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Fibre techniques in Nielsen Periodic point theory on nil and solvmanifolds

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Presented by S. Halperin, F.R.S.C.

Abstract

In this announcement we outline results and methods for evaluating the Nielsen type numbers $N\Phi_n(f)$ and $NP_n(f)$ for self maps f of nilmanifolds and solvmanifolds (which includes the Klein Bottle). Through the use of fibre space techniques we relate these numbers to the various $N(f^m)$ for $m|n$.

In this announcement we state a number of theorems that allow us to calculate the Nielsen type numbers $N\Phi_n(f)$ and $NP_n(f)$ (see [J,HPY,HY]) for self maps f on nilmanifolds (homogeneous spaces of nilpotent Lie groups), and solvmanifolds (homogeneous spaces of solvable Lie groups). Let $M\Phi_n(f)$ denote the least number of periodic points of all periods less than or equal to n for any map g homotopic to f , and let $MP_n(f)$ denote the least number of periodic points of period exactly n of any map g homotopic to f . The numbers $N\Phi_n(f)$ and $NP_n(f)$ are homotopy invariants of f which provide lower bounds for $M\Phi_n(f)$ and $MP_n(f)$ respectively.

It is a mistake (made for example in [Halp1,Halp2]) to think that $N\Phi_n(f)$ always coincides with $N(f^n)$ the ordinary Nielsen number of the n th iterate f^n of f (see example 6 and also the introduction to [HY] for a number of inadequate candidates for $N\Phi_n(f)$). An oversimplification that can be made for $NP_n(f)$ is to express it from Möbius inversion in terms of the $N(f^m)$ (see theorem 1). However, there are cases when these oversimplifications do give the correct answers. For example, it was demonstrated in [HPY] and [HY] that they hold for tori when $L(f) \neq 0$. The proofs there made heavy use of the commutativity of the fundamental groups of the spaces involved. In this

announcement we indicate two types of results. In the first using fibration techniques, we extend the results given in [HPY] and [HY] for tori to the highly non commutative situation of nilmanifolds and solvmanifolds. Our second result, which also uses fibration techniques, introduces an addition formula for periodic points. This handles the complications which occur on solvmanifolds when some of the maps on the fibres have Lefschetz number zero (this is explained more fully below). Many examples are forth coming (see [HK]), though here because of space we merely indicate the type of application that can be made.

Our first two theorems are useful because canonical fibrations associated with nilmanifolds or solvmanifolds can, up to changes in homotopy, allow any map f to be realized as a fibre preserving map $(f, \tilde{f})$ (see below). Additionally these fibrations satisfy the "naïve conditions" which means that $N(f)$ (hence also $N(f^n)$) can be calculated either as a product $([A, FH])$, or as the sum over various $N(f_x)$ for $x \in \Phi(\tilde{f})$. ([KMc, Mc, HKW]). (Here f_x denotes f restricted to the fibre over x , and for any $f : X \rightarrow X$, $\Phi(f) = \{x \in X | f(x) = x\}$).

Theorem 1 *Let $f : X \rightarrow X$ be a self map of a nilmanifold in which $L(f^n) \neq 0$. Then $N\Phi_n(f) = N(f^n)$, and $NP_n(f) = \sum_{r \in P(n)} (-1)^{\#r} N(f^{n:r})$ where $p(n)$ denotes the set of prime divisors of n and $n : r = n \prod_{p \in r} p^{-1}$. $\square$*

Example 2 (Baby Nil) Let G be the topological group that is represented by matrices over the reals of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

The binary operation is matrix multiplication. Let Γ be the discrete subgroup consisting of those elements of G with integer entries, and let $\phi : G \rightarrow G$ be the homomorphism which takes x to $2x$, y to $3y$, and z to $5z$. Then G/Γ is a nilmanifold, and ϕ induces a map f of $N = G/\Gamma$. Our aim is to calculate $N\Phi_n(f)$, and $NP_n(f)$.

We note first that N can be imbedded in the fibre sequence $S^1 \times S^1 \rightarrow N \xrightarrow{p} S^1$ where p is induced by the projection on the "y factor". Note that f induces a self map $\tilde{f}$ of degree 3 on the base, and a map f_0 , whose matrix representation is

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

on the fibre over the base point 0. Now the n th iterate f^n of f induces a map whose matrix representation is A^n on the fibre, and is a map of degree 3^n on the base. By [FH], p is orientable and satisfies the conditions for the naïve product formula ($N(f^n) = N(f_0^n)N(\tilde{f}^n)$) so by theorem 1

$$N\Phi_n(f) = N(f^n) = |(1 - 2^n)(1 - 5^n)(1 - 3^n)|.$$

As an example, for $n = 12$, theorem 1 also says that

$$\begin{aligned} NP_{12}(f) &= N(f^{12}) - N(f^6) - N(f^4) + N(f^2) \\ &= 531, 310, 251, 012, 675, 840. \end{aligned}$$

The fibre decomposition of example 2 is a special case of a general canonical decomposition of nilmanifolds (called the Fadell-Husseini fibration see [FH]) which associates to each nilmanifold N , a fibration $T \hookrightarrow N \rightarrow B$ where T is a torus, and B is a nilmanifold of smaller dimension than N . Since in dimensions 1 and 2 the only nilmanifolds are tori, the proofs of known results for tori extend, by induction on the dimension of the nilmanifold, to proofs of similar results for all nilmanifolds.

A similar "splitting" occurs for all maps of solvmanifolds. For a solvmanifold S the canonical fibration is of the form $N \rightarrow S \rightarrow T$ where N is a nilmanifold and T a torus. Unlike the canonical fibrations for nilmanifolds, these Mostow fibrations (see [Mc]) are non orientable. In practice this means that when $x, y \in \Phi(\tilde{f})$ one can have $L(f_x) \neq L(f_y)$. We need this information in order to state the hypothesis for the next theorem which gives a result similar to theorem 1, for some maps of solvmanifolds.

Theorem 3 *Let S be a solvmanifold, and $N \rightarrow S \rightarrow T$ be a fibration sequence in which N is a nilmanifold and T a torus. Let $f : S \rightarrow S$ be a fibre preserving map with the property that $L(f^n) \neq 0$, and for each $y \in \Phi(\tilde{f}^n)$ the map $(f^n)_y : F_y \rightarrow F_y$ has $L((f^n)_y) \neq 0$ (see [KM]). Then the conclusion of theorem 1 holds.*

There are examples of theorem 3 that have the flavour of example 2. The examples are more general because the non orientability of the fibration forces one to use the naïve addition formulas ([HKW, Mc]) in place of the product formulas to make the needed calculations of the ordinary Nielsen numbers. While there are many such examples, they are in general difficult

to describe briefly so we refer the reader to [HK] for more details. The one notable exception to this statement is the Klein bottle which we touch on now in lieu of more complicated endeavors. We remark that C. You mentioned to the first author a number of years ago, that the calculation of $N(f^n)$ in the example below might be possible by fibre techniques.

Example 4 Let K^2 denote the Klein bottle. Then $\mathbb{R}^2$ is the universal cover and K^2 is the quotient $\mathbb{R}^2/\Gamma$ where Γ is the group of automorphisms on $\mathbb{R}^2$ generated by $A(x, y) = (x, y + 1)$ and $B(x, y) = (x + 1, -y)$. By defining $p : \mathbb{R}^2/\Gamma \rightarrow S^1$ to be projection on the first factor we get the standard fibration $S^1 \hookrightarrow K^2 \xrightarrow{p} S^1$ of the Klein bottle.

Given any pair of integers (r, q) for which r is odd, or r is even and $q = 0$, the correspondence $(s, t) \rightarrow (rs, qt) \bmod \mathbb{Z}^2$ induces a well defined, fibre preserving map $(f, \bar{f})$ on K^2 . Here $\bar{f}$ is the standard map of degree r and the restriction f_0 of f has degree q . (There are actually many non homotopic maps with this specification but the degrees of f_0 and $\bar{f}$ are the only considerations in determining $N(f)$.) Thus $\Phi(\bar{f}) = \{x_j : j = 0, 1, \dots, |r - 1| - 1\}$ consists of points equally spaced on the circle with $x_0 = \{0, 1\}$ in $S^1 = I/[0 \sim 1]$. Moreover each x_j is in its own Nielsen class or, in the language of [HKW], the set $\Phi(\bar{f})$ is a set of essential representatives for $\bar{f}$. The key relationship between the various f_x , which is of interest here, is that f_x has degree $(-1)^j q$ (see [HKW; 4.6]). Hence for r odd the naïve addition formula ($N(f) = \sum_{x_j \in \Phi(\bar{f})} N(f_{x_j})$ [Mc] or [HKW]) implies that for $f = (r, q)$ we have $f^n = (r^n, q^n)$ and $N(f^n) = \frac{|r^n - 1|}{2} (|1 - q^n| + |1 + q^n|) = |q^n(r^n - 1)|$. In addition if $q \neq \pm 1$, one has by 3 that $N\Phi_n(f) = N(f^n)$ for all n . "Möbius inversion gives the $NP_n(f)$.

Thus for any pair of integers (r, q) for which r is odd, or r is even and $q = 0$, our calculation of $N(f^n)$ agrees with that of [Halp1, Har] (obtained by entirely different means). For this same range of integer pairs if $q \neq \pm 1$, we also agree with [Halp1] that $N(f^n)$ is an appropriate lower bound for $M\Phi_n(f)$, but not as example 6 shows when $q = \pm 1$. This latter case is handled by the next theorem. If x is a periodic point in the fixed point set of f^n , then let $\text{per}(x)$ denote the smallest $m|n$ such that x is in the fixed point set of f^m .

Theorem 5 Let S be a solvmanifold, and $N \rightarrow S \rightarrow T$ be a fibration sequence with N a nilmanifold and T a torus. Let $f : S \rightarrow S$ be a fibre preserving map inducing $\bar{f} : T \rightarrow T$. If n and f are such that $N(f^n) \neq 0$, we may assume without loss of generality that $\bar{f}^n$ has exactly $N(\bar{f}^n)$ fixed points. Then

$$NP_n(f) = \sum_{b \in \Phi(f^n)} NP_{\text{per}(b)}((f^{\text{per}(b)})_b) \text{ and } N\Phi_n(f) = \sum_{m|n} NP_m(f). \quad \square$$

If the $(f^k)_b$ for $b \in \Phi(\bar{f}^k)$ have the property that $NP_m((f^k)_b) = 0$ for all but finitely many m (this occurs for example when the $(f^k)_b$ are periodic) then the formula of the first part of theorem 5 simplifies because many values for $\text{per}(b)$ do not need to be considered. This is illustrated below.

Example 6 *The Klein Bottle revisited.* We consider a map f of the Klein bottle of type $(r, 1)$ with $|r| > 1$ as in example 4. Note that r must be odd. We will calculate $NP_n(f)$ for $n = 2^k$ for all positive integers k . Recall that on S^1 , the map of degree 1 has all $NP_n = 0$, whereas the map of degree -1 has $NP_1 = 2$ and all other $NP_n = 0$ (see [HPY]). Thus for all $n > 1$ we have that $NP_n = 0$ and so in order to compute $NP_n(f)$ by theorem 5 we need only consider those $b \in \Phi(\bar{f}^n)$ with $\text{per}(b) = n$. Thus we need to know which of the x_j for $j = 0, 1, \dots, |r^n - 1| - 1$ of $\Phi(\bar{f}^n)$ are irreducible. Recall from [HPY;1.8] that on the circle x_j is reducible if the index j is divisible by $r_{m,n} = 1 + r^m + r^{2m} + \dots + r^{n-m}$ for some $m|n$. Now $r_{m,n}$ consists of $\frac{n}{m}$ terms all of which are odd since r is odd. With $n = 2^k$, $\frac{n}{m}$ is even for all $m|n$. That is, for such m , $r_{m,n}$ is multiplication by an even integer. Thus every reducible x_j has j even. Moreover if j is even and irreducible then since from example 4 the degree of $(f^n)_{x_j}$ is $(-1)^j q$, then $NP_1((f^n)_{x_j}) = 0$. So we need take the sum in theorem 5 over the odd integers only. Thus we have that

$$NP_n(f) = \sum_{j \text{ odd}} NP_1((f^n)_{x_j}) = \sum_{x_j \in \Phi(f^n)} NP_1((f^n)_{x_j}) = \sum_{x_j \in \Phi(f^n)} N((f^n)_{x_j}) = N(f^n)$$

The second step is because over the even indices $NP_1((f^n)_{x_j})$ is zero, the third by definition, and the fourth by the naïve addition formula [HKW,Mc]. We note that in this case $N(f^n) = N(\bar{f}^n) = |1 - r^n|$.

Thus for $n = 2^k$ we get the perhaps surprising contrast to theorems 1 and 3 that this time it is $NP_n(f)$ rather than $N\Phi_n(f)$ that is equal to $N(f^n)$.

Note also that for this example

$$N\Phi_n(f) = \sum_{m|n} N(f^m) = |1 - r| + \sum_{i=1}^k (r^{2^i} - 1).$$

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A General Variational Inequality With Application

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ABSTRACT. We present a general multivalued quasi-variational inequality involving a general class of multifunctions with convex as well as non-convex values. A typical application to a minimization problem along trajectories of a differential inclusion is discussed.

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The purpose of this note is to present a general existence theorem for multivalued variational inequalities involving upper semicontinuous multifunctions that are approachable - in the sense of the graph - by continuous single-valued functions. This class of multifunctions is very broad: it contains upper semicontinuous multifunctions with convex, or contractible, or ∞ -proximally connected values (see for instance [1], [2], and references therein). Thus Theorem 3 below unifies classical theorems for convex multifunctions (see for instance [6] and problem 8.8.B there), as well as more recent results for contractible ones ([8]); it also applies to multifunctions with non-contractible values. A typical application to a minimization problem of a cost function along trajectories of a differential inclusion is briefly discussed. A detailed treatment of generalized multivalued quasi-variational inequalities with more applications to variational inequalities, complementarity theory, and optimization problems ([3]) will be published elsewhere. For details concerning the types of spaces and multifunctions considered here, as well as the properties relating them, the reader is referred to [1], [2].

Definition 1. ([2]) Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces. A multifunction $\Phi : X \rightarrow \mathcal{P}(Y)$ is said to be approachable if and only if $\forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \Phi$ admits a continuous (U, V) -approximative selection, that is a continuous single-valued function $s : X \rightarrow Y$ verifying $s(x) \in V[\Phi(U[x])], \forall x \in X$.

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Let us mention that if X is a paracompact topological space equipped with a compatible uniformity $\mathcal{U}$, Y is a convex subset of a locally convex topological vector space, then every upper semicontinuous (u.s.c. for short) multifunction $\Phi : X \rightarrow \mathcal{P}(Y)$ with non-empty convex values is approachable. Also, if X and Y are two ANRs with X compact, then every u.s.c. multifunction $\Phi : X \rightarrow \mathcal{P}(Y)$ with compact contractible values is approachable. More generally, if X is an approximative absolute neighborhood extension space for compact spaces, and Y is a uniform space, then every u.s.c. multifunction $\Phi : X \rightarrow \mathcal{P}(Y)$ with non-empty compact ∞ -proximally connected values in Y is approachable. It is shown in [2] that if X is an ANR, $(Y, \mathcal{V})$ is a uniform space, and $\Phi : X \rightarrow \mathcal{P}(Y)$ is a u.s.c. multifunction with non-empty values such that the restriction $\Phi|_P$ of Φ to any finite polyhedron $P \subset X$ is approachable. Then the restriction $\Phi|_K$ of Φ to any compact subset K of X is approachable. The main tool used in the proof of our main theorem is the following generalization of the fixed point theorem of Ky Fan ([5]) to approachable multifunctions.

Theorem 2. ([2]) *Let X be non-empty convex subset of a Hausdorff locally convex space E , and let $\Phi : X \rightarrow \mathcal{P}(X)$ be a u.s.c. multifunction with non-empty closed values. Assume that Φ is compact, that is there exists a compact subset Y of X such that $\Phi(X) \subseteq Y$. If one of the following conditions is satisfied: (i) Φ is approachable; or (ii) for each finite subset N of X , the multifunction $\Phi_N : \text{conv}\{N\} \rightarrow \mathcal{P}(Y)$ defined by $\Phi_N(x) = \Phi(x), x \in \text{conv}\{N\}$, is approachable. Then Φ has a fixed point, that is, $\exists x_0 \in X$ with $x_0 \in \Phi(x_0)$.*

Note that Definition 1, together with a simple compactness argument, implies immediately that if X is a topological space having the fixed point property for continuous single-valued mappings (e.g. X is an acyclic compact ANR), and if $\Phi : X \rightarrow \mathcal{P}(X)$ is a u.s.c. approachable compact multifunction with non-empty closed values, then Φ has a fixed point.

