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CONTENTS

On the Diophantine equation $x^4 - py^2 = z^p$ *

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§ 1 Introduction

Let p be an odd prime. In [7], Terai proved the following

Theorem 1. If $p \equiv 1 \pmod 4$ and the Bernoulli number $B_{(p-1)/2}$ is not divisible by p, then the Diophantine equation

$$x^4 - py^4 = z^p$$

has no integral solutions x, y, z with $(x, y) = 1, p \mid y$ and $2 \mid x$.

The proof of Theorem 1 needs a result of Rotkiewicz [5] on Lehmer's numbers. In fact, Theorem 1 holds for $x^2 - py^4 = z^p$ (it is easy to see from the proof in [7]).

Let $a, k \in \mathbb{Z}, a > 0, (a, k) = 1, a^2 - k > 0$, and $a^2 - k = Db^2$, where D is a square-free positive integer. From the ideas of Ko [4] and Terjanian [8], Rotkiewicz [5] consider Lehmer's number $L_p = (\alpha^p - \beta^p)/(\alpha - \beta) = z^2$ or pz^2 using properties of Jacobi's symbol, where $\alpha = a + b\sqrt{D}, \beta = a - b\sqrt{D}$. In [2], we proved the following

Theorem 2. If $k \equiv 1 \pmod 4$, then

$$(\alpha^p + \beta^p)/(\alpha + \beta) \neq pz^2, z \in \mathbb{Z}.$$

In this paper, we'll prove

Theorem 3. If $k \equiv 3 \pmod 4$, then

$$(\alpha^p + \beta^p)/(\alpha + \beta) \neq z^2, z \in \mathbb{Z}.$$

By Theorem 3 we obtain the following

Theorem 4. If $p \equiv 1 \pmod 4$ and $p \nmid B_{(p-1)/2}$, then the Diophantine equation

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$$x^4 - py^2 = z^p \tag{1}$$

has no integral solutions x, y, z with $(x, y) = 1$, $p \mid y$ and $2 \mid x$.

It is obvious that Theorem 4 contains Theorem 1.

§ 2 Properties of Jacobi's symbol $\left(\frac{E(m)}{E(n)}\right)$

Let m and n be coprime positive odd integers, $n < m$, and let $E(t) = (\alpha^t + \beta^t)/(\alpha + \beta)$ for $2 \nmid t$. We write the following sequence of equalities

$$\left.\begin{aligned} m &= 2l_1 n + e_1 r_1, 0 < r_1 < n, \\ n &= 2l_2 r_1 + e_2 r_2, 0 < r_2 < r_1, \\ r_1 &= 2l_3 r_2 + e_3 r_3, 0 < r_3 < r_2, \\ &\cdots\cdots \\ r_{s-1} &= 2l_{s+1} r_s + e_{s+1} r_{s+1}, r_{s+1} = 1, \end{aligned}\right\} \tag{2}$$

where $e_i = \pm 1, 2 \nmid r_i (i = 1, 2, \cdots, s+1)$. Then (see [6] p. 332) the following formula holds:

$$\left(\frac{m}{n}\right) = (-1)^{\frac{n-1}{2}\cdot\frac{e_1 r_1 - 1}{2} + \frac{r_1 - 1}{2}\cdot\frac{e_2 r_2 - 1}{2} + \cdots + \frac{r_s - 1}{2}\cdot\frac{e_{s+1} r_{s+1} - 1}{2}}.$$

Now we shall find a formula for the symbol $\left(\frac{E(m)}{E(n)}\right)$.

Theorem 5. If $k \equiv 3 \pmod 4$, then

$$\left(\frac{E(m)}{E(n)}\right) = \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}\left(\frac{m}{n}\right),$$

where $\lambda_i = l_i + \frac{e_i - 1}{2} (i = 1, 2, \cdots, s+1)$.

Proof. It is easy to see that $E(n) \equiv n \pmod 4$ and $(E(m), E(n)) = 1$ when $k \equiv 3 \pmod 4$, $(a, k) = 1$ and $(m, n) = 1$.

First we consider the case $m = 2l_1 n + r_1, 0 < r_1 < n, 2 \nmid r_1$. We have

$$E(m) \equiv \frac{(\alpha^n)^{2l_1} + (\beta^n)^{2l_1}}{2} \cdot E(r_1) (\bmod E(n)). \tag{3}$$

Let $a_n + b_n\sqrt{D} = \alpha^n$, $a_n, b_n \in \mathbb{Z}$. Then

$$\frac{(\alpha^n)^{2l_1} + (\beta^n)^{2l_1}}{2} = \sum_{0 \le i \le l_1} \binom{2l_1}{2i} a_n^{2l_1 - 2i} (b_n \sqrt{D})^{2i}$$

$$\equiv (b_n\sqrt{D})^{2l_1} \equiv (-k^n)^{l_1} \pmod{a_n}. \tag{4}$$

Since $a_n = (\alpha+\beta)/2 \cdot E(n)$, from (3) and (4) it follows that

$$\left(\frac{E(m)}{E(n)}\right) = \left(\frac{(-k^n)^{l_1}}{E(n)}\right)\left(\frac{E(r_1)}{E(n)}\right) = \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{\varepsilon_1 E(r_1)}{E(n)}\right)$$
$$= \left(\frac{-k}{E(n)}\right)^{\lambda_1}(-1)^{\frac{n-1}{2}\cdot\frac{\varepsilon_1 r_1-1}{2}}\left(\frac{E(n)}{E(r_1)}\right).$$

Next, we consider the case $m = 2l_1 n - r_1, 0 < r_1 < n, 2\nmid r_1$. We have

$$k^{r_1}E(m) = \frac{(\alpha^n)^{2l_1}\beta^{r_1} + (\beta^n)^{2l_1}\cdot\alpha^{r_1}}{\alpha+\beta} \equiv \frac{(\alpha^n)^{2l_1} + (\beta^n)^{2l_1}}{2}\cdot E(r_1)$$
$$\equiv (-k^n)^{l_1}E(r_1) \pmod{E(n)},$$

and so

$$\left(\frac{E(m)}{E(n)}\right) = \left(\frac{(-1)^{l_1}k^{nl_1-r_1}}{E(n)}\right)\left(\frac{E(r_1)}{E(n)}\right) = \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{\varepsilon_1 E(r_1)}{E(n)}\right)$$
$$= \left(\frac{-k}{E(n)}\right)^{\lambda_1}(-1)^{\frac{n-1}{2}\cdot\frac{\varepsilon_1 r_1-1}{2}}\left(\frac{E(n)}{E(r_1)}\right).$$

Now from (2) we have

$$\left(\frac{E(m)}{E(n)}\right) = \left(\frac{-k}{E(n)}\right)^{\lambda_1}(-1)^{\frac{n-1}{2}\cdot\frac{\varepsilon_1 r_1-1}{2}}\left(\frac{E(n)}{E(r_1)}\right)$$

$$= \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}(-1)^{\frac{n-1}{2}\cdot\frac{\varepsilon_1 r_1-1}{2}+\frac{r_1-1}{2}\cdot\frac{\varepsilon_2 r_2-1}{2}}\left(\frac{E(r_1)}{E(r_2)}\right) = \cdots$$

$$= \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}\cdot(-1)^{\frac{n-1}{2}\cdot\frac{\varepsilon_1 r_1-1}{2}+\frac{r_1-1}{2}\cdot\frac{\varepsilon_2 r_2-1}{2}+\cdots+\frac{r_s-1}{2}\cdot\frac{\varepsilon_{s+1}r_{s+1}-1}{2}}$$

$$= \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}\left(\frac{m}{n}\right).$$

This completes the proof of Theorem 5.

Remark 1. In [2], we proved that if $k \equiv 1 \pmod 4$ then

$$\left(\frac{E(m)}{E(n)}\right) = \left(\frac{-k}{E(n)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}.$$

§ 3 Proof of Theorems

Proof of Theorem 3. Suppose that $k \equiv 3 \pmod 4$ and $E(p) = z^2, z \in \mathbb{Z}$. By Theorem 5 we have

$$1=\left(\frac{E(p)}{E(q)}\right)=\left(\frac{-k}{E(q)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}\left(\frac{p}{q}\right) \tag{5}$$

for every odd $q < p$.

First we consider the case

I. $p \equiv 7 \pmod 8$. Then we have

$$\left.\begin{aligned} p &= 2q-(q-2),\\ q &= 2(q-2)-(q-4),\\ &\cdots\cdots\\ q-2(s-1) &= 2(q-2s)-(q-2(s+1)) \end{aligned}\right\} \tag{6}$$

for $q = p-2$, where $q-2(s+1)=1$. From (2) and (6), we see that $\varepsilon_i = -1, l_i = 1 (i=1,2,\cdots,s+1)$ and so $\lambda_i = 0 (i=1,2,\cdots,s+1)$. Thus, (5) gives

$$1=\left(\frac{p}{q}\right)=\left(\frac{q+2}{q}\right)=\left(\frac{2}{q}\right)=-1$$

which is impossible.

II. If $p \equiv 3 \pmod 8$, then we have

$$\begin{aligned} q &= 2p-(p-2),\\ p &= 2(p-2)-(p-4),\\ &\cdots\cdots\\ p-2(s-1) &= 2(p-2s)-(p-2(s+1)) \end{aligned}$$

for $p = q-2$, where $p-2(s+1)=1$. Thus, by Theorem 5 we have

$$\begin{aligned} 1 &= \left(\frac{E(p)}{E(q)}\right)=\left(\frac{E(q)}{E(p)}\right)=\left(\frac{-k}{E(p)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}\left(\frac{q}{p}\right)\\ &=\left(\frac{q}{p}\right)=\left(\frac{p+2}{p}\right)=\left(\frac{2}{p}\right)=-1 \end{aligned}$$

which is impossible.

III. Suppose that $p \equiv 1 \pmod 4$, $p > 5$. Let λ be a positive integer such that $\left(\frac{\lambda}{p}\right) = -1$ and $p-4\lambda > 0$ (It is easy to see that such a positive integer λ exists, see [3], Ch. 7). Then we have

$$q = 2p-(p-4\lambda),$$

$$p = 2(p - 4\lambda) - (p - 4\lambda \cdot 2),$$

$$\cdots\cdots$$

$$p - 4\lambda(s-1) = 2(p - 4\lambda \cdot s) - (p - 4\lambda \cdot (s+1))$$

for $q = p + 4\lambda$, where $p - 4\lambda \cdot (s+1) = 1$. Thus, by Theorem 5 we have

$$1 = \left(\frac{E(p)}{E(q)}\right) = \left(\frac{E(q)}{E(p)}\right) = \left(\frac{-k}{E(p)}\right)^{\lambda_1}\left(\frac{-k}{E(r_1)}\right)^{\lambda_2}\cdots\left(\frac{-k}{E(r_s)}\right)^{\lambda_{s+1}}\left(\frac{q}{p}\right)$$
$$= \left(\frac{q}{p}\right) = \left(\frac{p+4\lambda}{p}\right) = \left(\frac{\lambda}{p}\right) = -1$$

which is also impossible.

If $p = 5$, then from (5) since $5 = 2\times 3 - 1$ we have

$$1 = \left(\frac{E(5)}{E(3)}\right) = \left(\frac{5}{3}\right) = -1$$

which is also impossible.

This completes the proof of Theorem 3.

Remark 2. Let α, β be the different zeros of the trinomial $x^2 - \sqrt{L}x + M$, where $L, M \in \mathbb{Z}$, $L > 0$, $K = L - 4M > 0$ and $(L, M) = 1$. And let $Q_n = (\alpha^n + \beta^n)/(\alpha + \beta)$ for $2 \nmid n$, then Theorems 3 and 5 hold for Q_n.

Proof of Theorem 4. We use a result of Adachi to prove Theorem 2. In [1], Adachi proved that if p is an odd prime and a, b relatively prime and of opposite parity with $p \equiv 1 \pmod 4$, $p \nmid B_{(p-1)/2}$ and $p \mid b$, then

$$a^2 - pb^2 = c^p \Rightarrow a + b\sqrt{p} = (u + v\sqrt{p})^p$$

holds for some integers u and v.

Now suppose that the Diophantine equation (1) has integral solutions x, y, z with $(x, y) = 1$, $p \mid y$ and $2 \mid x$. Then it follows from the result of Adachi that

$$x^2 + y\sqrt{p} = (a + b\sqrt{p})^p \tag{7}$$

where $(a, b) = 1$. Let $\alpha = a + b\sqrt{p}$, $\beta = a - b\sqrt{p}$, then (7) gives

$$x^2 = \frac{\alpha^p + \beta^p}{2} = a \cdot \frac{\alpha^p + \beta^p}{\alpha + \beta} = a \cdot E(p). \tag{8}$$

It is easy to see that $a>0$ and $(a, E(p)) = (a, pb^{p-1}p^{(p-1)/2}) = (a,b) = 1$ since $p\mid y, (x,y)=1$ and

$$E(p) = \sum_{j=0}^{(p-1)/2} \binom{p}{2j} a^{p-(2j+1)} b^{2j} p^j > 0.$$

Thus from (8) we have

$$E(p) = c^2, c \in \mathbb{Z}. \tag{9}$$

Since $2\mid x, (x,y)=1$, we have $2\mid a$, $2\nmid b$. Hence by $p\equiv 1 \pmod 4$ we obtain $z = a^2 - pb^2 \equiv 3 \pmod 4$. Therefore (9) contradicts Theorem 3.

This completes the proof of Theorem 4.

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ON THE CONDITIONING OF A COMPOSITE POLYNOMIAL ZEROFINDING MATRIX ALGORITHM

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ABSTRACT. A new three-stage polynomial zerofinding matrix algorithm reduces a polynomial $p(x)$ to a polynomial $q(x)$ whose zeros are simple and coincide with the distinct zeros of p. One of three companion matrices of q is then constructed and its eigenvalues are computed to high accuracy by means of the QR algorithm. Finally, the multiplicity of each distinct zero of p is determined by means of Lagouanelle's limiting formula. The conditioning of Frobenius', Schmeisser's and Fiedler's companion matrices, and of their eigenvalues is compared. Numerical results illustrate the effectiveness of the composite algorithm.

Un nouvel algorithme matriciel tripartite réduit un polynôme p à un polynôme q dont les zéros sont simples et coïncident avec les zéros distincts de p. On construit une matrice compagnon de q et l'on calcule ses valeurs propres à une grande précision par l'algorithme QR. La multiplicité des zéros distincts de p s'obtient par la limite de Lagouanelle. On compare le conditionnement des matrices compagnons de Frobenius, de Schmeisser et de Fiedler, ainsi que le conditionnement des matrices formées de leurs vecteurs propres. On illustre l'efficacité de la méthode au moyen de résultats numériques.

1. Introduction. Given a monic polynomial, $p(x)$, of degree n,

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad a_k \in \mathbb{C}, \quad k = 0, \ldots, n-1, \tag{1}$$

of a complex variable $x \in \mathbb{C}$, a matrix $M \in \mathbb{C}^{n \times n}$ is called a *companion matrix* of $p(x)$ if

$$\det(M - \lambda I) = (-1)^n p(\lambda). \tag{2}$$

To find the n zeros of p, one needs only find the n eigenvalues of M. The sensitivity of the eigenvalues of M is measured by the condition number, $\kappa(X) = \|X\|_2\|X^{-1}\|_2$, of the matrix X of eigenvectors of M, according to Bauer–Fike's theorem [1], p. 87.

Three types of companion matrices will be considered.

Frobenius' companion matrix, C_p, of $p(x)$ is defined in terms of the coefficients of p, by

$$C_p := \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3}$$

Key words and phrases. Matrix condition number, condition number of eigenvalues, companion matrices, polynomial zero-finding matrix methods.

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The Matlab [2] ROOTS command uses C_p to find the zeros of p by means of the QR algorithm. with the option of balancing C_p. It is shown numerically, in [3], that p and C_p exhibit the same conditioning, that is, the pseudozeros of p and the pseudospectrum of balanced C_p are quite close to each other. In [4], the backwards normwise stability of ROOTS is established. In [5], it is shown that the QR algorithm of Eispack has considerable advantages over the standard algorithms of NAG (based on Laguerre's method) and IMSL (based on Jenkins and Traub's method) for finding the zeros of a polynomial.

A second form of companion matrix, T_p, of $p(x)$, due to Schmeisser [6], for polynomials with real zeros. is obtained by means of the following modification to the Euclidean algorithm.

(a) Let $\quad f_1(x) := p(x), \quad f_2(x) := p'(x)/n.$

(b) For $\nu = 1, 2, \ldots, n$:

- (b.1) If $f_{\nu+1}(x) \not\equiv 1$, set $\quad f_\nu(x) := q_\nu(x) f_{\nu+1}(x) - c_\nu f_{\nu+2}(x).$
- (b.2) If $f_{\nu+2}(x) \equiv 0$, set $\quad c_\nu := 0, \quad f_{\nu+2}(x) := f'_{\nu+1}(x)/c[f'_{\nu+1}].$
- (b.3) Else, stop and set $\quad q_\nu(x) := f_\nu(x).$

(c) End For. □

The numbers c_ν and $c[f'_{\nu+1}]$ are used to produce monic polynomials $f_{\nu+2}(x)$ in (b.1) and (b.2). The $n \times n$ tridiagonal matrix T_p, defined by

$$t_{ii} = -q_i(0), \qquad t_{i+1,i} = t_{i,i+1} = \sqrt{c_i}, \qquad t_{ij} = 0 \quad \text{otherwise}, \tag{4}$$

is real symmetric for real polynomials with real zeros. If T_p can be formed for a polynomial with complex zeros, then it is complex symmetric. A stable divide-and-conquer algorithm is proposed in [7] as an alternative to the QR algorithm for large (real) symmetric tridiagonal eigenvalue problems.

