hamiltonian systems. A review of these papers and their extended bibliography is contained in monograph [6].

One of the well-known unsolved problems in this area is

The Inverse Problem. *Are there supplemental invariant Poisson structures for a general completely integrable Hamiltonian system on a manifold M^n , $n = 2k$, with a non-degenerate Poisson structure P_1 ? Are they necessarily compatible with P_1 ?*

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In the present paper we solve this problem. We derive the general and previously unknown formula

$$\omega_c = d\left(\frac{\partial B(J)}{\partial J_\alpha}\right) \wedge d\varphi_\alpha + df_\alpha(I) \wedge dI_\alpha, \quad (1)$$

$$J_\alpha = \frac{\partial H(I)}{\partial I_\alpha}, \quad \alpha = 1, \dots, k \quad (2)$$

in the action-angle coordinates $I_1, \dots, I_k, \varphi_1, \dots, \varphi_k$. Here $B(J)$ and $f_\alpha(I)$ are arbitrary smooth functions of k variables. Formula (1) presents a continuum of the invariant closed 2-forms ω_c and the invariant Poisson structures $P_c = \omega_c^{-1}$ for an arbitrary completely integrable Hamiltonian system with the Hamiltonian function H .

In general the constructed Poisson structures P_c are incompatible with the original Poisson structure $P_1 = \omega_1^{-1}$, $\omega_1 = dI_\alpha \wedge d\varphi_\alpha$. Magri's definition [4] of compatibility states: Two Poisson structures P_1 and P_2 are compatible if their sum $P_1 + P_2$ is a Poisson structure. This definition is equivalent to the condition that the Schouten bracket $[P_1, P_2]$ vanishes. Later Gelfand and Dorfman in [3] and Magri and Morosi in [5] proved that the compatibility is equivalent to the condition that the Nijenhuis tensor N_A vanishes where A is the (1,1) tensor $A = P_1 P_2^{-1}$. Therefore we prove the incompatibility of the Poisson structures P_c and P_1 by a direct calculation of the non-zero components $N_{A^{ij}l}^i$ of the Nijenhuis tensor in the action-angle coordinates.

II. The second well-known unsolved problem in the theory of compatible Poisson structures is

The Stability Problem. *Is the property of compatibility with P_1 stable for the supplemental non-degenerate Poisson structure P_2 that is invariant with respect to the completely integrable non-degenerate Hamiltonian system?*

In Theorem 2, we prove that the compatibility property is unstable. Using the key formula (1) and a method of "toroidal surgeries" we construct a continuum of invariant Poisson structures P_C in any neighbourhood of P_2 which are incompatible with P_1 .

III. Let P_1^{ij} be a non-degenerate Poisson structure on a manifold M^n , $n = 2k$. A Hamiltonian system

$$\dot{x}^i = P_1^{i\alpha} H_{,\alpha} \quad , \quad H_{,\alpha} = \partial H / \partial x^\alpha \quad (3)$$

is called completely integrable in the Liouville's sense if it has $k = n/2$ independent involutive first integrals $F_1(x), \dots, F_k(x)$:

$$\{F_j, F_\ell\} = P_1^{\alpha\beta} F_{j,\alpha} F_{\ell,\beta} = 0. \quad (4)$$

The summation with respect to the repeated indices is taken everywhere in this paper.

The Liouville Theorem [1] implies that almost all points of the manifold M^n (excluding a set $S \subset M^n$, $\dim S \leq n - 1$) are covered by a system of open toroidal domains $\mathcal{O}_m \subset M^n$ with the action-angle coordinates $I_1, \dots, I_k, \varphi_1, \dots, \varphi_k$. In these coordinates the completely integrable system (3) has the form

$$\dot{I}_j = 0, \quad \dot{\varphi}_j = \frac{\partial H}{\partial I_j}, \quad H = H(I_1, \dots, I_k). \quad (5)$$

The symplectic structure ω_1 has the canonical form $\omega_1 = dI_\alpha \wedge d\varphi_\alpha$. The Hamiltonian system (5) preserves the symplectic structure ω_1 and the Poisson structure $P_1 = \omega_1^{-1}$: $\dot{\omega}_1 = 0$, $\dot{P}_1 = 0$.

The action coordinates I_1, \dots, I_k are defined in a ball

$$B_r: \sum_{j=1}^k (I_j - I_{j0})^2 < r^2. \quad (6)$$

The angle coordinates $\varphi_1, \dots, \varphi_k$ run over a torus \mathbb{T}^k , $0 \leq \varphi_j \leq 2\pi$, in the compact case or over a toroidal cylinder $\mathbb{T}^m \times \mathbb{R}^{k-m}$, $0 \leq m < k$ if the manifold $I_j(u) = I_{j0}$ is non-compact.

IV. Let us consider a completely integrable Hamiltonian system (3) for which the submanifolds of constant level of the k involutive first integrals are compact. Almost all of these invariant submanifolds are tori \mathbb{T}^k :

$$\mathbb{T}^k: \quad I_1 = c_1, \dots, \quad I_k = c_k, \quad 0 \leq \varphi_i \leq 2\pi. \quad (7)$$

The completely integrable Hamiltonian system (3), (5) is called non-degenerate if the condition for the Hessian

$$\det \left\| \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta} \right\| \neq 0 \quad (8)$$

is met almost everywhere in the action-angle coordinates

$$I_1, \dots, I_k, \quad \varphi_1, \dots, \varphi_k, \quad \varphi_i = \varphi_i \bmod(2\pi). \quad (9)$$

In this paper we study the problem of the existence of supplemental invariant non-degenerate Poisson structures for the completely integrable Hamiltonian system (3). This problem is equivalent to the investigation of all closed 2-forms ω_i which are invariant with respect to the system (3).

Theorem 1 *In the toroidal domain $\mathcal{O} \subset M^n$ defined by conditions (6) and (7) a closed 2-form ω_c is invariant with respect to the completely integrable non-degenerate Hamiltonian system (3), (5) having compact invariant submanifolds (7) if and only if it has the form (1), where $B(J_1, \dots, J_k)$ and $f_\alpha(I_1, \dots, I_k)$ are arbitrary functions of k arguments.*

Let us prove that any 2-form ω_c (1) is preserved by the Hamiltonian system (3), (5). Using classical properties of the Lie derivative $L_V \omega_c = \dot{\omega}_c$ with respect to the dynamical system (5) and substituting (2) we obtain

$$\dot{\omega}_c = d\left(\frac{\partial B(J)}{\partial J_\alpha}\right) \wedge d\left(\frac{\partial H}{\partial I_\alpha}\right) = \frac{\partial^2 B(J)}{\partial J_\alpha \partial J_\beta} dJ_\beta \wedge dJ_\alpha = 0. \quad (10)$$

Therefore all 2-forms ω_c (1) are invariant with respect to the completely integrable Hamiltonian system (3), (5).

The proof of the uniqueness of the invariant closed 2-forms (1) is presented in our paper [2].

The invariant 2-form ω_c (1) is non-degenerate if and only if

$$\det \left\| \frac{\partial^2 B(J)}{\partial J_\alpha \partial J_\beta} \right\| \neq 0, \quad \det \left\| \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta} \right\| \neq 0. \quad (11)$$

V. Theorem 1 implies that in the action-angle coordinates the recursion operator $A = P_1 P_c^{-1}$ has the $k \times k$ block structure

$$A = \begin{pmatrix} B^t & 0 \\ \sigma & B \end{pmatrix} \quad (12)$$

with the following entries (here and below we assume $1 \leq i, j, \ell, m \leq k$):

$$A_j^i = B_i^j(I), \quad A_{j+k}^i = 0, \quad A_j^{i+k} = \sigma_{ij}(I), \quad A_{j+k}^{i+k} = B_j^i(I), \quad (13)$$

$$B_{\ell}^i(I) = \frac{\partial B_{\ell}(I)}{\partial I_i}, \quad B_{\ell}(I) = \frac{\partial B(J(I))}{\partial J_{\ell}}, \quad \sigma_{ij}(I) = f_{j,i} - f_{i,j}.$$

For any (1,1) tensor A_{β}^{α} the Nijenhuis tensor $N_{\beta\gamma}^{\alpha}$ is defined by the formula

$$N_{\beta\gamma}^{\alpha} = A_{\gamma,\tau}^{\alpha} A_{\beta}^{\tau} - A_{\beta,\tau}^{\alpha} A_{\gamma}^{\tau} + (A_{\beta,\gamma}^{\tau} - A_{\gamma,\beta}^{\tau}) A_{\tau}^{\alpha}. \quad (14)$$

After substituting formulae (13) into (14) the compatibility condition [3,5] $N_A(u, v) = 0$ yields the system of partial differential equations

$$N_{j+k,\ell}^{i+k} = \frac{\partial^2 B_j(I)}{\partial I_{\ell} \partial I_m} \frac{\partial B_m(I)}{\partial I_i} - \frac{\partial^2 B_j(I)}{\partial I_i \partial I_m} \frac{\partial B_m(I)}{\partial I_{\ell}} = 0. \quad (15)$$

If function $B(J(I))$ is generic then equations (15) are not satisfied and therefore the supplemental invariant Poisson structure $P_c = \omega_c^{-1}$ (1) is incompatible with the original Poisson structure P_1 .

VI. Assume that a completely integrable non-degenerate Hamiltonian system (3) is given on a symplectic manifold M^n , $n = 2k$, with a symplectic form ω_1 and Poisson structure $P_1 = \omega_1^{-1}$. Assume there exists a second non-degenerate Poisson structure P_2 that is invariant with respect to the dynamical system (3) and is compatible with the original Poisson structure $P_1 = \omega_1^{-1}$. The following theorem proves that for the supplemental invariant Poisson structure P_2 the property of compatibility with P_1 is unstable.

Theorem 2 *In any neighbourhood of the Poisson structure P_2 there exists a non-degenerate Poisson structure P_C that is incompatible with the Poisson structure P_1 and invariant with respect to the Hamiltonian system (3).*

The proof is based on formula (1) and on a "toroidal surgeries" method that extends the symplectic structure (1) on the whole manifold M^n . The proof is presented in our paper [2].

VII. These results show that the notion of the compatibility of Poisson structures and its counterpart, incompatibility, are conceptually inadequate for a good insight into the diversity of the Poisson structure pairs. Therefore we introduce the following

Definition 1 *Two Poisson structures P_1 and P_2 on a manifold M^n are called dynamically compatible (DC) if there exists a non-trivial dynamical system that preserves both of them.*

Any two Poisson structures which are compatible in the Magri sense are dynamically compatible. But two dynamically compatible Poisson structures P_1 and P_2 are not compatible if the corresponding Nijenhuis tensor N_A is not equal to zero.

Definition 2 *Two Poisson structures P_1 and P_2 on a manifold M^n , $n = 2k$, are called dynamically compatible in the strong form if there exists a dynamical system that preserves both of them and is completely integrable and non-degenerate with respect to some non-degenerate Poisson structure P , and its invariant submanifolds are compact.*

Theorem 1 implies that all constructed invariant incompatible Poisson structures $P_c = \omega_c^{-1}$ are mutually dynamically compatible in the strong form.

References

- [1] Arnold, V. I.: *Mathematical methods of classical mechanics*. Berlin, Heidelberg, New York: Springer Verlag, 1978
- [2] Bogoyavlenskij, O. I.: *Theory of tensor invariants of integrable Hamiltonian systems. I. Incompatible Poisson structures*, to appear (1995)
- [3] Gelfand, I. M., and Dorfman, I. Ya.: *Hamiltonian operators and algebraic structures related to them*. *Funct. Anal. and Applic.* **13**, 248-262 (1979)
- [4] Magri, F.: *A simple model of an integrable Hamiltonian system*. *J. Math. Phys.* **19**, 1156-1162 (1978)
- [5] Magri, F., and Morosi, C.: *A geometrical characterization of integrable Hamiltonian systems through the theory of the Poisson-Nijenhuis manifolds*. *Quaderno S 19*, Universita di Milano (1984)
- [6] Olver, P. J.: *Applications of Lie groups to differential equations*. Berlin, Heidelberg, New York: Springer Verlag, 1993

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Lie Algebraic Invariants of Two Compatible Poisson Structures

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Abstract: An alternating (1,2) tensor is introduced for an arbitrary (1,1) tensor A having multiple eigenvalues. For a pair of compatible Poisson structures and the corresponding recursion operator A , this (1,2) tensor satisfies the Jacobi identity and defines a structure of solvable Lie algebra in each tangent space $T_x(M^n)$. Formulae for the commutator relations and the corresponding Cartan-Killing form are presented for a generic pair of compatible Poisson structures.

I. During the last two decades more than one hundred papers and several books were published devoted to the investigation of diverse properties of compatible pairs of Poisson structures and their applications in mathematical physics. Magri's definition [4] of compatibility states: Two Poisson structures P_1 and P_2 are compatible if their sum $P_1 + P_2$ is a Poisson structure. This definition is equivalent to the condition that the Schouten bracket $[P_1, P_2]$ vanishes. Later Gelfand & Dorfman in [3] and Magri & Morosi in [5] proved that the compatibility is equivalent to the condition that the Nijenhuis tensor N_A vanishes where A is the (1,1) tensor $A = P_1 P_2^{-1}$. These results show that for two compatible Poisson structures the main tensors which can be constructed from P_1 and P_2 are identically equal to zero.

One of the well-known unsolved problems in this area is

The Tensor Invariants Problem. *To find non-trivial tensor invariants of two compatible Poisson structures.*

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In this paper we introduce a new alternating (1,2) tensor $B_C(u, v)$. For two arbitrary compatible Poisson structures P_1 and P_2 this tensor defines a structure of Lie algebra \mathcal{A}_x in each tangent space $T_x(M^n)$, $n = 2k$. We prove that these Lie algebras always are solvable and present concrete non-trivial examples.

II. We proved in [2] that the (1,1) tensor

$$A_j^i(x) = P_1(x)^{i\alpha}(P_2^{-1}(x))_{\alpha j} \quad (1)$$

for two arbitrary Poisson structures P_1 and P_2 satisfies the algebraic equation of degree $n/2$:

$$Q(A, x) = \sum_{m=0}^{n/2} q_m(x) A^m(x) = 0, \quad (2)$$

$$Q(\lambda, x) = \text{Pf}(P_1(x) - \lambda P_2(x)) \text{Pf}(P_2^{-1}(x)).$$

Here polynomial $\text{Pf } S$ is the classical Pfaffian of an arbitrary skew-symmetric matrix S : $\det S = (\text{Pf } S)^2$. The characteristic polynomial $P(\lambda, x)$ of the (1,1) tensor $A_j^i(x)$ (1) is a perfect square $P(\lambda, x) = Q^2(\lambda, x)$.

Let $C(\lambda, x)$ be an arbitrary polynomial-valued function on the manifold M^n that has the same roots as the characteristic polynomial $P(\lambda, x)$ of the (1,1) tensor $A(x)$ and annihilates the operators $A(x)$

$$C(\lambda, x) = \sum_{\ell=0}^r c_\ell(x) \lambda^\ell, \quad c_r = 1, \quad C(A(x), x) = 0. \quad (3)$$

Obviously the minimal polynomial $m(\lambda, x)$ of $A(x)$ is a divisor of the polynomial $C(\lambda, x)$. The coefficients $c_\ell(x)$ are functions of the eigenvalues $\lambda_i(x)$ of the operator $A(x)$ or equivalently are functions of the invariants $f_\ell(x)$:

$$f_m(x) = \frac{1}{m} \text{Tr } A^m(x) = \frac{1}{m} \sum_{\ell=1}^n \lambda_\ell^m, \quad dc_\ell = \sum_{m=1}^n \frac{\partial c_\ell}{\partial f_m} df_m. \quad (4)$$

The integrable distribution $\mathcal{L}_x \subset T_x(M^n)$ is defined by two equivalent systems of equations

$$\mathcal{L}_x : \quad df_m = 0, \quad m = 1, \dots, n; \quad d\lambda_i = 0, \quad i = 1, \dots, n. \quad (5)$$

We introduce the following alternating (1,2) tensor

$$B_C(u, v) = \sum_{m=2}^r c_m(x) \sum_{p+q+s=m-2} A^p N_A(A^q u, A^s v) + \quad (6)$$

$$+ \sum_{m=0}^r (v(c_m) A^m u - u(c_m) A^m v).$$

This tensor plays a crucial role in the solution of the following problems:

- 1) Algebraic identities for the Nijenhuis tensor $N_A(u, v)$.
- 2) Necessary conditions for the existence of conservation laws for a quasi-linear first order system of n partial differential equations.
- 3) Necessary conditions for the dynamical compatibility of two incompatible Poisson structures.
- 4) Tensor invariants of two compatible Poisson structures.

