C.R. Math. Rep. Acad. Sci. Canada

Vol. XVI,No. 5 **October 1994 octobre**

CONTENTS

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

CONTINUOUS MULTISCALE ANALYSIS AND PARTIAL RECONSTRUCTIONS ON SEMISIMPLE LIE GROUPS AND ON CARTAN MOTION GROUPS

K. TRIMECHE

Presented by G.A. Elliott, F.R.S.C.

Abstract :

In this work we strudy continuous multiscale analysis and partial reconstructions on semisimple Lie groups G of real rank ℓ and on $p = G_o/K$ where G_o are Cartan motion groups associated with $X = G/K$.

I. Continuous multiscale analysis on G.

Let G be a noncompact connected real semisimple Lie group with finite center and real rank ℓ.

We consider in this paper the notations used in my paper [2].

Definition I.1 : A continuous multiscale analysis on G is defined by a net $\{V_\varepsilon\}_{\varepsilon>o}$ of closed subspaces of $L^2(K\backslash G/K)$ satisfying

i) If $o < \varepsilon_1 \leq \varepsilon_2$, then $V_{\varepsilon_1} \subseteq V_{\varepsilon_2}$ (inclusion property)

ii) If $f \in V_\varepsilon$, $\varepsilon > 0$, then for all $x \in G$, the function $T_x(f)$ is also in V_ε, $\varepsilon > 0$, where T_x is the generalized translation operator on G (translation invariance of the subspaces V_ε, $\varepsilon > 0$).

iii) A function $f \in V_{\varepsilon_1}$, $\varepsilon_1 > 0$, if and only if the function $f_{\varepsilon_2/\varepsilon_1}$ belongs to V_{ε_2}, $\varepsilon_2 > 0$, where $f_{\varepsilon_2/\varepsilon_1}$ is the function of $L^2(K\backslash G/K)$ defined by :

$$\forall \lambda \in \mathfrak{a}^*, \ \mathcal{F}(f_{\varepsilon_2/\varepsilon_1})(\lambda) = \mathcal{F}(f)\left(\frac{\varepsilon_2}{\varepsilon_1}\lambda\right)$$

(rescaling property)

iv) We have : $\lim\limits_{\varepsilon \to o} V_\varepsilon = L^2(K\backslash G/K)$ and $\lim\limits_{\varepsilon \to +\infty} V_\varepsilon = \{0\}$.

Definition I.1 : Let g be a wavelet on G in $(L^p \cap L^2)(K\backslash G/K)$, $p \in [1,2[$, such that for all $a > 0$, the dilated function g_a of g belongs to $L^p(K\backslash G/K)$, $p \in [1,2[$. We denote by V_ε, $\varepsilon > 0$, the subspace of $L^2(K\backslash G/K)$ defined by

$$V_\varepsilon = \{ f \in L^2(K\backslash G/K) \ / \ \Phi_g(f)(a,x) = 0, \text{ a.e. on }]o,\varepsilon[\times G\}$$

where Φ_g is the continuous wavelet transform on G, and the almost every where is meant with respect to the measure da dx.

Theorem I.1 : The net $\{V_\varepsilon\}_{\varepsilon>0}$ given by the definition I.1 is a continuous multiscale analysis on G, if and only if the support of $\mathcal{F}(g)$ (the spherical Fourier transform of the wavelet g on G) is bounded away from zero, i.e., there exists an $\eta > 0$ such that

$$\{\lambda \in \mathfrak{a}^* \ / \ \|\lambda\| \leq \eta\} \cap \operatorname{supp}\mathcal{F}(g) = 0$$

Corollary I.1 : Let g be a wavelet on G satisfying the conditions of the definition I.1 and the support of $\mathcal{F}(g)$ is bounded away from zero. Let γ denote the lower limit frequency of g i.e,

$$\gamma = \operatorname{Sup}\{\gamma > 0 \ / \ \operatorname{Supp}\mathcal{F}(g) \subset \{\lambda \in \mathfrak{a}^* \ / \ \|\lambda\| \geq \gamma\}\}$$

Then $V_\varepsilon, \varepsilon > 0$, consists of the band limited functions with upper limit frequency $\frac{\gamma}{\varepsilon}$, i.e.,

$$\operatorname{Supp}\mathcal{F}(f) \subset \{\lambda \in \mathfrak{a}^* \ / \ \|\lambda\| \leq \frac{\gamma}{\varepsilon}\}, \text{ for all } f \in V_\varepsilon.$$

Remarks

i) From the corollary I.1 we deduce that the projection P_{V_ε} on the subspace V_ε, $\varepsilon > 0$, is given by the relation

$$\forall \lambda \in \mathfrak{a}^*, \ \mathcal{F}(P_{V_\varepsilon}(f))(\lambda) = \chi_B(\lambda)\,\mathcal{F}(f)(\lambda)$$

where χ_B is the characteristic function of the ball $\{\lambda \in \mathfrak{a}^* \ / \ \|\lambda\| \leq \frac{\gamma}{\varepsilon}\}$.

ii) The condition that supp $\mathcal{F}(g)$ is bounded away from zero implies that wavelets on G which belongs to $\mathcal{D}(K\backslash G/K)$ cannot define a continuous multiscale analysis on G.

iii) H.G.Stark has studied in [1] continuous multiscale analysis associated with the affine group.

II. Partial reconstructions on G.

We suppose in this section that the rank of G is one. Let g be a wavelet on G satisfying the conditions of the corollary I.1.

Definition II.1 : let $\varepsilon > 0$ and $f \in L^2(\mathbb{R}_+, A(t)dt)$. The partial reconstructions f_ε of f "down to scale ε " from its continuous wavelet transform Φ_g on G, is defined by

$$f_\varepsilon(.) = \frac{1}{C_g}\int_\varepsilon^{+\infty}\left(\int_0^\infty \Phi_g(f)(a,y)\; g_{a,y}(.)\; A(y)dy\right)\frac{k(a)}{a^2}\, da$$

Proposition I.1 :

i) There exists a function $\theta \in L^2(\mathbb{R}_+, A(t)\, dt)$ such that

$$\forall \lambda \geq 0,\ \mathcal{F}(\theta)(\lambda) = \frac{1}{C_g}\int_\lambda^{+\infty} |\mathcal{F}(g)(u)|^2 \frac{du}{u}$$

ii) Let $\varepsilon > 0$. We denote θ_ε the function in $L^2(\mathbb{R}_+, A(t)dt)$ give by

$$\forall \lambda \geq 0,\ \mathcal{F}(\theta_\varepsilon)(\lambda) = \mathcal{F}(\theta)(\varepsilon\lambda)$$

iii) We have

$$\forall \lambda > 0,\ \mathcal{F}(\theta_\varepsilon)(\lambda) = \frac{1}{C_g}\int_\varepsilon^{+\infty} |\mathcal{F}(g)|\ (a\lambda)|^2 \frac{da}{a}$$

iv) - The even function $\mathcal{F}(\theta_\varepsilon)$ is continuous on $\mathbb{R}$, positive and monotonely decreasing on $[o,+\infty[$.

- $\forall \lambda \in [-\frac{\gamma}{\varepsilon}, \frac{\gamma}{\varepsilon}],\ \mathcal{F}(\theta_\varepsilon)(\lambda) = 1$

where γ is the lower limit frequency of g.

- $\lim_{\lambda \to +\infty} \mathcal{F}(\theta_\varepsilon)(\lambda) = 0$

Theorem II.1 : Let f be a continuous and bounded function on $[o,+\infty[$ which belongs to $(L^p \cap L^2)(\mathbb{R}_+, A(t)\, dt)$, $p \in [1,2[$, and such that $\mathcal{F}(f) \in L^1(\mathbb{R}_+, |C(\lambda)|^{-2} d\lambda)$ (the space of functions f on $\mathbb{R}_+$, mesurable and such that $\int_0^\infty |f(\lambda)|\ |C(\lambda)|^{-2} d\lambda < +\infty$). Then we have

$$\forall \lambda \geq 0,\ \mathcal{F}(f_\varepsilon)(\lambda) = \mathcal{F}(f)(\lambda).\mathcal{F}(\theta_\varepsilon)(\lambda)$$

Remarks

i) The projection of f onto V_ε, $\varepsilon > 0$, (in the sens of the preceding section) amounts simply to multiplying $\mathcal{F}(f)(\lambda)$ with a "brute force" window, namely the characteristic function of $[-\frac{\gamma}{\varepsilon}, \frac{\gamma}{\varepsilon}]$, whereas for f_ε this window function additionally has a "tail" containing contributions also from frequencies outside the intervall. The strength of these contributions, mesured by $\mathcal{F}(\theta_\varepsilon)(\lambda)$, depends on the localization properties of the wavelet g in the λ-space.

ii) H. G. Stark has studied in [1] partial reconstructions associated with the affine group.

III. Continuous multiscale analysis and partial reconstructions on $\mathfrak{p}$.

We define as for G a continuous multiscale analysis and partial reconstructions on $\mathfrak{p}$, and prove the same results as in sections I and II.

REFERENCES

[1] H. G. STARK : Continuous wavelet transform and continuous multiscale analysis. J. Math. Analy. and Appl. 169, 1992, pp 179 - 196.

[2] K. TRIMECHE : Continuous wavelet transforms on semisimple Lie groups and on Cartan motion groups. Math. Reports

Department of Mathematics, Faculty of Sciences of Tunis, 1060 Tunis, TUNISIA.

Received June 6, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre 177

A NEW THEORY OF DIFFERENTIABILITY FOR A CERTAIN CLASS OF MULTIVALUED MAPS

GILLES FOURNIER AND DONALD VIOLETTE

Presented by G.F.D. Duff, F.R.S.C.

Abstract. The aim of this work is to develop a theory of differentiability of single-valued type for a class of functions which are compositions of a multivalued map having an acyclic decomposition followed by a single-valued function. This new theory is an important improvement of the one given by the same authors in [3], since it is more convenient while working with homotopy.

Introduction

The theory of differentiability for multivalued maps has already been considered by several mathematicians from different points of view (see [1] to [5]). The principal aim of this paper is to develop a theory of differentiability for a class of multivalued functions which are compositions of a multivalued map having an acyclic decomposition followed by a continuous single-valued map. This new concept of differentiability allows us to have a differentiable map which is not continuous and to obtain a mean-value theorem. Also, the composition of multivalued differentiable maps is differentiable. Finally, this new theory is more convenient while working with homotopy since the convex homotopy of two differentiable maps is differentiable. We hope to apply this notion of differentiability to the optimal control theory in which the control that we can exert on the system depends, in a continuously differentiable manner, on the position we are at e.g. the control is restricted by the state of the system we occupy at a given time.

1. Notations and definitions

Let X and Y be two spaces. In this paper, all multivalued maps $F: X \to Y$ are assumed to satisfy $F(x)$ is compact for every $x \in X$. F is acyclic if (i) for every $x \in X$, $F(x)$ is acyclic with respect to Čech homology functor H with rationals coefficients i.e. $F(x)$ is non-empty, $H_0(F(x)) \simeq Q$ and $H_q(F(x)) = 0 \ \forall\, q \geq 1$, and if (ii) F is upper semi-continuous (u.s.c.) i.e. for each open subset V of Y, the set $F^{-1}(V) = \{x \in X \mid F(x) \subset V\}$ is an open subset of X. If $F: X \to Y$ and $G: Y \to Z$ are two u.s.c. multivalued maps, then the composition $G \circ F: X \to Z$, defined by $(G \circ F)(x) = G(F(x)) \ \forall\, x \in X$, is u.s.c.

Let $f = (F_n, \ldots, F_0)$ be a sequence of acyclic maps $F_i: X_i \to X_{i+1}$ where each space X_i is Hausdorff, $i = 0, 1, \ldots, n$. The sequence f is said to be an acyclic decomposition for the map $F = F_n \circ \ldots \circ F_0$: $X_0 \to X_{n+1}$.

2. A class of multivalued maps

Definition 2.1.

Let X, Y, Z be Hausdorff spaces. We denote by $MU(X, Y)$ the class of all acyclic multivalued maps $F: X \to Y$ which can be expressed as $F = \varphi \circ i$ where $i: X \to X \times Z$ is a multivalued map defined by $i(x) = \{x\} \times Z_x$ with $Z_x \subset Z$, having an acyclic decomposition and $\varphi: X \times Z \to Y$ is a single-valued continuous map.

Proposition 2.2.

All u.s.c. multivalued maps $F: X \to E$ with convex values where E is a strictly convex Banach space, belong to the class $MU(X, E)$.

Sketch of proof: It is clear that $F = \varphi \circ i$ where $i: X \to X \times E$ is defined by $i(x) = \{x\} \times F(x)$ and $\varphi: X \times E \to E$ is defined by $\varphi(x,y) = \rho_{F(x)}(y)$ such that $\rho_{F(x)}$ is the projection on the compact convex $F(x)$

$\subset E$ (e.g. $\rho_{F(x)}(y)$ is the unique point of $F(x)$ which is nearest to y since E is strictly convex and $F(x)$ is compact).

Example 2.3.

The map $F: [0,1] \to [0,1]$ defined by $F(x) = \begin{cases} 0 \text{ if } 0 \le x < ½ \\ [0,1] \text{ if } x = ½ \\ 1 \text{ if } ½ < x \le 1 \end{cases}$

belongs to the class $MU([0,1], [0,1])$ by proposition 2.2 .

3. A theory of differentiability for multivalued maps

In this section, we give a new definition of a differentiable multivalued map. In the following, E, E' are real Banach spaces and O is an open subset of E.

Definition 3.1.

Let $F \in MU(O, E')$; we have $F = \varphi \circ i$ where $i: O \to O \times Z$ is defined by $i(x) = \{x\} \times Z_x$, $Z_x \subset Z$, having an acyclic decomposition and $\varphi: O \times Z \to E'$ a single-valued continuous map. We say that F is differentiable at $x \in O$ if

(3.1.1) $\forall$ $(x, z) \in i(x)$, $\exists$ a single-valued continuous linear map $D_{x,z}\,\varphi: \underset{h \mapsto (D_{x,z}\varphi)(h)}{E \to E'}$ such that $\forall\ \epsilon > 0$, $\exists\ \delta = \delta(x,\epsilon) > 0$ such that if

$\|h\| < \delta$, then $\forall\ z' \in C_{x,h}(z) \subset \{z' \in Z \mid (x + h, z') \in i(x + h)\}$ we have

$\|\varphi(x + h, z') - \varphi(x, z) - (D_{x,z}\,\varphi)(h)\| < \epsilon\,\|h\|$ where the map $h \mapsto C_{x,h}(z)$ is u.s.c.,

(3.1.2) $\exists\ \delta_0 > 0$ such that

$$\varphi(\{x + h\} \times \bigcup_{(x,z) \in i(x)} C_{x,h}(z)) = F(x + h) \quad \text{for} \quad \|h\| < \delta_0.$$

The set $\{D_{x,z}\,\varphi \mid (x, z) \in i(x)\}$ is called a differential of F at x and the map $D_{x,z}\varphi$ is called a differential of φ at (x,z). If F is differentiable at every point of O, than F is said to be differentiable on O.

Example 3.2.

Let $F: [0,1] \to [0,1]$ be defined by $F(x) = \begin{cases} 0 \text{ if } 0 \le x < ½ \\ [0,1] \text{ if } x = ½ \\ 1 \text{ if } ½ < x \le 1 \end{cases}$

$F = \varphi \circ i$ where $\varphi(x, z) = \rho_{F(x)}(z)$ and $i(x) = \begin{cases} (x,0) \text{ if } 0 \le x < ½ \\ \{½\} \times [0,1] \text{ if } x = ½. \\ (x,1) \text{ if } ½ < x \le 1 \end{cases}$

The map $D_{½,z}\ \varphi: [0,1] \to [0,1]$ defined by $(D_{½,z}\varphi)(h)=0$ is a differential of φ at the point $(½,z) \in i(½)$ since $\forall\ \epsilon > 0$, $\exists\ \delta>0$ such that if $\|h\| < \delta$, then $|\ \varphi(½ + h,z') - \varphi(½,z) - (D_{½,z}\varphi)(h)\ | = 0 < \epsilon\ |h|$ for every $z' \in C_{½,h}(z) = \begin{cases} \{1\} \text{ if } z=1 \text{ and } h>0 \\ \phi \text{ if } z\in(0,1), z=1 \text{ and } h<0 \text{ or } z=0 \text{ and } h>0 \\ \{0\} \text{ if } z=0 \text{ and } h<0 \end{cases}$

Proposition 3.3.

