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EQUILIBRIUM FOR ABSTRACT NONCONVEX GAMES

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Presented by G. F. D. Duff, F.R.S.C.

ABSTRACT. We present an equilibrium theorem for abstract qualitative games where convexity is replaced by a contractibility condition.[1]

1. PRELIMINARIES

The purpose of this note is to present an equilibrium theorem for abstract games where convexity is replaced by a condition of contractibility.

Let I be a (possibly uncountable) set of agents. Each agent $i \in I$ has a strategy set X_i. Denote by X the cartesian product $\prod_{i \in I} X_i$ and by X^i the product $\prod_{j \in I \setminus \{i\}} X_j$. For each $i \in I$, let $A_i : X \longrightarrow \mathcal{P}(X_i)$ be a (preference) multifunction (as usual, given a set E, $\mathcal{P}(E)$ denotes the family of all subsets of E). Following Gale and Mas-Colell [3], the collection $\mathcal{G} = (X_i, A_i)_{i \in I}$ is called a *qualitative game*. An *equilibrium* for the game $\mathcal{G}$ is an element $x_* \in X$ such that $A_i(x_*) = \emptyset$ for all $i \in I$; (x_* is sometimes called a *maximal element* for the family of multifunctions $(A_i)_{i \in I}$).

2. THE MAIN THEOREM

Our main theorem (Theorem 3 below) is based on the following topological generalization of the Browder-Fan fixed point theorem.

Theorem 1. *(Horvath [4]) Let X be a compact contractible Hausdorff topological space and let $A : X \longrightarrow \mathcal{P}(X)$ be a multifunction satisfying the following conditions:*

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(i) $\forall y \in X, A^{-1}(y)$ *is open in* X; *(ii)* $\forall U$ *open subset of* X, $\bigcap_{x \in U} A(x)$ *is empty or contractible.*

Then one of the following properties holds:

(1) A *has a fixed point, that is,* $\exists x^* \in X$ *with* $x^* \in A(x^*)$; *or*

(2) A *has a maximal element, that is,* $\exists x_* \in X$ *with* $A(x_*) = \emptyset$.

Browder-Fan fixed point theorem corresponds to the particular instance when X is a compact convex subset of a topological vector space and A is convex-valued (in which case, condition (ii) is clearly satisfied). The following simple example suggests that Theorem 1 is a natural generalization of the Browder-Fan fixed point theorem.

Example 1. *Let* $X = [-1,1]^2 \subset \mathbf{R}^2$, *and let* $A : X \longrightarrow \mathcal{P}(X)$ *be defined as follows:*

$$A(x) := R_+(x) \cup R_-(x) \cup C(x), x = (x_1, x_2) \in X,$$

where $R_+(x)$ *is the open rectangle with vertices* x *and* 0, $R_-(x) = -R_+(x)$, *and* $C(x)$ *is the cross without endpoints* $(-|x_1|, |x_1|) \times \{0\} \cup \{0\} \times (-|x_2|, |x_2|)$.

One readily verifies that in addition of having contractible (or empty) values, the multifunction A *satisfies condition (ii) of Theorem 1. For, if* $U \subseteq X\backslash\{0\}$ *is open, then* $\bigcap_{x \in U} A(x)$ *is contractible at* 0; *and if* $0 \in U$ *open in* X, *then* $\bigcap_{x \in U} A(x)$ *is empty. Moreover,* A *is fixed point-free and has open counter-images (which are empty, or disjoint unions of open rectangles or open strips). Obviously,* 0 *is a maximal element for* A.

Our main theorem is in fact a generalization of Theorem 1 to arbitrary families of multifunctions $(A_i : X = \prod_{i \in I} X_i \longrightarrow \mathcal{P}(X_i))_{i \in I}$ based on a trick of Borglin and Keiding [1].

Theorem 2. *Let* $\mathcal{G} = (X_i, A_i)_{i \in I}$ *be a qualitative game with a (possibly uncountable) set* I *of agents. Assume that for each agent* $i \in I$, *the strategy set* X_i *is a compact contractible Hausdorff topological space, and that the multifunction* $A_i : X = \prod_{i \in I} X_i \longrightarrow \mathcal{P}(X_i)$ *satisfies the following properties:*

(i) $\forall y_i \in X_i$, $A_i^{-1}(y_i)$ *is open in* X;

(ii) $\forall U$ *open subset of* X, $\bigcap_{x\in U} A_i(x)$ *is empty or contractible;*

(iii) $\forall x = (x_i)_{i\in I} \in X, \exists i \in I(x) = \{i \in I : A_i(x) \neq \emptyset\}$ *such that* $x_i \notin A_i(x)$.

Then the game G *has an equilibrium.*

Proof. *Step 1.* For clarity, assume first that G is an N-person game, that is $I = \{1,...,N\}$ is a finite set. For every $i \in I$, define a multifunction $B_i : X \longrightarrow \mathcal{P}(X)$ by

$$y \in B_i(x) \Longleftrightarrow y_i \in A_i(x),\ x \in X.$$

Observe that this is equivalent to putting: $B_i(x) = \pi_i^{-1}(A_i(x)), x \in X, i \in I$, where π_i is the projection of X onto X_i.

Now, define a multifunction $A : X \longrightarrow \mathcal{P}(X)$ by putting, for each $x \in X$

$$A(x) = \begin{cases} \bigcap_{i\in I(x)} B_i(x) & \text{if } I(x) \neq \emptyset; \\ \emptyset & \textit{otherwise}; \end{cases}$$

We will show that the multifunction A satisfies conditions (i)-(ii) but not property (1) of Theorem 1. This will imply the existence of a maximal element $x_* \in X$ for A which is also an equilibrium for the game $\mathcal{G}$.

The fact that A is fixed point free readily follows from (iii).

Assume that $I(x) \neq \emptyset$ for all $x \in X$, the conclusion being otherwise immediate. Let U be a given open subset of X. Then.

$$\begin{aligned} \bigcap_{x\in U} A(x) &= \bigcap_{x\in U}\bigcap_{i\in I(x)} B_i(x) = \bigcap_{x\in U}[\prod_{i\in I(x)} A_i(x) \times \prod_{i\notin I(x)} X_i] \\ &= \bigcap_{x\in U}\prod_{i\in I} R_i(x) = \prod_{i\in I}\bigcap_{x\in U} R_i(x), \text{ where} \end{aligned}$$

$$R_i(x) = \begin{cases} A_i(x) & \text{if } i \in I(x), \\ X_i & \textit{otherwise.} \end{cases}$$

Let us fix $i \in I$. Divide U into two disjoint subsets $U_i = \{x \in U : i \in I(x)\}$ and $U_i^c = U \backslash U_i$. Clearly, U_i is an open subset of U; for, given $x \in U_i, y_i \in A_i(x)$, the set $A_i^{-1}(y_i) \cap U$ is an open neighborhood of x in U_i. Furthermore,

$$\bigcap_{x\in U} R_i(x) = \bigcap_{x\in U_i} A_i(x) \cap \bigcap_{x\in U_i^c} X_i = \bigcap_{x\in U_i} A_i(x)$$

which, by (ii), is empty or contractible. Hence, $\bigcap_{x \in U} A(x) = \prod_{i \in I} \bigcap_{x \in U} R_i(x)$ as a product of empty or contractible spaces is also empty or contractible.

Finally, the reader may verify easily that $x \in A^{-1}(y) \Longleftrightarrow x \in \bigcap_{i \in I(x)} A_i^{-1}(y_i)$ which is a finite intersection of open sets and hence open. Theorem 1 applied to the multifunction A completes the proof.

Step 2. We modify an argument introduced (in the convex case) in [1] in the case where I is arbitrary (possibly uncountable). As in Step 1, for every $i \in I$, define a multifunction $B_i : X \longrightarrow \mathcal{P}(X)$ by

$$y \in B_i(x) \Longleftrightarrow y_i \in A_i(x),\ x \in X.$$

For a given $x \in X$, let $I(x) = \{i \in I : A_i(x) \neq \emptyset\}$, and define a multifunction $A : X \longrightarrow \mathcal{P}(X)$ as before by putting for each $x \in X$,

$$A(x) = \begin{cases} \bigcap_{i \in I(x)} B_i(x) & \text{if } I(x) \neq \emptyset, \\ \emptyset & \textit{otherwise.} \end{cases}$$

The difference with the finite case above is that here, the multifunction A does not necessarily satisfy the hypotheses of Theorem 1. We shall show that A is a multiselection of a fixed point-free multifunction B that does satisfy those hypotheses. Once this is done, it is clear that a maximal element $x_* \in X$ for B will be a maximal element for A and of course an equilibrium for the game $\mathcal{G}$.

Assume that $\forall\, x \in X$, $I(x) \neq \emptyset$ (the conclusion of the theorem being otherwise immediate). In view of hypothesis (iii), for any given $x \in X$, choose $i_x \in I(x)$ so that $x_{i_x} \notin A_{i_x(x)}$ and choose $y_{i_x} \in A_{i_x}(x)$. The set $U_x = A_{i_x}^{-1}(y_{i_x})$ is an open neighborhood of x in X. By compactness, there exists a finite subset $\{x_1, ..., x_n\}$ of X such that $X = \bigcup_{k=1}^{n} U_{x_k}$. Let $K(x) = \{k \in \{1, ..., n\} : x \in U_{x_k}\}$. Define $B : X \longrightarrow \mathcal{P}(X)$ by

$$B(x) = \bigcap_{k \in K(x)} B_{i_{x_k}}(x), x \in X.$$

For any given $x \in X$, and any $k \in K(x)$, $x \in U_{x_k} = A_{i_{x_k}}^{-1}(y_{i_{x_k}}) \Longleftrightarrow y_{i_{x_k}} \in A_{i_{x_k}}(x)$, which implies that $i_{x_k} \in I(x)$; thus, $A(x) = \bigcap_{i \in I(x)} B_i(x) \subseteq B_{i_{x_k}}(x)$. Therefore A is a multiselection of B.

Assume that B has a fixed point $x \in B(x)$, that is, $x \in B_{i_{x_k}}(x)$ for all $k \in K(x)$. This is equivalent to $x_{i_{x_k}} \in A_{i_{x_k}}(x)$ for all $k \in K(x)$. Clearly, this contradicts the choice of the indices i_{x_k}. Therefore B is fixed point free.

One readily verifies that for any given $y \in X$, $B^{-1}(y) = \bigcap_{k \in K(x)} A^{-1}_{i_{x_k}}(y_{i_{x_k}})$ is open by (i).

It remains to show that given any open subset U of X, the set $\bigcap_{x \in U} B(x)$ is empty or contractible. To do this, observe that, if $J(x) = \{i_{x_k} : k \in K(x)\}$,

$$\begin{aligned}
\bigcap_{x \in U} B(x) &= \bigcap_{x \in U} \bigcap_{k \in K(x)} B_{i_{x_k}}(x) = \bigcap_{x \in U} [\prod_{i \in J(x)} A_i(x) \times \prod_{i \notin J(x)} X_i], \\
&= \bigcap_{x \in U} \prod_{i \in I} R_i(x) = \prod_{i \in I} \bigcap_{x \in U} R_i(x),
\end{aligned}$$

where

$$R_i(x) = \begin{cases} A_i(x) & \text{if } i \in J(x), \\ X_i & \textit{otherwise.} \end{cases}$$

Let $i \in I$ be fixed. Divide U into two disjoint subsets $U_i = \{x \in U : i \in J(x)\}$ and $U_i^c = U \setminus U_i$. Clearly, U_i is open in U, and

$$\bigcap_{x \in U} R_i(x) = \bigcap_{x \in U_i} A_i(x) \cap \bigcap_{x \in U_i^c} X_i = \bigcap_{x \in U_i} A_i(x)$$

is empty or contractible by (ii). Hence, $\bigcap_{x \in U} B(x) = \prod_{i \in I} \bigcap_{x \in U} R_i(x)$ as a product of empty or contractible spaces is also empty or contractible. Theorem 1 applied to the multifunction B completes the proof. □

3. Concluding Remarks

If the strategy sets are compact convex subsets of topological vector spaces and the preference multifunctions are convex-valued, we recover classical results of [1], [3], [6], and others.

Let us mention that in a standard and straightforward way (see for instance Yannelis-Prabhakar [7], Toussaint [6], Ionescu Tulcea [5], and Ding-Kim-Tan [2] for the convex case), our result extends to generalized games where constraint multifunctions enter into consideration.

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THE MEASURABLE SOLUTIONS OF A FUNCTIONAL EQUATION OF C. ALSINA AND J. L. GARCIA-ROIG[1]

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Presented by J. Aczel, F.R.S.C.

ABSTRACT. For all fixed $0 < p < 1$ the nowhere zero solutions of the functional equation

$$f(x)f(px+\bar{p}y)+f(y)f(\bar{p}x+py)=f(px+\bar{p}y)^2+f(\bar{p}x+py)^2 \qquad (\bar{p}=1-p)$$

are determined under measurability conditions.

Motivated by some classical results of de Saint-Vincent related to the "duplication of the cube" C. Alsina and J. L. Garcia-Roig studied the functional equation

$$(AG) \qquad f(x)f(px+\bar{p}y)+f(y)f(\bar{p}x+py)=f(px+\bar{p}y)^2+f(\bar{p}x+py)^2$$

for all $x, y \in \mathbb{R}$ where $0 < p < 1$ is fixed, $\bar{p} = 1-p$, and found its continuous solutions $f : \mathbb{R} \to]0,\infty[$ in the case $p = 1/3$ (see [1], p. 265). C. Alsina posed the problem to find the continuous solutions $f : \mathbb{R} \to]0,\infty[$ of (AG) for $0 < p < 1$, $p \neq 1/3$ (see [1], p. 302). This problem was solved by the authors and the result was announced without detailed proof during the 31. ISFE in Debrecen 1993 ([1], p. 302). In this note we find the nowhere zero solutions of (AG) for all fixed $0 < p < 1$ under measurability conditions.

Theorem. *Let $0 < p < 1$ be fixed, $\bar{p} = 1-p$, $I \subset \mathbb{R}$ be an open interval of positive length and $f : I \to \mathbb{R} \setminus \{0\}$ be measurable on a subset of I with positive Lebesgue measure. Then the functional equation (AG) has the following solutions and only these:*

$$(S) \qquad \begin{array}{lll} \textit{for all } p \in]0,1[& & f(x) = c_1 \quad (x \in I) \\ \textit{for } p = \dfrac{1}{3} & \textit{also} & f(x) = c_1 e^{c_2 x} \quad (x \in I) \\ \textit{for } p = \dfrac{1}{2} & \textit{also} & f(x) = c_1(x + c_3) \quad (x \in I) \end{array}$$

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where c_1, c_2, c_3 are arbitrary constants, $c_1 \neq 0$, $-c_3 \notin I$.

So, under the above conditions, (AG) has nonconstant solutions only for $p = 1/2$ and $p = 1/3$.

Proof. We prove first that every solution, which is measurable on a set A of positive Lebesgue measure, is infinitely many times differentiable. We rewrite the equation in the form

$$\text{(RE)} \qquad f(x) = f(px + \bar{p}y) + \frac{f(\bar{p}x + py)^2}{f(px + \bar{p}y)} - f(y)\frac{f(\bar{p}x + py)}{f(px + \bar{p}y)}.$$

Let us choose a q for which $1 > q > 1 - 1/(1/p + 1/\bar{p})$, i.e. $0 < (1-q)(1/p + 1/\bar{p}) < 1$. Using Lebesgue's density theorem we can find a point $c \in I$ and an $r > 0$ such that for $C = [c - r, c + r]$ we have $C \subset I$ and $\lambda(A \cap C) > q\lambda(C)$ where λ denotes the Lebesgue measure on $\mathbb{R}$. Let $g_{1,x}(y) = px + \bar{p}y$ and $g_{2,x}(y) = \bar{p}x + py$ whenever $x, y \in I$. We want to apply Theorem 3.3 from [2] to prove that f is continuous on a neighbourhood of c. The only nontrivial condition to be checked is that the Lebesgue measure of the set $g_{1,c}^{-1}(A) \cap g_{2,c}^{-1}(A)$ is positive. The mappings $g_{1,c}^{-1}$ and $g_{2,c}^{-1}$ are central enlargements with center c and factors $1/\bar{p}$ and $1/p$, respectively. Hence $C \setminus g_{i,c}^{-1}(A)$ is contained in $g_{i,c}^{-1}(C \setminus A)$ for $i = 1, 2$ and has Lebesgue measure less than $\lambda(C)(1-q)/p$ and $\lambda(C)(1-q)/\bar{p}$, respectively. This shows that the Lebesque measure of $C \cap g_{1,c}^{-1}(A) \cap g_{2,c}^{-1}(A)$ is at least $\lambda(C)(1 - (1-q)/p - (1-q)/\bar{p}) > 0$.

Let $c \in I$ and, as above, for an $r > 0$ let C denote the closed interval $[c-r, c+r] \subset I$. Let $1 < Q \leq \min\{(1-\bar{p}/2)/p, (1-p/2)/\bar{p}\}$. If $|y-c| \leq r/2$ and $|x-c| < Qr$ then $|px+\bar{p}y-c| < pQr + r\bar{p}/2 = r(pQ + \bar{p}/2) \leq r$ and similarly $|\bar{p}x + py - c| < r$. Hence, using equation (RE) we have that, if f is continuous on C, then f is continuous on $]c - Qr, c + Qr[\cap I$. Taking an appropriate increasing sequence of intervals we obtain that f is continuous on I.

Now we prove that f is a locally Lipschitz function on I. Theorem 2 from the paper [3] can be applied with the compact set C above, and we get that f is a locally Lipschitz function on $]c - Qr, c + Qr[\cap I$. Taking an appropriate C if I is bounded or choosing an appropriate sequence of C's if I is not bounded, we get that f is a locally Lipschitz function on I. Applying Theorem 1.5 from [2], equation (RE) implies that f is infinitely many times differentiable.