III. The first three problems are investigated in our papers [1,2]. In the present paper we study the fourth problem.

Theorem 1 *If the Nijenhuis tensor $N_A(u, v)$ vanishes then the (1,2) tensor $B_C(u, v)$ (6) defines a structure of solvable Lie algebra \mathcal{A}_x in each tangent space $T_x(M^n)$. The derivative Lie subalgebra \mathcal{A}'_x belongs to the subspace \mathcal{L}_x (5). This subspace is a commutative ideal in \mathcal{A}_x that is invariant with respect to the (1,1) tensor $A(x)$. Any linear subspace $\mathcal{E}_x \subset T_x(M^n)$ that is invariant with respect to the operator $A(x)$ is a Lie subalgebra in \mathcal{L}_x . This subalgebra is commutative if either $\mathcal{E}_x \subset \mathcal{L}_x$ or $\mathcal{E}_x \cap \mathcal{L}_x = \emptyset$.*

Proof. Let us prove that the Jacobi identity

$$B_C(B_C(u, v), w) + B_C(B_C(v, w), u) + B_C(B_C(w, u), v) = 0 \quad (7)$$

is satisfied for arbitrary tangent vectors $u, v, w \in T_x(M^n)$. Applying Theorem 2 of our paper [2] and the compatibility condition $N_A(u, v) = 0$ we obtain

$$df_m(B_C(u, v)) = 0, \quad dc_l(B_C(u, v)) = 0, \quad m, l \geq 1, \quad u, v \in T_x(M^n). \quad (8)$$

In view of $N_A(u, v) = 0$, the (1,2) tensor $B_C(u, v)$ (6) has the form

$$B_C(u, v) = \sum_{m=0}^r (v(c_m) A^m u - u(c_m) A^m v). \quad (9)$$

This expression and formulae (8) yield

$$\begin{aligned} B_C(B_C(u, v), w) &= \sum_{m=0}^r (w(c_m)A^m B_C(u, v) - B_C(u, v)(c_m)A^m w) = (10) \\ &= \sum_{\ell, m=0}^r (v(c_\ell)w(c_m)A^{m+\ell}u - u(c_\ell)w(c_m)A^{m+\ell}v). \end{aligned}$$

The Jacobi identity (7) follows after the substitution of the formulae (10) and the similar terms cancellation. Therefore the (1,2) tensor $B_C(u, v)$ (6) defines a structure of Lie algebra \mathcal{A}_x in each tangent space $T_x(M^n)$.

The key relations (8) prove that

$$B_C(u, v) \in \mathcal{L}_x, \quad \text{or} \quad \mathcal{A}'_x \subset \mathcal{L}_x. \quad (11)$$

If two vectors \tilde{u}, \tilde{v} belong to \mathcal{L}_x then formulae (4) and (5) yield

$$\tilde{u}(c_\ell) = dc_\ell(\tilde{u}) = 0, \quad \tilde{v}(c_\ell) = dc_\ell(\tilde{v}) = 0, \quad \ell = 0, \dots, r. \quad (12)$$

Substituting (12) into formula (9) we get $B_C(\tilde{u}, \tilde{v}) = 0$. Therefore subspace \mathcal{L}_x is a commutative ideal in the Lie algebra \mathcal{A}_x . Hence we obtain

$$\mathcal{A}''_x = [\mathcal{A}'_x, \mathcal{A}'_x] = 0. \quad (13)$$

That means that the Lie algebra \mathcal{A}_x is solvable.

For $N_A(u, v) = 0$, the conservation laws $Adf_m = f_{m+1}$ [7] imply

$$df_m(A\tilde{u}) = (Adf_m)(\tilde{u}) = df_{m+1}(\tilde{u}) = 0 \quad (14)$$

for any tangent vector $\tilde{u} \in \mathcal{L}_x$. Therefore the subspace \mathcal{L}_x is invariant with respect to the operator $A(x)$.

If a linear subspace $\mathcal{E}_x \subset T_x(M^n)$ is invariant with respect to the operator $A(x)$ then for any vectors $\tilde{u}, \tilde{v} \in \mathcal{E}_x$ formula (9) implies $B_C(\tilde{u}, \tilde{v}) \in \mathcal{E}_x$. Therefore the \mathcal{E}_x is a Lie subalgebra in \mathcal{A}_x . Using (11) we obtain $B_C(\tilde{u}, \tilde{v}) \in \mathcal{E}_x \cap \mathcal{L}_x$. Hence $\mathcal{E}'_x \subset \mathcal{E}_x \cap \mathcal{L}_x$ and the Lie subalgebra \mathcal{E}_x is commutative when $\mathcal{E}_x \cap \mathcal{L}_x = \emptyset$.

Corollary 1 Any two compatible Poisson structures P_1^{ij} and P_2^{ij} define by virtue of the (1,2) tensor $B_Q(u, v)$ a structure of solvable Lie algebra \mathcal{A}_x

in each tangent space $T_x(M^n)$. Any bi-Hamiltonian flow that preserves the two Poisson structures preserves the (1,2) tensor $B_Q(u, v)$ also. Therefore for any trajectory $x(t)$ of a bi-Hamiltonian flow the Lie algebras $\mathcal{A}_{x(t)}$ are isomorphic to each other.

Remark 1. Theorem 1 describes a new way of the appearance of the finite-dimensional Lie algebras in problems of differential geometry and mathematical physics: the (1,2) tensor $B_C(u, v)$ defines a structure of solvable Lie algebra in each tangent space $T_x(M^n)$. These Lie algebras are not connected with any symmetry of the initial problem in contrast to the classical symmetry-based Lie algebraic constructions of mathematical physics.

IV. Assume that for two compatible Poisson structures all eigenvalues of the recursion operator $A = P_1 P_2^{-1}$ are real and doubly degenerate. In view of the compatibility condition $N_A(u, v) = 0$ the Nijenhuis Theorem [6] implies that the local coordinates $x^1, y^1, \dots, x^k, y^k$ exist where the (1,1) tensor A_j^i has the form

$$A_j^i = \text{diag}(\lambda_1(x^1), \lambda_1(x^1), \dots, \lambda_k(x^k), \lambda_k(x^k)), \quad 2k = n. \quad (15)$$

Polynomial (2) for the (1,1) tensor A_j^i (15) has the form

$$Q(\lambda, x) = \prod_{i=1}^k (\lambda - \lambda_i(x_i)) = \sum_{m=0}^k q_m(x) \lambda^m. \quad (16)$$

In view of $N_A(u, v) = 0$, the (1,2) tensor $B_Q(u, v)$ (6) takes the form

$$B_Q(u, v) = \sum_{m=0}^k (v(q_m) A^m u - u(q_m) A^m v). \quad (17)$$

The formulae (15) - (17) imply

$$B_Q\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \delta_j^i Q'(\lambda_i) \lambda_i' \frac{\partial}{\partial y^i}, \quad \lambda_i' = \frac{d\lambda_i(x^i)}{dx^i}, \quad (18)$$

$$B_Q\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0, \quad B_Q\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

Formulae (18) provide an independent proof that the (1,2) tensor $B_Q(u, v)$ defines the structures of non-trivial solvable Lie algebras \mathcal{A}_x in the tangent

spaces $T_x(M^n)$. These Lie algebras are split into direct sums of the two-dimensional solvable Lie subalgebras in the planes spanned by vectors $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}$.

Using formulae (18) we readily find the corresponding Cartan-Killing form

$$g(u, v) = \text{Tr ad}_u \text{ad}_v = \sum_{i=1}^k (Q'(\lambda_i))^2 d\lambda_i(u) d\lambda_i(v). \quad (19)$$

Obviously all tangent vectors $u \in \mathcal{L}_x$ are null-vectors of this symmetric form.

The solvable Lie algebras \mathcal{A}_x defined by virtue of the (1,2) tensor $B_Q(u, v)$ have more complicated structure when the (1,1) tensor A has complex eigenvalues or when its Jordan normal form is non-diagonal. The tensor invariants $B_Q(u, v)$ and $g(u, v)$ introduced in the present paper can be applied for the classification of the non-isomorphic compatible pairs of Poisson structures.

References

- [1] Bogoyavlenskij, O. I.: Necessary conditions for existence of conservation laws, 75-91, In: Proceedings of the 2-nd Gaussian Symposia, Munich, 1993.
- [2] Bogoyavlenskij, O. I.: Algebraic identities for the Nijenhuis tensors and their applications, to appear (1995)
- [3] Gelfand, I. M., & Dorfman, I. Ya.: Hamiltonian operators and algebraic structures related to them. *Funct. Anal. and Applic.* **13**, 248-262 (1979)
- [4] Magri, F.: A simple model of an integrable Hamiltonian system. *J. Math. Phys.* **19**, 1156-1162 (1978)
- [5] Magri, F., & Morosi, C.: A geometrical characterization of integrable Hamiltonian systems through the theory of the Poisson-Nijenhuis manifolds. *Quaderno S 19*, Universita di Milano (1984)
- [6] Nijenhuis, A.: X_{n-1} -forming sets of eigenvectors. *Proc. Kon. Ned. Akad. Amsterdam* **54**, 200-212 (1951)
- [7] Stone, A.P.: Generalized conservation laws. *Proc. of Am. Math. Soc.* **18**, 868-873, (1967)

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The Dyer-Lashof Algebra in Bordism (extended abstract)

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We present a theory of Dyer-Lashof operations in unoriented bordism and investigate their behavior on the canonical splitting $N_*(X) \simeq N_* \otimes H_*(X)$, where $N_*()$ is unoriented bordism and $H_*()$ is homology mod 2. For any finite covering space we define a "polynomial functor" from the category of topological spaces to itself. If the covering space is a closed manifold we obtain an operation defined on the bordism of any E_∞ -space. A certain sequence of operations called squaring operations are defined from two-fold coverings; they satisfy the Cartan formula and also a generalization of the Adem relations that is formulated by using Lubin's theory of isogenies of formal group laws. We call a ring equipped with such a sequence of squaring operations a *D-ring*, and observe that the bordism ring of any free E_∞ -space is free as a *D-ring*. In particular, the bordism ring of finite covering manifolds is the free *D-ring* on one generator. In a second compte-rendu we discuss the (Nishida) relations between the Landweber-Novikov and the Dyer-Lashof operations, and show how to represent the Dyer-Lashof operations in terms of their actions on the characteristic numbers of manifolds.

1. The algebra of covering manifolds.

We begin with the observation that a covering space $p : T \rightarrow B$ can be used to define a functor $X \mapsto p(X)$ from the category of topological spaces to itself, where

$$p(X) = \{(u, b) \mid b \in B, u : p^{-1}(b) \rightarrow X\}.$$

Then $p(X)$ is the total space of a bundle over B with fibers $X^{p^{-1}(b)}$, and any continuous map $f : X \rightarrow Y$ induces a continuous map $p(f) : p(X) \rightarrow p(Y)$. We shall say that $p()$ is a *polynomial functor*. For functors F and G from the category of topological spaces to itself, we have functors $F + G$, $F \times G$ and $F \circ G$ given by $(F + G)(X) = F(X) + G(X)$, $(F \times G)(X) = F(X) \times G(X)$, and $(F \circ G)(X) = F(G(X))$. Polynomial functors happen to be closed under these operations, and we obtain well-defined operations $p + q$, $p \times q$ and $p \circ q$ on coverings. These operations satisfy the kinds of identities that one should expect for an algebra of polynomials.

We define the *derivative* p' of a covering $p : T \rightarrow B$ to be the covering whose base space is T and whose fiber over $t \in T$ is the set $p^{-1}(p(t)) - \{t\}$. The rules of differential calculus apply: $(p + q)' = p' + q'$, $(p \times q)' = p' \times q + p \times q'$ and $(p \circ q)' = (p' \circ q) \times q'$. If we observe that the total space of p is $p'(1)$ (where 1 denotes a single point) and that its base space is $p(1)$ the formula $(p \times q)'(1) = p'(1) \times q(1) + p(1) \times q'(1)$ expresses the total space of $(p \times q)$ in terms of the total and based spaces of p and q . Similarly for the formula $(p \circ q)'(1) = p'(q(1)) \times q'(1)$.

Remark 1: There is a parallel between this algebra of covering spaces and the algebra of combinatorial species developed in [9] and [10].

Remark 2: By using the Euler-Poincaré characteristic one can associate a polynomial $\chi(p)$ to any covering p of a finite complex. We have $\chi(p + q) = \chi(p) + \chi(q)$, $\chi(p \times q) = \chi(p) \times \chi(q)$, $\chi(p \circ q) = \chi(p) \circ \chi(q)$, and $\chi(p') = \chi(p)'$.

Remark 3: It is also possible to define various kinds of higher differential operators on coverings. For example, the group Σ_2 acts on any second derivative p'' by permuting the order of differentiation, and we can define

$$\frac{1}{2!} \frac{d^2 p}{dx^2} = p'' / \Sigma_2.$$

Higher divided derivatives can be handled similarly.

Remark 4: Polynomial functors of n variables are easily defined. They are obtained from n -tuples (p_1, \dots, p_n) where $p_i : T_i \rightarrow B$ is a finite covering for every i .

Let us now consider coverings of smooth compact manifolds. We say that two coverings of closed manifolds are *cobordant* if together they form the boundary of a covering. Let $N_*\Sigma$ denote the set of cobordism classes of closed coverings. Let $N_d\Sigma_n$ denote the set of cobordism classes of degree n (i.e. n -fold) coverings over closed manifolds of dimension d .

Proposition 1. *The operations of sum $+$, product \times , and composition \circ are compatible with the cobordism relation on closed coverings. They define on $N_*\Sigma$ the structure of a commutative \mathbb{Z}_2 algebra, graded by dimension.*

Notice that if $p \in N_k\Sigma_m$ and $q \in N_r\Sigma_n$ then $p \circ q \in N_{mr+k}\Sigma_{mn}$. This defines in particular an action of $N_*\Sigma$ on $N_*\Sigma_0 = N_*$. More generally, let us see that $N_*\Sigma$ acts on the bordism ring of any E_∞ -space.

Recall (see [1], [18]) that an E_∞ -space X has structure maps $E\Sigma_n \times_{\Sigma_n} X^n \rightarrow X$ for each n . These structure maps give rise to structure maps $p(X) \rightarrow X$ for every degree n covering space $p : T \rightarrow B$. To see this it suffices to express p as a pull back of the tautological n -fold covering u_n of $B\Sigma_n$ along some map $B \rightarrow B\Sigma_n$. This furnishes a map $p(X) \rightarrow u_n(X) = E\Sigma_n \times_{\Sigma_n} X^n$ and the structure map $p(X) \rightarrow X$ is then obtained by composing with $u_n(X) \rightarrow X$.

Recall (see [6] for instance) that an element of N_*X is the bordism class of a pair (M, f) where $f : M \rightarrow X$ and M is a compact manifold; then $p(M)$ is a compact manifold and the structure map for X gives $p(M) \rightarrow p(X) \rightarrow X$, representing an element in N_*X .

Proposition 2. *Let X be an E_∞ -space. Each covering of degree n and dimension d defines an operation $N_m X \rightarrow N_{nm+d} X$. Cobordant covering spaces give the same operation. Moreover, for double coverings these operations are additive.*

It should be noted that tom Dieck [7] and Alliston [3] develop bordism Dyer-Lashof operations which agree with ours; the relationship will be clearer after section 2.

Example: The classifying space for finite coverings is $B\Sigma_*$, the disjoint union of the classifying spaces of the symmetric groups $B\Sigma_n$. Then $N_*B\Sigma_* = N_*\Sigma$ and $B\Sigma_*$ has a natural E_∞ -space structure defined from disjoint sum. The covering operations on $N_*B\Sigma_*$ correspond to composition of coverings.

Remark: It is a classical result [19], [8], [12] that the inclusion $i : \Sigma_{n-1} \subset \Sigma_n$ defines a *split monomorphism* $i_* : N_*\Sigma_{n-1} \rightarrow N_*\Sigma_n$. In our setting i_* is the map $p \mapsto x \times p$. It is easy to see, by applying the rules of differential calculus, that the map

$$q \mapsto \frac{dq}{dx} + x \frac{1}{2!} \frac{d^2q}{dx^2} + x^2 \frac{1}{3!} \frac{d^3q}{dx^3} + \dots$$

is a splitting [11].

For any space X let $\epsilon : N_*(X) \rightarrow H_*(X)$ denote the Thom reduction, where $H_*(\)$ is mod 2 homology. If $(M, f) \in N_*(X)$ we have $\epsilon(M, f) = f_*(\mu_M)$ where μ_M denotes the fundamental homology class of M . If X is an E_∞ -space then each covering of degree n and dimension d defines an operation $H_n X \rightarrow H_{n+d} X$ which is the Thom reduction of the corresponding operation in bordism.

We now describe the sequence of cobordism class of double coverings that leads to the concept of D -rings. It is a classical result that $N^*(RP^\infty) = N_*[[t]]$. Let q_k in $N_*B\Sigma_2 = N_*(RP^\infty)$ be represented by the canonical inclusion $RP^k \hookrightarrow RP^\infty$. The sequence q_0, q_1, \dots is a basis of the N_* -module $N_*(RP^\infty)$. The Kronecker pairing $N^*(RP^\infty) \times N_*(RP^\infty) \rightarrow N_*$ defines an exact duality between $N^*(RP^\infty)$ and $N_*(RP^\infty)$. Let d_0, d_1, \dots be the basis dual to the basis t^0, t^1, t^2, \dots under the Kronecker pairing. The relation between the two bases of $N_*(RP^\infty)$ can be expressed as an equality of generating series

$$\left(\sum_{i \geq 0} [RP^i] x^i \right) \left(\sum_{k \geq 0} d_k x^k \right) = \left(\sum_{n \geq 0} q_n x^n \right),$$

where x is a formal indeterminate. We have $d_0 = q_0$, and $d_1 = q_1$ since $[RP^0] = 1$ and $[RP^1] = 0$. It turns out (see [2] for instance) that d_n can be represented by the Milnor hypersurface $H(n, 1) \hookrightarrow RP^n \times RP^1 \rightarrow RP^n$. The coverings d_n and q_n give operations which are distinct in bordism but agree in mod 2 homology.

2. D -rings and Dyer-Lashof operations

Recall that a formal group law over a commutative ring R is a formal power series $F(x, y) \in R[[x, y]]$ which satisfies identities corresponding to associativity and unit; (see Quillen [21] or Lazard [13] for instance). We say that a formal group law F has order two if $F(x, x) = 0$.