A third form of companion matrix. A_p, of $p(x)$, due to Fiedler [8], is derived as follows: Let $b_1, b_2, \ldots, b_n \in \mathbb{C}$ be n distinct numbers such that $p(b_k) \neq 0$ for $k = 1, \ldots, n$. Set

$$v(x) = \prod_{k=1}^{n} (x - b_k) \quad \text{and} \quad B = \operatorname{diag}(b_k) \in \mathbb{C}^{n\times n},$$

and define the rank-one perturbation, $A_p = B - \sigma dd^T \in \mathbb{C}^{n\times n}$, of the diagonal matrix B by

$$a_{ik} = -\sigma d_i d_k, \quad \text{if } i \neq k, \qquad a_{kk} = b_k - \sigma d_k^2, \qquad i, k = 1, \ldots, n, \tag{5}$$

where the number $\sigma \neq 0$ is fixed and d_k is a root of the quadratic equation

$$\sigma v'(b_k) d_k^2 - p(b_k) = 0.$$

In [9] and [10], Fiedler's algorithm is used iteratively and its fast convergence to simple zeros is exploited.

This note reports on a new composite algorithm [10] described in the next section. The algorithm is tested on a set of polynomials, p, and condition numbers of the companion matrices C_q, T_q, A_q of the reduced polynomials, q, and of the matrices of their eigenvectors. VC_q, VT_q and VA_q, are listed in Tables 1, 2 and 3 below.

2. The composite algorithm. The composite algorithm is divided into three stages.

2.1. *Polynomial reduction.* Let $p(x) = (x - r_1)^{\mu_1}(x - r_2)^{\mu_2}\dots(x - r_N)^{\mu_N}$ be a complex polynomial of degree n with N distinct zeros. r_i of multiplicity μ_i, for $i = 1, \dots, N$, where $\sum_{i=1}^{N} \mu_i = n$. The monic greatest common divisor of $p(x)$ and $p'(x)$,

$$g(x) := \gcd\big(p(x), p'(x)\big) = (x - r_1)^{\mu_1-1}(x - r_2)^{\mu_2-1}\cdots(x - r_N)^{\mu_N-1}, \tag{6}$$

is obtained by means of the classic Euclidean algorithm (a), (b), which is stopped as soon as the remainder $f_{\nu+2}(x)$ in (b.2) is the zero polynomial. Numerically, $f_{\nu+2}(x)$ is set to zero if all its coefficients are smaller than a chosen tolerance. The zeros of the reduced polynomial,

$$q(x) := \frac{p(x)}{g(x)} = (x - r_1)(x - r_2)\cdots(x - r_N),$$

are simple and coincide with the distinct zeros of $p(x)$ (see [11, 12]).

2.2. *Computing distinct roots.* Once $q(x)$ is formed, one of the companion matrices, C_q, T_q, or A_q, associated with q, with simple eigenvalues, is constructed (if C_q is used, it is balanced), and its eigenvalues, which coincide with the distinct zeros of $p(x)$, are computed by the QR algorithm.

2.3. *Computing root multiplicities.* The multiplicity $\mu_1 > 1$, of a zero x_1 of a given polynomial $u(x) = (x - x_1)^{\mu_1}h(x)$, where $h(x_1) \neq 0$, can be obtained by Lagouanelle's limiting formula [13],

$$\mu_1 = \lim_{x \to x_1} \frac{1}{[u(x)/u'(x)]'}. \tag{7}$$

To overcome the numerical difficulty caused by the vanishing of both u and u' at a multiple zero of u, one needs only set $v(x) = u(x)/g(x)$ and $w(x) = u'(x)/g(x)$, where $g(x) = \gcd\big(u(x), u'(x)\big)$, and rewrite Lagouanelle's limiting formula in the non-indeterminate form:

$$\mu_1 = \lim_{x \to x_1} \frac{w(x)}{v'(x)}. \tag{8}$$

In numerical computation, μ_1, given by (8), is rounded to the nearest (real) integer.

3. Numerical results. The present composite algorithm has been tested with polynomials found in the literature, some of which deemed to be ill-conditioned. In all cases, the results have been extremely successful. The algorithm has been first implemented in Mathematica [14] and then converted to Fortran 77, on a Sun Microsystems Sparc 10 workstation. Quadruple precision was used everywhere except for the QR algorithm which used double precision arithmetic.

The following sample polynomials were solved by the proposed algorithm, where, for Fiedler's algorithm, the initial values for b_k were chosen on a circle of radius 25 centered at the origin and the number of iterations was set at five.

TABLE 1. For given p, the Table lists the precisions used in evaluating p and in the QR algorithm, and the minimum number of correct digits in the computed eigenvalues of the companion matrices for p and q.

Polynomial	Precision in		Min. no. of corr. dig. in eigenval. of					
p	p	QR	C_p	C_q	T_p	T_q	A_p	A_q
JW20	32	16	0	0	15	15	15	15
MR05	32	16	3	16	16	16	7	16
MR10	32	16	1	13	15	15	3	16
MR14	32	16	2	7	10	10	8	7
DUN1	32	16	3	14	15	15	7	15
DUN2	32	16	4	15	16	16	8	15
DUN3	32	16	6	15	15	15	11	15
DUN4	32	16	8	15	15	15	15	16
BT1	32	16	3	14	15	15	7	15
BT2	32	16	5	15	15	15	10	15
BT3	32	16	2	11	16	16	6	16

JW20: [1, 15] of degree 20, $(x-1)(x-2)\cdots(x-20)$;

TT32: [3] of degree 32, $\prod_{k=1}^{32}[(x+2-4(k-1)]/31$;

MR05: with a zero of multiplicity five. $(x+\pi/3)^5$;

MR10: with a zero of multiplicity ten, $(x-\pi/3)^{10}(x-5)(x-1)(x+1)^2$;

MR14: of degree forteen with multiple zeros,

$$(x-5-6i)^4(x^2-10^{-6})(x^2+10^{-6})(x-6-i)(x-6+2i)$$
$$\cdot(x-8-7i)(x-8+9i)(x-10-15i)(x-10+17i);$$

HM40: [16, 17] of degree 40, $\left\{\prod_{k=1}^{19}\left[x-e^{i\pi(k-10)/20}\right]\right\}\left\{\prod_{k=20}^{40}\left[x-(9/10)e^{i\pi(k-10)/20}\right]\right\}$;

DUN1: [11] of degree 16, $(x^2+x+2)^4(x^2+x+3)^4$;

DUN2: [11] of degree 16, $[(x-1.7)(x+1.7)(x-1.3)(x+1.3)]^4$;

DUN3: [11] of degree 12, $(x^6-2^6)(x+2)(x^2+3^2)(x-1)^3$;

DUN4: [11] of degree 5, $(x+1)(x-1.150016-3.57064i)^2(x-1.150016+3.57064i)^2$;

BT1: [12] of degree 19, $(x-1)(x-2)^2(x-3)^3(x-4)^4(x-0.25)^4(x-0.5)^5$;

BT2: [12] of degree 17,

$$(x-1)^3(x+1)^4(x-0.5+i)^3(x-0.5-i)^3(x-0.5-0.5i)^2(x-0.5+0.5i)^2$$

BT3: [12] of degree 11, $(x-1)^4(x-1.1)(x-0.9)(x-1+0.1i)(x-1-0.1i)(x-1.05)^3$.

Table 1 illustrates the effectiveness of the first stage of the composite algorithm in improving the accuracy of multiple zeros of p. For JW20 with $C_q = C_p$, the 16-digit QR algorithm produced complex conjugates zeros for $x = 14$ and $x = 15$. For MR14, the low precision of the four small zeros can be improved by using the reciprocal polynomial (see [10]).

4. Condition numbers of C, T and A and of their eigenvalues for p and q. The composite algorithm was used with the companion matrices, C, T and A, for the above

TABLE 2. For p with real zeros, the Table lists the condition numbers of C_q, T_q and A_q, and of the corresponding matrices of eigenvectors, VC_q, VT_q and VA_q, respectively.

p	$\kappa(C_q)$	$\kappa(T_q)$	$\kappa(A_q)$	$\kappa(VC_q)$	$\kappa(VT_q)$	$\kappa(VA_q)$
JW20	2.2×10^{19}	20	20	7.4×10^{13}	1	1
TT32	1.2×10^{13}	31	31	1.5×10^{9}	1	1
MR05	1	1	1	1	1	1
MR10	22.7	5	5	90.0	1	1
DUN2	9.2	3	1.3	14.8	1	1
BT1	4292	16	16	1323.8	1	1

TABLE 3. For p with complex zeros, the Table lists the condition numbers of C_q, T_q and A_q, and of the corresponding matrices of eigenvectors, VC_q, VT_q and VA_q, respectively.

p	$\kappa(C_q)$	$\kappa(T_q)$	$\kappa(A_q)$	$\kappa(VC_q)$	$\kappa(VT_q)$	$\kappa(VA_q)$
HM40	31.5	1.1×10^{7}	1.1	44.0	5.0×10^{13}	1
MR14	1.5×10^{7}	2.0×10^{8}	19724.1	1.5×10^{9}	5.9×10^{6}	1
DUN1	16.9	1.2	1.2	29.8	1	1
DUN3	1166.5	13.4	3	507.0	15.5	1
DUN4	24.4	6.5	3.8	5.6	1.8	1
BT2	25.7	109.3	1.5	13.6	23.6	1
BT3	924.6	1.7	1.2	3.0×10^{7}	18.2	1

polynomials. Although the prime interest of this note lies in the condition number of the eigenvalues of a companion matrix, the condition number of the companion matrix itself was also computed since, in practice, these two numbers are often both large or both small. The condition number of a matrix was computed by means of its singular value decomposition. It could have been estimated by an algorithm found in [18]. The results are illustrated in Tables 2 and 3 for polynomials with real and complex zeros, respectively.

4.1. *Polynomials with only real zeros.* If $q(x)$ has only real zeros, then T_q is a real symmetric tridiagonal matrix with distinct eigenvalues. Therefore it is diagonalizable by a unitary matrix VT_q, whose condition number is $\kappa(VT_q) = 1$. Since Fiedler's matrix, A_q, eventually converges to a diagonal matrix after a certain number of iterations (see [9, 10]), then finally $\kappa(VA_q) = 1$. Therefore, in this case, T_q and A_q are usually better conditioned than C_q, as seen in Table 2.

4.2. *Polynomials with complex zeros.* If $q(x)$ has complex zeros, then T_q is a complex symmetric tridiagonal matrix, whose condition number, in general, compares favourably with the condition number of C_q. Since A_q converges to a diagonal matrix after a certain number of iterations, its final conditioning is superior to both the conditioning of T_q and C_q. The results are shown in Table 3.

5. Conclusion. If a (reduced) polynomial has only *real* simple zeros, the eigenvalues of Schmeisser's matrix are better conditioned than those of Frobenius' matrix; however, in the presence of complex zeros, their conditioning is comparable. Since Fiedler's matrix converges to a diagonal matrix, its eigenvalues eventually becomes very well conditioned. A companion matrix of a reduced polynomial q can be used to find the multiple zeros of a given polynomial p to higher accuracy than with the corresponding companion matrix of p. In all cases, the multiplicity of the zeros was computed correctly by Lagouanelle's modified formula.

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GENERALIZED CONTINUOUS WAVELET TRANSFORM AND GENERALIZED CONTINUOUS MULTISCALE ANALYSIS ASSOCIATED WITH LAGUERRE FUNCTIONS

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Abstract : In this work we provide $K = [0,+\infty[\times \mathbb{R}$ with an hypergroup structure associated with a system of partial differential operators D_1, D_2. wich admits Laguerre functions as eigenfunctions. We study an Harmonic analysis on this hypergroup. Next we define and study generalized wavelets and generalized continuous wavelet transform on K. Moerover we study a generalized continuous multiscale analysis and partial reconstructions on K.

I. Hypergroup structure on K.

We consider the partial differential operators on $K = [0,+\infty[\times \mathbb{R}$. $D_1 = \frac{\partial}{\partial t}$ and

$D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}$ where $(x,t) \in\ ]0,+\infty[\times \mathbb{R}$ and $\alpha \in \mathbb{R}$, $\alpha \geq 0$. For $\alpha = n-1$, $n \in \mathbb{N} \setminus \{0\}$, the operators D_2 is the radial part of the subLaplacian on the Heisenberg group H^n . Let $\varphi_{\lambda,m}(x,t)$, $(\lambda,m) \in \mathbb{R} \times \mathbb{N}$, be the function defined by $\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^{\alpha}(|\lambda|x^2)$ where $\mathcal{L}_m^{\alpha}$ is the Laguerre defined by $\mathcal{L}_m^{\alpha}(x) = e^{-\frac{x}{2}}\frac{L_m^{(\alpha)}(x)}{L_m^{(\alpha)}(0)}$, L_m^{α} being the Laguerre polynomial of degree m and with order α. The function $\varphi_{\lambda,m}(x,t)$, $(\lambda,m) \in \mathbb{R} \times \mathbb{N}$, is the unique solution of

$$\begin{cases} D_1 u(x,t) = i\lambda u(x,t) \\ D_2 u(x,t) = -4|\lambda|(m + \frac{\alpha+1}{2})u(x,t) \\ u(0,0) = 1,\ \frac{\partial u}{\partial x}(0,t) = 0\ ,\ \text{for all } t \in \mathbb{R} \end{cases}$$

and satisfies the product formula $T_{(x,t)}\varphi_{\lambda,m}(y,s) = \varphi_{\lambda,m}(x,t)\,\varphi_{\lambda,m}(y,s)$, where $T_{(x,t)}$, $(x,t) \in K$, is the generalized translation operators associated with D_1, D_2. defined by

$$T_{(x,t)}f(y,s) = \begin{cases} \frac{\alpha}{\pi}\int_0^1\int_0^{\pi} f(\sqrt{x^2+y^2+2xyr\cos\theta},\ t+s+xyr\sin\theta)\,r(1-r^2)^{\alpha-1}\,dr\,d\theta\ , & \text{if } \alpha > 0 \\ \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta}\ ,\ t+s+xy\sin\theta)\,d\theta & ,\ \text{if } \alpha = 0 \end{cases}$$

Definition I.1 : For (x,t), $(y,s) \in K$ and f a continuous function on K we put

$$\delta_{(x,t)} * \delta_{(y,s)}(f) = T_{(x,t)}f(y,s)$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t). The product $*$ is called the generalized convolution on K associated with the operators D_1,D_2.

Let i be the involution on K defined by $i(x,t) = (x,t)$.

Theorem I.1 : $(K,*,i)$ is an hypergroup in the sense of Jewett (see [1]).

Notations : We denote by

- $L^p_\alpha(K)$, $p \in [1,+\infty[$, the space of functions on K, measurable and such that $\|f\|_{p,\alpha} = (\int_K |f(x,t)|^p \, dm_\alpha(x,t))^{1/p} < +\infty$, where dm_α is the measure defined on K by

$$dm_\alpha(x,t) = \frac{1}{\pi\,\Gamma(\alpha+1)}\, x^{2\alpha+1}\, dx\, dt$$

- $\mathcal{S}(K)$ the space of C^∞-functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives and even with respect to the first variable.
- $d\gamma_\alpha$ the measure on $\mathbb{R} \times \mathbb{N}$ defined by $d\gamma_\alpha(\lambda,m) = \pi \dfrac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)} |\lambda|^{\alpha+1} d\lambda \otimes \delta_m$ where δ_m is the Dirac measure at m.
- $L^p_\alpha(\mathbb{R} \times \mathbb{N})$, $p \in [1,+\infty[$, the space of functions g defined on $\mathbb{R} \times \mathbb{N}$, mesurable and such that $\|g\|_{L^p_\alpha} = \int_{\mathbb{R}\times\mathbb{N}} |g(\lambda,m)|^p \, d\gamma_\alpha(\lambda,m))^{1/p} < +\infty$.
- Δ_+, Δ_-, Λ_1, Λ_2 the operators defined on the space of functions defined on $(\mathbb{R} \setminus \{0\}) \times \mathbb{N}$ by

$$\Delta_+ g(\lambda,m) = g(\lambda,m+1) - g(\lambda,m)\,,\quad \Delta_- g(\lambda,m) = \begin{cases} g(\lambda,m) - g(\lambda,m-1)\,, & \text{if } m \geq 1 \\ g(\lambda,0)\,, & \text{if } m = 0 \end{cases}$$

$$\Lambda_1 g(\lambda,m) = \frac{1}{|\lambda|}(\Delta_+\Delta_- g(\lambda,m) + (\alpha+1)\ \Delta_- g(\lambda,m))$$

$$\Lambda_2 g(\lambda,m) = \frac{1}{2\lambda}\,((\alpha+m+1)\ \Delta_+ g(\lambda,m) + m\ \Delta_- g(\lambda,m)).$$

- $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ the space of functions $g : \mathbb{R} \times \mathbb{N} \longrightarrow \mathbb{C}$ satisfying

i) $\forall\ m,p,q,r,s \in \mathbb{N}$, the function $\lambda \longrightarrow \lambda^p (|\lambda|\,(m + \frac{\alpha+1}{2}))^q\ \Lambda_1^r (\Lambda_2 + \frac{\partial}{\partial\lambda})^s g(\lambda,m)$

is continuous and bounded on $\mathbb{R}$, C^∞ on $\mathbb{R} \setminus \{0\}$ and such that the left and the right derivatives at o exist.

ii) $\forall\ k,p,q \in \mathbb{N}$, we have

$$N_{k,p,q}(g) = \sup_{(\lambda,m)\in(\mathbb{R}\setminus\{0\})\times\mathbb{N}} \{(1+\lambda^2(1+m^2))^k\, |\Lambda_1^p (\Lambda_2 + \frac{\partial}{\partial\lambda})^q g(\lambda,m)|\} < +\infty$$

- $\mathcal{D}(K)$ the space of C^∞-functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support.
- $\mathcal{S}_H(\mathbb{R} \times \mathbb{N})$ the space of functions g in $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ satisfying

$$\exists\ A > 0,\ \exists\ B > 0 \,/\, \forall\, k \in \mathbb{N},\ \forall\,(\lambda,m) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{N}$$

we have

$$1 + \lambda^2\{1+[4|\lambda|\,(m + \frac{\alpha+1}{2})]^2\}^{p_\alpha}\, |\Lambda_3^k g(\lambda,m)| \leq A\ B^{2k}$$

where $\Lambda_3 = -\Lambda_1 - (\Lambda_2 + \frac{\partial}{\partial\lambda})^2$, $p_\alpha = [\frac{\alpha+2}{4}] + 1$, and $[\ ]$ denote the integer part.