The main result of this note is:

Theorem 3. *Let C be a non-empty compact convex subset in a locally convex space E , and let Y be a non-empty complete convex subset of a locally convex space F . Let $\Phi : C \rightarrow \mathcal{P}(Y)$ be a u.s.c. multifunction with non-empty compact values such that one of the following equivalent conditions is satisfied: (i)₁ Φ is approachable, or (i)₂ for each finite subset N of C , the restriction*

$\Phi[\text{conv}\{N\}]$ to the convex hull $\text{conv}\{N\}$ is approachable. Let $\Psi : C \longrightarrow \mathcal{P}(C)$ be a continuous multifunction with non-empty compact convex values, and let $\varphi : C \times Y \times C \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ be a continuous extended proper real function satisfying: (ii) $\forall (x, y) \in C \times Y$, the function $\varphi(x, y, \cdot)$ is quasiconvex on C . Then the problem (1) below has a solution,

$$\begin{cases} \exists x_0 \in \Psi(x_0), \exists y_0 \in \Phi(x_0), \text{ such that} \\ \varphi(x_0, y_0, x) \geq \varphi(x_0, y_0, x_0), \forall x \in \Psi(x_0). \end{cases} \quad (1)$$

Proof. Define the marginal multifunction $M_{\varphi, \Psi} : C \times Y \longrightarrow \mathcal{P}(C)$ by putting:

$$M_{\varphi, \Psi}(x, y) := \{u \in \Psi(x); \varphi(x, y, u) = \inf_{z \in \Psi(x)} \varphi(x, y, z)\}, (x, y) \in C \times Y. \quad (2)$$

The compactness of the values of Ψ , together with the continuity of φ , implies that $M_{\varphi, \Psi}$ has non-empty compact values. The convexity of the values of Ψ , together with (ii), implies that $M_{\varphi, \Psi}$ has convex values. We verify that $M_{\varphi, \Psi}$ is u.s.c.. To do this, observe that $M_{\varphi, \Psi}(x, y) = \Psi(x) \cap \tilde{M}_{\varphi, \Psi}(x, y)$ where $\tilde{M}_{\varphi, \Psi}(x, y) := \{u \in C; \varphi(x, y, u) = \inf_{z \in \Psi(x)} \varphi(x, y, z)\}$. Since Ψ is u.s.c. and has compact values, it suffices to verify that the graph of $\tilde{M}_{\varphi, \Psi}$ is closed. To do this, let $(x_\alpha, y_\alpha, u_\alpha)_\alpha$ be a net in $\text{graph}(\tilde{M}_{\varphi, \Psi})$ converging to $(x, y, u) \in C \times Y \times C$. Then,

$$\begin{aligned} \varphi(x, y, u) &= \lim_\alpha \varphi(x_\alpha, y_\alpha, u_\alpha) = \lim_\alpha \inf_{z \in \Psi(x_\alpha)} \varphi(x_\alpha, y_\alpha, z) \\ &= \limsup_\alpha \inf_{z \in \Psi(x_\alpha)} \varphi(x_\alpha, y_\alpha, z) \leq \inf_{z \in \Psi(x)} \varphi(x, y, z), \end{aligned}$$

where the inequality above follows from the upper semicontinuity of the marginal function $\inf_{z \in \Psi(\cdot)} \varphi(\cdot, \cdot, z)$ (this follows from the facts that φ is lower semicontinuous as a real function and that Ψ is lower semicontinuous as a multifunction). Hence, $(x, y, u) \in \text{graph}(\tilde{M}_{\varphi, \Psi})$.

Now, since Y is convex and complete, $\overline{\text{conv}}\Phi(C)$ is a convex compact subset of Y . Since the product $C \times \overline{\text{conv}}\Phi(C)$ is compact, the restriction of the multifunction $M_{\varphi, \Psi}$ to $C \times \overline{\text{conv}}\Phi(C)$ is approachable. Define a multifunction $\Gamma : C \times \overline{\text{conv}}\Phi(C) \longrightarrow \mathcal{P}(C \times \overline{\text{conv}}\Phi(C))$ by putting:

$$\Gamma(x, y) := M_{\varphi, \Psi}(x, y) \times \Phi(x), (x, y) \in C \times \overline{\text{conv}}\Phi(C).$$

Being the product of compact-valued u.s.c. approachable multifunctions, the multifunction Γ is also u.s.c., approachable and has non-empty compact values (see [1]). All conditions of Theorem 2 are thus satisfied. Therefore, Γ has a fixed point $(x_0, y_0) \in \Gamma(x_0, y_0)$, that is, $x_0 \in \Psi(x_0)$, $y_0 \in \Phi(x_0)$ and $\varphi(x_0, y_0, x_0) \leq \varphi(x_0, y_0, x), \forall x \in \Psi(x_0)$. $\square$

Remarks. (a) If $\forall x \in X$ with $x \in \Psi(x), \forall y \in \Phi(x)$ one has $\varphi(x, y, x) \geq 0$, then the inequality in (1) becomes $\varphi(x_0, y_0, x) \geq 0, \forall x \in \Psi(x_0)$. (b) If $\Psi(x) = C, \forall x \in C$, the continuity assumptions on φ can be slightly relaxed to: φ is l.s.c. and $\varphi(\cdot, \cdot, u)$ is u.s.c.. (c) The following purely topological formulation of Theorem 3 generalizes the main abstract existence result in [8]. Let X be an acyclic compact ANR, and let Y be an ANR. Let $\Phi : X \rightarrow \mathcal{P}(Y)$ be a u.s.c. approachable multifunction with non-empty compact values. Let $\Psi : X \rightarrow \mathcal{P}(X)$ be a continuous multifunction with non-empty compact values, and let $\varphi : X \times Y \times X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be a continuous extended proper real function. Assume that for any finite polyhedron P contained in $X \times Y$, the restriction of the marginal multifunction $M_{\varphi, \Psi}$ defined by (2) to P is approachable. Then problem (1) has a solution.

A classical result of Aronszajn asserts that the solution set of the Cauchy problem with continuous right hand side $y' = f(t, y), y(0) = x$, is an R_δ set (i.e. a countable intersection of a decreasing sequence of compact contractible spaces), hence ∞ -proximally connected in the space of continuous functions (see for instance [1], [3] and references there). This qualitative property of solution sets was extended by many authors to differential inclusions. Let K be a non-empty subset of $\mathbb{R}^n$ and $F : [0, T] \times K \rightarrow \mathcal{P}(\mathbb{R}^n)$ be a multifunction with non-empty compact values. Denote by $S_F(x; K)$ ($S_F(x)$ for $S_F(x; \mathbb{R}^n)$) the sets of Carathéodory solutions viable in K (i.e. $y \in S(x; K)$ if and only if $y(t) \in K, \forall t \in [0, T]$) of the Cauchy problem with initial value x :

$$\begin{cases} y'(t) \in F(t, y(t)) \\ y(0) = x. \end{cases} \quad \text{a.e. in } [0, T], \quad (3)$$

Let C be a non-empty subset of K . Assume that for any given $x \in C$ there corresponds a subset $\Psi(x) \subset C$ of possible return points. Starting at an arbitrary point $x \in C$, we travel along a trajectory y of problem (3). We then follow a return path to a point $z \in \Psi(x)$. Assume that a cost $\varphi(x, y, z)$ is associated to this journey (for instance, $\varphi(x, y, z)$ could be the sum of an attack cost $\varphi_1(x, y)$ and a retreat cost $\varphi_2(y(T), z)$). We are interested in the problem:

$$\begin{cases} \text{Find } x_0 \in C, x_0 \in \Psi(x_0), y_0 \in S_F(x_0; K), \text{ such that} \\ \varphi(x_0, y_0, x_0) = \inf_{z \in \Psi(x_0)} \varphi(x_0, y_0, z). \end{cases} \quad (4)$$

Let us recall that F is said to be a Carathéodory multifunction if the following conditions

are satisfied: (i) F has convex values; (ii) $y \mapsto F(t, y)$ is u.s.c. a.e. $t \in [0, T]$; (iii) $\forall y \in K, t \mapsto F(t, y)$ is measurable; (iv) $\sup\{|v|; v \in F(t, y), y \in K\} < \mu(t)$ where $\mu : [0, T] \rightarrow [0, +\infty)$ is an integrable function. Let us also recall that a non-empty closed subset K of $\mathbb{R}^n$ is said to be a *proximate retract* ([7]) if there exists an open neighborhood U of K in $\mathbb{R}^n$ and a continuous mapping $r : U \rightarrow K$ (called neighborhood retraction) such that the following two conditions are satisfied: (i) $r(x) = x, \forall x \in K$; (ii) $\|r(x) - x\| = \text{dist}(x, K) = \inf_{u \in K} \|x - u\|, \forall x \in U$. Any closed convex subset of $\mathbb{R}^n$, and any C^2 -submanifold of $\mathbb{R}^n$ is a proximate retract ([7]).

Theorem 4. Assume that C is convex compact, Ψ is continuous with non-empty convex compact values, φ is continuous on $C \times C([0, T], \mathbb{R}^n) \times C$, and φ is quasiconvex with respect to the return variable z . Then problem (4) has a solution provided K is a proximate retract and F is a Carathéodory multifunction satisfying the tangency condition

$$F(t, y) \cap T_K(y) \neq \emptyset, \forall (t, y) \in [0, T] \times K, \quad (5)$$

where $T_K(y) := \{v \in \mathbb{R}^n; \liminf_{t \downarrow 0^+} \frac{d(y+tv, K)}{t} = 0\}$ is the Bouligand contingent cone to K at y .

Proof. In view of Theorem 1.1 in [7], the multifunction $\Phi : K \rightarrow \mathcal{P}(C([0, T], K))$ defined by $\Phi(x) := S_F(x; K)$, $x \in K$, has R_δ values. Moreover, one can show that Φ is u.s.c.. Indeed, the multifunction F extends to a multifunction $\hat{F} : [0, T] \times \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}^n)$ in such a way that $S_{\hat{F}}(x; K) = \Phi(x)$ (see [7] for details). One then invokes well-known results to obtain that the solution set multifunction $S_{\hat{F}} : \mathbb{R}^n \rightarrow \mathcal{P}(C([0, T], \mathbb{R}^n))$ is u.s.c.. Being a closed-graph multiselection of $S_{\hat{F}}$, the multifunction $S_F^K : K \rightarrow \mathcal{P}(C([0, T], K))$ given by $S_F^K(x) := S_{\hat{F}}(x; K) = \Phi(x), x \in K$, is also u.s.c.. Consequently, Φ is approachable. The conclusion immediately follows from Theorem 3. $\square$

Remarks. (a) Observe that in case K is an open subset of $\mathbb{R}^n$, then $T_K(y) = \mathbb{R}^n$, the inwardness condition (5) being automatically satisfied. (b) Theorem 4 also holds for differential inclusions of order k in a Banach space E .

When F is l.s.c., the multifunction $S_F(\cdot)$ is generally neither u.s.c. or l.s.c., nor are its values always closed. However, $S_F(\cdot)$ admits a u.s.c. multiselection Φ with compact values ([4]) This fact leads to a lower semicontinuous version of the preceding theorem.

Theorem 5. Assume that C is a closed disk $D(u_0, b)$ and that K is an open subset of $\mathbb{R}^n$ containing the closed disk $D(u_0, b + LT)$, where $b, L > 0$. Assume also that Ψ and φ are as in Theorem 4. If F is l.s.c. with values in the open ball $B(0, L)$, then problem (4) has a solution.

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ON SOME FUNCTIONAL INEQUALITIES CONNECTED WITH QUASICONVEX FUNCTIONS

Jacek Smolarz

Presented by J. Aczel, F.R.S.C.

Abstract. We prove that functions $f, g: I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, satisfy the inequality

$$f(\lambda x + (1-\lambda)y) \leq \max\{g(x), g(y)\}, \quad x, y \in I, \lambda \in [0, 1]$$

if and only if there exists a quasiconvex function $h: I \rightarrow \mathbb{R}$ such that $f \leq h \leq g$ on I . Using this theorem we characterize solutions of a similar functional inequality connected with quasiconvex functions. As a corollary of this result we obtain also a theorem on approximately quasiconvex functions.

In [1] K. Baron, J. Matkowski and K. Nikodem have proved that two real functions f, g defined on an interval $I \subseteq \mathbb{R}$ can be separated by a convex function if and only if they satisfy the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y), \quad x, y \in I, \lambda \in [0, 1].$$

In this paper we prove an analogous result for quasiconvex functions. We present also some applications of this result.

Let us recall that a function $f: D \rightarrow \mathbb{R}$, where D is a convex set, is said to be quasiconvex if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad x, y \in D, \lambda \in [0, 1].$$

Equivalently, f is quasiconvex iff for every $a \in \mathbb{R}$ the level set $\{x \in D : f(x) \leq a\}$ is convex.

Theorem 1. Let $I \subseteq \mathbb{R}$ be an interval. Functions $f, g: I \rightarrow \mathbb{R}$ fulfil the inequality

$$(1) \quad f(\lambda x + (1-\lambda)y) \leq \max\{g(x), g(y)\}, \quad x, y \in I, \lambda \in [0, 1]$$

if and only if there exists a quasiconvex function $h: I \rightarrow \mathbb{R}$ such that $f \leq h \leq g$.

Proof. Assume that $f, g: I \rightarrow \mathbb{R}$ satisfy the inequality (1) and consider a function $h: I \rightarrow \mathbb{R}$ defined by the formula

$$h(x) := \inf\{a : x \in \text{conv } g^{-1}(-\infty, a]\}.$$

The inequality $h \leq g$ is trivial. Let $x \in I$. Fix arbitrarily $b \in \mathbb{R}$ such that $h(x) < b$. From the definition of h we have $x \in \text{conv } g^{-1}(-\infty, b]$. In view of Carathéodory's theorem (cf. [4, Th. 31E])

$$x = \lambda_1 x_1 + \lambda_2 x_2$$

with some $x_1, x_2 \in g^{-1}(-\infty, b]$ and $\lambda_1, \lambda_2 \in [0, 1]$, $\lambda_1 + \lambda_2 = 1$.

Hence

$$f(x) = f(\lambda_1 x_1 + \lambda_2 x_2) \leq \max\{g(x_1), g(x_2)\} \leq b.$$

Thus passing to the infimum we obtain $f(x) \leq h(x)$.

Now we will show that h is quasiconvex. Let $x, y \in I$ and $\lambda \in [0, 1]$. Suppose that $h(x) \leq h(y)$ and fix $b \in \mathbb{R}$ such that $h(y) < b$. By the definition of h we have $x, y \in \text{conv } g^{-1}(-\infty, b]$. Hence

$$\lambda x + (1-\lambda)y \in \text{conv } g^{-1}(-\infty, b]$$

and consequently $h(\lambda x + (1-\lambda)y) \leq b$. Therefore

$$h(\lambda x + (1-\lambda)y) \leq h(y) = \max\{h(x), h(y)\}.$$

The proof of the first implication is complete. The converse implication is obvious.

The following example shows that an analogous statement is not true for functions defined on the plane.

Example 1. Let $K = \{ x \in \mathbb{R}^2 : \|x\| < 2 \}$ and

$$e_i = (\cos(2i\pi/3), \sin(2i\pi/3)) , \quad i = 1, 2, 3 .$$

Define functions $f, g: K \rightarrow \mathbb{R}$ putting

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} , \quad g(x) = \begin{cases} 0 & x \in \{e_1, e_2, e_3\} \\ 1 & x \notin \{e_1, e_2, e_3\} \end{cases}$$

It is easy to observe that inequality (1) holds.

Suppose that there exists a quasiconvex function h between f and g . Then $\text{conv}\{e_1, e_2, e_3\} \subseteq h^{-1}(-\infty, 0]$ but that is not possible because $h(0) \geq 1$.

In the same way as in the proof of Theorem 1 we can get the following

Theorem 2. Let D be a convex subset of a real linear space. If functions $f, g: D \rightarrow \mathbb{R}$ fulfil the inequality

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \max\{g(x_i) : i=1, \dots, n\} ,$$

$$x_i \in D, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1$$

for each $n \in \mathbb{N}$, then there exists a quasiconvex function $h: D \rightarrow \mathbb{R}$ such that $f \leq h \leq g$.

As an immediate consequence of Theorem 1 we can get the following theorem. An analogous result connected with convex functions was proved in [1] (cf also [2]).

Theorem 3. Let $T > 0$. A function $f: [0, \infty) \rightarrow \mathbb{R}$ satisfies the inequality

$$(2) \quad f(\lambda x + (T - \lambda)y) \leq \max\{f(x), f(y)\}, \quad \lambda \in [0, T], \quad x, y \in [0, \infty)$$

if and only if there exists a quasiconvex function $h: [0, \infty) \rightarrow \mathbb{R}$ such that

$$(3) \quad h \leq f \leq \hat{h},$$

where $\hat{h}(x) = h(\frac{1}{T}x)$.

Proof. Putting $T\lambda$ instead of λ in (2) we obtain

$$f(T\lambda x + (T - T\lambda)y) \leq \max\{f(x), f(y)\}, \quad \lambda \in [0, 1].$$

Hence

$$(4) \quad \bar{f}(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \quad \lambda \in [0, 1], \quad x, y \in [0, \infty)$$

where $\bar{f}$ is defined by the formula $\bar{f}(x) = f(Tx)$, $x \in [0, \infty)$.

Therefore by Theorem 1 there exists a quasiconvex function $h: [0, \infty) \rightarrow \mathbb{R}$ such that

$$f(Tx) \leq h(x) \leq f(x).$$

Putting $\frac{1}{T}x$ instead of x we get also

$$f(x) \leq h(\frac{1}{T}x) = \hat{h}(x).$$

Conversely, if f satisfies (3) with a quasiconvex function h then $\bar{f}$ and f satisfy (4) which is equivalent to (2).

As a corollary of Theorem 1 we can obtain the following (one dimensional) stability theorem for quasiconvex functions due to K. Nikodem [3].