So it is enough to find the C^∞ solutions of the functional equation (AG). Differentiating first with respect to x then with respect to y we get

$$\begin{aligned} &f'(x)f'(px + \bar{p}y)\bar{p} + f(x)f''(px + \bar{p}y)p\bar{p} \\ &\quad + f'(y)f'(\bar{p}x + py)\bar{p} + f(y)f''(\bar{p}x + py)p\bar{p} \\ &\quad = 2f'(px + \bar{p}y)^2 p\bar{p} + 2f(px + \bar{p}y)f''(px + \bar{p}y)p\bar{p} \\ &\quad + 2f'(\bar{p}x + py)^2 p\bar{p} + 2f(\bar{p}x + py)f''(\bar{p}x + py)p\bar{p} \end{aligned}$$

for all $x, y \in I$. With the substitution $y = x$ this equation implies that

$$\text{(1)} \qquad pf''(x)f(x) + (2p - 1)f'(x)^2 = 0$$

for all $x \in I$. Define the function g on I by

$$g = \frac{f'}{f}. \tag{2}$$

Then it follows from (1) that

$$pg'(x) + (3p-1)g(x)^2 = 0, \qquad x \in I. \tag{3}$$

If $g(x_0) = 0$ for some $x_0 \in I$ then, by (3), $g'(x_0) = 0$ too, and (3) has only one solutions satisfying these initial conditions. But the zero function is a solution of (3) on I satisfying the same conditions, so g is identically zero on I. Hence (2) implies (S). If $g(x) \neq 0$ for all $x \in I$ then define h on I by $h = 1/g$. It follows from (3) that $h'(x) = (3p-1)/p$ for all $x \in I$ therefore

$$h(x) = \frac{3p-1}{p}x + c \tag{4}$$

for some $c \in \mathbb{R}$ and for all $x \in I$. If $p = 1/3$ then by the definition of h we have that $c \neq 0$ and (2) implies that $f'(x) - (1/c)f(x) = 0$ for all $x \in I$. Thus we have (S). From now on we suppose $p \neq 1/3$. Then $-pc/(3p-1) \notin I$ and, again by the definition of h, equation (2) implies that

$$f'(x) - \frac{1}{\frac{3p-1}{p}x + c}f(x) = 0 \qquad \text{for all } x \in I.$$

Therefore, for some $d \in \mathbb{R} \setminus \{0\}$ we have

$$f(x) = d\left(\frac{3p-1}{p}x + c\right)^{p/(3p-1)} \qquad \text{for all } x \in I. \tag{5}$$

We denote $p/(3p-1)$ by α. Then the function f is a solution of (AG) if and only if the equation

$$\begin{aligned}\left(\frac{1}{\alpha}x + c\right)^{\alpha}\left(\frac{1}{\alpha}(px + \bar{p}y) + c\right)^{\alpha} + \left(\frac{1}{\alpha}y + c\right)^{\alpha}\left(\frac{1}{\alpha}(\bar{p}x + py) + c\right)^{\alpha}\\ = \left(\frac{1}{\alpha}(px + \bar{p}y) + c\right)^{2\alpha} + \left(\frac{1}{\alpha}(\bar{p}x + py) + c\right)^{2\alpha}\end{aligned}$$

is satisfied for all $x, y \in I$. With linear substitutions we get that this is true exatly when the equation

$$x^{\alpha}(px + \bar{p}y)^{\alpha} + y^{\alpha}(\bar{p}x + py)^{\alpha} = (px + \bar{p}y)^{2\alpha} + (\bar{p}x + py)^{2\alpha}$$

is satisfied for all $x, y \in \frac{1}{\alpha}I + c$. If this equation holds then the function

$$F(t) = (p + \bar{p}t)^{\alpha} + t^{\alpha}(\bar{p} + pt)^{\alpha} - (p + \bar{p}t)^{2\alpha} - (\bar{p} + pt)^{2\alpha}$$

is 0 in a neighbourhood of 1. We will show that this is possible only if $\alpha = 1$ and hence $p = 1/2$. Indeed, if F were 0 in a neighbourhood of 1 then we would have

$$F^{(n)}(1) = 0 \qquad \text{for all } n = 0, 1, \ldots.$$

So we calculate the derivatives of F at 1. It is clear that all these derivatives have the form $F^{(n)}(1) = P_n(p)(3p-1)^{-n}$ for some polynomial P_n of p. If (6) were true we would have $P_n = 0$ for $n = 0, 1, \ldots$. Using the computer algebra system Maple® for calculations we get that $P_0(p) = P_1(p) = P_2(p) = P_3(p) = 0$ for all p but

$$P_4(p) = -48p^8 + 124p^7 - 90p^6 - 12p^5 + 40p^4 - 16p^3 + 2p^2$$

has the roots

$$0, 0, \frac{1}{3}, \frac{1}{2}, -\frac{1}{8} + \frac{1}{8}\sqrt{17}, -\frac{1}{8} - \frac{1}{8}\sqrt{17}, 1, 1.$$

Since $p \neq 1/3$, $0 < p < 1$ were supposed, the only remaining possibilities are $p = 1/2$ and $p = -1/8 + \sqrt{17}/8$. The latter value is a root of P_5, but not of

$$\begin{aligned} P_6(p) = &-10080p^{12} + 45192p^{11} - 109044p^{10} + 165872p^9 - 141426p^8 + 43526p^7 \\ &+ 26452p^6 - 30532p^5 + 12166p^4 - 2298p^3 + 172p^2, \end{aligned}$$

because of the equality

$$P_6\left(-\frac{1}{8} + \frac{1}{8}\sqrt{17}\right) = -\frac{718978689}{524288} + \frac{174377945}{524288}\sqrt{17}.$$

Hence we have $p = 1/2$ and thus $\alpha = 1$. In this case the function f in (5) clearly satisfies (AG), and this concludes the proof of the Theorem.

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SERIES SOLUTION FOR IMPEDANCE CHANGE INDUCED BY A DOUBLE-LAYERED SPHERE WITH VARYING PROPERTIES

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ABSTRACT. This note presents an analytical solution for the change in impedance in a coil due to the presence of a conducting double-layered sphere symmetrically situated with respect to the coil. In the case the relative magnetic permeability, $\mu(r) = r^\alpha$, of the outer sphere is a function of the distance r from the centre of the sphere, where α is an arbitrary real number, the solution is expressed in the form of a series containing Bessel functions and can be used to compute the change of impedance. Numerical results for the case $\alpha = -2$ are presented.

RÉSUMÉ. On présente une solution analytique du changement d'impédance d'une bobine induit par une sphère à deux couches en position symétrique par rapport à la bobine. La perméabilité magnétique relative de la couche extrérieure $\mu(r) = r^\alpha$ est fonction de la distance radiale du centre, où α est une constante réelle. On exprime la solution sous forme d'une série qui contient des fonctions de Bessel. On présente des résultats numériques pour le cas où $\alpha = -2$.

1. Introduction. Eddy current testing is widely used to control the quality of materials of spherical shape [1]. Analytical formulae for the change of impedance, Z, in a single-turn coil situated above a conducting spherical body are presented in [2]. In these solutions, the parameters of the medium are assumed to be constant. If the conductivity and/or magnetic permeability of the sphere vary with the spatial coordinates, as it often happens in practice, no analytical solutions are known to the authors. In the general case, the medium is usually divided into a large number of thin layers where it is assumed that all the parameters of the medium are constant within a given layer, and the technique for a multilayered sphere is used. The number of layers, which may be large, depends on the properties of the medium.

This note presents a solution [3] for Z, in the form of an infinite series, to a problem of a double-layered sphere in the case the relative permeability, $\mu(r) = r^\alpha$, of the medium is a function of the radial coordinate of the system, where α is an arbitrary real number. Numerical results are obtained for the case $\alpha = -2$.

2. Formulation of the problem. Consider a coil of radius ρ_c situated horizontally above two concentric spheres of radii ρ_2 and ρ_1, respectively, where $\rho_2 < \rho_1$, and the axis of the coil passes through the centre of the spheres (see Fig. 1(a)).

Let the density of the external current in the coil be described, in the spherical coordinates (r, θ, φ), with centre at 0, by the formula

$$I = I_\varphi e^{j\omega t} e_\varphi,$$

Key words and phrases. Eddy current NDE, medium with varying magnetic properties, Bessel functions.

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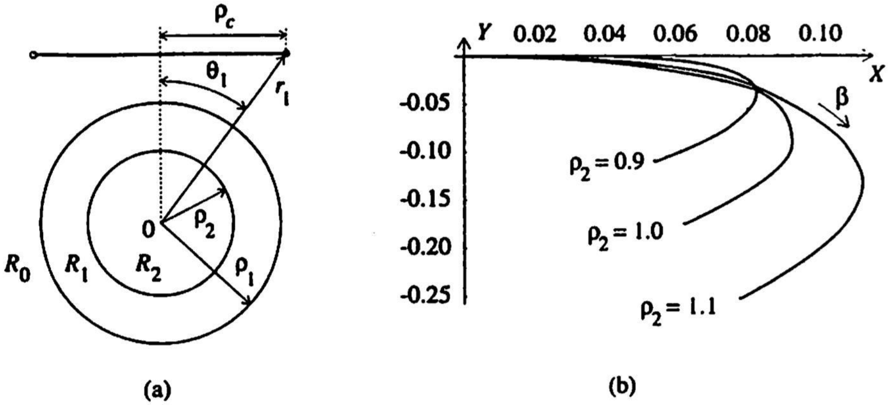

FIGURE 1. (a) Single-turn coil above double-layered sphere. (b) The change in impedance, Z_0, as a function of β, for $\rho_2 = 1.1, 1.0, 0.9$.

where $\boldsymbol{e}_\varphi$ is a unit vector in the azimuthal φ-direction.

We let R_0, R_1 and R_2 denote empty space: $r > \rho_1$, the outer spherical shell: $\rho_2 < r < \rho_1$, and the inner ball: $0 \le r < \rho_2$, respectively.

The outer spherical shell is a conducting medium with constant conductivity σ_1 and variable relative magnetic permeability, $\mu_1(r)$, and the inner ball is a conducting medium with constant conductivity σ_2 and constant relative magnetic permeability μ_2.

The electromagnetic field in a region with variable relative magnetic permeability, $\mu_1(r)$, is described by the following subset of the full set of Maxwell's equations:

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \operatorname{curl} \boldsymbol{H} = \boldsymbol{I} + \boldsymbol{I}^e, \quad \operatorname{div} \boldsymbol{E} = 0, \quad \boldsymbol{I} = \sigma \boldsymbol{E}, \quad \boldsymbol{B} = \mu_0 \mu_1(r) \boldsymbol{H}, \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field strengths, respectively, $\boldsymbol{B}$ is the magnetic induction vector, $\boldsymbol{I}$ is the current density, $\boldsymbol{I}^e$ is the external current density, σ is the conductivity, μ_0 and μ_1 are, respectively, the magnetic constant and the relative magnetic permeability of the medium. In (1) the displacement current is neglected, as it is usual in problems of eddy current testing.

Introducing the vector potential $\boldsymbol{A}$ and the scalar potential ψ, by the relations

$$\operatorname{curl} \boldsymbol{A} = \boldsymbol{B}, \qquad \boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\operatorname{grad} \psi, \tag{2}$$

and using (1)–(2), we obtain

$$\operatorname{curl}\left[\frac{1}{\mu_0 \mu_1(r)} \operatorname{curl} \boldsymbol{A}\right] = -\sigma \operatorname{grad} \psi - \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{I}^e. \tag{3}$$

Because of axial symmetry, the vector potential $\boldsymbol{A}$ has only one nonzero component, in the azimuthal direction,

$$\boldsymbol{A} = (0, A(r, \theta), 0)\, e^{j\omega t}. \tag{4}$$

By means of basic operations of vector analysis, (3) and (4) become

$$\frac{1}{\mu_0\mu_1(r)}\operatorname{grad}\operatorname{div} \boldsymbol{A} - \frac{1}{\mu_0\mu_1(r)}\Delta \boldsymbol{A} + \frac{1}{\mu_0\mu_1^2(r)}\frac{d\mu_1}{dr}\left(\frac{A}{r}+\frac{\partial A}{\partial r}\right)\boldsymbol{e}_\varphi = -\sigma \operatorname{grad}\psi - \sigma\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{I}^e. \quad (5)$$

Since $\operatorname{div}\boldsymbol{A} = 0$, it follows from (1) and (2) that $\psi = 0$.

Then, using (4) and taking the φ-component of (5), we obtain the scalar equation

$$\frac{\partial^2 A}{\partial r^2} + \frac{2}{r}\frac{\partial A}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial A}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 A}{\partial\theta^2} - \frac{A}{r^2\sin^2\theta} - \frac{1}{\mu_1(r)}\frac{d\mu_1}{dr}\left(\frac{A}{r}+\frac{\partial A}{\partial r}\right) - j\omega\sigma\mu_0\mu_1(r)A = -\mu_0\mu_1(r)I_\varphi, \quad (6)$$

where I_φ is the φ-component of the vector $\boldsymbol{I}^e$.

3. Mathematical analysis. Since an analytical solution to the general equation (6) cannot be found in general, we suppose that $\mu_1(r)$ is of the form

$$\mu(r) = \left(\frac{r}{\rho_c}\right)^\alpha, \qquad \alpha\in\mathbf{R}. \quad (7)$$

Substituting (7) into (6) and using the dimensionless radial coordinate $r_d = r/\rho_c$ we obtain the following system of equations in each of the regions R_0, R_1 and R_2 (the subscript d to r_d is omitted and all the geometric quantities are measured in units of ρ_c):

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{2}{r}\frac{\partial A_0}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial A_0}{\partial\theta} + \frac{1}{r^2}\frac{\partial A_0}{\partial\theta^2} - \frac{A_0}{r^2\sin^2\theta} = -\mu_0 I_\varphi, \quad (8)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{2-\alpha}{r}\frac{\partial A_1}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial A_1}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 A_1}{\partial\theta^2} - \frac{A_1}{r^2\sin^2\theta} - \left(j\beta_1^2 r^\alpha + \frac{\alpha}{r^2}\right)A_1 = 0, \quad (9)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{2}{r}\frac{\partial A_2}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial A_2}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 A_2}{\partial\theta^2} - \frac{A_2}{r^2\sin^2\theta} - j\beta_2^2 A_2 = 0, \quad (10)$$

where $\beta_1 = \rho_c\sqrt{\omega\sigma_1\mu_0}$ and $\beta_2 = \rho_c\sqrt{\omega\sigma_2\mu_0\mu_2}$, and $A_i(r,\theta)$ denotes the vector potential in the region R_i, for $i = 0,1,2$.

The boundary conditions have the form

$$A_0\Big|_{r=\rho_1} = A_1\Big|_{r=\rho_1}, \qquad \frac{\partial A_0}{\partial r}\Big|_{r=\rho_1} = \frac{1}{\mu_{11}}\frac{\partial A_1}{\partial r}\Big|_{r=\rho_1},$$
$$A_1\Big|_{r=\rho_2} = A_2\Big|_{r=\rho_2}, \qquad \frac{1}{\mu_{12}}\frac{\partial A_1}{\partial r}\Big|_{r=\rho_2} = \frac{1}{\mu_2}\frac{\partial A_2}{\partial r}\Big|_{r=\rho_2}, \quad (11)$$

where $\mu_{11} = \mu_1(\rho_1)$ and $\mu_{12} = \mu_1(\rho_2)$. The current, I_φ, in the coil is represented in the form

$$I_\varphi = I\delta(r-r_1)\delta(\theta-\theta_1), \quad (12)$$

where I is the amplitude of the current and $\delta(x)$ is the Dirac delta function.

Introducing the variable $\xi = \cos\theta$, we express the solution of problem (8)–(12) by the following integral transform:

$$\tilde{A}_i(r,n) = \frac{1}{D_n}\int_{-1}^{1}\hat{A}_i(r,\xi)P_n^{(1)}(\xi)\,d\xi, \qquad i = 0,1,2, \quad (13)$$

where $\widehat{A}_i(r,\xi) = A_i(r,\theta)$, $P_n^{(1)}(\xi)$ is an associated Legendre function of the first kind and

$$D_n = \int_{-1}^{1} \left[P_n^{(1)}(\xi)\right]^2 d\xi = \frac{2n(n+1)}{2n+1}.$$

By (13) problem (8)–(12) reduces to

$$\frac{d^2\tilde{A}_0}{dr^2} + \frac{2}{r}\frac{d\tilde{A}_0}{dr} - \frac{n(n+1)}{r^2}\tilde{A}_0 = -\mu_0 I\frac{2n+1}{2n(n+1)}P_n^{(1)}(\cos\theta_1)\sin\theta_1\delta(r-r_1), \tag{14}$$

$$\frac{d^2\tilde{A}_1}{dr^2} + \frac{2-\alpha}{r}\frac{d\tilde{A}_1}{dr} - \frac{n(n+1)}{r^2}\tilde{A}_1 - \left(j\beta_1^2 r^\alpha + \frac{\alpha}{r^2}\right)\tilde{A}_1 = 0, \tag{15}$$

$$\frac{d^2\tilde{A}_2}{dr^2} + \frac{2}{r}\frac{d\tilde{A}_2}{dr} - \frac{n(n+1)}{r^2}\tilde{A}_2 - j\beta_2^2\tilde{A}_2 = 0, \tag{16}$$

with the boundary conditions

$$\begin{aligned} \tilde{A}_0\Big|_{r=\rho_1} &= \tilde{A}_1\Big|_{r=\rho_1}, & \frac{d\tilde{A}_0}{dr}\Big|_{r=\rho_1} &= \frac{1}{\mu_{11}}\frac{d\tilde{A}_1}{dr}\Big|_{r=\rho_1}, \\ \tilde{A}_1\Big|_{r=\rho_2} &= \tilde{A}_2\Big|_{r=\rho_2}, & \frac{1}{\mu_{12}}\frac{d\tilde{A}_1}{dr}\Big|_{r=\rho_2} &= \frac{1}{\mu_2}\frac{d\tilde{A}_2}{dr}\Big|_{r=\rho_2}. \end{aligned} \tag{17}$$

It is convenient to consider the solution of equation (14) in the two subregions R_{01} and R_{02} of R_0, corresponding to the intervals $\rho_1 < r < r_1$ and $r > r_1$, respectively. We denote the solution in R_{01} and R_{02} by $\tilde{A}_{01}$ and $\tilde{A}_{02}$, respectively. The bounded general solutions of (14) in R_{01} and R_{02} are

$$\tilde{A}_{01}(r,n) = C_1 r^n + C_2 r^{-n-1}, \qquad \tilde{A}_{02}(r,n) = C_3 r^{-n-1}, \tag{18}$$

respectively. Since the vector potential is continuous at $r = r_1$, then

$$\tilde{A}_{01}\Big|_{r=r_1} = \tilde{A}_{02}\Big|_{r=r_1}. \tag{19}$$

Multiplying equation (14) by r^2, integrating from $r = r_1 - \varepsilon$ to $r = r_1 + \varepsilon$, and considering the limit as $\varepsilon \to +0$, we obtain

$$\frac{d\tilde{A}_{02}}{dr}\Big|_{r=r_1} - \frac{d\tilde{A}_{01}}{dr}\Big|_{r=r_1} = -\mu_0 I\frac{2n+1}{2n(n+1)}P_n^{(1)}(\cos\theta_1)\sin\theta_1. \tag{20}$$

The general solution to equation (15), in terms of Bessel functions, is

$$\tilde{A}_1(r,n) = C_4 r^a J_p(\beta r^\gamma) + C_5 r^a Y_p(\beta r^\gamma), \tag{21}$$

where

$$\gamma = \frac{\alpha+2}{2}, \qquad a = \frac{\alpha-1}{2}, \qquad \beta = \frac{\beta_1\sqrt{-j}}{\gamma}, \qquad p = \frac{\sqrt{(\alpha+1)^2 + 4n(n+1)}}{\alpha+2}.$$

The solution to equation (16) which remains bounded as $r \to 0$, has the form

$$\tilde{A}_2(r,n) = \frac{C_6}{\sqrt{r}}J_{n+1/2}(k_2 r), \qquad k_2 = \beta_2\sqrt{-j}. \tag{22}$$

Using (18), (21) and (22), determining the constants C_1 to C_6 from conditions (17), (19) and (20), and inverting the integral transform (13), we obtain the series

$$A_i(r,\theta) = \sum_{n=1}^{\infty} \tilde{A}_i(r,n) P_n^{(1)}(\cos\theta), \qquad i = 0,1,2. \tag{23}$$

Formula (23) gives the complete solution to problem (8)–(12).