The Lazard ring (for formal group laws of order two) is the commutative ring with generators $a_{i,j}$ and relations making $F(x, y) = \sum a_{i,j} x^i y^j$ a formal group law of order two. Let us temporarily denote this Lazard ring by L . Then for any ring R and any formal group law $G(x, y) \in R[[x, y]]$ of order two there is a unique ring homomorphism $\phi : L \rightarrow R$ such that $(\phi F)(x, y) = G(x, y)$. Quillen [21] showed that L is naturally isomorphic to $N_* = N_*(pt)$. This provides a beautiful interpretation of Thom's original calculation of the unoriented cobordism ring.

Let R be a commutative ring and let $F \in R[[x, y]]$ be a formal group law of order two (this implies that R is a \mathbb{Z}_2 -algebra). According to Lubin [14] there exists a unique formal group law F_t defined over $R[[t]]$ such that $h_t(x) = xF(x, t)$ is a morphism $h_t : F \rightarrow F_t$. The

kernel of h_t is $\{0, t\}$, which is a group under the F -addition $x +_F y = F(x, y)$. We will refer to F_t as the *Lubin quotient* of F by $\{0, t\}$ and to h_t as the *isogeny*. The construction can be iterated and a Lubin quotient $F_{t,s}$ of F_t can be obtained by further killing $h_t(s) \in R[[t, s]]$. The composite isogeny $F \rightarrow F_t \rightarrow F_{t,s}$ is

$$h_{t,s}(x) = h_t(x)F_t(h_t(x), h_t(s)) = xF(x, t)F(x, s)F(x, F(s, t))$$

Its kernel consists of $\{0, t, s, F(s, t)\}$, which is an elementary abelian 2-group under the F -addition. By doing the construction in a different order we obtain $F_{s,t}$ but it turns out that $F_{t,s} = F_{s,t}$.

Definition: A D -ring is a commutative ring R together with a formal group law of order two F defined over R and a ring homomorphism $D_t : R \rightarrow R[[t]]$ called the *total square*, satisfying the following conditions:

- i) $D_0(a) = a^2$ for every a in R ;
- ii) $D_t(F) = F_t$;
- iii) $D_t \circ D_s$ is symmetric in t and s . Here we have extended $D_t : R \rightarrow R[[t]]$ to $D_t : R[[s]] \rightarrow R[[s, t]]$ by defining $D_t(s) = h_t(s) = sF(s, t)$.

A *morphism* of D -rings is a ring homomorphism which preserves the formal group laws and the total squares. A D -ring is also an algebra over the Lazard ring N_* , and a morphism of D -rings is a morphism of N_* -algebras.

A D -ring is *graded* if R is graded and F is homogeneous in grade -1 and $D_t(x)$ has grading $2i$ in $R[[t]]$ for each element of grading i in R (where t and s have grading -1).

Example: The Lazard ring N_* has a unique ring homomorphism $D_t : N_* \rightarrow N_*[[t]]$ such that $D_t(F) = F_t$, and this defines a D -structure on N_* . Thus N_* is initial in the category of D -rings.

Proposition. If X is an E_{∞} -space then N_*X is a commutative ring under Pontryagin product; it is also an N_* -algebra. If d_0, d_1, \dots are the double coverings described in the previous section then the total squaring

$$D_t(x) = \sum_n d_n(x)t^n$$

gives an D -structure on N_*X .

Example: BO_* , the disjoint union of the classifying spaces of the orthogonal groups $BO(n)$, is an E_{∞} -space with $N_*BO_* = N_*[b_0, b_1, \dots]$. It forms a D -ring with F given by the cobordism formal group law over N_* and with D_t determined by

$$D_t(b)(xF(x, t)) = b(x)b(F(x, t))$$

where $b(x) = \sum b_i x^i$.

We shall refer to any D -ring R with $F = (+)$ as a Q -ring. The mod 2 homology of an E_{∞} -space E is a Q -ring, and the Thom reduction $\epsilon : N_*(E) \rightarrow H_*(E)$ is a morphism of D -rings.

Proposition. A commutative ring R is a Q -ring if and only if it has a sequence of additive operations $q_n : R \rightarrow R$ which satisfy the following three conditions:

i) Squaring: $q_0(x) = x^2$ for all $x \in R$.

ii) Cartan formula: $q_n(xy) = \sum_{i+j=n} q_i(x)q_j(y)$ for all $x, y \in R$.

iii) Adem relations: $q_m(q_n(x)) = \sum_i \binom{i-n-1}{2i-m-n} q_{m+2n-2i}(q_i(x))$ for all $x \in R$.

In the graded case, $\text{grade}(q_n(x)) = 2 \cdot \text{grade}(x) + n$.

This is exactly an action of the classical Dyer-Lashof algebra on R . This idea of writing Adem relations via generating series is suggested by [4] and by Bullett and MacDonald [5]. See [17], [15], [16] for background on Dyer-Lashof operations.

Example: The Q -structure on $H_*BO_* = \mathbb{Z}_2[b_0, b_1, \dots]$ is characterized by

$$Q_i(b)(x(x+t)) = b(x)b(x+t)$$

where $b(x) = \sum b_i x^i$. This expresses via generating series a calculation of Priddy's in [20].

Notice that if A and B are Q -rings then $A \otimes_{\mathbb{Z}_2} B = A \otimes_{\mathbb{Z}_2} B$ is a Q -ring. Let $Q(M)$ denote the free Q -ring generated by a \mathbb{Z}_2 -vector space M . If M has a comultiplication, then $Q(M)$ has a comultiplication extending it which is a morphism of Q -rings.

Recall that $E_\infty(X)$ is the free E_∞ -space generated by X (see [18] or [1] for background). The following is a classical result:

Theorem 1. (May [17]) For any space X the canonical map

$$Q(H_*X) \rightarrow H_*E_\infty(X)$$

is an isomorphism which preserves the comultiplication. In particular, $H_*B\mathbb{Z}_2 = Q(x)$ is the free Q -ring on one generator.

If A and B are D -rings then $A \otimes_{N_*} B$ is naturally a D -ring. Let us denote $D(M)$ denote the D -ring freely generated by an N_* -module M . If M is a coalgebra in the category of N_* -modules, then $D(M)$ has a comultiplication.

Theorem 2. The bordism of an E_∞ -space is an D -ring. Moreover, for any space X the canonical map

$$D(N_*X) \rightarrow N_*E_\infty(X)$$

is an isomorphism which preserves the comultiplication. In particular, $N_*\Sigma = N_*(B\mathbb{Z}_2) = D(x)$ is the free D -ring on one generator.

Thus, both $D(x)$ and $N_*\Sigma$ are algebras equipped with operations of substitution; the former because it is the set of unary operations in the theory of D -rings and the latter because we have defined a substitution operation among coverings of manifolds. The above theorem says that the canonical isomorphism of D -rings $D(x) \rightarrow N_*\Sigma$ which sends the generator x to the unique non-zero element x in $N_0(B\mathbb{Z}_2)$ preserves the operations of substitution.

References

- [1] J. F. Adams. Infinite loop spaces. *Ann. of Math. Studies* no. 90, Princeton 1978.
 - [2] M. A. Aguilar, Generators for the bordism algebra of immersions, *Trans. Amer. Math. Soc.* 316 (1989).
 - [3] R. M. Alliston, Dyer-Lashof operations and bordism. Ph.D. Thesis, University of Virginia, 1976.
 - [4] T. P. Bisson, Divided sequences and bialgebras of homology operations. Ph.D. Thesis, Duke 1977.
 - [5] S. R. Bullett, I. G. Macdonald, On the Adem relations, *Topology* 21 (1982), 329-332.
 - [6] P. E. Conner (and E. E. Floyd). *Differentiable Periodic Maps* (2nd ed.), *Lect. Notes in Math.* no. 738, Springer 1979.
 - [7] T. tom Dieck, Steenrod operationen in Kobordismen-Theorie, *Math. Z.* 107 (1968), 380-401.
 - [8] A. Dold, Decomposition Theorem for $S(n)$ -complexes, *Ann. of Math.*(2) 75 (1962), 8-16.
 - [9] A. Joyal, Une théorie combinatoire des séries formelles, *Adv. in Math.* 42 (1981), 1-82.
 - [10] A. Joyal, Foncteurs analytiques et espèces de structures, *Lect. Notes in Math.* no. 1234, Springer 1985.
 - [11] A. Joyal, Calcul Integral Combinatoire et homologie des groupes symétriques, *C. R. Math. Rep. Acad. Sci. Canada* 12 (1985).
 - [12] D. S. Kahn, S. B. Priddy, Application of transfer to stable homotopy theory, *Bull. AMS* 78 (1972), 981-987.
 - [13] M. Lazard. *Commutative Formal Groups*, *Lect. Notes in Math.* no. 443, Springer 1975.
 - [14] J. Lubin, Finite subgroups and isogenies of one-parameter formal Lie groups, *Ann. of Math.* 85 (1967) 296-302.
 - [15] I. Madsen, On the action of the Dyer-Lashof algebra in $H_*(G)$, *Pacific J. Math.* 60 (1975), 235-275.
 - [16] I. Madsen, R. J. Milgram. The classifying spaces for surgery and cobordism of manifolds, *Ann. of Math. Studies* no. 92, Princeton 1979.
 - [17] J. P. May, Homology operations on infinite loop spaces, in *Proc. Symp. Pure Math.* 22, Amer. Math. Soc. 1971, 171-185.
 - [18] J. P. May, Infinite loop space theory, *Bull. Amer. Math. Soc.* 83 (1977), 456-494.
 - [19] M. Nakaoka, Decomposition theorems for the homology of the symmetric groups, *Ann. of Math.* 71 (1960), 16-42.
 - [20] S. B. Priddy, Dyer-Lashof operations for the classifying spaces of certain matrix groups, *Quart. J. Math. Oxford* 26 (1975), 179-193.
 - [21] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, *Adv. in Math.* 7 (1971), 29-56.
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Nishida Relations in Bordism and Homology
(extended abstract)

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This is the second of a series of *Compte Rendus*. In the first [1] we have presented a theory of Dyer-Lashof operations in unoriented bordism. Here we shall discuss the (Nishida) relations between Dyer-Lashof and Landweber-Novikov operations. They are used to represent the algebra $N_*\Sigma$ of covering manifolds in terms of their homology characteristic numbers. The proofs are based on the properties of the covering space operations and the notions of D -ring and Q -ring introduced in [1].

1. The Nishida relations in homology.

In homology mod 2 the Nishida relations are commutation relations between Dyer-Lashof and Steenrod operations (see [4] for instance). We shall express the Nishida relations as a commutative square which involves the Milnor coaction and the Q -structure on the homology of any E_∞ -space discussed in [1].

Recall that the Milnor Hopf algebra \mathcal{A}_* is the dual of the Steenrod algebra; see [6] for instance. As a graded algebra it is $\mathcal{A}_* = \mathbb{Z}_2[\xi_0^\pm, \xi_1, \xi_2, \dots] = \xi_0^{-1} \mathbb{Z}_2[\xi_0, \xi_1, \xi_2, \dots]$ with $\text{grade}(\xi_i) = 2^i - 1$; the diagonal $\delta : \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$ is the unique ring homomorphism such that $\delta(\xi) = (\xi \otimes 1) \circ (1 \otimes \xi)$ where $\xi(x) = \sum \xi_i x^{2^i}$. We are diverging from the usual convention that puts $\xi_0 = 1$.

The homology of any space has a natural (left) coaction

$$\alpha : H_*(X) \rightarrow \mathcal{A}_* \otimes H_*(X) = H_*(X)[\xi_0^\pm, \xi_1, \xi_2, \dots]$$

which restricts to the usual (left) coaction when we put $\xi_0 = 1$ and to the "grading" coaction $\alpha(x_n) = \xi_0^{-n} x_n$ for $x \in H_n(X)$ when we put $0 = \xi_1 = \xi_2 = \dots$.

For example, for $X = RP^\infty$ we have $\alpha(b)(x) = b(\xi^{(-1)}(x))$ where $b(x) = \sum_i b_i x^i$ and b_0, b_1, \dots is the canonical basis of $H_* RP^\infty$ and $\xi^{(-1)}(x)$ is the composition inverse of the power series $\xi(x)$.

Proposition. *If R is a Q -ring, then there is a Q -structure on $\mathcal{A}_* \otimes R$ determined by $Q_t(\xi)(x(x+t)) = \xi(x)\xi(x+t)$. In particular, there is a unique Q -structure on \mathcal{A}_* such that $Q_t(\xi)(x(x+t)) = \xi(x)\xi(x+t)$.*

Remark: Since the definition forces

$$Q_t(\xi_0) = \xi_0 \sum_i \xi_i t^{2^i - 1}$$

we see that the Q -structure on \mathcal{A}_* cannot survive if we try to insist that $\xi_0 = 1$.

Suppose now that R is both a Q -ring and has a Milnor coaction. We give $\mathcal{A}_* \otimes R$ the Q -structure from the proposition.

Definition: The *Nishida relations* hold for a Q -ring R if the following diagram

$$\begin{array}{ccc}
 R & \xrightarrow{Q_t} & R[[t]] \\
 \downarrow \alpha & & \downarrow \alpha \\
 \mathcal{A}_* \otimes R & \xrightarrow{Q_t} & \mathcal{A}_* \otimes R[[t]]
 \end{array}$$

commutes, where α is extended to $R[[t]]$ by putting $\alpha(t) = \xi(t)$.

Theorem. *If X is an E_∞ -space, then $H_*(X)$ is a Q -ring with a Milnor coaction, and the Nishida relations hold for $H_*(X)$.*

This is a complete description of the Nishida relations in mod 2 homology. We want to give a similar description in unoriented bordism

2. The Nishida relations in unoriented bordism.

For any \mathbb{Z}_2 -algebra R let $\mathcal{B}_*(R)$ be the group of invertible formal power series $f(x) \in xR[[x]]$ under substitution. The functor $R \mapsto \mathcal{B}_*(R)$ is representable by a Hopf algebra \mathcal{B}_* called the *Faa di Bruno Hopf algebra* (see example 49 in [2] for instance). We have $\mathcal{B}_* = \mathbb{Z}_2[h_0^\pm, h_1, \dots] = h_0^{-1}\mathbb{Z}_2[h_0, h_1, \dots]$ and the diagonal of \mathcal{B}_* is given by

$$\delta(h) = (h \otimes 1) \circ (1 \otimes h)$$

where $h(x) = \sum_n h_n x^{n+1}$. We want to consider left comodules over \mathcal{B}_* . Recall that the tensor product of left comodules over a bialgebra is a left comodule.

Example: Let N_* denote the Lazard ring for formal power series of order two (see [5] for example). There is a natural (left) coaction $\phi : N_* \rightarrow \mathcal{B}_* \otimes N_*$ encoding change of parameters in formal group laws. We have $\phi(F)(x, y) = h(F(h^{-1}x, h^{-1}y))$ where $\phi : N_* \rightarrow N_*[h_0^\pm, h_1, \dots]$.

Notice that $N_* \otimes N_* \rightarrow N_*$ is a morphism of left \mathcal{B}_* comodules, so that N_* is a monoid in the category of left \mathcal{B}_* comodules.

Definition: A *Landweber-Novikov coaction* is a module over N_* in the category of \mathcal{B}_* -comodules. More explicitly, this is a module over N_* with a comodule structure over \mathcal{B}_* such that the module structure map $N_* \otimes M \rightarrow M$ is a map of \mathcal{B}_* -comodules.

In order to describe the natural Landweber-Novikov coaction on unoriented bordism, we first need to recall the theory of cobordism characteristic classes for vector bundles, as sketched by Quillen in [5] for instance; we use a slightly unstable version of the usual definitions, however. For any space X the total characteristic class of a real vector bundle V on X is the element $c(V) \in N^*(X)[b_0, b_1, \dots]$ having the following properties:

- 1) the map $V \rightarrow c(V)$ is a natural transformation $c : Vect(X) \rightarrow N^*(X)[b_0, b_1, \dots]$
- 2) $c(V_1 \oplus V_2) = c(V_1)c(V_2)$

3) $c(L) = \sum b_i e(L)^i$ if L is a line bundle with euler class $e(L) \in N^1(X)$. We have an expansion

$$c(V) = \sum_R c_R(V) b^R$$

where $b^R = b_0^{r_0} b_1^{r_1} \dots$ for $R = (r_0, r_1, \dots)$. For virtual bundles the total characteristic class is the element of $N^*(X)[b_0^\pm, b_1, \dots]$ defined by putting $c(V - W) = c(V)c(W)^{-1}$.

Example: The unoriented bordism group $N_*(X)$ of any space X has a Landweber-Novikov coaction given by the Landweber-Novikov (total) operation

$$\phi : N_*(X) \rightarrow \mathcal{B}_* \otimes N_*(X) = N_*(X)[h_0^\pm, h_1, \dots].$$

It can be defined via the following explicit formula. If $f : M \rightarrow X$ represents an element of $N_*(X)$ then

$$\phi(M, f) = \sum_R f_*(c_R(\nu_M) \cap \mu_M) h^R$$

where $\nu_M = -\tau_M$ is the normal bundle of M and μ_M is the fundamental class of M .

Example: The Landweber-Novikov coaction on $N_*(RP^\infty)$ is characterised by the identity $\phi(b)(x) = b(h^{(-1)}(x))$ where $b(x) = \sum_i b_i x^i$.

Proposition. If R is a D -ring, then there is a D -structure on $\mathcal{B}_* \otimes R = R[h_0^\pm, h_1, \dots]$ determined by $D_t(h)(x(x+t)) = h(x)h(F(x,t))$.