Definition I.2 : The generalized Fourier transform $\mathcal{F}$ is definied on $L^1_\alpha(K)$ by

$$\mathcal{F}(f)(\lambda,m) = \int_K f(x,t)\,\varphi_{-\lambda,m}(x,t)\, dm_\alpha(x,t),\ (\lambda,m) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{N}.$$

Proposition I.1 : i) The space $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ is dense in $L^2_\alpha(\mathbb{R} \times \mathbb{N})$.

ii) The generalized Fourier transform $\mathcal{F}$ is an isomorphism between $\mathcal{S}(K)$ and $\mathcal{S}(\mathbb{R} \times \mathbb{N})$.

Theorem I.2 : i) (Inversion formula) : Let f be in $L^2_\alpha(K)$ such that $\mathcal{F}(f)$ is in $L^1_\alpha(\mathbb{R} \times \mathbb{N})$ then

$$f(x,t) = \int_{\mathbb{R}\times\mathbb{N}} \mathcal{F}(f)(\lambda,m)\, \varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m) \ ; \ \text{a.e.}$$

ii) For all $f \in \mathcal{S}(K)$, we have the Plancherel formula

$$\int_K |f(x,t)|^2 dm_\alpha(x,t) = \int_{\mathbb{R}\times\mathbb{N}} |\mathcal{F}(f)(\lambda,m)|^2\, d\gamma_\alpha(\lambda,m)$$

ii) The generalized Fourier transform can be extended to an isometric isomorphism from $L^2_\alpha(K)$ onto $L^2_\alpha(\mathbb{R}\times\mathbb{N})$.

Theorem I.3 : (Paley-Wiener Theorem). The generalized Fourier transform $\mathcal{F}$ is an isomorphism from $\mathcal{D}(K)$ onto $\mathcal{S}_H(\mathbb{R} \times \mathbb{N})$.

II. Wavelets and continuous wavelet transform on K associated with the operators D_1, D_2.

Definition II.1 : We say that a function $g \in L^2_\alpha(K)$ is a generalized wavelet on K associated with the operators D_1, D_2 if there is a constant C_g with the property $0 < C_g < +\infty$, and for all $m \in \mathbb{N}$, for λ almost everywhere on $\mathbb{R}$, $C_g = \int_0^{+\infty} |\mathcal{F}(g)(a\lambda,m)|^2 \frac{da}{a}$.

Example. Let g be in $L^2_\alpha(K)$ defined by $\mathcal{F}(g)(\lambda,m) = \lambda^2(m + \frac{\alpha+1}{2})^2 e^{-\lambda^2(m+\frac{\alpha+1}{2})^2}$, then g is a generalized wavelet on K.

Proposition II.1 : Let $g \neq 0$ be a function in $L^2_\alpha(K)$ such that for all $m \in \mathbb{N}$ the function $\lambda \longrightarrow \mathcal{F}(f)(\lambda,m)$ is continuous at 0, g is real valued and for each $\varepsilon > 0$, there exist $\beta,\delta > 0$ such that

$$|\mathcal{F}(g)(\lambda,m) - \mathcal{F}(g)(0,m)| \leq \varepsilon |\lambda|^\beta \text{ for all } (\lambda,m) \in [-\delta,\delta] \times \mathbb{N}.$$

Then g is a generalized wavelet iff $\mathcal{F}(g)(0,m) = 0$ for all $m \in \mathbb{N}$.

Theorem II.1 . Let g be a generalized wavelet on K and $a \in \,]0,+\infty[$. Then the function $(\lambda,m) \longrightarrow \mathcal{F}(g)(a\lambda,m)$ belongs to $L^2_\alpha(\mathbb{R} \times \mathbb{N})$ and we have

$$\|\mathcal{F}(g)(a,.)\|_{L^2_\alpha} = \frac{1}{a^{\frac{\alpha}{2}+1}} \|g\|_{2,\alpha}.$$

Mereover the function g_a defined by $g_a(x,t) = \frac{1}{a^{\alpha+2}}\, g(\frac{x}{\sqrt{a}}, \frac{t}{a})$ satisfies

$$\mathcal{F}(g_a)(\lambda,m) = \mathcal{F}(g)(a\lambda,m), \text{ for all } (\lambda,m) \in \mathbb{R} \times \mathbb{N}.$$

<u>**Definition**</u> **II.2** : Let g be a generalized wavelet on K in $\mathcal{S}(K)$. We define the generalized continuous wavelet transform associated with the operators D_1, D_2, for $f \in \mathcal{S}(K)$ by

$$\Phi_g(f)(a,(x,t)) = \int_K f(y,s)\, \bar{g}_{a,(x,t)}(y,s)\, dm_\alpha(y,s)$$

where $g_{a,(x,t)}$ is the function defined by

$$g_{a,(x,t)}(y,s) = T_{(x,t)} g_a(y,s).$$

<u>**Theorem**</u> **II.2** : Let g be a generalized wavelet on K in $L^2_\alpha(K)$. Then for all $f \in L^2_\alpha(K)$, we have

Plancherel formula for Φ_g : $\|f\|^2_{2,\alpha} = \frac{1}{C_g} \int_0^{+\infty} \int_K |\Phi_g(f)(a,(x,t))|^2 \frac{da}{a^{\alpha+3}} dm_\alpha(x,t)$

Inversion formula for Φ_g : $f(.) = \frac{1}{C_g} \int_0^{+\infty} \int_K \Phi_g(f)(a,(x,t)) g_{a,(x,t)}(.) \frac{da}{a^{\alpha+3}} dm_\alpha(x,t)$ Weakly in $L^2_\alpha(K)$

By Theorem II.2 the generalized continuous wavelet transform Φ_g is an isometry from the Hilbert space $L^2_\alpha(K)$ into the Hilbert space $L^{2\,\#}_\alpha(]0,\infty[\times K)$ of the square integrable functions on $]0,+\infty[\times K$ with respect to the measure $\frac{da}{a^{\alpha+3} C_g} dm_\alpha(x,t)$. For the characterization of the image of Φ_g we interpred the vectors $g_{a,(x,t)}$, $(a,(x,t)) \in]0,+\infty[\times K$ as a set of coherent states in the Hilbert space $L^2_\alpha(K)$. (See [2] section 1.2).

<u>**Theorem**</u> **II.3** : Let Φ_g be the generalized continuous wavelet transform associated with the operators D_1, D_2, with g a generalized wavelet on K in $L^2_\alpha(K)$. Let F be in $L^{2\,\#}_\alpha(]0,+\infty[\times K)$. Then there exists a function f in $L^2_\alpha(K)$ such that $F = \Phi_g(f)$ if and only if

$$F(a,(x,t)) = \frac{1}{C_g} \int_0^{+\infty} \int_K F(a',(x',t')) \left(\int_K g_{a',(x',t')}(y,s)\, g_{a,(x,t)}(y,s)\, dm_\alpha(y,s)\right) \frac{da'}{(a')^{\alpha+3}} dm_\alpha(x',t').$$

We give now another inversion formula for the generalized continuous wavelet transform associated with the operators D_1, D_2.

<u>**Theorem**</u> **II.4** : Let g be a generalized waveled on K associated with the operators D_1,D_2 in $L^2_\alpha(K)$. If $f \in L^1_\alpha(K)$ such that $\mathcal{F}(f) \in L^1_\alpha(\mathbb{R} \times \mathbb{N})$. Then for (x,t) almost everywhere in K we have

$$f(x,t) = \frac{1}{C_g} \int_0^{+\infty} \left(\int_K \Phi_g(f)(a,(y,s)) g_{a,(y,s)}(x,t)\, dm_\alpha(y,s)\right) \frac{da}{a^{\alpha+3}}$$

III. Generalized continuous multiscal analysis and partial reconstructions on K associated with the operators D_1, D_2.

Definition III.1 : A generalized continuous multiscale analysis on K is defined by a net $\{v_\varepsilon\}_{\varepsilon\in]0,+\infty[}$ of closed subspace of $L^2_\alpha(K)$, with the properties that $V_{\varepsilon_2} \subset V_{\varepsilon_1}$ if $0 < \varepsilon_1 < \varepsilon_2$, if $f \in V_\varepsilon$ then $T_{(x,t)} f \in V_\varepsilon$ a function f is in V_{ε_1} if and only if $f_{\frac{\varepsilon_2}{\varepsilon_1}} \in V_{\varepsilon_2}$, where $f_{\frac{\varepsilon_2}{\varepsilon_1}}$ is the function defined by $f_{\frac{\varepsilon_2}{\varepsilon_1}}(x) = (\frac{\varepsilon_1}{\varepsilon_2})^{\alpha+2} f(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\, x, \frac{\varepsilon_1}{\varepsilon_2}t)$, $\lim\limits_{\varepsilon\to 0} V_\varepsilon = L^2_\alpha(K)$ and $\lim\limits_{\varepsilon\to+\infty} V_\varepsilon = \{0\}$.

Theorem III.1 : The net $\{V_\varepsilon\}_{\varepsilon\in]0,+\infty[}$ given by

$$V_\varepsilon = \{f \in L^2_\alpha(K) / \Phi_g(f)(a,(x,t)) = 0 \text{ a.e. on }]0,+\infty[\times K\}$$

where g is a generalized wavelet on K in $(L^1_\alpha \cap L^2_\alpha)(K)$, is a generalized continuous multiscale analysis if and only if the support of $\mathcal{F}(g)$ is bounded away from zero i.e. there exists $\gamma > 0$ such that $\{\lambda \in \mathbb{R} / |\lambda| \leq \gamma\} \cap \sup \mathcal{F}(g)(.,m)$ is a subset measurable on $\mathbb{R}$ whose Lebesgue measure vanishes for all $m \in \mathbb{N}$.

Definition III.2 : Let g be a generalized wavelet in $(L^1_\alpha \cap L^2_\alpha)(K)$ such that supp $\mathcal{F}(g)$ is bounded away from zero. Let $\varepsilon > 0$ and $f \in L^2_\alpha(K)$. The partial reconstruction f_ε of f from its generalized continuous wavelet transform $\Phi_g(f)$ is defined by

$$f_\varepsilon(.) = \frac{1}{C_g}\int_\varepsilon^{+\infty} (\int_K \Phi_g(f)(a,(y,s))\, g_{a,(y,s)}(.)\, dm_\alpha(y,s)) \frac{da}{a^{\alpha+3}}$$

Theorem III.2 . Let f be a function on K with the properties $f \in (L^1_\alpha \cap L^2_\alpha)(K)$ and $\mathcal{F}(f) \in L^1_\alpha(\mathbb{R} \times \mathbb{N})$ then we have

$$\mathcal{F}(f_\varepsilon)(\lambda,m) = \mathcal{F}(f)(\lambda,m)\ F_\varepsilon(\lambda,m),\ (\lambda,m) \in \mathbb{R} \times \mathbb{N}$$

where F_ε the function defined by

$$F_\varepsilon(\lambda,m) = \frac{1}{C_g}\int_\varepsilon^{+\infty} |\mathcal{F}(g_a)(\lambda,m)|^2 \frac{da}{a}$$

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THE RADON TRANSFORM ON THE LAGUERRE HYPERGROUP

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Abstract. *We consider the Radon transform* R_α, $\alpha \geq 0$, *on the Laguerre hypergroup* $K = [0,+\infty[\times \mathbb{R}$. *We give a space of rapidly decreasing functions on which* R_α *is bijective. We establish an inversion formula and a Plancherel theorem for the operator* R_α. *Next, by using the continuous wavelet transform on the Laguerre hypergroup* K, *we deduce an other expression of the inverse operator* R_α^{-1} *of* R_α.

I. Radon transform on the Laguerre hypergroup.

Notations. We use the following notations

- $K = [0,+\infty[\times \mathbb{R}$
- $\mathcal{C}(K)$ the space of continuous functions on K.
- $M_b(K)$ the space of bounded Radon measures on K.
- $\mathcal{S}(K)$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives and even with respect to the first variable.
- $L^p_\alpha(K)$, $p \in [1,+\infty[$, the space of functions defined on K, measurable and such that $\int_K |f(x,t)|^p \, dm_\alpha(x,t) < +\infty$, where dm_α is the measure defined on K by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi.\,\Gamma(\alpha+1)} dx dt.$$

Definition I.1. Let f be in $\mathcal{C}(K)$, we put for all (x,t), (y,s) $\in$ K

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f\left(\sqrt{x^2+y^2+2xy\cos\theta}\,,\ s+t+xy\sin\theta\right) d\theta, & \text{if } \alpha = 0, \\ \dfrac{\alpha}{\pi} \displaystyle\int_0^1\int_0^{2\pi} f\left(\sqrt{x^2+y^2+2xyr\cos\theta}\,,\ s+t+xyr\sin\theta\right) r(1-r^2)^{\alpha-1} dr d\theta, & \text{if } \alpha > 0. \end{cases}$$

The operators $T^{\alpha}_{(x,t)}$, $(x,t) \in K$, are called generalized translation operators associated with Laguerre functions.

Definition I.2

i) For all (x,t), (y,s) $\in$ K and $f \in \mathcal{C}(K)$ we put

$$\delta_{(x,t)} * \delta_{(y,s)}(f) = T^{\alpha}_{(x,t)} f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t).

The product * is called the generalized convolution product on K.

ii) The convolution product * on $M_b(K)$ is defined by

$$v_1 * v_2(f) = \int_{K\times K} T^{\alpha}_{(x,t)} f(y,s)\, dv_1(x,t)\, dv_2(y,s).$$

Theorem I.1 : (K,*,i) is a hypergroup in the sense of Jewett ([3]), where i is the involution on K defined by

$$i(x,t) = (x,-t).$$

Definition I.3 . The Radon transform on the Laguerre hypergroup K is defined by

$$R_\alpha(f)(x,t) = \frac{2\,\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^{+\infty} T^{\alpha}_{(x,t)} f(y,o)\, y^{2\alpha+1}\, dy \; ; \; f \in \mathcal{S}(K) \qquad (I.1)$$

Remark I.1 : In [5], p 384, the Radon transform on the Heisenberg group $\mathbb{H}^n$ is defined by

$$R(g)(z,t) = \int_{\mathbb{C}^n} g(\xi+z, t - \frac{1}{2}\,\mathrm{Im}(z/\xi))\, d\xi \;;\; (z,t) \in \mathbb{H}^n,$$

where $d\xi$ is the Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and (./.) is the usual positive definite Hermitian form : $(z/z') = \sum_{i=1}^{n} z_i \bar{z}'_i$. If the function g is radial i.e. $g(z,t) = f(\|z\|,t)$, where $\|z\| = \sum_{i=1}^{n} |z_i|^2$ is the usual norm on $\mathbb{C}^n$, the function R(g) is also radial and we have : $R(g)(z,t) = R_{n-1}(f)(2\|z\|,t)$.