If we set $A_0(r,\theta) = A_0^{\text{ind}}(r,\theta)e_\varphi$, where $A_0^{\text{ind}}(r,\theta)$ is the induced part of the vector potential, then the change of impedance in the coil caused by a two-layered conducting sphere is given by the integral

$$Z = \frac{j\omega}{I}\oint_L A_0(r,\theta)\cdot d\boldsymbol{l}, \tag{24}$$

where L is the coil contour. More precisely, $A_0^{\text{ind}}(r,\theta)$ is described by formula (23) where $\tilde{A}_0(r,n)$ is replaced by $\tilde{A}_0^{\text{ind}}(r,n) = C_2 r^{-n-1}$.

Substituting (23) into (24), we obtain the change in impedance in the form

$$Z = j\pi\omega\mu_0\rho_1^2 \sin^2\theta_1 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{\rho_1}{r_1}\right)^{2n-1} [P_n^{(1)}(\cos\theta_1)]^2 \frac{F_1}{F_2}, \tag{25}$$

where

$$\begin{aligned}
F_1 &= [dJ_p(\beta\rho_1^\gamma) + Y_p(\beta\rho_1^\gamma)](\mu_{11}n - a) - \beta\gamma\rho_1^\gamma[dJ_p'(\beta\rho_1^\gamma) + Y_p'(\beta\rho_1^\gamma)], \\
F_2 &= [dJ_p(\beta\rho_1^\gamma) + Y_p(\beta\rho_1^\gamma)][a + \mu_{11}(n+1)] + \beta\gamma\rho_1^\gamma[dJ_p'(\beta\rho_1^\gamma) + Y_p'(\beta\rho_1^\gamma)],
\end{aligned}$$

$d = d_{11}/d_{12}$, with

$$\begin{aligned}
d_{11} &= -\rho_2^a\mu_{12}[-J_{n+1/2}(k_2\rho_2) + 2k_2\rho_2 J_{n+1/2}'(k_2\rho_2)]Y_p(\beta\rho_2^\gamma) \\
&\quad + 2\rho_2\mu_2 J_{n+1/2}(k_2\rho_2)[a\rho_2^{a-1}Y_p(\beta\rho_2^\gamma) + \beta\gamma\rho_2^{a+\gamma-1}Y_p'(\beta\rho_2^\gamma)], \\
d_{12} &= \rho_2^a\mu_{12}[-J_{n+1/2}(k_2\rho_2) + 2k_2\rho_2 J_{n+1/2}'(k_2\rho_2)]J_p(\beta\rho_2^\gamma) \\
&\quad - 2\rho_2\mu_2 J_{n+1/2}(k_2\rho_2)[a\rho_2^{a-1}J_p(\beta\rho_2^\gamma) + \beta\gamma\rho_2^{a+\gamma-1}J_p'(\beta\rho_2^\gamma)];
\end{aligned}$$

here $'$ denotes ordinary derivative of a function of one variable.

For $\alpha = -2$, equation (15) becomes Euler's equation, whose general solution is

$$\tilde{A}_1(r,n) = C_4 r^{\gamma_1} + C_5 r^{\gamma_2}, \qquad \gamma_{1,2} = \frac{-3 \pm \sqrt{1 + 4n(n+1) + 4j\beta_1^2}}{2}. \tag{26}$$

In this case, the change in impedance is

$$Z = \omega\pi\mu_0\rho_1^2\sin^2\theta_1 Z_0,$$

where

$$Z_0 = j\sum_{n=1}^{\infty}\left(\frac{\rho_1}{r_1}\right)^{2n-1} \frac{P_n^{(1)}(\cos\theta_1)]^2}{n(n+1)} \frac{n\mu_{11}(g\rho_1^{\gamma_1} + \rho_1^{\gamma_2}) - g\gamma_1\rho_1^{\gamma_1} - \gamma_2\rho_1^{\gamma_2}}{g\gamma_1\rho_1^{\gamma_1} + \gamma_2\rho_1^{\gamma_2} + (n+1)\mu_{11}(g\rho_1^{\gamma_1} + \rho_1^{\gamma_2})}, \tag{27}$$

and $g = -g_{11}/g_{12}$, with

$$g_{11} = \mu_{12}\rho_2^{\gamma_2}[-J_{n+1/2}(k_2\rho_2) + 2k_2\rho_2 J'_{n+1/2}(k_2\rho_2)] - 2\mu_2\gamma_2\rho_2^{\gamma_2} J_{n+1/2}(k_2\rho_2),$$

$$g_{12} = \mu_{12}\rho_2^{\gamma_1}[-J_{n+1/2}(k_2\rho_2) + 2k_2\rho_2 J'_{n+1/2}(k_2\rho_2)] - 2\mu_2\gamma_1\rho_2^{\gamma_1} J_{n+1/2}(k_2\rho_2).$$

4. Numerical results. Formula (27) was used to compute the change of impedance, $Z_0 = X + jY$, in the case the relative magnetic permeability of media R_1 and R_2 are $\mu_1(r) = r^{-2}$ and a constant μ_2, respectively. The computational results presented in Fig. 1(b) show Z_0 as a function of β for three values, $\rho_2 = 0.9, 1.0, 1.1$, of the radius of the inner sphere with variable magnetic permeability. Increasing values of β are indicated by the arrow. The remaining parameters are set at $r_1 = 1.5$, $\rho_1 = 1.3$, $\mu_2 = 1$, $\mu_{11} = 1/\rho_1^2$, $\mu_{12} = 1/\rho_2^2$ and $\beta_2 = 1$.

It is seen that a small change in the radius of the internal sphere has a large influence on Z_0, a fact which is used for controlling the thickness of coverings with variable magnetic permeability μ.

Note that the present technique applies to more general problems. For example, one can take a multilayered sphere where the magnetic permeability of the ith layer is given by $\mu_i(r) = r^{\alpha_i}$, and α_i are distinct constants. Moreover, an analytical solution can also be found in the case where the conductivity of the ith layer, $\sigma_i(r)$, has the form $\sigma_i(r) = r^{\delta_i}$, where δ_i are distinct constants. Finally, one can find an analytical solution for a multilayered sphere, where both $\mu_i(r)$ and $\sigma_i(r)$ are given as above, with possibly $\alpha_i \neq \alpha_j$ and $\delta_i \neq \delta_j$ for $i \neq j$.

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RANGES AND INVERSION FORMULAS FOR SPHERICAL MEAN OPERATOR AND ITS DUAL

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Abstract. *We consider the spherical mean operator* R *and its dual* ${}^tR$. *We establish some Harmonic Analysis results for the generalized Fourier transform associated with the operator* R. *We deduce that the operators* R *and* ${}^tR$ *are transmutation operators. Next, we give inversion formulas for these operators and a Plancherel theorem for the operator* ${}^tR$.

I. Spherical mean operator and its dual on $\mathbb{R}^{n+1}$.

Notations. We denote by

- $\mathcal{E}_*(\mathbb{R}^{n+1})$ the space of C^∞-functions on $\mathbb{R}^{n+1}$, even with respect to the first variable.
- $\mathcal{S}_*(\mathbb{R}^{n+1})$ the space of C^∞-functions, even with respect to the first variable and rapidly decreasing together with all their derivatives.
- $\mathcal{D}_*(\mathbb{R}^{n+1})$ the space of C^∞-functions, even with respect to the first variable and with compact support.

Definition I.1. The spherical mean operator is defined on $\mathcal{E}_*(\mathbb{R}^{n+1})$ by

$$R(f)(r,x) = \int_{S^n} f(r\eta, x+r\xi)\, d\sigma_n(\eta,\xi)\ ;\ (r,x) \in [0,\infty[\times \mathbb{R}^n,$$

where S^n is the unit sphere : $\{(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n ; \eta^2 + \|\xi\|^2 = 1\}$ in $\mathbb{R}^{n+1}$ and σ_n is the surface measure on S^n normalized to have total measure one.

Definition I.2. The dual ${}^tR$ of the spherical mean operator R is defined on $\mathcal{D}_*(\mathbb{R}^{n+1})$ by

$${}^tR(f)(r,x) = \frac{\Gamma((n+1)/2)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f\left(\sqrt{r^2+\|x-y\|^2}, y\right)dy.$$

Remark. For all $f \in \mathcal{E}_*(\mathbb{R}^{n+1})$ and $g \in \mathcal{D}_*(\mathbb{R}^{n+1})$, we have

$$\int_{]0,\infty[\times\mathbb{R}^n} f(r,x)R(g)(r,x)r^n\,dr\,dx = \int_{]0,\infty[\times\mathbb{R}^n} {}^tR(f)(r,x)g(r,x)\,dr\,dx.$$

II. Generalized Fourier transform associated with the operator R.

Notations. We denote by

- $\Gamma_{cone} = \Gamma^o_{cone} \cup \mathbb{R}^{n+1}$, where $\Gamma^o_{cone} = \{(it,x) ; t \in \mathbb{R},\ x \in \mathbb{R}^n \text{ and } |t| \le \|x\|\}$
- $L^2([0,\infty[\times\mathbb{R}^n, r^n\,dr\,dx)$ the space of square integrable functions on $[0,\infty[\times\mathbb{R}^n$ with respect to the measure $r^n\,dr\,dx$.
- $\mathcal{S}_*(\Gamma_{cone})$ the space of C^∞-function on Γ_{cone}, even with respect to the first variable and rapidly decreasing together with all their derivatives.
- $\mathbb{H}_{*,cone}(\mathbb{C}^{n+1})$ the space of entire functions on $\mathbb{C}^{n+1}$, even with respect to the frist variable, rapidly decreasing of exponential type and such that for all $k \in \mathbb{N}$ we have

$$\sup\{(1-\mu^2+2\|\lambda\|^2)^k|f(i\mu,\lambda)| ; (i\mu,\lambda) \in \Gamma^o_{cone}\} < +\infty$$

- $\mathcal{E}'_*(\mathbb{R}^{n+1})$ the space of distributions on $\mathbb{R}^{n+1}$, even with respect to the first variable and with compact support.

- $\mathcal{H}_{*,cone}(\mathbb{C}^{n+1})$ the space of entire functions on $\mathbb{C}^{n+1}$, even with respect to the first variable, slowly increasing of exponential type and such that there exists $k \in \mathbb{N}$:

$$\sup\{(1-\mu^2+2\|\lambda\|^2)^{-k}|f(i\mu,\lambda)| ; (i\mu,\lambda) \in \Gamma^{\upsilon}_{cone}\} < +\infty$$

- $L^2(\Gamma_{cone}, d\gamma)$ the space of square integrable functions on Γ_{cone} with respect to the measure $d\gamma$ defined by

$$\int_{\Gamma_{cone}} f(z,\lambda)d\gamma(z,\lambda) = \frac{1}{2^{2n-1}\pi^n[\Gamma(\frac{n+1}{2})]^2}\cdot\{\int_{]0,\infty[\times\mathbb{R}^n} f(t,x)(t^2+\|x\|^2)^{\frac{n-1}{2}} tdtdx +$$

$$\int_{\mathbb{R}^n}(\int_0^{\|x\|} f(it,x)(\|x\|^2-t^2)^{\frac{n-1}{2}} tdt)dx\}$$

We consider the function $\varphi_{\mu,\lambda}$; $(\mu,\lambda) \in \mathbb{C} \times \mathbb{C}^n$, defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$\varphi_{\mu,\lambda}(r,x) = R(e^{-i<\lambda,.>}\cos(\mu.))(r,x).$$

Proposition II.1.

i) The function $\varphi_{\mu,\lambda}$ is given by

$$\varphi_{\mu,\lambda}(r,x) = e^{-i<\lambda,x>} j_{\frac{n-1}{2}}(r\sqrt{\mu^2+\lambda_1^2+...+\lambda_n^2}).$$

where $j_{\frac{n-1}{2}}(s) = (\frac{s}{2})^{-(\frac{n-1}{2})}\Gamma(\frac{n+1}{2}) J_{\frac{n-1}{2}}(s)$, $J_{\frac{n-1}{2}}$ being the Bessel function of first kind and order $\frac{n-1}{2}$.

ii) The function $\varphi_{\mu,\lambda}$ is the unique C^∞-function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable and satisfying the following system

$$\begin{cases} D_j u(r,x) = -i\lambda_j u(r,x) \; ; \; 1 \le j \le n, \\ L_1 u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1 \; ; \; \frac{\partial u}{\partial r}(0,x) = 0, \text{ for all } x \in \mathbb{R}^n, \end{cases}$$

where $D_j = \frac{\partial}{\partial x_j}$; $1 \le j \le n$ and $L_1 = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} - \sum_{j=1}^{n} D_j^2$.

Definition II.1. The generalized Fourier transform $\mathcal{F}$ associated with the operator R is defined on $\mathcal{D}_*(\mathbb{R}^{n+1})$ by

$$\mathcal{F}(f)(\mu,\lambda) = \int_{]0,\infty[\times\mathbb{R}^n} f(r,x)\varphi_{\mu,\lambda}(r,x)r^n drdx \; ; (\mu,\lambda) \in \mathbb{C} \times \mathbb{C}^n.$$

and on $\mathcal{E}'_*(\mathbb{R}^{n+1})$ by

$$\mathcal{F}(T)(\mu,\lambda) = <T,\varphi_{\mu,\lambda}> \; ; (\mu,\lambda) \in \mathbb{C} \times \mathbb{C}^n.$$

Theorem II.1. (Plancherel theorem)

i) The generalized Fourier transform $\mathcal{F}$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^{n+1})$ onto $\mathcal{S}_*(\Gamma_{cone})$. The isomorphism inverse is given by

$$\mathcal{F}^{-1}(g)(r,x) = \int_{\Gamma_{cone}} g(z,\lambda)\bar{\varphi}_{z,\lambda}(r,x)\, d\gamma(z,\lambda) \;;\; g \in \mathcal{S}_*(\Gamma_{cone}).$$

ii) For every $f \in \mathcal{S}_*(\mathbb{R}^{n+1})$ we have the Plancherel formula

$$\int_{]0,\infty[\times\mathbb{R}^n} |f(r,x)|^2 r^n drdx = \int_{\Gamma_{cone}} |\mathcal{F}(f)(z,\lambda)|^2 d\gamma(z,\lambda).$$

iii) The generalized Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(]0,\infty[\times \mathbb{R}^n, r^n drdx)$ onto $L^2(\Gamma_{cone}, d\gamma)$.

Theorem II.2. (Paley-Wiener theorem) The generalized Fourier transform $\mathcal{F}$ is bijective

i) from $\mathcal{D}_*(\mathbb{R}^{n+1})$ onto $\mathbb{H}_{*,cone}(\mathbb{C}^{n+1})$.

ii) from $\mathcal{E}'_*(\mathbb{R}^{n+1})$ onto $\mathcal{H}_{*,cone}(\mathbb{C}^{n+1})$.

III. Transmutation operators.

Notations. We denote by

- $\mathcal{S}_{*,o}(\mathbb{R}^{n+1})$ the subspace of $\mathcal{S}_*(\mathbb{R}^{n+1})$ consisting of functions f such that $\int_0^\infty P(r)f(r,x)dr = 0$, for all $x \in \mathbb{R}^n$ and all one variable polynomial P.

- $\mathcal{S}_{*,cone}(\mathbb{R}^{n+1})$ the subspace of $\mathcal{S}_*(\mathbb{R}^{n+1})$ consisting of functions f such that for all $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $|\alpha| < 1$, we have

$$\int_{\mathbb{R}^n} g(\|z\|, u-\alpha\, \tilde{z})\, dz = 0 \;;\; z = (z_1, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^n.$$

Theorem III.1.

i) The operator ${}^tR$ is linear continuous, injective and not surjective from $\mathcal{D}_*(\mathbb{R}^{n+1})$ into itself.

ii) The operator ${}^tR$ is linear continuous, surjective and not injective from $\mathcal{S}_*(\mathbb{R}^{n+1})$ onto itself.

Remark. The property ii) has been proved by L.E.Anderssan ([1], p 218).

Theorem III.2.

i) The operator R is bijective from $\mathcal{S}_{*,o}(\mathbb{R}^{n+1})$ onto $\mathcal{S}_{*,cone}(\mathbb{R}^{n+1})$, satisfying the permutation relations

$$L_1(R(f)) = R\left(\frac{\partial^2 f}{\partial r^2}\right) \quad ; \quad D_j(R(f)) = R(D_j f) \;;\; 1 \le j \le n.$$

ii) The operator ${}^tR$ is bijective from $\mathcal{S}_{*,cone}(\mathbb{R}^{n+1})$ onto $\mathcal{S}_{*,o}(\mathbb{R}^{n+1})$, satisfying the permutation relations

$${}^tR(L_1 f) = \frac{\partial^2}{\partial r^2}\, {}^tR(f) \quad ; \quad {}^tR(D_j f) = D_j({}^tR(f)) \;;\; 1 \le j \le n,$$

Remark. From the theorem III.2 we deduce that the operators R and ${}^tR$ are transmutation operators of L_1, D_j ; $1 \le j \le n$, into $\frac{\partial^2}{\partial r^2}, D_j$, $1 \le j \le n$.