Suppose now that R is a D -ring which also has a Landweber-Novikov coaction. We give $\mathcal{B}_* \otimes R$ the D -structure from the proposition.

Definition: The Nishida relations hold for a D -ring R if the diagram

$$\begin{array}{ccc} R & \xrightarrow{D_t} & R[[t]] \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{B}_* \otimes R & \xrightarrow{D_t} & \mathcal{B}_* \otimes R[[t]] \end{array}$$

commutes, where ϕ is extended to $R[[t]]$ by putting $\phi(t) = h(t)$.

Suppose that X is an E_∞ -space. Then N_*X is a D -ring together with a Landweber-Novikov coaction.

Theorem 2. If X is an E_∞ -space, then the Nishida relations hold for N_*X in that the following diagram commutes:

$$\begin{array}{ccc} N_*(X) & \xrightarrow{D_t} & N_*(X)[[t]] \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{B}_* \otimes N_*(X) & \xrightarrow{D_t} & \mathcal{B}_* \otimes N_*(X)[[t]] \end{array}$$

Here ϕ has been extended to $N_*(X)[[t]]$ by putting $\phi(t) = h(t)$.

Recall from [1] that for any space X we have $N_*E_\infty(X) = D(N_*(X))$, the free D -ring generated by $N_*(X)$. Of course, $N_*(X)$ also has a Landweber-Novikov coaction.

Theorem 3. *Suppose that M has a Landweber-Novikov coaction, and let $D(M)$ be the free D -ring generated by M . Then there is a unique Landweber-Novikov coaction on $D(M)$ which satisfies all of the following:*

- i) *the canonical map $M \rightarrow D(M)$ is a comodule map;*
- ii) *$D(M)$ is an algebra over its Landweber-Novikov coaction;*
- iii) *$D(M)$ satisfies the Nishida relations.*

The theorem shows that the Landweber-Novikov coaction on $N_*E_\infty(X)$ is fully determined by its values on N_*X . In particular, the Landweber-Novikov coaction on $N_*(B\Sigma_*) = D(x)$ is determined by the relation $\phi(x) = x$.

It is well known that the Thom reduction $\epsilon : N_*(X) \rightarrow H_*(X)$ induces an isomorphism

$$N_*(X) \otimes_{N_*} \mathbb{Z}_2 \simeq H_*(X)$$

We shall write $\epsilon : \mathcal{B}_* \rightarrow \mathcal{A}_*$ for the ring homomorphism such that $\epsilon(h) = \xi$. We will characterize the behavior of the Thom reduction with respect to the usual Landweber-Novikov operations in unoriented cobordism and the Steenrod operations in mod 2 homology by showing that the Thom reduction gives rise to a simple equivalence of categories.

Let $[\mathcal{B}_*\mathcal{N}_*]$ denote the category of Landweber-Novikov coactions (left \mathcal{B}_* -comodules which are modules over N_*) and let $[\mathcal{A}_*]$ denote the category of Milnor coactions (left \mathcal{A}_* -comodules).

For any $M \in [\mathcal{B}_*\mathcal{N}_*]$ let us put $T(M) = M \otimes_{N_*} \mathbb{Z}_2$. By composing the coaction $M \rightarrow \mathcal{B}_* \otimes M$ with $\epsilon : \mathcal{B}_* \rightarrow \mathcal{A}_*$ we obtain a coaction $M \rightarrow \mathcal{A}_* \otimes M$. By further tensoring with \mathbb{Z}_2 we obtain a coaction $T(M) \rightarrow \mathcal{A}_* \otimes T(M)$. This defines a functor $T : [\mathcal{B}_*\mathcal{N}_*] \rightarrow [\mathcal{A}_*]$.

Proposition. *The functor T defines an equivalence of categories $[\mathcal{B}_*\mathcal{N}_*] \simeq [\mathcal{A}_*]$.*

In fact, we get a similar result even when we take the Dyer-Lashof operations into account, which we do by using monads which encode the Dyer-Lashof operations in bordism and homology. For background on monads and their category of actions see [3] for instance.

The functor $M \mapsto D(M)$ from $[\mathcal{B}_*\mathcal{N}_*]$ to itself is a monad that we shall denote D (it is a monad since $D(M)$ is a free structure). The D -actions are exactly the D -rings with a Landweber-Novikov coaction which satisfies the Nishida relations. Let us denote this category by $[\mathcal{B}_*\mathcal{N}_*]^D$. Similarly, the functor $M \mapsto Q(M)$ from $[\mathcal{A}_*]$ to itself is a monad that we shall denote Q . The Q -actions are exactly the Q -rings with a Milnor coaction which satisfies the Nishida relations. Let us denote this category by $[\mathcal{A}_*]^Q$.

Proposition 2. *The functor T transforms the D -actions into the Q -actions and defines an equivalence of categories*

$$[\mathcal{B}_*\mathcal{N}_*]^D \simeq [\mathcal{A}_*]^Q$$

This result shows that for any E_∞ -space X , the D -ring $N_*(X)$ can be recovered entirely from the Q -ring $H_*(X)$ as long as the coaction of \mathcal{A}_* on $H_*(X)$ is known.

3. Characteristic numbers and covering space operations

The bordism classes of manifolds are determined by their tangential characteristic numbers, or equally by their normal characteristic numbers.

Recall the discussion in section 2 of characteristic classes in unoriented cobordism. The total Stiefel-Whitney class $w(V) \in H^*(X)[b_0, b_1, \dots]$ of a vector bundle V on X is the Thom reduction of the total cobordism characteristic class. Then $w(\)$ is multiplicative in that $w(V_1 \oplus V_2) = w(V_1)w(V_2)$, and $w(L) = \sum_i b_i e(L)^i$ for any line bundle over X , where $e(L) \in H^1(X)$ is the euler class of L .

The tangential characteristic numbers can be grouped together as the coefficients of a polynomial $\beta^r(M) \in \mathbb{Z}_2[h_0, h_1, \dots]$. We have

$$\beta^r(M) = \sum_R (w_R(\tau_M), \mu_M) h^R$$

where $w(\tau_M) = \sum_R w_R(\tau_M) b^R$ is the total Stiefel-Whitney class of the tangent bundle of M and μ_M is the fundamental class of M .

In keeping with the Landweber-Novikov coaction, we can work instead with characteristic number polynomials for the normal bundle, the virtual bundle $\nu_M = -\tau_M$. More generally, as in [5] for instance, we can define a *Boardman map*

$$\beta : N_*(X) \rightarrow \mathcal{B}_* \otimes H_*(X) = H_*(X)[h_0^\pm, h_1, \dots]$$

by the explicit formula

$$\beta(M, f) = \sum_R f_*(w_R(\nu_M) \cap \mu_M) h^R.$$

Let $B\Sigma_*$ denote the classifying space for finite covering spaces. It is an E_∞ -space. Let $N_*\Sigma$ denote $N_*B\Sigma_*$ and $H_*\Sigma$ denote $H_*B\Sigma_*$. The Boardman map $\beta : N_*\Sigma \rightarrow \mathcal{B}_* \otimes H_*\Sigma$ obviously preserves sums and products. In fact, it also preserves the operation of substitution, when substitution is properly defined on $\mathcal{B}_* \otimes H_*\Sigma$.

Proposition. *If R is a Q -ring then there is a Q -structure on $\mathcal{B}_* \otimes R$ determined by $Q_t(h)(x(x+t)) = h(x)h(x+t)$. In particular, there is a unique Q -structure on \mathcal{B}_* such that $Q_t(h)(x(x+t)) = h(x)h(x+t)$.*

From [1] we have that $H_*\Sigma = Q(x)$ is the free Q -ring on one generator. Since $\mathcal{B}_* \otimes H_*\Sigma$ is the coproduct in the category of Q -rings, we can view $\mathcal{B}_* \otimes H_*\Sigma$ as the collection of unary operations in the theory of Q -rings which are extensions of \mathcal{B}_* . Therefore $H_*\Sigma[h_0^\pm, h_1, \dots] = \mathcal{B}_* \otimes H_*\Sigma$ admits an operation of substitution. Here is an explicit formula for substituting two elements of $H_*\Sigma[h_0^\pm, h_1, \dots]$:

$$\left(\sum_R p_R h^R\right) \circ \left(\sum_R q_R h^R\right) = \sum_{R,S} p_R (q_S h^S) h^R$$

where the operations p_R are applied to the elements $q_S h^S$ of the Q -ring $\mathcal{B}_* \otimes H_*\Sigma$.

Theorem 4. *The Boardman map $\beta : N_*\Sigma \rightarrow H_*\Sigma[h_0^\pm, h_1, \dots]$ preserves sums, products and substitutions.*

We view this as describing for each covering p of closed manifolds how the normal characteristic numbers of $p(M)$ are determined by the characteristic numbers of M . The complete description is summarized by the formula

$$\beta(p(M)) = (\beta(p))(\beta(M)),$$

where the right-hand substitution takes place in $H_*\Sigma[h_0^\pm, h_1, \dots]$.

References

- [1] T. P. Bisson, A. Joyal, The Dyer-Lashof algebra in bordism, C. R. Math. Rep. Acad. Sci. Canada (to appear).
- [2] A. Joyal, Une théorie combinatoire des séries formelles, Adv. in Math. 42 (1981), 1-82.
- [3] S. MacLane. Categories for the Working Mathematician, Springer 1971.
- [4] J. P. May, Homology operations on infinite loop spaces, in Proc. Symp. Pure Math. 22, A. M. S. 1971, 171-185.
- [5] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, Adv. in Math. 7 (1971), 29-56.
- [6] N. E. Steenrod, D. B. A. Epstein, Cohomology operations. Ann. of Math. Studies no. 50, Princeton 1962.
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**A duality for the category of
directed multigraphs**

by R. Squire

Presented by J. Lambek, F.R.S.C.

Abstract. The category, Grph, of presheaves over the category $\overrightarrow{\downarrow}$ can be interpreted as a category of graphs; more exactly, of directed multigraphs. The subgraphs of any such graph form a complete completely distributive Heyting algebra; moreover a map between graphs induces, by pulling back, a function which preserves all joins, all meets, and the binary implication operation. We add to these operations a single unary operation, and to the equations one further equation. The category of algebras which model the enlarged equational theory is dually equivalent, via the contravariant subgraphs functor, to the category Grph.

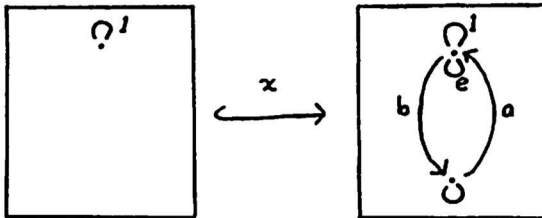
Introduction. The prototype of the duality to be described is that induced between Set and caBool, the category of complete atomic Boolean algebras, by the two element algebra 2. The topos Set is replaced here by the topos $\text{Set}^{\overrightarrow{\downarrow}}$, the algebra 2 by a new internal algebraic structure $\underline{\Omega}$ on the subobject classifier Ω , and caBool by a category of algebras $\underline{\text{Alg}}$, equivalent to the category $\text{caBool}^{\overrightarrow{\downarrow}}$. Much of what is done can be carried out under the hypotheses that the replacement for Set is a topos $\text{Set}^{\mathbf{C}}$ with \mathbf{C} finite and Ω a cogenerator. In the case to be considered the duality can be made particularly transparent.

1. **The category of directed multigraph.** The exponent category of $\text{Set}^{\overrightarrow{\downarrow}}$ consists of two objects A and V and two parallel non-identity morphisms t and h from A to V . The letters are the initials of Arrow, Vertex, tail and head; they indicate how a functor from $\overrightarrow{\downarrow}$ to Set is to be interpreted as a directed multigraph. We follow the conventions of category theory [5], denoting $\text{Set}^{\overrightarrow{\downarrow}}$ by Grph and calling its objects graphs. When the context is clear we abbreviate $G(\ell)$ to ℓ , where G is a graph and $\ell \in \{t, h\}$.

Since Grph is a topos it possesses a subgraph classifier true: $\mathbf{I} \rightarrow \Omega$. In terms appropriate to the duality this means that the subgraph $x \rightarrow \Omega$ corresponding to true is generic, in the sense that for any $K \hookrightarrow G$, there is a uniquely determined $f : G \rightarrow \Omega$ for which

the function $f^* : \text{Sub}(\Omega) \rightarrow \text{Sub}(G)$, induced by pulling back subgraphs of Ω along f , satisfies $f^*(x) = K$. The bijection between $\text{Sub}(G)$ and $[G, \Omega]$ so determined extends to an isomorphism between the contravariant functors Sub and $[-, \Omega]$. This isomorphism lifts and reflects any limit defined structure on Ω to and from a similar structure on Sub .

The procedure for calculating Ω and its Heyting algebra structure is standard ([2],[3]). The opposite of the exponent category for Grph becomes by the Yoneda embedding a full subcategory $V \rightrightarrows A$. The graph V consists of a single vertex, A consists of a single arrow between distinct vertices, and the parallel graph maps t and h send the vertex to the tail and head of the arrow. The graph Ω_{\leq} in the category of posets is given by $\Omega_{\leq}(V) \approx [V, \Omega_{\leq}] \approx \underline{\text{Sub}}_{\leq} V \approx 2_{\leq}$, a two element chain; $\Omega_{\leq}(A) \approx \underline{\text{Sub}}_{\leq}(A) \approx (2_{\leq})^2 \oplus 1$, a poset arising from the addition of a new top element to the 4 element poset $(2_{\leq})^2$; $\Omega(V) = \{0, 1\}$ with $0 < 1$ and $\Omega(A) = \{0, b, a, e, 1\}$ with $0 < a < e < 1$ and $0 < b < e$. The graph maps t and h induce by pulling back two order preserving maps t^* and h^* from $\Omega_{\leq}(A)$ to $\Omega_{\leq}(V)$; they are determined by the principal up-sets $(t^*)^{-1}\{1\} = \{b, e, 1\}$ and $(h^*)^{-1}\{1\} = \{a, e, 1\}$. The maps t^* and h^* organize the two posets into a graph Ω , which, together with its generic subgraph is depicted below



The corresponding po object Ω_{\leq} in Grph has a uniquely determined Heyting algebra structure

$$\Omega_H = (\Omega; 0, 1, \wedge, \vee, \Rightarrow) .$$

The arrow b generates a smallest containing subgraph $\langle b \rangle$; this subgraph is classified by an endomorphism τ of Ω which has the explicit description

$$\begin{aligned} \text{for all } x \in \Omega(V) \quad \tau_V(x) &= 1 \\ \text{for all } y \in \Omega(A) \quad \tau_A(y) &= (y \Leftrightarrow b) \vee e \\ &= \begin{cases} 1 & \text{if } y = b \\ e & \text{if } y \neq b \end{cases} . \end{aligned}$$

The structure

$$\Omega = (\Omega; 0, 1, \tau, \wedge, \vee, \Rightarrow)$$

will play the role of a dualizing object for finite graphs.

The finite powers of Ω together with the operations of Ω generate an algebraic theory (or abstract clone), whose category of set valued models, $\underline{\text{Alg}}$, we call graphic algebras; thus Ω itself is a graphic algebra in Grph . An equation is valued for $\underline{\Omega}$ iff it is valid for both $\underline{\Omega}(V)$ and $\underline{\Omega}(A)$, but since the former is a homomorphic image of the latter we may drop $\underline{\Omega}(V)$. Thus by Birkhoff's theorem ([6]) $\underline{\text{Alg}}$ is the category of algebras whose objects form the variety $HSP \{ \underline{\Omega}(A) \}$.

The contravariant functor $\underline{\text{Sub}}: \text{Grph} \rightarrow \underline{\text{Alg}}$ sends a graph G to a graphic algebra

$$\underline{\text{Sub}}(G) = (\text{Sub}(G); 0, G, \tau, \wedge, \vee, \Rightarrow)$$

where, for K a subgraph, $\tau(K)$ is given by

$$\begin{aligned} \tau(K)(V) &= G(V) \\ \tau(K)(A) &= t^{-1}(K(V)) - h^{-1}(K(V)) \\ &= \left\{ \begin{array}{l} \text{all arrows with tail in } K \\ \text{and head outside } K \end{array} \right\}. \end{aligned}$$

2. Axiomatization of $\underline{\text{Alg}}$. We first give an equational basis for $HSP \{ \underline{\Omega}_H(A) \}$. The reduct algebra $\underline{\Omega}_H(A)$ satisfies some finite set Σ_H of equations for Heyting algebras - for example [1] p.177. Any further finite set of equations (even if it includes τ) can be converted to a single one thus: $t = s$ can be converted to $t \Leftrightarrow s = 1$; $\psi = 1$ and $\theta = 1$ to $\psi \wedge \theta = 1$. A basis for $HSP \{ \underline{\Omega}_H(A) \}$ ([7],[8]) is $\Sigma_H \cup \{ M \wedge L = 1 \}$ where

$$\begin{aligned} M &\equiv x \vee (x \Rightarrow (y \vee \neg y)) \quad \text{and} \\ L &\equiv \neg(x \wedge y) \vee \neg(x \wedge \neg y) \vee \neg(\neg x \wedge y). \end{aligned}$$

A basis for $HSP \{ \underline{\Omega}(A) \}$ consists of the above basis for $HSP \{ \underline{\Omega}_H(A) \}$ together with a new finite list involving the unary operation τ . We first define $e \equiv \tau 1$. The new list is:

$$\begin{aligned} e \Rightarrow (x \vee \neg x) = 1, \quad \tau 0 = \tau e = e, \quad \tau x \wedge \tau \neg x = e \\ \text{and } x \vee (e \Rightarrow x) \vee \tau x \vee \tau \neg x = 1. \end{aligned}$$

3. The finitary duality.

The structure Ω is a graphic algebra in Grph, thus the contravariant functor $[-, \Omega]$ has as values graphic algebras (in Set). The corresponding structure $\underline{\Omega}$ is a graph in the category Alg, thus $[-, \underline{\Omega}]$ has as values graphs. The latter functor is calculated at a graphic algebra \mathcal{A} by $[\mathcal{A}, \underline{\Omega}](B) = [\mathcal{A}, \Omega(B)]$ for $B \in \{V, A\}$. The two functors are adjoint on the right

$$\text{Grph} \begin{array}{c} [-, \Omega] \\ \leftarrow \circlearrowleft \\ [-, \underline{\Omega}] \end{array} \underline{\text{Alg}} \tag{1}$$

The bijection $[G, [\mathcal{A}, \underline{\Omega}]] \approx [\mathcal{A}, [G, \Omega]]$ relates a map $\theta : G \rightarrow [\mathcal{A}, \Omega]$ of graphs to a homomorphism $f : \mathcal{A} \rightarrow [G, \Omega]$ by

$$(\theta_B(x))(a) = (f(a))_B(x)$$

where $B \in \{V, A\}$, $x \in G(B)$ and $a \in |\mathcal{A}|$.