If $F,G \in MU(O,E')$ are differentiable maps on O, then the convex homotopy of F and G, $tF + (1 - t)G$, $t \in [0,1]$, belongs to $MU(O,E')$ and is differentiable on O.

4. A mean-value theorem

In this section, we give a mean-value theorem.

Definition 4.1.

Let $F = \varphi \circ i \in MU(O,E')$. If F is differentiable at $x \in O$ with $D_{x,z}\varphi: E \to E'$ a differential of φ at the point (x,z), put $\|D_{x,z}\varphi\| = \inf \{k > 0\ |\ \|(D_{x,z}\varphi)(h)\| \le k\|h\|,\ (x,z) \in i(x),\ h \in E\}$. $\|D_{x,z}\varphi\|$

is a norm on the space of all single-valued continuous linear maps.

Theorem 4.2. (Mean-value theorem)

Let $F \in MU(O,E')$ be a differentiable map on $J = co(x_1,x_2) \subset O$ where $co(x_1,x_2)$ denotes the line segment between the points x_1 and x_2 in E. Then

$d_H(F(x_1),F(x_2)) \leq k \|x_1 - x_2\| + \text{Var } F$ where $k = \sup_{x \in J} \|D_{x,z}\varphi\|$ and

$\text{Var } F = \sup\left\{\sum_{i=1}^{n} d_H(B_{y_i,x_1-x_1}, B_{y_i,x_2-x_1}) \mid y_i \in co(x_1,x_2),\ i=1,\ldots,n\right\}$ with

$B_{x,h} = \{\varphi(x,z) \mid C_{x,h}(z) \neq \phi\}$ and d_H the Hausdorff metric.

Example 4.3.

The map defined in the example 3.2 is differentiable on [0,1] and we have $1 = |F(1)-F(0)| = 0\cdot|1-0| + \text{Var } F$ where $\text{Var } F = 1$.

5. Composition of differentiable maps

Let E, E′ and E″ be real Banach spaces, V_1 an open subset of E and V_2 an open subset of E′.

Proposition 5.1.

Let $F = \varphi \circ i \in MU(V_1,V_2)$ and $G = \psi \circ j \in MU(V_2,E'')$ such that F is differentiable on V_1 and G is differentiable on V_2. Suppose that $\sup_{(x,z) \in K} \| D_{x,z}\varphi \| < \infty$ and $\sup_{(y,z') \in K} \| D_{y,z'}\psi \| < \infty$ for any compact K,

where $D_{x,z}\varphi$ is a differential of φ and $D_{y,z'}\psi$ is a differential of ψ. Then $G \circ F$ belongs to $MU(V_1,E'')$, and $G \circ F$ is differentiable at $x \in V_1$ and the map $D_{x,z,z'}\alpha: E \to E''$ defined by

$(D_{x,z,z'}\alpha)(h) = ((D_{\varphi(x,z),z'}\psi)((D_{x,z}\varphi))(h))$ is a differential of α at the point (x,z,z').

REFERENCES

[1] H. T. Banks and M. Q. Jacobs, A differential calculus for multifunctions, *J. Math Anal. Appli.*,29(1970), 246-272.

[2] F. S. De Blasi, On the differentiability of multifunctions, *Pacific J. Math.*, 66(1976), 67-81.

[3] G. Fournier and D. Violette, A fixed point theorem for a class of multivalued continuously differentiable maps, *Annalès Polonici Matematici*, 47(1987), 381-402.

[4] R. Guzzardi, Differential theory of multivalued maps and applications, proceedings of Ryad Conference 1982.

[5] M. Martelli and A. Vignoli, On differentiability of multivalued maps, *Bolletino U.M.I.* (4) (10) (1974), 701-712.

Donald Violette
Département de mathématiques
Université de Moncton
Moncton (Nouveau-Brunswick)
E1A 3E9

Gilles Fournier
Département de mathématiques et d'informatique
Université de Sherbrooke
Sherbrooke (Québec)
J1K 2R1

Received June 21, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

ON THE UNITIZATION OF UNIFORMLY A-CONVEX ALGEBRAS [1]

M. Akkar[(*)], L. Oubbi[(**)] and M. Oudadess[(**)]

Presented by P.G. Rooney, F.R.S.C.

Abstract : *It is shown that the unitization of a locally uniformly A-convex algebra is not, in general, a locally uniformly A-convex algebra.*

Introduction : Let E be an algebra over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). The unitization of E is the algebra $E^1 := E \times \mathbb{K}$, with the product :

$$(x,\alpha)(y,\beta) = (xy + \alpha y + \beta x, \alpha\beta).$$

If τ is a locally convex topology on E, where τ is generated by a family $(P_\lambda)_{\lambda\in\Lambda}$ of semi-norms, one endows E^1 with the topology τ^1 generated by the family $(P_\lambda^1)_{\lambda\in\Lambda}$ of semi-norms, with $P_\lambda^1((x,\alpha)) = P_\lambda(x) + |\alpha|$, $\lambda \in \Lambda$. In this note, we give necessary and sufficient conditions for the unitization (E^1,τ^1) of a locally uniformly A-convex algebra (E,τ) to be of the same type. We will give a class of such algebras (E,τ) for which (E^1,τ^1) are not locally uniformly A-convex despite a claim in [1]. This is an unexpected phenomenon because the adjunction of unity is always possible for locally m-convex and locally A-convex algebras.

Unitization of uniformly A-convex algebras

Let (E,τ) be a locally convex algebra (l. c. a.) and $(P_\lambda)_{\lambda\in\Lambda}$ a family of semi-norms defining τ.

[1] 1980 A. M. S. subject classification: 46H05

Key words and phrases : locally Uniformly A-convex algebra, Unitization.

Definition 1 : *We say that (E,τ) is a locally uniformly A-convex algebra (l. u. A-c. a.) if for every $x \in E$ there is $M(x) > 0$ such that :*

$P_\lambda(xy) \leq M(x)P_\lambda(y)$ and $P_\lambda(yx) \leq M(x)P_\lambda(y)$ $y \in E$, $\lambda \in \Lambda$.

Such a family $(P_\lambda)_{\lambda\in\Lambda}$ is said to be uniformly A-convex (u. A-c.).

We then have :

Proposition 2 : *Let (E,τ) be a l. u. A-c. a. and E^1 its unitization. The following assertions are equivalent :*

1. E^1 is a l. u. A-c. a.,

2. There is a linear norm on E whose topology is stronger than τ.

3. The topology τ can be defined by a u. A-c. family $(Q_\lambda)_{\lambda\in\Lambda}$ of semi-norms so that

$$\forall x \in E.\ \sup\{Q_\lambda(x),\ \lambda \in \Lambda\} < +\infty.$$

Proof : 1. $\Longrightarrow$ 2. Since E^1 is supposed to be a l. u. A-c. a., there is, for every $x \in E$, some $M(x) > 0$ such that :

$P_\lambda^1((x,0)(y,\alpha)) \leq M(x)P_\lambda^1((y,\alpha))$ and $P_\lambda^1((y,\alpha)(x,0)) \leq M(x)P_\lambda^1((y,\alpha))$ $(y,\alpha) \in E^1$, $\lambda \in \Lambda$.

In particular :

$P_\lambda^1((x,0)) \leq M(x)P_\lambda^1((0,1))$ and $P_\lambda^1((x,0)) \leq M(x)P_\lambda^1((0,1))$ $\lambda \in \Lambda$.

Putting $\|x\| := \sup_{\lambda\in\Lambda} \frac{P_\lambda^1((x,0))}{P_\lambda^1((0,1))}$, we get the required norm.

2. $\Longrightarrow$ 3. If τ is given by a u. A-c. family $(P_\lambda)_{\lambda\in\Lambda}$, we have :

$$\forall\lambda \in \Lambda,\ \exists M(\lambda) > 0 :\ P_\lambda(x) \leq M(\lambda)\|x\|,\ \ \forall x \in E.$$

where $\|\ \|$ is the norm stronger than τ. Put $Q_\lambda(x) := \frac{P_\lambda(x)}{M(\lambda)}$. Then $(Q_\lambda)_{\lambda\in\Lambda}$ is a u. A-c. family and $\sup\{Q_\lambda(x),\ \lambda \in \Lambda\} \leq \|x\| < +\infty$.

3. $\Longrightarrow$ 1. Assume that $(Q_\lambda)_{\lambda\in\Lambda}$ is as in 3. Let $(x,\alpha) \in E^1$. One has :

$$\begin{aligned} Q_\lambda^1((x.\alpha)(y,\beta)) &= Q_\lambda^1((xy+\alpha y+\beta x,\ \alpha\beta)) \\ &= Q_\lambda(xy+\alpha y+\beta x) + |\alpha\beta| \end{aligned}$$

$$\leq M_1(x)Q_\lambda(y)+|\alpha|Q_\lambda(y)+|\beta|Q_\lambda(x)+|\alpha||\beta|$$
$$\leq (M_1(x)+|\alpha|)Q_\lambda(y)+(Q_\lambda(x)+|\alpha|)|\beta|$$
$$\leq (M(x)+|\alpha|)Q^1_\lambda((y,\beta)),$$

where $M_1(x)$ is such that :

$Q_\lambda(xy)\leq M_1(x)Q_\lambda(y)$, and $Q_\lambda(yx)\leq M_1(x)Q_\lambda(y)$, $y\in E$ $\lambda\in\Lambda$,

and $M(x):=\max(M_1(x),\ \sup\{Q_\lambda(x),\ \lambda\in\Lambda\})$.

Similarly we get that :

$$Q^1_\lambda((y,\beta)(x,\alpha))\leq(M(x)+|\alpha|)Q^1_\lambda((y,\beta)).$$

Hence E^1 is a l. u. A-c. a.

Corollary 3. *Let X be a σ-compact locally compact space, and $E=\mathcal{K}(X)$ the algebra of all continuous functions on X with compact support, endowed with the measure topology τ_L. Then E^1 is a l. u. A-c. a. if and only if X is compact.*

Proof : The sufficiency is obvious. We have just to show the necessity. If E^1 is l. u. A-c., there is a linear norm $\|\ \|$ on E whose topology is stronger then τ_L. Hence the unit ball B of $\|\ \|$ is bounded in $\mathcal{K}(X)$. But since every bounded set of $\mathcal{K}(X)$ is contained in some $\mathcal{K}_K(X)$, the subalgebra of all functions with support in the compact K of X, we have $\mathcal{K}(X)\subset\mathcal{C}(K)$. But this is true only if $X=K$.

Examples 4 :

1. By Corollry 3. if X is a non compact σ-compact locally compact space, the algebra $\mathcal{K}(X)$ is a l. u. A-c. a. which is complete, bornological but its unitization is not a l. u. A-c. a.

2. We now exhibit an example of a Fréchet l. u. A-c. a. E whose unitization is not a l. u. A-c. a.

For every $n\in\mathbb{N}^*$, let w_n be the function defined by $w_n(x):=e^{n|x|}$, $x\in\mathbb{R}$. Take

$$E:=\{f:\ \mathbb{R}\longrightarrow\mathbb{C},\text{continuous}:\ \sup_{t\in\mathbb{R}}|f(t)|\,w_n(t)<+\infty,\forall n\in\mathbb{N}\}$$

and endow E with the topology τ defined by the semi-norms

$$P_n(f) := \sup_{t\in\mathbb{R}} |f(t)|\, w_n(t).$$

Then (E,τ) is a non unital Fréchet l. u. A-c. a. which is not normable. If E^1 were l. u. A-c., then there would be a linear norm $\|\ \|$ on E (Proposition 2), whose topology is stronger than τ. If we put $A = \{f \in E : \|f\| \leq 1\}$ and $B = \overline{A}^{\tau}$, then B is a bounded barrel in (E,τ). Hence a bounded 0-neighborhood. This contradicts the non normability of E.

Remark 5 : Examples 1. and 2. show that the unitization of a weighted algebra is not always a weighted algebra. Indeed in both of them, if E^1 were a weighted algebra, it would necessarily be a l. u. A-c. a., for $E^1 \subset C_b(X)$ (cf. [2]).

References

[1] A. C. COCHRAN: Representation of A-convex algebras. Proc. Amer. Math. Soc, 41, (1973), 473-479.

[2] L. OUBBI : Weighted algebras of continuous functions. Res. Math., 24 (1993), 298-307.

Authors' addresses:

(*) Université de Bordeaux I, U.F.R. de Mathématiques et d'informatique

351, Cours de la Libération, 33405 Talence Cedex (France)

(**) Département de Mathématiques, Ecole Normale Supérieure Takaddoum,

BP. 5118, 10 000 Rabat (Morocco).

Received July 25, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

Divided Differences and Polynomials

PL. KANNAPPAN

Presented by J. Aczel. F.R.S.C.

Divided differences and Peano derivatives were used to characterize polynomials through functional equations, [1, 5, 9, 11, 7]. In the present paper we extend this notion of divided differences to characterize polynomials.

1 Introduction

It is well known that, if $f : \mathbb{R} \to \mathbb{R}$, then the following two functional equations

$$\frac{f(x)-f(y)}{x-y} = \frac{f'(x)+f'(y)}{2}, \tag{1.1}$$

and

$$\frac{f(x)-f(y)}{x-y} = f'\left(\frac{x+y}{2}\right), \tag{1.2}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$, characterize quadratic polynomials [1, 2, 8, 10]. Generalizations of (1.1) and (1.2) connected with quadratic polynomials were treated by several authors.

It is not clear what the generalization of (1.1) and (1.2) should be for higher order polynomials. Ideas came from the notions of Peano derivatives [7] and of divided differences [5, 6, 9]. In this paper, we give a generalization of the divided difference to characterize polynomials with the help of functional equations.

Notation

We write for the distinct points $x_1, x_2, \ldots, x_n$

$$[x_1, x_2, \ldots, x_n] = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{vmatrix} \tag{1.3}$$

$$[x_1, x_2, \ldots, x_n; f] = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-2} & f(x_1) \\ \ldots & \ldots & & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-2} & f(x_n) \end{vmatrix}. \tag{1.4}$$

With this notation the *divided difference* is defined as [9]

$$[x_1, x_2, \ldots, x_n; f]/[x_1, \ldots, x_n]. \tag{1.5}$$

The equations

$$[x_1,x_2,\ldots,x_n;f]/[x_1,\ldots,x_n]=\sum_{k=1}^{n}g_k(x_k), \tag{1.6}$$

and

$$[x_1,x_2,\ldots,x_n;f]/[x_1,\ldots,x_n]=g\left(\sum_{k=1}^{n}x_k\right), \tag{1.7}$$

were treated in [6] and [11, 13] respectively. Without being aware of the result on (1.7) in [6], Bailey posed in [3] the question whether every continuous (or differentiable) f satisfying (1.7) is a polynomial of degree at most n (see also [11, 13] for an answer). The divided difference denoted by $f[x_1,x_2,\ldots,x_n]$ and defined recursively by

$$\begin{aligned} f[x_0]=f(x_0), f[x_0,x_1] &= \frac{f(x_0)-f(x_1)}{x_0-x_1},\ldots,\\ f[x_0,x_1,\ldots,x_{n+1}] &= \frac{f[x_0,x_1,\ldots,x_n]-f[x_1,\ldots,x_{n+1}]}{x_0-x_{n+1}} \end{aligned}$$

in [5] is same as the divided difference defined in (1.5) [9].

For distinct points $x_1,x_2,\ldots,x_n$ we define

$$[x_1,\ldots,x_n;f_1,\ldots,f_n]=\begin{vmatrix} 1 & x_1\ldots x_1^{n-2} & f_1(x_1)\\ 1 & x_2\ldots x_2^{n-2} & f_2(x_2)\\ \ldots & \ldots\ldots & \ldots\\ 1 & x_n\ldots x_n^{n-2} & f_n(x_n)\end{vmatrix} \tag{1.8}$$

and the *generalized difference* by

$$[x_1,\ldots,x_n;f_1,\ldots,f_n]/[x_1,\ldots,x_n] \tag{1.9}$$

and we characterize polynomials by determining the general solution of the functional equations

$$[x_1,x_2,\ldots,x_n;f_1,\ldots,f_n]/[x_1,\ldots,x_n]=\sum_{k=1}^{n}g_k(x_k), \tag{1.10}$$

and

$$[x_1,\ldots,x_n;f_1,\ldots,f_n]/[x_1,\ldots,x_n]=g\left(\sum_{k=1}^{n}x_k\right). \tag{1.11}$$

2 Solution of the equations (1.10) and (1.11)

First we treat the case $n=2$ and then the general case $n\geq 2$. For the $n=2$ case of (1.10) we have the following.