IV. Inversion formulas for the operators R and ${}^tR$ and Plancherel theorem for the operator ${}^tR$.

Notations. We denote by

- $\mathcal{S}_*'(\mathbb{R}^{n+1})$ the space of tempered distributions on $\mathbb{R}^{n+1}$, even with respect to the first variable.

- $L^2_{cone}([0,\infty[\times \mathbb{R}^n)$ the space of square integrable functions f on $]0,\infty[\times \mathbb{R}^n$ with respect to the measure $r^n drdx$ and such that the function $\tilde{\mathcal{F}}(f)(r,x) = \int_{]0,\infty[\times\mathbb{R}^n} f(t,y)e^{-i<x,y>} j_{\frac{n-1}{2}}(rt)t^n dtdy$ is supported in $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n;\ |t| \geq \|x\|\}$.

a) Fractional power of the Laplacian operator Δ on $\mathbb{R}^{n+1}$.

For $\gamma \in \mathbb{C}_{n+1} = \{\gamma \in \mathbb{C}\ ;\ \gamma - (n+1) \notin 2\mathbb{N}\}$ and $f \in \mathcal{S}_*(\mathbb{R}^{n+1})$ we put

$$I^{\gamma}_{n+1}(f)(r,x) = \frac{\Gamma(\frac{n+1-\gamma}{2})}{2^{\gamma}\, \pi^{\frac{n+1}{2}}\, \Gamma(\frac{\gamma}{2})} \int_{\mathbb{R}^{n+1}} \frac{f(t,y)}{\|(t,y)-(r,x)\|^{n+1-\gamma}} dtdy.$$

When $\mathrm{Re}(\gamma) \leq 0$, this is interpreted as an analytic continuation (see [3], [4] p 67).

For any real p such that $-2p \notin \mathbb{C}_{n+1}$, the fractional power of the Laplacian $\Delta = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^{n+1}$, is defined by

$$(-\Delta)^p f(r,x) = I^{-2p}_{n+1}(f)(r,x).$$

b) Inversion formulas for the operators R and ${}^tR$ and Plancherel theorem for ${}^tR$.

Notations. We denote by

- $\mathcal{H}$ the Hilbert transform given by

$$\mathcal{H}(f)(t,x) = \frac{-i}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \mathrm{sgn}(r) f(s,x) e^{-i(t-s)r} ds\right) dr$$

- S and T the tempered distributions, even with respect to the first variable and defined by

$$<S,\varphi> = \int_{\mathbb{R}^n} \varphi(\|z\|,z)dz\ ;\ \varphi \in \mathcal{S}_*(\mathbb{R}^{n+1}),$$

$$T = L^{n-1}(L + \tilde{\Delta}).S$$

where L is the Bessel operator : $L = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r}$ and $\tilde{\Delta}$ is the Laplacian operator on $\mathbb{R}^n$:

$$\tilde{\Delta} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

- K_1, K_2, K_3 the operators defined by

$$K_1(g)(r,x) = \frac{\pi^2}{2^n [\Gamma(\frac{n+1}{2})]^2} \mathcal{H}\left(\frac{\partial}{\partial r}(-\Delta^{\frac{n-1}{2}})\right)(g)(r,x).$$

$$K_2(g)(r,x) = \frac{\pi^{\frac{n+1}{2}}}{2^n [\Gamma(\frac{n+1}{2})]^3} \langle T, \mathcal{T}_{(r,x)}(g) \rangle,$$

where

$$\mathcal{T}_{(r,x)}g(t,y) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n}{2})} \int_0^\pi g(\sqrt{r^2+t^2+2rt\cos\theta}, x+y)(\sin\theta)^{n-1} d\theta.$$

$$K_3(g)(r,x) = \frac{\pi}{2^n [\Gamma(\frac{n+1}{2})]^2} I_1^{-\frac{1}{2}} (-\Delta)^{\frac{n-1}{4}} (g)(r,x).$$

<u>Theorem</u> IV.1. (Inversion formulas for R and ${}^tR$)

i) For any $f \in \mathcal{S}_*(\mathbb{R}^{n+1})$ we have

(IV.1) $$f = K_1\, {}^tR\, R(f) \;\; ; \;\; f = {}^tR\, K_2 R(f).$$

ii) For any $g \in \mathcal{S}_{*,cone}(\mathbb{R}^{n+1})$ we have

(IV.2) $$g = R\, K_1 {}^tR(g) \;\; ; \;\; g = K_2 R\, {}^tR(g).$$

<u>Remark</u>. The inversion formulas (IV.1) and (IV.2) are more precise than those announced by L.E.Andersson ([1]) and J. Fawcett ([2]), since we characterize the spaces of functions on which these formulas hold and we give the explicit form of the operators which are in these formulas.

<u>Theorem</u> IV.2. (Plancherel theorem for ${}^tR$)

i) For any $g \in \mathcal{S}_{*,cone}(\mathbb{R}^{n+1})$ we have the following Plancherel formula

$$\int_{]0,\infty[\times\mathbb{R}^n} |g(r,x)|^2 r^n dr dx = \frac{1}{2^{2n-1}\pi^n [\Gamma(\frac{n+1}{2})]^2} \int_{]0,\infty[\times\mathbb{R}^n} |K_3{}^tR(g)(r,x)|^2 dr dx$$

ii) The operator $K_3{}^tR$ can be extended to an isometric isomorphism from $L^2_{cone}(]0,\infty[\times \mathbb{R}^n, r^n dr dx)$ onto $L^2([0,\infty[\times \mathbb{R}^n, dr dx)$.

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On the existence of standing waves for the nonlinear Schrödinger equation

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Abstract. *We present a simple method to show that the nonlinear Schrödinger equation* $iu_t = -\Delta u + f(u)$ *has standing waves for a large class of functions* f.

1. In quantum mechanics a variety of phenomena are described by the nonlinear Schrödinger equation

$$iu_t = -\Delta u + f(u), \qquad t \geq 0,\ x \in R^3,\ |x| > A > 0, \tag{1}$$

with the function f satisfying $f(0) = 0$ and $f(re^{i\theta}) = e^{i\theta}f(r)$ for $\theta, r \in R$.

A standing wave for (1) is a solution $u(x,t)$ of (1) of the form $u(x,t) = e^{i\lambda t}v(x)$ where λ is a real constant and v a real function. The classical condition on such a solution (see [2,3]) is that $v(x)$ is exponentially small at infinity.

It is easy to see that u of the above form is a solution of (1) if and only if the real function v satisfies

$$\Delta v - f(v) = \lambda v, \qquad x \in R^3,\ |x| > A > 0. \tag{2}$$

Nonlinear eigenvalue problems of the type (2) have been studied by using variational methods (see [3] and the references there). By using ODE techniques we will show that if $f \in C^1(R, R)$ is such that $f'(0) = 0$, then for each $\lambda > 0$, equation (2) has a positive solution in $\{x \in R^3 : |x| > A\}$ which is exponentially small at infinity. Each such solution provides a standing wave for the nonlinear Schrödinger equation (1).

Key Words and Phrases: Schrödinger equation, standing wave.
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2. In our analysis we will need the following result regarding the differential equation

$$y'' = F(t, y), \qquad A \le t < \infty, \tag{3}$$

with the condition

$$y(A) = B \tag{4}$$

where $B > 0$ and $F \in C([A, \infty) \times R, R)$:

Lemma [1]. *Assume that $F(t, y)$ is strictly increasing in y for each fixed t and that there is a constant $M > 0$ such that $yF(t, y) > 0$ for $|y| > M$. If there are bounded functions $y_0(t)$ and $y_1(t)$, defined for $t \ge A$, such that $y_0(A) \ge B \ge y_1(A)$ and*

$$y_0''(t) \le F(t, y_0(t)), \qquad t \ge A,$$
$$y_1''(t) \ge F(t, y_1(t)), \qquad t \ge A,$$

then the problem (3) – (4) has at least a solution $y(t)$ with $y_1(t) \le y(t) \le t_0(t)$, $t \ge A$.

Theorem. *If $f \in C^1(R, R)$ is such that $f'(0) = 0$, then for every $\lambda > 0$ there is a standing wave of (1) of the form $u(x, t) = e^{i\lambda t} v(x)$ defined for all $t \ge 0$ and $x \in R^3$ with $|x| > A$.*

Proof. Let $\lambda \in R$ with $\lambda > 0$.

There is a constant $L > 0$ with

$$|f'(v)| \le \frac{1}{2}\lambda, \qquad |v| \le L.$$

Let $\delta_2 > \delta_1 > 0$ be such that $\delta_2 > \sqrt{2\lambda}$ and $\delta_1^2 \le \sqrt{\frac{\lambda}{2}}$ and let us define the functions

$$y_0(t) = \frac{LA}{2} e^{-\delta_1 t + \delta_1 A}, \qquad t \ge A,$$
$$y_1(t) = \frac{LA}{2} e^{-\delta_2 t + \delta_2 A}, \qquad t \ge A,$$

and let $F : [A, \infty) \times R \to R$ be given by

$$F(t, y) = \begin{cases} tf(\frac{y_1(t)}{t}) + \lambda y_1(t) + y - y_1(t), & y \le y_1(t), \\ tf(\frac{y}{t}) + \lambda y, & y_1(t) \le y \le y_0(t), \\ tf(\frac{y_0(t)}{t}) + \lambda y_0(t) + y - y_0(t), & y \ge y_0(t). \end{cases}$$

One can easily check that

$$y_0''(t) \leq F(t, y_0(t)), \qquad t \geq A,$$
$$y_1''(t) \geq F(t, y_1(t)), \qquad t \geq A,$$

so that an application of the Lemma ensures the existence of a solution $y(t)$ to the corresponding problem (3) – (4) (with $B = \frac{LA}{2}$) satisfying $y_1(t) \leq y(t) \leq y_0(t)$ for $t \geq A$. Taking into account the particular form of $F(t, y)$, we have that $y(t)$ is a solution of

$$y''(t) = tf(\frac{y(t)}{t}) + \lambda y(t), \qquad t \geq A.$$

Let $h(r) = \frac{y(r)}{r}$ for $r \geq A$. We define the radial function $v(x) = h(r)$ for $r = |x| > A$ and observe that

$$r^2\{\Delta v(x) - \lambda v(x) - f(v(x))\} = \frac{d}{dr}\{r^2 \frac{d\, h(r)}{dr}\} - r^2[\lambda h(r) + f(h(r))].$$

This equation is the same as

$$y''(r) = rf(\frac{y(r)}{r}) + \lambda y(r), \qquad r \geq A,$$

and we proved just before that $y(r)$ is a solution of this equation.

The proof is completed in view of the fact that

$$0 < v(x) \leq \frac{LA}{2} e^{-\delta_1 r + \delta_1 A}, \qquad r \geq A.$$

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The second Chinburg invariant for cyclotomic fields via the Hom-description

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1 Introduction

Let E/K be a Galois extension of number fields with group $G(E/K)$. The Chinburg invariants, $\Omega(E/K,i)$ for $1 \leq i \leq 3$, are important elements of the projective class-group, $C\mathcal{L}(\mathbf{Z}[G(E/K)])$, which were introduced in [1]. The second Chinburg invariant, $\Omega(E/K,2)$ is equal to the class of the ring of integers of E in the tamely ramified case and in this case it was shown in [8] to be equal to the Cassou-Noguès-Fröhlich class made from the Artin root numbers of the irreducible symplectic representations of $G(E/K)$. Further details concerning these invariants may be found in ([5] Chapter 7; [7]). The second Chinburg conjecture asserts that this holds in general and, since an abelian group has no irreducible symplectic representations, predicts that $\Omega(E/K,2)$ vanishes in this case.

In this note we shall prove the following result.

Theorem 1.1 Let p be an odd prime and let $\xi_{p^{s+1}} = exp(2\pi i/p^{s+1})$. Then

$$0 = \Omega(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q},2) \in C\mathcal{L}(\mathbf{Z}[G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})]).$$

This result is not new. When p is a regular odd prime similar vanishing results were first obtained in [6] (see [5] for a less terse account). The most general abelian results are to be found in [3], proved Iwasawa's analysis of the image of the p-adic logarithm [4] and its generalisation in [2].

The Hom-description of the class-group is an isomorphism between the class-group, $C\mathcal{L}(\mathbf{Z}[G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})])$, and ([5] §4.2; [7] §2.1.3)

$$\frac{Hom_{G(N/\mathbf{Q})}(R(G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})), J^*(N))}{Hom_{G(N/\mathbf{Q})}(R(G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})), N^*) \cdot Det(U(\mathbf{Z}[G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})]))}$$

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where $J^*(N)$ denotes the idèles of N, a cyclotomic field the containing $p^s(p-1)$-th roots of unity, $Det(U(\mathbf{Z}[G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})]))$ consists of those functions, f, whose q-adic coomponent is given by the determinant of a unit in $\mathbf{Z}_q[G(\mathbf{Q}(\xi_{p^{s+1}})/\mathbf{Q})]^*$ and $R(G)$ is the complex representation ring of G.

In ([5] Chapter 7) a formula for $\Omega(E/K,2)$ was given in terms of the Hom-description, in the case when all the wild decomposition groups are cyclic. Theorem 1.1 is the simplest such case and the purpose of this note is to how the results of [4], used module-theoretically to the same end in [3], may be used in conjuction with the Hom-description formulae of ([5] Chapter 7). Needless to say, this paper owes its existence to the analysis given in [3] and several aspects of that analysis have Hom-description paraphrases in our proof.

2 The cyclotomic Chinburg invariant

We adopt the notation of ([5] p.379 et seq). In particular, for brevity, we shall use the formula of ([5] Chapter 7; see Proposition 2.1) as our definition of the Chinburg invariant. Let p be an odd prime and let ξ_n denote a primitive n-th root of unity. Set $\pi_t = 1 - \xi_{p^t}$. For each $s \geq 0$ write $E_{s+1} = \mathbf{Q}(\xi_{p^{s+1}})$ and $L_{s+1} = \mathbf{Q}_p(\xi_{p^{s+1}})$. Hence the Galois groups, $G(E_{s+1}/\mathbf{Q})$ and $G(L_{s+1}/\mathbf{Q}_p)$, are both isomorphic to $(\mathbf{Z}/p^{s+1})^*$. If $j \in (\mathbf{Z}/p^{s+1})^*$ then σ_j will denote the Galois automorphism characterised by $\sigma_j(\xi_{p^{s+1}}) = \xi^j_{p^{s+1}}$. Let $a \in \mathbf{Z}_p^*$ be the topological generator, $a = \xi_{p-1}(1+p)$.

As in ([5] §7.4.37), we define $\mathbf{Z}[G(L_{s+1}/\mathbf{Q}_p)]$-modules

$$W_p(s+1) = Tors(L^*_{s+1}/ < x >) \text{ and } V_p(s+1) = (L^*_{s+1}/ < x >)/W_p(s+1).$$

From ([5] §§7.4.39-7.4.41) we know that $W_p(s+1))$ is a cohomologically trivial $\mathbf{Z}[G(L_{s+1}/\mathbf{Q}_p)]$-module of order $p^{s+1}(p-1)^2$, whose p-Sylow subgroup is the module of p-primary roots of unity, and that $V_p(s+1)$ is a free $\mathbf{Z}_p[G(L_{s+1}/\mathbf{Q}_p)]$-module of rank one. In addition, there is a short exact sequence of the form

$$0 \longrightarrow (U^1_{L_{s+1}})/Tors(U^1_{L_{s+1}}) \longrightarrow V_p(s+1) \xrightarrow{val} \mathbf{Z}/p^s \longrightarrow 0$$

in which $U^i_{L_{s+1}} = 1 + \pi^i_{s+1}\mathbf{Z}_p[\xi_{p^{s+1}}]$ and val is induced by the valuation, so that $val(\pi_{s+1}) \equiv 1$ $(modulo\ p^s)$.

There is a $\mathbf{Z}_p[G(L_{s+1}/\mathbf{Q}_p)]$-module homorphism ([5] p.384)

$$Xog_p : V_p(s+1) \longrightarrow L_{s+1}$$

which is characterised by the fact that Xog_p is equal to the map induced by the p-adic logarithm, log_p, on $(U^1_{L_{s+1}})/Tors(U^1_{L_{s+1}})$.

Now let us recall the formula for the second Chinburg invariant, $\Omega(E_{s+1}/\mathbf{Q}, 2)$, which is established in ([5] §7.4.35 and §7.4.57). Let $\alpha_{s+1} = \xi_{p^{s+1}} + \xi_{p^s} + \ldots + \xi_p \in L_{s+1}$ so that α_{s+1} is a normal basis generator for $L_{s+1}/\mathbf{Q}_p$. Let Π'_{s+1} be a generator for $V_p(s+1)$ as a $\mathbf{Z}_p[G(L_{s+1}/\mathbf{Q}_p)]$-module and set $\Pi_{s+1} = \mathcal{X}og_p(\Pi'_{s+1}) \in L_{s+1}$. Define an idèlic-valued function

$$\lambda_{s+1} \in Hom_{G(N/\mathbf{Q})}(R(G(E_{s+1}/\mathbf{Q})), J^*(N))$$

by the following formulae for the local components of $\lambda_{s+1}(\chi)$

$$\lambda_{s+1}(\chi) = \begin{cases} (\Pi_{s+1}|\chi)(\alpha_{s+1}|\chi)^{-1} & \textit{at the prime } p, \\ \\ 1 & \textit{at other primes.} \end{cases}$$

Here, when χ is a one-dimensional representation, $(z|\chi)$ denotes the resolvent homomorphism which is given by ([5] p.390)

$$(z|\chi) = \sum_{g \in G(L_{s+1}/\mathbf{Q}_p)} g(z)\chi(g)^{-1} = \sum_{g \in G(L_{s+1}/\mathbf{Q}_p)} g(z)\overline{\chi}(g).$$

Proposition 2.1 *([5] Chapter 7)*
Let p be an odd prime. In the class-group, $\mathcal{CL}(\mathbf{Z}[G(E_{s+1}/\mathbf{Q})])$, the second Chinburg invariant is given by the formula

$$\Omega(E_{s+1}/\mathbf{Q}, 2) = [W_p(s+1)] + [\lambda_{s+1}] \in \mathcal{CL}(\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]),$$

where $[\lambda_{s+1}]$ is the class represented by the homomorphism, λ_{s+1}, in the Hom-description of the class-group.