Theorem 4. Let $I \subseteq \mathbb{R}$ be an interval and c be a positive constant. If a function $f: I \rightarrow \mathbb{R}$ satisfies the condition

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} + c, \quad x, y \in I, \lambda \in [0, 1].$$

then there exists a quasiconvex function $h: I \rightarrow \mathbb{R}$ such that

$$f \leq h \leq f + c.$$

Proof. It is enough to apply Theorem 1 to the functions f and $f+c$.

Remark. We say that a function $h: I \rightarrow \mathbb{R}$ is J-quasiconvex iff

$$h\left(\frac{x+y}{2}\right) \leq \max\{h(x), h(y)\}, \quad x, y \in I.$$

Clearly, if $f \leq h \leq g$ on I and h is J-quasiconvex, then

$$(5) \quad f\left(\frac{x+y}{2}\right) \leq \max\{g(x), g(y)\}, \quad x, y \in I.$$

However the converse implication is not true. Namely, consider the following

Example 2. Let H be a Hamel base of $\mathbb{R}$ over $\mathbb{Q}$, h_1, h_2, h_3, h_4 be different elements of H and

$$x_0 = \frac{1}{4}(h_1 + h_2 + h_3 + h_4).$$

Define functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \neq x_0 \\ 1, & x = x_0 \end{cases}, \quad g(x) = \begin{cases} 0, & x \in H \\ 1, & x \notin H \end{cases}$$

Then f, g satisfy the inequality (5). Suppose that there is a J-quasiconvex function $h: \mathbb{R} \rightarrow \mathbb{R}$ between f and g . Then we get

a contradiction:

$$\begin{aligned}
 1 = f(x_0) &= h\left(\frac{h_1 + h_2 + h_3 + h_4}{4}\right) \leq \\
 &\leq \max\{h(h_1), h(h_2), h(h_3), h(h_4)\} \leq \\
 &\leq \max\{g(h_1), g(h_2), g(h_3), g(h_4)\} = 0.
 \end{aligned}$$

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ON THE CONSTRUCTION OF BIORTHOGONAL WAVELET BASES OF $L^2(\mathbb{R}^2)$ BY MCCLELLAN'S TRANSFORMATIONS

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ABSTRACT. Bidimensional wavelet bases are constructed by means of McClellan's transformation, M , applied to a pair of one-dimensional biorthogonal wavelet filters. Under appropriate conditions on the transfer function $F(\omega_1, \omega_2)$ associated to M and on the dilation matrix D , one can construct symmetric compactly supported biorthogonal wavelet bases of $L^2(\mathbb{R}^2)$. The method is illustrated by a numerical example.

RÉSUMÉ. On construit des bases d'ondelettes bidimensionnelles au moyen de la transformation de McClellan appliquée à un paire de filtres unidimensionnels biorthogonaux. Sous des conditions appropriées sur la fonction de transfert $F(\omega_1, \omega_2)$ de McClellan et sur la matrice de dilatation D , on construit des bases d'ondelettes de $L^2(\mathbb{R}^2)$ biorthogonales symétriques et à support compact. On illustre la méthode au moyen d'un exemple numérique.

1. Introduction. The design of nonseparable multidimensional wavelets [1] is complicated because many one-dimensional (1-D) techniques and results do not generalize to higher dimensions; in fact, 2-D wavelets are usually built by tensor product of 1-D wavelets.

McClellan's transformation [2] is used to generate and implement multidimensional finite impulse response (FIR) filters from a 1-D zero-phase (i.e. real) FIR filter $h(\omega) = \sum_{k \in \mathbb{Z}} c_k e^{-ik\omega}$. In this note, we report on an adaptation [3] of this transformation to the construction of nonseparable wavelet bases of $\mathbb{R}^2$, which preserves the number of vanishing moments.

2. Biorthogonal multiresolution analysis of $L^2(\mathbb{R}^2)$.

Definition 1. A matrix $D \in \mathbb{Z}^{2 \times 2}$ with singular values $\sigma_1 \geq \sigma_2 > 1$ is said to be a dilation matrix.

By Definition 1, $D\mathbb{Z}^2 \subseteq \mathbb{Z}^2$, $\|D^{-1}\|_2 = 1/\sigma_2 < 1$, and every direction is dilated.

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Definition 2. A 2-D multiresolution analysis is a decreasing sequence of closed linear subspaces of $L^2(\mathbb{R}^2)$,

$$\{0\} \subset \cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(\mathbb{R}^2), \quad (2.1)$$

with the following properties:

(P1) $\forall f \in L^2(\mathbb{R}^2)$ and $\forall j \in \mathbb{Z}$, $f(x) \in V_{j+1} \iff f(Dx) \in V_j$;

(P2) $\exists \phi \in V_0 \subset L^2(\mathbb{R}^2)$ such that, $\forall k \in \mathbb{Z}^2$, $\phi_{0,k}(x) = \phi(x - k)$ is an orthonormal basis of V_0 .

Scaling functions are defined by

$$\phi_{j,k}(x) = |\det D|^{-j/2} \phi(D^{-j}x - k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^2, \quad (2.2)$$

By Definition 2, for fixed $j \in \mathbb{Z}$, the family $\phi_{j,k}(x)$, $k \in \mathbb{Z}^2$, form an orthonormal basis of V_j , and a sampling rate of $|\det D|$ has to be achieved in order to go from one approximation level to the next, because of the geometry of the sampling grid, $\Gamma = \mathbb{Z}^2/D\mathbb{Z}^2$. Thus, to achieve exact reconstruction, one needs to construct one scaling function and d elementary wavelets $\psi^{(l)}(x)$, $l = 1, 2, \dots, d-1$, where $d = |\det D| - 1$.

If $|\det D| = 2$ and if in (P2) we take $f = \phi$, then there exists a finite sequence of real numbers α_k such that $\phi(x)$ satisfies the multiple-scale identity

$$\phi(x) = \sum_{k \in \mathbb{Z}^2} \alpha_k \phi(Dx - k). \quad (2.3)$$

In this note, we consider only the dilation matrix

$$D_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{with } |\det D_1| = 2, \quad (2.4)$$

which generates the quincunx decimation with sampling sublattice

$$Q = \mathbb{Z}^2/D_1\mathbb{Z}^2 = \{(a, b)^t \in \mathbb{Z}^2; \quad a, b \text{ have the same parity}\}.$$

To use different analysing and synthetizing families of biorthogonal wavelets we use a 2-D biorthogonal multiresolution analysis.

Definition 3. A 2-D biorthogonal multiresolution analysis is a decreasing pair of families, $(V_j)_{j \in \mathbb{Z}}$, and $(\tilde{V}_j)_{j \in \mathbb{Z}}$, of linear subspaces of $L^2(\mathbb{R}^2)$, each satisfying (2.1) and property (P1) of Definition 2, and the following biorthogonality conditions:

$$\tilde{W}_j \perp V_j, \quad W_j \perp \tilde{V}_j, \quad (2.5)$$

where W_j and $\tilde{W}_j$ are the (generally non-orthogonal) complements of V_j and $\tilde{V}_j$ in V_{j-1} and $\tilde{V}_{j-1}$, respectively.

The elementary biorthogonal wavelets for the quincunx decimation [5] are given by

$$\psi(x) = \sum_{k \in \mathbb{Z}^2} (-1)^{k_1} \tilde{\alpha}_{1-k_1, -k_2} \phi(Dx - k), \quad \tilde{\psi}(x) = \sum_{k \in \mathbb{Z}^2} (-1)^{k_1} \alpha_{1-k_1, -k_2} \tilde{\phi}(Dx - k).$$

3. Designing 2-D Nonseparable Wavelets.

Definition 4. Let

$$h(\omega) = \sum_{n=0}^N \alpha_n \cos(n\omega) = \sum_{n=0}^N \alpha_n T_n(\cos(\omega))$$

be the frequency response of a 1-D zero-phase FIR filter, where T_n is Chebyshev polynomial of degree n . If $F(\omega)$ is the frequency response of a 2-D zero-phase FIR filter, then McClellan's transformation associated to $F(\omega)$ and applied to $h(\omega)$ is

$$M_h(\omega) = \sum_{n=0}^N \alpha_n T_n(F(\omega)). \quad (3.1)$$

Given the Fourier transforms,

$$h(\omega) = \sum_{n=0}^N \alpha_n \cos(n\omega), \quad \tilde{h}(\omega) = \sum_{n=0}^{\tilde{N}} \tilde{\alpha}_n \cos(n\omega), \quad (3.2)$$

of a pair of dual filters, then $M_h(\omega)$ satisfies the following two properties:

(α) The identity

$$h(\omega)\tilde{h}(\omega) + h(\omega + \pi)\tilde{h}(\omega + \pi) = 1, \quad \forall \omega \in [0, \pi], \quad (3.3)$$

holds if and only if, for all $\omega \in [0, \pi]$,

$$\left[\sum_{n=0}^N \alpha_n T_n(\cos \omega) \right] \left[\sum_{n=0}^{\tilde{N}} \tilde{\alpha}_n T_n(\cos \omega) \right] + \left[\sum_{n=0}^N \alpha_n T_n(\cos(\omega + \pi)) \right] \left[\sum_{n=0}^{\tilde{N}} \tilde{\alpha}_n T_n(\cos(\omega + \pi)) \right] = 1. \quad (3.4)$$

(β) The function $h(\omega)$ has a zero of order $2m$ at π if and only if, for $0 \leq x \leq 1$,

$$\sum_{n=0}^N \alpha_n T_n(2x^2 - 1) = x^{2m} P_{2(N-m)}(x), \quad (3.5)$$

where $P_{2(N-m)}(x)$ is a polynomial of degree $2(N-m)$.

Property (α) provides necessary conditions on $F(\omega)$ to have exact 2-D reconstruction. Property (β) shows how to preserve the number of vanishing moments. In fact, if

$$F(\omega) = 2f^2(\omega) - 1 \quad (3.6)$$

and $h(\omega)$ has a zero of order $2m$ at π , then the corresponding filter factors as

$$H(\omega) = f^{2m}(\omega) P_{2(N-m)}(f(\omega)), \quad (3.7)$$

where $f^m(\omega) := [f(\omega)]^m$. By choosing an appropriate auxiliary transformation function $f(\omega)$, the 1-D and corresponding 2-D wavelets will have the same number of vanishing moments.

We now assume that $|\det D| = 2$. For quincunx decimation, the construction of 2-D biorthogonal wavelets reduces to the design of a pair of 2-D low-pass filters whose frequency responses, $H(\omega_1, \omega_2)$ and $\tilde{H}(\omega_1, \omega_2)$, satisfy the identities

$$H(\omega_1, \omega_2)\tilde{H}(\omega_1, \omega_2) + H(\omega_1 + \pi, \omega_2 + \pi)\tilde{H}(\omega_1 + \pi, \omega_2 + \pi) = 1, \quad \forall \omega_1, \omega_2 \in [0, \pi]. \quad (3.8)$$

Note that H and $\tilde{H}$ are obtained by applying McClellan's transformation on a 1-D biorthogonal filter and its dual, respectively. If (3.2) denotes the Fourier transforms of the 1-D filter and its dual, respectively, then (3.3) is satisfied. In this case, H and $\tilde{H}$ will satisfy (3.8) if the transformation function $F(\omega_1, \omega_2)$ satisfies

$$F(\omega_1 + \pi, \omega_2 + \pi) = -F(\omega_1, \omega_2). \quad (3.9)$$

The wavelets will be in $L^2(\mathbb{R}^2)$ only if the infinite product $\prod_{j=1}^{\infty} H([D^{-j}]^t \omega)$, converges; thus necessarily

$$F(0, 0) = 1. \quad (3.10)$$

In terms of the auxiliary transformation function $f(\omega_1, \omega_2)$, conditions (3.9) and (3.10) are written as

$$f^2(\omega_1 + \pi, \omega_2 + \pi) = 1 - f^2(\omega_1, \omega_2), \quad f(0, 0) = 1. \quad (3.11)$$

Conditions of type (3.11) are necessary, but not sufficient, for exact reconstruction. In fact, they do not ensure that the constructed wavelets are regular or even in $L^2(\mathbb{R}^2)$.

The construction of 2-D biorthogonal wavelets bases is ensured by the following theorem [5] which generalizes Theorem 3.2 of [6] to n dimensions.

Theorem 1. *Let $D \in \mathbb{Z}^{n \times n}$ be a dilation matrix whose n singular values satisfy $\sigma_l > 1$, $l = 1, \dots, n$, and set $d = \sigma_1 \sigma_2 \cdots \sigma_n$. Assume that for some positive numbers, $\epsilon, \tilde{\epsilon} > 0$, the dual scaling functions, $\psi^{(0)}$, $\tilde{\psi}^{(0)}$, and the $(d-1)$ different dual mother wavelets, $\psi^{(i)}$, $\tilde{\psi}^{(i)}$, $i = 1, \dots, d-1$, satisfy the inequalities*

$$\|\hat{\psi}^{(i)}(\omega)\| < (1 + \|\omega\|^2)^{-\epsilon-n/4}, \quad \|\hat{\tilde{\psi}}^{(i)}(\omega)\| < (1 + \|\omega\|^2)^{-\tilde{\epsilon}-n/4}. \quad (3.12)$$

For $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $i = 0, \dots, d-1$, define

$$\psi_{j,k}^{(i)}(x) := |\det D|^{-j/2} \psi^{(i)}(D^{-j}x - k), \quad \tilde{\psi}_{j,k}^{(i)}(x) := |\det D|^{-j/2} \tilde{\psi}^{(i)}(D^{-j}x - k).$$

Moreover, assume that

$$\sum_{i=0}^{d-1} \sum_{l \in \mathbb{Z}^n} \alpha_{l-Dj}^{(i)} \tilde{\alpha}_{l-Dk}^{(i)} = \delta_{j-k}. \quad (3.13)$$

Then, if $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^n)$, we have the wavelet expansion

$$f = \sum_{i=1}^{d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j,k}^{(i)} \rangle \psi_{j,k}^{(i)} = \sum_{i=1}^{d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{j,k}^{(i)} \rangle \tilde{\psi}_{j,k}^{(i)}, \quad f \in L^2(\mathbb{R}^n),$$

where the series converge in the L^2 norm.

A trivial choice for the auxiliary transformation function $f_Q(\omega_1, \omega_2)$ for the construction of 2-D biorthogonal wavelet bases associated to matrix (2.4) is

$$f_Q^2(\omega_1, \omega_2) = a_0 + a_1 \cos \omega_1 + a_2 \cos \omega_2 + a_3 \cos \omega_1 \cos \omega_2,$$

where the coefficients a_0, a_1, a_2 and a_3 are to be determined. For quincunx decimation, f_Q needs to satisfy the identity

$$f_Q^2(\omega_1, \omega_2) = 1 - f_Q^2(\omega_1 + \pi, \omega_2 + \pi), \quad 0 \leq \omega_1 \leq \pi, \quad 0 \leq \omega_2 \leq \pi.$$

By direct computation, $a_0 = \frac{1}{2}$ and $a_1 + a_2 = \frac{1}{2}$, and, consequently,

$$F_Q(\omega_1, \omega_2) = 2f_Q^2(\omega_1, \omega_2) - 1 = a_1 \cos \omega_1 + (1 - a_1) \cos \omega_2, \quad a_1 \in \left[0, \frac{1}{2}\right].$$

Choosing $a_1 = \frac{1}{2}$ for symmetry, we finally obtain

$$f_Q^2(\omega_1, \omega_2) = 1 - \frac{1}{2} \sin^2 \left(\frac{\omega_1}{2} \right) - \frac{1}{2} \sin^2 \left(\frac{\omega_2}{2} \right).$$

4. Numerical Results. The 2-D biorthogonal scaling functions $\phi(x)$ and $\tilde{\phi}(x)$, and the corresponding wavelets, $\psi(x)$ and $\tilde{\psi}(x)$, shown in Fig. 1, for the quincunx decimation, with D_1 given in (2.4), were approximated numerically by six iterations of the cascade algorithm given in [5]. McClellan's transformation function is

$$F_Q(\omega_1, \omega_2) = \frac{1}{2}(\cos \omega_1 + \cos \omega_2).$$

The coefficients of the 1-D biorthogonal wavelet filters, $h(\omega)$ and $\tilde{h}(\omega)$, used in this example, are given in Table 2 in [3].