Theorem 1. Let $f,g,h,k:F\to F$ where F is a field with more than three elements and char $F\neq 2$, satisfy the generalized divided difference equation

$$\frac{f(x)-g(y)}{x-y}=h(x)+k(y), \quad x\neq y. \tag{2.1}$$

The general solution of (2.1) is given by

$$f(x)=\alpha x^2+(c+d)x+a=g(x), h(x)=\alpha x+c, k(x)=\alpha x+d \tag{2.2}$$

where α, a, c, d are arbitrary constants (refer to [3] for a special case of (2.1)).

Proof. Letting $y=0$ and $x=0$ separately in (2.1), we obtain

$$f(x)=x[h(x)+d]+b, \quad x\neq 0, \tag{2.3}$$

$$g(y)=y[k(y)+c]+a, \quad y\neq 0, \tag{2.4}$$

where $f(0)=a, g(0)=b, h(0)=c, k(0)=d$.

Put (2.3) and (2.4) into (2.1) to get

$$x(h(x)+d)+b-y(k(y)+c)-a=(x-y)(h(x)+k(y)), x\neq y, x, y\neq 0. \tag{2.5}$$

Interchange x and y in (2.5) to obtain

$$x(k(x)+c)+a-y(h(y)+d)-b=(x-y)(h(y)+k(x)), x\neq y, x, y\neq 0. \tag{2.6}$$

Adding (2.5) and (2.6), we get

$$x[l(y)-l(0)]=y[l(x)-l(0)], x\neq y, x, y\neq 0$$

where

$$l(x)=h(x)+k(x), \tag{2.7}$$

so that

$$l(x)=\alpha x+c+d, \quad x\neq 0 \tag{2.8}$$

where α is a constant. (It is used in the above argument that F has more than three elements).

Similarly, subtracting (2.5) from (2.6), we have

$$x(n(y)-n(0))+y(n(x)-n(0))+2(b-a)=0, x\neq y, x, y\neq 0$$

where

$$n(x)=h(x)-k(x), \tag{2.9}$$

so that

$$n(x)=\beta x+\delta \tag{2.10}$$

where β, δ are constants. (Again it is used in the above argument that F has more than three elements). Consequently, from (2.7), (2.9), (2.8) and (2.10), we get

$$h(x)=a_1x+a_2, k(x)=b_1x+b_2, x\neq 0 \tag{2.11}$$

for some constants a_1, a_2, b_1, b_2 (here divisibility by 2 is used). Now, (2.11) and (2.6) give $a_1 = b_1, b_2 = d, a_2 = c, a = b$. Note now (2.11) holds for $x = 0$ also. These h and k together with (2.3) and (2.4) give the sought for solution (2.2).

Remark 1. For fields of characteristic 2, Theorem 1 is not true. There are 16 sets of solutions when $F = \mathbb{Z}_2$ - Some of the solutions which are not of the form (2.2) are: $f(x) = x$, $g(x) = x + 1$, $h(x) = x$, $k(x) = x + 1$; $f(x) = 0$, $g(y) = 1, h(x) = x$, $k(y) = y$ and $f(x) = 0$, $g(x) = x + 1$, $h(x) = x$ and $k(x) = 0$.

Remark 2. When F has three elements in Theorem 1, say $F = \mathbb{Z}_3$, the general solution of (2.1) is not of the form (2.2). As a matter of fact, the solution is given by $f(x) = a_1x^2 + (a_2 + d)x + b$, $g(x) = b_1x^2 + (b_2 + c)x + a$, $h(x) = a_1x + a_2$, $k(x) = b_1x + b_2$ for $x \neq 0$ where $a = f(0)$, $b = g(0)$, $c = h(0), d = k(0)$ with $a_1 - b_1 = b - a$ and $d + c = a_2 + b_2$.

Indeed, from (2.5), we have

$$x(k(y) - d) + a = y(h(x) - c) + b, \quad x, y \neq 0, x \neq y.$$

By fixing $x \neq 0$ and separately fixing $y \neq 0$ in the above we get

$$k(y) = b_1y + b_2, h(x) = a_1x + a_2$$

and $x(b_1y + b_2 - d) + a = y(a_1x + a_2 - c) + b$. Since $\mathbb{Z}_3$ has only 3 elements 0, 1, 2 and x, y are different from 0 and $x \neq y$, the possibilities are $x = 1, y = 2$ or $x = 2, y = 1$, so that we obtain $2b_1 + b_2 - d + a = 2(a_1 + a_2 - c) + b$ and $2(b_1 + b_2 - d) + a = 2a_1 + a_2 - c + b$. This leads to the conditions $a_1 - b_1 = b - a$ and $d + c = a_2 + b_2$.

Thus, it is possible to have solutions of (2.1) with $f \neq g$ when $F = \mathbb{Z}_3$.

For completeness, we quote Aczél's following result [1].

Theorem 2.[1, 12] The general solution of

$$\frac{f(x) - g(y)}{x - y} = h(x + y), \quad x \neq y$$

in a field of characteristic different from 2 is given by
$f(x) = g(x) = ax + bx + c$, $h(x) = ax + b$.

This is, of course, the case $n = 2$ of (1.11). We deal now with the general case $n \geq 2$ of (1.10) and (1.11).

It is easy to check that the generalized divided difference (1.9) is given by

$$\frac{[x_1, x_2, \ldots, x_n; f_1, \ldots, f_n]}{[x_1, x_2, \ldots, x_n]} = \sum_{i=1}^{n} \frac{f_i(x_i)}{(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}.$$

Theorem 3. Let $f_i, g_k : \mathbb{R} \to \mathbb{R}$ ($_{i,k} = 1, 2, \ldots, _n$) satisfy the functional equation (1.10). Then all $f_i^{'}s$ are equal, f_1 is a polynomial of degree at most

n and $g_k(x) = a_n x + b_k (k = 1,2,\ldots,n)$, with $\sum_{k=1}^{n} b_k = a_{n-1}$, where a_n, a_{n-1} are the coefficients of x^n and x^{n-1} respectively in f_1.

Proof. Let x_1 and x_2 be any two distinct reals. Choose non-zero distinct reals $x_3,\ldots,x_n$ different from x_1 and x_2. Obviously there are plenty of choices. In (1.10) and (2.12) interchange x_1 and x_2 and subtract the result from (1.10) to get

$$F(x_1) + F(x_2) = (x_2 - x_1)(G(x_1) - G(x_2)), x_1 \neq x_2 \tag{2.13}$$

where

$$F(x) = \frac{f_1(x) - f_2(x)}{(x_3 - x)\ldots(x_n - x)}, \quad G(y) = g_1(y) - g_2(y) \tag{2.14}$$

for $x, y \in \mathbb{R}$ with $x \neq x_3, \ldots x_n$. Setting $x_2 = 0$ in (2.13) we get $F(x_1) = -F(0) - x_1(G(x_1) - G(0))$, for $x_1 \neq 0$, which in (2.13) gives

$$x_1(G(x_2)-G(0))+x_2(G(x_1)-G(0))+2F(0) = 0, x_1, x_2 \neq 0, x_1 \neq x_2. \tag{2.15}$$

Consequently, $G(x) = \alpha x + b$ for all $x \neq 0$ which in (2.15) gives $\alpha = 0, b = G(0), F(0) = 0$, that is, $G(x) =$ constant and from (2.13) we see that $F(x) = 0$ for $x \neq x_3, \ldots, x_n$ and from (2.14) that $f_1(x) = f_2(x)$ for $x \neq x_3, \ldots, x_n$. Since there are plenty of choices for $x_3, \ldots, x_n$, $f_1(x) = f_2(x)$ for all x. (Here the divisibility by 2 is used). Similarly, it follows that the $f_i's$ are equal and the generalized divided difference becomes the divided difference and the result follows from [9].

Theorem 4. Let $f_i, g : \mathbb{R} \to \mathbb{R}(i = 1,2,\ldots,n)$ satisfy the functional equation (1.11). Then all $f_i's$ are equal, f_1 is a polynomial of degree at most n and g is linear, that is, $g(x) = a_n x + a_{n-1}$ where a_n, a_{n-1} are the coefficients of x^n and x^{n-1} respectively in f_1.

Proof. Let x_1 and x_2 be any two distinct reals. Choose non-zero distinct reals $x_3,\ldots,x_n$ different from x_1 and x_2. Obviously there are plenty of choices. In (1.11) and (2.12) interchange x_1 and x_2 and subtract the result from (1.12) to obtain

$$F(x_1) + F(x_2) = 0$$

where F is given by (2.14). Thus, as in Theorem 3, $F = 0$ and $f_1 = f_2$. Now the result follows from [6, 11].

Remark 3. The proofs of theorems 3 and 4 work also on fields of characteristic different from 2 with enough distinct points.

Remark 4. Note that no regularity condition was assumed in theorems 1 - 4.

Acknowledgment: I express my thanks to Professor J. Aczél and the referee for their comments and suggestions which improved the form and the

content of the paper.

References

1. J. Aczél, A mean value property of the derivatives of quadratic polynomials – without mean values and derivatives, Math. Mag., 58(1985),42-45.

2. J. Aczél and M. Kuczma, On two mean value properties and functional equations associated with them, Aequationes Math., 38(1989), 216-234.

3. J. Aczél, Remark 23, Proceedings of the 23rd International Symposium on Functional Equations, Ctre. Inf. Th., Waterloo, ON., 1985, 66-67.

4. D.F. Bailey, A mean-value property of cubic polynomials – without mean values, Math. Mag., 65(1992), 123-124.

5. D.F. Bailey and G.F. Fix, A generalization of the mean value theorem, Appl. Math. Lett., 1 (1988), 327-330.

6. B. Cristici and M. Neagu, About a functional characterization of polynomials, Inst. Politech. "Traian Vuia" Timisoara. Lucrar Sem. Mat. Fiz. mai-noi 1987, 59-62.

7. G.E. Cross and PL. Kannappan, A functional identity characterizing polynomials, Aequationes Math., 34 (1987), 147-152.

8. S. Haruki, A property of quadratic polynomials, Amer. Math. Monthly, 86 (1979), 577-579.

9. PL. Kannappan and B. Cristici, Two functional identities characterizing polynomials, Itinerant Seminar on Functional Equations., Approximations and Convexity, Cluj - Napoca, 1989, 175-180.

10. PL. Kannappan and P.G. Laird, Functional equations and second degree polynomials, J. Ramanujan Math. Soc. 8 (1993), 95-110.

11. PL. Kannappan and P.K. Sahoo, Characterizations of polynomials and the divided difference (submitted).

12. M. Kuczma, On a Stamate-type functional equation, Publ. Math. Debrecen, 39 (1991), 325-338.

13. J. Schwaiger, On a characterization of polynomials by divided differences (to appear in Aequationes Math.)

PL. Kannappan
Department of Pure Mathematics
University of Waterloo
Waterloo, ON N2L 3G1
CANADA

Received May 28, 1994
in revision from July28, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

Integrable dynamical systems on the classical Lie algebras connected with dynamics of interacting rigid bodies

Eduard V. Vlasov*

Presented by G.F.D. Duff, F.R.S.C.

Abstract

The Lax representations are found for dynamical systems on classical matrix Lie algebras generalizing dynamics of two interacting N-dimensional rigid bodies. The integrable dynamical systems are constructed which generalize particular cases of dynamics of n interacting rigid bodies. New integrable cases are pointed out for dynamics of interacting 3-dimensional rigid bodies having ellipsoidal cavities filled with an incompressible fluid. The physical parameters are indicated when these integrable cases realize.

1. In the recent past the Lax representation [7] has been applied in the numerous problems of mathematical physics and mechanics. A profound connection between the theory of integrable systems and the theory of Lie groups and algebras appears in a possibility to write a Lax type equation for many kind of integrable systems. As examples we shall point out the following ones: for the Toda lattice obtained in [3,4] and for a one-dimensional particle system generalizing the Calogero systems in [10]. A general scheme of construction of integrable systems on an orbit in Lie algebras was suggested in [1,6].

We shall consider a Lax representation of the problem of n interacting rigid bodies connected with the Manakov case [8]. The Lax equation depending on a spectral parameter was used in [8] to obtain commutative integrals

*Supported by NSERC grant OGPIN 337.

of the Euler equation of a N-dimensional rigid body. An extention to the case of all semisimple Lie algebras was obtained in [9]. The Manakov analog of a problem of two interacting rigid bodies was derived in [11] and for n N-dimensional rigid bodies in [5]. In the papers [5,11] Lie algebras $so(N)$ and $sl(N)$ were examined respectively and question of generalization of those results.

In this paper we consider some generalizations on the classical matrix Lie algebras.

2. We assume that equations of dynamics of a system of many interacting rigid bodies have the Lax representation

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)], \tag{1.1}$$

with

$$L(\lambda) = L_0 + \lambda\hat{A}\ , \ \ M(\lambda) = M_0 + \lambda\hat{B}, \tag{1.2}$$

where λ is a spectral parameter, L_0 has the block diagonal form $L_0 = diag(L^{(1)}, \ldots, L^{(n)})$ and $\hat{A} = D \otimes A$ is tensor product of the matrices D and A. The matrices M_0 and $\hat{B} = \tilde{D} \otimes B$ have analogous forms. The matrices $L^{(i)} \in gl(N)$ and $M^{(i)}$ are of the dimension $N \times N$. The constant matrices D and $\tilde{D}$ have dimension $n \times n$. The diagonal matrices $A = diag(a_1, \ldots, a_N)$ and $B = diag(b_1, \ldots, b_N)$ have only constant entries and $(a_i)^n \neq (a_j)^n$ if $i \neq j$.

According to this condition the Lax representation (1.1) splits into the following three equations:

$$\dot{L}_0 = [M_0, L_0], \tag{1.3}$$

$$[\hat{A}, M_0] = [\hat{B}, L_0], \ \ [\hat{A}, \hat{B}] = 0. \tag{1.4}$$

It follows from these equations (see for example [5]) that without loss of generality we can assume $D = \tilde{D}$ and the matrix D must have at most n nonzero entries. They are placed in a different rows and columns. We also can assume that all nonzero entries are equal to unit. The choice of order of the blocks $L^{(i)}$ in L_0 is arbitrary and by rearranging their numeration we can reduce all the cases to the following: $d_{i,i+1} = 1$, $d_{n,1} = 1$ for $i = 1, \ldots, n-1$. From this stage the order of blocks in matrix L_0 is strictly fixed.

Using the method of [2,5] we derive the relation

$$M_{ij}^{(m)} = \frac{\beta_i a_i^n - \beta_j a_j^n}{a_i^n - a_j^n} L_{ij}^{(m)} + \frac{\beta_i - \beta_j}{a_i^n - a_j^n} \sum_{s=1}^{n-1} a_i^s a_j^{n-s} L_{ij}^{(k)} \tag{1.5}$$

with $\beta_j = b_i a_i^{-1}$; $i,j = 1,\dots,N$; $m = 1,\dots,n$;

$$k = \begin{cases} m+s & \text{if } m+s<n \\ m+s-n+1 & \text{otherwise.} \end{cases}$$

The formula (1.5) is correct for any matrix $L^{(i)}$ but the Lax representation (1.1) will be equivalent to the system (1.3) - (1.4) only in the case when the commutator $[L,A]$ belongs to the same Lie algebra as L. This is true if $L^{(i)} \in sl(N)$ [5] and $L^{(i)} \in so(N)$ but $n=2$ [11].

3. At first we consider two interacting N-dimensional rigid bodies i.e. $n=2$. Then the formula (1.5) can be rewritten as

$$M_{ij}^{(1)} = c_{ij}L_{ij}^{(1)} + d_{ij}L_{ij}^{(2)} \, , \; M_{ij}^{(2)} = d_{ij}L_{ij}^{(1)} + c_{ij}L_{ij}^{(2)} \tag{1.6}$$

where

$$c_{ij} = \frac{a_i b_i - a_j b_j}{a_i^2 - a_j^2} \, , \; d_{ij} = \frac{a_j b_i - a_i b_j}{a_i^2 - a_j^2}.$$

Let us note that $c_{ij} = c_{ji}$, $d_{ij} = d_{ji}$ and analyze in detail the algebraic closure of the system (1.3) only for the matrix $M^{(1)}$, since for the matrix $M^{(2)}$ all arguments can be conducted in an analogous way.