Remark I.2. The relation (I.1) can also be written as follows

$$R_\alpha(f)(x,t) = \frac{2\,\pi^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_0^{+\infty} \left(\int_u^{\infty} (v^2-u^2)^{\alpha-1/2} [f(v,t+xu) + f(v,t-xu)]\, v dv \right) du$$

II. Range of some subspace of $\mathcal{S}(K)$ by the Radon transfor R_α.

Notations. We denote by

- $\mathcal{S}_1(K)$ the subspace of $\mathcal{S}(K)$, consisting of functions g such that g vanishes with all its derivatives on $\mathbb{R} \times \{0\}$.
- $\mathcal{S}_2(K)$: the subspace of $\mathcal{S}(K)$, consisting of functions g such that

$$\forall k \in \mathbb{N},\ \forall x \in \mathbb{R},\ \int_{\mathbb{R}} t^k g(x,t)\, dt = 0$$

- $\mathcal{F}_1$, $\mathcal{F}_2$ the operators defined on $L^1_\alpha(K)$ by

$$\mathcal{F}_1(f)(x,\lambda) = \int_{\mathbb{R}} f(x,t) e^{-i\lambda t}\, dt \;;\; (x,\lambda) \in K,$$

$$\mathcal{F}_2(f)(\mu,\lambda) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_K f(x,t) e^{-i\lambda t} j_\alpha(\mu.x)\, x^{2\alpha+1} dx dt \;;\; (\mu,\lambda) \in K,$$

where $j_\alpha(x) = \Gamma(\alpha+1)\,(\frac{x}{2})^{-\alpha} J_\alpha(x)$, J_α being the Bessel function of first kind and with order α.

Proposition II.1.

i) We have

$$\mathcal{S}_2(K) = \{f \in \mathcal{S}(K) / \mathcal{F}_1(f) \in \mathcal{S}_1(K)\} = \{f \in \mathcal{S}(K) / \mathcal{F}_2(f) \in \mathcal{S}_1(K)\}$$

ii) For all $f \in \mathcal{S}(K)$ we have

$$\mathcal{F}_1(R_\alpha f)(x,\lambda) = \mathcal{F}_2(f)(x\lambda,\lambda)\ ;\ (x,\lambda) \in K$$

Theorem II.1. The Radon transform R_α is a bijection from $\mathcal{S}_2(K)$ onto itself.

III. Inversion formula and Plancherel theorem for the operator R_α.

Notations : For $\alpha \geq 0$, we denote by L_α the operator defined on $\mathcal{S}(K)$ by

$$L_\alpha f(x,t) = \begin{cases} D_2^{\alpha+1} f(x,t) & , \quad \text{if } (\alpha+1) \in 2\mathbb{N}, \\ I^{-(\alpha+1)} f(x,t) & , \quad \text{otherwise,} \end{cases}$$

where D_2 is the partial differential operator defined by : $D_2 f(x,t) = \dfrac{\partial f(x,t)}{\partial t}$ and I^γ , $\gamma \in \mathbb{C}$, such that $\gamma - 1 \notin 2\mathbb{N}$, is the operator defined by

$$I^\gamma f(x,t) = \frac{\Gamma(\frac{1-\gamma}{2})}{\pi . 2^\gamma . \Gamma(\frac{\gamma}{2})} \int_{\mathbb{R}} \frac{f(x,u)}{|t-u|^{1-\gamma}}\, du$$

For $Re(\gamma) \leq 0$, this is interpreted as an analytic continuation (see [1], [2] p 67).
The space $\mathcal{S}_2(K)$ is invariant under I^γ and we have

$$\mathcal{F}_1(I^\gamma f)(x,\lambda) = |\lambda|^{-2\gamma}\, \mathcal{F}_1(f)(x,\lambda)\ ;\ (x,\lambda) \in K \text{ and } f \in \mathcal{S}(K).$$

Definition III.1. The generalized Fourier transform $\mathcal{F}$ is defined on $L^1_\alpha(K)$ by

$$\forall\ (\lambda,m) \in \mathbb{R} \times \mathbb{N},\ \mathcal{F}(f)(\lambda,m) = \int_K f(x,t)\, e^{-i\lambda t}\, \mathcal{L}_m^{(\alpha)}(|\lambda| x^2)\, dm_\alpha(x,t),$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre function defined by $\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2}(L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0))$, $L_m^{(\alpha)}$ being the Laguerre polynomial of degree m and order α.

Proposition III.1. Let f be in $\mathcal{S}_2(K)$. Then $L_\alpha R_\alpha(f)$ belongs to $\mathcal{S}_2(K)$ and we have

$$\mathcal{F}(L_\alpha R_\alpha f)(\lambda,m) = (2\pi)^{\alpha+1}\, (-1)^m\, \mathcal{F}(f)(\lambda,m)\ ;\ (\lambda,m) \in \mathbb{R} \times \mathbb{N}.$$

Theorem III.1. For all $f \in \mathcal{S}_2(K)$, we have the following inversion formula

$$(L_\alpha R_\alpha)^2 f = (2\pi)^{2(\alpha+1)} f,$$

$$R_\alpha (L_\alpha)^2 R_\alpha f = (2\pi)^{2(\alpha+1)} f,$$

where $(L_\alpha R_\alpha)^2 = L_\alpha R_\alpha\, L_\alpha R_\alpha$ and $(L_\alpha)^2 = L_\alpha L_\alpha$.

Theorem III.2. (Plancherel formula and theorem for R_α).

i) The operator $L_\alpha R_\alpha$ is an isomorphism from $\mathcal{S}_2(K)$ onto itself. Furthermore, we have the following Plancherel formula

$$\forall f \in \mathcal{S}_2(K), \int_K |f(x,t)|^2 dm_\alpha(x,t) = (2\pi)^{-2(\alpha+1)} \int_K |L_\alpha R_\alpha(f)(x,t)|^2 dm_\alpha(x,t).$$

ii) The operator $\frac{1}{(2\pi)^{\alpha+1}} L_\alpha R_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(K)$ onto itself. Furthermore, we have

$$\forall f \in L^2_\alpha(K), (L_\alpha R_\alpha)^2 f = (2\pi)^{2(\alpha+1)} f$$

IV. Inversion of the Radon transform using generalized wavelets on K.

We define for regular function g the generalized continuous wavelet transform on the Laguerre hypergroup K by

$$\Phi_g(f)(a,(x,t)) = \int_K f(y,s)\bar{g}_{a,(x,t)}(y,s)\, dm_\alpha(y,s)) , a \in]o,+\infty[, (x,t) \in K,$$

where $g_{a,(x,t)}$ is given by

$$g_{a,(x,t)}(y,s) = a^{\frac{\alpha}{2}+1} T^{(\alpha)}_{(x,t)} g_a(y,s)$$

and

$$g_a(x,t) = \frac{1}{a^{\alpha+2}} g(\frac{x}{\sqrt{a}}, \frac{t}{a})$$

The function g is a generalized wavelet on K i.e. a function on K satisfying the condition that there exists a constant $o < C_g < \infty$, such that for all $m \in \mathbb{N}$ and for λ almost everywhere on $\mathbb{R}$, we have

$$C_g = \int_o^\infty |\mathcal{F}(g)(a\lambda,m)|^2 \frac{da}{a}$$

The results obtained for the generalized wavelet transform on the hypergroup K are proved in [4]. In particular, we have the following inversion formula :

$$\text{(IV.2)} \qquad f(x,t) = \frac{1}{C_g} \int_{]o,\infty[} (\int_K \Phi_g(f)(a,(y,s))g_{a,(y,s)}(x,t)dm_\alpha(y,s)) \frac{da}{a^{\alpha+3}} ,$$

when $f \in L^1_\alpha(K)$ and $\mathcal{F}(f)$ integrable on $\mathbb{R} \times \mathbb{N}$ with respect to the measure $d\gamma_\alpha$ defined by : $d\gamma_\alpha(\lambda,m) = \pi . \frac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)} |\lambda|^{\alpha+1} d\lambda \otimes \delta_m$, where δ_m is the Dirac measure at m and $d\lambda$ is the Lebesgue measure on $\mathbb{R}$.

Theorem IV.1. For all $f,g \in \mathcal{S}_2(K)$ we have

$$\Phi_{L_\alpha R_\alpha L_\alpha g}(R_\alpha f) = (2\pi)^{2(\alpha+1)} \Phi_g(f).$$

From this theorem and the relation (IV.1) we deduce the following theorem

Theorem IV.2 . Let $g \in \mathcal{S}(K)$ be a generalized wavelet on K. Then for all $f \in \mathcal{S}_2(K)$, we have

$$R_\alpha^{-1} f(x,t) = \frac{1}{(2\pi)^{2(\alpha+1)}.C_g} \int_{]0,\infty[} \left(\int_K \Phi_{L_\alpha R_\alpha L_\alpha g}(f)(a,(y,s)) g_{a,(y,s)}(x,t) dm_\alpha(y,s) \right) \frac{da}{a^{\alpha+3}}$$

Remark IV.1. The function $L_\alpha R_\alpha L_\alpha g$ is not necessarily a generalized wavelet on K. But, if we suppose that $L_\alpha g$ is a generalized wavelet together with g, $L_\alpha R_\alpha L_\alpha g$ becomes also a generalized wavelet on K.

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CONTINUOUS WAVELET TRANSFORM ON SEMISIMPLE LIE GROUPS AND INVERSION OF THE ABEL TRANSFORM AND ITS DUAL

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Abstract

In this work we consider the Abel transform $\mathcal{A}$ on semisimple Lie groups G of real rank ℓ , and its dual $\mathcal{A}^*$. Using these transforms we give relations between the continuous wavelet transform on G and the classical continuous wavelet transform on $\mathbb{R}^\ell$, and we deduce the inverse operators of the operators $\mathcal{A}$ and $\mathcal{A}^*$.

Let G be a noncompact connected real semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{g}$ = k+p be the Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a} \subset$ p be a maximal abelian subspace, $\mathfrak{a}^*$ its (real) dual. The dimension ℓ of $\mathfrak{a}$ is called the real rank of G. We can identify $\mathfrak{a}$ with $\mathbb{R}^\ell$. We denote by Σ the set of restricted roots. Let W be the Weyl group associated with Σ and |W| its cardinality. Let Σ^+ be the set of positive roots, and put $\Sigma_o^+ = \Sigma^+ \cap \{\alpha \in \Sigma / \frac{1}{2}\alpha \notin \Sigma\}$. Put $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$, with m_α the multiplicity of α. Let K(resp.A) be the analytic subgroup of G with Lie algebra k(resp.$\mathfrak{a}$), and H : G --> $\mathfrak{a}$ the Iwasawa projection according to the Iwasawa décomposition G = K A N i.e. if g $\in$ G then H(g) is the unique element in $\mathfrak{a}$ such that g $\in$ K (exp H(g))N. On the compact group K the Haar measure dk is normalized by $\int_K dk = 1$. For the normalization of the Harr measures dx and dn respectively on the groups G and N see [1] p 273.

I. The Abel transform on G and its dual.

The Abel transform on G denoted $\mathcal{A}$ is defined on $\mathcal{D}$(K\G/K) (the space of C^∞-functions on G which are bi-invariant under K and with compact support) by

$$\forall\, H \in \mathfrak{a},\ \mathcal{A}(f)(H) = e^{\rho(H)}\int_N f(\exp H(n))dn$$

(See [2] p 450-454 and 51] p 107-108 and p 264 - 265).

For f in $\mathcal{D}$(K\G/K) we have the relation : $\mathcal{F}(f) = \mathcal{F}_0 \circ \mathcal{A}(f)$ where $\mathcal{F}$ is the spherical Fourier transform on G and $\mathcal{F}_0$ the classical Fourier transform on $\mathfrak{a}$.

The dual Abel transform on G denoted $\mathcal{A}^*$ is defined on $\mathcal{E}(\mathfrak{a})^W$ (the space of C^∞-function $\mathfrak{a}$ which are W-invariant) by

$$\forall\, x \in G,\ \mathcal{A}^*(f)(x) = \int_K f(H(xk))e^{-\rho H(xk)}dk$$

Notations : We denote by

- $\mathcal{S}(\mathfrak{a})^W$ (resp $\mathcal{S}(\mathfrak{a}^*)^W$) the space of C^∞-functions on $\mathfrak{a}$ (resp. $\mathfrak{a}^*$) which are W-invariant and rapidly decreasing as their derivatives.

- $\mathcal{S}^o(\mathfrak{a}^*)^W$ the subspace of $\mathcal{S}(\mathfrak{a}^*)^W$ consists of functions f such that :

$$\forall\, \alpha \in \mathbb{N}^\ell,\ \frac{\partial^{|\alpha|}}{\partial\lambda_1^{\alpha_1}\ldots\partial\lambda_\ell^{\alpha_\ell}} f(\lambda)/_{\lambda=0} = 0.$$

- $\mathcal{A}_0(\mathfrak{a})^W$ (resp. $\mathcal{C}_0(K\backslash G/K)$) the subspace of $\mathcal{A}(\mathfrak{a})^W$ (resp. the spherical Schwartz space $\mathcal{C}(K\backslash G/K)$ (see [1] p 252-257)) consists of functions f such that $\mathcal{F}_0(f)$ (resp. $\mathcal{F}(f)$) belongs to $\mathcal{A}^0(\mathfrak{a}^*)^W$.

Proposition 1.1 : The operator $\mathcal{K}_0$ (resp $\mathcal{K}_1$) defined on $\mathcal{A}_0(\mathfrak{a})^W$ (resp. $\mathcal{C}_0(K\backslash G/K)$) by

$$\forall H \in \mathfrak{a},\ \mathcal{K}_0(f)(H) = \mathcal{F}_0^{-1}[|W|^{-1}|C(\lambda)|^{-2}\mathcal{F}_0(f)](H)$$

$$(\text{resp. } \forall x \in G,\ \mathcal{K}_1(f)(x) = \mathcal{F}^{-1}[|W|^{-1}|C(\lambda)|^{-2}\mathcal{F}(f)](x))$$

is a topological isomorphism from $\mathcal{A}_0(\mathfrak{a})^W$ (resp. $\mathcal{C}_0(K\backslash G/K)$ onto itself.

Theorem 1.1.

i) The Abel transform $\mathcal{A}$ is a topological isomorphism from $\mathcal{C}_0(K\backslash G/K)$) onto $\mathcal{A}_0(\mathfrak{a})^W$.

ii) The dual Abel transform $\mathcal{A}^*$ is a topological isomorphism from $\mathcal{A}_0(\mathfrak{a})^W$ onto $\mathcal{C}_0(K\backslash G/K)$.

iii) We have the following inversion formulas

- For $f \in \mathcal{C}_0(K\backslash G/K)$, $f = \mathcal{A}^* \mathcal{K}_0 \mathcal{A}(f)$, $f = \mathcal{K}_1 \mathcal{A}^*\mathcal{A}(f)$
- For $f \in \mathcal{A}_0(\mathfrak{a})^W$, $f = \mathcal{A}\, \mathcal{K}_1 \mathcal{A}^*(f)$, $f = \mathcal{K}_0 \mathcal{A}\mathcal{A}^*(f)$.

II. Inversion of the transforms $\mathcal{A}$ and $\mathcal{A}^*$ using wavelets on G.

1 - Classical continuous wavelet transform on $\mathbb{R}^\ell$.

We define for regular functions on $\mathbb{R}^\ell$, the classical continuous wavelet transform S on $\mathbb{R}^\ell$ by :

$$S_{g^o}(f)(a,y) = \int_o^\infty\int_{\mathbb{R}^\ell} f(x)\, \overline{g^o_{a,y}}(x)dx,\ a \in]o,+\infty[,\ y \in \mathbb{R}^\ell$$

where $g^o_{a,y}(x) = a^{\ell/2}\, \mathcal{T}_x(g_a)(-y)$, with $\mathcal{T}_x$ is the translation operator defined by : $\mathcal{T}_x(h)(y) = h(x+y)$, and $g^o_a(x) = \frac{1}{a^\ell} g^o(\frac{x}{a})$, the function g^o is a classical wavelet on $\mathbb{R}^\ell$, i.e. a function on $\mathbb{R}^\ell$ satisfying the condition : There exists a constant C_{g^o} such that $o < C_{g^o} < +\infty$ and, for λ almost every where on $\mathbb{R}^\ell$ we have :

$$C_{g^o} = \int_o^\infty |\mathcal{F}_o(g)(a\lambda)|^2 \frac{da}{a}$$

The results obtained for the transform S are given in [3]. In particular we have the following inversion formula

$$f(x) = \frac{1}{C_{g^o}} \int_o^\infty (\int_{\mathbb{R}^\ell} S_{g^o}(f)(a,y)\, g^o_{a,y}(x)dy)\, \frac{da}{a^{\ell+1}},\ \text{a.e.} \tag{II.1}$$

when g^o is square integrable on $\mathbb{R}^\ell$ with respect to the Lebesgue measure and the functions f and $\mathcal{F}_o(f)$ are integrable on $\mathbb{R}^\ell$ with respect to the Lebesgue measure.

2 - Continuous wavelet transform on G.

We said that a function g on G, measurable, bi-invariant under K, is a wavelet on G if there is a constant C_g with the property that $o < C_g < +\infty$ and, for λ almost every where on $\mathfrak{a}^*$,

$$C_g = \int_o^\infty |\mathcal{F}(g)(a\lambda)|^2 \frac{da}{a}$$

Let g be a wavelet on G in $L^2(K\backslash G/K)$ (the space of functions on G measurable, bi-invariante under K and square integrable with respect to the Haar measure of G) and $a \in]o,+\infty[$. Then there exists a function $g_a \in L^2(K\backslash G/K)$ such that

(II.2) $$\forall \lambda \in \mathfrak{a}^*, \ \mathcal{F}(g_a)(\lambda) = \mathcal{F}(g)(a\lambda)$$

and we have

$$||g_a||_{L^2} \leq (k(a)a^{-\ell})^{1/2} \, ||g||_{L^2}$$

where

$$k(a) = \sup_{\lambda \in \mathfrak{a}^*\backslash\{o\}} \frac{|C(\lambda/a)|^{-2}}{|C(\lambda)|^{-2}} = \prod_{\alpha \in \Sigma_o^+} \begin{cases} \dfrac{1}{a^2} & , \ if \ a \geq 1 \\ \dfrac{1}{a^{m_\alpha + m_{2\alpha}}} & , \ if \ a < 1 \end{cases}$$

Where C is the Harish-Chandra's function (See [1] p 175).