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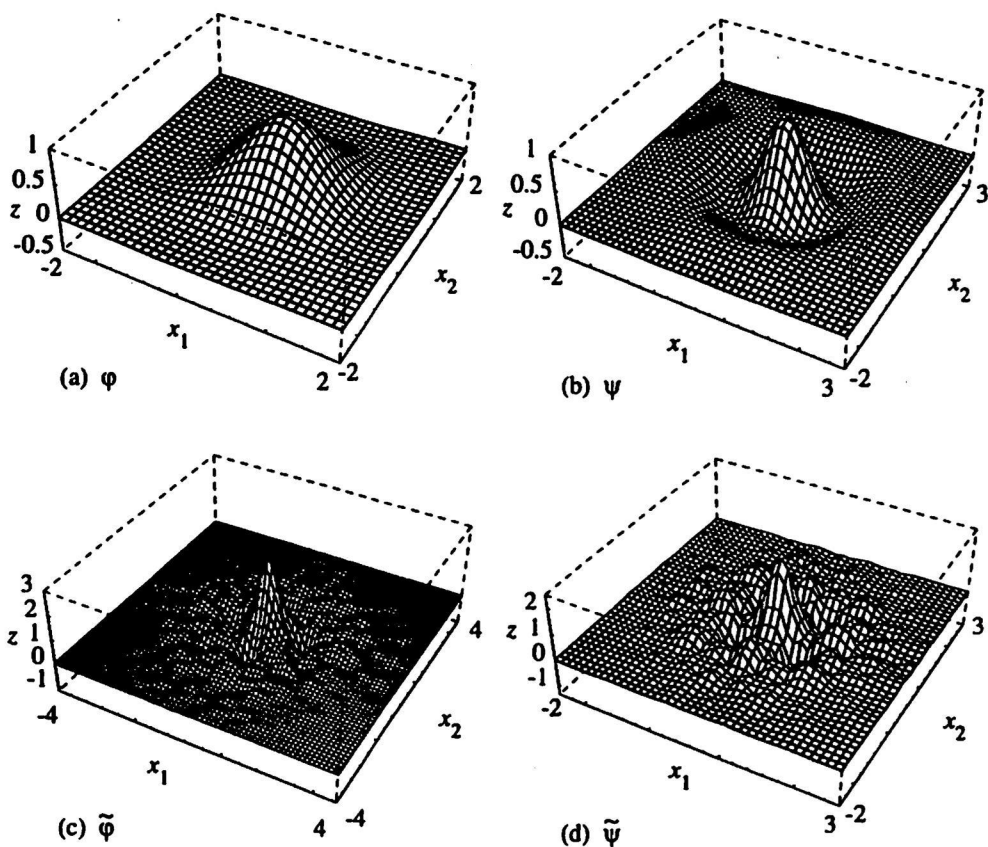


FIGURE 1. Biorthogonal dual scaling functions, $\phi(x)$ and $\tilde{\phi}(x)$, and corresponding wavelets, $\psi(x)$ and $\tilde{\psi}(x)$.

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A GENERALIZATION OF NAPOLEON'S THEOREM TO n -GONS

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On the sides of an arbitrary triangle in the Euclidean plane, construct similar copies of an equilateral triangle and its centre. Napoleon's theorem then states that the copied centres form an equilateral triangle.

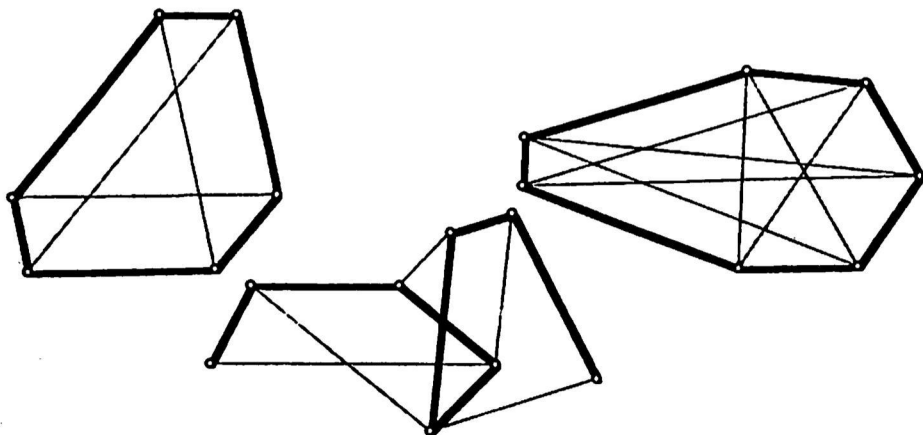
The Napoleon-Barlotti theorem [2] generalizes this result to polygons. Construct similar copies of a regular n -gon and its centre onto the sides of an affinely regular n -gon; then the copied centres form a regular n -gon. (An affinely regular n -gon is an affine transformation of a regular n -gon.)

A theorem of Rigby [4] generalizes Napoleon's theorem in a different direction: replace the equilateral triangle and its centre by an arbitrary triangle Δjkm and an arbitrary, non-vertex point e . On the sides of a variable triangle Δabc , construct triangles Δpcb , Δacq and Δarb all similar to Δjkm , and points s , t and u in the same position relative to these triangles that e is to Δjkm . Then for all triangles Δabc , the triangles Δstu are similar. In fact, they all turn out to be anti-similar to the pedal triangle of e with respect to Δjkm [3].

We prove here a further generalization of Napoleon's theorem which encompasses both the Barlotti and Rigby theorems. The theorem deals with similar copies of one polygon plus an arbitrary non-vertex point constructed on the sides of another polygon. The polygons need only satisfy a very weak form of affine regularity - in fact, they need not even be closed.

To begin with, some terminology. We take a *polygon* to be a finite sequence of points (vertices) in the Euclidean plane such that any two consecutive vertices are distinct. Its *sides* are the lines joining consecutive vertices; it is *non-degenerate* whenever no two consecutive sides coincide. (We do not assume that polygons are necessarily closed; if they are, we interpret the word "consecutive" cyclically.) The *pedal polygon* of any non-vertex point with respect to a given non-degenerate polygon has as vertices the feet of the perpendiculars from the point to the sides of the given polygon.

A polygon will be called trapezoidal if for every four consecutive vertices a, b, c and d , the first diagonal ad is parallel to the side bc . This relation is preserved by affine transformations, thus since regular polygons are trapezoidal, so are affinely regular polygons. We illustrate some trapezoidal polygons which are not affinely regular; any affine transformation of these is also trapezoidal.



Our theorem is as follows.

Theorem. Let $\mathcal{Q}$ be a non-degenerate trapezoidal polygon, m any non-vertex point and $\mathcal{P}$ an affine transformation of $\mathcal{Q}$. On the sides of $\mathcal{P}$, construct appropriately oriented copies of $\mathcal{Q}$ and m , i.e. construct for each side of $\mathcal{P}$ a polygon similar to $\mathcal{Q}$ so situated that the corresponding sides of $\mathcal{P}$ and $\mathcal{Q}$ are coincident, and a point in the same position relative to the copy of $\mathcal{Q}$ that m is to the original. Let $\mathcal{R}$ be the polygon formed by joining the copies of m in order. Then $\mathcal{R}$ is anti-similar to the pedal n -gon of m with respect to $\mathcal{Q}$.

We prove this theorem by extending the notion of the *shape* of a triangle developed in [3] to polygons. Some notation. For any complex number z different from 0 and 1, define $z' := (1 - z)^{-1}$. This "cycle notation" obeys the following calculation rules:

$$z' = \frac{1}{1-z}, \quad z'' = \frac{z-1}{z}, \quad z''' = z, \quad zz'z'' = -1.$$

Another useful rule, which we leave to the reader for verification: for any non-zero θ , φ , ρ and σ ,

$$\theta + \varphi^{-1} = \rho + \sigma^{-1} \quad \text{if and only if} \quad \left(\frac{\theta}{\rho}\right)\left(\frac{\varphi}{\sigma}\right)^* = -\rho\sigma.$$

Identify the Euclidean plane with the complex numbers $\mathbb{C}$. The shape of any triangle Δ_{abc} is defined to be the complex number

$$\Delta_{abc} = \frac{a-b}{a-b}.$$

The argument and modulus of Δ_{abc} give the angle between sides ab and ac and the ratio of their lengths, so two triangles are similar whenever they have the same shape and anti-similar when they have conjugate shapes. The cycle notation gives the effect of cycling the vertices of the triangle: if $\Delta_{abc} = \Delta$, then $\Delta_{bca} = \Delta'$ and $\Delta_{cab} = \Delta''$ (an easy calculation).

The shape of a polygon can be defined in terms of the shapes of its sub-triangles. There are various ways to do this: see [1] for an alternative. Here, we define the shape of a polygon to be the sequence of shapes of the triangles formed by triples of consecutive vertices: if $(\dots, a, b, c, \dots)$ denotes a polygon with typical consecutive vertices a, b, c , then the shape component of the polygon at vertex b is the number $\beta = \Delta_{bca}$. We denote the shape component at each vertex by the corresponding Greek letter, so a typical polygon $(\dots, a, b, c, \dots)$ has shape $(\dots, \alpha, \beta, \gamma, \dots)$. (Note that open polygons have no shape components at their end-points. Since we use only "local" arguments below, this does not affect the proof.) As with triangles, shapes determine polygons up to similarity: similar polygons have the same shape and anti-similar polygons have conjugate shapes. Note that, as a polygon, a triangle with shape Δ has polygon shape $(\Delta, \Delta', \Delta'')$.

The effect on shapes of adding and deleting vertices is easy to determine.

Lemma. Let $(\dots, a, b, c, d, e, \dots)$ be any polygon with shape $(\dots, \alpha, \beta, \gamma, \delta, \epsilon, \dots)$.

a) If a vertex p is added between b and c , then for $\rho = \Delta_{pcb}$, the resulting polygon has shape

$$(\dots, \alpha, \beta\rho'', \rho, \rho'\gamma, \delta, \epsilon, \dots).$$

b) If vertex c is deleted, then the resulting polygon has shape

$$(\dots, \alpha, \beta/\gamma'', \delta/\gamma', \epsilon, \dots).$$

Proof. a) Only the shape components at b and c change; these become

$$\Delta_{bpa} = \Delta_{bca} \Delta_{bpc} = \beta\rho'' \quad \text{and} \quad \Delta_{cdp} = \Delta_{cbp} \Delta_{cdb} = \rho'\gamma.$$

b) Only the shape components at b and d change; these become

$$\Delta_{bda} = \Delta_{bca} / \Delta_{bcd} = \beta/\gamma'' \quad \text{and} \quad \Delta_{deb} = \Delta_{dec} \Delta_{deb} = \delta/\gamma'.$$

A few words about trapezoidal polygons and their shapes. For an arbitrary polygon $(\dots, a, b, c, d, \dots)$, a simple calculation shows that

$$\frac{a-d}{b-c} = 1 - (\beta + \gamma^{-1}).$$

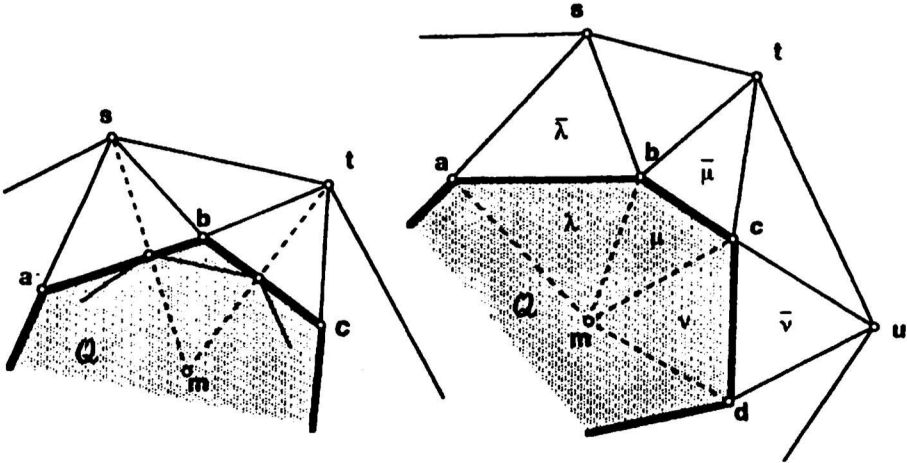
The argument of the left-hand side gives the angle between side bc and diagonal ad , while its modulus gives the ratio of their lengths. Thus the polygon is trapezoidal if and only if for every two consecutive shape components β and γ , $\beta + \gamma^{-1}$ is real, i.e. if and only if $\beta + \gamma^{-1} = \bar{\beta} + \bar{\gamma}^{-1}$. From rule $\ast$, then, a polygon is trapezoidal if and only if

$$(\beta/\bar{\beta})(\gamma/\bar{\gamma})'' = -\bar{\beta}\bar{\gamma}$$

for every pair β, γ of consecutive shape components. Furthermore, since any affine transformation preserves parallel line segments and the ratio of their lengths, if $(\dots, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \dots)$ is an affine transformation of the trapezoidal polygon $(\dots, a, b, c, d, \dots)$, then for corresponding pairs of consecutive shape components β, γ and $\hat{\beta}, \hat{\gamma}$, we have $\hat{\beta} + \hat{\gamma}^{-1} = \beta + \gamma^{-1}$, or equivalently (from rule $\ast$),

$$(\hat{\beta}/\bar{\hat{\beta}})(\hat{\gamma}/\bar{\hat{\gamma}})'' = -\bar{\beta}\bar{\gamma}.$$

We begin the proof of the theorem by finding the shape of the pedal polygon of m with respect to $Q = (\dots, a, b, c, d, \dots)$. The pedal polygon is similar to the polygon $(\dots, s, t, u, \dots)$ obtained by reflecting m through the sides of Q (dilate the pedal polygon by a factor 2 about centre m). We use the lemma to find its shape: we add the vertices $\dots, s, t, u, \dots$ to Q and then delete the vertices $\dots, a, b, c, d, \dots$



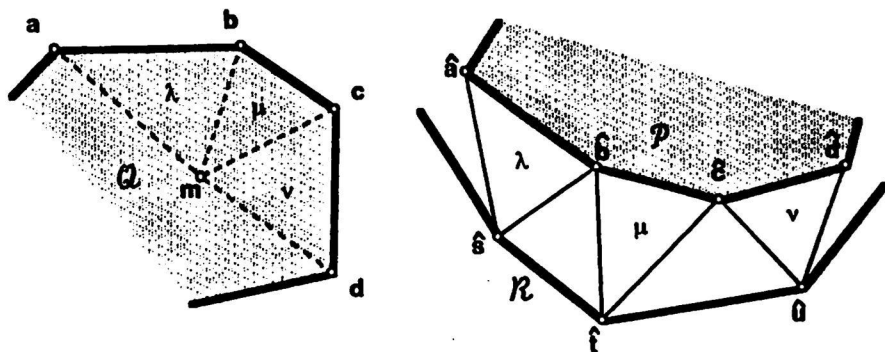
Set $\lambda := \Delta_{mba}$, $\mu := \Delta_{mcb}$ and $\nu := \Delta_{mdc}$; then $\Delta_{sba} = \bar{\lambda}$, $\Delta_{tcb} = \bar{\mu}$ and $\Delta_{udc} = \bar{\nu}$, since the reflected triangles are anti-similar to their originals. A simple calculation relates λ, μ and ν to β and γ : $\lambda\mu'' = \beta^{-1}$ and $\mu\nu'' = \gamma^{-1}$. Then (lemma) since Q has shape $(\dots, \alpha, \beta, \gamma, \delta, \dots)$, the "star-shaped" polygon $(\dots, s, b, t, c, u, \dots)$ has shape $(\dots, \bar{\lambda}, \bar{\lambda}\bar{\beta}\bar{\mu}'', \bar{\mu}, \bar{\mu}\bar{\gamma}\bar{\nu}'', \bar{\nu}, \dots) = (\dots, \bar{\lambda}, \beta/\bar{\beta}, \bar{\mu}, \gamma/\bar{\gamma}, \bar{\nu}, \dots)$.

Now delete $\dots, a, b, c, d, \dots$. At vertex t , the resulting polygon has shape component (lemma)

$$\frac{\bar{\mu}}{(\beta/\bar{\beta})(\gamma/\bar{\gamma})''} = -\frac{\bar{\mu}}{\bar{\beta}\bar{\gamma}},$$

since Q is trapezoidal.

We now find the corresponding shape component of the polygon $\mathcal{R}$. Suppose $\hat{a}, \hat{b}, \hat{c}$, etc. are the vertices of $\mathcal{P}$ corresponding to the vertices a, b, c , etc. of $\mathcal{Q}$. The only parts of the copies of $\mathcal{Q}$ and m that matter are the triangles with bases on $\mathcal{P}$ and apex vertices at the copies $\dots, \hat{s}, \hat{t}, \hat{u}, \dots$ of m . These triangles have shapes $\Delta_{sba}^{\hat{a}\hat{b}\hat{a}} = \Delta_{mba}^{\hat{a}\hat{b}\hat{a}} = \lambda$, $\Delta_{tcb}^{\hat{a}\hat{b}\hat{a}} = \mu$ and $\Delta_{udc}^{\hat{a}\hat{b}\hat{a}} = \nu$. We find the shape of $\mathcal{R}$ by adding the vertices $\dots, \hat{s}, \hat{t}, \hat{u}, \dots$ to $\mathcal{P}$ and then deleting $\dots, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \dots$.



From the lemma, since $\mathcal{P}$ has shape $(\dots, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \dots)$, the star-shaped polygon $(\dots, \hat{s}, \hat{t}, \hat{c}, \hat{u}, \dots)$ has shape

$$(\dots, \lambda, \lambda \hat{\beta} \mu, \mu, \mu \hat{\gamma} \nu, \nu, \dots) = (\dots, \lambda, \hat{\beta}/\beta, \mu, \hat{\gamma}/\gamma, \nu, \dots).$$

Then at vertex $\hat{t}$, $\mathcal{R}$ has shape component

$$\frac{\mu}{(\hat{\beta}/\beta)(\hat{\gamma}/\gamma)} = -\frac{\mu}{\beta\gamma},$$

since $\mathcal{P}$ is an affine transformation of the trapezoidal polygon $\mathcal{Q}$.

Thus, since they have conjugate shape components at typical corresponding vertices t and $\hat{t}$, the pedal polygon of m with respect to $\mathcal{Q}$ and the polygon $\mathcal{R}$ are anti-similar, and we are done.