If we restrict to the full subcategories Alg_f and Grph_f, of finite algebras and finite graphs, respectively, we get

Theorem 1. The functors Sub and $[-, \Omega]$ constitute a dual equivalence.

$$\text{Grph}_f \begin{array}{c} \underline{\text{Sub}} \\ \leftarrow \circlearrowleft \\ [-, \Omega] \end{array} \underline{\text{Alg}}_f \tag{2}$$

Using this dual equivalence we can calculate the free graphic algebra on n generators as Sub(Ω^n); in particular the initial graphic algebra is the three element chain Sub(Ω), and the free algebra in one generator x is a 39 element algebra, Sub(Ω), with $x \mapsto \Omega$ as generator.

The dual adjointness (1) fails to be a dual equivalence; however using Gabriel-Ulmer duality we can derive a new equivalence from (2). For finite toposes and locally finite varieties "finite"(f) coincides with "finitely presentable"(fp), thus Alg_f \approx $(\text{Set}_{fp}^A \xrightarrow{\rightarrow} V)^{op} \approx (\text{Bool}^V \xrightarrow{\rightarrow} A)_{fp}$, so Alg \approx $\text{Bool}^V \xrightarrow{\rightarrow} A$. Explicitly, if \mathcal{A} is a graphic algebra then $\hat{\mathcal{A}}$, the corresponding graph in Bool, is given by

$$\begin{aligned} \hat{\mathcal{A}}(V)_{\leq} &= (\{x \in |\mathcal{A}| : x \leq e\}, \leq) \\ \hat{\mathcal{A}}(A)_{\leq} &= (\{y \in |\mathcal{A}| : e \leq y\}, \leq) \end{aligned}$$

where the partial order is that induced by \mathcal{A} ; (The Boolean algebra structure is then uniquely determined)

$$t(x) = \neg \neg x \vee \tau x \quad \text{and} \quad h(x) = \neg \neg x \vee \tau \neg x .$$

For a homomorphism, $f : \mathcal{A} \rightarrow \mathcal{A}'$, the map $\hat{f} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}'$ is given simply by restricting f to the principal filter $\uparrow e$ and the principal ideal $\downarrow e$.

The inverse functor, assigning a graphic algebra \hat{B} to a graph B in Bool, is a poset $\hat{B}_{\leq} \hookrightarrow B(V)_{\leq} \times B(A)_{\leq}$ with the induced order, and

$$|\hat{B}| = \{(x, y) : y \leq tx \wedge hx\} .$$

The Heyting algebra structure of \hat{B}_{\leq} is then uniquely determined. The operation τ is given by $\tau(x, y) = (1, tx \wedge \neg hx)$. The extension of the inverse functor to morphisms, $\theta : B \rightarrow B'$, is by $\hat{\theta}(x, y) = (\theta_V(x), \theta_A(y))$.

4. The infinitary duality. We now enrich $\underline{\Omega}$ to $\underline{\Omega}$ by introducing, for each small cardinal α , the meet and join operations $\bigwedge, \bigvee : \Omega^\alpha \rightarrow \Omega$. The algebra $\underline{\Omega}$ then satisfies, in addition to (0) the equations already given for $\underline{\Omega}$, (1) a class of equations stating that \bigwedge and \bigvee are meet and join for each cardinal α , and (2) a class of equations stating that $\underline{\Omega}$ is completely distributive. A model ((4)) of these operations and equations is simply a complete completely distributive graphic algebra (a ccd algebra). The category of all such algebras with join and meet preserving homomorphism will be denoted by Alg.

Theorem 2. The functors Sub and $[-, \underline{\Omega}]$ constitute a dual equivalence

$$\text{Grph} \begin{matrix} \xrightarrow{\text{Sub}} \\ \xleftrightarrow{[-, \underline{\Omega}]} \end{matrix} \underline{\text{Alg}} . \tag{3}$$

The functor $[-, \underline{\Omega}]$ can be replaced by a more concrete equivalent functor, Spec, using the join and meet operations. The new functor is described as follows.

An element p of a ccd algebra \mathcal{A} is called a *prime* if for all $U \subset | \mathcal{A} |$,

$$(p \leq \bigvee U) \Rightarrow \exists x(x \in U \ \& \ p \leq x) .$$

For \mathcal{A} a ccd algebra we define a graph $\text{Spec}(\mathcal{A})$ by $\text{Spec}(\mathcal{A})(V) = \{\text{atoms of } \mathcal{A}\}$, and $\text{Spec}(\mathcal{A})(A) = \{\text{non atomic primes of } \mathcal{A}\}$; the tail and head maps are

$$\begin{aligned} t(a) &= \bigwedge \{x : a \leq \neg\neg x \vee \tau x\} \\ h(a) &= \bigwedge \{x : a \leq \neg\neg x \vee \tau\neg x\} . \end{aligned}$$

For $f : \mathcal{A} \rightarrow \mathcal{A}'$ in Alg, the left adjoint $\bar{f} : \mathcal{A}'_{\leq} \rightarrow \mathcal{A}_{\leq}$ preserves atoms and preserves non atomic primes, in such a way that tails and heads are preserved. Thus \bar{f} restricts to a

graph map $\text{Spec}(f) : \text{Spec } \mathcal{A}' \rightarrow \text{Spec } \mathcal{A}$. This makes Spec a functor naturally equivalent to $[-, \Omega]$; that is the dual equivalence (3) is:

$$\text{Grph} \begin{array}{c} \xrightarrow{\text{Sub}} \\ \xleftrightarrow{\text{Spec}} \\ \xleftarrow{\text{Alg}} \end{array} \underline{\text{Alg}} \quad (4)$$

The prototypical equivalence is included in (4) by viewing sets as graphs with no arrows and complete atomic Boolean algebras as ccd graphic algebras satisfying $e = 1$.

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References

- [1] Balbes, R. & Dwinger, P. *Distributive Lattices*. University of Missouri Press, 1974.
- [2] Johnstone, P.T. *Topos Theory*, Academic Press, 1977.
- [3] Lawvere, F.W. Qualitative distinctions between some toposes of generalized graphs. In *Contemporary Mathematics*, pp. 261-299. *Categories in Computer Science and Logic, Proceedings of Summer Research Conference held June 14-20, 1987*. American Mathematics Society Volume 92.
- [4] Linton, F.E.J. *Some Aspects of Equational Categories*, Proceedings of the Conference on Categorical Algebra. Springer-Verlag 1966 pp. 84-94.
- [5] MacLane, S. *Categories for the Working Mathematician*. Graduate Texts in Mathematics 5. Springer Verlag, 1971.
- [6] McKenzie, R., McNulty, G. & Taylor, W. *Algebras, Lattices, Varieties I*, Wadsworth and Brooks/Cole, 1987.
- [7] Maksimova, L. Craig's Theorem in Superintuitionistic Logics and Amalgamable Varieties of Pseudo-Boolean Algebras. *Algebra i Logika*, v.16, 6, 1977, 643-681.
- [8] Squire, R. *Presheaf toposes & toposes propositional logic*, Ph.D. Thesis, McMaster University, 1990.

FIXED POINT ISHIKAWA ITERATION IN A CONVEX METRIC SPACE

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Presented by E. Bierstone, F.R.S.C.

ABSTRACT. An Ishikawa type iteration scheme for a family of self mappings of a convex metric space is defined. Subsequently, the above scheme is used to prove common fixed point theorems which, in turn, generalise certain theorems of Ćirić [1], Ding [2] and Rhoades [7,8].

1. DEFINITIONS AND NOTATIONS. For the sake of completeness we recall the following definitions on convex metric spaces. For details we refer to Takahashi [9].

Definition 1.1. Let (X, d) be a metric space and I be the closed unit interval. A mapping $W: X \times X \times I \rightarrow X$ is called a convex structure on X if for all $x, y \in X, \lambda \in I$, $d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y)$ for all $u \in X$. X together with a convex structure is called a convex metric space.

All Banach spaces and their convex subsets are convex metric spaces. However there are many convex metric spaces which are not embedded in any normed space (cf. Takahashi [9]).

Definition 1.2. A nonempty subset C of a convex metric space X is called convex, if $W(x, y, \lambda) \in C$ whenever $x, y \in C$ and $\lambda \in I$.

2. COMMON FIXED POINTS OF ϕ - CONTRACTIVE MAPPINGS.

Throughout this section, let \mathbb{R}_+ denote the set of non-negative reals. Let U denote the family of all mappings $\phi: (\mathbb{R}_+)^5 \rightarrow \mathbb{R}_+$ which are upper semicontinuous and non-decreasing in each coordinate variable. Any pair of mappings satisfying condition (1) below will be called ϕ - contractive. Now we have the following.

Theorem 2.1. Let f, g be self mappings of a non-empty closed convex subset C of a complete convex metric space X . Suppose there exists a $\phi \in U$ such that for all $x, y \in C$,

$$(1) \quad d(fx, gy) \leq \phi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)),$$

where ϕ satisfies the condition

$$(2) \quad \max\{ \phi(t, t, t, 0, t), \phi(0, 0, t, t, 0), \phi(0, t, 0, 0, t) \} < t \text{ for all } t > 0.$$

Suppose that $\{x_n\}$ is an Ishikawa type iteration scheme defined by

$$(3) \quad x_0 \in C, y_n = W(fx_n, x_n, \beta_n), x_{n+1} = W(gy_n, x_n, \alpha_n), n \geq 0 \text{ where } \{\alpha_n\} \text{ and } \{\beta_n\} \\ \text{satisfy } 0 \leq \alpha_n, \beta_n \leq 1.$$

If $\{\alpha_n\}$ is bounded away from zero and $\{x_n\}$ converges to p , then p is a common fixed point of f and g .

Proof. From (3) and Takahashi[9, p 145], it follows that $d(x_{n+1}, x_n) = d(W(gy_n, x_n, \alpha_n), x_n) = \alpha_n d(gy_n, x_n)$. Since $x_n \rightarrow p$, $d(x_{n+1}, x_n) \rightarrow 0$ as $n \rightarrow \infty$. Also $\{\alpha_n\}$ is bounded away from zero, it follows that $d(gy_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$. By condition (1) we have

$$(4) \quad d(fx_n, gy_n) \leq \phi(d(x_n, y_n), d(x_n, fx_n), d(y_n, gy_n), d(x_n, gy_n), d(y_n, fx_n))$$

Further, $d(x_n, y_n) \leq d(x_n, W(fx_n, x_n, \beta_n)) \leq \beta_n d(x_n, fx_n) \leq \beta_n [d(x_n, gy_n) + d(gy_n, fx_n)] \leq d(x_n, gy_n) + d(gy_n, fx_n)$. Similarly we have $d(x_n, fx_n) \leq d(x_n, gy_n) + d(gy_n, fx_n)$, $d(y_n, gy_n) \leq d(fx_n, gy_n) + d(x_n, gy_n)$ and $d(y_n, fx_n) \leq d(x_n, fx_n)$. Suppose $r = \limsup_n d(fx_n, gy_n)$.

Then $r = 0$. Otherwise, from (4) it follows that $r \leq \phi(r, r, r, 0, r) < r$, which is absurd. Hence $r = 0$ and $\lim_n d(fx_n, gy_n) = 0$.

Since $d(x_n, fx_n) \leq d(x_n, gy_n) + d(gy_n, fx_n) \rightarrow 0$ as $n \rightarrow \infty$, it follows that $d(p, fx_n) \rightarrow 0$ as $n \rightarrow \infty$. Again by setting $x = x_n$ and $y = p$ in (1) and using $d(p, gp) \leq d(p, x_n) + d(x_n, gp)$, $d(x_n, gp) \leq d(x_n, fx_n) + d(fx_n, gp)$, it follows that $s = \limsup_n d(fx_n, gp) = 0$. Otherwise, $s \leq \phi(0, 0, s, s, 0) < s$, a contradiction. Therefore $\limsup_n d(fx_n, gp) = 0$ and thus $\lim_n d(fx_n, gp) = 0$.

Further, since $d(p, gp) \leq d(p, x_n) + d(x_n, fx_n) + d(fx_n, gp) \rightarrow 0$ as $n \rightarrow \infty$, it follows that $p = gp$. A similar argument will lead to $p = fp$. Hence p is a common fixed point of f and g .

Corollary 2.2. Let X and C be as in Theorem 2.1 and let f and g be self mappings of C . Suppose there is a constant $k \in [0,1)$ such that for all $x,y \in C$,

$$d(fx,gy) \leq k \max\{d(x,y), d(x,fx), d(y,gy), d(x,gy)+d(y,gx)\}.$$

If $\{x_n\}$ is an Ishikawa type iterative scheme satisfying the conditions of Theorem 2.1 and converging to p , then p is a common fixed point of f and g .

Proof. If $\phi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, t_4 + t_5\}$ for all $(t_1, t_2, t_3, t_4, t_5) \in (\mathbb{R}_+)^5$, then ϕ satisfies all the requirements of Theorem 2.1 and hence the conclusion follows.

Remark 2.3. For $f = g$, the results of Ding [1, Theorem 2.3 and Corollary 2.1] follow as special cases of our above results, which in turn, generalise the results of Naimpally and Singh [6, Theorem 1.2] and Rhoades [8, Theorem 9].

The following two theorems follow from Theorem 2.1 .

Theorem 2.4. Let X and C be as in Theorem 2.1. Let F be the family of all self mappings of C such that for any pair $f, g \in F$, there exists a $\phi = \phi(f,g) \in U$ such that conditions of Theorem 2.1 are satisfied. If for any pair $f, g \in F$, the corresponding sequence $\{x_n\}$ of Ishikawa type iteration converges to p , then p is a common fixed point of f and g .

Theorem 2.5. Let X and C be as in Theorem 2.1. Let F be the family of all self mappings of C such that for any pair $f, g \in F$, there exist positive integers $m = m(f,g)$, $n = n(f,g)$ and a $\phi = \phi(f,g)$ such that conditions of Theorem 2.1 are satisfied with f and g being replaced by f^m and g^n respectively. If the corresponding sequence of Ishikawa type iteration $\{x_n\}$ converges to p , then p is a common fixed point of f and g .

3. COMMON FIXED POINTS OF QUASI-CONTRACTIVE MAPPINGS.

These mappings were introduced by Ćirić [1]. While studying the convergence of Ishikawa iteration procedure[3], Rhoades [8] asked an open question that if the above procedure could be extended to quasi-contractive mappings. This question was answered in affirmative, among others by Liu [4, 5] in Hilbert spaces and by Ding [2] in convex metric

spaces. In this section we continue to generalize above results.

Let \mathbb{Z}_+ denote the set of non-negative integers. For $m, n \in \mathbb{Z}_+$ with $m > n$, let $O_{n,m} = \cup \{x_j, y_j, fx_j, gy_j; j = n, n+1, \dots, n+m\}$, where f and g are as defined in Theorem 3.1 below. Further let $\delta(O_{n,m})$ denote the diameter of $O_{n,m}$. Now we have the following.

Theorem 3.1. Let f and g be self mappings of a non-empty closed convex subset C of a complete convex metric space X . Suppose there exists a constant $k \in (0,1)$ such that for all $x, y \in C$,

$$(1) \quad d(fx, gy) \leq k \max \{ d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx) \},$$

$$(2) \quad \max \{ d(fx, fy), d(gx, gy) \} \leq k \delta(O_{n,m}).$$

Suppose $\{x_n\}$ is an Ishikawa type iteration scheme defined by

$$(3) \quad x_0 \in C, y_n = W(fx_n, x_n, \beta_n), x_{n+1} = W(gy_n, x_n, \alpha_n), n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \leq \alpha_n, \beta_n \leq 1$ and $\sum \alpha_n$ diverges.

Then $\{x_n\}$ converges to a unique common fixed point of f and g .

Proof. From conditions (1) and (2) for $p, q \in \mathbb{Z}_+$ with $n \leq p, q \leq m$, it follows that

$$(4) \quad \max \{ d(fx_p, gy_q), d(gy_p, fx_q), d(fx_p, fx_q), d(gy_p, gy_q) \} \leq k \delta(O_{n,m})$$

Further, using conditions (1), (2) and proof techniques of Ding [2], it can be easily verified that

$$(5) \quad \delta(O_{n,m}) = \max \{ d(x_n, fx_j), d(x_n, gy_j); n \leq j \leq m \}.$$

Hence for any $m \in \mathbb{Z}_+$, $\delta(O_{0,m}) = \max \{ d(x_0, fx_j), d(x_0, gy_j); 0 \leq j \leq m \} \leq d(x_0, gx_0) + \max \{ d(gx_0, fx_j), d(gx_0, gy_j); 0 \leq j \leq m \} \leq d(x_0, gx_0) + k \delta(O_{0,m})$. Therefore $\delta(O_{0,m}) \leq (1/1-k) d(x_0, gx_0)$. A similar argument will lead to $\delta(O_{0,m}) \leq (1/1-k) d(x_0, fx_0)$.