Let $N_{s+1} = \sum_{g \in G(L_{s+1}/\mathbf{Q}_p)} g \in \mathbf{Z}_p[G(L_{s+1}/\mathbf{Q}_p)]$.
In ([4] p.164), as explained in ([3] §6), the logarithm

$$log_p : (U^1_{L_{s+1}})/Tors(U^1_{L_{s+1}}) \longrightarrow L_{s+1}$$

induces an isomorphism of the form

$$log_p : (U^1_{L_{s+1}})/Tors(U^1_{L_{s+1}}) \xrightarrow{\cong} p\mathbf{Z}_p \oplus C_{s+1}$$

where C_{s+1} is the free $\mathbf{Z}_p[G(L_{s+1}/\mathbf{Q}_p)]/(N_{s+1})$-module generated by $y'_{s+1} = (1 - a^{-1}\sigma_a)y_{s+1} \in L_{s+1}$. and $y_{s+1} = p(1-p)^{-1} + z_{s+1} \in L_{s+1}$ where $z_{s+1} = \pi_{s+1} + p^{-1}\pi_s + p^{-2}\pi_{s-1} + \ldots + p^{-s}\pi_1 \in L_{s+1}$.

We may use this result to find $\Pi_{s+1} = \mathcal{X}og_p(\Pi'_{s+1})$ and hence to evaluate $\lambda_{s+1}(\chi)$.

Henceforth we shall identify $V_p(s+1)$ with its image under $\mathcal{X}og_p$. Therefore we have a short exact sequence of $\mathbf{Z}_p[G(L_{s+1}/\mathbf{Q}_p)]$-modules of the form

$$p\mathbf{Z}_p \oplus C_{s+1} \longrightarrow V_p(s+1) \xrightarrow{val} \mathbf{Z}/p^s.$$

The group, $G(L_{s+1}/\mathbf{Q}_p)$, acts trivially on $\mathbf{Z}/p^s$. Tate cohomology groups are computed from the chain complex

$$\ldots \longrightarrow \mathbf{Z}/p^s \xrightarrow{1-\sigma_a} \mathbf{Z}/p^s \xrightarrow{N_{s+1}} \mathbf{Z}/p^s \xrightarrow{1-\sigma_a} \mathbf{Z}/p^s \longrightarrow \ldots$$

so that $\hat{H}^i(G(L_{s+1}/\mathbf{Q}_p);\mathbf{Z}/p^s) \cong \mathbf{Z}/p^s$ for all i. The coinvariants of C_{s+1}, $C_{s+1}/((1-\sigma_a)C_{s+1})$ is isomorphic to $\mathbf{Z}/p^s$, generated by the image of y'_{s+1}. The map $1-\sigma_a : C_{s+1} \longrightarrow C_{s+1}$ is injective while $N_{s+1} : C_{s+1} \longrightarrow C_{s+1}$ is zero so that $\hat{H}^i(G(L_{s+1}/\mathbf{Q}_p);C_{s+1}) \cong \mathbf{Z}/p^s$ when i is odd and is zero otherwise. On the trivial module, $p\mathbf{Z}_p$, N_{s+1} is multiplication by $p^s(p-1)$ so that $\hat{H}^i(G(L_{s+1}/\mathbf{Q}_p);p\mathbf{Z}_p) \cong \mathbf{Z}/p^s$ when i is even and is zero otherwise.

Proposition 2.2

The element, $\Pi_{s+1} = Xog_p(\Pi'_{s+1}) \in L_{s+1}$, *is given by*

$$\Pi_{s+1} = \lambda + a^{-1}z_{s+1} + (1-a^{-1})p^{-s}(1-p)^{-1}\sum_{j=1}^{p^s(p-1)-1} j\sigma_{a^j}(z_{s+1})$$

where $\lambda \in \mathbf{Q}_p$ *is chosen so that* $u_{s+1} = p^{-1}Trace_{L_{s+1}/\mathbf{Q}_p}(\Pi_{s+1}) \in \mathbf{Z}_p^*$.

Proof

The Tate cohomology groups of V_{s+1} are all trivial. Therefore the coboundary maps from $\hat{H}^i(G(L_{s+1}/\mathbf{Q}_p);\mathbf{Z}/p^s)$ to $\hat{H}^{i+1}(G(L_{s+1}/\mathbf{Q}_p);p\mathbf{Z}_p \oplus C_{s+1})$ are all isomorphisms. When i is even this implies that

$$(1-\sigma_a)\Pi_{s+1} = y'_{s+1} = (1-a^{-1}\sigma_a)y_{s+1} \in L_{s+1}.$$

This equation defines Π_{s+1} up to addition of elements from $\mathbf{Q}_p$. When i is odd we must have $Trace_{L_{s+1}/\mathbf{Q}_p}(\Pi_{s+1}) = N_{s+1}\Pi_{s+1} = pu_{s+1}$ with $u_{s+1} \in \mathbf{Z}_p^*$. Since λ can be chosen to ensure that the latter condition holds, it suffices to verify that $(1-\sigma_a)\Pi_{s+1} = (1-a^{-1}\sigma_a)y_{s+1}$, which is straightforward.□

Now suppose that $\chi : (\mathbf{Z}/p^{s+1})^* \longrightarrow \mathbf{C}^*$ is a non-trivial Dirichlet character with conductor, $f_\chi = p^e$. Therefore χ factorises as reduction modulo p^e followed by an injective character, $\chi : (\mathbf{Z}/p^e)^* \longrightarrow \mathbf{C}^*$. Recall from ([5] §7.4.54) that the associated local Gauss sum, $\tau(\chi)$, is defined by $\tau(\chi) = \sum_{b\in(\mathbf{Z}/p^e)^*}\chi(b)\xi_{p^e}^b$. If $\chi = 1$ we set $\tau(1) = 1$. The following result is elementary.

Proposition 2.3

Suppose that $\chi : G(L_{s+1}/\mathbf{Q}_p) \cong (\mathbf{Z}/p^{s+1})^* \longrightarrow \mathbf{C}^*$ *is a Dirichlet character with conductor,* $f_\chi = p^e$. *Then the values of the resolvent,* $(\xi_{p^v}|\chi)$, *are given by the following table:*

	$v=0$	$v=1$	$v>1$
$e=0$	$p^s(p-1)$	$-p^s$	0
$0<e<v$	$--$	$--$	0
$v<e$	0	0	0
$0<e=v$	$--$	$p^{s+1}\tau(\overline{\chi})f_\chi^{-1}$	$p^{s+1}\tau(\overline{\chi})f_\chi^{-1}$

2.4 *Proof of Theorem 1.1*

Following ([5] §7.4.59) it is straightforward to calculate $(\Pi_{s+1}|\chi)(\alpha_{s+1}|\chi)^{-1}$ using Propositions 2.2 and 2.3. For one-dimensional representations, χ, of $G(E_{s+1}/\mathbf{Q})$, one finds that

$$\lambda_{s+1}(\chi) = \begin{cases} (-f_\chi p^{-(s+1)})(1-a^{-1}\chi(\sigma_a))(1-\chi(\sigma_a))^{-1} & \textit{at } p, \textit{ if } \chi \neq 1, \\ (-u_{s+1})p^{1-s} & \textit{at } p, \textit{ if } \chi = 1, \\ 1 & \textit{at other primes.} \end{cases}$$

In the class-group, write $[W_p(s+1)] = [\mu_{p^{s+1}}] + [W'_p(s+1)]$ – the sum of the p-primary roots of unity and the prime-to-p submodule. As explained in ([5] §5.6.4), the Hom-description of $[\mu_{p^{s+1}}]$ is given by $\chi \mapsto (1-a^{-1}\chi(\sigma_a))^{-1}$ at p and is trivial elsewhere. We may also remove the conductor and unit factors to show that the Chinburg invariant is equal to the sum of $[W'_p(s+1)]$ and the element whose Hom-description representative, ρ_{s+1}, is given by $\rho_{s+1}(\chi) = (1-\chi(\sigma_a))^{-1}$ at p, if $\chi \neq 1$ and is trivial otherwise. This is accomplished by dividing by the global function $(-f_\chi p^{-(s+1)})$ (or p^{1-s} when $\chi = 1$) to put these factors at all other finite primes, q, where they are easily seen to be determinants (c.f. [5] Chapter 5) by using the characters of $\mathbf{Z}/p^s$ to decompose the q-adic group-ring. We use the fact that the class of a Swan modules is trivial in the class-group of a cyclic group to remove the factor, $(-u_{s+1})$. For, by q-adic approximation (c.f. [5] §7.4) and the Chinese Remainder Theorem, one can show that the function which is trivial except for sending 1 to $(-u_{s+1})$ at p represents a Swan module.

This discussion shows that

$$\Omega(E_{s+1}/\mathbf{Q}, 2) = [\rho_{s+1}] + [W'_p(s+1)].$$

We shall show that this element is zero by considering its image under the canonical map from $C\mathcal{L}(\mathbf{Z}[G(E_{s+1}/\mathbf{Q})])$ to $C\mathcal{L}(\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]/ < N_{s+1} >)$ where, as usual, N_{s+1} is the sum of all the group elements. In general, the kernel of this map is the Swan subgroup but in the case of a cyclic group the Swan subgroup is trivial and the map is injective.

The Hom-description of $C\mathcal{L}(\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]/ < N_{s+1} >)$ is identical to that of the class-group of the integral group-ring except that the functions are only defined on non-trivial one-dimensional representations. The module, $W'_p(s+1)$, has order $(p-1)^2$ and is isomorphic to the quotient of $L_{s+1}/ < p(1+p)\xi_{p-1} >$ by the submodule generated by $\mu_{p^{s+1}}$ and Π'_{s+1}. It is generated by the image of $\pi^{p^s}_{s+1}$. However, $W'_p(s+1)$ maps to $W'_p(s+1)/N_{s+1}W'_p(s+1)$ and $N_{s+1}(\pi^{p^s}_{s+1}) = p^{p^s})$ which is equal to ξ^{-1}_{p-1} in $W'_p(s+1)$, since the image of $1+p$ is trivial in this quotient. Hence the image of $W_p'(s+1)$ is represented by the trivial module given by $W_p'(s+1)/ < \xi_{p-1} > \cong \mathbf{Z}/(p-1)$. However, the quotient of $\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]/ < N_{s+1} >$ by the principal ideal $(1-\sigma_a)\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]/ < N_{s+1} >$ is isomorphic to the trivial module given by the cyclic group of order $p^s(p-1)$, which therefore represents the trivial class in the class-group of $\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]/ < N_{s+1} >$. The Hom-description of this trivial element sends $\chi \neq 1$ to $(1-\chi(\sigma_a))^{-1}$ at primes dividing $p^s(p-1)$ and is trivial otherwise. Similarly, the Hom-description of the prime-to-p part of this module – the image of $W_p'(s+1)$ – has a Hom-description which is given by $(1-\chi(\sigma_a))^{-1}$ at primes dividing $p-1$ and is trivial otherwise.

Hence the complete Hom-description representative of the second Chinburg invariant in $C\mathcal{L}(\mathbf{Z}[G(E_{s+1}/\mathbf{Q})]/ < \Delta_{s+1} >)$ is the same as that for the element given by the cyclic group of order $p^s(p-1)$ – namely $(1-\chi(\sigma_a))^{-1}$ at all primes dividing $p^s(p-1)$ and trivial otherwise – which we have seen to be trivial.□

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A MATHEMATICAL FORMULA FOR DETERMINING THE USEFUL MEASURING TIME FOR THE TRANSIENT HOT-WIRE PROBLEM

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Presented by G. F. D. Duff, F.R.S.C.

ABSTRACT. A new simplified formula for the average temperature of a thin metal wire of finite length situated coaxially in a gas-filled cylindrical cavity of same length is used numerically to determine the region of measurement of thermal conductivity and diffusivity of gases. The theoretical results agree with experimental data found in the literature.

RÉSUMÉ. On présente une formule simplifiée pour la température moyenne d'un fil métallique fin de longueur finie en milieu gazeux cylindrique de même longueur et d'axe commun pour déterminer la région utile de la mesure de la conductivité thermique et de la diffusivité des gaz. Les résultats théoriques concordent avec des données expérimentales publiées.

1. Introduction. There has been sustained study [1] of the transient hot-wire method for the determination of thermal properties of physical substances. Transient methods, unlike steady methods, introduce only short duration disturbances in a medium and allow simultaneous measurements of thermal conductivity and thermal diffusivity.

A new mathematical formula [2] is proposed for determining the useful measuring time for the transient hot-wire problem. This formula takes care of important corrections to the experimental line source in order to approximate the ideal solution of the continuous line source during measurement of thermal conductivity and diffusivity of gases. Until now, several corrections were done experimentally and a more complex double-wire apparatus was used to take care of some of the corrections. It is hoped that the new formula will allow the practical use of a simpler one-wire apparatus.

The idealized physical model for the transient hot-wire method is an infinite continuous line source of constant heat flux, q, per unit time per unit length applied stepwise at time $\tau = 0$. The line is situated along the z axis in an unbounded incompressible medium of constant density, ρ_2, thermal conductivity, λ_2, and specific heat, c_{p2}. Heat is lost from the source only by radial conduction, thus increasing the temperature, T, of the medium. The ideal solution [3] is

$$T(\tau) = \frac{q}{4\pi\lambda_2}\ln\frac{4\kappa_2\tau}{r^2} - \frac{\gamma q}{4\pi\lambda_2}, \tag{1}$$

provided $\kappa_2\tau/r^2$ is sufficiently large, where r is the radial distance from the source, τ is the time, $\kappa_2 = \lambda_2/(\rho_2 c_{p_2})$ is the thermal diffusivity of the medium and $\gamma = 0.5772\ldots$ is Euler's

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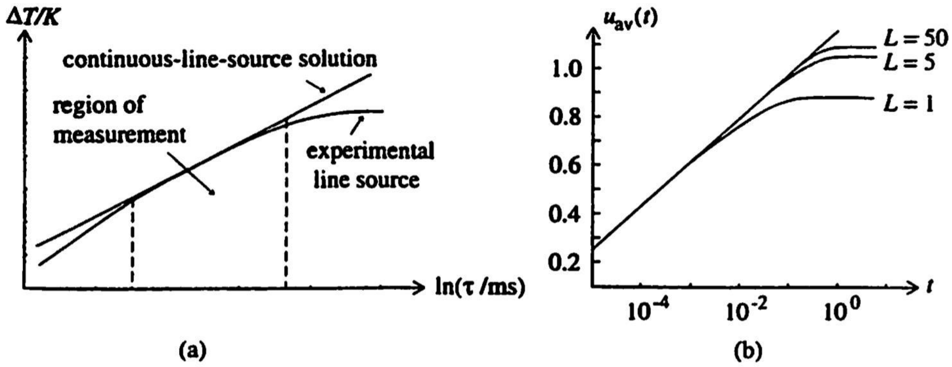

FIGURE 1. (a) Typical graph of temperature rise against time for a hot-wire experiment. (b) Average wire temperature, $\lambda_2 u_{av}(t)/q$, for shown values of L. The straight line represents the ideal solution $T_{ideal}(t) = \lambda_2 T(t)/q$.

constant. In the $(\ln \tau, T)$-plane, (1) represents a straight line. The parameters λ_2 and c_{p2} can be obtained from the slope and intercept of this line, respectively.

In practice, the experimental line source differs from the continuous line source as shown in Fig. 1(a) (see [4]). The temperature rise is less than that of the ideal case at small times because of the finite heat capacity of the wire and at long times, firstly, because of the effect of the finite outer boundary of the cell, and secondly, at even longer times, because of natural convection. The central portions of the curves are close, however, and it is this so-called *region of measurement* that is utilized in the transient hot-wire method.

A two-component convective motion of the fluid exists from the start due to the radial temperature gradient and the axial temperature gradient at the end of the wire. Experimental [5] and theoretical [6] papers take convection effects into account. Experiments [5] indicate that solution (1) agrees well with experimental data for τ smaller than a characteristic time τ_c; for example, for toluene at 0°C and q=0.004 W/cm, convection becomes experimentally significant for $\tau > \tau_c \approx 2$s. In [6] the influence of convection on thermal conductivity measurements is investigated numerically for several values of the Prandtl and Grashof numbers, and it is shown that convection effects are negligible if $\tau < 3s$.

2. Mathematical formulation of the problem. We consider a cylinder of radius R_2 and finite length l, filled with a gas whose thermal properties are to be determined. A thin metal wire of radius R_1 and length l is coaxially situated inside the cylinder. The initial temperature of the wire and medium is set to zero. Heat is supplied to the wire at the constant rate $q/(\pi R_1^2)$ per unit length per unit time, where q is a constant.

The formulation of the physical model makes use of the following experimentally justified assumptions: (a) the thermal properties of the gas and wire do not depend upon the temperature range, (b) the temperature distribution is axisymmetric, (c) the radiation flux from the surface of the wire and the convection effects are negligible, and (d) the external boundary of the cylinder and both ends of the cylinder and wire are kept at constant zero temperature.

The validity of these assumptions is discussed in [4] where it is shown that the corrections to (a)–(c) are negligible during the time interval of measurements. Assumption (d) is an approximation to the conditions in a hot-wire cell.

The physical model is mathematically described, in the present novel approach, by means of the initial boundary value problem for the heat equation for both the wire and the cell as a whole:

$$\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial z^2} - a^2\frac{\partial u_1}{\partial t} = -Q, \qquad 0 < r < R, \tag{2}$$

$$\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} + \frac{\partial^2 u_2}{\partial z^2} - \frac{\partial u_2}{\partial t} = 0, \qquad R < r < 1, \tag{3}$$

$$u_2|_{r=1} = 0, \qquad u_1|_{r=R} = u_2|_{r=R}, \qquad K\frac{\partial u_1}{\partial r}\bigg|_{r=R} = \frac{\partial u_2}{\partial r}\bigg|_{r=R}, \tag{4}$$

$$u_i|_{z=0} = 0, \qquad u_i|_{z=L} = 0, \qquad u_i|_{t=0} = 0. \qquad i = 1, 2, \tag{5}$$

where u_1 and u_2 are the wire and gas temperatures, respectively, and $Q = q/(\pi\lambda_1 R^2)$. The numbers $a^2 = \kappa_2/\kappa_1$, $K = \lambda_1/\lambda_2$, $R = R_1/R_2$ and $L = l/R_2$ are dimensionless quantities, where λ_i, c_{p_i} and ρ_i are the thermal conductivity, the heat capacity and the density of the ith medium, respectively, and $\kappa_i = \lambda_i/(c_{p_i}\rho_i)$, $i = 1, 2$. Finally, t is dimensionless time related to real time τ by the relation $t = \kappa_2\tau/R_2^2$.