We shall suppose that matrices $L^{(i)}$ and $M^{(i)}$ have the block form

$$\begin{pmatrix} t & k \\ g & h \end{pmatrix}.$$

Let $L^{(i)} \in so(p,q)$, $p+q=N$, $i,j=1,2$. It is easy to prove that in this case $t \in so(p)$, $h \in so(q)$, $g = k^T$, where T indicates transposing. The condition of closure is fulfilled for blocks t and h [11]. For blocks k and $g = k^T$ having noted that $L_{ij}^{(1)} = L_{ji}^{(1)}$, $L_{ij}^{(2)} = L_{ji}^{(2)}$; $i=1,\dots,p$; $j=p+1,\dots,n$ that is $L_{ij}^{(1)}, L_{ij}^{(2)} \in k$; $L_{ji}^{(1)}, L_{ji}^{(2)} \in g$ we have

$$M_{ji}^{(1)} = c_{ji}L_{ji}^{(1)} + d_{ji}L_{ji}^{(2)} = c_{ij}L_{ij}^{(1)} + d_{ij}L_{ij}^{(2)} = M_{ij}^{(1)}.$$

Hence the matrices $M^{(1)}$ and $L^{(1)}$ have the same form. It means $M^{(1)} \in so(p,q)$ and equation (1.3) is closed.

If $L^{(i)} \in sp(N)$, $i=1,2$ then matrix t will be an arbitrary quadratic matrix of the order N, $h := -t^T$, k and g are matrices satisfying conditions $k^T = k$, $g^T = g$.

Obviously that $L^{(1)}$ and $L^{(2)}$ are $2N \times 2N$ matrices then A and B must have the same dimension. For $i,j = 1,\dots,n$; $k = 1,2$: $L^{(k)}_{i+N,j+N} \in h$. Since the block t is arbitrary the condition of closure will not be fulfilled for this block. For the block h

$$\begin{aligned} M^{(1)}_{j+N,i+N} &= c_{j+N,i+N}L^{(1)}_{j+N,i+N} + d_{j+N,i+N}L^{(2)}_{j+N,i+N} \\ &= c_{j+N,i+N}(-L^{(1)}_{ij}) + d_{j+N,i+N}(-L^{(2)}_{ij}) \\ &= -(c_{i+N,j+N}L^{(1)}_{ij} + d_{i+N,j+N}L^{(2)}_{ij}) = -M^{(1)}_{ij} \end{aligned}$$

if $c_{i+N,j+N} = c_{ij}$ and $d_{i+N,j+N} = d_{ij}$. The last conditions will be satisfied when $a_{i+N} = a_i$, $b_{i+N} = b_i$; $i = 1,\dots,N$. It means, that we must consider the matrices $A = diag(a_1,\dots,a_N,a_1,\dots,a_N)$ and $B = diag(b_1,\dots,b_N, b_1,\dots,b_N)$ to ensure the system (1.3) is closed.

On the other hand blocks k and g are symmetric matrices and coefficients c and d are symmetric too. Moreover: $a_{i+N} = a_i$, $b_{i+N} = b_i$; $i = 1,\dots,N$ so the blocks $\tilde{k}$ and $\tilde{g}$ of matrices $M^{(1)}$ and $M^{(2)}$ are symmetric. This means, that $M^{(i)} \in sp(N)$, $i = 1,2$.

The ideas of the proof of conditions of closure for the cases $L^{(i)} \in su(N)$ and $L^{(i)} \in su(p,q)$, $p+q = N$, $i = 1,2$ are analogous to the ideas, which were analyzed above. Thus, we have proved the theorem:

Theorem 1 *The dynamical system $\dot{L}_0 = [M_0, L_0]$ with $L_0 = diag(L^{(1)}, L^{(2)})$ and $M_0 = diag(M^{(1)}, M^{(2)})$, where matrices $L^{(1)}$ and $L^{(2)}$ simultaneously belong to the one of the following classes of matrices: $gl(N)$, $sl(N)$, $so(N)$, $so(p,q)$, $sp(N)$, $su(N)$, $su(p,q)$, $p+q = N$ under conditions (1.6) admits an equivalent Lax-type representation depending on a spectral parameter.*

4. For an arbitrary n we can not conduct an argument analogous to the previous one, because for $n > 2$ there is no symmetry of the kind $c_{ij} = c_{ji}$. However, it is possible in a partial case. We have assumed that the order of blocks $L^{(i)}$ of the matrix L_0 is strictly fixed. Let us consider $n = 2k$ and suppose that

$$L^{(1)} = L^{(3)} = \dots = L^{(2k-1)}\,;\ L^{(2)} = L^{(4)} = \dots = L^{(2k)}. \qquad (1.7)$$

If we denote

$$\hat{c}_{ij} = \frac{\beta_i a_i^n - \beta_j a_j^n}{a_i^n - a_j^n}\,;\ \hat{d}_{ij} = \frac{\beta_j - \beta_i}{a_i^n - a_j^n} a_i a_j$$

then the relation (1.6) takes the form

$$M_{ij}^{(1)} = \gamma_{ij}L_{ij}^{(1)} + \mu_{ij}L_{ij}^{(2)} \; ; \; M_{ij}^{(2)} = \mu_{ij}L_{ij}^{(1)} + \gamma_{ij}L_{ij}^{(2)} \; ; \tag{1.8}$$

$$M^{(1)} = M^{(3)} = \ldots = M^{(2k-1)} \; ; \; M^{(2)} = M^{(4)} = \ldots = M^{(2k)}$$

where

$$\gamma_{ij} = \hat{c}_{ij} + \hat{d}_{ij}(a_i a_j^{2k-3} + a_i^3 a_j^{2k-5} + \ldots + a_i^{2k-3} a_j),$$

$$\mu_{ij} = \hat{d}_{ij}(a_j^{2k-2} + a_i^2 a_j^{2k-4} + \ldots + a_i^{2k-4} a_j^2 + a_i^{2k-2})$$

(i.e. $\gamma_{ij} = \gamma_{ji}$ and $\mu_{ij} = \mu_{ji}$) and we have to check the algebraic closure of the system of the same form as was analyzed before. Thus the following theorem is true.

Theorem 2 *The dynamical system (1.3) under conditions (1.5) and (1.7) and when matrices $L^{(1)}$ and $L^{(2)}$ simultaneously belong to one of the following classical Lie algebras: $gl(N)$, $sl(N)$, $so(N)$, $so(p,q)$, $sp(N)$, $su(N)$, $su(p,q)$, $p+q = N$ admits an equivalent Lax-type representation depending on a spectral parameter.*

Remark. In the case when $L_i \in sp(N)$, $i = 1, \ldots, 2k$ we have to consider $A = diag(a_1, \ldots, a_{2k}, a_1, \ldots, a_{2k})$ and $B = diag(b_1, \ldots, b_{2k}, b_1, \ldots, b_{2k})$.

The last theorem is related to a number of real physical problems. We shall describe only two of them. Let $n = 2k$ is arbitrary and $N = 3$. Then under these assumptions Lie algebras $so(N)$ considered in the Theorem 2 are reduced to the direct sum of $2k$ components: $so(3)+\ldots+so(3)$. The presented equations describe a motion of $2k$ interacting 3-dimensional rigid bodies, k of which are identical, and the other k are also identical but different from the first group. Taking into account that $so(3) + so(3) = so(4)$ we can consider the previous case as the direct sum of k components: $so(4) + \ldots + so(4)$.

Acknowledgements. The author is grateful to Oleg Bogoyavlenskij for the useful discussions.

References

[1] Adler, M. On a trace functional for formal pseudo-differential operators and the symplectic structure for Korteweg - de Vriese type equations. Invent. Math. **50**, 219 - 248, (1979).

[2] Bogoyavlenskij, O. I. A theorem on two commuting automorfisms, and integrable differential equations. Math. USSR Isvestiya **36**, N 2, 263 - 279, (1991).

[3] Bogoyavlenskij, O. I. On perturbations of the periodic Toda lattice. Commun. Math. Phys. **51**, 201 - 209, (1976).

[4] Flaschka, H.: The Toda lattice. II - Inverse-scattering solution. Progr. Theor. Phys. **51**, 703 - 716, (1974).

[5] Jiang, Z., Wojciechowsky, S. Integrable system of Many Interacting Rigid Bodies. Nuovo Cimento. B. **101**, N 4, 415 - 427, (1988).

[6] Kostant, B. Quantization and representation theory. Proc. of the Reseach Symp. on Representation of Lie groups. Oxford 1977. London Math. Soc. Lect. Notes Series **34**, (1979).

[7] Lax, P.P. Integrals of nonlinear equations of evolution and solitary waves. Commun. Pure Appl. Math. **21**, 467 - 490, (1968).

[8] Manakov, S. V. A remark on integration of the Euler equations dynamics of an n-dimensional rigid body. Functional Anal. Appl. **10**, (1976).

[9] Mishchenko, A.S., Fomenko, A.T. The Integrability of Euler equations on semisimple Lie algebras. (In Russian). Tr. Semin. Vectorn. Tenzorn. Anal. **19**, 3 - 94, (1979).

[10] Moser, J. Three integrable Hamiltonian systems connected with isospectral deformations. Adv. Math. **16**, 197 - 220, (1975).

[11] Perelomov, A.M., Ragnisco, O., Wojciechowsky, S. Integrability of Two Interacting N-Dimensional Rigid Bodies. Comm. Math. Phys **102**, 573 - 583, (1986).

Department of Mathematics and Statistics
Queen's University, Kingston, K7L 3N6, Canada
Kursk Pedagogical University,
33, Radistcheva ul, Kursk, 305000, Russia

Received July 29, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

POLYNOMIAL INTERPOLATION IN HIGHER DIMENSION

Presented by J. Aczel, F.R.S.C.

ALBERT BORBÉLY

ABSTRACT. Let f be a complex valued function defined on a finite collection of hyperplanes of $\mathbf{C}^n$ in "general position", that is, no three hyperplanes contain the same 2-codimensional complex subspace. Suppose, that on each of the hyperplanes f is polynomial, that is, f is the restriction of some polynomial. Then we show that f extends to a polynomial in $\mathbf{C}^n$. The extension is of course not unique.

In a recent issue of the Arch. Math. [2], Z. Sasvári proved the following theorem. Let S be the union of a finite collection of horizontal and vertical lines in $\mathbf{R}^2$ and $f : S \to \mathbf{R}$ be a function such that the restriction of f to each of these lines is a polynomial of one variable. Then there is a polynomial extension of f to the whole $\mathbf{R}^2$.

In the present note we are going to generalise this theorem for an arbitrary collection of lines in "general position" and in higher dimensions for an arbitrary collection of hyperplanes in "general position".

The complexified version of this theorem reads as follows:

Theorem 1. *Let $H_i = \{z \in \mathbf{C}^n : g_i(z) = 0\}$ $(i = 1, ..., N)$ be different hyperplanes such that*

$$(*) \qquad \text{if} \quad H_i \cap H_k \neq \emptyset \quad \text{then} \quad H_i \cap H_k \neq H_j \cap H_k, \quad \text{for all different } i, j, k.$$

Let $S = \cup_{i=1}^{N} H_i$ and suppose that $f : S \to \mathbf{C}$ is a function such that for each $i = 1, ..., N$ there is a polynomial $F_i \in \mathbf{C}[z_1, ..., z_n]$ for which $F_i |_{H_i} = f |_{H_i}$. Then there is a polynomial $F \in \mathbf{C}[z_1, ..., z_n]$ such that $F |_S = f |_S$.

This theorem could be thought of as a higher dimensional version of polynomial interpolation.

The condition $(*)$ is necessary as the following example shows: In $\mathbf{C}^2$ consider the following lines $H_1 = \{(x, y) : x = 0\}, H_2 = \{(x, y) : y = 0\}$ and $H_3 = \{(x, y) : x - y = 0\}$. Condition $(*)$ is clearly not satisfied. Let us now set $F_1(x, y) = y$, $F_2(x, y) = x$, $F_3(x, y) = x$ and let $f : H_1 \cup H_2 \cup H_3 \to \mathbf{C}$ be the restriction of F_1, F_2 and F_3 to H_1, H_2 and H_3 respectively. It is easy to see that f has no polynomial extension F to $\mathbf{C}^2$. For $F(x, y)$ has to be of the form $F(x, y) = x + y + xyQ(x, y)$ on account of F_1 and F_2 but then its restriction to H_3 cannot be equal to $F_3(x, x) = x$.

We will use notions and theorems from elementary algebraic geometry all of which can be found in most introductory books, for example [1].

Theorem 1 will be a simple consequence of the following more general version.

Proposition 2. *Let $V_i = \{z \in \mathbb{C}^n : g_i(z) = 0\}$ $(i = 1, \ldots, N)$ be affine algebraic varieties defined by the polynomials $g_i \in \mathbb{C}[z_1, \ldots, z_n]$ such that the following condition holds:*

(**) *the ideals $I(g_1, g_2), I(g_1g_2, g_3), \ldots, I(g_1g_2 \cdots g_{N-1}, g_N)$ are all radical ideals.*

Let $S = \cup_{i=1}^N V_i$ and suppose that $f : S \to \mathbb{C}$ is a function such that for each $i = 1, \ldots, N$ there is a polynomial $F_i \in \mathbb{C}[z_1, \ldots, z_n]$ for which $F_i|_{V_i} = f|_{V_i}$. Then there is a polynomial $F \in \mathbb{C}[z_1, \ldots, z_n]$ such that $F|_S = f|_S$.

Proof of Proposition 2. We prove the proposition by induction on N. For $N = 1$ it is obviously true. Suppose that it has already been proved for fewer than N varieties satisfying condition (**).

From the compatibility conditions on F_i we have that

$$\{z \in \mathbb{C}^n : F_1(z) - F_2(z) = 0\} \supset \{z \in \mathbb{C}^n : g_1(z) = 0 \quad \text{and} \quad g_2(z) = 0\}.$$

Then Hilbert's Nullstellensatz and condition (**) implies that

$$F_1 - F_2 \in I(g_1, g_2).$$

That is, there are polynomials $a, b \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$F_1 + ag_1 = F_2 + bg_2 = F_0.$$

$F_0|_{V_1 \cup V_2} = f|_{V_1 \cup V_2}$ therefore applying the induction to the varieties $V_1 \cup V_2, V_3, \ldots, V_N$ with polynomials $F_0, F_3, \ldots, F_N$ completes the proof.

Of course, condition (**) of Proposition 2 may not be always easy to verify but in the special case of Theorem 1 it is simple.

Proof of Theorem 1. It is enough to show that the ideal $I(g_1g_2 \cdots g_{k-1}, g_k)$ is radical for $k = 2, \ldots, N$. The defining functions g_i, $i = 1, \ldots, N$ are all linear so with a change of variables, if necessary, we may assume that $g_k(z) = g_k(z_1, \ldots, z_n) = z_n$.

Suppose that

$$h^m = ag_1g_2 \cdots g_{k-1} + bg_k, \quad \text{for some} \quad h, a, b \in \mathbb{C}[z_1, \ldots, z_n].$$

Let $\varphi : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]/_{I(g_k)} \cong \mathbb{C}[z_1, \ldots, z_{n-1}]$ denote the canonical map. Then we have

$$\varphi(h)^m = \varphi(a)\varphi(g_1) \cdots \varphi(g_{k-1}).$$

We want to show that $\varphi(g_i)$ $(i = 1, ..., k-1)$ divides $\varphi(h)$. Without loss of generality we may assume that $\varphi(g_i)$ $(i = 1, ..., r)$ are all constant and $\varphi(g_j)$ $(j = r+1, ..., k-1)$ are all non-constant linear polynomials. It is clear that every constant $\varphi(g_i)$ $(i = 1, ..., r)$ divides $\varphi(h)$, so we have only to show that every non-constant linear polynomial $\varphi(g_j)$ $(j = r+1, ..., k-1)$ divides $\varphi(h)$ as well.