Remark : Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{S}(K\backslash G/K)$) and g_a the function given by the relation (II.2). Then we have the relation : $\forall H \in \mathfrak{a}, \ \mathcal{A}(g_a) = a^{-\ell} \mathcal{A}(g)(H/a)$ and the function $\mathcal{A}(g)$ is a classical wavelet on $\mathbb{R}^\ell$.

We define the continuous wavelet transform on G for $f \in L^2(K\backslash G/K)$ by

$$\Phi_g(f)(a,y) = \int_G f(x) \, \bar{g}_{a,y}(x)dx \ , \quad y \in G$$

where $g_{a,y}(x) = (k(a)a^{-\ell})^{-1/2} \, T_x(g_a)(y)$, where T_x is the generalized translation operator on G given by : $T_x(h)(y) = \int_K h(xky) \, dk$.

The results obtained for the transform Φ are given in [4]. In particular we have the following inversion formula

(II.3) $$f(x) = \frac{1}{C_g} \int_o^\infty (\int_G \Phi_g(f)(a,y) g_{a,y}(x)dy) \frac{k(a)}{a^{\ell+1}} da, \quad x \in G$$

when g and f are in $\mathcal{S}(K\backslash G/K)$.

3 - Inversion of the transforms $\mathcal{A}$ and $\mathcal{A}^*$

Notations : We denote by $\tilde{S}_{\mathcal{K}_o(G_a)}$ and $\tilde{\Phi}_{\mathcal{K}_1(G_a)}$ the operators defined by

$$\forall H \in \mathfrak{a}, \ \tilde{S}_{\mathcal{K}_o(G_a)}(F)(H) = a^{\ell/2} \, F *_o \overline{\mathcal{K}_o(G_a)} \, (H)$$

$$\forall y \in G, \ \tilde{\Phi}_{\mathcal{K}_1(G_a)}(F)(y) = (k(a)a^{-\ell})^{-1/2} \, F * \overline{\mathcal{K}_1(G_a)}(y)$$

where $*_o$ (resp. $*$) the convolution of functions on $\mathbb{R}^\ell$ (resp. of functions on G bi-invariant under K).

Remark : The transform $\bar{S}_{\mathcal{K}_o(G_a)}$(resp. $\bar{\Phi}_{\mathcal{K}_1(G_a)}$) is not necessarily a classical continuous wavelet transform on $\mathbb{R}^\ell$ (resp. a continuous wavelet transform on G) because in general we have not : $\mathcal{K}_0(G_a) = h_0(a)\, \mathcal{K}_0(G)_a$ (resp. $\mathcal{K}_1(G_a) = h_1(a)\mathcal{K}_1(G)_a$),where h_0(resp. h_1) is a function on $]0,+\infty[$.

Theorem II.1 : Let g be a wavelet on G in $\mathcal{D}_0(K\backslash G/K)$.

i) For all $f \in \mathcal{D}_0(K\backslash G/K)$ we have for $y \in G$

$$\Phi_g(f)(a,y) = (k(a))^{-1/2}\, \mathcal{A}^*[\bar{S}_{\mathcal{K}_o(\mathcal{A}(g))_a}(\mathcal{A}(f))](y)$$

ii) For all $f \in \mathcal{S}_0(\mathfrak{a})^W$ we have for $H \in \mathfrak{a}$

$$S_{\mathcal{A}(y)}(f)(a,H) = (k(a))^{1/2}\, \mathcal{A}[\bar{\Phi}_{\mathcal{K}_1(g_a)}(\mathcal{A}^*(f))](H).$$

From this theorem and the relations (II.1), (II.3) we deduce the following theorem which gives the inverse operators of the operators $\mathcal{A}$ and $\mathcal{A}^*$ using wavelets on G.

Theorem II.2 : Let g be a wavelet on G in $\mathcal{D}_0(K\backslash G/K)$

i) For all $f \in \mathcal{S}_0(\mathfrak{a})^W$ we have for $x \in G$

$$\mathcal{A}^{-1}(f)(x) = \frac{1}{C_g}\int_0^\infty \left(\int_G \mathcal{A}^*[\bar{S}_{\mathcal{K}_o(\mathcal{A}(g)_a)}(f)](y)\, g_{a,y}(x)dy\right) \frac{(k(a))^{1/2}}{a^{\ell+1}}\, da$$

ii) For all $f \in \mathcal{D}_0(K\backslash G/K)$ we have for $H' \in \mathfrak{a}$

$$(\mathcal{A}^*)^{-1}(f)(H') = \frac{1}{C_{\mathcal{A}(g)}}\int_0^\infty \left(\int_{\mathfrak{a}} \mathcal{A}[\bar{\Phi}_{\mathcal{K}_1(g_a)}(f)](H)\, g_{a,H}(H')dH\right) \frac{(k(a))^{1/2}}{a^{\ell+1}}\, da$$

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SOLUTION METHOD OF FORWARD EDDY CURRENT TESTING PROBLEMS

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ABSTRACT. An analytical formula for the impedance change, Z, in the coil of an eddy current probe is obtained for the forward problem in the case the conductivities, σ_2 and σ_1, of the flaw and surrounding medium, respectively, are almost the same. The vector-valued solution is expanded in power series in the small parameter $\varepsilon = 1 - \sigma_2/\sigma_1$ and a first-order approximation is obtained by a perturbation method. Two examples are treated, where the flaw is not in a symmetric position with respect to the coil axis. It is shown that, in some cases, only one component of the vector potential (or even a part of this component) needs to be known in order to compute Z.

RÉSUMÉ. On présente une solution analytique du changement d'impédance, Z, d'une bobine d'une sonde à courants de Foucault pour le problème direct si la conductivité d'une paille et celle du milieu environnant, resp. σ_2 et σ_1, diffèrent de peu. On développe le vecteur solution en série de puissance de $\varepsilon = 1 - \sigma_2/\sigma_1$ et l'on considère l'approximation du premier ordre par une méthode de perturbation. On traite deux exemples où la paille est en position asymétrique par rapport à l'axe de la bobine. On montre que dans certains cas il suffit de connaitre une composante du vecteur solution, ou même une partie de cette composante, pour calculer Z.

1. Introduction. To control properties of materials, eddy currents are induced in a conducting medium in the presence of a time-varying magnetic field. Inhomogeneities in the medium produce an impedance change, Z, in the detector coil of an eddy current probe. This change can be used to estimate properties of the flaw (for example, its depth, cross-section, conductivity).

Perturbation methods have been used, in [1], to solve forward eddy current testing problems under the condition that the conductivities of the flaw and surrounding medium do not differ by much. For example, in quality control of spot welding, experimental data show that the difference in conductivities of the cast core and the surrounding medium varies between 10% and 20%.

In this note, a general scheme for the solution of forward eddy current testing problems is obtained by a perturbation method and is illustrated by two particular problems, namely, a double conductor line situated above a cylinder with a flaw and a single-turn coil situated above a ball with a flaw, where the flaw is asymmetrically positioned with respect to the coil axis.

Key words and phrases. Eddy current NDE, medium with varying magnetic properties, Bessel functions.
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In some cases, it is found that one does not need to know the complete distribution of the electromagnetic field in order to find Z, but it is enough to know a single component of the vector potential, or even a part of this component (for example, the first term of the Fourier series expansion of the solution, as in the second example of this note).

2. Mathematical analysis. Neglecting the displacement current in Maxwell's equations [2] and assuming that all the functions which describe the electromagnetic field are periodic in time t, with frequency ω, we obtain the following equation for the vector potential $\boldsymbol{A}$:

$$\Delta \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu_0 \boldsymbol{I}^e, \tag{1}$$

where $k^2 = -j\omega\sigma\mu_0$, $j = \sqrt{-1}$, $\boldsymbol{I}^e$ is the external current density, σ is the conductivity of the medium and μ_0 is the magnetic constant.

We consider the following physical model. Let R_0 denote free space and $\boldsymbol{I}^e$ be the density of the external currents in R_0. We assume that region R_1 is a uniform conducting medium with conductivity σ_1. The interior region R_2, with conductivity σ_2, is surrounded by R_1.

Let $\boldsymbol{A}_i$, $i = 0, 1, 2$, denote the vector potentials in R_0, R_1, R_2, respectively, so that (1) becomes

$$\Delta \boldsymbol{A}_0 = -\mu_0 \boldsymbol{I}^e, \qquad \Delta \boldsymbol{A}_1 + k_1^2 \boldsymbol{A}_1 = 0, \qquad \Delta \boldsymbol{A}_2 + k_2^2 \boldsymbol{A}_2 = 0, \tag{2}$$

where $k_i^2 = -j\omega\sigma_i\mu_0$, $i = 1, 2$.

Assuming that the conductivities $\sigma_1 > \sigma_2$ do not differ by much, we introduce the small parameter $\varepsilon = 1 - \sigma_2/\sigma_1$ so that k_1 and k_2 are related by the formula $k_2^2 = k_1^2(1-\varepsilon)$.

Using a perturbation method, we represent the vector potential in the series

$$\boldsymbol{A}_i = \boldsymbol{A}_i^{(0)} + \varepsilon \boldsymbol{A}_i^{(1)} + \varepsilon^2 \boldsymbol{A}_i^{(2)} + \dots, \qquad i = 0, 1, 2, \tag{3}$$

since, in this case, we have a regular perturbed problem.

By standard perturbations techniques, we obtain the following system of equations for $\boldsymbol{A}_i^{(0)}$ and $\boldsymbol{A}_j^{(n)}$, respectively, where $i = 0, 1$, $j = 0, 1, 2$ and $n = 1, 2, \dots$:

$$\Delta \boldsymbol{A}_0^{(0)} = -\mu_0 \boldsymbol{I}^e, \qquad \Delta \boldsymbol{A}_1^{(0)} + k_1^2 \boldsymbol{A}_1^{(0)} = 0, \tag{4}$$

$$\Delta \boldsymbol{A}_0^{(n)} = 0, \qquad \Delta \boldsymbol{A}_1^{(n)} + k_1^2 \boldsymbol{A}_1^{(n)} = 0, \qquad \Delta \boldsymbol{A}_2^{(n)} + k_1^2 \boldsymbol{A}_2^{(n)} = k_1^2 \boldsymbol{A}_2^{(n-1)}, \tag{5}$$

where $\boldsymbol{A}_1^{(0)}$ represents the vector potential in $R_1 \cup R_2$, since the properties of R_1 and R_2 coincide when $\varepsilon = 0$.

Thus, (4) describes the electromagnetic field when a current source $\boldsymbol{I}^e$ is located in R_0 above flawless R_1, i.e. $R_2 = \emptyset$.

We shall consider first-order approximate solutions to two particular cases of the above problem.

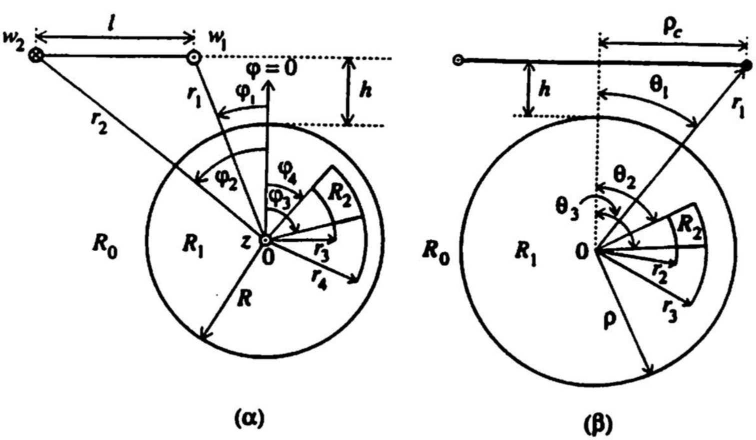

FIGURE 1. (α) Horizontal double conductor line in free space, R_0, above and parallel to a metallic circular cylinder, R_1, with a flaw, R_2. (β) Central section of single-turn coil in free space, R_0, above a ball, R_1, of radius ρ with a flaw, R_2.

3. Double conductor line above an infinitely long cylinder with a flaw. Consider a double conductor line, w_1, w_2, in R_0, parallel to a metallic circular cylinder, R_1, of radius R with an interior cylindrical flaw, R_2 (see Fig. 1(α)). We introduce cylindical polar coordinates (r, φ, z) centered at 0 and let the z-axis coincide with the cylinder axis. The flaw cross-section is the "curvilinear rectangle" : $r = r_3$, $r = r_4$, $\varphi = \varphi_3$, $\varphi = \varphi_4$. We assume that the parameter ε is small.

The zeroth-order approximate solution in $R_1 \cup R_2$, found in [3], is

$$A_1^{(0)}(r,\varphi) = \frac{1}{2\pi}\frac{\mu_0 I l^2(r_2 - r_1)}{\rho k_1 J_1(k_1\rho)} J_0(k_1 r) + \frac{\mu_0 I l^2}{\pi k_1}\sum_{m=1}^{\infty}\left[\left(\frac{\rho}{r_1}\right)^{m-1}\cos m(\varphi-\varphi_1) - \left(\frac{\rho}{r_2}\right)^{m-1}\cos m(\varphi-\varphi_2)\right]\frac{J_m(k_1 r)}{J_{m-1}(k_1\rho)}, \quad (6)$$

where $J_m(\xi)$ is the Bessel function of the first kind of order m; r is a dimensionless coordinate; l, chosen as unit length, is the distance between the wires; $\rho = R/l$; and r_1 and r_2 are measered in units of l.

We now consider the first-order approximation to the solution. By (5) with $n = 1$, we have the following system of equations for the z-component, $A_i^{(1)}(r,\varphi)$, $i = 0, 1$, of the vector potential,

$$\frac{\partial^2 A_0^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial A_0^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_0^{(1)}}{\partial \varphi^2} = 0, \qquad \text{in } R_0, \quad (7)$$

$$\frac{\partial^2 A_1^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial A_1^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_1^{(1)}}{\partial \varphi^2} + k_1^2 A_1^{(1)} = \begin{cases} 0, & \text{in } R_1, \\ k_1^2 A_1^{(0)}(r,\varphi), & \text{in } R_2, \end{cases} \quad (8)$$

where $A_1^{(1)}$ denotes the solution in $R_1 \cup R_2$. The boundary conditions are

$$A_0^{(1)}\Big|_{r=\rho} = A_1^{(1)}\Big|_{r=\rho}, \qquad \frac{\partial A_0^{(1)}}{\partial r}\Big|_{r=\rho} = \frac{\partial A_1^{(1)}}{\partial r}\Big|_{r=\rho}. \tag{9}$$

The solution to (7)–(9) can be found by separation of variables. If we restrict ourselves to the first two terms on the right-hand side of (3), then the first-order perturbation of $A_0(r,\varphi)$ due to the flaw is equal to $\varepsilon A_0^{(1)}(r,\varphi)$. Thus, the impedance change, Z_C^{ind}, of the double conductor line, per unit length, due to the flaw, is

$$Z_C^{\text{ind}} = \frac{j\omega\varepsilon}{I} \oint_C A_0^{(1)}(r,\varphi) e_z \cdot dl, \tag{10}$$

where C is the contour of unit length of the double conductor line.

Using (10) and the solution $A_0^{(1)}(r,\varphi)$ to (7)–(9) in R_0, we obtain

$$Z_C^{\text{ind}} = \frac{\mu_0 I^2 \omega\varepsilon}{\pi^2} Z_C,$$

where

$$Z_C = \frac{j(r_2 - r_1)}{2\rho^2 J_1(k_1\rho)} \sum_{m=1}^{\infty} \frac{1}{m J_{m-1}(k_1\rho)} \int_{r_3}^{r_4} s J_0(k_1 s) J_m(k_1 s)\, ds \left[\left(\frac{\rho}{r_2}\right)^m d_{0m}(\varphi_2) - \left(\frac{\rho}{r_1}\right)^m d_{0m}(\varphi_1)\right]$$
$$+ \frac{j}{\rho} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{J_{m-1}(k_1\rho)} \int_{r_3}^{r_4} s J_n(k_1 s) J_m(k_1 s)\, ds \left[\left(\frac{\rho}{r_2}\right)^m d_{mn}(\varphi_2) - \left(\frac{\rho}{r_1}\right)^m d_{mn}(\varphi_1)\right], \tag{11}$$

$$d_{0n}(\varphi) = \sin(n\varphi_4 - n\varphi) - \sin(n\varphi_3 - n\varphi),$$

$$d_{mm}(\varphi) = \sum_{i=1}^{2} (-1)^{i-1} \left(\frac{\rho}{r_i}\right)^{m-1} \frac{1}{J_{m-1}(k_1\rho)} \left\{\frac{1}{2}(\varphi_4 - \varphi_3)\cos m(\varphi - \varphi_i) + \frac{1}{4m}[\sin(2m\varphi_4 - m\varphi_i - m\varphi) - \sin(2m\varphi_3 - m\varphi_i - m\varphi)]\right\}, \qquad m = n,$$

$$d_{mn}(\varphi) = \sum_{i=1}^{2} (-1)^{i-1} \left(\frac{\rho}{r_i}\right)^{m-1} \frac{1}{J_{m-1}(k_1\rho)} \times \left\{\frac{1}{2(m-n)}[\sin(m\varphi_4 - n\varphi_4 - m\varphi_i + n\varphi) - \sin(m\varphi_3 - n\varphi_3 - m\varphi_i + n\varphi)] + \frac{1}{2(m+n)}[\sin(m\varphi_4 + n\varphi_4 - m\varphi_i - n\varphi) - \sin(m\varphi_3 + n\varphi_3 - m\varphi_i - n\varphi)]\right\}, \qquad m \neq n,$$

Formula (11) was used to compute $|Z_C|$ for different values of the parameters of the problem by means of Mathematica, version 2.2.2, on a Sun Sparc 10.