Setting $\mu = \max \{ d(x_0, fx_0), d(x_0, gx_0) \}$, we have $\delta(O_{0,m}) \leq \mu / 1-k$.

Further, by (5) $\delta(O_{n,m}) = \max \{ d(x_n, fx_j), d(x_n, gy_j); n \leq j \leq m \}$. Now the following cases arise. (a) $\delta(O_{n,m}) = d(x_n, fx_j)$. In this case for any $m, n \in \mathbb{N}_+$ with $0 < n < m$ we have $\delta(O_{n,m}) = d(x_n, fx_j) = d(fx_j, x_n) = d(fx_j, W(gy_{n-1}, x_{n-1}, \alpha_{n-1})) \leq \alpha_{n-1} d(fx_j, gy_{n-1}) + (1-\alpha_{n-1}) d(fx_j, x_{n-1}) = \alpha_{n-1} k \delta(O_{n-1,m}) + (1-\alpha_{n-1}) \delta(O_{n-1,m}) = [1 - (1-k)\alpha_{n-1}] \delta(O_{n-1,m})$. (b) $\delta(O_{n,m}) = d(x_n, gy_j)$. Now using a similar argument as in (a), it can be easily verified that $\delta(O_{n,m}) \leq [1 - (1-k)\alpha_{n-1}] \delta(O_{n-1,m})$ for $m, n \in \mathbb{N}_+$ with $0 < n < m$. Hence in either case

$$\begin{aligned} \delta(O_{n,m}) &\leq [1 - (1-k)\alpha_{n-1}] \delta(O_{n-1,m}) \\ &\leq \dots\dots\dots \\ &\leq \prod_{j=0}^{n-1} [1 - (1-k)\alpha_j] \delta(O_{0,m}) \leq (\mu / 1-k) \prod_{j=0}^{n-1} [1 - (1-k)\alpha_j] \end{aligned}$$

Since $1-k > 0$ and $\sum_j \alpha_j$ diverges, $\prod_j [1 - (1-k)\alpha_j] = 0$. Therefore $\delta(O_{n,m}) \rightarrow 0$ as $n, m \rightarrow \infty$ and hence $\{x_n\}$ is a Cauchy sequence. We also note that $d(x_n, fx_n) \rightarrow 0$ and $d(x_n, gx_n) \rightarrow 0$ as $n \rightarrow \infty$. Since X is complete, $x_n \rightarrow s$ for some $s \in X$. Therefore $fx_n \rightarrow s$ and $gx_n \rightarrow s$ as $n \rightarrow \infty$. Now using condition (1) with $x = x_n, y = s$ and making $n \rightarrow \infty$, it follows that $s = gs$. A similar argument will result in $s = fs$. Hence s is a common fixed point of f and g . The uniqueness of s as a common fixed point of f and g can be easily verified.

Remark 3.2. Our above theorem generalizes a result of Rhoades [7, Theorem 7] among other results. In particular, for $f = g$, we have a result of Ding [2, Theorem 2.1].

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REFERENCES

- [1] Ćirić, L.B., A generalisation of Banach's contraction principle, Proc. Amer. Math. Soc. 45(1974), 267-273.
- [2] Ding, X.P., Iteration processes for nonlinear mappings in convex metric spaces, J. Math. Anal. Appl. 132(1988), 114-122.
- [3] Ishikawa, S., Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44(1974), 147-150.
- [4] Liu, Q., On Naimpally and Singh's open questions, J. Math. Anal. Appl. 124(1987), 157-164.
- [5] ———, A convergence theorem of Ishikawa iterates for quasi-contractive mappings, J. Math. Anal. Appl. 146(1990), 301-305.
- [6] Naimpally, S.A. and Singh K.L., Extensions of some fixed point theorems of Rhoades, J. Math. Anal. Appl. 96(1983), 437-446.
- [7] Rhoades, B.E., Fixed point iterations using infinite matrices, Trans. Amer. Math. Soc. 196(1974), 161-176.
- [8] ———, Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56(1976), 741-750.
- [9] Takahashi, W., A convexity in metric space and nonexpansive mappings I, Kodai Math. Sem. Rep. 22(1970), 142-149.

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FERMAT AND EULER TYPE QUOTIENTS

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1. INTRODUCTION

Let p be a prime and n be a positive integer ≥ 2 . For an integer a with $(a, p) = (a, n) = 1$, we denote by $q_p(a)$ and $q(a, n)$ the Fermat and Euler quotients with base a , respectively. That is,

$$q_p(a) = \frac{a^{p-1} - 1}{p} \quad \text{and} \quad q(a, n) = \frac{a^{\varphi(n)} - 1}{n},$$

where φ is the Euler totient function.

It is easily seen that the above quotients satisfy the "logarithmic property" as follows: if $(ab, p) = (ab, n) = 1$, then

$$\begin{aligned} q_p(ab) &\equiv q_p(a) + q_p(b) \pmod{p}, \\ q(ab, n) &\equiv q(a, n) + q(a, n) \pmod{n}. \end{aligned}$$

Many interesting arithmetic properties of these quotients are known in connection with Bernoulli numbers and Wilson quotients for prime and composite numbers (see, e.g., [1, 2]).

In his classical papers [3, 4], Lerch introduced the following beautiful expressions for Fermat and Euler quotients:

$$\begin{aligned} q_p(a) &\equiv \sum_{k=1}^{p-1} \frac{1}{ak} \left[\frac{ak}{p} \right] \pmod{p}, \\ q(a, n) &\equiv \sum_{\substack{k=1 \\ (k, n)=1}}^{n-1} \frac{1}{ak} \left[\frac{ak}{n} \right] \pmod{n}, \end{aligned}$$

where $[x]$ means the greatest integer $\leq x$.

For a and n with $(a, n) = 1$, we now define the following special quotient:

$$Q(a, n) = \frac{a^e - 1}{n},$$

where $e = \text{ord}_n a$ (the order of a modulo n), i.e., e is the least positive exponent of a such that $a^e \equiv 1 \pmod{n}$.

This quotient possesses similar properties to Fermat and Euler quotients. Clearly, if a is a primitive root modulo n , then $Q(a, n)$ coincides with $q(a, n)$.

In the present paper we shall introduce basic properties of $Q(a, n)$ and discuss some expressions of the Lerch type for $Q(a, n)$ and $q(a, n)$.

2. BASIC PROPERTIES OF $Q(a, n)$

First, we deal with a relation between $q(a, n)$ and $Q(a, n)$:

Proposition 1. *Let a, n be integers with $(a, n) = 1$ and $e = \text{ord}_n a$. Then*

$$q(a, n) \equiv \frac{\varphi(n)}{e} Q(a, n) \pmod{n}.$$

Proof. We let $u = \varphi(n)/e$. Then

$$\begin{aligned} a^{\varphi(n)} - 1 &= ((a^e - 1) + 1)^u - 1 \\ &= \prod_{i=1}^u \binom{u}{i} (a^e - 1)^i \\ &\equiv u(a^e - 1) \pmod{n^2}, \end{aligned}$$

which proves the assertion. \square

The following proposition corresponds to the logarithmic properties of $q_p(a)$ and $q(a, n)$ mentioned in the Introduction.

Proposition 2. *Let a, b, n be integers with $(ab, n) = 1$ and let $e = \text{ord}_n a, l = \text{ord}_n b, k = \text{ord}_n ab$. Then*

$$elQ(ab, n) \equiv klQ(a, n) + ekQ(b, n) \pmod{n}.$$

Proof. We shall use an identity

$$(ab)^X - 1 = (a^X - 1)(b^X - 1) + (a^X - 1) + (b^X - 1),$$

where X is any integer. Here, take $X = ekl$. Then both sides become

$$(ab)^{ekl} - 1 = \{((ab)^k - 1) + 1\}^{el} - 1 \equiv el\{(ab)^k - 1\} \pmod{n^2}$$

and

$$\begin{aligned} &(a^{ekl} - 1)(b^{ekl} - 1) + (a^{ekl} - 1) + (b^{ekl} - 1) \\ &\equiv \{((a^e - 1) + 1)^{kl} - 1\} + \{((b^l - 1) + 1)^{ek} - 1\} \\ &\equiv kl(a^e - 1) + ek(b^l - 1) \pmod{n^2}, \end{aligned}$$

which deduce the congruence as desired. \square

Proposition 3. *Let a, n be integers with $(a, n) = 1$, x be any integer and $e = \text{ord}_n a$. Then*

$$Q(a + xn, n) \equiv Q(a, n) + \frac{xe}{a} \pmod{n}.$$

Proof. Noting that $\text{ord}_n a = \text{ord}_n(a + xn)$, we have

$$\begin{aligned} Q(a + xn, n) &= \frac{(a + xn)^e - 1}{n} \equiv \frac{a^e - 1}{n} + \binom{e}{1} a^{e-1} x \\ &\equiv Q(a, n) + \frac{xe}{a} \pmod{n}, \end{aligned}$$

as indicated. \square

Proposition 4. *Let a, m, n be integers with $(a, mn) = (m, n) = 1$ and let $d = \text{ord}_m a, e = \text{ord}_n a, f = \text{ord}_{mn} a$. Then*

$$Q(a, mn) \equiv \frac{f}{dn} Q(a, m) \pmod{m},$$

$$Q(a, mn) \equiv \frac{f}{em} Q(a, n) \pmod{n}.$$

Proof. We can easily deduce

$$\begin{aligned} Q(a, mn) &= \frac{a^f - 1}{mn} = \frac{1}{mn} \{((a^d - 1) + 1)^{f/d} - 1\} \\ &\equiv \frac{1}{mn} \binom{f/d}{1} (a^d - 1) \\ &\equiv \frac{f}{dn} Q(a, m) \pmod{m}, \end{aligned}$$

which proves the first formula. The second one can be given by interchanging m and n (and so, d and e) in the first. \square

Proposition 5. *Let a, m, n, d, e, f be as in Proposition 4 and let X, Y be integers satisfying $m^2 X + n^2 Y = 1$. Then*

$$Q(a, mn) \equiv \frac{n f}{d} Y Q(a, m) + \frac{m f}{e} X Q(a, n) \pmod{mn}.$$

Proof. For brevity, set $A = (f/d)Q(a, m), B = (f/e)Q(a, n)$ and $C = Q(a, mn)$. From the first congruence of Proposition 4 there exists an integer t such that $nC = A + tm$. Since the second congruence gives $mnC \equiv Bn \pmod{n^2}$, we have $m(A + tm) \equiv Bn \pmod{n^2}$, hence $m^2 t \equiv nB - mA \pmod{n^2}$. Multiplying by X , it follows that $m^2 tX = (1 - n^2 Y)t \equiv t \equiv (nB - mA)X \pmod{n^2}$. Hence, we may write $t = (nB - mA)X + n^2 s$, where s is some integer. Using this we obtain

$$\begin{aligned} nC &= A + mt = A + m\{(nB - mA)X + n^2 s\} \\ &\equiv (1 - m^2 X)A + mnXB \\ &\equiv n^2 YA + mnXB \pmod{mn^2}, \end{aligned}$$

hence $C \equiv nYA + mXB \pmod{mn}$. This completes the proof. \square

We note that the original result for Euler quotients corresponding to Proposition 5 has been proved by Lerch [4]. In fact, he mentioned that, with the same notations as above, if $(a, mn) = (m, n) = 1$, then

$$q(a, mn) \equiv n\varphi(n)Yq(a, m) + m\varphi(m)Xq(a, n) \pmod{mn}.$$

3. LERCH TYPE EXPRESSIONS

In the following, assume that a, n are positive integers with $n \geq 2$ and $(a, n) = 1$. Let $\Gamma(n) = \{k \mid 1 \leq k \leq n - 1, (k, n) = 1\}$. We now consider a nonempty subset $A(a)$ of $\Gamma(n)$ satisfying the following properties:

- (I) For any $x \in A(a)$, there exists an element $y \in A(a)$ such that $ax = [ax/n]n + y$.
- (II) The set $A(a)$ contains no proper subset satisfying (I).

Note that for any given positive integers a and $n \geq 2$ with $(a, n) = 1$, the existence of such a subset $A(a)$ satisfying (I) and (II) is obviously assured. Also, note that the set $A(a)$ is not determined uniquely. In fact, if Γ^* is the multiplicative group of irreducible system of residues modulo n , then $A(a)$ is any reduced residue class of Γ^*/\sim , where the relation \sim is defined by " $x \sim y$ (for $x, y \in \Gamma^*$) if and only if there exists an integer i such that $x = a^i y$ ".

As an example, if we take $n = p = 13$ and $1 \leq a \leq 12$, then the subsets $A(a)$ of $\Gamma(n)$ satisfying the above (I) and (II) are:

$$\begin{aligned} A(1) &: \{1\}, \{2\}, \dots, \{12\}; \\ A(12) &: \{1, 12\}, \{2, 11\}, \{3, 10\}, \{4, 9\}, \{5, 8\}, \{6, 7\}; \\ A(3), A(9) &: \{1, 3, 9\}, \{2, 5, 6\}, \{4, 10, 12\}, \{7, 8, 11\}; \\ A(5), A(8) &: \{1, 5, 8, 12\}, \{2, 3, 10, 11\}, \{4, 6, 7, 9\}; \\ A(4), A(10) &: \{1, 3, 4, 9, 10, 12\}, \{2, 5, 6, 7, 8, 11\}; \\ A(2), A(6), A(7), A(11) &: \{1, 2, \dots, 12\}. \end{aligned}$$

Let $A(a) = \{r_1, r_2, \dots, r_e\}$ be any one of subsets of $\Gamma(n)$ satisfying (I) and (II). Then, by changing order of elements of $A(a)$ appropriately we can give the following cyclic relations:

$$\begin{aligned} ar_1 &= \left[\frac{ar_1}{n} \right] n + r_2, \\ ar_2 &= \left[\frac{ar_2}{n} \right] n + r_3, \\ &\dots\dots\dots \\ ar_{e-1} &= \left[\frac{ar_{e-1}}{n} \right] n + r_e, \\ ar_e &= \left[\frac{ar_e}{n} \right] n + r_1, \end{aligned} \tag{3.1}$$

where $r_i \neq r_j$ if $i \neq j$, $1 \leq i, j \leq e$.

Conversely, if $A(a) = \{r_1, r_2, \dots, r_e\}$ is the set generated by the procedure (3.1), then it is clear that $A(a)$ satisfies (I) and (II).

It is also easily seen that the number $e = \#A(a)$ of elements of $A(a)$ is equal to $\text{ord}_n a$. In fact, since $ar_i \equiv r_{i+1} \pmod{n}$ for $i = 1, 2, \dots, e$, where $r_{e+1} = r_1$, we have $a^e \pi \equiv \pi \pmod{n}$, hence $(a^e - 1)\pi \equiv 0 \pmod{n}$, where $\pi = r_1 r_2 \dots r_e$. Noting that $(\pi, n) = 1$, we obtain $a^e \equiv 1 \pmod{n}$. Simultaneously, we know from (3.1) that the number e is the least positive exponent of a satisfying this congruence, and so $e = \text{ord}_n a$.

In the following, we fix an ordered subset $A(a)$ of $\Gamma(n)$ with $e = \#A(a)$ satisfying (I) and (II) (hence (3.1)), and write it as $A(a) = \{r_1, r_2, \dots, r_e\}$.

Proposition 6. *If $(a, n) = 1$, then*

$$Q(a, n) \equiv \sum_{x \in A(a)} \frac{1}{ax} \left[\frac{ax}{n} \right] \pmod{n}$$

and

$$q(a, n) \equiv \frac{\varphi(n)}{e} \sum_{x \in A(n)} \frac{1}{ax} \left[\frac{ax}{n} \right] \pmod{n}.$$

Proof. Since $A(a)$ satisfies (3.1), we have

$$\begin{aligned} a^e \pi &= \prod_{j=1}^e \left(\left[\frac{ar_j}{n} \right] n + r_{j+1} \right) \\ &\equiv \pi + n \sum_{i=1}^e \frac{\pi}{r_{i+1}} \left[\frac{ar_i}{n} \right] \pmod{n^2}, \end{aligned}$$

where $\pi = r_1 r_2 \cdots r_e$ and $r_{e+1} = r_1$. Since $(a, \pi) = 1$, we can deduce

$$Q(a, n) \equiv \sum_{i=1}^e \frac{1}{r_{i+1}} \left[\frac{ar_i}{n} \right] \pmod{n},$$

which shows the first congruence, because $r_{i+1} \equiv ar_i \pmod{n}$ for $i = 1, 2, \dots, e$. The second congruence for $q(a, n)$ immediately follows from Proposition 1. \square

Proposition 7. *If $(a, n) = 1$, then*

$$Q(a, n) \equiv \frac{1}{n} \left(a \sum_{i=1}^e \frac{r_i}{r_{i+1}} - e \right) \pmod{n}$$

and

$$q(a, n) \equiv \frac{\varphi(n)}{en} \left(a \sum_{i=1}^e \frac{r_i}{r_{i+1}} - e \right) \pmod{n},$$

where $r_{e+1} = r_1$.