First we show that $\varphi(g_j)$ $(j = r+1, ..., k-1)$ are all pairwise relatively prime linear polynomials in $\mathbb{C}[z_1, ..., z_{n-1}]$. Suppose that on the contrary $\varphi(g_i) = \alpha\varphi(g_j)$ for some $\alpha \in \mathbb{C}$. This implies that their zero sets are the same, that is $H_i \cap H_k = H_j \cap H_k$. These polynomials are non-constant, therefore their zero sets are also non-empty, that is $\emptyset \neq H_i \cap H_k = H_j \cap H_k$. This contradicts condition $(*)$.

The fact that $\mathbb{C}[z_1, ..., z_{n-1}]$ is a unique factorisation domain and $\varphi(g_j)$ $(j = r+1, ..., k-1)$ are all pairwise relatively prime irreducible polynomials, implies that every $\varphi(g_j)$ $(j = r+1, ..., k-1)$ must divide $\varphi(h)$. Therefore $\varphi(h)$ has the following form

$$\varphi(h) = c\varphi(g_1)\cdots\varphi(g_{k-1}) = c\varphi(g_1\cdots g_{k-1}), \quad c \in \mathbb{C}[z_1, ..., z_{n-1}],$$

that is,

$$h - cg_1\cdots g_{k-1} \in I(g_k),$$

which shows that

$$h \in I(g_1g_2\cdots g_{k-1}, g_k).$$

This proves that the ideal $I(g_1g_2\cdots g_{k-1}, g_k)$ is radical and concludes the proof of the theorem.

It remains to show that the real version of the theorem is also valid.

Corollary 3. *Let* $H_i = \{x \in \mathbb{R}^n : g_i(x) = 0\}$ $(i = 1, ..., N)$ *be different hyperplanes such that*

$$(*) \qquad \text{if} \quad H_i \cap H_j \neq \emptyset \quad \text{then} \quad H_i \cap H_j \neq H_i \cap H_k, \quad \text{for all different} \quad i, j, k.$$

Let $S = \cup_{i=1}^N H_i$ *and suppose that* $f : S \to \mathbb{R}$ *is a function such that for each* $i = 1, ..., N$ *there is a polynomial* $F_i \in \mathbb{R}[x_1, ..., x_n]$ *for which* $F_i|_{H_i} = f|_{H_i}$. *Then there is a polynomial* $F \in \mathbb{R}[x_1, ..., x_n]$ *such that* $F|_S = f|_S$.

Proof. By Theorem 1 there is a complex polynomial $G \in \mathbb{C}[z_1, ..., z_n]$ such that $G|_S = f|_S$. Because the function f is real and each point of the set S has real co-ordinates we also have $\overline{G(\bar{z})}|_S = f|_S$, that is

$$F(z)|_S = \frac{1}{2}(G(z) + \overline{G(\bar{z})})|_S = f|_S$$

and the polynomial $F(z)$ has real coefficients.

References

[1] J. Harris, *Algebraic Geometry*, Springer-Verlag, Berlin-New York, 1992.
[2] Z. Sasvári, *On separately polynomial functions*, Arch. Math. 59 (1992), 91-94.

DEPARTMENT OF MATHEMETICS, KUWAIT UNIVERSITY, P.O. BOX 5969, SAFAT 13060, KUWAIT

Received August 6, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

ON THE CUNTZ-QUILLEN BOUNDARY MAP

Victor Nistor[1]

Presented by P.A. Fillmore, F.R.S.C.

Abstract

We show that the Cuntz-Quillen boundary map in periodic cyclic cohomology satisfies properties similar to the properties of the boundary map in simplicial homology. We also prove that their boundary map is compatible with the boundary map in algebraic and topological K-Theory, which leads to index theorems. As an application one obtains a new proof of the Connes-Moscovici index theorem for coverings.

1991 AMS Subject Classification: Primary 19D55,46L80,58G12, Secondary 19K56,46L87,55N91

Key words: Cyclic cohomology, Chern character, K-Theory, Index morphisms.

Introduction. In a recent break-through paper Cuntz and Quillen have shown that periodic cyclic cohomology groups HC^{i}_{per} introduced by Connes [2] satisfy excision [4, 5] and hence for any two-sided ideal I of algebra A it gives rise to a periodic six-term exact sequence

$$\ldots \to \mathrm{HC}^{i}_{per}(A) \to \mathrm{HC}^{i}_{per}(I) \xrightarrow{\partial} \mathrm{HC}^{i+1}_{per}(A/I) \to \mathrm{HC}^{i+1}_{per}(A) \to \ldots$$

[1] Partially supported by NSF grant DMS 92-05548

$i \in \mathbb{Z}_2$ similar to the topological K-theory exact sequence. Thus periodic cyclic cohomology defines a generalized cohomology theory for algebras.

If $A = C^\infty(M)$ for M a smooth compact manifold and I is the ideal of functions vanishing on a closed submanifold $N \subset M$ then for continuous periodic cyclic cohomology we have $\mathrm{HC}^i_{per}(I) = \oplus_k \mathrm{H}_{i+2k}(M, N) \otimes \mathbb{C}$ (=singular homology with complex coefficients made $\mathbb{Z}_2$-periodic) and the Cuntz-Quillen exact sequence identifies with the (periodized) homology exact sequence of the pair of spaces (N, M).

The purpose of this note is to describe some applications of the methods developed in [6] to the study the properties of the boundary map ∂ in the Cuntz-Quillen six-term exact sequence. One of the main theorems (Theorem 1), suitable interpreted, states that the Chern-Connes character is a natural transformation of cohomology theories.

An application of these results is a proof of the Connes-Moscovici index theorem for coverings which is very much in the spirit of Algebraic Topology. The original proof was based on heat kernels and their analysis. Our proof, using the properties we establish for the boundary map, reduces the index theorem for coverings to the Atiyah-Singer index theorem for a single elliptic operator.

We hope that the method we developed will lead to new index theorems.

Compatibility with the index morphism. One of the main results is the compatibility between the boundary morphisms in algebraic and topological K-Theory and the boundary morphism in cyclic cohomology (Theorem 1). These results extend the results in [6]. We now proceed to state this result.

All the algebras considered bellow will be complex algebras.

Recall that for any exact sequence of algebras $0 \to I \to A \to A/I \to 0$ there is a five term exact sequence of algebraic K-Theory groups:

$$K_1^{alg}(I) \to K_1^{alg}(A) \to K_1^{alg}(A/I) \xrightarrow{\partial} K_0^{alg}(I) \to K_0^{alg}(A) \to K_0^{alg}(A/I).$$

The boundary map $\partial : K_1^{alg}(A/I) \to K_0^{alg}(I)$, also called *the index map* is defined as in [3].

We will also use the Chern-Connes character in cyclic homology

$$ch_{2n+i} : K_i^{alg}(A) \to \mathrm{HC}_{2n+i}(A),\ i = 0, 1.$$

The Chern-Connes character is uniquely determined by the following properties, Morita invariance of cyclic cohomology, $Sch_{2n+2+i} = ch_{2n+i}$ (where S is Connes' periodicity operator [2]), the normalizations $ch_0([e]) = e$ for an idempotent e, and $ch_1([u]) = u \otimes u^{-1}$ for an invertible element u and, most importantly, functoriality. By functoriality we mean the following. Let $f : A \to B$ be an algebra morphism and denote by $f_* : K_i^{alg}(A) \to K_i^{alg}(B)$ and $f_* : \mathrm{HC}_i(A) \to \mathrm{HC}_i(B)$ the induced morphisms, then $f_*(ch_i(x)) = ch_i(f_*(x))$ for any $x \in K_j^{alg}(A)$, $j = 0, 1$ of the same parity as i.

We obtain pairings [2] $K_i^{alg}(A) \otimes \mathrm{HC}^i_{per}(A) \to \mathbb{C}$, $i = 0, 1$. In particular, for any $\varphi \in \mathrm{HC}^i_{per}(A)$ we obtain a morphism $\varphi_* : K_i^{alg}(A) \to \mathbb{C}$.

The following theorem describes the relation between the boundary map in algebraic K-Theory and the Cuntz-Quillen boundary map in periodic cyclic cohomology. The results contained in the following theorem reduce the computation of a "cohomological index formula" to that of the computation of a Cuntz-Quillen boundary map.

Theorem. 1 *Let $0 \to I \to A \to A/I \to 0$ be an exact sequence of complex algebras. Then for any $\varphi \in \mathrm{HC}^0_{per}(A)$ and $[u] \in K_1^{alg}(A)$ we have*

$$\varphi_*(\mathrm{Ind}[u]) = (\partial\varphi)_*[u]$$

where $\mathrm{Ind} = \partial : K_1^{alg}(A/I) \to K_0^{alg}(I)$ *is the boundary map in algebraic K-theory and* $\partial : \mathrm{HC}^0_{per}(I) \to \mathrm{HC}^1_{per}(A/I)$ *is the Cuntz-Quillen boundary map in periodic cyclic cohomology.*

The above Theorem generalizes Theorem 5.4 from [6].

The proof of the above theorem is based on ideas similar to those in [6]. The proof uses the algebraic Toeplitz extension, the naturality of the boundary morphisms in algebraic K-Theory and cyclic cohomology and the Cuntz-Quillen exact sequence applied to the exact sequence

$$0 \longrightarrow J \to \mathbb{C}\langle a,b\rangle \to \mathbb{C}[\mathbb{Z}] \longrightarrow 0$$

where $\mathbb{C}\langle a,b\rangle$ is the free unital algebra on two generators.

Properties of the Cuntz-Quillen boundary morphism. We proceed to explain some properties of this boundary map which are useful in computations.

Recall [2, 6] that for any complex algebras A_0 and B_0 there exists a natural product

$$\mathrm{HC}^*_{per}(A_0) \otimes \mathrm{HC}^*_{per}(B_0) \xrightarrow{\otimes} \mathrm{HC}^*_{per}(A_0 \otimes B_0).$$

The following theorem shows the relation between this product and the Cuntz-Quillen boundary map.

Theorem. 2 *Let $I \subset A$ and B be complex algebras, I a two sided ideal of A. Denote by* $\partial_{A,I} : \mathrm{HC}^*_{per}(I) \to \mathrm{HC}^{*+1}_{per}(A/I)$ *and* $\partial_{A\otimes B, I\otimes B} : \mathrm{HC}^*_{per}(I \otimes B) \to \mathrm{HC}^{*+1}_{per}((A/I)\otimes B)$ *the Cuntz-Quillen boundary morphisms associated to $I \subset A$ and $I \otimes B \subset A \otimes B$. Then for any $\xi \in \mathrm{HC}^*_{per}(I)$ and $\zeta \in \mathrm{HC}^*_{per}(B)$ we have*

$$\partial_{A\otimes B, I\otimes B}(\xi \otimes \zeta) = \partial_{A,I}(\xi) \otimes \zeta\,.$$

The proof of this theorem is based on the description of the $\otimes$-product above using cyclic objects [6]. This theorem generalizes the corresponding properties of the $\times$-product of the homology of manifolds. Using this theorem one can prove further properties of the Cuntz-Quillen boundary map for particular classes of exact sequences related to group actions.

Consider a group Γ acting on A by $\Gamma \times A \ni (\gamma, a) \to \alpha_\gamma(a) \in A$. We are going to work with the algebraic crossed product $A \rtimes \Gamma$ consisting of finite linear combinations of elements of the form $a\gamma$, with the product rule $(a\gamma)(b\gamma_1) = a\alpha_\gamma(b)\gamma\gamma_1$. If the two-sided ideal I is invariant we obtain an exact sequence of crossed product algebras

$$0 \to I \rtimes \Gamma \to A \rtimes \Gamma \to (A/I) \rtimes \Gamma \to 0.$$

The periodic cyclic cohomology of a crossed product has further structure, $\mathrm{HC}^*_{per}(A \rtimes \Gamma)$ is a $\mathrm{HC}^*_{per}(\mathbb{C}[\Gamma])$ module. For commutative algebras $A = C^\infty(X)$ this module structure coincides with the $\mathrm{H}^*(\mathrm{B}\Gamma)$ module structure on the cohomology of the homotopy quotient $\mathrm{E}\Gamma \times_\Gamma X$. See [1] for details. The module structure alluded to above can be defined as the composition

$$\mathrm{HC}^*_{per}(A \rtimes \Gamma) \otimes \mathrm{HC}^*_{per}(\mathbb{C}[\Gamma]) \to \mathrm{HC}^*_{per}((A \rtimes \Gamma) \otimes \mathbb{C}[\Gamma]) \xrightarrow{\delta^*} \mathrm{HC}^*_{per}(A \rtimes \Gamma)$$

where the first map is the product we have defined above and $\delta(ag) = ag \otimes g$ is the standard coalgebra structure of $A \rtimes \Gamma$ over the Hopf algebra $\mathbb{C}[\Gamma]$.

We have the following theorem

Theorem. 3 *Let Γ be a discrete group acting on the unital algebra A, and I be a Γ-invariant ideal. Then the boundary map*

$$\partial_{A \rtimes \Gamma, I \rtimes \Gamma} : \mathrm{HC}^*_{per}(I \rtimes \Gamma) \to \mathrm{HC}^{*+1}_{per}((A/I) \rtimes \Gamma)$$

is $\mathrm{HC}^*_{per}(\mathbb{C}[\Gamma])$-linear.

One can also show that the Cuntz-Quillen boundary map coincides with the morphism induced by the bivariant Chern-Connes character whenever the later is defined [6]. This is in analogy with the fact that the boundary map in the topological K-Theory of C^*-algebras is given by a Kasparov product whenever the later is defined.

References

[1] J.-L. Brylinski and V. Nistor. Cyclic cohomology of etale groupoids. Preprint 1993, to appear in K-Theory.

[2] A. Connes. Non-commutative differential geometry. *Publ. Math. IHES*, 62:41–144, 1985.

[3] A. Connes and H. Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology*, 29:345–388, 1990.

[4] J. Cuntz and D. Quillen. On excision in periodic cyclic cohomology. *C. R. Acad. Sci. Paris*, 317:917–922, 1993.

[5] J. Cuntz and D. Quillen. On excision in periodic cyclic cohomology II, the general case. *C. R. Acad. Sci. Paris*, 318:11–12, 1994.

[6] V. Nistor. A bivariant Chern-Connes character. *Ann. of Math.*, 138:555–590, 1993.

Mailing Address: Harvard University
Mathematics Department, One Oxford Street
Cambridge, MA 02138, USA.
(On leave from Penn State.)

Received August 12, 1994

SOME RESULTS RELEVANT TO BARR'S CODENSITY THEOREM

Hongde Hu

Presented by J. Lambek, F.R.S.C.

Abstract. Let **C** be a small exact category, $Lex(\mathbf{C}, \mathbf{Set})$ the category of functors from **C** into the category of small sets preserving finite limits. We show that an accessible full subcategory **A** of $Lex(\mathbf{C}, \mathbf{Set})$ closed under filtered colimits and products is codense iff every object of $Lex(\mathbf{C}, \mathbf{Set})$ has a regular mono embedding into an object of **A**. Further, we prove that $Reg(\mathbf{C}, \mathbf{Set})$, the category of regular functors, is the smallest codense full subcategory of $Lex(\mathbf{C}, \mathbf{Set})$ with certain closedness properties. This is based on a stronger version of the strong conceptual completeness theorem for exact categories.

Introduction

Exact categories were originally described as a non-Abelian generalization of the theory of Abelian categories by M.Barr in [3]. The precise logical analysis of the notion was given by M.Makkai [20], who introduced the term regular theory. The important feature of the notion of exact category is its elegance and robustness. In fact, many important mathematical situations can be axiomatized by exact categories satisfying some additional conditions (see [4], [20], [7] and [22]). On the other hand, the category of models of exact categories are accessible (with products), a subject that has been widely investigated in recent years (see [10], [21] and [2]).

One of the fundamental theorems in the theory of representation of categories states that a small exact category has a full, regular embedding into a category of set-valued functors (see [3]). An equivalent formulation is that for **C** small exact, with $Reg(\mathbf{C}, \mathbf{Set})$ the category of regular functors from **C** into the category **Set** of small sets, then the evaluation functor $\mathbf{C} \to (Reg(\mathbf{C}, \mathbf{Set}), \mathbf{Set})$ is full and faithful (see [19] and [20]). The main difficulty of the proof of the full embedding theorem lies in the fact that in general the functor category does not have genuine injectives, unlike the case of Grothendieck's theorem that every AB5 category with a generator has an injective cogenerator [11]. However, the paper of Barr [5] showed that every functor $\mathbf{C} \to \mathbf{Set}$ preserving finite limits has a regular mono embedding into a functor of $Reg(\mathbf{C}, \mathbf{Set})$. The consequence of this is Barr's codensity theorem, which says that

1991 Mathematics Subject Classification. 03G30, 18B25, 18E10, 18G05.

This work was partially supported by a special research grant of the Faculty of Arts at York University.