Measurements are usually restricted to short time intervals (200 ms $< \tau <$ 1100 ms in [7] and 40 ms $< \tau <$ 930 ms in [4]) to avoid convection effects [5], [6]. The radius of the wire is sufficiently small ($R_1 = 3.5\,\mu$m in [4] and $R_1 = 2.5\,\mu$m in [5] so that $R = 0.467 \times 10^{-3}$ and 2.5×10^{-3}, respectively). In that case, the problem was analytically solved in [8] for an infinitely long wire. In [2], it is found that the most important parameters affecting the difference between the ideal and real situations (except, perhaps, at very short times) are L and R. The ratio of the heat capacities, which is small, appears to second order in the asymptotic expansion of the solution.

3. Simplification of the model. The solution to problem (2)–(5) may be used for computing the average temperature of the wire. However, in applications, it is too unwieldy because it is expressed in terms of the modified Bessel functions, $I_\nu(s)$ and $K_\nu(s)$, $\nu = 0, 1$, with complex arguments. However, if $R \approx 10^{-3}$, then experimental conditions approximate to a continuous line source [4], [5], and the solution found in [2] can be greatly simplified.

Since the temperature of the wire is obtained by measuring its resistance, it is important to determine the average temperature, $u_{av}(t)$, of the wire. Using the complete solution found in [2], we obtain the Laplace transform of $u_{av}(t)$ in the form

$$\bar{u}_{av}(p) = -\frac{4Q}{\pi^2 R}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]^2}{n^2\beta^2 p}\left\{\frac{\alpha[K_0(\alpha)I_1(\alpha R) + I_0(\alpha)K_1(\alpha R)]}{\beta\Delta}I_1(\beta R) - \frac{R}{2}\right\}. \tag{6}$$

where $\alpha = \sqrt{p + n^2\pi^2/L^2}$, $\beta = \sqrt{a^2 p + n^2\pi^2/L^2}$, and

$$\Delta = \alpha I_0(\beta R)[I_1(\alpha R)K_0(\alpha) + I_0(\alpha)K_1(\alpha R)] + K\beta I_1(\beta R)[I_0(\alpha)K_0(\alpha R) - I_0(\alpha R)K_0(\alpha)].$$

Setting $\alpha = \sqrt{p + (2k+1)^2\pi^2/L^2}$, we have the asymptotic expansion

$$\bar{u}_{av}(p) = -\frac{4q}{\pi^3\lambda_2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left[\frac{1}{p}\ln\frac{\alpha R}{2} - \frac{1}{4Kp} + \frac{\gamma}{p} + \frac{K_0(\alpha)}{pI_0(\alpha)} + O(R^2\ln R)\right]. \tag{7}$$

Firstly, we consider the case of small t, corresponding to large $|p|$. Using asymptotic expansions for Bessel functions for large values of their argument, the term $K_0(\alpha)/pI_0(\alpha)$ in (7) may be written in the form

$$\frac{K_0(\alpha)}{pI_0(\alpha)} \sim \frac{\pi\, e^{-2\alpha}}{p}\left[1 - \frac{1}{4\alpha} + O\left(\frac{1}{\alpha^2}\right)\right]. \tag{8}$$

By (8), formulae (2.4.119), (2.4.132) and (2.4.127) in [9], and the sum $\sum_{k=0}^{\infty} 1/(2k+1)^2 = \pi^2/8$, the inverse Laplace transform of (7) becomes

$$u_{av}(t) = \frac{q}{4\pi\lambda_2}\left[\ln\frac{4t}{R^2C} - \varphi(K,L,t)\right], \tag{9}$$

at small times, where $C = e^\gamma$,

$$\begin{aligned}\varphi(K,L,t) = \gamma - \frac{1}{2K} + 2\ln\frac{\pi}{L} + \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{\ln(2k+1)}{(2k+1)^2} + \ln t\\ + \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left\{-\frac{1}{2}\operatorname{Ei}(-\delta_k^2 t) + \frac{\pi}{2}\left[e^{-2\delta_k}\operatorname{erfc}\left(\frac{1}{\sqrt{t}} - \delta_k\sqrt{t}\right)\right.\right.\\ \left.\left. + e^{2\delta_k}\operatorname{erfc}\left(\frac{1}{\sqrt{t}} + \delta_k\sqrt{t}\right)\right] + \dots\right\},\end{aligned} \tag{10}$$

$\delta_k = (2k+1)\pi/L$, $-\operatorname{Ei}(-s) = \int_s^\infty 1/(x\,e^x)\,dx$ and $\operatorname{erfc}(s)$ is the complementary error function. The second series in (10) is rapidly convergent at small times. Expression (10) contains only the indicated terms of the asymptotic expansions (7)–(8) since computations show that adding other terms practically does not change the results at short times.

Secondly, the residue theorem is used to obtain the solution at long times. As it can be seen from (7) the only singular points of the function $\bar{u}_{av}(p)$ are first order poles at the points $p = 0$ and $p_j = i\alpha_j$, where α_j are the positive zeros of Bessel's function $J_0(s)$. Thus the solution is

$$u_{av}(t) = \frac{q}{4\pi\lambda_2}\left[\ln\frac{4t}{R^2C} - \psi(K,L,t)\right], \tag{11}$$

where $\psi(K,L,t)$ is given by

$$\begin{aligned}\psi(K,L,t) = \gamma - \frac{1}{2K} + 2\ln\frac{\pi}{L} + \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{\ln(2k+1)}{(2k+1)^2}\\ + \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{K_0(\delta_k)}{(2k+1)^2 I_0(\delta_k)} + \ln t\\ + \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left[\sum_{j=1}^{\infty}\frac{\alpha_j Y_0(\alpha_j)\exp\left[\left(-\alpha_j^2 - \delta_k^2\right)t\right]}{(\alpha_j^2 + \delta_k^2)J_1(\alpha_j)}\right].\end{aligned} \tag{12}$$

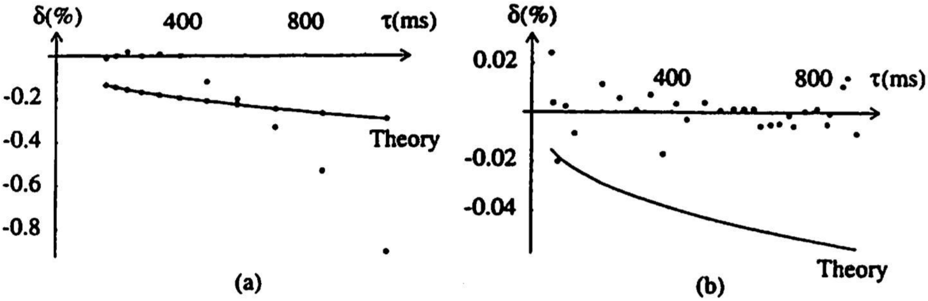

FIGURE 2. Percentage departure form linearity, $\delta(\tau)$, for theoretical values (lines) and experimental data (unjoined dots) for (a) ammonia and (b) liquid argon.

4. Numerical results. Formulae (9)–(12) were used in [2] to determine numerically the dependance of $u_{av}(t)$ upon each of the parameters of the problem and to evaluate the region of measurement. If (10) and (12) are added to the real average temperature of the wire, then the ideal solution is obtained within the limits of our model. It is seen from (9)–(12) that L and R are the important parameters.

In Fig. 1(b), $u_{av}(t)$ is plotted for different values of L. It is seen that steady state is reached sooner for short wires, and a longer wire allows a longer region of measurement.

With $T(t)$ given by (1), the percentage departure from linearity is defined by

$$\delta(t) = \frac{u_{av}(t) - T(t)}{T(t)} \times 100. \tag{13}$$

In Fig. 2, experimental data (isolated dots) and theoretical curves (solid lines) are shown against real time, τ. In Fig. 2(a), the experimental data [4] are for ammonia at temperature $T = 383$ K and density $\rho = 41$ kg·m^{-3}, and a cell with $R = 0.5 \cdot 10^{-3}$ and $L = 21.4$. In Fig. 2(b), the experimental data [10] are for liquid argon at temperature $T = 124$ K, pressure $p = 7.47$ MPa and density $\rho = 1177$ kg·m^{-3}, and a cell with $R = 0.68 \cdot 10^{-3}$ and $L = 33.8$.

The satisfactory agreement between our theoretical solution and the experimental data is explained in more detail in the following three points.

First, the percentage departure from linearity in experiments was computed by the formula

$$\delta(t) = \frac{T_{exp}(t) - T_{fit}(t)}{T_{fit}(t)} \times 100, \tag{14}$$

where $T_{exp}(t)$ are experimental temperature data and $T_{fit}(t)$ are values obtained by least-square approximation to experimental data. Since the departure from linearity in Fig. 2 is only of part of one percent, it is obvious that a change in the fitting procedure (variation of the number of points, for example) can lead to a change in the slope of the line represented by $T_{fit}(t)$ and, as a consequence, (14) will produce different values of $\delta(t)$.

Second, since no simple analytical correction was known in the previous theoretical studies of the transient hot-wire method, the correction for the finite wire length was done experimentally. In modern transient hot-wire apparatus two wires of different lengths are used to correct experimentally the effect of the finite length of the wire [1], [4], and, in this case, the actual boundary conditions at $z = 0$ and $z = L$ are more complicated than the ones used in (5). This may also explain the discrepancy between theoretical and experimental data.

Third, the accuracy of the experimental determination of λ_2 is also within a fraction of one percent (± 0.5, for example, in [10]). As can be seen, in Fig. 2, the theoretical curves lie within these limits if $\tau < 1\,\mathrm{s}$ (the considerable difference between theory and experiment in Fig. 2(a) for $\tau > 1\,\mathrm{s}$ can be explained [4] by convection effects).

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ON THE INFLUENCE OF RADIALLY NONUNIFORM INTERNAL HEAT SOURCES ON THE STABILITY OF COUETTE FLOW

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presented by K. B. Ranger, F.R.S.C.

ABSTRACT. The influence of an axial convective motion caused by radially nonuniform heat sources on the stability of a Couette flow between cylinders, if the inner one is rotating with constant angular velocity and the outer one is fixed, is studied in this paper. The axisymmetric and the first few asymmetric modes are studied. It is found that, for wide gaps, there exist regions of stabilization of a Couette flow. These regions, which correspond to asymmetric modes, decrease as the Prandtl number grows.

RÉSUMÉ. On étudie l'influence d'une convection axiale sous sources calorifiques nonuniformes sur la stabilité d'un écoulement de Couette entre un cylindre intérieur de vitesse angulaire constante et un cylindre extérieur fixe. On se restreint au mode axisymétrique et aux premiers modes asymétriques. On démontre l'existence de régions de stabilisation pour les grands écarts. Ces régions correspondant aux modes asymétriques décroissent si le nombre de Prandtl croit.

1. Introduction. Recently, the classical hydrodynamic problem of the stability of Couette flow between two rotating cylinders is has been investigated under additional factors such as axial isothermal flow in the vertical direction, and radially nonuniform temperature distribution in the fluid [1]–[3] in view of applications to complicated convective heat transfer in thermal systems such as gas turbines and rotating machinery, crystal growth [4] and the design of photochemical reactors for the purification of industrial waste water [2].

The stability of nonisothermal Couette flow between rotating cylinders under radial heating of the fluid was initially studied in [5]–[6], without taking the vertical component of the fluid base velocity caused by a nonuniform temperature distribution through the fluid into account. However, experimental studies [7] indicated that this component greatly influences the stability and must be taken into account. A corresponding theoretical stability analysis is presented in [3], where it is shown that, at certain values of the parameters, a decrease of the Taylor number leads to a sequence of transitions from the axisymmetric mode to asymmetric modes with an increasing number of azimuthal modes. Computational results in [3] compare satisfactorily with experiments [7].

A theoretical investigation of the stability of a convective motion caused by internal heat sources uniformly, or nonuniformly, distributed through the fluid was done in [8]–[11]. The

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influence of an axial convective motion, caused by uniformly distributed internal heat sources. on the stability of a circular Couette flow between two rotating cylinders is studied in [12].

In this note, the stability of a Couette flow with nonuniform heat sources is studied in the region between an inner rotating cylinder and an outer cylinder at rest. Both the axisymmetric (toroidal) and the asymmetric (spiral) modes are studied for different values of the free parameters of the problem [13].

2. Mathematical analysis. We consider an infinitely long vertical annular channel of radii R_1 and R_2 ($R_1 < R_2$). The inner cylinder rotates with constant angular velocity, ω, and the outer cylinder is at rest. The channel is filled with a viscous incompressible fluid. The temperature of both cylinders is constant and equal. We introduce a system of cylindrical polar coordinates $(\tilde{r}, \varphi, \tilde{z})$ with the origin on the axis of the cylinders. The $\tilde{z}$-axis is directed upwards (opposite to gravity) and coincides with the cylinders' axes.

Heat sources of volume density $Q(r) = Q_0\, e^{-\alpha(r-r_1)}$ are distributed within the fluid, where Q_0 and α are constants, $r_1 \le r \le r_2$ and r_1 and r_2 are the dimensionless radii of the cylinders and $h = (R_2 - R_1)/2$ is taken as unit length.

The Navier–Stokes equations in the Boussinesq approximation have the base flow steady solution:

$$v_r = 0,\ v_\varphi = V_0(r),\ v_z = W_0(r),\ T = T_0(r),\ p_0(r,z) = p_{01}(z) + p_{02}(r), \tag{1}$$

where

$$T_0(r) = C_1 \ln r + C_2 + A(r) + B(r), \qquad V_0(r) = \frac{r_1 r_2^2}{(r_2^2 - r_1^2)r} - \frac{r_1 r}{r_2^2 - r_1^2},$$

$$W_0(r) = C_3 \ln r + C_4 + C_5 r^2 + \int_{r_1}^{r} \Big\{ \xi \ln\frac{\xi}{r}\, [C_1 \ln\xi + C_2 + A(\xi)] - 2\xi \ln\xi\, e^{-\alpha(\xi - r_1)} \left(\frac{\xi^2}{2} \ln\frac{\xi}{r} + \frac{r^2 - \xi^2}{4} \right) \Big\}\, d\xi,$$

and the constants C_1 to C_5 and the functions $A(r)$ and $B(r)$ are given in [11].

We consider the stability of the flow (1) by the method of normal perturbations. The solutions in a neighbourhood of the base flow are sought in the form

$$[v_r, v_\varphi, v_z, T, p] = [0, V_0(r), W_0(r), T_0(r), p_0(r,z)] + [u(r), v(r), w(r), \theta(r), q(r)]\, e^{-\lambda t + ikz + in\varphi}. \tag{2}$$

where $n = 0$ and $n \neq 0$ correspond to toroidal and spiral disturbances, respectively. Substituting (2) into the approximated Navier–Stokes equations and linearizing the equations in a neighbourhood of the base flow, we obtain a boundary value system of ordinary differential equations in r. This system is solved by the pseudospectral collocation method and the stability of the flow (1) is determined by the eigenvalues, $\lambda_m = a_m + ib_m$, of the problem. The flow is stable if $a_m > 0$ for all m, and is unstable if $a_m < 0$, for at least one

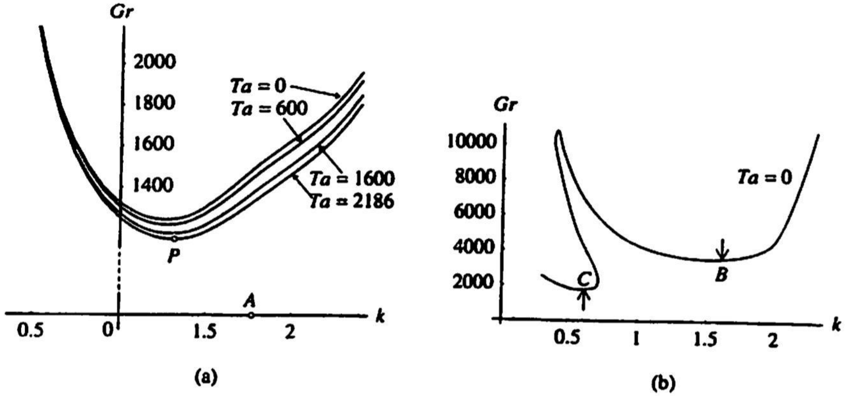

FIGURE 1. Neutral stability curves: (a) for $n = 0$, $Pr = 1$, $\eta = 0.7$, $\alpha = 0.5$ and four values of Ta; (b) for $n = 2$, $Pr = 5$, $\eta = 0.4$, $\alpha = 2$ and $Ta = 0$.

m. The generalized eigenvalue problem which is obtained from the discretized problem is solved numerically.

3. Numerical results and discussion. For an easy comparison with the results for the isothermal case [6], the computational results are given in terms of the Taylor number Ta and the Grashof number Gr:

$$Ta = \frac{64\omega^2 h^4 \eta^4}{\nu^2(1-\eta^2)}, \qquad Gr = \frac{g\beta Q_0 h^5}{2\nu^2 \kappa \rho c_p}. \tag{3}$$

Let α describe the nonuniformity of the heat sources, η denote the radius ratio and Pr be the Prandtl number. Computations were done for $\eta = 0.7$ and 0.4, $\alpha = 0$, 0.5 and 2, and $Pr = 1$, 5 and 20, since the importance of Pr on stability characteristics is known [9], [14]. The axisymmetric and the first two asymmetric modes, with azimuthal wave numbers $n = 0, 1, 2$, respectively, were studied.

The shape of the neutral stability curves changes considerably if the values of some parameters of the problem are changed. We consider samples of such curves.

The four curves shown in Fig. 1(a) correspond to the axisymmetric mode ($n = 0$) with $\alpha = 0.5$, $Pr = 1$, $\eta = 0.7$. Each curve has only one minimum corresponding to the critical values Gr_c and k_c, respectively, of Gr and k. Computations show that rotation has a destabilizing effect on the flow; thus Gr_c decreases as Ta grows. Instability in this case is of hydrodynamical nature (sometimes called *thermal-shear instability* [14]) since the role of thermal factors is relatively small for small Pr and energy is transmitted to the perturbations basically from the main flow.