If $\mathcal{Q}$ is a triangle, we recover Rigby's theorem (triangles are trapezoidal by default). If $\mathcal{Q}$ is regular and m is its centre, we recover Barlotti's theorem (the orientation of $\mathcal{Q}$ is irrelevant and the pedal triangle of m with respect to $\mathcal{Q}$ is regular).

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Résolution des équations quadratiques sur les corps $\mathbb{F}_2((G))$

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Introduction :

Dans la suite G est un groupe commutatif totalement ordonné, $G^+ = \{ \alpha \in G ; \alpha > 0 \}$, $\mathbb{F}_q$ le corps fini à q éléments et $\mathbb{F}_q((G))$ le corps des séries formelles généralisées. Un élément de $\mathbb{F}_q((G))$ sera noté $f = \sum f_\alpha T^\alpha$ avec $\text{Supp } f = \{ \alpha \in G ; f_\alpha \neq 0 \}$. On considère l'équation : $aY^2 + bY + c = 0$, où $a \neq 0, b, c \in \mathbb{F}_2((G))$. Si $b = 0$, cette équation admet une solution dans $\mathbb{F}_2((G))$, définie par $\sum (\frac{c}{a})_{2\alpha} T^\alpha$, ssi pour tout $\alpha \in G$ non 2-divisible, on a $(\frac{c}{a})_\alpha = 0$.

Si $b \neq 0$, en posant : $X = \frac{a}{b} Y$ et $d = \frac{ac}{b^2}$, l'équation devient : $X^2 + X + d = 0$.

Plus généralement, on s'intéresse à l'équation : (*) $X^q - X - d = 0$ avec $d \in \mathbb{F}_q((G))$.

1- Remarques :

- Si (*) admet une solution f dans $\mathbb{F}_q((G))$, les autres solutions sont $f-x$, où $x \in \mathbb{F}_q$. L'une d'entre elles est de terme constant nul. On la notera $\underline{d}$.

- Si d et e existent dans $\mathbb{F}_q((G))$ et $x \in \mathbb{F}_q$, alors $\underline{xd} = x\underline{d}$ et $\underline{d+e} = \underline{d} + \underline{e}$.

2- Proposition :

a) Si $\text{Supp } d \subset G^+$, alors $\underline{d} = \sum_{n=0}^{\infty} d^{q^n} \in \mathbb{F}_q((G))$.

b) Si les éléments de $\text{Supp } d$ sont < 0 et de q -hauteurs infinies, alors :

$$\underline{d} = \sum_{n=1}^{\infty} d^{q^n} \in \mathbb{F}_q((G)).$$

Démonstration :

b) $\text{Supp } \underline{d} \subset \bigcup_{n=1}^{\infty} \frac{1}{q^n} \text{Supp } d$, qui est bien ordonné, d'après le cas particulier q premier. Voir [2] p. 133.

Exemples :

- Une racine de $X^q - X + T = 0$ dans $\mathbb{F}_q((T))$ est $\sqrt[q]{T} = \sum_{n=0}^{\infty} T^{qn}$.

- On suppose que q est premier. Soit $f = \sum_{n=0}^{\infty} f_n T^n \in \mathbb{F}_q((T))$ avec f_n la somme des chiffres de n en base q . D'après [1] p. 281, on a : $\frac{X}{1-X} = (X-1)f$.

3- Corollaire:

Toute extension quadratique de $\mathbb{F}_2((G))$ est obtenue par adjonction d'un élément de l'une des formes suivantes :

- a) $\sqrt{f}$ avec $f \in \mathbb{F}_2((G))$ tel que pour tout $\alpha \in \text{Supp } f$, α n'est pas 2-divisible.
- b) $\sqrt[d]{d}$ avec $d \in \mathbb{F}_2((G))$ tel que pour tout $\alpha \in \text{Supp } d$, $\alpha < 0$ et de 2-hauteur finie.

4- Lemme :

Si $\text{Supp } d$ est une partie finie de G^- , dont les éléments sont de q -hauteurs finies, alors

$\sqrt[d]{d}$ existe dans $\mathbb{F}_q((G))$ ssi $d = \sum_{i=1}^n a_i (T^{q^{\alpha_i}} - T^{\alpha_i})$, avec $n \in \mathbb{N}$, $a_i \in \mathbb{F}_q$ et $\alpha_i \in G$.

Démonstration:

$$" \Leftarrow " \quad \sqrt[d]{d} = \sum_{i=1}^n a_i T^{\alpha_i}.$$

" $\Rightarrow$ " Remarquons que si $d \in \mathbb{F}_q((G))$ est de valuation $v(d) < 0$ tel que

$\sqrt[d]{d} \in \mathbb{F}_q((G))$, alors $v(d) \in qG$. Posons : $d = a T^{q\alpha} + \dots$ et $d_1 = d - a T^{q\alpha} + a T^{\alpha}$.

On a : $\sqrt[d_1]{d_1} = \sqrt[d]{d} - a T^{\alpha}$. Si $v(d_1) < 0$, on recommence. L'opération doit s'arrêter après un nombre fini d'étapes à cause des hypothèses sur $\text{Supp } d$. D'où le résultat.

5- Lemme :

Soient C et D deux parties non vides de G . On suppose que D est une partie bien ordonnée de G^- , et que pour tout $d \in D$, la partie :

$V_d = \{c \in C ; \exists i \in \mathbb{N}, d = q^i c\}$ est finie non vide.

On pose : $X_d = \bigcup_{c \in V_d} \{q^j c ; 0 \leq j \leq i, q^i c = d\}$.

Alors : $X = \bigcup_{d \in D} X_d$ est une partie bien ordonnée de G .

Démonstration :

Supposons que X contient une suite (x_i) strictement décroissante.

On a : $x_i \in X_{d_i}$, $d_i \in D$, pour $i \geq 1$. L'ensemble $\{d_i, i \geq 1\}$ est infini car les X_d sont finis. Comme D est bien ordonnée, quitte à considérer une sous suite extraite, on peut supposer que $d_1 < d_2 < \dots$. On a : $d_i \leq x_i$, pour tout i . L'inégalité est toujours stricte car si

$d_n = x_n$, alors $x_{n+1} < x_n = d_n < d_{n+1} \leq x_{n+1}$: absurde. Pour tout $i \geq 1$, il existe $c_i \in C$ et des entiers $0 \leq n_i \leq m_i$ tels que :

$x_i = q^{n_i} c_i$ et $d_i = q^{m_i} c_i = q^{m_i - n_i} x_i$. Il résulte que $m_i - n_i \geq 1$. Les inégalités :

$0 > x_i > x_{i+1} > d_{i+1} > d_i$ s'écrivent : $0 > x_i > x_{i+1} > q^{m_{i+1} - n_{i+1}} x_{i+1} > q^{m_i - n_i} x_i$.

Donc : $m_{i+1} - n_{i+1} < m_i - n_i$. On obtient une suite d'entiers : $m_1 - n_1 > m_2 - n_2 > \dots \geq 1$: absurde.

Notations :

On désigne par G_q l'ensemble des éléments de G qui ne sont pas q -divisibles. Si $d \in \mathbb{F}_q((G))$, on désigne par $\tilde{d}$ la partie de d dont le support est formé par des éléments $\alpha < 0$ et de q -hauteurs finies ; et pour tout $\alpha \in G_q$, on pose :

$\tau_\alpha(d) = \text{card} \{ i \geq 0 ; q^i \alpha \in \text{Supp } d \} \in \mathbb{N}$.

6- Théorème :

$\lfloor d \rfloor$ existe dans $\mathbb{F}_q((G))$ ssi $d_0 = 0$ et $\tilde{d} = \sum_{\alpha \in P} \sum_{i \in I_\alpha} a_i (T^{q^i \alpha} - T^{q^{i+1} \alpha})$, avec

$P \subset G_q$; I_α une partie finie de $\mathbb{N}$; $i < i'$ et $a_i \in \mathbb{F}_q$.

Démonstration :

" $\Rightarrow$ " D'après remarque 1 et proposition 2, on peut supposer que :

$$d = \tilde{d} = \sum_{\alpha \in G_q} \sum_{i \in \mathbb{N}} d_{q^i \alpha} T^{q^i \alpha}. \quad \left| \sum_{i \in \mathbb{N}} d_{q^i \alpha} T^{q^i \alpha} \right.$$

existe dans $\mathbb{F}_q((G))$. On conclut par le lemme 4.

" $\Leftarrow$ " $\lfloor \tilde{d} \rfloor = - \sum_{\alpha \in P} \sum_{i \in I_\alpha} a_i \sum_{k=i}^{i-1} T^{q^k \alpha} \in \mathbb{F}_q((G))$, d'après le lemme 5.

7- Corollaire:

$\lfloor d \rfloor$ existe dans $\mathbb{F}_2((G))$ ssi $d_0 = 0$ et pour tout $\alpha \in G_q$, on a : $\tau_\alpha(d) \equiv 0 \pmod{2}$.

Définition :

On pose : $q = p^c$, où p est un nombre premier.

Pour tout $n \geq 0$, on définit le polynôme : $P_n(X) = \sum_{m=0}^n (-1)^m C_n^m X^{q^m} \in \mathbb{F}_p[X]$.

On a : $P_{n-1}^q - P_{n-1} = -P_n$.

8- Lemme :

Soient K un corps de caract p et $a \in K$. On définit la suite (a_n) par $a_0 = a$ et a_n une racine du polynôme : $X^q - X - a_{n-1}$ dans une clôture algébrique $\overline{K}$ de K .

Alors a_n est aussi racine du polynôme : $P_n(X) + (-1)^{n+1} a$.

Démonstration :

$$\begin{aligned} P_n(a_n) + (-1)^{n+1} a &= -P_{n-1}^q(a_n) + P_{n-1}(a_n) + (-1)^{n+1} a \\ &= -P_{n-1}(a_n^q - a_n - a_{n-1}) - P_{n-1}(a_{n-1}) + (-1)^{n+1} a = -P_{n-1}(a_{n-1}) - (-1)^n a. \end{aligned}$$

On termine par récurrence.

9- Remarques :

- Notons par S_n l'ensemble des racines du polynôme $P_n(X)$ dans $\overline{K}$, alors l'ensemble des racines de $P_n(X) + (-1)^{n+1} a$ est égal à $\{a_n + \theta; \theta \in S_n\}$.

$$\text{- On a : } P_{n-1}(1 - P_{n-1}^{q^{-1}}) = P_n. \text{ Donc } P_m \text{ divise } P_n \text{ pour } m < n \text{ et } P_n = X \prod_{m=1}^{n-1} (1 - P_m^{q^{-1}}).$$

En particulier, si $n \leq q^k$, le polynôme P_n divise $P_{q^k} = X - X^{q^{q^k}}$.

Donc : $S_n = \{\theta; \theta = 0 \text{ ou } P_m^{q^{-1}}(\theta) = 1; 1 \leq m < n\} \subset \mathbb{F}_{q^k}$ avec égalité si $n = q^k$.

Définition :

Pour $d \in \mathbb{F}_q((G))$, on pose $\lfloor^1 d = \lfloor d$ et $\lfloor^n d = \lfloor \lfloor^{n-1} d$.

On suppose dans la suite que q est un nombre premier.

10- Lemme :

Soient F un sous-corps de $\mathbb{F}_q((G))$ et $d \in F$ tels que $\lfloor^2 d$ existe dans $\mathbb{F}_q((G))$.

Si $[F(\lfloor d) : F] = q$, alors $[F(\lfloor^2 d) : F(\lfloor d)] = q$.

Démonstration :

Supposons que $\lfloor^2 d \in F(\lfloor d)$. Alors il existe $P(X) \in F[X]$ tel que $\lfloor^2 d = P(\lfloor d)$.

Donc : $P^q(\lfloor d) - P(\lfloor d) - \lfloor d = 0$. Puisque $X^q - X - d$ est irréductible sur F et admet $\lfloor d$ et $\lfloor d - 1$ comme racines, alors l'application $\lfloor d \longrightarrow \lfloor d - 1$ définit un F -automorphisme ϕ de $F(\lfloor d)$. En appliquant ϕ à l'égalité précédente, on trouve :

$P^q(\lfloor d - 1) - P(\lfloor d - 1) - \lfloor d + 1 = 0$. Or puisque $(\lfloor d - 1)_0 \neq 0$, d'après le théorème 6, l'équation $X^q - X - \lfloor d + 1 = 0$ n'a pas de solution dans $\mathbb{F}_q((G))$.

11- Théorème :

Soient $n \in \mathbb{N}^*$, F un sous-corps de $\mathbb{F}_q((G))$ et $d \in F$ tels que $\lfloor^n d$ existe dans

$\mathbb{F}_q((G))$ et $\underline{d} \notin F$.

Alors le polynôme minimal de $\underline{d}$ sur F est égal à :

$$P_n(X) + (-1)^{n+1} d = \sum_{m=0}^n (-1)^m C_n^m X^{q^m} + (-1)^{n+1} d.$$

12- Remarque :

Si $\text{Supp } d \subset G^+$ ou les éléments de $\text{Supp } d$ sont < 0 et de q -hauteurs infinies, alors

$\underline{d}$ existe dans $\mathbb{F}_q((G))$, pour tout $n \geq 1$.

Exemples :

a) La série de Morse $f \in \mathbb{F}_2((T))$, définie par $f_1 = 0$, $f_2 = 1$ et les formules :

$$f_{2i-1} = f_i, f_{2i} = f_i + 1, \text{ pour } i \geq 1, \text{ satisfait : } (1+T)^3 f^2 + (1+T)^2 f + T^3 = 0.$$

Posons : $g = \frac{1+T}{T} f$. Alors $g^2 + g + \frac{T}{1+T} = 0$. Le polynôme minimal de $\underline{g} \left(\frac{T}{1+T} \right)$

sur $\mathbb{F}_2(T)$ est égal à $\sum_{m=0}^n C_n^m X^{2^m} + \frac{T}{1+T}$.

b) La série $f = \sum_{i=1}^{\infty} T^{\frac{-1}{q^i}}$ de $\mathbb{F}_q((\mathbb{Q}))$ est racine du polynôme : $X^q - X - T^{-1}$;

Le polynôme minimal de $\underline{T^{-1}}$ sur $\mathbb{F}_q((T))$ est égal à : $\sum_{m=0}^n (-1)^m C_n^m X^{q^m} + (-1)^{n+1} T^{-1}$.

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AN EMBEDDING INVARIANT FOR OPERATOR SPACES

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Presented by M.-D. Choi, F.R.S.C.

Abstract

We investigate the question of whether the maximal and minimal C^* -norms agree on the tensor product of two C^* -algebras A and B . An isomorphism invariant $\omega(E)$ for an operator space is introduced, and we show that these C^* -norms are distinct when $\omega(A)$ and $\omega(B)$ are finite. In particular our results apply to the tensor product of von Neumann algebras.

Introduction. It is well known that if A and B are C^* -algebras then there are two distinguished C^* -norms on the algebraic tensor product $A \otimes B$:

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\min} = \sup \left\| \sum_{i=1}^n \pi(a_i) \otimes \rho(b_i) \right\|_{B(H \otimes K)}$$

over all representations $\pi: A \rightarrow B(H)$, $\rho: B \rightarrow B(K)$ and

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\max} = \sup \left\| \sum_{i=1}^n \pi(a_i) \rho(b_i) \right\|$$

over all commuting representations π and ρ of A and B . For every other C^* -norm β on $A \otimes B$ one has

$$\| \cdot \|_{\min} \leq \beta \leq \| \cdot \|_{\max}.$$

A C^* -algebra is called nuclear if for every C^* -algebra B

$$\| \cdot \|_{\min} = \| \cdot \|_{\max} \text{ on } A \otimes B.$$

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Although the theory of nuclear C^* -algebra has had a deep and broad development, the problem of deciding whether the min and max norms are equal on $A \otimes B$ for a given pair of C^* -algebras A and B is still not well understood.

S. Wassermann proved [6] that if $C(H)$ denotes the Calkin algebra, then on $C(H) \otimes C(H)$ the min and max norms are different. E. Kirchberg [3] showed that the min and max norms are equal on $B(H) \otimes C^*(F_\infty)$ where $C^*(F_\infty)$ is the full C^* -algebra of the free group F_∞ on countably many generators. We refer to [4] for more results on equality of the two tensor norms.

An old question of Guichardet concerned the case $B(H) \otimes B(H)$ and recently M. Junge and G. Pisier [2] proved that the min and max norms are different on $B(H) \otimes B(H)$.

In this note we prove a similar result for $M \otimes N$ where M and N are von Neumann algebras not finite of type I. Our result is in fact a consequence of a more general one (Theorem 3) which involves an embedding invariant for operator spaces and it is our belief that this invariant is likely to have further applications.

Preliminary results. Denote by OS_n the set of all n -dimensional operator spaces. We identify two operator spaces if they are completely isometrically isomorphic. For E and F in OS_n define

$$d_{cb}(E, F) = \inf \{ \|u\|_{cb} \cdot \|u^{-1}\|_{cb} : u: E \rightarrow F \text{ completely bounded isomorphism} \}.$$

It has been shown [5] that OS_n together with $\delta_{cb}(E, F) = \log d_{cb}(E, F)$ is a complete metric space. Moreover ([2]), this space is nonseparable for $n \geq 3$.