In Fig. 2(α), $|Z_C|$ is plotted against the dimensionless coordinate $\xi = 0.5 + r_1 \sin\varphi_1$ which measures the horizontal shift of the double conductor line from the vertical axis $\varphi = 0$. The diameter, $2\rho = 0.8$, of the cylinder is smaller than the distance between the wires, and the remaining parameters were set at $h = 0.05$, $r_3 = 0.3$, $r_4 = 0.35$, $\beta = 2$, where $\beta = l\sqrt{\omega\sigma_1\mu_0}$. In the three cases, the curves have four maxima. This large number of maxima can be explained by the relatively small diameter of the cylinder as compared with

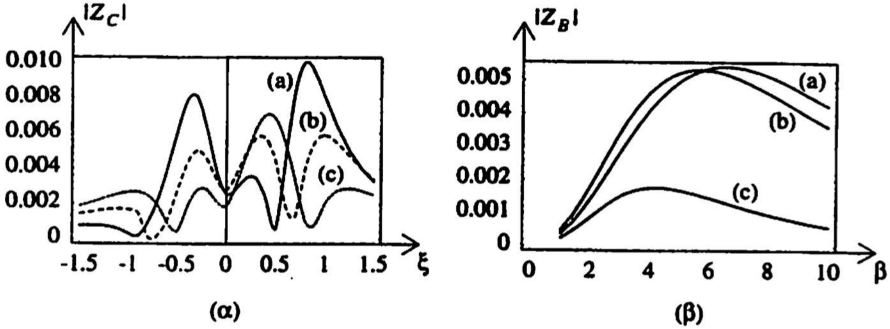

FIGURE 2. (α) $|Z_C|$ against ξ, for $\varphi_4 = \varphi_3 + \pi/8$ and flaw orientations (a) $\varphi_3 = 3\pi/16$, (b) $\varphi_3 = 5\pi/16$, (c) $\varphi_3 = 7\pi/16$. (β) $|Z_B|$ against β, for $\theta_3 = \theta_2 + \pi/4$ and flaw orientations (a) $\theta_2 = 0$, (b) $\theta_2 = \pi/8$, (c) $\theta_2 = \pi/4$.

the distance between the wires so that the curvature of R_1 is important. Similar curves were experimentally observed in crack inspection problems (see, for example, [4]). The complicated behaviour of the curves can help reconstruct the parameters of the flaw and be used to solve the inverse problem since one can hope to establish a correspondence between the abscissa and magnitude of the extremal points of the curves and the parameters of the flaw.

4. A single-turn coil above a ball with a flaw. We consider a conducting ball with an interior flaw. Let R_2 denote the flaw and R_1 denote the remaining part of the ball.

A single-turn circular coil, of radius ρ_c, is situated in R_0 at height h above the ball and the axis of the coil goes through the centre, 0, of the sphere (see Fig. 1(β)). We introduce spherical coordinates, (r, θ, φ), centered at 0. We suppose that R_2 is the "curvilinear rectangular parallelepiped" $r = r_2$, $r = r_3$, $\theta = \theta_2$, $\theta = \theta_3$, $\varphi = \varphi_1$, $\varphi = \varphi_2$, so that the final formula for the coil impedance change will have a simple form.

By the approach used in the previous problem, we obtain a first-order approximation to the change in impedance in the form

$$Z_B^{\text{ind}} = \frac{j\omega\varepsilon}{I} \oint_C \left(A_{0,r}^{(1)}\, dr + r A_{0,\theta}^{(1)}\, d\theta + A_{0,\varphi}^{(1)} r \sin\theta\, d\varphi \right), \tag{12}$$

where integration is taken along the contour, C, of the coil, and $A_{0,r}^{(1)}$, $A_{0,\theta}^{(1)}$ and $A_{0,\varphi}^{(1)}$ are the r-, θ- and φ-components of $A_0^{(1)}$ in R_0. However, we need only know $A_{0,\varphi}^{(1)}$, since $r = r_1 = \text{const}$ and $\theta = \theta_1 = \text{const}$ on C (see Fig. 1(β)).

We seek a solution in the case $n = 1$ as a pair of Fourier series,

$$A_{i,\varphi}^{(1)}(r, \theta, \varphi) = a_i(r, \theta) + \sum_{n=1}^{\infty} [a_{in}(r, \theta) \cos n\varphi + b_{in}(r, \theta) \sin n\varphi], \qquad i = 0, 1. \tag{13}$$

By formula (12), where integration with respect to φ is over $[0, 2\pi]$, the terms in (13) containing sines and cosines do not contribute to the integral. Therefore, in order to evaluate this integral we need only know $a_0(r,\theta)$, which is independent of φ. This fact considerably simplifies the solution.

The first-order approximation to the coil impedance change, Z_B^{ind}, due to the flaw, is

$$Z_B^{\text{ind}} = \frac{\mu_0\omega\rho_c^2\varepsilon}{8\pi} Z_B, \qquad \text{where} \quad Z_B = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \widehat{\gamma}_{mn}, \tag{14}$$

and

$$\widehat{\gamma}_{mn} = -j(\varphi_2-\varphi_1)\sin^2\theta_1 \frac{(2n+1)(2m+1)P_m^{(1)}(\cos\theta_1)P_n^{(1)}(\cos\theta_1)}{n(n+1)m(m+1)J_{m-1/2}(k\rho)J_{n-1/2}(k\rho)}$$
$$\times\left(\frac{\rho}{r_1}\right)^{m+n-1}\int_{\theta_2}^{\theta_3} P_m^{(1)}(\cos\theta)P_n^{(1)}(\cos\theta)\sin\theta\, d\theta \int_{r_2}^{r_3} uJ_{n+1/2}(ku)J_{m+1/2}(ku)\, du,$$

where $k = \rho_c\sqrt{-j\omega\sigma_1\mu_0}$, σ_1 is the conductivity of the ball and $P_m^{(1)}(\eta)$ is an associated Legendre function of the first kind.

In Fig. 2(β), $|Z_B|$, computed by means of (14), is plotted against the dimensionless parameter $\beta = \rho_c\sqrt{\omega\sigma_1\mu_0}$ for $\theta_3 = \theta_2 + \pi/4$ and three orientations of the flaw: (a) $\theta_2 = 0$, (b) $\theta_2 = \pi/8$, (c) $\theta_2 = \pi/4$. The remaining parameters were set at $\rho = 0.9$, $r_2 = 0.7$, $r_3 = 0.8$, $h = 0.1$, $\varphi_1 = 0$, $\varphi_2 = \pi/4$. It is seen, from the figure, that $|Z_B|$ decreases as θ_2 increases.

In conclusion, it is to be noted that the above method is rather general. If the distribution of the electromagnetic field in a conducting medium in the absence of a flaw is given in closed form, then, in many cases, a closed form solution for the corresponding problem with a flaw can be obtained even if the conducting medium is not uniform, where the conductivity, σ, and the magnetic permeability, μ, are functions of the spatial coordinates. In this case the general perturbation method can still be applied.

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SUR L'IDENTIFICATION EN DIMENSION UN D'UNE PERTURBATION LINEAIRE DE L'EQUATION DES ONDES PAR UNE OBSERVATION FRONTIERE.

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Presented by G.F.D. Duff, F.R.S.C.

Résumé

On montre pour l'équation des ondes définie sur un segment, par une méthode directe, que l'observation de la réponse de la dérivée normale à des actions de type Dirichlet sur l'une des extrêmités du segment détermine de manière unique toute perturbation lineaire.

Abstract

We prove by direct method the identification of a linear perturbation of a wave equation defined in a segment by boundary observations.

I. Notations

Soit T un réel strictement positif, on pose

$$Q =]0,1[\times]0,T[\qquad \text{et} \qquad \Delta = \left\{(x,s)\in \mathbb{R}^2 \setminus 0\leq s\leq x\leq 1\right\}.$$

On note a et b des fonctions positives de $C^1([0,1],\mathbb{R})$.

Soit $h \in L^2([0,T])$, on dit que y est solution du problème $P(a,h)$ ou $y \in P(a,h)$ si on a:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \dfrac{\partial^2 y}{\partial x^2} + a(x)y = 0 & \text{dans } Q \\ y(0,t)=0 & \text{sur }]0,T[\\ y(1,t)=h(t) & \text{sur }]0,T[\\ y(x,0)=\dfrac{\partial y}{\partial t}(x,0)=0 & \text{dans }]0,1[\end{cases}$$

II. POSITION DU PROBLEME

Soit T>2, on considère le problème suivant:

Si pour tout $h \in L^2(0,T)$, la solution y de $P(a,h)$ et la solution z de $P(b,h)$ vérifient :

$$\forall\ t \in \,]0,T[, \quad \left(\frac{\partial y}{\partial x}(0,t)\ ,\ \frac{\partial y}{\partial x}(1,t) \right) = \left(\frac{\partial z}{\partial x}(0,t)\ ,\ \frac{\partial z}{\partial x}(1,t) \right)$$

a -t-on a = b ?

On donne une réponse positive à cette question et on améliore les résultats de A. EL BADIA [1],qui obtient l'identification de la perturbation linéaire de l'équation des ondes pour le même type d'action au bord, par une observation frontière et l'observation de l'état du système à l'instant T.

Dans notre cas l'observation est entièrement frontière de même que le mode d'action.

La preuve repose sur une méthode de résolution des équations hyperboliques introduite par E. Picard [4] et sur la contrôlabilité exacte de l'équation des ondes avec une perturbation linéaire [2].

III. Résultat principal

Théorème

Soit $T>2$, *si pour tout* $h \in L^2(0,T)$ *les solutions* y *de* $P(a,h)$ *et* z *de* $P(b,h)$ *vérifient:*

pour tout $t \in \,]0,T[$, $\left(\frac{\partial y}{\partial x}(0,t)\ ,\ \frac{\partial y}{\partial x}(1,t) \right) = \left(\frac{\partial z}{\partial x}(0,t)\ ,\ \frac{\partial z}{\partial x}(1,t) \right)$,

alors $a = b$.

Remarque:

On agit sur une seule extrémité du segment $[0\ ,\ 1]$.■

IV. Preuve du résultat principal

Elle se fait en plusieurs étapes:

Lemme1

Soient Φ_1 *et* Φ_2 *deux fonctions de* $C^2([0,1],\mathbb{R})$ *telles que* $\Phi_1(0) = \Phi_2(0)$ *et* c *une fonction de* $C^2(\,[0,1] \times [0,1],\ \mathbb{R}\,)$, *alors le problème*

$$(P)\ \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,s) - \dfrac{\partial^2 u}{\partial s^2}(x,s) + c(x,s)u(x,s) = 0 \quad \text{sur } \Delta & (1) \\ u(x,0) = \Phi_1(x) \quad \text{sur } [0,1] & (2) \\ u(x,x) = \Phi_2(x) \quad \text{sur } [0,1] & (3) \end{cases}$$

admet une solution unique deux fois continûment dérivable sur Δ.

<u>Preuve</u>

On construit, comme dans E. Picard [4], une suite $(u_n)_{n\in\mathbb{N}^*}$ de fonctions de $C^2(\Delta, \mathbb{R})$ telle que la série $u = \sum_{n\in\mathbb{N}^*} u_n$ converge vers la solution du problème (P).

Soit u_1 la solution de:

$$\begin{cases} \frac{\partial^2 u_1}{\partial x^2}(x,s) - \frac{\partial^2 u_1}{\partial s^2}(x,s)=0 \quad \text{sur } \Delta \\ u_1(x,0)=\Phi_1(x) \quad \text{sur } [0,1] \\ u_1(x,x)=\Phi_2(x) \quad \text{sur } [0,1] \end{cases}$$

alors pour tout $(x,y) \in \Delta$,

$$u_1(x,y)=\Phi_2((x+y)/2)-\Phi_2((x-y)/2)+\Phi_1(x-y)$$

Pour $n\geq 2$, on note u_n la solution du problème:

$$\begin{cases} \frac{\partial^2 u_n}{\partial x^2}(x,s) - \frac{\partial^2 u_n}{\partial s^2}(x,s)+c(x,s)u_{n-1}(x,s)=0 \quad \text{sur } \Delta \\ u(x,0)=0 \quad \text{sur } [0,1] \\ u(x,x)=0 \quad \text{sur } [0,1] \end{cases}$$

alors pour $(x,y) \in \Delta$,

$$u_n(x,y)=-\frac{1}{4}\int_0^{x-y}\int_0^{x-y} c((u+v)/2,(u-v)/2)u_{n-1}((u+v)/2,(u-v)/2))dudv$$

$$+\frac{1}{4}\int_0^{x+y}\int_0^{x-y} c((u+v)/2,(u-v)/2)u_{n-1}((u+v)/2,(u-v)/2))dudv$$

La série $\sum u_n$ et celles des dérivées partielles de u_n jusqu'à l'ordre 2, sont normalement convergentes; il en résulte que

$u = \sum_{n=1}^{+\infty} u_n$ est de classe C^2 sur Δ et comme

$u(x,x) = u_1(x,x) = \Phi_2(x)$ et $u(x,0)=u_1(x,0)=\Phi_1(x)$,u est solution de (P).

Lemme2

Soit T>2 , si y est solution du problème

$$(P')\begin{cases}\dfrac{\partial^2 y}{\partial t^2}(x,t) - \dfrac{\partial^2 y}{\partial x^2}(x,t) + a(x)\, y(x,t) = 0 & \text{sur } Q\\ y(0,t)=\dfrac{\partial y}{\partial x}(0,t)=0 & \text{sur }]0,1[\\ y(x,0)=\dfrac{\partial y}{\partial t}(x,0)=0 & \text{sur }]0,1[\end{cases}$$

alors y est identiquement nulle.

Preuve

On prolonge l'équation sur]0 , 2[en x et on écrit la décomposition spectrale de la solution.■

On note K la solution de (P) correspondant à $c(x,s) = -(b(x) - a(s))$, $\Phi_1=0$ et $\Phi_2(x) = \frac{1}{2}\int_0^x (b(t) - a(t))\,dt$.

On dit que y est solution de P(a) ou y ∈ P(a) si y vérifie:

$$\begin{cases}\dfrac{\partial^2 y}{\partial t^2} - \dfrac{\partial^2 y}{\partial x^2} + a(x)\, y = 0 & \text{dans } Q\\ y(0,t)=0 & \text{sur }]0,T[\\ y(x,0)=\dfrac{\partial y}{\partial t}(x,0)=0 & \text{dans }]0,1[\end{cases}$$

Lemme3

Soit T>2, si y ∈ P(a) alors la fonction

$$z(x,t) = y(x,t)+\int_0^x K(x,s)\, y(s,t)\, dt \tag{4}$$

est l'unique solution de P(b) qui satisfait:

$$\frac{\partial z}{\partial x}(0,t) = \frac{\partial y}{\partial x}(0,t) .$$

Preuve

On vérifie que

$$\frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) + b(x)\, z(x,t)=0;$$

et de (4), on tire

$$\frac{\partial z}{\partial x}(0,t)=\frac{\partial y}{\partial x}(0,t) \quad \text{et} \quad z(0,t)=y(0,t)=0;$$

on achève la preuve du lemme3 en appliquant le lemme 2 ■

Lemme 4

Si pour tout $s \in [0,1]$ $K(1,s) = \frac{\partial K}{\partial x}(1,s) = 0$ *alors* $a = b$.