When $\eta = 0.7$, it is known that the classical isothermal Couette flow is unstable at $k_c = 1.57$, $Ta_c = 2186$ (see [6]). In this case, $Gr_c = 0$; thus the point $A : (k_c, Gr_c, Ta_c) = (1.57, 0, 2186)$ lies on the k-axis in the (k, Gr)-plane. One would expect that if the Ta grew from 0 to Ta_c then the Grashof number would decrease monotonically to 0 and the neutral curve at $Ta = Ta_c$ would be tangent to the k-axis at A. We shall see, later, that such a *continuous transition* to isothermal Couette flow occurs at larger Prandtl numbers. However, for the parameter values used in Fig. 1(a), our computations show that the limit position of the neutral stability curve is reached as Ta increases to the limiting value $Ta_c = 2186$; but Gr decreases only to the limiting value $Gr_* = 1147.9 > 0$.

Hence, the stability boundary can be described as follows. If $Ta < Ta_c$, stability is determined by the convective mode associated with the convective flow in the vertical direction (see Fig. 1(a)). When $Ta > Ta_c$, the flow is unstable with respect to the pure centrifugal mode for any $Gr > 0$. This situation resembles the uniform heat generation treated in detail in [12]. Therefore a small increase of Ta beyond Ta_c produces a "jump" from point $P : (k_*, Gr_*, Ta_c) = (1.31, 1147.9, 2186)$ to point A, that is, a *discontinuous transition* to isothermal Couette flow. In [15], such abrupt transition from one regime to another is observed for $Pr = 1$, where convection is studied in a region between two horizontal rotating cylinders; but, if the Rayleigh number exceeds some value, hysteresis effects are observed, that is, the characteristics of the flow depend considerably on the direction in which the parameters vary.

Fig. 1(b) shows a typical curve corresponding to $n = 2$, $Pr = 5$, $\eta = 0.4$, $\alpha = 2$, $Ta = 0$. This curve has two minima (points B and C) and one cusp. In general, the curve can have a cusp or a closed loop (or even several closed loops; see [9] and [10]). For small Prandtl numbers, the neutral stability curves have the typical shape shown in Fig. 1(a). These curves undergo a continuous deformation as Pr grows. Such a resulting curve is shown in Fig. 1(b) for $Pr = 5$.

The right minimum (point B) corresponds to thermal-shear instability, that is, instability due to the interaction of convective flows moving in opposite directions. Computations show that the position of B does not change considerably as Pr varies; hence, this minimum can be associated with thermal-shear instability. However, a second minimum (point C) appears as the result of deformation of the neutral curve; in this case C corresponds to the absolute minimum of the neutral stability curve. This minimum is shifted to the region of smaller k and Gr_c decreases. Perturbations in the form of thermal running waves moving downstream with high phase velocity correspond to the lower part of the neutral curve in Fig. 1(b); hence, this minimum can be associated with thermal-buoyant instability. Therefore, depending on Pr, two kinds of instability can occur: thermal-shear instability and the instability in the form of thermal running waves (thermal-buoyant instability).

If Ta changes, our computations show that, in some cases, a deformation of a neutral curve takes place in the direction indicated by the arrows in Fig. 1(b). This explains "jump" transitions to different wave numbers (see [12] and the results given below). Physically,

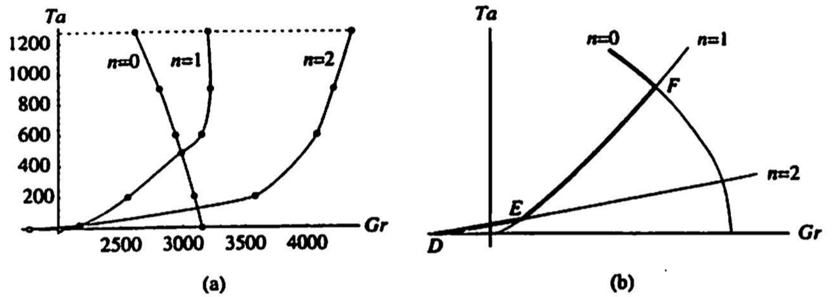

FIGURE 2. (a) Stability diagrams for $Pr = 5$, $\eta = 0.4$, $n = 0, 1, 2$ and $\alpha = 2$.
(b) Not-to-scale magnification of lower left-hand part of (a).

this means that the rotating cells change vertical size. Note that this phenomenon was also observed for Couette flow with radial heating [3].

In Fig. 2, the case $Pr = 5$ is characterized by the concurrence of different modes as α and Ta change. In the presence of strongly nonuniform heat sources ($\alpha = 2$) there is a sequence of transitions from the most unstable mode ($n = 2$) for $Ta = 0$, to the axisymmetric mode for large Ta (Fig. 2(a)).

To determine the stability boundary we can draw horizontal lines Ta = const and find the point of intersection, closest to the Ta-axis, of these lines and the neutral curves with $n = 0$, 1 and 2. Therefore the stability boundary of the flow for $Ta \in (0, 23)$ (the curve DE) is determined by the spiral mode $n = 2$, and $Gr_c \in (1774, 2168)$. Then there is a transition to the spiral mode $n = 1$ (curve EF); this mode determines the stability boundary in the rectangle $(2168 < Gr < 2985) \times (23 < Ta < 484)$. Our computations show that the values of the wave number, k, and the dimensionless phase velocity, $c = \Im(\lambda)/(kGrv_{0\,\max})$, where $v_{0\,\max}$ is the maximum value of the vertical component of the base velocity, are almost the same at the point E, namely, $k_2 = 0.52$, $c_2 = -0.78$, $k_1 = 0.50$, $c_1 = -0.80$ (the subscripts 2 and 1 correspond to the cases $n = 2$ and $n = 1$, respectively). If Ta grows, the asymmetric instability of the mode $n = 1$ changes to the axisymmetric one ($n = 0$) at the point $(Gr, Ta) = (2985, 484)$. But, in this case, there is a jump to the axisymmetric mode, and both k and c change from $k_1 = 0.37$, $c_1 = -0.70$ to $k_0 = 1.65$, $c_0 = 0.063$, respectively. Thus. if $Ta > 484$, spiral instability changes to axisymmetric stability; the vertical sizes of the new rotating cells suddently shrink to a vertical size close to the size of the cells for the case of isothermal Taylor vortices. Moreover, the phase velocity decreases considerably and rotating cells move slightly in the positive z-direction.

It can be seen from Figs. 1 and 2, that, in all cases, the axisymmetric mode ($n = 0$) corresponds to flow destabilization (Ta decreases as Gr increases), and, on the other hand, in many cases. the asymmetric modes ($n = 1, 2$) correspond to flow stabilization (Ta increases as Gr increases). For small Prandtl numbers, stabilization will not be observed experimentally: the values of Ta_c for asymmetric modes are higher then those for axisymmetric modes. However, an increase of the Prandtl number ($Pr = 5$ and $Pr = 20$ in our computations) leads to the appearance of regions of stabilization (see, for example, curve DF in Fig. 2(b)) which can be observed experimentally. Moreover, as previously mentioned, stabilization of Couette flow by radial heating was observed experimentally [7].

We have also shown that the region of stabilization decreases as Pr increases. This fact was also found in [12] but for very wide gaps ($\eta = 0.1$).

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DUAL FOR A LINEAR FRACTIONAL PROGRAM WITH VARIABLE COEFFICIENTS

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Presented by L. A. Lorch, F.R.S.C.

ABSTRACT. In this paper a maximization linear fractional program when activity vectors belong to bounded polyhedral convex sets, is taken as a primal program. These bounded polyhedral convex sets are called activity sets. An inexact linear program is proposed as its Dual Problem. A special case, when activity sets are parallelepipeds, is also considered.

Key Words: linear fractional programming, inexact linear programming, duality

AMS Classification: 90C32 (Fractional Programming)

1. INTRODUCTION

The linear fractional programs of the form

$$\text{Maximize } [(Cx + c_0)/(Dx + d_0) \mid Ax \le b,\ x \ge 0]$$

have wide applications in many management situations, production planning and scheduling, education administration, and the analysis of financial enterprises and undertakings. In the usual applications it is assumed that the coefficients a_{ij}, c_j, d_j and b_i are exactly known. Unfortunately, this frequently is not an accurate assumption. In this paper a linear fractional program is considered in which there is some freedom in the choice of coefficients of an activity.

The linear fractional program with variable coefficients is stated as

$$\text{Maximize } \frac{\sum_{j=1}^{n} c_j x_j + c_o}{\sum_{j=1}^{n} d_j x_j + d_0}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, \ldots m \tag{1}$$

or

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \le b$$
$$x_j \ge 0.$$

where each column vector $a_j \in \beta_j$, and the β_j's are bounded polyhedral convex sets in R^m: $j = 1, 2, \dots, n$.

When the objective function of (1) is linear, Dantzig [3] calls it a generalized linear program. An optimizing approach for solving (1) has been proposed by Chadha [5]. The intention here is to present an inexact linear programming approach to solve problems of type (1). First, a dual of problem (1) is proposed; this dual, which turns out to be a linear program, can be solved using the same computational methods as suggested by Thente [2]. The derivation of the dual problem for a linear fractional objective function with linear inequality constraints, and the proof of the duality theorem for the fixed coefficient version of the problem precede this paper in [4] and [6].

2. THE PRIMAL AND DUAL PROGRAMS

In matrix form the linear fractional program with variable coefficients (primal program), can be stated as

$$\text{Maximize } \frac{Cx + c_0}{Dx + d_0} \tag{2}$$

subject to

$$Ax \le b$$
$$x \ge 0,$$

where $Ax \le b$ for some $a_j \in \beta_j$. Here C and D are row vectors with n components, b and x are column vectors with m and n components respectively, c_0 and d_0 are scalars, and $A = (a_{ij})$ is an m by n matrix. An optimal solution to (2) will be a pair $(\hat{A}, \hat{x})$ such the $\hat{A}\hat{x} \le b$, $\hat{x} \ge 0$, and $\dfrac{C\hat{x} + c_0}{D\hat{x} + d_0} \ge \dfrac{Cx + c_0}{Dx + d_0}$ for all other pairs (A, x) such that $Ax \le b$ and $x \ge 0$.

The dual program of (1) and (2) is the following inexact linear program

Minimize z

subject to

$$yA + zD \geq C \tag{3}$$

$$-yb + zd_0 \geq c_0$$

$y \geq 0 \in R^m$, $z \in R$ for all $a_j \in \beta_j$.

The following theorem establishes the desired duality relationship between the two programs.

Theorem 2.1: If either the primal or the dual progam has a finite optimal solution, then the other program has a finite optimal solution and their extreme values are the same. If either program is unbounded, then the other program has no feasible solution.

Proof: Let $(a_j^1, a_j^2, \ldots, a_j^{k_j})$ be the extreme points of the bounded convex polyhedral convex set β_j. Any point in β_j can be written as a convex combination of its extreme points. Let $x_j^i = \lambda_i x_j$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k_j} \lambda_i = 1$. For any column vector $a_j \in \beta_j$, $a_j x_j$ can be written as

$$a_j x_j = \sum_{i=1}^{k_j} \lambda_i a_j^i x_j = \sum_{i=1}^{k_j} a_j^i x_j^i \tag{4}$$

Thus $x_j = \sum_{i=1}^{k_j} x_j^i$, and, hence, program (1) or (2) can be written as

$$\text{Maximize} \quad \frac{c_1 \sum_{i=1}^{k_1} x_1^i + c_2 \sum_{i=1}^{k_2} x_2^i + \ldots + c_n \sum_{i=1}^{k_n} x_n^i + c_0}{d_1 \sum_{i=1}^{k_1} x_1^i + d_2 \sum_{i=1}^{k_2} x_2^i + \ldots + d_n \sum_{i=1}^{k_n} x_n^i + d_0}$$

subject to

$$\sum_{i=1}^{k_1} a_1^i x_1^i + \sum_{i=1}^{k_2} a_2^i x_2^i + \ldots + \sum_{i=1}^{k_n} a_n^i x_n^i \leq b \tag{5}$$

$$x_j^i \geq 0,\ i = 1, 2, \ldots, k_j;\ j = 1, 2, \ldots, n$$

The dual of (5), as proposed by Chadha [6] or Kaska [4] is to

Minimize z
subject to

$$y\,a_j^i + zd_j \geq c_j, \; i = 1, 2, \ldots, k_j; \; j = 1, 2, \ldots, n.$$
$$-yb + zd_0 \geq c_0 \tag{6}$$
$$y \geq 0 \in R^m, \; z \in R.$$

Since $(a_j^1, a_j^2, \ldots, a_j^{k_j})$ spans the convex set β_j, the first set of constraints in (6) can be written as

$$yA + zD \geq C \text{ for all } a_j \in \beta_j, \; j = 1, 2, \ldots, n.$$

This program (6) is the same as (3). The duality results of linear fractional programming complete the proof.

Corollary: In case the activity sets β_j are parallelepipeds in R^m, then the dual of

$$\text{Maximize } \frac{Cx + c_0}{Dx + d_0}$$
subject to
$$Ax = b, \text{ where } Ax = b \text{ for some } a_j \in \beta_j \tag{7}$$
$$x \geq 0$$

is given by

Minimize $z_1 - z_2$
subject to

$$y_1\underline{M} - y_2\overline{M} \geq E \tag{8}$$
$$z_1, z_2, y_1, y_2 \geq 0.$$

where $\underline{M}$, $\overline{M}$, y_1, y_2, and E are defined in the proof that follows.

Proof: As β_j is a parallelpiped in R^m, each technological coefficient a_{ij} is bounded by $\underline{a}_{ij} < a_{ij} < \overline{a}_{ij}$. For this special case, the dual of (7) is given by

Minimize z

subject to

$$uA + zD \geq C, \text{ for all } a_j \in \beta_j$$

$$-ub + zd_o \geq c_o$$

$$u \in R^m, \; z \in R.$$

Setting $u = u_1 - u_2$, and $z = z_1 - z_2$, the above dual program reduces to

Minimize $z_1 - z_2$

subject to

$$(u_1 - u_2)A + (z_1 - z_2)D \geq C; \text{ for all } a_j \in \beta_j$$

$$-(u_1 - u_2)b + (z_1 - z_2)d_o \geq c_o$$

$$u_1, u_2, z_1, z_2 \geq 0.$$

This is the same as to

Minimize $z_1 - z_2$

subject to

$$(u_1, z_1, u_2, z_2) \begin{pmatrix} A & -b \\ D & d_o \\ -A & b \\ -D & -d_c \end{pmatrix} \geq (C, c_o) \text{ for all } a_j \in \beta_j$$

$$u_1, u_2, z_1, z_2 \geq 0$$

Let $q_j = (a_j, d_j)$ and $q_{n+1} = (-b, d_o)$.

By taking

$$M = \begin{pmatrix} A & -b \\ D & d_o \end{pmatrix}$$

and by letting m_j be the jth column of M, program (9) reduces to

Minimize $z_1 - z_2$

subject to

$$(y_1, y_2)\begin{pmatrix} M \\ -M \end{pmatrix} \geq E \text{ for all } m_j, \; j = 1, 2, \ldots, n+1 \qquad (10)$$

$$y_1, y_2, z_1, z_2 \geq 0$$

Here $y_1 = (u_1, z_1)$, $y_2 = (u_2, z_2)$ and $E = (C, c_o)$.

By taking support functionals for q_j and by utilizing the transported form of an inexact linear programming theorem of Soyster [1], program (10) becomes

$$\begin{aligned} &\text{Minimize} \quad z_1 - z_2 \\ &\text{subject to} \\ &\qquad (y_1, y_2) \begin{pmatrix} \underline{M} \\ -\overline{M} \end{pmatrix} \geq E \\ &\qquad y_1, y_2, z_1, z_2 \geq 0. \end{aligned} \tag{11}$$

Here $\underline{M}$ and $\overline{M}$ denote matrices with columns $\underline{a}_j^1$ and $\overline{a}_j^1$ given by

$$\underline{a}_j^1 = \begin{pmatrix} \underline{a}_j \\ d_j \end{pmatrix} \text{ and } \overline{a}_j^1 = \begin{pmatrix} \overline{a}_j \\ d_j \end{pmatrix}.$$

Thus, (11) is the same as (8).

One observes here that if program (3) is used, one must consider all 2^m extreme points of each β_j with the corresonding number of variables or constraints. However, in (8), one only needs to consider the largest and smallest extreme points (corners) of each q_j. Programs of the type (8) are called inexact linear programs and can be solved with the help of the algorithm suggested by Thuente [2].

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ON ANALYTIC SOLUTIONS OF THE EQUATION OF INVARIANT CURVES

Si Jianguo

Presented by J. Aczel, F.R.S.C.

ABSTRACT

In this paper we apply the method of majorant series to find an existence theorem on analytic solutions of the equation

$$\varphi(z+\varphi(z))=\rho(\varphi(z)), \quad z\in\mathbb{C}.$$

The discussion of the functional equation

$$(*) \qquad \varphi(x+\varphi(x))=\rho(\varphi(x))$$

arises directly from the theory of invariant curves. In 1991 W. Jarczyk [1] studied equation (*). He generalized some previous conclusions of J. Dhombres (see [1] and the references cited therein). The purpose of this paper is to prove a theorem concerning the existence of analytic solutions of equation (*) in the complex field. Namely, we consider the equation

$$(1) \qquad \varphi(z+\varphi(z))=\rho(\varphi(z)),$$

where $\varphi(z)$ is the unknown function and $\rho(z)$ is a given complex-valued function of a complex variable.

Assume that $\rho(z)$ is analytic on a neighbourhood of zero, $\rho(0)=0$ and denote by α the derivative $\rho'(0)$. Consider the following three hypotheses regarding α:

(i) $\quad |\alpha|\geqslant\frac{1+\sqrt{5}}{2}$;

(ii) $\quad 0<|\alpha|<1$;

(iii) $\quad |\alpha|=1$, α is not a root of unity, and there exists a positive constant k such that $\log|\alpha^n-1|^{-1}\leqslant k\log n$, $\quad n=2,3,\ldots$

Our main result reads as follows:

THEOREM. Assume that one of the conditions (i)-(iii) is fulfilled. Then equation (1) has an analytic solution on a neighbourhood of zero.