Proposition 1 ([2]). *Let A be a separable, infinite dimensional C^* -algebra A . For every $\delta > 1$ there is an integer $n(\delta)$ such that for every $n \geq n(\delta)$ there exists an operator space $E_0 \in OS_n$ such that for any n -dimensional subspace $E \subset A$ we have $d_{cb}(E, E_0) > \delta$.*

Lemma 2 ([1], 5.2). *Let E and F be operator spaces. If E is finite dimensional, then the operator spaces $CB(E, F)$ and $E^* \underset{\min}{\otimes} F$ are completely isometrically isomorphic.*

Here $CB(E, F)$ denotes the space of all completely bounded linear maps between E and F . Let us now define the embedding invariant. Let E be an *infinite dimensional*

operator space. Define

$$\omega(E) = \sup_n \sup_{F \in OS_n} \inf \{ \|u\|_{cb} \cdot \|u^{-1}\|_{cb} : u: F \rightarrow u(F) \subset E \text{ completely bounded embedding.} \}$$

It is clear that ω is invariant under completely isometric isomorphisms. Proposition 1 shows that for every separable infinite dimensional C^* -algebra A , $\omega(A) = \infty$. It is obvious that $\omega(B(H)) = 1$. We obtain more properties of ω in Proposition 4.

The main result.

Theorem 3. *Let A and B be infinite dimensional C^* -algebras. If $\omega(A) < \infty$ and $\omega(B) < \infty$ then $A \otimes_{\min} B \neq A \otimes_{\max} B$.*

Proof: To get a contradiction, assume that $A \otimes_{\min} B = A \otimes_{\max} B$. Fix $\delta > 0$ and choose, by Proposition 1, $E_0 \in OS_n$ such that $d_{cb}(E_0, E) > \delta$ for all n -dimensional subspaces $E \subset C^*(F_\infty)$, where F_∞ denotes the free group on countably many generators. It is easy to see that if Λ is any infinite set, then also $d_{cb}(E, E_0) > \delta$ for all n -dimensional subspaces $E \subset C^*(F_\Lambda)$. Let then Λ be such that there exists a surjective $*$ -homomorphism $\pi: C^*(F_\Lambda) \rightarrow B$. If $\mathcal{J} = \ker \pi$ then $B = C^*(F_\Lambda)/\mathcal{J}$. The complete contraction

$$A \otimes_{\min} C^*(F_\Lambda) \longrightarrow A \otimes_{\min} C^*(F_\Lambda)/\mathcal{J} = A \otimes_{\min} B$$

vanishes on $A \otimes_{\min} \mathcal{J}$, therefore it induces a complete contraction

$$A \otimes_{\min} C^*(F_\Lambda)/A \otimes_{\min} \mathcal{J} \longrightarrow A \otimes_{\min} B.$$

It follows that the quotient norm on $A \otimes_{\min} C^*(F_\Lambda)/A \otimes_{\min} \mathcal{J}$ induces a C^* -norm on $A \otimes B$.

Uniqueness of the norm on $A \otimes B$ implies that the C^* -algebras

$$A \otimes_{\min} B \quad \text{and} \quad A \otimes_{\min} C^*(F_\Lambda)/A \otimes_{\min} \mathcal{J} \quad \text{are isomorphic.}$$

There is a completely bounded embedding $j: E_0 \rightarrow j(E_0) \subset B$ such that both $\|j\|_{cb}$ and $\|j^{-1}\|_{cb}$ are at most $\omega(B)^{1/2}$. By Lemma 2, j can be viewed as an element $j_0 \in E_0^* \otimes_{\min} B$ of norm $\|j_0\| \leq \omega(B)^{1/2}$. Moreover, j_0 can be viewed as an element $j_1 \in A \otimes_{\min} B$ of norm $\|j_1\| \leq (\omega(A)\omega(B))^{1/2}$. We claim that for every $\varepsilon > 0$ there is a lifting $\tilde{j}_0$ of j_0 to $E_0^* \otimes_{\min} C^*(F_\Lambda)$ of norm $\|\tilde{j}_0\| < (\omega(A)\omega(B))^{1/2} + \varepsilon$ and $(Id_{E_0^*} \otimes \pi)(\tilde{j}_0) = j_0$.

Indeed, if $j_0 = \sum_{i=1}^m e_i \otimes \hat{x}_i$, then j_1 has a lifting $\tilde{j}_1 = \sum_{i=1}^m e_i \otimes x_i + \sum_{i=1}^p a_i \otimes h_i$ of norm $\|\tilde{j}_1\| < (\omega(A)\omega(B))^{1/2} + \varepsilon$ such that $(Id_A \otimes \pi)(\tilde{j}_1) = j_1$, where

$$e_i \in E_0^*, \quad a_i \in A, \quad x_i \in C^*(F_\Lambda), \quad h_i \in \mathcal{J}.$$

Let $(u_\alpha) \subset \mathcal{J}$ be an approximate unit for $\mathcal{J}$. Then

$$\begin{aligned} \|\tilde{j}_1(I \otimes (I - u_\alpha))\| &< (\omega(A)\omega(B))^{1/2} + \varepsilon \\ \left\| \sum_{i=1}^m e_i \otimes x_i(I - u_\alpha) + \sum_{i=1}^p a_i \otimes h_i(I - u_\alpha) \right\| &< (\omega(A)\omega(B))^{1/2} + \varepsilon. \end{aligned}$$

For α large enough we get that

$$\left\| \sum_{i=1}^m e_i \otimes x_i(I - u_\alpha) \right\| < (\omega(A)\omega(B))^{1/2} + \varepsilon.$$

If we denote $\tilde{j}_0 = \sum_{i=1}^m e_i \otimes x_i(I - u_\alpha)$ then

$$\|\tilde{j}_0\| < (\omega(A)\omega(B))^{1/2} + \varepsilon \quad \text{and} \quad (Id_{E_0^*} \otimes \pi)(\tilde{j}_0) = j_0.$$

Let $\tilde{j}: E_0 \rightarrow C^*(F_\Lambda)$ be the operator associated with $\tilde{j}_0$ (Lemma 2). Then $\|\tilde{j}\|_{cb} < (\omega(A)\omega(B))^{1/2} + \varepsilon$ and $\pi \circ \tilde{j} = j$. It follows that $\tilde{j}$ is an embedding and

$$d_{cb}(E_0, \tilde{j}(E_0)) \leq \|\pi\|_{cb} \cdot \|\tilde{j}\|_{cb} \|\tilde{j}^{-1}\|_{cb} < ((\omega(A)\omega(B))^{1/2} + \varepsilon)\omega(B)^{1/2}$$

which is a contradiction if we choose $\delta > \omega(A)^{1/2}\omega(B)$ and ε small enough. This concludes the proof.

Proposition 4.

- a) Let A be an infinite dimensional C^* -algebra. Then $\omega(A) = \infty$ if A is either separable, nuclear or the full C^* -algebra of a free group.
- b) $\omega(C(H)) = 1$ where $C(H)$ is the Calkin algebra.
- c) $\omega(A) = 1$ if A is a von Neumann algebra not finite of type I.

Proof: Note that $A \subset B$ implies $\omega(B) \leq \omega(A)$.

- a) This was proved earlier for separable C^* -algebras, the nuclear case follows from Theorem 3 and for free group algebras it follows from Kirchberg's result and the above remark.
- b) Let $(p_n)_{n \geq 1}$ be an increasing sequence of finite dimensional projections in $B(H)$, $\lim_n p_n = I$. Since for every $T \in B(H)$ one has

$$\|T\| = \sup_n \|p_n T p_n\|$$

it follows that the map $T \rightarrow \bigoplus_n p_n T p_n$ establishes a completely isometric embedding $B(H) \rightarrow \bigoplus_n M_n \subset B(H)$. The distance from $\bigoplus_n p_n T p_n$ to the compact operators is equal to $\|\bigoplus_n p_n T p_n\| = \|T\|$, therefore $B(H)$ embeds completely isometrically into $C(H)$, hence $\omega(C(H)) = 1$. This recaptures Wassermann's result on $C(H) \otimes C(H)$ [6].

- c) Let R be the hyperfinite type II_1 factor. We first show that $\omega(R) = 1$. Let $(e_n)_{n \geq 1}$ be a sequence of mutually orthogonal projections in R , $\sum_{n \geq 1} e_n = I$. Then $B(H) \subset \bigoplus_n M_n \subset \bigoplus_n e_n R e_n \subset R$ where the above inclusions are completely isometric embeddings, therefore $\omega(R) = 1$. If A is of type I_∞, II_∞ or III , then A contains a copy of $B(H)$ so $\omega(A) = 1$. If A is of type II_1 then A contains a copy of R and since $\omega(R) = 1$ we get $\omega(A) = 1$.

Remark. In all known cases the value of ω is either ∞ or 1. Is there any C^* -algebra A for which $1 < \omega(A) < \infty$?

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On positive solutions of Emden - Fowler equations

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Presented by G.F.D. Duff, F.R.S.C.

Abstract. *We present a necessary and sufficient condition for the nonexistence of any positive solution of the equation $\Delta u + p(x)|u|^\gamma \text{sgn}(u) = 0$ in exterior domains of R^2 .*

Let us consider the Emden-Fowler equation

$$(1) \quad \Delta u + p(x)|u|^\gamma \text{sgn}(u) = 0, \quad \gamma > 0,$$

where $p : \Omega \rightarrow R_+$ is continuous and nonnegative in some exterior domain $\Omega \subset R^2$ (Ω is called exterior if $\{x \in R^2 : |x| > a\} \subset \Omega$ for some $a > 0$). A solution of (1) is a function $u \in C^2(\Omega, R)$ satisfying (1) in Ω .

Since the linear case $\gamma = 1$ is well-understood (see [1]), we will consider the problem of the nonexistence of positive global solutions of (1) in the sublinear ($0 < \gamma < 1$) and superlinear ($\gamma > 1$) case.

Let us denote

$$p_*(t) = \inf_{|x|=t} \{p(x)\}, \quad p^*(t) = \sup_{|x|=t} \{p(x)\}, \quad t > 0.$$

Theorem 1 [2]. *If equation (1) has no positive solution, then*

$$(2) \quad \int_1^\infty t(\log t)^\gamma p^*(t) dt = \infty, \quad 0 < \gamma < 1,$$

$$(3) \quad \int_1^\infty t(\log t) p^*(t) dt = \infty, \quad \gamma > 1.$$

It turns out (see again [2]) that these conditions are also sufficient if $\limsup_{t \rightarrow \infty} \left\{ \frac{p^*(t)}{p_*(t)} \right\}$ is finite.

Key Words and Phrases: Emden-Fowler equation, positive solution.
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This assumption permits only small fluctuations of p on the sphere $S_r = \{x \in \mathbb{R}^2 : |x| = r\}$ as $r \rightarrow \infty$, and we would like to show that one can allow

$$\limsup_{t \rightarrow \infty} \left\{ \frac{p^*(t)}{p_*(t)} \right\} = \infty$$

provided we control the growth.

Let $\mathfrak{R}$ be the class of nondecreasing functions $w \in C(R_+, R_+)$ with $w(t) > 0$ for $t > 0$ and $\lim_{t \rightarrow \infty} w(t) = \infty$, satisfying $\int_1^\infty \frac{dt}{w(t)} = \infty$. Examples of functions $w \in \mathfrak{R}$ are linear functions and $w(t) = t \ln(t+1)$, $t \geq 0$.

Theorem 2. *Let $\gamma > 1$. If there is a $K \geq 1$ and a function $w \in \mathfrak{R}$ such that*

$$(4) \quad \frac{p^*(t)}{w(\int_1^t s(\log s)p^*(s)ds)} \leq p_*(t), \quad t \geq K,$$

then (3) is the necessary and sufficient condition for the nonexistence of positive global solutions to (1).

Proof. The necessity follows by Theorem 1.

In order to prove the sufficiency, we intend to show that if (3) holds, then

$$(5) \quad \int_1^\infty t(\log t)p_*(t)dt = \infty.$$

Let us denote

$$V(t) = \int_1^t s(\log s)p_*(s)ds, \quad W(t) = \int_1^t s(\log s)p^*(s)ds, \quad t \geq K.$$

By (4) we deduce that

$$V'(t) \geq t(\log t)p_*(t) \geq \frac{t(\log t)p^*(t)}{w(\int_1^t s(\log s)p^*(s)ds) + 1}, \quad t \geq K,$$

and an integration yields

$$V(t) \geq V(K) + \int_{W(K)}^{W(t)} \frac{ds}{w(s) + 1}, \quad t \geq K.$$

Let $A > W(K)$ be such that $w(t) \geq 1$ for $t \geq A$. Since $\lim_{t \rightarrow \infty} W(t) = \infty$ in view of (3), there is an $M \geq K$ with $W(t) \geq A$ for $t \geq M$. We obtain

$$V(t) \geq V(K) + \int_A^{W(t)} \frac{ds}{w(s) + 1} \geq V(K) + \frac{1}{2} \int_A^{W(t)} \frac{ds}{w(s)}, \quad t \geq M.$$

Remembering the fact that $w \in \mathfrak{R}$, this yields (letting $t \rightarrow \infty$) $\lim_{t \rightarrow \infty} V(t) = \infty$, so that (5) holds.

Referring now to another result of [2], condition (5) is sufficient to guarantee that there are no positive global solutions to (1). $\square$

Similarly, we can prove

Theorem 3. *Let $\gamma \in (0, 1)$. If there is a $K \geq e$ and a function $w \in \mathfrak{R}$ such that*

$$\frac{p^*(t)}{w(\int_e^t s(\log s)^\gamma p^*(s) ds)} \leq p_*(t), \quad t \geq K,$$

then (2) is the necessary and sufficient condition for the nonexistence of positive global solutions to (1).

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SERIES SOLUTION OF FORWARD EDDY CURRENT PROBLEM FOR A CYLINDER WITH NONCONSTANT WALL PROPERTIES

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ABSTRACT. The change of impedance in a double conductor line parallel to an infinitely long metallic circular cylinder consisting of a homogeneous inner core and an outer layer is found in the form of an infinite series. The relative magnetic permeability, $\mu(r) = r^\alpha$, and the conductivity, $\sigma(r) = \sigma^{(0)}r^\kappa$, of the outer layer vary with respect to the radial coordinate, r , and α and κ are real constants. Numerical results are presented in the form of figures.

On exprime le changement d'impédance d'un double fil conducteur parallèle à un cylindre circulaire infiniment long formé d'une double couche métallique au moyen d'une série infinie. Le noyau cylindrique est homogène, mais la perméabilité magnétique relative $\mu(r) = r^\alpha$ et la conductivité $\sigma(r) = \sigma^{(0)}r^\kappa$ de la couche extérieure sont fonctions de la coordonnée radiale r , où α et κ sont des constantes. On présente les résultats numériques sous forme de figures.

1. Introduction. Since analytical solutions to eddy current testing problems usually assume constant properties of materials [1], analytical solutions are needed for media with spatially varying properties. Particular cases are solved in [2]-[4].

This note reports on a series solution [5] for the change of impedance in a double conductor line parallel to a double-layered metallic cylinder for which the magnetic permeability, $\mu(r)$, and the conductivity, $\sigma(r)$, of the outer cylinder are functions of the radial coordinate, r , of the form $\mu(r) = r^\alpha$, $\sigma(r) = \sigma^{(0)}r^\kappa$, where α and κ are arbitrary real numbers. The solution is found for single wires and then superposed for double wires. Numerical results are presented in the form of curves in Fig. 2.

2. Governing equations. Consider a double conductor line, w_1 and w_2 , parallel to a two-layered metallic cylinder. The radii of the inner core and outer annular shell are $\bar{\rho}_2$ and $\bar{\rho}_1$, respectively, as shown in Fig. 1. The conductivity, σ_2 , and the relative magnetic permeability, μ_2 , of the inner cylinder are constant, but for the outer shell,

$$\sigma_1(r) = \sigma^{(0)} \left(\frac{r}{l} \right)^\kappa, \quad \mu_1(r) = \left(\frac{r}{l} \right)^\alpha, \quad (1)$$

where $\alpha, \kappa \in \mathbb{R}$, $\sigma^{(0)} = \text{const}$ and l is the distance between the wires. A solution to a similar problem for constant σ_1 and μ_1 is found in [6]. Let (r, φ, z) be cylindrical polar coordinates centered at O with the z -axis parallel to the cylinder axis.

Since we neglect the displacement current, Maxwell's equations reduce to

$$(a) \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (b) \text{curl } \mathbf{H} = \sigma(r)\mathbf{E} + \mathbf{I}^e, \quad (c) \mathbf{B} = \mu_0\mu(r)\mathbf{H}, \quad (2)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths, respectively, $\mathbf{B}$ is the magnetic induction vector, $\mathbf{I}^e$ is the external current density, $\sigma(r)$ is the conductivity of the

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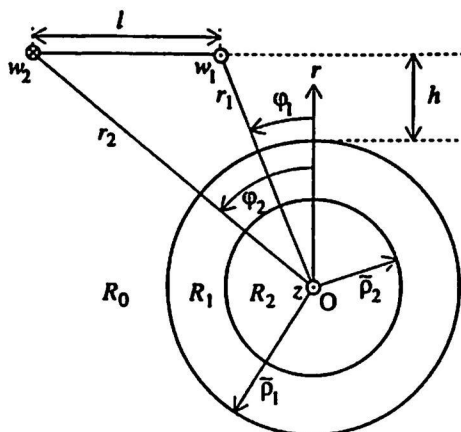


FIGURE 1. Double conductor line, in free space, lying in a horizontal plane above, and parallel to, a two-layered conducting circular cylinder.

medium and μ_0 and $\mu(r)$ are, respectively, the magnetic constant and the relative magnetic permeability of the medium.