$Reg(\mathbf{C}, \mathbf{Set})$ is codense in the category $Lex(\mathbf{C}, \mathbf{Set})$ of functor preserving finite limits. This argument yields a more conceptual proof of the full embedding theorem.

The present paper is concerned with the codensity theorem above. We first show that for any accessible full subcategory of a locally presentable category $\mathbf{B}$ which is closed under filtered colimits and products, if $\mathbf{A}$ is codense in $\mathbf{B}$, then every object of $\mathbf{B}$ has a regular mono embbedding into an object of $\mathbf{A}$. As a corollary, for $\mathbf{C}$ small exact and $\mathbf{B} = Lex(\mathbf{C}, \mathbf{Set})$, $\mathbf{A}$ is codense in $\mathbf{B}$ iff every object of $\mathbf{B}$ has a regular mono embedding into an object of $\mathbf{A}$. Furthermore, we consider an accessible full subcategory $\mathbf{A}$ of $Reg(\mathbf{C}, \mathbf{Set})$ closed under filtered colimits and products. Suppose that the evaluation functor $\mathbf{A} \to (\mathbf{A}^+, \mathbf{Set})$ is full and faithful; here $\mathbf{A}^+ = \prod Filt(\mathbf{A}, \mathbf{Set})$ is the category of functors preserving filtered colimits and products. We prove that if $\mathbf{A}$ is codense in $Lex(\mathbf{C}, \mathbf{Set})$, then $\mathbf{C}$ is equivalent to $\mathbf{A}^+$; moreover, we have that $Reg(\mathbf{C}, \mathbf{Set})$ is the smallest codense subcategory of $Lex(\mathbf{C}, \mathbf{Set})$ in the categories of the form $Reg(\mathbf{D}, \mathbf{Set})$ with closedness properties above.

The central concept we need here is the notion of quotient morphism between exact categories. A regular functor $F : \mathbf{C} \to \mathbf{D}$ between small exact categories is a quotient morphism if F is full on subobjects and for any object D of $\mathbf{D}$, there is a regular epimorphism $F(C) \to D$ with $C \in \mathbf{C}$ (see [18]). For above $\mathbf{A}$, the categories of the form $\mathbf{A}^+$ are known to be small exact (see [20] and [12]). In order to obtain above mentioned results, we prove that the evaluation functor $\mathbf{C} \to \mathbf{A}^+$ is a quotient morphism iff the evaluation functor $\mathbf{A} \to (\mathbf{A}^+, \mathbf{Set}))$ is full and faithful. This is a stronger version of the strong conceptual completeness theorem for exact categories in the sense of Makkai [18] (see Theorem 2.1).

Throughout the paper "exact category" is replaced by "κ-Barr-exact category", with an arbitrary regular cardinal number κ.

1 Codensity theorem on exact categories

Recall from [3] that a category is exact if it has finite limits and stable quotients of equivalence relations. A functor between exact categories is regular if it preserves finite limits and quotients of equivalence relations. The notions of κ-Barr-exact category and κ-regular functor are introduced in [20], for κ any infinite regular cardinal. They are a natural generalization of the notions of exact category and regular functor.

Definition 1.1 *([20]) A category $\mathbf{C}$ is κ-Barr-exact if it is exact, has κ-limits, and satisfies the principle of $< \kappa$ dependent choices (DC_κ): let α be an ordinal less than κ, and let $\Gamma = < A_\beta, f_{\beta,\gamma} : A_\beta \to A_\gamma >_{\gamma\leq\beta<\alpha}$ be an inverse diagram of type α in $\mathbf{C}$ such that*

(i) $f_{\beta+1,\beta}$ is a regular epi, for every β with $\beta + 1 < \alpha$; and

(ii) the restriction $\Gamma \mid \leq \beta$ of Γ to the domain consisting of all ordinals $\gamma \leq \beta$ is a limit diagram: A_β is a limit of $\Gamma \mid < \beta$ (Γ restricted to ordinals $< \beta$) with limit projections $f_{\beta,\gamma} : A_\beta \to A_\gamma (\gamma < \beta)$, for every limit ordinal $\beta < \alpha$.

Then every $f_{\beta,\gamma}$ is a regular epi, for all $\gamma \leq \beta < \alpha$.

A functor between κ-Barr-exact categories is κ-regular, if it preserves all regular epis and all κ-limits. Let $\mathbf{C}$ and $\mathbf{D}$ be two κ-Barr-exact categories, we write κ-$Reg(\mathbf{C}, \mathbf{D})$ for the category of κ-regular functors from $\mathbf{C}$ into $\mathbf{D}$.

Recall from [9] that an object A of a category $\mathbf{A}$ is said to be κ-presentable if the representable functor $\mathbf{A}(A,-)$ preserves κ-filtered colimits existing in $\mathbf{A}$. $\mathbf{A}$ is κ-accessible if: (i) $\mathbf{A}$ has κ-filtered colimits; (ii) there is a small subcategory $\mathbf{C}$ of $\mathbf{A}$ consisting of κ-presentable objects such that every object of $\mathbf{A}$ is a κ-filtered colimit of a diagram of objects in $\mathbf{C}$. A category is accessible if it is κ-accessible for some κ. A functor between accessible categories is accessible if it preserves κ-filtered colimits for some κ (see [21] and [2]). The full subcategory of $\mathbf{A}$ whose objects are κ-presentable is denoted by $\mathbf{A}_\kappa$.

A κ-accessible category with products is called weakly locally κ-presentable in [AR]. We record here some properties on accessible categories with products which are found in [1], [2], [20], [12] and [13].

(1) An accessible category has products iff it has weak colimits.

(2) A κ-accessible category $\mathbf{A}$ has products iff $\mathbf{A}_\kappa$ has weak κ-colimits.

(3) For a small κ-Barr-exact category $\mathbf{C}$, $\mathbf{C}^*$ $(= \kappa\text{-}Reg(\mathbf{C},\mathbf{Set}))$ is accessible with κ-filtered colimits and products.

(4) For $\mathbf{C}$ small κ-Barr-exact having enough projectives, $\mathbf{C}^*$ is κ-accessible with products. Conversely, for $\mathbf{A}$ κ-accessible with products, $\mathbf{A}^+ = \prod F_\kappa(\mathbf{A},\mathbf{Set})$, the category of functors preserving κ-filtered colimits and products, is a small κ-Barr-exact category having enough projectives.

(5) For $\mathbf{A}$ accessible with products, $\prod Acc(\mathbf{A},\mathbf{Set})$, the category of all accessible functors preserving products, is an exact category having enough projectives $\mathbf{A}(A,-)$ for $A \in \mathbf{A}$; moreover, every functor of $\prod Acc(\mathbf{A},\mathbf{Set})$ is the codomain of a coequalizer of a pair of morphisms between representable functors.

Let $\mathbf{C}$ be a small κ-Barr-exact category. $L_\kappa(\mathbf{C},\mathbf{Set})^{op}$, the opposite category of the category of set-vlaued functors that preserve κ-limits, is a κ-Barr-exact category, and every $M \in L_\kappa(\mathbf{C},\mathbf{Set})$ is the domain of an equalizer of a pair of morphisms in $\mathbf{C}^*$ (see [5] and [20]). These properties are sufficient to give the proof of the codensity theorem: the inclusion $I : \mathbf{C}^* \to L_\kappa(\mathbf{C},\mathbf{Set})$ is codense (see [5]). This yields the full embedding theorem: the evaluation functor $ev_\mathbf{C} : \mathbf{C} \to (\mathbf{C}^*,\mathbf{Set})$ is full and faithful, since $ev_\mathbf{C}$ is given by the composite of the Yoneda embedding $\mathbf{C} \to L_\kappa(\mathbf{C},\mathbf{Set})^{op}$ with the functor $\Sigma : L_\kappa(\mathbf{C},\mathbf{Set})^{op} \to (\mathbf{C}^*,\mathbf{Set})$ induced by I.

Proposition 1.2 *Let $\mathbf{A}$ be an accessible full subcategory $\mathbf{A}$ of a locally presentable category $\mathbf{B}$ closed under κ-filtered colimits and products for some κ. If $\mathbf{A}$ is codense in $\mathbf{B}$, then every object of $\mathbf{B}$ is the domain of an equalizer of a pair of morphisms of objects of $\mathbf{A}$.*

The codensity of $\mathbf{A}$ in $\mathbf{B}$ gives that the functor $\Sigma : \mathbf{B}^{op} \to (\mathbf{A},\mathbf{Set})$ $(B \mapsto \mathbf{B}(B,I(-)))$ induced by I is full and faithful. But $\mathbf{B}(B,I(-))$ is in $\prod Acc(\mathbf{A},\mathbf{Set})$. Therefore, Proposition 1.2 follows from (5) above.

For $\mathbf{B} = L_\kappa(\mathbf{C},\mathbf{Set})$ with $\mathbf{C}$ small κ-Barr-exact, if every object of $\mathbf{B}$ is the domain of an equalizer of a pair of morphisms of objects of $\mathbf{A}$, then $\mathbf{A}$ is codense in $\mathbf{B}$ (see [5] and [12]). Therefore, we have

Corollary 1.3 *Let $\mathbf{C}$ be a small κ-Barr-exact category, and $\mathbf{B} = Lex(\mathbf{C},\mathbf{Set})$. Then $\mathbf{A}$ of Proposition 1.2 is codense in $\mathbf{B}$ iff every object of $\mathbf{B}$ is the domain of an equalizer of a pair of morphisms of objects of $\mathbf{A}$.*

2 A stronger version of the strong conceptual completeness theorem

For any κ-regular functor $F : \mathbf{C} \to \mathbf{D}$, write $Inv(F)$ for the collection of all those morphisms f in $\mathbf{C}$ for which $F(f)$ is an isomorphism in $\mathbf{D}$. Given a collection U of morphisms in $\mathbf{C}$, recall from [18] that F is said to be obtained by inverting the morphisms in U if we have the following universal property: for any κ-Barr-exact category $\mathbf{B}$, the functor $F^* : \kappa\text{-}Reg(\mathbf{D}, \mathbf{B}) \to \kappa\text{-}Reg(\mathbf{C}, \mathbf{B})$ induced by F induces an equivalence of $\kappa\text{-}Reg(\mathbf{D}, \mathbf{B})$ onto the full subcategory of $\kappa\text{-}Reg(\mathbf{C}, \mathbf{B})$ consisting of those $G : \mathbf{C} \to \mathbf{B}$ for which $U \subset Inv(G)$. F is a quotient morphism if it is obtained by inverting the morphisms in $Inv(F)$. In case that both of $\mathbf{C}$ and $\mathbf{D}$ are small categories, it was shown in [18] that F is a quotient morphism iff it is full on subobjects and for any $D \in \mathbf{D}$, there is a regular epimorphism $F(C) \to D$ with $C \in \mathbf{C}$. We recall here that a functor $F : \mathbf{C} \to \mathbf{D}$ is full on subobjects if for every $C \in \mathbf{C}$, the poset morphism $F_C : Sub_{\mathbf{C}}(C) \to Sub_{\mathbf{D}}(F(C))$ induced by F is surjective. Here $Sub_{\mathbf{C}}(C)$ is the poset of subobjects of C; to the subobject determined by the monomorphism $m : B \to C$, F_C assigns the subobject determined by $F(m)$.

Let $\mathbf{A}$ be an accessible full subcategory of $\mathbf{C}^*$ closed under κ-filtered colimits and products. We have the evaluation functor $e : \mathbf{C} \to \mathbf{A}^+$. It is clear that e is κ-regular. If e is a quotient morphism, by the definition above, the functor $e^* : \mathbf{A}^{+*} \to \mathbf{C}^*$ is full and faithful. The initial observation is that the evaluation functor $e_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^{+*}$ is fully faithful, since the inclusion $\mathbf{A} \to \mathbf{C}^*$ is given by the composite of $e_{\mathbf{A}}$ with e^*. We have

Theorem 2.1 *e above is a quotient morphism iff the evaluation functor $e_{\mathbf{A}}$ is full and faithful.*

Only the "if" part still needs to be proved. that e is full on subobjects can be obtained by a similar argument of Lemma 4.2(ii) in [19] (without using that $e_{\mathbf{A}}$ is full and faithful). Let $F : \mathbf{C}^{*+} \to \mathbf{A}^+$ be the functor induced by the inclusion $\mathbf{A} \to \mathbf{C}^*$. Note that $e = F \circ e_{\mathbf{C}}$ with the equivalence $e_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}^{*+}$ of Theorem 5.1 in [20]. In order to prove that e is a quotient morphism, it suffices to show that for any $M \in \mathbf{A}^+$, there is a regular epimorphism $F(N) \to M$ with $N \in \mathbf{C}^{*+}$. The idea in proving this is to analysis the dual of κ-accessibility of $L_\kappa(\mathbf{A}^+, \mathbf{Set})$ (in $(\mathbf{A}^{+*}, \mathbf{Set})$), and to show

Lemma 2.2 *Let $e^{\#}_{\mathbf{A}} : (\mathbf{A}^{+*}, \mathbf{Set}) \to (\mathbf{A}, \mathbf{Set})$ be the functor induced by $e_{\mathbf{A}}$. Then*

(i)For any $M \in \mathbf{A}^+$, there are $A \in \mathbf{A}$ and a morphism $f : \mathbf{A}^{+*}(e_{\mathbf{A}}(A), -) \to e_{\mathbf{A}^+}(M)$ such that $e^{\#}_{\mathbf{A}}(f)$ is regular epi.

(ii)For any $A \in \mathbf{A}$, $\mathbf{A}^{+*}(e_{\mathbf{A}}(A), -)$ is a κ-cofiltered limit of objects consisting of the image of $e_{\mathbf{A}^+}$ in $(\mathbf{A}^{+*}, \mathbf{Set})$; here $e_{\mathbf{A}^+}$ is the evaluation functor $\mathbf{A}^+ \to (\mathbf{A}^{+*}, \mathbf{Set})$.

Lemma 2.2 can be proved from that $e_{\mathbf{A}}$ is full and faithful by using the codensity theorem and the properties mentioned in Section 1. Full details appear in [14].

Corollary 2.3 *If $\mathbf{A}$ is codense in $L_\kappa(\mathbf{C}, \mathbf{Set})$, and $e_{\mathbf{A}}$ is full and faithful, then $\mathbf{C}$ is equivalent to $\mathbf{A}^+$.*

The codensity of A gives that the functor $\Sigma : L_\kappa(\mathbf{C}, \mathbf{Set})^{op} \to (\mathbf{A}, \mathbf{Set})$ induced by the inclusion is full and faithful. But $e = \Sigma \circ Y$ ($Y : \mathbf{C} \to L_\kappa(\mathbf{C}, \mathbf{Set})^{op}$ is given by the Yoneda embedding), so e is a full and faithful. That e is an equivalence follows from that e is a quotient morphism (see Lemma 1.4.9 in [17]).

Remark 2.4 (i) For a κ-regular functor $F : \mathbf{C} \to \mathbf{D}$ between small categories, the strong conceptual completeness theorem says that F is a quotient morphism iff the induced functor $F^* : \mathbf{D}^* \to \mathbf{C}^*$ is full and faithful (see [18]). This is an immediate consequence of Theorem 2.1 by taking $\mathbf{A} = \mathbf{D}^*$ (that $e_{\mathbf{D}^*}$ is full and faithful follows from Theorem 5.1 of [20]).

(ii) Suppose that $\mathbf{A}$ is a category of the form $\mathbf{D}^*$ with $\mathbf{D}$ small κ-Barr-exact, and $\mathbf{A}$ is codense in $L_\kappa(\mathbf{C}, \mathbf{Set})$. According to Corollary 2.3, $\mathbf{C}$ is equivalent to $\mathbf{D}^{*+}$, hence $\mathbf{C}$ is equivalent to $\mathbf{D}$. We conclude that $\mathbf{C}^*$ is the smallest codense full subcategory of $L_\kappa(\mathbf{C}, \mathbf{Set})$ in the categories of the form $\mathbf{D}^*$ with the closedness properties above.