Proof. From (3.1) we have

$$a \frac{r_i}{r_{i+1}} = \frac{1}{r_{i+1}} \left[\frac{ar_i}{n} \right] n + 1$$

for $i = 1, 2, \dots, e$. Hence, summing over $i = 1, 2, \dots, e$ we get

$$a \sum_{i=1}^e \frac{r_i}{r_{i+1}} - e = n \sum_{i=1}^e \frac{1}{r_{i+1}} \left[\frac{ar_i}{n} \right],$$

which concludes the statement by means of Proposition 6. \square

Next proposition deals with explicit expressions of $Q(a, n)$ and $q(a, n)$:

Proposition 8. *If $(a, n) = 1$, then*

$$Q(a, n) = \frac{1}{r_1} \sum_{i=1}^e a^{e-i} \left[\frac{ar_i}{n} \right].$$

In particular, if a is a primitive root mod n , then

$$q(a, n) = \frac{1}{r_1} \sum_{i=1}^{\varphi(n)} a^{\varphi(n)-i} \left[\frac{ar_i}{n} \right].$$

Proof. We obtain from (3.1)

$$\begin{aligned} & a^e r_1 + a^{e-1} r_2 + \cdots + a^2 r_{e-1} + a r_e \\ &= a^{e-1} R_1 + a^{e-2} R_2 + \cdots + a R_{e-1} + R_e, \end{aligned}$$

where

$$R_i = \left[\frac{ar_i}{n} \right] n + r_{i+1}, \quad i = 1, 2, \dots, e,$$

and $r_{e+1} = r_1$. This relation immediately yields

$$(a^e - 1)r_1 = \left(\sum_{i=1}^e a^{e-i} \left[\frac{ar_i}{n} \right] \right) n,$$

which proves the assertion for $Q(a, n)$. For the second equality for $q(a, n)$ we have only to put $e = \varphi(n)$. \square

We may choose any integer r_1 so far as $(r_1, n) = 1$ in the first procedure in (3.1). Therefore, if we take especially $r_1 = 1$, then each r_i ($i = 1, 2, \dots, e$) can be expressed as follows:

$$r_i = a^{i-1} - \left[\frac{a^{i-1}}{n} \right] n.$$

REFERENCES

1. AGOH, T., *On Fermat and Wilson quotients*, Preprint (1995).
2. AGOH, T., SKULA, L. AND DILCHER, K., *Fermat and Wilson quotients for composite moduli*, Preprint (1995).
3. LERCH, M., *Zur Theorie des Fermatschen Quotienten $\frac{a^{p-1}-1}{p} = q(a)$* , Math. Annalen 60 (1905), 471-490.
4. LERCH, M., *Sur les théorèmes de Sylvester concernant le quotient de Fermat*, C.R. Acad. Sci. Paris 142 (1906), 35-38.

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Convolution algebras and factorization of measures on Chébli-Trimèche hypergroups.

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ABSTRACT. We consider the Chébli-Trimèche hypergroup $(\mathbb{R}_+, *(A))$ associated with the function A on \mathbb{R}_+ , its Haar measure m_A , the algebra $(M_b(\mathbb{R}_+), *(A))$ of bounded complex measures on \mathbb{R}_+ and $(L^1(\mathbb{R}_+), *(A))$ the subalgebra of integrable functions on \mathbb{R}_+ with respect to the measure m_A . In this paper we prove that the maximal ideal space of the algebras of $(L^1(\mathbb{R}_+), *(A))$ (respectively $(M_b(\mathbb{R}_+), *(A))$) is homeomorphic to the set $\Sigma = \{ \lambda \in \mathbb{C} : |Im(\lambda)| \leq \rho \}$ with $\rho = \frac{1}{2} \lim_{r \rightarrow +\infty} \frac{A'(r)}{A(r)}$ (respectively $\tilde{\Sigma} = \Sigma \cup \{+\infty\}$). We also establish some results on the factorization of functions and measures on the hypergroup $(\mathbb{R}_+, *(A))$.

0. Introduction

Throughout this paper $(\mathbb{R}_+, *(A))$ will be the Chébli-Trimèche hypergroup (see [1] p:201-248, [6] and [7]) associated with the function A on $\mathbb{R}_+ = [0, +\infty[$ such that $A(r) = r^{2\alpha+1} B(r)$ with $\alpha > -1/2$ and B is an positive and even C^∞ -function on \mathbb{R} satisfying $B(0) = 1$. We assume A not decreasing, $\lim_{r \rightarrow +\infty} A(r) = +\infty$, A'/A not decreasing, $\lim_{r \rightarrow +\infty} A'(r)/A(r) = 2\rho \geq 0$ and for all $a > 0$ there exist $\delta > 0$ such that for all $r \geq a$, we have $B'(r)/B(r) = 2\rho - (2\alpha + 1)r^{-1} + e^{-\delta r} F(r)$ if $\rho > 0$ or $B'(r)/B(r) = c^{-\delta r} F(r)$ if $\rho = 0$, where F is a C^∞ -function on $[a, +\infty[$, bounded together with its derivatives.

As particular cases of Chébli-Trimèche hypergroups we have Bessel-Kingman and Jacobi hypergroups which correspond respectively to $A(r) = r^{2\alpha+1}$, $\alpha > -1/2$, and $A(r) = 2^{2(\alpha+\beta+1)} sh(r)^{2\alpha+1} ch(r)^{2\beta+1}$, $\alpha \geq \beta \geq -1/2$, $\alpha > -1/2$ (see [1] p:234-235, [6] and [7]).

We note that $(\mathbb{R}_+, *(A))$ is commutative with neutral element 0 and the identity mapping as the involution (see [1], [6] and [7] for general details on hypergroups). Indeed $M_b(\mathbb{R}_+)$ the space of bounded complex measures on \mathbb{R}_+ , is a commutative Banach algebra with norm $\| \mu \| = \int_{\mathbb{R}_+} d | \mu |$, convolution $*(A)$ and identity δ_0 where δ_r is the unit point mass at $r \in \mathbb{R}_+$. The Haar measure m_A on $(\mathbb{R}_+, *(A))$ is absolutely continuous with respect to the Lebesgue measure, and can be chosen to have density A .

1. Fourier transform and convolution on Chébli-Trimèche hypergroups.

Let L_A be the differential operator defined on \mathbb{R}_+^* by

$$L_A u(r) = \frac{1}{A(r)} \left\{ \frac{d}{dr} (A(r) \frac{du}{dr}(r)) \right\},$$

for each function u twice differentiable on \mathbb{R}_+^* .

The solutions $\varphi_\lambda, \lambda \in \mathbb{C}$, of the differential equation

$$\begin{cases} L_A \varphi_\lambda(r) = -(\lambda^2 + \rho^2) \varphi_\lambda(r), \\ \varphi_\lambda(0) = 1, \quad \varphi'_\lambda(0) = 0. \end{cases}$$

are multiplicative on $(\mathbb{R}_+, \ast(A))$ in the sense that

$$\int_{\mathbb{R}_+} \varphi_\lambda(t) \delta_r \ast \delta_s(dt) = \varphi_\lambda(r) \varphi_\lambda(s),$$

and they give all multiplicative functions on the hypergroup. The dual is given by $\hat{\mathbb{R}}_+ = \{\varphi_\lambda \mid \lambda \in \mathbb{R}_+ \cup i[0, \rho]\}$. It is convenient to identify $\hat{\mathbb{R}}_+$ with the set $\mathbb{R}_+ \cup i[0, \rho]$.

The function $\varphi_\lambda, \lambda \in \mathbb{C}$, satisfies the following properties

i) For all r in \mathbb{R}_+ we have $|\varphi_\lambda(r)| \leq 1$ if and only if λ belongs to the set $\Sigma = \{\lambda \in \mathbb{C} \mid |\text{Im}(\lambda)| \leq \rho\}$.

ii) For all λ in \mathbb{C} , the function φ_λ admits the following Mehler integral representation: $\forall r \in \mathbb{R}_+^*, \varphi_\lambda(r) = \int_0^r K(r, u) \cos(\lambda u) du$, where $K(r, \cdot)$ is supported in $[-r, r]$ and is infinitely differentiable on $]-r, r[$.

iii) We suppose $\rho = 0$ and $\alpha \geq 3/2$. Then for all integers $k, 0 \leq k \leq \alpha - 1/2$, there exists a constant $D_k > 0$ such that

$$\forall r \geq 1, \forall \lambda > 0, \quad \left| \left(\frac{d}{d\lambda} \right)^k |\varphi_\lambda(r)| \right| \leq \frac{D_k}{\lambda^k}.$$

(This inequality can be deduced from [4] Theorem 3.1 p:60).

Notation.

For $p \in [1, +\infty[$ we write $L^p(m_A) = \{f \mid \|f\|_p < +\infty\}$, where

$$\|f\|_p = \left(\int_{\mathbb{R}_+} |f(r)|^p dm_A(r) \right)^{1/p}.$$

Definition 1.1.

i) The Fourier transform of a function f in $L^1(m_A)$ is defined by

$$\forall \lambda \in \Sigma, \quad \mathcal{F}_A(f)(\lambda) = \int_{\mathbb{R}_+} f(r) \varphi_\lambda(r) A(r) dr.$$

ii) For r in \mathbb{R}_+ and f in $L^1(m_A)$ the r -translate of f is defined by

$$T_r(f)(s) = f(r \ast s) = \int_{\mathbb{R}_+} f(u) d(\delta_r \ast \delta_s)(u).$$

Properties. (See [2], [4], [5], [6] and [7]).

i) For all r, s in \mathbb{R}_+^* there exists a nonnegative function $W(r, s, \cdot)$ supported in $]\mid r - s \mid, r + s[$, continuous on $]\mid r - s \mid, r + s[$, and belongs to $L^1(m_A)$ with $\|W(r, s, \cdot)\|_1 = 1$ and $\delta_r \ast \delta_s(du) = W(r, s, u) dm_A(u)$.

ii) The function $W(r, s, \cdot)$ is the kernel of the translation in the sense that

$$\forall r, s \in \mathbb{R}_+^*, T_r(f)(s) = f(r * s) = \int_{|r-s|}^{r+s} f(u)W(r, s, u)dm_A(u).$$

iii) For all r, s in \mathbb{R}_+ , and for all λ in Σ , $\mathcal{F}_A(W(r, s, \cdot))(\lambda) = \varphi_\lambda(r)\varphi_\lambda(s)$.

iv) For all f in $L^1(m_A)$ and r in \mathbb{R}_+ we have

$$\forall \lambda \in \Sigma, \mathcal{F}_A(T_r(f))(\lambda) = \varphi_\lambda(r)\mathcal{F}_A(f)(\lambda).$$

Definition 1.2. The convolution of two functions f and g in $L^1(m_A)$ is the function $f * g$ in $L^1(m_A)$ defined by: $f * g(r) = \int_{\mathbb{R}_+} T_r(f)(s)g(s)dm_A(s)$.

Properties.

- i) For all f, g in $L^1(m_A)$ we have: $\forall \lambda \in \Sigma, \mathcal{F}_A(f * g)(\lambda) = \mathcal{F}_A(f)(\lambda)\mathcal{F}_A(g)(\lambda)$.
- ii) The space $(L^1(m_A), *)$ is a commutative Banach algebra.

2. The maximal ideal space of $L^1(m_A)$.

Theorem 2.1. To each homomorphism χ from $L^1(m_A)$ into \mathbb{C} there corresponds a unique element λ in Σ such that: $\chi(f) = \mathcal{F}_A(f)(\lambda)$, for all f in $L^1(m_A)$.

Remark. The theorem 2.1 proves that the Fourier transform \mathcal{F}_A coincides with the Gelfand transform defined on $L^1(m_A)$, by: $\hat{f}(\chi) = \chi(f)$, $\chi \in S$, where S denotes the set of all homomorphisms χ from $L^1(m_A)$ into \mathbb{C} .

Let $L^1(m_A)^\wedge$ the set of all \hat{f} where $f \in L^1(m_A)$. The Gelfand topology of S is the weak topology induced by $L^1(m_A)^\wedge$, that is the weakest topology that makes every \hat{f} continuous. Then we have: $L^1(m_A)^\wedge \subset C(S)$, where $C(S)$ is the space of all complex continuous functions on S .

The set S equipped with its Gelfand topology is usually called the maximal ideal space of $L^1(m_A)$.

Theorem 2.2. The maximal ideal space S of $L^1(m_A)$ is homeomorphic to Σ equipped with the usual topology.

Notation. Let I be a closed ideal of $L^1(m_A)$. The zero set $Z(I)$ of I is defined by: $Z(I) = \{\lambda \in \Sigma \mid \mathcal{F}_A(f)(\lambda) = 0, \text{ for every } f \in I\}$.

Theorem 2.3. We suppose $\alpha > 3/2$. Then for all $\lambda_0 > 0$ the set $\{\lambda_0\}$ is the zero set of at least two distinct ideals of $L^1(m_A)$.

3. The maximal ideal space of $M_b(\mathbb{R}_+)$.

Definition 3.1. We define the Fourier transform of a measure μ in $M_b(\mathbb{R}_+)$ by

$$\forall \lambda \in \Sigma, \mathcal{F}_A(\mu)(\lambda) = \int_{\mathbb{R}_+} \varphi_\lambda(r)d\mu(r).$$

For two measures $\mu, \nu \in M_b(\mathbb{R}_+)$ the convolution is defined by

$$\mu * \nu(f) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} T_r f(s) d\mu(r) d\nu(s),$$

where f is a continuous function on \mathbb{R} , with compact support.

In particular if $\mu = gm_A$ and $\nu = hm_A$ with g and h are $L^1(m_A)$, then $\mu * \nu = (g * h)m_A$ where $g * h$ is the function in $L^1(m_A)$ given by the Definition 1.2.

The convolution $\mu * \nu$ satisfies the relation

$$\forall \lambda \in \Sigma, \mathcal{F}_A(\mu * \nu)(\lambda) = \mathcal{F}_A(\mu)(\lambda)\mathcal{F}_A(\nu)(\lambda).$$

Remark: Let μ be a measure in $M_b(\mathbb{R}_+)$. We put $\mathcal{F}_A(\mu)(+\infty) = \mu(\{0\}) = \lim_{\lambda \rightarrow +\infty} \mathcal{F}_A(\mu)(\lambda)$, then the Fourier transform $\mathcal{F}_A(\mu)$ can be extended on $\tilde{\Sigma} = \Sigma \cup \{+\infty\}$.

Theorem 3.2. To each homomorphism $\chi : M_b(\mathbb{R}_+) \rightarrow \mathbb{C}$ corresponds a unique element λ in $\tilde{\Sigma}$ such that: $\chi(\mu) = \mathcal{F}_A(\mu)(\lambda)$, for all μ in $M_b(\mathbb{R}_+)$.

Remark. The theorem 3.2 shows that the Fourier transform \mathcal{F}_A coincide with the Gelfand transform defined on $M_b(\mathbb{R}_+)$, by: $\hat{\mu}(\chi) = \chi(\mu)$, $\chi \in S^*$, where S^* denote the set of all homomorphisms χ from $M_b(\mathbb{R}_+)$ into \mathbb{C} .

Let $M_b(\mathbb{R}_+)^*$ be the set of all $\hat{\mu}$ where $\mu \in M_b(\mathbb{R}_+)$. The Gelfand topology of S^* is the weak topology induced by $M_b(\mathbb{R}_+)^*$, that is the weakest topology that makes every $\hat{\mu}$ continuous. Then we have: $M_b(\mathbb{R}_+)^* \subset \mathcal{C}(S^*)$.

The set S^* equipped with its Gelfand topology is usually called the maximal ideal space of $M_b(\mathbb{R}_+)$.

Theorem 3.3. The maximal ideal space S^* of $M_b(\mathbb{R}_+)$ is homeomorphic to Σ equipped with the usual topology.

4. Factorization of functions and measures.

In this paragraph we suppose $\rho = 0$. We have : $\Sigma = [0, +\infty[$ and $\tilde{\Sigma} = [0, +\infty]$.

Theorem 4.1. For all f in $L^1(m_A)$ there are functions f_1 and f_2 in $L^1(m_A)$ such that $f = f_1 * f_2$.

Theorem 4.2. If K is a compact set in $\tilde{\Sigma}$ and $\mu \in M_b(\mathbb{R}_+)$ satisfying $\mathcal{F}_A(\mu)(\lambda) \neq 0$, for all λ in K , then there exists $\nu \in M_b(\mathbb{R}_+)$ such that

$$\forall \lambda \in K, \mathcal{F}_A(\mu)(\lambda)\mathcal{F}(\nu)(\lambda) = 1.$$

In particular, if $K = \tilde{\Sigma}$ then $\mu * \nu = \delta_0$.

Definition 4.3. A measure μ in $M_b(\mathbb{R}_+)$ will be called

- i) invertible if there exists a measure ν in $M_b(\mathbb{R}_+)$ such that $\mu * \nu = \delta_0$,
- ii) reducible if we can write $\mu = \mu_1 * \mu_2$ where μ_1 and μ_2 are in $M_b(\mathbb{R}_+)$ and neither is invertible,
- iii) irreducible if it is not reducible.

Definition 4.4. The zero set of a measure μ in $M_b(\mathbb{R}_+)$ is the set $\mathcal{L}(\mu)$ defined by: $\mathcal{L}(\mu) = \{\lambda \in \Sigma \mid \mathcal{F}_\lambda(\mu)(\lambda) = 0\}$.

Theorem 4.5. Let μ be a measure in $M_b(\mathbb{R}_+)$.