Introducing the vector potential A by the relation $B = \text{curl } A$ and using (2)(a) we obtain

$$E = -\frac{\partial A}{\partial t} - \text{grad } \psi. \quad (3)$$

where ψ is the scalar potential. We assume that A , ψ and I^e are periodic in t ,

$$A = \tilde{A}(r, \varphi, z) e^{j\omega t}, \quad \psi = \tilde{\psi}(r, \varphi, z) e^{j\omega t}, \quad I^e = \tilde{I}^e(r, \varphi, z) e^{j\omega t}, \quad (4)$$

where $j = \sqrt{-1}$. By symmetry, $\tilde{A}$ is parallel to the z -axis and independent of z ,

$$\tilde{A}(r, \varphi, z) = (0, 0, A(r, \varphi)). \quad (5)$$

Using (2)-(5) and basic formulae of vector calculus we obtain

$$\begin{aligned} \frac{1}{\mu_0 \mu(r)} \text{grad div } \tilde{A} - \frac{1}{\mu_0 \mu(r)} \Delta \tilde{A} - \frac{1}{\mu_0 \mu(r)^2} \frac{d\mu}{dr} e_r \times \text{curl } \tilde{A} \\ = -j\omega\sigma(r) \tilde{A} - \sigma(r) \text{grad } \tilde{\psi} + \tilde{I}^e. \end{aligned} \quad (6)$$

We need to solve equation (6) in three cylindrical regions (see Fig. 1), free space R_0 : $\tilde{\rho}_1 < r < \infty$, outer metallic layer R_1 : $\tilde{\rho}_2 < r < \tilde{\rho}_1$, and inner metallic core R_2 : $0 \leq r < \tilde{\rho}_2$.

Taking the z -component of (6), using the gauge

$$\frac{1}{\mu_0 \mu(r)} \text{grad div } \tilde{A} = -\sigma(r) \text{grad } \tilde{\psi},$$

and denoting $A(r, \varphi)$, by $A_i(r, \varphi)$ in R_i , $i = 0, 1, 2$, we obtain

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_0}{\partial \varphi^2} = -\mu_0 I \delta(r - r_1) \delta(\varphi - \varphi_1) + \mu_0 I \delta(r - r_2) \delta(\varphi - \varphi_2), \quad (7)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \left[\frac{1}{r} - \frac{1}{\mu_1(r)} \frac{d\mu_1}{dr} \right] \frac{\partial A_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_1}{\partial \varphi^2} - j\omega\sigma_1(r)\mu_0\mu_1(r)A_1 = 0, \quad (8)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_2}{\partial \varphi^2} - j\omega\sigma_2\mu_0\mu_2 A_2 = 0. \quad (9)$$

3. Mathematical analysis. We introduce the dimensionless variable $r_d = r/l$, where the distance, l , between the wires is chosen as unit length, and note, from (1), that $\mu_1(r_d) = r_d^\alpha$ and $\sigma_1(r_d) = \sigma^{(0)} r_d^\alpha$. Henceforth, the subscript d will be omitted, and r_1 and r_2 are dimensionless variables.

From (7)–(9), we obtain the following system of equations:

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_0}{\partial \varphi^2} = -\mu_0 I l^2 \delta(r - r_1) \delta(\varphi - \varphi_1) + \mu_0 I l^2 \delta(r - r_2) \delta(\varphi - \varphi_2), \quad (10)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1 - \alpha}{r} \frac{\partial A_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_1}{\partial \varphi^2} - j\beta_1^2 r^{\alpha+\kappa} A_1 = 0, \quad (11)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_2}{\partial \varphi^2} - j\beta_2^2 A_2 = 0, \quad (12)$$

where $\beta_1 = l\sqrt{\omega\sigma^{(0)}\mu_0}$ and $\beta_2 = l\sqrt{\omega\sigma_2\mu_0\mu_2}$. The boundary conditions are

$$A_0|_{r=\rho_1} = A_1|_{r=\rho_1}, \quad \frac{\partial A_0}{\partial r}|_{r=\rho_1} = \frac{1}{\mu_{11}} \frac{\partial A_1}{\partial r}|_{r=\rho_1}, \quad (13)$$

$$A_1|_{r=\rho_2} = A_2|_{r=\rho_2}, \quad \frac{1}{\mu_{12}} \frac{\partial A_1}{\partial r}|_{r=\rho_2} = \frac{1}{\mu_2} \frac{\partial A_2}{\partial r}|_{r=\rho_2}, \quad (14)$$

where $\mu_{11} = \rho_1^\alpha$, $\mu_{12} = \rho_2^\alpha$ and $\rho_1 = \tilde{\rho}_1/l$, $\rho_2 = \tilde{\rho}_2/l$.

Since system (10)–(14) is linear, its solution can be expressed as the sum of two solutions. For the first solution, only the first term on the right-hand side of (10) is present. For the second solution, only the second term on the right-hand side of (10) is present and we replace r_1 by r_2 , φ_1 by φ_2 and I by $-I$ in the first solution.

Since we are interested in the change of impedance in the double conductor line due to the presence of the conducting cylinder, we shall consider only the induced potential, $A_0^{\text{ind}}(r, \varphi)$, which, in R_0 , can be written as the sum of two parts:

$$A_0^{\text{ind}}(r, \varphi) = A_0^{(w_1)}(r, \varphi) + A_0^{(w_2)}(r, \varphi),$$

where the first and second terms on the right-hand side correspond to the wires w_1 and w_2 , with coordinates $r = r_i$, $\varphi = \varphi_i$, $-\infty < z < \infty$ for $i = 1, 2$, respectively.

To obtain the first solution, we expand Dirac's delta function in a Fourier series,

$$\delta(\varphi - \varphi_1) = \frac{1}{\pi} \sum_{n=0}^{\infty} \delta_n \cos n(\varphi - \varphi_1), \quad \text{where } \delta_n = \begin{cases} 1, & n > 0, \\ 1/2, & n = 0, \end{cases}$$

and seek the solution to (10)–(14) in a Fourier series,

$$A_i(r, \varphi) = \frac{1}{\pi} \sum_{n=0}^{\infty} \delta_n a_{in}(r) \cos n(\varphi - \varphi_1). \quad (15)$$

Substituting (15) into (10)–(14) we obtain a system of ordinary differential equations,

$$\frac{d^2 a_{0n}}{dr^2} + \frac{1}{r} \frac{da_{0n}}{dr} - \frac{n^2}{r^2} a_{0n} = -\mu_0 I l^2 \delta(r - r_1), \quad (16)$$

$$\frac{d^2 a_{1n}}{dr^2} + \frac{1-\alpha}{r} \frac{da_{1n}}{dr} - \frac{n^2}{r^2} a_{1n} - j\beta_1^2 r^{\alpha+\kappa} a_{1n} = 0, \quad (17)$$

$$\frac{d^2 a_{2n}}{dr^2} + \frac{1}{r} \frac{da_{2n}}{dr} - \frac{n^2}{r^2} a_{2n} - j\beta_2^2 a_{2n} = 0, \quad (18)$$

with the boundary conditions

$$a_{0n}|_{r=\rho_1} = a_{1n}|_{r=\rho_1}, \quad \left. \frac{da_{0n}}{dr} \right|_{r=\rho_1} = \frac{1}{\mu_{11}} \left. \frac{da_{1n}}{dr} \right|_{r=\rho_1}, \quad (19)$$

$$a_{1n}|_{r=\rho_2} = a_{2n}|_{r=\rho_2}, \quad \left. \frac{1}{\mu_{12}} \frac{da_{1n}}{dr} \right|_{r=\rho_2} = \left. \frac{1}{\mu_2} \frac{da_{2n}}{dr} \right|_{r=\rho_2}. \quad (20)$$

The structure of the solution to equation (16) depends on the value of n .

It is convenient to consider two cylindrical subregions, R_{00} : $\rho_1 < r < r_1$ and R_{01} : $r > r_1$, of region R_0 , where $0 \leq \varphi \leq 2\pi$, $-\infty < z < \infty$.

First, if $n = 0$, we denote the solution of (16) in R_{00} and R_{01} by $a_{00}^{(0)}$ and $a_{00}^{(1)}$, respectively. Bounded general solutions of (16) in R_{00} and R_{01} are

$$a_{00}^{(0)}(r) = C_{10} \ln r + C_{20}, \quad a_{00}^{(1)}(r) = C_{30}, \quad (21)$$

respectively. Second, if $n \neq 0$, bounded general solutions to (16) in R_{00} and R_{01} are

$$a_{0n}^{(0)}(r) = C_{1n} r^n + C_{2n} r^{-n}, \quad a_{0n}^{(1)}(r) = C_{3n} r^{-n}, \quad (22)$$

respectively. The general solution to equation (17) is (see [7])

$$a_{1n}(r) = C_{4n} r^a J_p(br^c) + C_{5n} r^a Y_p(br^c), \quad (23)$$

where $J_p(s)$ and $Y_p(s)$ are Bessel functions and

$$a = \frac{\alpha}{2}, \quad c = \frac{\alpha + \kappa}{2} + 1, \quad b = \frac{\beta_1 \sqrt{-j}}{c}, \quad p = \frac{\sqrt{\alpha^2 + 4n^2}}{\kappa + \alpha + 2}.$$

A bounded general solution to (18) is

$$a_2(r) = C_{6n} J_n(kr), \quad (24)$$

where $k = \beta_2 \sqrt{-j}$. The structure of (23) and (24) remains the same for $n = 0, 1, 2, \dots$, except for the case $\alpha + \kappa = -2$ where (17) degenerates into Euler's equation.

We solve (16)–(20) in the case $n = 0$. Since the vector potential is continuous at $r = r_1$ we have $a_{00}^{(0)}(r_1) = a_{00}^{(1)}(r_1)$, that is

$$C_{10} \ln r_1 + C_{20} = C_{30}. \quad (25)$$

Multiplying (16) by r , integrating with respect to r from $r_1 - \varepsilon$ to $r_1 + \varepsilon$ and taking the limit as $\varepsilon \rightarrow 0$, we obtain

$$\left. \frac{da_{02}}{dr} \right|_{r=r_1} - \left. \frac{da_{01}}{dr} \right|_{r=r_1} = -\mu_0 I l^2. \quad (26)$$

Using (21), (23) and (24) (for $n = 0$) and determining the constants $C_{10}, C_{20}, \dots, C_{60}$ from (19), (20), (25) and (26), we obtain the first term of the Fourier series of $A_0^{(w_1)}(r, \varphi)$ as

$$a_0^{(w_1)}(r) = \frac{\mu_0 \mu_{11} I l^2 r_1 \rho_2^{-a} \rho_1^a [d_0 J_p(b\rho_1^c) + Y_p(b\rho_1^c)]}{d_0 [a J_p(b\rho_1^c) + bc \rho_1^c J_p'(b\rho_1^c)] + a Y_p(b\rho_1^c) + bc \rho_1^c Y_p'(b\rho_1^c)}, \quad (27)$$

where

$$d_0 = -\frac{\mu_{12} k \rho_2 J_0'(k\rho_2) Y_p(b\rho_2^c) - \mu_2 J_0(k\rho_2) [a Y_p(b\rho_2^c) + bc \rho_2^c Y_p'(b\rho_2^c)]}{\mu_{12} k \rho_2 J_0'(k\rho_2) J_p(b\rho_2^c) - \mu_2 J_0(k\rho_2) [a J_p(b\rho_2^c) + bc \rho_2^c J_p'(b\rho_2^c)]},$$

and ' denotes ordinary derivative of a function of one variable.

The solution to (16)–(20) for $n = 1, 2, \dots$, can be found in a similar way. Thus,

$$A_0^{(w_1)}(r, \varphi) = \frac{a_0^{(w_1)}(r)}{2\pi} + \frac{\mu_0 I l^2 r_1}{2\pi} \sum_{n=1}^{\infty} \frac{\rho_1^{2n}}{n r_1^n r^n} \frac{b_{11}}{b_{12}} \cos n(\varphi - \varphi_1), \quad (28)$$

where

$$\begin{aligned} b_{11} &= \mu_{11} n [d_n J_p(b\rho_1^c) + Y_p(b\rho_1^c)] - d_n [a J_p(b\rho_1^c) + bc \rho_1^c J_p'(b\rho_1^c)] \\ &\quad - a Y_p(b\rho_1^c) - bc \rho_1^c Y_p'(b\rho_1^c), \\ b_{12} &= \mu_{11} n [d_n J_p(b\rho_1^c) + Y_p(b\rho_1^c)] + d_n [a J_p(b\rho_1^c) + bc \rho_1^c J_p'(b\rho_1^c)] \\ &\quad + a Y_p(b\rho_1^c) + bc \rho_1^c Y_p'(b\rho_1^c), \\ d_n &= -\frac{\mu_{12} k \rho_2 J_n'(k\rho_2) Y_p(b\rho_2^c) - \mu_2 J_n(k\rho_2) [a Y_p(b\rho_2^c) + bc \rho_2^c Y_p'(b\rho_2^c)]}{\mu_{12} k \rho_2 J_n'(k\rho_2) J_p(b\rho_2^c) - \mu_2 J_n(k\rho_2) [a J_p(b\rho_2^c) + bc \rho_2^c J_p'(b\rho_2^c)]}, \end{aligned}$$

and a, b, c and p , in the first term (27) on the right-hand side of (28), correspond to $n = 0$. To obtain $A_0^{(w_2)}(r, \varphi)$, we replace r_1, φ_1 and I in (28) by r_2, φ_2 and $-I$, respectively. Adding $A_0^{(w_1)}(r, \varphi)$ and $A_0^{(w_2)}(r, \varphi)$ we have

$$\begin{aligned} A_0^{\text{ind}}(r, \varphi) &= \frac{\mu_0 I l^2 (r_1 - r_2)}{2\pi} \frac{\mu_{11} \rho_2^{-a} \rho_1^a [d_0 J_p(b\rho_1^c) + Y_p(b\rho_1^c)]}{d_0 [a J_p(b\rho_1^c) + bc \rho_1^c J_p'(b\rho_1^c)] + a Y_p(b\rho_1^c) + bc \rho_1^c Y_p'(b\rho_1^c)} \\ &\quad + \frac{\mu_0 I l^2}{2\pi} \left[r_1 \sum_{n=1}^{\infty} \frac{\rho_1^{2n}}{n r_1^n r^n} \frac{b_{11}}{b_{12}} \cos n(\varphi - \varphi_1) - r_2 \sum_{n=1}^{\infty} \frac{\rho_1^{2n}}{n r_1^n r^n} \frac{b_{11}}{b_{12}} \cos n(\varphi - \varphi_2) \right]. \end{aligned} \quad (29)$$

The induced change of impedance is given by the formula

$$Z^{\text{ind}} = \frac{j\omega}{I} \oint_L A_0^{\text{ind}}(r, \varphi) dl = \frac{\omega \mu_0 l^2}{2\pi} Z_0, \quad (30)$$

where L is the contour of integration along the two wires in the opposite directions over one unit of length. Hence, from (29) and (30) we obtain

$$Z_0 = j \sum_{n=1}^{\infty} \frac{b_{11}}{n b_{12}} \left[r_1 \left(\frac{\rho_1}{r_1} \right)^{2n} + r_2 \left(\frac{\rho_1}{r_2} \right)^{2n} - (r_1 + r_2) \left(\frac{\rho_1}{r_1} \right)^n \left(\frac{\rho_1}{r_2} \right)^n \cos n(\varphi_1 - \varphi_2) \right]. \quad (31)$$

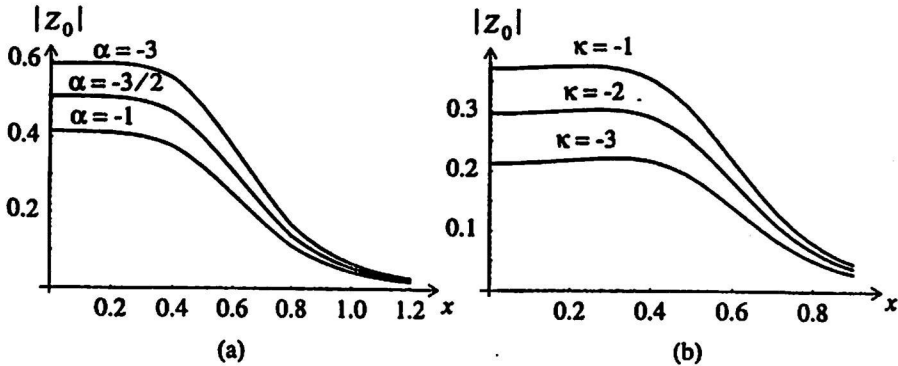


FIGURE 2. $|Z_0|$ against x (a) for magnetic permeability $\mu_1(r) = r^\alpha$ and (b) for conductivity $\sigma_1(r) = \sigma^{(0)}r^\kappa$, of medium R_1 .

4. Numerical results. Formula (31) was used for computing Z_0 for different values of the parameters of the problem by means of Mathematica, version 2.2.2, on a Sun Microsystems Sparc 10, which can evaluate Bessel functions of fractional order and of a complex argument.