(iii) Assuming Vopěnka's principle, the accessibility of $\mathbf{A}$ can be removed. As shown in [2], under Vopěnka's principle, each full subcategory of a locally presentable category closed under κ-filtered colimits (for some κ) is accessible.

Acknowledgments

This work was done under the direction of M.Makkai. The author thanks for his guidance. The author would also like to thank W.Tholen for helpful conversations.

References

[1] J.Adámek and J.Rosický, On injectivity in locally presentable categories, Trans. Amer. Math. Soc. 336(1993), 785-804

[2] J.Adámek and J.Rosický, Accessible Categories and Locally Presentable Categories, Cambridge University Press, 1994

[3] M.Barr, Exact categories, Lecture Notes in Math., no.236, Springer-Verlag, 1971, 1-120

[4] M.Barr, P.A.Grillet and D.H.van Osdol, Exact categories and categories of sheaves, Lecture Notes in Math., no.236, Springer-Verlag, 1971

[5] M.Barr, Representations of categories, J. Pure Appl. Algebra 41 (1986), 113-137

[6] W.H.Boshuck, Reduced products and regular logic, preprint

[7] A.Carboni, Some free constructions in realizability and proof theory, to appear in J. Pure Appl. Algebra

[8] P.Freyd, Abelian Categories, Harper and Row, New York, 1964

[9] P.Gabriel and F.Ulmer, Lokal präsentierbare Kategorien, Lecture Notes in Math., no.221, Springer-Verlag, 1971

[10] R.Guitart and C.Lair, Calcul syntaxique des modéles et calcul des formules intérnes, Diagrammes 4(1980)

[11] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohôku Math. J. 9(1957), 119-221

[12] H.Hu, Dualities for accessible categories, Can. Math. Soc. Conference Proceeding, AMS, Providence, 13(1992), 211-242

[13] H.Hu, Flat functors and free exact completions, to appear J. Austral. Math. Soc. (Series A)

[14] H.Hu, Categories of models of exact categories, preprint, 1994

[15] G.M. Kelly, Basic Concepts of Enriched Category Theory, Cambridge University Press, (64)1982

[16] S.Mac Lane, Categories for the Working Mathematician, Springer-Verlag, 1971

[17] M.Makkai and G.Reyes, First order categorical logic, Lecture Notes in Math., no. 611, Springer-Verlag, 1977

[18] M.Makkai, Ultraproducts and categorical logic, Lecture Notes in Math., no.1130, Springer-Verlag 1985, 222-309

[19] M.Makkai, Stone duality for first order logic, Adv. in Math. 65 (1987), 97-170

[20] M.Makkai, A theorem on Barr-exact categories, with an infinitary generalization, Ann. Pure Appl. Logic 47(1990), 225-268

[21] M.Makkai and R.Paré, Accessible categories: the foundations of categorical model theory, Contemporary Math. 104 (1990)

[22] E.M.Vitale, Left covering functors, Ph.D. thesis, Universitè Catholique de Louvain, 1994

Honode Hu, Department of Mathematics and Statistics, York University, North York, Ont., CANADA, M3J 1P3
E-mail address: hhu@mathstat.yorku.ca

Received August 17, 1994

C.R. Math. Rep. Acad. Sci. Canada - Vol. XVI, No. 5 October 1995 octobre

EXT-ALGEBRAS

ISTVÁN ÁGOSTON[1], VLASTIMIL DLAB[2] AND ERZSÉBET LUKÁCS[1,2]
F.R.S.C.

ABSTRACT. The Ext-algebra A^* of a finite dimensional associative K-algebra A is studied with a motivation to establish conditions under which (i) the species of A and A^{*op} coincide and (ii) the quasi-heredity of A (or A^*) yields the quasi-heredity of A^* (or A, respectively). These questions are closely related to the Kazhdan–Lusztig Theory as presented by [CPS2].

1. Introduction

Throughout the paper A will denote a finite dimensional basic algebra over an arbitrary field K. Let us recall that the K-*species* $\mathcal{S}(A)$ of A is the system $(D_i : i \in I; {}_iW_j : i,j \in I)$ of finitely many division algebras D_i and D_i-D_j-bimodules ${}_iW_j$ so that $A/\operatorname{rad} A \simeq \prod_{i\in I} D_i$ and $\operatorname{rad} A/\operatorname{rad}^2 A \simeq \sum_{i,j\in I} {}_iW_j$. Thus, if $\{ e_i \mid i \in I \}$ is a complete set of primitive orthogonal idempotents in A, and $\bar{e}_i$ denotes the image of e_i in $A/\operatorname{rad} A$, then $D_i = \bar{e}_i(A/radA)\bar{e}_i$ and ${}_iW_j = \bar{e}_i(\operatorname{rad} A/\operatorname{rad}^2 A)\bar{e}_j$. Notice that if $S(i)$ is the simple right A-module $e_iA/e_i \operatorname{rad} A$ then $D_i \simeq \operatorname{End}_A(S(i))$ and ${}_iW_j \simeq \operatorname{Ext}^1_A(S^\circ(j), S^\circ(i))$. If the field K is algebraically closed then one may speak about the *quiver* of the algebra A. For, all the division algebras are equal to K and the bimodules ${}_iW_j$ are just direct sums of copies of the regular bimodule K; hence, the complete information is contained in an oriented graph having I as its vertex set and $\dim_K {}_iW_j$ arrows from i to j.

Given an algebra A one may define the so-called *Ext-algebra* of A, denoted by A^*. This is a K-algebra whose underlying vector space is

$$\bigoplus_{k\geq 0}\bigoplus_{i,j\in I} \operatorname{Ext}^k_A(S(i), S(j)),$$

with the multiplication defined via the Yoneda-product of exact sequences. Observe that A^* is finite dimensional if and only if $gl.dim\, A < \infty$; moreover the identity element of A^* is the sum of the primitive orthogonal idempotents

1991 Mathematics Subject Classification. Primary 16E99. Secondary 16S99, 17B10

[1] Research partially supported by Hungarian NFSR grant no. T4265

[2] Research partially supported by NSERC of Canada

$f_i = \mathrm{id}_{S(i)}$, $i \in I$. In analogy to $S(i)$ and $P(i) = e_iA$, denote by $S^{\bullet o}(i)$ and $P^{\bullet o}(i)$ the corresponding simple and indecomposable projective left $A^{\bullet}$-modules.

Our principal objective is to study the connection between some of the properties of A and $A^{\bullet}$, respectively. Some of our results are parallel to those of [CPS2] although our approach is somewhat different.

Most results presented here were reported by the authors on several occasions (Sherbrooke: May 1994, Prague: June 1994, Mexico City: August 1994). The proofs of the statements, together with some examples and further references to the graded situation will appear in a more detailed version elsewhere.

2. The species of Ext-algebras

First we will be dealing with the question of the species of $A^{\bullet}$ (more precisely, of $A^{\bullet op}$). It is easy to see, that $\mathcal{S}(A) \subseteq \mathcal{S}(A^{\bullet op})$. We will show that the fact that the species of these two algebras coincide is equivalent to some easy-to-describe property of the projective resolutions of the simple A-modules.

To this end we recall that a submodule X of Y is a *top submodule* (denoted by $X \overset{t}{\subseteq} Y$) if $\mathrm{rad}\, X = X \cap \mathrm{rad}\, Y$, i.e. the embedding of X into Y induces an embedding of $\mathrm{top}\, X$ into $\mathrm{top}\, Y$ (see [ADL1]). A filtration $X = X_1 \supseteq X_2 \supseteq \dots \supseteq X_m$ of a module X is called a *top filtration* if $X_i \overset{t}{\subseteq} X$ for $1 \leq i \leq m$.

We shall also use the following notation. For an arbitrary module $X \in$ mod-A

$$\dots \xrightarrow{d_{j+1}} \mathcal{P}_j(X) \xrightarrow{d_j} \dots \xrightarrow{d_2} \mathcal{P}_1(X) \xrightarrow{d_1} \mathcal{P}_0(X) \xrightarrow{d_0} X \to 0$$

will denote a minimal projective resolution of X, with the corresponding syzygies $\Omega_{j+1}(X) = \mathrm{Ker}\, d_j$ for $j = 0, 1, \dots$.

Now we may introduce the following subcategory of the category of finitely generated right A-modules mod-A.

DEFINITION 2.1. We say that a module $X \in$ mod-A belongs to $\mathcal{C}^{(i)} = \mathcal{C}_A^{(i)}$ for some $i \in \mathbb{N}$ if $\Omega_j(X) \overset{t}{\subseteq} \mathrm{rad}\, \mathcal{P}_{j-1}(X)$ for $j = 1, 2, \dots, i$. We may also define $\mathcal{C}^{(0)} =$ mod-A. The intersection of these subcategories will be denoted by $\mathcal{C}$; thus $\mathcal{C} = \mathcal{C}_A = \bigcap_{i=0}^{\infty} \mathcal{C}^{(i)}$. - Similarly, one may define the subcategory $\mathcal{C}_A^{o} \subset$ A-mod of left A-modules.

It is easy to see, that the definition does not depend on which particular minimal projective resolution of X was chosen.

The following proposition gives an important homological property of the elements of $\mathcal{C}^{(i)}$.

PROPOSITION 2.2. *If $X \in \mathcal{C}^{(i)}$ then the natural maps $\mathrm{Ext}_A^k(\mathrm{top}\, X, S) \to \mathrm{Ext}_A^k(X, S)$ are surjective for every $0 \leq k \leq i$ and every simple module S.*

It turns out that with the addition of an easy necessary assumption, this property fully characterizes the elements of $C^{(i)}$.

PROPOSITION 2.3. *Assume that every simple A-module S is in C_A. Then a module X is an element of $C_A^{(i)}$ if and only if the natural maps $\mathrm{Ext}_A^k(\mathrm{top}\, X, S) \to \mathrm{Ext}_A^k(X,S)$ are surjective for every $0 \le k \le i$ and S simple module.*

Proposition 2.2 leads to a full answer as to when the species of A and $A^{\bullet op}$ coincide.

THEOREM 2.4. *The following are equivalent for an algebra A.*
(a) $S \in C_A$ for every simple right module S;
(b) $S^\circ \in C_A^\circ$ for every simple left module S°;
(c) $\mathcal{S}(A) = \mathcal{S}(A^{\bullet op})$.

3. The functor $\mathrm{Ext}^\bullet : \text{mod-}A \to A^\bullet\text{-mod}$

We shall assume in this section that the Ext-algebra $A^\bullet$ of the finite dimensional algebra A is itself finite dimensional, i.e. $gl.dim\, A < \infty$.

Let $\hat{S}$ denote the direct sum of all simple right A-modules, i.e. $\hat{S} = \bigoplus_{i \in I} S(i)$. Then we may define a contravariant functor $\mathrm{Ext}^\bullet : \text{mod-}A \to A^\bullet\text{-mod}$ by taking the direct sum of the functors $\mathrm{Ext}^k(-,\hat{S})$ for $k \ge 0$. Actually, the modules $\mathrm{Ext}^\bullet(X)$ will have a natural grading, with the morphisms $\mathrm{Ext}^\bullet(f)$ preserving this grading, hence we have a functor into $A^\bullet\text{-mod}_{gr}$. For a module $X \in A^\bullet\text{-mod}_{gr}$, let $X[j]$ denote the shifted graded module, i.e. $X[j]_i = X_{i-j}$. We have the following exactness properties of $\mathrm{Ext}^\bullet$.

LEMMA 3.1. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in* mod-A.
(a) Assume $X \overset{t}{\subseteq} Y$. If $X \in C_A$ then the sequence $0 \to \mathrm{Ext}^\bullet(Z) \to \mathrm{Ext}^\bullet(Y) \to \mathrm{Ext}^\bullet(X) \to 0$ is exact; if in addition $Z \in C_A$, then $\mathrm{Ext}^\bullet(Z) \overset{t}{\subseteq} \mathrm{Ext}^\bullet(Y)$.
(b) Assume $X \subseteq \mathrm{rad}\, Y$. If $Y \in C_A$ then the sequence $0 \to \mathrm{Ext}^\bullet(X)[1] \to \mathrm{Ext}^\bullet(Z) \to \mathrm{Ext}^\bullet(Y) \to 0$ is exact; if in addition $Z \in C_A$, then $\mathrm{Ext}^\bullet(X)[1] \subseteq \mathrm{rad}\,\mathrm{Ext}^\bullet(Z)$.

Based on this lemma, we get the following propositions.

PROPOSITION 3.2. *If $X, \mathrm{rad}\, X \in C_A$ then $\mathrm{Ext}^\bullet(X) \in C_{A^\bullet}^{(1)\circ}$. Thus if $\mathrm{rad}^i X \in C_A$ for every i then $\mathrm{Ext}^\bullet(X) \in C_{A^\bullet}^\circ$.*

PROPOSITION 3.3. *(a)* $\mathrm{Ext}^\bullet\left(S(i)\right) = P^{\bullet\circ}(i)$.
(b) $\mathrm{Ext}^\bullet\left(P(i)\right) = S^{\bullet\circ}(i)$.
(c) $\mathrm{Ext}^\bullet\left(\mathrm{rad}\, P(i)\right)[1] = \mathrm{rad}\, P^{\bullet\circ}(i)$.

4. Ext-algebras and quasi-heredity

To speak about the quasi-heredity of an algebra A, one must impose a (partial) order on the set $\{S(i) \mid i \in I\}$ of simple right A-modules (or equivalently, on the given complete set of primitive orthogonal idempotents). Actually, without loss of generality we may assume that we have a total order on the index set I. Thus assume that $I = \{1, 2, \ldots, n\}$ with the natural order. We shall write $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ for the corresponding ordered set of primitive orthogonal idempotents and we define $\varepsilon_i = e_i + e_{i+1} + \ldots + e_n$, $\varepsilon_{n+1} = 0$. Recall that $P(i)$ denotes the projective cover of the simple module $S(i)$. Consider the *trace filtration* of A:

$$A = A\varepsilon_1 A \supseteq A\varepsilon_2 A \supseteq \ldots \supseteq A\varepsilon_n A \supseteq 0.$$

We say that A is *quasi-hereditary* with respect to I (or briefly, $(A, \mathbf{e})$ is quasi-hereditary) if each of the so called *standard right modules* $e_i A / e_i A \varepsilon_{i+1} A$, denoted by $\Delta(i)$ is *Schurian* (i.e. it has a semisimple endomorphism ring) and the quotients of the trace filtration $A\varepsilon_i A / A\varepsilon_{i+1} A$ as right modules are direct sums of the corresponding standard modules. In addition, we say that A is *lean* with respect to this order if $\Delta(i) \in \mathcal{C}_A^{(1)}$ and $\Delta^\circ(i) \in \mathcal{C}_A^{(1)^\circ}$ for all $i \in I$. (Here $\Delta^\circ(i)$ stands for the corresponding standard left module.) We consider the following canonical exact sequences:

$$0 \to V(i) \to P(i) \to \Delta(i) \to 0 \qquad \text{and} \qquad 0 \to U(i) \to \Delta(i) \to S(i) \to 0.$$

For the basic properties of quasi-hereditary algebras, we refer to [CPS1], [DR1], [DR2] or [DK] and of lean algebras to [ADL1], [ADL2]. Canonical constructions for the so-called *shallow*, *replete* and *medial algebras* are also described there.

We have already noticed that the simple types of (right) A-modules are in one-to-one correspondence with the simple types of (left) A^*-modules; the corresponding idempotent to the primitive idempotent $e_i \in A$ is the element $f_i = \mathrm{id}_{S(i)} \in A^*$. Having fixed the order $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ for A we shall consider the reverse order $\mathbf{f} = (f_n, f_{n-1}, \ldots, f_1)$ for A^*; write $\varphi = f_i + f_{i-1} + \ldots + f_1$ and $\varphi_0 = 0$.

One of the key observations in recognizing the quasi-heredity of A^* is the following lemma.

LEMMA 4.1. *Assume that* $(A, \mathbf{e})$ *is quasi-hereditary with* $\Delta(i) \in \mathcal{C}_A$ *and* $U(i) \in \mathcal{C}_A$ *for* $1 \le i \le n$. *Then the left standard module* $\Delta^{*\circ}(i)$ *of* $(A^*, \mathbf{f})$ *is Schurian and* $\Delta^{*\circ}(i) \simeq \mathrm{Ext}^*(\Delta(i))$. *Furthermore, with similar notation,* $\mathrm{Ext}^*(U(i))[1] \simeq V^{*\circ}(i)$ *and* $\mathrm{Ext}^*(V(i))[1] \simeq U^{*\circ}(i)$.