- i) If $\mathcal{L}(\mu) = \emptyset$ then μ is irreducible.
- ii) If $\mathcal{L}(\mu) = \{+\infty\}$ then μ is reducible if and only if μ is absolutely continuous with respect to the Lebesgue measure.
- iii) If $\mathcal{L}(\mu)$ contains at least two points then μ is reducible.

Notation. We put $I_\lambda = \{\mu \mid \mathcal{L}(\mu) = \{\lambda\}\}$, $\lambda \in \bar{\Sigma}$.

Theorem 4.6. For all λ in $\Sigma \setminus \{0\}$ the set I_λ contains reducible and irreducible measures.

Remark. A. Schwartz has obtained in [3] the corresponding results for the Bessel-Kingman hypergroup.

References

1. W.R. BLOOM and H. HEYER, *Harmonic Analysis for Probability Measures on hypergroups*. de Gruyter Studies in Mathematics, volume 20, Editors: H. Bauer - J.L. Kazdan - E. Zehnder. de Gruyter Berlin - New-York 1994.
2. H. CHÉBLI, *Opérateurs de translation généralisée et semi-groupes de convolution*, Springer Lectures notes in Maths 404 (1974).
3. A. SCHWARTZ, *The structure of the algebra of Hankel transforms and the algebra of Hankel-Stieljes transforms*, Can. J. Math. Vol 23, N°2, 1971, p 236-246.
4. K. TRIMÈCHE, *Transformation intégrale de Weyl et théorème de Paley-Wiener associés à un opérateur différentiel singulier sur $(0, \infty)$* , J. Math. Pures et Appl. 60 (1981), pp 51-98.
5. K. TRIMÈCHE, *Transmutation operators and Mean-periodic Functions Associated with Differential operators*, Mathematical Reports 4, Part 1, 1988, pp 1-282.
6. Z. XU, *Harmonic Analysis on Chébli-Trimèche hypergroup*, PhD thesis, Murdoch University, Australia, 1994.
7. Hm. ZEUNER, *One-dimensional hypergroups*, Adv. Math. 76, 1989, N°1, pp 1-18.

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The Number of Ambiguous Cycles of Reduced Ideals Having No Ambiguous Ideals in Real Quadratic Orders

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For arbitrary real quadratic orders, we completely determine the number of ambiguous cycles of ideals having no ambiguous ideals in the cycle.

§1. Introduction.

In this paper we classify, and determine the exact number of those ambiguous cycles (which are not necessarily classes) of ideals (which are not necessarily invertible) in real quadratic orders (which are not necessarily Dedekind Domains). The motivation behind the paper is an assertion in [1] about such cycles in the maximal order of a real quadratic field — an assertion which turns out to be incorrect. We do not claim that our results are new, but rather that what is set down in the literature is often confusing, and that this topic deserves a proper elucidation from a strictly ideal-theoretic point of view in a concise and simple manner.

§2. Notation and Preliminaries.

Let $D_0 > 1$ be a square-free positive integer and set $\sigma_0 = 2$ if $D_0 \equiv 1 \pmod{4}$ and $\sigma_0 = 1$ otherwise. Define $\omega_0 = (\sigma_0 - 1 + \sqrt{D_0})/\sigma_0$ and $\Delta_0 = (\omega_0 - \omega'_0)^2 = 4D_0/\sigma_0^2$ where ω'_0 is the algebraic conjugate of ω_0 . Let $\omega_\Delta = f\omega_0 + h$ for some $f, h \in \mathbb{Z}$, and $D = (f/g)^2 D_0$ where $g = \gcd(f, \sigma_0)$. Thus, if $\Delta = (\omega_\Delta - \omega'_\Delta)^2$, then $\Delta = f^2 \Delta_0 = 4D/\sigma^2$ where $\sigma = \sigma_0/g$. D_0 is called the *radicand* associated with the *discriminant* Δ .

If $[\alpha, \beta] = \alpha\mathbb{Z} \oplus \beta\mathbb{Z}$, then $O_\Delta = [1, f\omega_0] = [1, \omega_\Delta]$. This is an *order* in K having *conductor* f and *fundamental discriminant* Δ_0 . Let $I = [a, b + c\omega_\Delta]$, with $a > 0$. It is well-known (eg. see [2]) that $I \not\subseteq \mathbb{Z}$ is an ideal in O_Δ if and only if $c|a, c|b$ and $ac|N(b + c\omega_\Delta)$, where N is the norm from $\mathbb{Q}(\sqrt{\Delta})$ to \mathbb{Q} , i.e. $N(\alpha) = \alpha\alpha'$. For a given ideal I in O_Δ with $I \not\subseteq \mathbb{Z}$, the integers a and c are unique, and a is in fact the least positive rational integer in I . We denote the *least positive rational integer in I* by $L(I)$ and we denote the value of $cL(I)$ by $N(I)$, which we call the *norm of I* . An ideal $I = [a, b + c\omega_\Delta]$ is called *primitive* if $c = 1$. Moreover, if $I = [a, b + \omega_\Delta]$ is primitive then so is its *conjugate* $I' = [a, b + \omega'_\Delta]$. Two ideals I and J of O_Δ are *equivalent* (denoted $I \sim J$) if there exist non-zero $\alpha, \beta \in O_\Delta$ such that $(\alpha)I = (\beta)J$ (where (x) denotes the principal ideal generated by x).

Now we give an elucidation of the theory of continued fractions as it pertains to the above. The details and proofs may also be found in [2].

If $I = [N(I), b + \omega_\Delta]$ is a primitive ideal in O_Δ we denote the continued fraction expansion of $(b + \omega_\Delta)/N(I)$ by $(a, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_l)$ with period length $l = l(I)$ where

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$$(P_0, Q_0) = ((\sigma_0 b + f(\sigma_0 - 1) + h\sigma_0)/g\sigma_0, ar/g),$$

and (for $i \geq 0$),

$$D = P_{i+1}^2 + Q_i Q_{i+1},$$

$$P_{i+1} = a_i Q_i - P_i,$$

and

$$a_i = \lfloor (P_i + \sqrt{D})/Q_i \rfloor,$$

with $\lfloor \cdot \rfloor$ being the greatest integer function.

From the continued fraction factoring algorithm (as given, for example, in [2]), we get all reduced ideals equivalent to a given reduced ideal $I = [N(I), b + \omega_\Delta]$, i. e. in the continued fraction expansion of $(b + \omega_\Delta)/N(I)$, we have

$$I = I_1 = [Q_0/\sigma, (P_0 + \sqrt{D})/\sigma] \sim I_2 = [Q_1/\sigma, (P_1 + \sqrt{D})/\sigma] \sim \dots \sim I_l = [Q_{l-1}/\sigma, (P_{l-1} + \sqrt{D})/\sigma].$$

Finally, $I_l = I_0 = I$ for a complete cycle of reduced ideals of length $l(I) = l$. Therefore, the $(P_i + \sqrt{D})/Q_i$ are the complete quotients of $(b + \omega_\Delta)/N(I)$, and the Q_i/σ 's represent the norms of all reduced ideals equivalent to I .

The following is a well-known result dating back to Lagrange, a proof of which may be found in [2], along with historical documentation.

Lemma 2.1 *Let $I = [a, b + \omega_\Delta]$ be a reduced ideal, where ϵ_Δ is the fundamental unit of O_Δ . If P_i and Q_i , for $i = 1, 2, \dots, l(I) = l$ appear in the continued fraction expansion of $(b + \omega_\Delta)/a$, then*

$$\epsilon_\Delta = \prod_{i=1}^l (P_i + \sqrt{D})/Q_i,$$

and

$$N(\epsilon_\Delta) = (-1)^l.$$

§3. Ambiguous Cycles without Ambiguous Ideals.

We begin by defining the basic notion of ambiguity.

Definition 3.1 *Let $\Delta > 0$ be a discriminant. If I is a reduced ideal in O_Δ , then I is said to be an ambiguous cycle if $I_j = I'$ for some integer j with $0 < j \leq l$. In particular, if $I = I'$, then I is called an ambiguous ideal.*

It is possible to have an ambiguous cycle of ideals having no ambiguous ideals, and this is our object of study. To determine the exact number of such cycles, we first need to determine when such cycles exist. To do this we need the following useful tool which is actually our classification device.

Definition 3.2 *Let $\Delta > 0$ be a discriminant and let $I = [a, (b + \sqrt{\Delta})/2]$ be a reduced ideal in O_Δ with $0 < (\sqrt{\Delta} - b)/(2a) < 1$. If I is in an ambiguous cycle, then $I' = I_p$ for some $p \in \mathbb{Z}$ with $0 < p \leq l$. We call $p = p(I)$ the palindromic index of I .*

First we need a couple of technical lemmas the proofs of the which may be found in [2].

Lemma 3.1. *If $I = [a, (b + \sqrt{D})/2]$ is a reduced ideal in O_Δ with $0 < (\sqrt{D} - b)/(2a) < 1$ then, in the continued fraction expansion of $(b + \sqrt{D})/(2a)$, we have that $\alpha_i = [(P_i + \sqrt{D})/Q_i]$, and*

$$(I_{i+1})' = [Q_i/\sigma, (P_i + \sqrt{D})/\sigma]' = [Q_i/\sigma, (P_{i+1} + \sqrt{D})/\sigma].$$

Lemma 3.2 *Let l, i and p be as in Definition 3.2, then*

$$(I_{i+1})' = I_{p+1-i}$$

for $0 \leq i \leq p$, and

$$\alpha_i = a_{p-i}, \quad Q_i = Q_{p-i}, \quad P_{i+1} = P_{p-i}.$$

Moreover, if $l \geq p + 2$, then

$$(I_{i+1})' = I_{l+p+1-i}$$

for $p + 1 \leq i \leq l - 1$, and

$$\alpha_i = \alpha_{l+p-i}, \quad Q_i = Q_{l+p-i}, \quad P_i = P_{l+p-i+1}.$$

Now we are in a position to give our criterion for the existence of an ambiguous cycle of reduced ideals, having no ambiguous ideals in it.

Theorem 3.1 *Let $\Delta > 0$ be a discriminant, then there exists an ambiguous cycle of reduced ideals without ambiguous ideals in O_Δ if and only if there is a reduced ideal I in an ambiguous cycle of O_Δ with $p = p(I)$ odd and $l = l(I)$ even.*

Proof. By Lemma 3.2, $(I_i)' = I_i$ for some i with $0 < i \leq l$ if and only if $i = p - i$ or $i = l + p - i$, i.e. $p = 2i$ or $l + p = 2i$, where $p = p(I)$. ■

Theorem 3.2 *Let $\Delta > 0$ be a discriminant. In O_Δ there exists an ambiguous cycle of reduced ideals containing no ambiguous ideal if and only if $N(\epsilon_\Delta) = 1$ and $D = \sigma^2/4$ is a sum of two squares.*

Proof. Let I be an ambiguous cycle without any ambiguous ideals. By Theorem 3.1, $l(I) = l$ is even and $p(I) = p$ is odd. By Lemma 2.1, $N(\epsilon_\Delta) = (-1)^l = 1$ and by Lemma 3.2, $Q_{(p-1)/2} = Q_{(p+1)/2}$ thereby forcing $D = P_{(p+1)/2}^2 + Q_{(p+1)/2}^2$. Conversely, if $N(\epsilon_\Delta) = 1$, then $l(I)$ is even for any reduced ideal I , by Lemma 2.1. If $D = a^2 + b^2$, then $I = [a/\sigma, (b + \sqrt{D})/\sigma]$ is clearly reduced, and $\sigma b/2 = P_0 = P_1$. Thus, $D = \sigma^2 a^2 + P_1^2$. However, $D = Q_1 Q_{l-1} + P_1^2$, so $Q_{l-1} = Q_1 = \sigma a$. It follows that $p = l - 1$ and the result follows from Theorem 3.1 ■

Remark 3.1 *Now we need a definition which will lead us into the result which basically says that, if we have one ambiguous cycle of reduced ideals without ambiguous ideals then we have the maximum possible. For maximal orders, this means that if we have one such class, then we may generate the class group entirely by such classes.*

Definition 3.3 Let $\Delta > 0$ be a discriminant and let s be one half the excess of the number of divisors of $D = \sigma^2\Delta/4$ of the form $4j + 1$ over those divisors of the form $4j + 3$. In other words, $s = (n_1 - n_2)/2$, where n_1 is the number of divisors of D of the form $4j + 1$ and n_2 is the number of divisors of D of the form $4j + 3$. Thus, s corresponds to the number of distinct sums of squares $D = a^2 + b^2$ (where distinct here means that $1 \leq a \leq b$). For example, the eight solutions of $x^2 + y^2 = 5$: $(1, 2), (-1, 2), (1, -2), (-1, -2), (2, 1), (2, -1), (-2, 1)$ and $(-2, -1)$ are considered as only one solution.

Theorem 3.3 Let $\Delta > 0$ be a discriminant. If there exists an ambiguous cycle of reduced ideals without ambiguous ideals, then there are s such cycles when $D = \sigma^2\Delta/4$ is even, and there are $s/2$ of them when D is odd.

Proof. It is well-known that $s \geq 0$ and that $s = 0$ if and only if Δ is divisible by the odd power of some prime congruent to 3 modulo 4 in its canonical prime factorization. By Theorem 3.2, the hypothesis implies that $s > 0$, and from the proof of Theorem 3.2, we see that we may choose an ambiguous cycle with an ideal I satisfying $I' = I_{t-1} = I_p$. Moreover, by Lemma 3.2,

$$D = \sigma^2\Delta/4 = P_{(p+1)/2}^2 + Q_{(p+1)/2}^2 = P_1^2 + Q_1^2.$$

If D is odd, then these two representations are distinct (since the only way there could be equal is if $P_1 = Q_{(p+1)/2}$, which cannot happen since the parity of the P_i 's and Q_i 's differ when D is odd). Thus, we get a pair of equivalent reduced ideals

$$[Q_1/\sigma, (P_1 + \sqrt{D})/\sigma] \sim [Q_{(p+1)/2}/\sigma, (P_{(p+1)/2} + \sqrt{D})/\sigma]$$

for each such pair of sums of squares. Hence, there are exactly $s/2$ ambiguous cycles of reduced ideals without ambiguous ideals. (Clearly s is even since each ambiguous cycle of reduced ideals without ambiguous ideals yields a distinct pair of two sums of squares by Theorem 3.2.)

If D is even and $D = a^2 + b^2$ with $[a, b + \sqrt{\Delta}] \sim [b, a + \sqrt{\Delta}]$, for each ambiguous cycle without ambiguous ideals, then each such cycle corresponds to a sum of two squares. In other words, there are exactly s ambiguous cycles without ambiguous ideals. If, on the other hand, $[a, b + \sqrt{\Delta}]$ is not equivalent to $[b, a + \sqrt{\Delta}]$, then there is another sum of two squares $D = c^2 + d^2$ and $[a, b + \sqrt{\Delta}] \sim [c, d + \sqrt{\Delta}]$. Also, in the cycle of $[b, a + \sqrt{\Delta}] = I$, we have $I \sim [d, c + \sqrt{\Delta}]$. Hence, each pair of sums of two squares yields two cycles as above, so there are exactly s ambiguous cycles of reduced ideals without ambiguous ideals. ■

Remark 3.2 In [1, Ex. 9, p.190], it is claimed that in the class group of a real quadratic field there is at most one ambiguous class without an ambiguous ideal. What Theorem 3.3 shows, for maximal orders, is that if there is one ambiguous class without ambiguous ideals, then the class group C_Δ can be generated entirely by such classes. For instance,

Example 3.1 Let $\Delta = 1640 = 2^3 \cdot 5 \cdot 41$ and $D = 410 = 2 \cdot 5 \cdot 41 = 19^2 + 7^2 = 11^2 + 17^2$. In this case $s = 2$ and the ideals $I = [7, 19 + \sqrt{410}]$ and $J = [11, 17 + \sqrt{410}]$ are inequivalent and are in ambiguous classes without ambiguous ideals. In fact, $C_\Delta = (I) \times (J)$.

Finally, we illustrate how, in the more general situation, we may have no ambiguous classes without ambiguous ideals, yet still have ambiguous cycles without ambiguous ideals.

Example 3.2 Let $\Delta = 1224 = 2^3 \cdot 3^2 \cdot 17$, $D = 306 = 2 \cdot 3^2 \cdot 17$, $D_0 = 2 \cdot 17 = 34$ and $\Delta_0 = 2^3 \cdot 17$. Therefore, $f = 3$, $\sigma = \tau = g = 1$ and $O_\Delta = [1, \sqrt{306}]$. In this case, $s = 1$ and the ideal $I = [9, 15 + \sqrt{306}]$ is in an ambiguous cycle without ambiguous ideals. Furthermore, I is the only such cycle representing the only sum of two squares, $D = 9^2 + 15^2$.

Finally, we present the classification.

Theorem 3.4 Let $\Delta > 0$ be a discriminant, then the following are equivalent:

- (1) $D = \sigma^2 \Delta / 4$ is a sum of two squares and $N(\epsilon_\Delta) = 1$.
- (2) There are s ambiguous cycles of reduced ideals without ambiguous ideals in O_Δ when D is even, and there are $s/2$ of them when D is odd.
- (3) There exists an ambiguous cycle of reduced ideals without ambiguous ideals in O_Δ .

Proof. (1) \Rightarrow (2) by Theorem 3.3. That (2) \Rightarrow (3) is clear, and (3) \Rightarrow (1) by Theorem 3.2. ■

We trust that this presentation has made the inherent beauty of this topic manifest.

References.

- [1] Cohn, H., A SECOND COURSE IN NUMBER THEORY, John Wiley and sons Inc., New York/London (1962).
- [2] Mollin, R.A., QUADRATICS, C.R.C. Press, Boca Raton, Florida (1995).

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