In Fig. 2, $\mu_1(r) = r^\alpha$, $\sigma_1(r) = \sigma^{(0)}r^\kappa$, $\mu_{11} = \rho_1^\alpha$, $\mu_{12} = \rho_2^\alpha$, $\mu_2 = 1$, $\beta_1 = \beta_2 = 1$ and $h = 0.1$, and $|Z_0|$ is plotted against the horizontal shift, $x = r_1 \sin \varphi_1 + 1/2$, of the centre of the double conductor line measured from the vertical axis $\varphi = 0$. In Fig. 2(a), $\kappa = 0$, $\rho_1 = 0.4$ and $\rho_2 = 0.3$. It is seen that $|Z_0|$ increases as the change of magnetic permeability across the layer $\rho_2 < r < \rho_1$ becomes stronger, a fact which is important in qualitative analysis of eddy current testing of media with varying properties. In Fig. 2(b), $\mu_1(r) = r^{-2}$, $\rho_1 = 0.3$ and $\rho_2 = 0.2$. It is seen that $|Z_0|$ decreases with α and increases with κ .

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QUASI-HEREDITY OF ENDOMORPHISM ALGEBRAS

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ABSTRACT. Quasi-hereditary algebras were introduced by Cline-Parshall-Scott (see [CPS] or [PS]) to deal with highest weight categories which occur in the study of semi-simple complex Lie algebras and algebraic groups. In fact, the quasi-hereditary algebras which appear in these applications enjoy a number of additional properties. The objective of this brief note is to describe a class of lean quasi-hereditary algebras [ADL] which possess such typical characteristics. The complete proofs of the statements will appear elsewhere.

Let A be a finite dimensional (associative) algebra. Let $\{S(\lambda) | \lambda \in \Lambda\}$ be the set of all non-isomorphic (left) simple A -modules indexed by a partially ordered set Λ . For every λ , denote by $P(\lambda)$ the projective cover of $S(\lambda)$ and by $\Delta(\lambda)$ the corresponding standard module, i.e. the maximal factor module of $P(\lambda)$ with composition factors of the form $S(\kappa)$ for $\kappa \leq \lambda$.

We say that A is quasi-hereditary with respect to Λ if there is a linear order $\lambda_1 < \lambda_2 < \dots < \lambda_n$ on Λ refining the given partial order and satisfying the following conditions: for each $1 \leq i \leq n$,

(i) the standard module defined above equals

$$\Delta(\lambda_i) = P(\lambda_i) / \text{trace}(\bigoplus_{j>i} P(\lambda_j) \rightarrow P(\lambda_i)),$$

(ii) the endomorphism algebra of $\Delta(\lambda_i)$ is a division algebra and

(iii) $P(\lambda_i)$ can be filtered by $\Delta(\lambda_j)$'s, $j \geq i$.

Here, $\text{trace}(X \rightarrow Y)$ denotes the submodule of Y generated by all homomorphic images of X in Y . The latter condition is equivalent to the fact that the factors

$$\text{trace}(\bigoplus_{j=k}^n P(\lambda_j) \rightarrow P(\lambda_i)) / \text{trace}(\bigoplus_{j=k+1}^n P(\lambda_j) \rightarrow P(\lambda_i))$$

of the trace filtration of $P(\lambda_i)$ are direct sums of $\Delta(\lambda_k)$'s ($i \leq k \leq n$) [D].

Throughout the note, R denotes a (finite dimensional) commutative local self-injective K -algebra with a splitting field K , and A the endomorphism algebra of a (finite) direct sum $X = \bigoplus_{\lambda \in \Lambda} X(\lambda)$ of pair-wise non-isomorphic (finite dimensional)

local-colocal R -modules $X(\lambda)$, i.e. such that both $X(\lambda)/\text{rad}X(\lambda)$ and $\text{soc}X(\lambda)$ are simple. Write, for each λ , $e_\lambda = m_\lambda p_\lambda$, where $p_\lambda : X \rightarrow X(\lambda)$ and $m_\lambda : X(\lambda) \rightarrow X$

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Typeset by $\mathcal{A}_M\mathcal{S}\text{-}\mathcal{T}_E\mathcal{X}$

are the canonical projection and embedding, respectively. Thus, for all $\lambda \in \Lambda$, $S(\lambda) = Ae_\lambda / \text{rad} Ae_\lambda$ are the (pair-wise non-isomorphic) left simple A -modules, $P(\lambda) = Ae_\lambda$ their projective covers and $I(\lambda) = \text{Hom}_K(e_\lambda A, K)$ their injective hulls. Write also $\varepsilon_t = \sum_{i=1}^n e_i$ for $1 \leq t \leq n$ and $\varepsilon_{n+1} = 0$.

Observe that, for each $X(\lambda)$, there is a (unique) embedding into R and that every R -homomorphism $f: X(\lambda) \rightarrow X(\kappa)$ is induced by multiplication by an element $r \in R$: Given f , there is an extension $\bar{f}: R \rightarrow R$ and every endomorphism of R_R is given by multiplication,

$$\begin{array}{ccccc} 0 & \longrightarrow & X(\lambda) & \longrightarrow & R_R \\ & & \downarrow f & & \downarrow \bar{f}=r \\ 0 & \longrightarrow & X(\kappa) & \longrightarrow & R_R \end{array}$$

Thus, in particular the image $\text{Im} f$ is isomorphic to a submodule of $X(\lambda)$. As a result, the following three statements are equivalent:

- (a) $R \supseteq X(\kappa) \supseteq X(\lambda)$;
- (b) there is a monomorphism from $X(\lambda)$ to $X(\kappa)$;
- (c) there is an epimorphism from $X(\kappa)$ to $X(\lambda)$.

Furthermore, each $X(\lambda)$ is a factor module of R and as such has a natural structure of a local commutative selfinjective K -algebra; thus $\text{Hom}_K(X(\lambda), K) \simeq X(\lambda)$. As a consequence, $A = \text{End}_R X$ is an algebra with involution and thus there is a duality functor $D: A\text{-mod} \rightarrow A\text{-mod}$ satisfying $D(S) \simeq S$ for all simple A -modules S . Indeed, the map $*$: $A \rightarrow A$ defined for

$$f: X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{f_{\lambda\lambda}} X(\kappa) \xrightarrow{m_\kappa} X$$

by

$$f^*: X \xrightarrow{p_\kappa} X(\kappa) \simeq \text{Hom}_K(X(\kappa), K) \xrightarrow{\text{Hom}(f_{\lambda\lambda}, K)} \text{Hom}_K(X(\lambda), K) \simeq X(\lambda) \xrightarrow{m_\lambda} X$$

is an involution. In addition to the relations $(ab)^* = b^*a^*$ and $(a^*)^* = a$, we have also $e_\lambda^* = e_\lambda$ for all $\lambda \in \Lambda$. Hence, we get a duality functor D if, for every right A -module Y_A we define the left module ${}_A Y^*$ by putting $Y^* = Y$ and $ay = ya^*$, and set $D(Y_A) = \text{Hom}_K({}_A Y^*, K)$. Thus $D(P(\lambda)) \simeq I(\lambda)$ and $D(S(\lambda)) \simeq S(\lambda)$.

The main result of this paper is the following theorem.

THEOREM. *Let R be a commutative local selfinjective K -algebra over a splitting field K ; $\dim_K R = n$. Let $\mathcal{X} = \{X(\lambda) | \lambda \in \Lambda\}$ be a set of local ideals of R indexed by a finite partially ordered set Λ reflecting inclusions: $X(\lambda') \subset X(\lambda'')$ if and only if $\lambda' > \lambda''$. Let $R = X(\lambda_1)$ belong to $\mathcal{X}$. Then $A = \text{End}(\bigoplus_{\lambda \in \Lambda} X(\lambda))$ is a quasi-hereditary algebra with respect to Λ if and only if*

- (i) $\text{card}(\Lambda) = n$ and
- (ii) $\text{rad} X(\lambda) = \sum_{\lambda < \kappa} X(\kappa)$.

Let us add that under the conditions of the theorem, we can easily verify the following facts:

- (a) as mentioned earlier, there is a duality functor on the category of A -modules which fixes the simple modules $S(\lambda)$, $\lambda \in \Lambda$;
- (b) the algebra A is lean (see [ADL]) and every standard module $\Delta(\lambda)$ has a simple socle isomorphic to $S(\lambda_1)$;
- (c) $[\Delta(\lambda) : S(\kappa)] \leq 1$ for all $\lambda, \kappa \in \Lambda$; in fact, $[\Delta(\lambda) : S(\kappa)] = 1$ if and only if $\kappa \leq \lambda$, and thus $\dim_K \Delta(\lambda) = \text{card}\{\kappa | \kappa \leq \lambda\}$;
- (d) $R/\text{rad} R \simeq X(\lambda_n) \in \mathcal{X}$, $\dim_K P(\lambda_n) = n$ and generally

$$\dim_K P(\kappa) = \sum_{\lambda \leq \kappa} \dim_K \Delta(\lambda);$$

$$\text{thus } \dim_K A = \sum_{\lambda \in \Lambda} (\dim_K \Delta(\lambda))^2;$$

- (e) the dominant dimension of A is ≥ 2 (see [T]).

The proof of the sufficiency of the theorem is based on the following four lemmas:

Lemma 1. The set $\{x_\lambda | \lambda \in \Lambda\}$, where $X(\lambda) = x_\lambda R$, is a K -basis of the vector space R_K , and the set of all ideals $X(I) \subseteq R$ generated by $\{x_\lambda | \lambda \in I\}$, for every subset I of Λ , forms a distributive lattice with respect to addition and intersection.

Lemma 2. Every R -homomorphism $f : X(\lambda) \rightarrow \sum_{\mu \in I} X(\mu) \subseteq R$ for some $I \subseteq \Lambda$, factors through the canonical (summation) map $p : \bigoplus_{\mu \in I} X(\mu) \rightarrow \sum_{\mu \in I} X(\mu)$. In particular, every R -homomorphism $f : X(\lambda) \rightarrow \text{rad} X(\kappa)$ factors through the canonical map $\bigoplus_{\mu > \kappa} X(\mu) \rightarrow \text{rad} X(\kappa)$.

Lemma 3. For every $\lambda \in \Lambda$,

$$\{m_\kappa m_{\kappa\lambda} p_\lambda | X(\lambda) \subseteq X(\kappa)\},$$

where $m_{\kappa\lambda}$ denotes the embedding $X(\lambda) \subseteq X(\kappa)$, is a K -basis for the (left) standard module $\Delta(\lambda)$. In fact,

$$\Delta(\lambda) = P(\lambda) / \text{trace}(\bigoplus_{\mu > \lambda} P(\mu) \rightarrow P(\lambda)).$$

Let us point out that Lemma 3 describes the structure of the standard modules: the factorizations $m_{\kappa\lambda} = m_{\kappa\rho} m_{\rho\lambda}$ correspond to the embeddings $X(\lambda) \subseteq X(\rho) \subseteq X(\kappa)$. In particular, every standard module $\Delta(\lambda)$ has a simple socle generated by $m_{\lambda_1\lambda}$, and hence is isomorphic to $S(\lambda_1)$.

Lemma 4. If $X(j) \subseteq X(i)$, then $Ae_j Ae_i / Ae_{j+1} Ae_i \simeq \Delta(j)$. If $X(j) \not\subseteq X(i)$, then $Ae_j Ae_i = Ae_{j+1} Ae_i$.

On the other hand, the proof that the conditions (i) and (ii) of the Theorem are necessary uses the following lemma together with the previously mentioned duality $D: A\text{-mod} \rightarrow A\text{-mod}$ fixing simple modules and Bernstein-Gelfand-Gelfand reciprocity law.

Lemma 5. Let $f: X(i) \rightarrow X(k)$ be an R -homomorphism. If f is a monomorphism, then $m_k f p_i \notin Ae_{i+1} Ae_i$. If f is not a monomorphism, and A is quasi-hereditary, then $m_k f p_i \in Ae_{i+1} Ae_i$. Therefore, if A is quasi-hereditary, then the multiplicity $[\Delta(i): S(k)] = 1$ for $X(i) \subseteq X(k)$ and $[\Delta(i): S(k)] = 0$ otherwise.

Remarks

Let us conclude this note with a few observations and examples.

First, the (ordered) quiver Q_A of the algebra A is given by the monomorphisms and epimorphisms between the direct summands of X . To be more explicit, let $(1, 2, \dots, n)$ be the sequence of the vertices of Q_A corresponding to a (linear) order of the direct summands $X(1) = R, X(2), \dots, X(n) = R/\text{rad} R$ of the module X (which refines the partial order Λ of the theorem). Then, for $i > j$, there is an arrow $i \rightarrow j$ in Q_A if and only if $X(i) \subset X(j) \subseteq R$ and there is no $X(k)$ satisfying $X(i) \subset X(k) \subset X(j) \subseteq R$ for $k \neq i, j$. Furthermore, in that case, there is an arrow $i \leftarrow j$ corresponding to an epimorphism $X(j) \rightarrow X(i)$ which cannot be factored through any $X(k)$, $k \neq i, j$. Thus, Q_A is a connected quiver with single arrows which appear in pairs: either there are no arrows between two vertices i and j of Q_A or there is a pair of arrows, $i \rightleftharpoons j$. From here, we can easily read the structure of the standard modules established earlier: each $\Delta(i)$ is given by the subquiver of Q_A consisting of all sequences of arrows

$$i = j_0 \rightarrow j_1 \rightarrow \dots \rightarrow j_{t-1} \rightarrow j_t = j, \quad i = j_0 > j_1 > \dots > j_{t-1} > j_t = j,$$

and the respective vertices.

Recall that the trace filtration of the projective-injective indecomposable module

$$P(1) = Ae_1 = Ae_1 Ae_1 \supset Ae_2 Ae_1 \supset \dots \supset Ae_n Ae_1 \supset 0$$

has the property that $Ae_i Ae_1 / Ae_{i+1} Ae_1 \simeq \Delta(i)$ for every $1 \leq i \leq n$. Here, the extensions

$$0 \rightarrow Ae_{i+1} Ae_1 \rightarrow Ae_i Ae_1 \rightarrow \Delta(i) \rightarrow 0$$

are determined by the arrows of Q_A corresponding to the epimorphisms. Observe that there is a (unique) embedding of $P(i)$ in $P(1)$, for every $1 \leq i \leq n$.

The following examples should serve as simple illustrations of the theorem, as well as indications of its limitations.

1. $R = K[x]/(x^t)$, $t \geq 1$. There is a unique choice of X (the direct sum of all indecomposable R -modules) and thus A is the respective Auslander algebra. The quiver Q_A is as follows:

$$1 \rightleftharpoons 2 \rightleftharpoons \dots \rightleftharpoons t-1 \rightleftharpoons t.$$

2. $R = K[x, y]/(xy, x^t - y^t)$, $t \geq 2$. Here, for $t \geq 3$, we have several choices for X ; for instance, we get the following forms of Q_A :

$$\begin{array}{ccccccc} 2 & \rightleftharpoons & 4 & \rightleftharpoons & \dots & \rightleftharpoons & 2s & \rightleftharpoons & 2s+2 & \rightleftharpoons & 2s+4 & \rightleftharpoons & \dots & \rightleftharpoons & 2t \\ \updownarrow & & \updownarrow & & \dots & & \updownarrow & & & & & & \dots & & \updownarrow \\ 1 & \rightleftharpoons & 3 & \rightleftharpoons & \dots & \rightleftharpoons & 2s-1 & \rightleftharpoons & 2s+1 & \rightleftharpoons & 2s+3 & \rightleftharpoons & \dots & \rightleftharpoons & 2t-1 \end{array},$$

$1 \leq s \leq t$.

3. $R = K[x, y]/(x^2 - y^3, x^3 - y^4, x^4)$. Here, the algebra is 8-dimensional. Write $\bar{p}$ for the canonical image of $p \in K[x, y]$ in R , and consider

$$X = R \oplus \bar{x}R \oplus \bar{y}R \oplus \bar{x}\bar{y}R \oplus \bar{y}^2R \oplus \bar{x}\bar{y}^2R \oplus \bar{x}^2R \oplus \bar{x}^3R$$

(in that linear order). Then $A = \text{End}_R X$ is a 159-dimensional algebra whose quiver Q_A has the form

$$\begin{array}{ccccccc} 2 & \rightleftharpoons & 4 & \rightleftharpoons & 6 & \rightleftharpoons & 8 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ 1 & \rightleftharpoons & 3 & \rightleftharpoons & 5 & \rightleftharpoons & 7 \end{array}.$$

4. Consider again the 4-dimensional algebra $R = K[x, y]/(xy, x^2 - y^2)$. Taking

$$X = R \oplus R/\bar{x}^2R \oplus \bar{x}^2R$$

(thus only 3 direct summands, not all local-colocal), or

$$X' = R \oplus (\bar{x}R \oplus R/\bar{x}^2R)/(\bar{x}^2 - (\bar{y} + \bar{x}^2R)) \oplus \bar{x}R \oplus \bar{x}^2R$$

(thus not all direct summands are local-colocal), the respective endomorphism algebras are still quasi-hereditary. The first one $A = \text{End}_R X$ is a 19-dimensional algebra (without duality) whose quiver Q_A is

$$1 \rightleftharpoons 2 \rightleftharpoons 3.$$

The algebra $A' = \text{End}_R X'$ is a 39-dimensional algebra with duality (and uniserial standard modules whose socles are isomorphic to $S(1)$, $[\Delta(4) : S(2)] = 2$) with $Q_{A'}$ of the form

$$\begin{array}{ccc} 1 & \rightleftharpoons & 2 & \rightleftharpoons & 4 \\ & & \updownarrow & & \\ & & 3 & & \end{array}.$$

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