Preuve

Soit u la solution du problème (P) correspondant à

$c(x,s)=-(b(x)-a(s))$, $\Phi_1(x)=0$ et $\Phi_2(x) =\int_0^x K(t,t)dt$ sur [0,1],

en multipliant (1) par $K(x,s)$ et en intégrant par parties sur Δ ,on obtient:

$$\int_0^1 K(x,x)\frac{d}{dx}(u(x,x))dx-\int_0^1 u(x,x)\frac{d}{dx}(K(x,x))dx +\int_0^1 \frac{\partial K}{\partial s}(x,0)u(x,0)dx$$
$$-\int_0^1 \left(K(1,x) \frac{\partial u}{\partial x}(1,x)-u(1,x) \frac{\partial K}{\partial x}(1,x) \right)dx =0$$

D'où en tenant compte des conditions aux limites ,on a :

$$\int_0^1 K^2(x,x)dx = 0$$

Ainsi $K(x,x)=0$, ce qui entraine que $a = b$.■

Preuve du théorème

Pour achever la preuve du théorème, on considère:

$h \in L^2([0,T])$, $y \in P(a,h)$ et $z \in P(b,h)$

alors $\frac{\partial z}{\partial x}(0,t)= \frac{\partial y}{\partial x}(0,t)$, $\frac{\partial z}{\partial x}(1,t)= \frac{\partial y}{\partial x}(1,t)$ et $y(1,t)=z(1,t)$. (5)

D'après le lemme 3,

$$z(x,t)=y(x,t)+\int_0^x K(x,s)y(s,t)dt$$

Cette relation donne en tenant compte de (5) que :

$\forall\, t \in]0,T[$, $\int_0^1 K(1,s)y(s,t)\, ds =0$ (6)

$$\text{et} \quad K(1,1)y(1,t)+\int_0^1 \frac{\partial K}{\partial x}(1,s)y(s,t)ds = 0 \qquad (7)$$

De (6) on tire:

$$\int_0^1 K(1,s)y(s,T)\, ds = 0. \qquad (8)$$

La contrôlabilité exacte frontière entraine que :

$$\left\{ y(.,T) \ / \ h \in L^2([0,T]) \right\} = L^2([0,1]),$$

donc

$$\forall \ s \in [0,1], \ K(1,s) = 0.$$

De même

$$\forall \ s \in [0,1], \ \frac{\partial K}{\partial x}(1,s) = 0 \ .$$

D'après le lemme 4, on a donc a = b .■

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A Class of Interacting Measure-valued Branching Diffusions and Their Spatial Structures

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Abstract

A class of interacting measure-valued branching diffusions with local Lipschitz coefficients are constructed and characterized. Their spatial structures are investigated. The effective state space of these processes is contained in the set of purely atomic measures or the set of absolutely continuous measures according as $\epsilon = 0$ or $\epsilon \neq 0$ if the coefficients are smooth functions. In particular under the assumption that the coefficients are local Lipschitz functions and $\epsilon = 0$ it is proved that the processes take values in the set of purely atomic measures.

The purpose of this paper is to outline the construction and study of a new class of measure-valued diffusion processes which arise as limits in distribution of a sequence of interacting branching particle systems. The particle systems studied are as follows. For any integer n, we consider a system of particles initially with m_0^n particles which move, die and produce offspring in a Brownian medium on $\bar{\mathbb{R}} := \mathbb{R} \cup \{\Delta\}$, one point compactification of $\mathbb{R}$. Such a system has the form

$$dx_i(t) = \int_{\mathbb{R}} g(y - x_i(t))W(dy, dt) + \epsilon dB_t^i, \tag{0.1}$$

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where $x_i(t)$ is the location of i^{th} particle , $W(\cdot,\cdot)$ is a cylindrical Brownian motion (refer to Dawson [1]), $\{B_t^i\}$ are independent one-dimensional Brownian motions which are independent of W, ϵ is a real constant, and g is a symmetric function on $\bar{\mathbb{R}}$ such that

$$(0.2) \qquad \lim_{x\to\pm\infty} g(x) = g(\Delta) = 0, \qquad \int_{\mathbb{R}} g^2(x)dx < \infty.$$

The quadratic variational process

$$(0.3) \qquad \langle x_i, x_j\rangle_t = \int_0^t \rho(x_i(s) - x_j(s))ds + \epsilon^2 t\delta_{ij}, \quad \rho(x) = \int_{\mathbb{R}} g(z-y)g(y)dy,$$

where δ_{ij} is equal to 1 or 0 according as i is equal to j or not. Here we assume that each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants. In the case of critical branching, when a particle dies, it produces k particles with probability $p_k; k = 0, 1, 2, \cdots$. Also we assume

$$(0.4) \qquad \sum_{k=0}^{\infty} kp_k = 1, \qquad m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty.$$

After branching, the resulting set of particles evolve in the same way and they start off from the parent particle's branching site. Let m_t^n denote the total number of particles at time t . Define

$$(0.5) \qquad \mu_t^n(\cdot) := \frac{1}{\theta^n}\sum_{i=1}^{m_t^n} \delta_{x_i^n(t)}(\cdot).$$

We assume that $m_0^n/\theta^n \leq \xi$, where $\xi > 0$ is a fixed constant, $\mu_0^n \Rightarrow \bar{\mu}$ (weak convergence), where $\bar{\mu}$ is a finite measure with compact support and there are no interactions in the branching mechanism. Let $E := M_F(\bar{\mathbb{R}})$ be the space of all bounded Radon measures on $\bar{\mathbb{R}}$ with weak topology. Then E is a Polish space. By Ito's formula, we can formally get the generators of the limiting measure-valued processes as following

$$(0.6) \qquad \mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu)$$

where

$$(0.7) \qquad \mathcal{B}F(\mu) := \tfrac{1}{2}\gamma(m_2 - 1)\int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx),$$

$$\mathcal{A}F(\mu) := \frac{1}{2}\int_{\mathbb{R}} \rho_\epsilon (\frac{d^2}{dx^2}) \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) \tag{0.8}$$

$$+\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x-y)(\frac{d}{dx})(\frac{d}{dy})\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy),$$

$F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(E)$. ρ_ϵ is a constant satisfying $\rho_\epsilon = \rho(0) + \epsilon^2$. Clearly our model includes super-Brownian motion as a special case.

In the following, we will present our main results. Following each main theorem, we will outline the main ideas to prove it. The detailed proofs will appear elsewhere.

Theorem 0.1 *Assume that*

$$\rho(z) = \int_{\mathbb{R}} g(z-y)g(y)dy, \qquad \int_{\mathbb{R}} g^2(x)dx < \infty, \tag{0.9}$$

and $g(x)$ is a symmetric function. Suppose $\rho(\cdot)$ is a local Lipschitz function (i.e. Lipschitz outside each interval $(-c,c)$, $c>0$), then for any finite measure μ with compact support, the $(\mathcal{L},\delta_\mu)$-martingale problem(MP) *has a unique solution.*

Outline of the Proof of Theorem 0.1:

The proof of this theorem is divided into two steps. In the first step, we suppose $\rho(\cdot)$ is a smooth function. For the existence, first we prove that for any finite measure μ with compact support, the $(\mathcal{A},\delta_\mu)$-MP has a unique solution, so $\mathcal{A}$ is a dissipative operator. The fact that $\mathcal{A}$ is a generator of a Feller semigroup follows from the Hille-Yosida's theorem. Then we can construct our finite particle systems by operator perturbation technique and the existence for the smooth function $\rho(\cdot)$ follows from a tightness argument. To prove the uniqueness, for monomial functions $< f, \mu^m >$ we have

$$\mathcal{L}F(\mu,f) =< G^m f, \mu^m > +\frac{1}{2}\gamma(m_2-1)\sum_{j,k=1,j\neq k}^m \{< \Phi_{jk} f, \mu^{m-1} > \tag{0.10}$$

$$- < f, \mu^m >\} + \frac{1}{2}\gamma(m_2-1)m(m-1) < f, \mu^m >,$$

$$G^N f(x_1,\cdots,x_N) := \frac{1}{2}\sum_{i,j=1}^N \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j} f(x_1,\cdots,x_N). \tag{0.11}$$

$$(0.12)\quad \Phi_{j,k}f(x_1,\cdots,x_m) := \begin{cases} f(x_1,\cdots,x_{k-1},x_j,x_{k+1},\cdots,x_m) & \text{if } j<k \\ f(x_1,\cdots,x_{j-1},x_k,x_{j+1},\cdots,x_m) & \text{if } j>k \end{cases}$$

Since the closure of G^m is a generator of a Feller semigroup, the existence of the function-valued dual process is obvious and which implies the uniqueness for smooth function $\rho(\cdot)$. In the second step, we consider $\rho(\cdot)$ to be a local Lipschitz function. In this case, it is easy to construct a sequence of smooth function $\rho_n(\cdot)$ such that ρ_n uniformly converges to $\rho(\cdot)$. Then the existence can be obtained by tightness argument. However the uniqueness has a difficulty since in this case, we cannot show that the closure of G^m is the generator of a Feller semigroup. The idea to overcome this difficulty is to construct the dual process by exploiting the martingale problem method and using C-diffusions to directly prove the dual identity. ■.

Theorem 0.2 *Suppose $\rho(\cdot)$ is a local Lipschitz function and μ is any finite measure on $\mathbb{R}$ with compact support. Let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P}_\mu,\mu_t)$ be the interacting measure-valued branching Markov process which is the unique solution to $(\mathcal{L},\delta_\mu)$-MP, then $\{\mu_t\}$ is a diffusion process.*

Outline of the Proof of Theorem 0.2:

This theorem can be proved by using a theorem in Jacod-Memin [3]. ■

Theorem 0.3 (A) *Suppose $\epsilon = 0$, then for any finite measure ν on $\mathbb{R}$ with compact support, the unique solution $\{\nu_t\}$ to $(\mathcal{A}+\mathcal{B},\delta_\nu)$-MP has form $\nu_t = \sum_{i\in I(t)} a_i(t)\delta_{x_i(t)}$, where $a_i(t)\geq 0, x_i(t)$ are continuous in t and $I(t)$ is a random subset of $\mathbb{N}$.*

(A.1) *If $\rho(\cdot)$ is a smooth function, then $x_i(t)\neq x_j(t)$ for $t\geq 0$ and $i\neq j$.*

(A.2) *If $\rho(\cdot)$ is a local Lipschitz function, then $\{x_i(t)\}$ may coalesce.*

(B) *Suppose $\epsilon\neq 0$ and $\rho(\cdot)$ is a smooth function, then for any finite measure ν on $\mathbb{R}$ with compact support, the unique solution $\{\mu_t\}$ to $(\mathcal{A}+\mathcal{B},\delta_\nu)$-MP is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ for almost all $t>0$, $\mathbb{P}_\nu$-almost surely.*

Outline of the Proof of Theorem 0.3:

First, let us consider the case in which $\epsilon = 0$ and $\rho(\cdot)$ is a smooth function. Suppose

μ and ν are any finite measures on $\mathbb{R}$ with compact support and $\{\nu_t\}$, $\{\mu_t\}$ denote the unique solutions to $(\mathcal{B}, \delta_\nu)$-MP, $(\mathcal{A}, \delta_\mu)$-MP on a probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ respectively. By a random time change and a transformation if necessary, we can change $\{\nu_t\}$ into a probability measure-valued process with support $[0,1]$. Define $\phi_\beta : \mathcal{P}[0,1] \to [0,\infty]$ for each $\beta > 0$ by

$$\phi_\beta(\mu) := \begin{cases} \sum_{0\le x\le 1} \mu(\{x\})^\beta & \text{if } \beta \neq 1 \\ 1 & \text{if } \beta = 1. \end{cases} \tag{0.13}$$

By exploiting martingale problem method and using function $\phi_\beta(\mu)$, we can prove $\{\nu_t\}$ is a purely atomic measure for any $t > 0$ and

$$\nu_t = \sum_{i\in I(t)} a_i(t)\delta_{x_i(0)}, \quad t > 0, \quad x_i(0) \in \mathbb{R}, \quad x_i(0) \neq x_j(0) \quad \text{for } i \neq j \tag{0.14}$$

where $I(t)$ is a random subset of $\mathbb{N}$, $a_i(t) \geq 0$ is the unique solution to $(\frac{1}{2}\gamma(m_2 - 1)x\frac{d^2}{dx^2}, \delta_{a_i(0)})$-MP. This also shows that the action of operator $\mathcal{B}$ only changes the masses of atoms. On the other hand, if μ is a purely atomic measure and has the form

$$\mu = \sum_{i\in J} a_i(0)\delta_{x_i(0)}, \quad J \subset \mathbb{N}.$$

Then we can prove that $\{\mu_t\}$ is continuous in weak atomic topology on E which was introduced by Ethier-Kurtz [2]. It follows that $\{\mu_t\}$ has following form

$$\mu_t = \sum_{i\in J} a_i(0)\delta_{x_i(t)},$$

and $x_i(t)$ is continuous in t for $i \geq 1$. This result demonstrates that the action of operator $\mathcal{A}$ only changes the positions of atoms. Let $\mathbb{P}^1_\nu$, $\mathbb{P}^2_\mu$ be solutions to $(\mathcal{B}, \delta_\nu)$-MP, and $(\mathcal{A}, \delta_\mu)$-MP respectively. Define

$$S_tF(\nu) := \int F(\omega^1(t))\mathbb{P}^1_\nu(d\omega^1) \tag{0.15}$$

$$T_tF(\mu) := \int F(\omega^2(t))\mathbb{P}^2_\mu(d\omega^2).$$

By direct calculation and above results, we can prove that the semigroup $\{S_t\}$ commutates with $\{T_t\}$, which implies (A) of the Theorem 0.3.
Now, let us turn to consider (B). In this case, we can use duality method together with an application of Malliavin calculus to directly construct the density process of the solution to the $(\mathcal{A}+\mathcal{B},\delta_\nu)$-MP and the conclusion (B) of Theorem 0.3 is the direct consequence of this result. ■

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TOPOLOGIE - Sur une définition non-standard de la dimension d'un espace topologique par B. BRUNET.

Presente par P. Ribenboim, F.R.S.C.

Dans ce qui suit, reprenant une idée de J.P. REVEILLES (6), nous proposons une définition non-standard de la dimension d'un espace topologique et nous établissons que, dans le cas des espaces métriques séparables, cette définition coïncide avec les définitions classiques.

TOPOLOGY - On a nonstandard definition of topological dimension.

Coming back on an idea of J.P. REVEILLES (6), *we give a nonstandard definition of the topological dimension and we prove this definition coincides with the classical definitions in the class of separable metric spaces.*

Dans (6), J.P. REVEILLES propose une définition non-standard - qu'il appelle *épaisseur* et note *ep* - de la dimension d'un espace topologique reposant sur la longueur des chaînes des « halos basiques » emboîtés et montre que, dans le cas des espaces $\mathbb{R}^n$ et plus généralement des espaces métriques séparables, cette définition coïncide avec les définitions classiques - petite dimension inductive (*ind*), grande dimension inductive (*Ind*), dimension de LEBESGUE (*dim*) . Les démonstrations qu'il propose de ces résultats (Corollaire 3, 4 et 5) sont malheureusement incorrectes.

Dans ce qui suit, en reprenant l'idée de J.P. REVEILLES et après avoir légèrement modifié la définition de la longueur des chaînes des halos basiques, nous établissons notamment que, pour tout espace métrique X, on a $ind\, X \leq ep\, X \leq Ind\, X$, ce qui permet, en utilisant un théorème classique concernant la dimension des espaces métriques séparables, d'établir le résultat annoncé.

1 - Epaisseur d'un espace topologique.

Dans ce qui suit, on désigne par X un espace topologique et par $\mathcal{E}$ un élargissement (voir par exemple (4)) de cet espace et on appelle base de X toute base d'ouverts de X supposée stable par intersections finies.

1.1 : Etant donné un élément a de ${}^*X$, on appelle *halo en base $\mathcal{B}$ de a* l'ensemble, noté $h_{\mathcal{B}}(a)$ défini par $h_{\mathcal{B}}(a) = \bigcap_{B \in \mathcal{B}_a} {}^*B$ où $\mathcal{B}_a = \{B \in \mathcal{B} : a \in {}^*B\}$.

1.2 : A toute base $\mathcal{B}$ de X, on peut associer une relation de préordre $\leq$ sur ${}^*X$ par : $a \leq b$ si et seulement si $a \in h_{\mathcal{B}}(b)$.

(Dans la suite on écrira $a < b$ si et seulement si $a \leq b$ mais pas $b \leq a$).

1.3 : Etant donné un point x de X, on dit que l'épaisseur en x d'une base $\mathcal{B}$ de X est inférieure à n, ce que l'on note $ep\,(x, \mathcal{B}) \leq n$, si et seulement si toute chaîne de type $a_p < ... < a_1 < {}^*x$ est de longueur $p \leq n$.

Notons que cette définition est équivalente à celle donnée dans (6) à condition de corriger la notion de « halos consécutifs » page 707.

1.4 : On appelle *épaisseur d'une base* $\mathcal{B}$ de X, l'élément de $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, noté $ep\ \mathcal{B}$, défini par $ep\ \mathcal{B} = sup\{\ ep\ (x, \mathcal{B}) : x \in X\}$.

1.5 : On appelle *épaisseur de* X, l'élément de $\overline{\mathbb{N}}$, noté epX, défini par $ep\ X = inf\{ep\ \mathcal{B} : \mathcal{B} \in \mathcal{B}(X)\}$, $\mathcal{B}(X)$ désignant l'ensemble des bases de X.

On déduit alors de ces définitions les propriétés suivantes :

1.6 : a) $ep\ X = 0$ si et seulement si X possède une base d'ouverts - fermés.

b) Pour tout espace totalement ordonné X, $ep\ X \leq 1$ (en effet, si $\mathcal{B}_0$ désigne la base formée des intervalles ouverts de X, on a $ep\ \mathcal{B}_0 \leq 1$). En particulier, puisque $\mathbb{R}$ est connexe, $ep\ \mathbb{R} = 1$.

c) Pour tout espace topologique X et toute partie A de X, $ep\ A \leq ep\ X$.

d) Pour tout couple (X, Y) d'espaces topologiques, $ep\ (X \times Y) \leq ep\ X + ep\ Y$.

e) Pour tout entier $n \geq 1$, $ep\ \mathbb{R}^n \leq n$.