Observe that, if $f(z)$ is an analytic solution of the equation

$$(2) \qquad f(\alpha^2 z)-f(\alpha z)=\rho[f(\alpha z)-f(z)]$$

and $f'(0)\neq 0$, then the formula

$$\varphi(z)=f[\alpha f^{-1}(z)]-z$$

defines an analytic function satisfying equation (1) on a neighbourhood of the origin. Thus the Theorem follows immediately from the lemma below.

LEMMA 1. Assume that one of the conditions (i)-(iii) is fulfilled. For any $\eta\in\mathbb{C}$ in the cases (i), (ii) and for $\eta=1$ in the case (iii), equation (2) has an analytic solution $f(z)$ on a neighbourhood of zero such that $f(0)=0$ *and* $f'(0)=\eta$.

In the proof the following fact ([2], Ch. VI) will be useful.

LEMMA 2. Assume that condition (iii) is fulfilled. Then there is a positive δ such that $|\alpha^n-1|^{-1}<(2n)^\delta$, $n=1,2,\ldots$ If, moreover,

$$\begin{cases} d_1=1, \\ d_n=|\alpha^n-1|^{-1}\max(d_{l_1}\cdots d_{l_t}), \quad n=2,3,\ldots, \end{cases}$$

(max being taken over all decompositions $n=l_1+\ldots+l_t$, where $0<l_1\leqslant l_2\leqslant\ldots\leqslant l_t$ are integers and $t\geqslant 2$) then $d_n\leqslant N^{n-1}n^{-2\delta}$, $n=1,2,\ldots$, where $N=2^{5\delta+1}$.

PROOF OF LEMMA 1. Fix an $\eta\in\mathbf{C}$. If $\eta=0$ then the zero function satisfies the assertion. So assume that $\eta\neq 0$ and, in addition, $\eta=1$ in the case (iii). Let

(3) $$\rho(z)=\sum_{n=1}^{\infty}c_nz^n, \quad c_1=\alpha.$$

Since $\rho(z)$ is analytic on a neighbourhood of zero, there exists a positive β such that $|c_n|\leqslant\beta^{n-1}$ for $n=2,3,\ldots$ Observe that (2) is invariant with respect to the transformation $f(z)=\tilde{f}(\beta z)/\beta$ and $\rho(z)=\tilde{\rho}(\beta z)/\beta$. Consequently, in the sequel we may assume that

(4) $$|c_n|\leqslant 1, \quad n=1,2,\ldots$$

Let

(5) $$f(z)=\sum_{n=1}^{\infty}b_nz^n$$

be the expansion of a formal solution $f(z)$ of equation (2). Inserting (3) and (5) into (2) and equating the coefficients we obtain the relations

(6) $$\begin{cases} (\alpha^2-(1+c_1)\alpha+c_1)b_1=0, \\ (\alpha^{2n}-(1+c_1)\alpha^n+c_1)b_n=\displaystyle\sum_{\substack{l_1+\ldots+l_t=n \\ t=2,3,\ldots,n}} c_t\prod_{k=1}^{t}(\alpha^{l_k}-1)b_{l_k}, \quad n=2,3,\ldots \end{cases}$$

Noting that $c_1=\alpha$ we see that $\alpha^2-(1+c_1)\alpha+c_1=0$. Hence we may choose $b_1=\eta$ and by (6) we get the relations

(7) $$\begin{cases} a_1=\eta, \\ (\alpha^n-\alpha)a_n=\displaystyle\sum_{\substack{l_1+\ldots+l_t=n \\ t=2,3,\ldots,n}} c_ta_{l_1}\cdots a_{l_t}, \quad n=2,3,\ldots, \end{cases}$$

where $a_1=b_1=\eta$ and

(8) $$a_n=(\alpha^n-1)b_n, \quad n=2,3,\ldots$$

The numbers a_n, $n=1,2,\ldots$, are uniquely determined by (7). We shall prove the convergence of the series $\sum_{n=1}^{\infty}a_nz^n$ in a neighbourhood of zero.

Put $q=\gamma=1$ if either (i) or (iii) is fulfilled. Otherwise $0<|\alpha|<1$, so taking $\gamma\in(0,|\alpha|)$ we can find a positive integer q such that $|\alpha|^n\leqslant|\alpha|-\gamma$ for every $n\geqslant q$. Let

$$R(z,\omega)=\omega-|\eta|z-\sum_{n=2}^{q}|a_n|z^n-\frac{1}{\gamma}\frac{\omega^{q+1}}{1-\omega}$$

for z and ω from a neighbourhood of zero. Since $R(0,0)=0$, $R'_\omega(0,0)=1\neq 0$ and $R'_z(0,0)=-|\eta|$, there exists a function $w(z)$, analytic on a neighbourhood of zero, such that $w(0)=0$, $w'(0)=|\eta|$ and satisfying the equality $R(z,w(z))=0$, that is,

$$w(z)=|\eta|z+\sum_{n=2}^{q}|a_n|z^n+\frac{1}{\gamma}\frac{[w(z)]^{q+1}}{1-w(z)}.$$

Hence, writing $w(z)=|\eta|z+\sum_{n=2}^{\infty}u_n z^n$, we obtain the recurrence formula

$$u_n=\begin{cases}|a_n|, & 2\leqslant n\leqslant q,\\ \dfrac{1}{\gamma}\displaystyle\sum_{\substack{l_1+\dots+l_t=n\\ t=2,3,\dots,n}}u_{l_1}\cdots u_{l_t}, & n>q.\end{cases}\tag{9}$$

In the case (i) $|\alpha^n-\alpha|\geqslant|\alpha|^n-|\alpha|\geqslant|\alpha|^2-|\alpha|\geqslant 1$ for $n=2,3,\dots$, whereas in the case (ii) $|\alpha^n-\alpha|\geqslant|\alpha|-|\alpha|^n>\gamma$ for $n=q+1$, $q+2,\dots$ Hence, using induction and the inequality (4), we infer that $|a_n|\leqslant u_n$, $n=1,2,\dots$ Thus the series $\sum_{n=1}^{\infty}a_n z^n$ and, consequently (cf. (8)), the series $\sum_{n=1}^{\infty}b_n z^n$ are convergent in a neighbourhood of zero.

Now consider the case (iii). Since the series $\sum_{n=1}^{\infty}u_n z^n$ converges in a neighbourhood of the origin, there is a positive A such that $u_n\leqslant A^n$ for $n=1,2,\dots$ On account of (7) and (9) we easily obtain, by induction, the inequality

$$|a_n|\leqslant u_n d_n,\quad n=1,2,\dots,$$

where the d_n are defined in Lemma 2. Hence, making use of that lemma, we infer the existence of positive numbers δ and N such that $|a_n|\leqslant A^n N^{n-1} n^{-2\delta}$, $n=1,2,\dots$, *and*

$$|b_n|=|\alpha^n-\alpha|^{-1}|a_n|\leqslant(2(n-1))^{\delta}A^n N^{n-1}n^{-2\delta},\quad n=1,2,\dots$$

Thus the series $\sum_{n=1}^{\infty}b_n z^n$ converges in a neighbourhood of zero also in this case. This completes the proof.

REMARK. Clearly, assuming one of the conditions (i)-(iii), we have also proved the uniqueness of the local solutions of equation (2): If $f_1(z)$ and $f_2(z)$ are solutions of (2) analytic on neighbourhoods U_1 and U_2 of zero, respectively, then $f_1(z)=f_2(z)$ for $z\in U_1\cap U_2$.

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Non Commutative Versions of the Burau Representation

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Presented by K. Murasugi, F.R.S.C.

ABSTRACT. We discuss natural extensions of the classical Burau representation of braids, and show that they are actually not more general than the original version.

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The Burau representation is the oldest and presumably the simplest linear representation of the braid groups (see [1], [3], [4], [5], [6], [8], [10]). It appears very naturally when one tries to construct matrices which satisfy the braid relations. The components of the Burau matrices are Laurent polynomials with integer coefficients involving one single variable. A two variable version also exists, but it is well known that it is not more powerful than the one variable initial version. On the other hand considering *non-commutative* polynomials involving variables that are not supposed to mutually commute has proved in several branches to be a powerful tool that is not reducible to the commutative case (see for instance [7] for theory of languages, [2] or [9] for free Lie algebras). It is therefore very natural to consider possible non-commutative versions of the classical braid representations that involve polynomials.

The aim of this note is to investigate the case of Burau representation. Using an elementary approach, one obtains a non-commutative version of the Burau representation that involves n non-commutative variables for representing n strand braids, as was noted by Ch. Reutenauer (unpublished). We show here that this representation is, in some sense, the only one that can be obtained in this way. Our main observation is then that, like in the case of the above mentioned two variable version, the latter representation is *not* more general than the original Burau representation: for every braid, the image under the new representation can be retrieved from its standard Burau image.

In the sequel we denote by B_n the n strand braid group: B_n admits $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ submitted to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for every i, j satisfying $|i-j| \geq 2$. Assume that $\Sigma_1, \ldots, \Sigma_{n-1}$ are matrices in some fixed linear group. One obtains a linear representation of B_n by mapping σ_i to Σ_i for $i = 1, \ldots, n-1$ if and only if the matrices Σ_i satisfy the counterpart of the above relations, namely

$$\begin{cases} \Sigma_i\Sigma_{i+1}\Sigma_i = \Sigma_{i+1}\Sigma_i\Sigma_{i+1} & (E_1) \\ \Sigma_i\Sigma_j = \Sigma_j\Sigma_i \qquad \text{for } |i-j| \geq 2 & (E_2) \end{cases}$$

We consider in the sequel the question of directly solving (E_1) considered as a family of equations for the entries of the matrices Σ_i, and we restrict to the case where the matrix Σ_i is trivial outside a 2×2-block lying on rows and columns i and $i+1$. In this way Equations (E_2) are automatically solved. Let us first consider the case $n = 3$: then the matrices have the form

$$\Sigma_1 = \begin{pmatrix} a & b & 0 \\ c & 1-d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\delta & \gamma \\ 0 & \beta & \alpha \end{pmatrix}.$$

We suppose that the components live in some group algebra of the form $\mathbf{Z}[G]$, and that the variables a, ..., δ have inverses. With the previous notations, the relation $\Sigma_1\Sigma_2\Sigma_1 = \Sigma_2\Sigma_1\Sigma_2$ is equivalent to the following system

$$\begin{cases} a(1-a) = b(1-\delta)c & (1) \\ ab = b(1-\delta)d & (2) \\ ca = d(1-\delta)c & (3) \\ cb + d(d-1) - d\delta d = \gamma\beta + \delta(\delta-1) - \delta d\delta & (4) \\ \gamma\alpha = \delta(1-d)\gamma & (5) \\ \alpha\beta = \beta(1-d)\delta & (6) \\ \alpha(1-\alpha) = \beta(1-d)\gamma & (7) \end{cases}$$

This system comprises two groups of three equations respectively that involve respectively the components of Σ_1 (and δ) and the components of Σ_2 (and d), and a 'linking' equation (4).

Lemma 1. *The 5-uples (a, b, c, d, δ) for which Σ_1 has an inverse and Equations (1) to (3) are satisfied are exactly the following ones:*

- *type (i):* $(0, b, c, d, 1)$ *with* $cb \neq 0$;
- *type (ii):* $(1 - bd^{-1}c, b, c, d, 1 - d^{-1} + d^{-1}cbd^{-1})$ *with* $d \neq 0$ *and* $cb \neq d(1-d)$;
- *type (iii):* $(1, 0, 0, d, \delta)$ *with* $d \neq 1$.

The proof is an easy computation. Observe that the above types intersect and that the constraints are compatible in this case. In the sequel we exclude type (iii) solutions which only give rise to trivial representations (the matrices Σ_1 and Σ_2 have trivial first row and column so that the associated representation of B_3 is only 2-dimensional). Resolution of Equations (5) to (7) leads to similar values for the components of Σ_2 and d, and it remains to amalgamate the possible values in order to obtain a representation of B_3.

Case 1: "type (i) + type (i)". Then δ and d are necessarily equal to 1. One obtains the values

$$\Sigma_1 = \begin{pmatrix} 0 & b & 0 \\ c & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \beta & 0 \end{pmatrix},$$

with the constraints $cb \neq 0$, $\gamma\beta \neq 0$. The link equation (4) then reduces to $cb = \gamma\beta$.

Case 2: "type (ii) + type (i)". Then d is 1, and definition of type (ii) then gives $\delta = cb$. So one obtains the values

$$\Sigma_1 = \begin{pmatrix} 1-bc & b & 0 \\ c & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-cb & \gamma \\ 0 & \beta & 0 \end{pmatrix}.$$

The constraints are again $cb \neq 0$, $\gamma\beta \neq 0$, and the link equation (4) still reduces to $cb = \gamma\beta$.

Case 3: "type (i) + type (ii)". This case is clearly similar to Case 2 up to a symmetry reversing both the orders of the rows and of the columns of the matrices.

Case 4: "type (ii) + type (ii)". The values are

$$\Sigma_1 = \begin{pmatrix} 1-bd^{-1}c & b & 0 \\ c & 1-d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\delta & \gamma \\ 0 & \beta & 1-\beta\delta^{-1}\gamma \end{pmatrix},$$

with constraints that can be written as

$$\begin{cases} cb \neq d(d-1), \qquad \gamma\beta \neq \delta(\delta-1) \\ \delta + \delta^{-1}\gamma\beta = 1 + d\delta = d + cbd^{-1} \end{cases} \tag{8}$$

One verifies that, if (8) is satisfied, then the last equation (4) is automatically satisfied. It is easy to eliminate say δ from (8), thus obtaining a unique relation linking cb, $\gamma\beta$ and d, namely

$$\begin{aligned} \gamma\beta = 2d^{-1}cb + cbd^{-1} - 3d^{-1}cbd^{-1} + d^{-2}cbd^{-1} + d^{-1}cbd^{-1}cbd^{-1} \\ - d^{-1}cbd^{-2}cbd^{-1} - d^{-2} + 2d^{-1} - 2 + d \end{aligned} \tag{9}$$

(which however reduces to $\gamma\beta = cb$ for $d = 1$, *i.e.*, when Case 4 intersects Case 2).

It seems uneasy to completely solve Equation (9), and we leave the description of the associated representations pending. But for the three other cases, the link equation $cb = \gamma\beta$ admits a most general solution of the form

$$b = x, \quad c = yz, \quad \beta = zx, \quad \gamma = y,$$

which thus gives rise to well defined representations of B_3.

The generalization to a higher number of strands is obvious. The general solution is obtained by taking for the matrices Σ_i type (i) or (ii) matrices (conveniently adapted to the required number of rows and columns) so that every one is linked to its neighbours as indicated above. If M is a 2×2 matrix, we denote by $e_{i,n}(M)$ the $n \times n$ matrix obtained from M by adding $i-1$ initial rows and columns from the identity matrix, and similarly $n-i-1$ final rows and columns. Then the most general solutions (in the sense that any other solution is the image of the present ones under a homomorphism) corresponding to the type decompositions "(i), (i), ..., (i)" and "(ii), (i), ..., (i)" are as follows:

Proposition 2. *Let F_n be the free group generated by $x_1, \ldots, x_n$. Then the mappings*

$$\widehat{\rho} : \sigma_i \mapsto e_{i,n}\begin{pmatrix} 0 & x_i \\ x_{i+1}\ldots x_n x_1 \ldots x_{i-1} & 0 \end{pmatrix},$$

$$\widetilde{\rho} : \sigma_i \mapsto e_{i,n}\begin{pmatrix} 1 - x_i x_{i+1}\ldots x_n x_1 \ldots x_{i-1} & x_i \\ x_{i+1}\ldots x_n x_1 \ldots x_{i-1} & 0 \end{pmatrix}$$

induce linear representations of the braid group B_n into $\mathrm{GL}_n(\mathbb{Z}[F_n])$.

We see that $\widetilde{\rho}$ extends the classical Burau representation, which is obtained for $x_1 = \dots = x_{n-1} = t$, $x_n = t^{-n+2}$, and that $\widehat{\rho}$ extends the "permutation" representation which associates to every braid the permutation matrix of its image under the canonical projection of B_n onto the symmetric group S_n (which is itself the particular case of Burau representation obtained for $t = 1$).

Now the natural question is as to whether the representations $\widetilde{\rho}$ and $\widehat{\rho}$ really extend the classical Burau representation. In other words the question is as to whether the additional information provided by the order of the noncommutative variables is meaningful. The answer is negative.

Lemma 3. *Let G be any group and, assume that $\vec{w}$ is a fixed n-uple in G^n. Define a mapping $\Theta_{\vec{w}}$ of the $n \times n$ matrices with entries in $\mathbb{Z}[G]$ into themselves by*

$$c_{i,j}(\Theta_{\vec{w}}(M)) = w_i c_{i,j}(M) w_j^{-1}$$

(where $c_{i,j}(M)$ denotes the i,j-component of M). Then $\Theta_{\vec{w}}$ is a ring homomorphism.

Proof. Obvious from the definition of matrix multiplication:

$$\begin{aligned} c_{i,j}(\Theta_{\vec{w}}(M)\Theta_{\vec{w}}(M')) &= \sum_k w_i c_{i,k}(M) w_k^{-1} w_k c_{k,j}(M') w_j^{-1} \\ &= c_{i,j}(\Theta_{\vec{w}}(MM')). \end{aligned}$$

(Applying $\Theta_{\vec{w}}$ is conjugating with the diagonal matrix associated with $\vec{w}$.) ∎

The result is now that the value of $\widetilde{\rho}$ for a given braid is obtained from its Burau image by substituting $x_1 \dots x_n$ for the variable t, and then applying a convenient homomorphism $\Theta_{\vec{w}}$.

Proposition 4. *The representation $\widetilde{\rho}$ is connected to the classical Burau representation ρ by the formula*

$$\widetilde{\rho} = \Theta_{\vec{w}} \circ \rho_{(t = x_1 \dots x_n)},$$

where $\vec{w}$ is defined by $w_1 = 1$ and $w_{i+1} = x_{i+1} \dots x_n x_1 \dots x_{i-1} w_i$ for $i \geq 1$.

Proof. By Lemma 3, it suffices to verify that the relation holds in the case of the matrices $\widetilde{\rho}(\sigma_i)$, which is elementary. ∎

A similar argument shows that the representation $\widehat{\rho}$ can be constructed from the permutation representation. So the conclusion of this note is that, as far as one restricts to the extensions of Burau representations (a framework that excludes type (ii) + (ii) representations which we have left uncompletely studied), the introduction of noncommutative variables does not strengthen the power of the representation.

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