Il convient de noter qu'à la différence des définitions classiques de la dimension les propriétés c) et d) sont valables sans aucune hypothèse particulière sur les espaces considérés.

2 - Comparaison avec les définitions classiques de la dimension.

Théorème 2.1 : *Pour tout espace topologique X, on a :*

a) $ep\ X = 0$ si et seulement si $ind\ X = 0$,

b) $ind\ X \leq ep\ X$.

a) est immédiat puisque les deux assertions équivalent à dire que X possède une base d'ouverts-fermés.

b) Ecartons le cas trivial où $ep\ X < +\infty$ et prouvons le résultat par récurrence sur $n = ep\ X$. Le résultat est vrai pour $n = 0$ d'après a). Supposons le montré pour tout espace d'épaisseur $n-1$. Comme par hypothèse $ep\ X = n$, il existe une base $\mathcal{B}$ de X telle que $ep\ \mathcal{B} = n$. Soit B un élément de $\mathcal{B}$ et soit $F = FrB$ (FrB désignant le frontière de B). Désignons par $\mathcal{C}$ la trace sur F de $\mathcal{B}$. Soit x un élément de F et soit $\{a_p, \ldots, a_1\}$ une chaine de $h_{\mathcal{C}}({}^*x)$. Comme $a_p \in^* F$, a_p n'est pas minimal (voir (1)) pour le préordre associé à $\mathcal{B}$. Il existe donc un élément a_{p+1} de ${}^*X$ tel que $\{a_{p+1}, a_p, \ldots, a_1\}$ soit une chaine de $h_{\mathcal{B}}({}^*x)$. Comme $ep\ \mathcal{B} = n$, on en déduit que $p \leq n-1$ et donc que $ep\ \mathcal{C} \leq n-1$. Il résulte alors de la définition de $ep\ F$ et de l'hypothèse de récurrence que $ind\ F \leq n-1$ ce qui entraine que $ind\ X \leq n$.

Corollaire 2.2 : *Pour tout entier $n \geq 1$, $ep\ \mathbb{R}^n = n$.*

On sait en effet que $ind\ \mathbb{R}^n = n$. Le résultat découle alors de 1.6 e).

Il convient de noter que, contrairement à ce qui est dit dans (6), il n'est pas possible de montrer ce dernier résultat sans utiliser le théorème du point fixe de BROUWER, la démonstration proposée reposant sur le résultat non établi suivant : pour toutes parties Y et Z d'un espace topologique X, on a $ep\ (Y \cup Z) \leq\ ep\ Y +\ ep\ Z + 1$.

De plus, et toujours contrairement à ce qui est dit dans (6), il n'est pas possible de déduire de 2.2 et des théorèmes classiques de plongement que, pour tout espace métrique séparable X, on a $ep\ X = ind\ X (= Ind\ X = dim\ X)$, sauf à établir que, pour toute partie A de $\mathbb{R}^n$, $ep\ A = ind\ A$.

2.3 *Un exemple d'espace X pour lequel $ind\ X < ep\ X$ et $Ind\ X < ep X$.*

Dans (3), V.V. FILIPPOV a montré qu'il existait deux espaces topologiques compacts non métriques X_1 et X_2 tels que $ind\ X_1 = Ind X_1 = 1$, $ind\ X_2 = Ind\ X_2 = 2$ et $ind\ (X_1 \times X_2) \geq 4$. Il résulte alors de cet exemple et de la propriété 1.6. d), que l'un de ces deux espaces X_i (et on peut montrer que c'est X_1) est tel que $ind\ X_i = Ind\ X_i < ep\ X_i$.

2.4 : *Un exemple d'espace X pour lequel $ep\ X = ind\ X < Ind\ X = dim X$.*

Dans (7), P. ROY a montré qu'il existait un espace métrique X tel que $ind\ X = 0$ et $Ind\ X = dim\ X = 1$. Cet espace X est donc tel que $ep\ X = ind\ X < Ind\ X = dim\ X$.

2.5 : *Un exemple d'espace X pour lequel $dim\ X < ep\ X$.*

Dans (5), O.V. LOKUCIEVSKII a prouvé qu'il existe un espace compact (non métrique) tel que $dim\ X = 1 < 2 = ind\ X = Ind\ X$. Cet espace X est donc tel que $dim\ X < ep\ X$.

3 - Cas des espaces métriques.

Théorème 3.1 : *Pour tout espace métrique X, on a $ind\ X \leq ep\ X \leq dim\ X = Ind\ X$.*

Puisque, pour tout espace métrique Y, on a $dim\ Y = Ind\ Y$ (voir par exemple (2)), il suffit, d'après 2.1 b), d'établir que, pour tout espace métrique X, on a $ep\ X \leq dim\ X$. Dans ce qui suit, on posera, pour toute famille indexée $\mathcal{F} = (F_i)_{i \in I}$ de parties de X et tout point x de X, $ord\ (x, \mathcal{F}) = |\{i \in I : x \in F_i\}| - 1$ ($|\mathcal{A}|$ désignant le cardinal de $\mathcal{A}$) et $ord\ \mathcal{F} = sup\{ord\ (x, \mathcal{F}) : x \in X\}$.

Lemme 3.1.1. : *Pour toute base $\mathcal{B}$ de X, si $\mathcal{F} = (Fr\ B)_{B \in \mathcal{B}}$, on a $ep\ \mathcal{B} \leq ord\ \mathcal{F} + 1$.*

Soit x un élément de X et soit $\{a_p, ... a_1\}$ une chaîne de $h_{\mathcal{B}}({}^*x)$. Il existe alors p éléments distincts de $\mathcal{B}$, $B_1, ..., B_p$, tels que, pour tout $i \in \{1, ..., p\}, a_j \in {}^*B_i$ si et seulement si $j \geq i$ et tels que $x \in Fr B_i$. Par conséquent, par définition de $ord\ (x, \mathcal{F})$, on a $p \leq ord\ (x, \mathcal{F}) + 1$ ce qui entraine que $ep\ (x, \mathcal{B}) \leq ord\ (x, \mathcal{F}) + 1$. Il résulte alors des définitions de $ep\ \mathcal{B}$ et de $ord\ \mathcal{F}$ que $ep\ \mathcal{B} \leq ord\ \mathcal{F} + 1$.

3.1.2 : démonstration de 3.1.

L'assertion est triviale si $dim\ X = +\infty$. Supposons donc que $dim\ X = n$. Il existe alors (voir par exemple (2), 4.2.2) une base σ-localement finie $\mathcal{B}$ de X telle que, si on pose $\mathcal{F} = (Fr\ B)_{B \in \mathcal{B}}$, on a $ord\ \mathcal{F} \leq n - 1$. Cette base $\mathcal{B}$ est donc, d'après 3.1.1, telle que $ep\ \mathcal{B} \leq n$, ce qui entraine que $ep\ X \leq n$.

3.2 : Notons que l'espace de P. ROY déjà cité est un exemple d'espace métrique tel que $ind\ X = ep\ X = 0 < dim\ X = Ind\ X = 1$.

3.3 : **Théorème de coïncidence pour les espaces métriques séparables.**
Pour tout espace métrique séparable X, on a $ep\ X = ind\ X = Ind\ X = dim\ X$.
Ce résultat est alors une conséquence immédiate de 3.1 et du théorème classique suivant : « Pour tout espace métrique séparable X, on a $ind\ X = Ind\ X = dim\ X$ ».

3.4 : *Un exemple d'espace métrique non séparable X pour lequel on a* $ep\ X = ind\ X = Ind\ X = dim\ X$.
Dans (8), E.K. VAN DOUWEN a montré l'existence d'un espace métrique non séparable X tel que $ind\ X = Ind\ X = dim\ X = 1$.
Cet espace X est donc tel que $ep\ X = ind\ X = Ind\ X = dim\ X$.

3.5 : Question : Existe-t-il un espace métrique non séparable X tel que $ind\ X < ep\ X$?

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ON THE PROBLEM OF GLOBAL REGULARITY FOR 3-DIMENSIONAL NAVIER-STOKES EQUATIONS.

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Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. The classical *small data regularity* result known in the theory of 3-dimensional Navier-Stokes equations for about 50 years states that, for any smooth bounded domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, there exists a *small open* set of initial data and forces in the space $D(A^{\frac{1}{4}}) \times L^\infty(0,\infty; D(A^0))$ generating globally regular solutions (here A is the Stokes operator). We describe, for any smooth bounded domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, a *large, unbounded open*, set of initial data and forces in $D(A^{\frac{1}{4}}) \times L^\infty(0,\infty; D(A^0))$ generating globally regular solutions to 3-dimensional Navier-Stokes equations.

We consider the 3-dimensional Navier-Stokes equations [1]-[2] with the Dirichlet non-slip boundary conditions $u|_{\partial\Omega} = 0$ on a smooth bounded domain $\Omega \subset R^3$:

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $A = -P\Delta$ is the Stokes operator, $P : L^2(\Omega) \to H = Cl_{L^2(\Omega)}\{u \in C_0^\infty(\Omega) : \nabla \cdot u = 0\}$ is the orthogonal projection, and $B(u,v) = P(u \cdot \nabla)v$. Despite the fact that the modern mathematical theory of these equations was begun more than 60 years ago with the works of J. Leray [3]–[5] and E. Hopf [6] (for a review of some results, see, e.g., [1]-[23]), a complete description of initial functions u_0 and forces f generating globally regular solutions is still unknown. (A precise definition of the notion of globally regular solutions is given below). The difficulty of finding a solution which is generated by smooth data but is not globally regular is addressed in [10], [17]. Most results establishing global regularity are obtained under some *a priori assumptions* on the solutions; for instance, [18] *assumes some partial regularity* *of the solution*, a result of [7] *assumes that the solution is symmetric* which reduces the problem to the 2-dimensional one, the result of [19] yields the persistence of the global regularity property in a *small* neighborhood of the initial function which *is assumed to generate a globally regular solution* $u(t)$ satisfying $\int_0^\infty ||\nabla u(t)||^4\,dt < \infty$.

A noteworthy exception, which does not assume a priori hypotheses on the flow, is the classical small data regularity result which establishes the existence of a *small* set of data $(u_0, f) \in V \times L^\infty(0,\infty;H)$ generating globally regular solutions. (Here $V = Cl_{H^1(\Omega)}\{u \in C_0^\infty(\Omega) :\ \nabla\cdot u = 0\}$.) In [20]–[23], G. Raugel and G. R. Sell employed a perturbation argument to establish a large data regularity result for the case of periodic boundary conditions on a thin domain $\Omega_\epsilon = (0,l_1) \times (0,l_2) \times (0,\epsilon) \subset R^3$, where $\epsilon > 0$ is small. Using the fact that Ω_ϵ is somehow close to the 2-dimensional domain $(0,l_1)\times(0,l_2)$ for which the global regularity problem is solved, they show the existence of a bounded open set of data $(u_0, f) \in V \times L^\infty(0,\infty;H)$ generating globally regular solutions and with radius proportional to ϵ^{-s}, for some $s > 0$. In particular, the radius is large for small $\epsilon > 0$, which motivates the name *large data regularity* result.

In this note, we make no a priori hypotheses on the flow nor do we assume the thinness of the domain $\Omega \subset R^3$. We present an arbitrarily large data regularity result which states that there is an *unbounded open* set of initial data and forces $(u_0, f) \in V \times L^\infty(0,\infty;H)$ generating globally regular solutions to the 3-dimensional Navier-Stokes equations. We emphasize that the words *arbitrarily large data regularity* refer to the *existence* of arbitrarily large data forming an unbounded open neighborhood of the origin in $V \times L^\infty(0,\infty;H)$ and generating globally regular solutions (by no means do we claim that *all* arbitrarily large data in $V \times L^\infty(0,\infty;H)$ generate such solutions).

We will use the letters D and $\mathcal{R}$ for the domain and the range of an operator, respectively. As in [1]–[2], by a globally regular solution of the Navier-Stokes equations we understand a function $u = u(t)$ such that

$$u \in C([0,\infty); V) \cap L^\infty(0,\infty; V) \cap L^2_{loc}(0,\infty; D(A)),$$
$$u_t \in L^2_{loc}(0,\infty; H)$$

and u satisfies Eq. (1) almost everywhere and Eq. (2). In the sequel we denote the norm in the space H by $\|\cdot\|$ and the norm in $L^\infty(0,\infty;H)$ by $\|\cdot\|_\infty$. In particular, the norm $\|u\|_V$ in the space V is equivalent to $\|A^{\frac{1}{2}}u\|$ and to $\|u\|_{H^1(\Omega)}$, the norm $\|u\|$ is equivalent to $\|u\|_{L^2(\Omega)}$ (hence, $H = D(A^0)$, $V = D(A^{\frac{1}{2}})$).

Now we state our result:

Theorem 1. *For any 3-dimensional bounded smooth domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, there exists an unbounded open star-shaped set of data $(u_0, f) \in V \times L^\infty(0,\infty;H)$ generating globally regular solutions to the 3-dimensional Navier-Stokes equations (1)–(2). Moreover, there exists a number $\kappa_1 > 0$ such that, for any $f \in L^\infty(0,\infty;H)$ with $\|f\|_{L^\infty(0,\infty;H)} < \kappa_1$, all $u_0 \in V$ for which pairs (u_0, f) belong to this set form an unbounded open star-shaped subset of V. Similarly, there exists a number $\kappa_2 > 0$ such that, for any $u_0 \in V$ with $\|u_0\|_V < \kappa_2$, all $f \in L^\infty(0,\infty;H)$ for which pairs (u_0, f) belong to this set form an unbounded open star-shaped subset of $L^\infty(0,\infty;H)$.*

The proof of the Theorem 1 is given by taking the union over $\{r \geq r_0\}$ of the Cartesian products of the sets (3) and (4) described in the Theorem 2 below.

In order to formulate Theorem 2 we define, for arbitrary self-adjoint operator $\mathcal{A} = \int_{-\infty}^{\infty} \lambda\, dE_\lambda$ in a Hilbert space X, the projection L_r, $r > 0$:

$$L_r \stackrel{\text{def}}{=} \int_{\delta_r} dE_\lambda, \qquad r > 0,$$

where $\delta_r = [-r, r]$. Note that in the particular case when $\mathcal{A}$ is the Laplacian operator $-\Delta$ on a bounded domain Ω, the projection L_r coincides with the projections $\mathcal{P}_r$, $G_{\leq r}$, and $P_{\leq r}$ used in [24], [25] (see also [26]), and [27], respectively. The use of the projection L_r now allows one to prove the following theorem.

Theorem 2. *For any smooth bounded domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, there exist numbers $N_0 > 0$ and $N_1 > 0$ such that if $\alpha > 0$ and $\beta > 0$ satisfy the inequality $\alpha + \beta < N_0$ then there exists a number $r_0 = r_0(\nu, \Omega, \alpha, \beta) > 0$ such that the 3-dimensional Navier-Stokes equations (1)–(2) have a unique globally regular solution $u = u(t)$ whenever $u_0 \in V$, $f \in L^\infty(0,\infty; H)$ and $r \geq r_0$ satisfy*

$$(3) \qquad ||L_r u_0||_V^2 < \alpha, \qquad ||(I - L_r)u_0||_V^2 < \ln r,$$

$$(4) \qquad ||L_r f||_\infty^2 < \beta, \qquad ||(I - L_r)f||_\infty^2 < \ln r.$$

This solution satisfies the following global H^1-norm estimates

$$||L_r u(t)||_V^2 \leq N_1(\alpha + \beta), \qquad t \in [0, \infty),$$
$$||(I - L_r)u(t)||_V^2 \leq \frac{3}{2} \ln r, \qquad t \in [0, \infty).$$

Moreover, there exists a time $T_1 = T_1(r) = T_1(r; \alpha, \beta)$ independent of u_0 and f such that $T_1 \to 0$ as $r \to \infty$ and

$$||(I - L_r)u(t)||_V^2 \leq 2D_2 r^{-2} \left[\ln r + r(\alpha + \beta)^2\right], \qquad t \in [T_1, \infty),$$

where D_2 is a constant depending only on ν, Ω.

Theorem 2 follows from the short-time existence of strong solutions to the 3-dimensional Navier-Stokes equations and from the property of the projection L_r stated in the following simple lemma.

Lemma. *For any self-adjoint operator $\mathcal{A}$ in a Hilbert space X and any $r > 0$, the following pair of inequalities holds:*

$$||w||_X^2 \leq \frac{1}{r^2} ||\mathcal{A}w||_X^2, \qquad ||\mathcal{A}v||_X^2 \leq r^2 ||v||_X^2,$$

for all $w \in D(\mathcal{A}) \cap \mathcal{R}(I - L_r)$ and all $v \in \mathcal{R}(L_r)$.

We remark that same results hold for other boundary conditions.

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Equilibrium for abstract nonconvex games

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Erratum (See page 4)

The *Step 2* of the proof of Theorem 2 remains the same with the following correct definition of the map $B : X \longrightarrow \mathcal{P}(X)$:

$$B(x) := \Big(\bigcap_{k\in K(x)} B_{i_{x_k}}(x)\Big) \cap B_{i_x}(x), x \in X.$$

and that of the set $J(x) := \{i_{x_k} : k \in K(x)\} \cup \{i_x\}$.

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