We can now state the following sufficient condition for a quasi-hereditary algebra to have a quasi-hereditary Ext-algebra.

DEFINITION 4.2. An algebra $(A, \mathbf{e})$ is said to be *solid*, if the following conditions are satisfied:

(1) $\Delta(i)$ is Schurian;

(2) $V(i) \overset{t}{\subsetneq} \operatorname{rad} P(i)$;

(3) $U(i)$ has a top filtration by $S(j)$'s and $\Delta(j)$'s for $j < i$;

(4) $V(i)$ has a top filtration by $\Delta(j)$'s and $P(j)$'s for $j > i$.

LEMMA 4.3. *If $(A, \mathbf{e})$ is solid then it is a lean quasi-hereditary algebra with $S(i), \Delta(i), U(i) \in \mathcal{C}_A$ for $1 \leq i \leq n$.*

THEOREM 4.4. *Let $(A, \mathbf{e})$ be a solid algebra. Then:*

(a) $(A^{*op}, \mathbf{f})$ is a solid algebra (hence quasi-hereditary), and

(b) $\mathcal{S}(A) = \mathcal{S}(A^{*op})$, $\dim_K A^{**} = \dim_K A$, $(\varepsilon_i A \varepsilon_i)^* \simeq A^*/(A^* \varphi_{i-1} A^*)$ and $\left(A/(A\varepsilon_i A)\right)^* \simeq \varphi_{i-1} A^* \varphi_{i-1}$.

COROLLARY 4.5. *If the algebra $(A, \mathbf{e})$ is shallow (left medial, right medial or replete) then $(A^{*op}, \mathbf{f})$ is replete (left medial, right medial or shallow, respectively) on the same species.*

5. Ext-algebras of monomial algebras

We can get a more complete picture of the situation in the case of monomial algebras. Here the principal tool in the understanding is the existence of a multiplicative basis for A^*, consisting of some paths in the quiver of A (see [GZ]). Thus we shall assume now that A is *monomial*, i.e. $A = K\Gamma/R$, where Γ is a quiver with R the set of relations which is generated by some paths of length at least 2. First, we have an extension of Theorem 2.4 about the quiver of A^{*op}.

THEOREM 5.1. *Let $A \simeq K\Gamma/R$ be a monomial algebra. Then the following are equivalent:*

(a) $S(i) \in \mathcal{C}_A$ for $1 \leq i \leq n$;

(b) A and A^{*op} have the same quiver;

(c) A is quadratic (i.e. the set of relations R is generated by paths of length 2);

(d) $\operatorname{Ext}^2_A(\hat{S}, \hat{S}) \subseteq \operatorname{rad}^2(A^*)$.

If $(A, \mathbf{e})$ is in addition lean with Schurian standard modules, then conditions (a)–(d) are all equivalent to:

(e) $\Delta(i) \in \mathcal{C}$, $\Delta^\circ(i) \in \mathcal{C}^\circ$ for $1 \leq i \leq n$.

On the question of quasi-heredity we have the following results.

THEOREM 5.2. *Let $A = K\Gamma/R$ be a monomial algebra with $gl.dim\, A < \infty$. Then $(A^*, \mathbf{f})$ is quasi-hereditary if and only if $(A, \mathbf{e})$ is lean with Schurian standard modules.*

THEOREM 5.3. *Let $A = K\Gamma/R$ be a monomial algebra. If $(A, \mathbf{e})$ is quasi-hereditary then either $(A^*, \mathbf{f})$ is lean with Schurian standard modules or the quiver of A^* has a loop.*

Thus from the previous two theorems we get the following corollary.

COROLLARY 5.4. *Let $A = K\Gamma/R$ be a monomial algebra. Then if $(A, \mathbf{e})$ is lean and quasi-hereditary, then so is $(A^*, \mathbf{f})$.*

REFERENCES

[ADL1] Ágoston, I., Dlab, V., Lukács, E., Lean quasi-hereditary algebras, in: *Representations of Algebras. Sixth International Conference, 1992, Ottawa.* CMS Conference Proceedings 14, 1–14.

[ADL2] Ágoston, I., Dlab, V., Lukács, E., Homological characterization of lean algebras, *Manuscripta Mathematica* 81 (1993) 141–147

[CPS1] Cline, E., Parshall, B.J., Scott, L.L., Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* 391 (1988), 85–99.

[CPS2] Cline, E., Parshall, B.J., Scott, L.L., The homological dual of a highest weight category, *Proc. London Math. Soc.* 68 (1994), 294–316.

[DR1] Dlab, V., Ringel, C.M., Quasi-hereditary algebras, Illinois J. of Math. 35 (1989), 280–291.

[DR2] Dlab, V., Ringel, C.M., The module theoretical approach to quasi-hereditary algebras, in: *Representations of Algebras and Related Topics* London Math. Soc. Lecture Note Series 168, Cambridge Univ. Press 1992, 200–224.

[DK] Drozd, Yu.A., Kirichenko, V.V., *Finite dimensional algebras*, Appendix by V. Dlab, Springer-Verlag 1994.

[GZ] Green, E., Zacharia, D., The cohomology ring of a monomial algebra, preprint

MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, P.O.Box 127, 1364 BUDAPEST, HUNGARY
E-mail address: agoston@konig.elte.hu

DEPARTMENT OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, K1S 5B6, CANADA
E-mail address: vdlab@math.carleton.ca

DEPARTMENT OF MATHEMATICS, FACULTY OF TRANSPORT ENGINEERING, TECHNICAL UNIVERSITY OF BUDAPEST, 1111 BUDAPEST, HUNGARY
E-mail address: lukacs@euromath.vma.bme.hu

Received September 25, 1994

EMPIRICAL-TYPE PROCESSES IN WEIGHTED METRICS AND CHANGE-POINT PROBLEMS

Barbara Szyszkowicz

Presented by Miklós Csörgő, F.R.S.C.

Abstract: We consider the asymptotic behaviour of L_p-approximations and functionals of weighted two-time parameter empirical type processes based on observations of independent, identically distributed random variables, their ranks and sequential ranks. We outline a possible application of our results to the change-point problem.

1. Introduction. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with the same continuous distribution F. We consider the two-time parameter empirical processes

$$\alpha_n(s,t) = n^{-1/2} \sum_{i=1}^{[nt]} \Big(1\{F(X_i) \le s\} - s\Big), \quad 0 \le s,t \le 1, \tag{1.1}$$

$$\beta_n(s,t) = n^{-1/2} \sum_{i=1}^{[nt]} \Big(1\{R_{in} \le s\} - \frac{[ns]}{n}\Big), \quad 0 \le s,t \le 1, \tag{1.2}$$

$$\gamma_n(s,t) = n^{-1/2} \sum_{i=1}^{[nt]} \Big(1\{\xi_i \le s\} - \frac{[is]}{i}\Big), \quad 0 \le s,t \le 1, \tag{1.3}$$

where $R_{1n}, \ldots, R_{nn}$ denote the normalized ranks, $R_{kn} = n^{-1}\sum_{i=1}^{n} 1\{X_i \le X_k\}$, $k = 1, \ldots, n$, and $\xi_i, \ldots, \xi_n$ are the normalized sequential ranks, $\xi_k = k^{-1}\sum_{i=1}^{k} 1\{X_i < X_k\}$, $k = 1, \ldots, n$, of the first n of the random variables $X_1, X_2, \ldots$. The limiting distributions of these processes are well known. The three processes (1.1)–(1.3) were studied by Pardzhanadze and Khmaladze (1986) under a class of contiguous alternatives which accommodate the possible occurrence of a changepoint in a series of measurements. Szyszkowicz (1991, 1994) studied these processes, as well as the "bridge type" (tied down at $t = 1$) versions of $\alpha_n(s,t)$ and $\gamma_n(s,t)$, in weighted supremum metrics in the time variable $t \in (0,1)$ under the null assumption of no change and also under a sequence of contiguous alternatives. Here we consider the asymptotic behaviour of these processes in weighted L_p-metrics. We succeed in establishing their asymptotic theory in such metrics for the optimal class of weight functions. This class of weight functions is also bigger than the ones we studied in the case of weighted supremum norms.

A natural way to test for the possibility of having a change in the distribution of a sequence of chronologically observed data up to time n is to compare their empirical distributions before and after the possibly unknown time of change $1 \le k < n$, via studying the asymptotic distribution of the sequence of processes

$$\begin{aligned} & n^{1/2}\left|\frac{1}{k}\sum_{i=1}^{k} 1\{X_i \le x\} - \frac{1}{n-k}\sum_{i=k+1}^{n} 1\{X_i \le x\}\right| \\ &= \left|\sum_{i=1}^{k} 1\{X_i \le x\} - \frac{k}{n}\sum_{i=1}^{n} 1\{X_i \le x\}\right| \Big/ n^{1/2}(k/n(1-k/n)), \\ & x \in \mathbb{R},\ 1 \le k < n,\ n = 1,2,\dots . \end{aligned} \tag{1.4}$$

This leads us to study weighted, tied down at $t = 1$, two-time parameter empirical processes. The naturally appearing weight functions have to be modified in order to ensure convergence of the whole process in sup-norm, or that of its sup- functional (cf. Picard, 1985; Deshayes and Picard, 1986; and Szyszkowicz, 1991, 1994). On the other hand, if we were to study the problem of the asymptotic behaviour of the process in (1.4) say in L_1, then, for having a non-degenerate limit, there is no need to replace the naturally arrived at weight function $((k/n)(1-k/n))$ in there by any other function that would be milder on the tails. Indeed, rewriting a bit the sequence of stochastic processes of (1.4) as $|\hat{\alpha}_n(s,t)|/t(1-t)$, where

$$\hat{\alpha}_n(s,t) := \begin{cases} n^{-1/2}\Big(\sum_{i=1}^{[(n+1)t]} 1\{F(X_i) \le s\} - \frac{[(n+1)t]}{n}\sum_{i=1}^{n} 1\{F(X_i) \le s\}\Big), & 0 \le t < 1,\ 0 \le s \le 1 \\ 0, & t = 1,\ 0 \le s \le 1, \end{cases}$$

it will follow from our Theorem 2.2 that, as $n \to \infty$,

$$\int_0^1\int_0^1 |\hat{\alpha}_n(s,t)|/(t(1-t))dsdt \xrightarrow{\mathcal{D}} \int_0^1\int_0^1 |K(s,t) - tK(s,1)|/(t(1-t))dsdt, \tag{1.5}$$

where $\{K(s,t);\ 0 \le s,t \le 1\}$ is a Kiefer process (cf. (2.1)). Moreover, according to Theorem 2.2, in (1.5) we can even have $(t(1-t))^{\nu}$ instead of $t(1-t)$ with any $\nu < 3/2$.

In order to better relate the result in (1.5) to the change-point problem as summarized by the sequence of stochastic processes in (1.4), we put $s = F(x)$ and consider $\hat{\alpha}_n(F(x),t) = \hat{\hat{\alpha}}_n(x,t)$, which does not depend on the possibly unknown distribution function F.

Then (1.5) translates into

$$\int_0^1\int_{-\infty}^{\infty} |\hat{\hat{\alpha}}_n(x,t)|/(t(1-t))dF(x)dt \xrightarrow{\mathcal{D}} \int_0^1\int_0^1 |K(s,t) - tK(s,1)|/(t(1-t))dsdt, \tag{1.6}$$

i.e., now we have a sequence of "statistics" (incomputable) converging to a distribution free (of F) random variable.

It will follow from our Theorem 3.1, that replacing the measure of integration dF on the left hand side of (1.6) by dx, will result in having

$$\int_0^1\int_{-\infty}^{\infty} |\widehat{\alpha}_n(x,t)|/(t(1-t))dxdt \xrightarrow{\mathcal{D}} \int_0^1\int_{-\infty}^{\infty} |K(F(x),t) - tK(F(x),1)|/(t(1-t))dxdt,$$

on assuming a bit more than two moments for F. That is to say, we now have a sequence of statistics (computable) converging to a non distribution free (a function of F) Gaussian random variable whose distribution can be simulated for each unknown F via repeated large samples, or by bootstrapping a given one.

For proofs of all presented results we refer to a forthcoming paper by the author (cf. also Szyszkowicz, 1993).

2. Main Results. Let $\{K(s,t);\ 0 \le s \le 1,\ 0 \le t < \infty\}$ denote a Kiefer process, i.e. a two-time parameter separable Gaussian process with mean zero and covariance function

$$EK(s_1,t_1)K(s_2,t_2) = (s_1 \wedge s_2 - s_1 s_2)(t_1 \wedge t_2). \tag{2.1}$$

Theorem 2.1. *Let $0 < p < \infty$ and q be a positive function on $(0,1]$. Then the following five statements are equivalent.*

(a) *We have*

$$\int_0^1 t^{p/2}/q(t)dt < \infty.$$

(b) *One can construct a Kiefer process $\{K(s,t);\ 0 \le s,t \le 1\}$ such that we have, as $n \to \infty$,*

$$\int_0^1\int_0^1 |\alpha_n(s,t) - n^{-1/2}K(s,nt)|^p/q(t)dsdt = o_P(1).$$

(c) *We have, as $n \to \infty$,*

$$\int_0^1\int_0^1 |\alpha_n(s,t)|^p/q(t)dsdt \xrightarrow{\mathcal{D}} \int_0^1\int_0^1 |K(s,t)|^p/q(t)dsdt.$$

(d) *There exists a sequence of Kiefer processes $\{K_n(s,t); 0 \le s,t \le 1\}$ such that, as $n \to \infty$*

$$\int_0^1\int_0^1 |\gamma_n(s,t) - K_n(s,t)|^p/q(t)dsdt = o_P(1).$$

(e) *We have, as $n \to \infty$*

$$\int_0^1\int_0^1 |\gamma_n(s,t)|^p/q(t)dsdt \xrightarrow{\mathcal{D}} \int_0^1\int_0^1 |K(s,t)|^p/q(t)dsdt.$$

Now we present our results for tied down at $t = 1$ empirical process, namely $\hat{\alpha}_n(s,t)$ as defined in the Introduction and $\beta_n(s,t)$ process, which by itself has "bridge" structure.

With $\{K(s,t); 0 \leq s,t \leq 1\}$ being a Kiefer process, we define $\Gamma(\cdot,\cdot)$ by

$$\Gamma(s,t) = K(s,t) - tK(s,1), \quad 0 \leq s,t \leq 1.$$

Consequently $\{\Gamma(s,t);\ 0 \leq s,t \leq 1\}$ is a separable Gaussian process with mean zero and covariance function

$$E\Gamma(s_1,t_1)\Gamma(s_2,t_2) = (s_1 \wedge s_2 - s_1 s_2)(t_1 \wedge t_2 - t_1 t_2). \tag{2.2}$$

We define

$$\hat{\beta}_n(s,t) = \begin{cases} n^{-1/2} \sum_{i=1}^{[(n+1)t]} \left(1\{R_{in} \leq s\} - \frac{[ns]}{n}\right) & , 0 \leq s \leq 1,\ 0 \leq t < 1 \\ 0 & , 0 \leq s \leq 1,\ t = 1. \end{cases}$$

Theorem 2.2. *Let $0 < p < \infty$ and q be a positive function on $(0,1)$. Then the following five statements are equivalent.*

(a) *We have*

$$\int_0^1 \left(t(1-t)\right)^{p/2} \Big/ q(t)dt < \infty. \tag{2.3}$$

(b) *There exists a Kiefer process $\{K(s,t);\ 0 \leq s \leq 1, t \geq 0\}$ such that with the sequence of stochastic processes $\Gamma_n(\cdot,\cdot)$*

$$\left\{\Gamma_n(s,t); 0 \leq s,t \leq 1\right\} = \left\{n^{1/2}(K(s,nt) - tK(s,n)); 0 \leq s,t \leq 1\right\}$$
$$\stackrel{\mathcal{D}}{=} \left\{\Gamma(s,t); 0 \leq s,t \leq 1\right\} \text{ for each } n \geq 1,$$

we have, as $n \to \infty$,

$$\int_0^1 \int_0^1 \left|\hat{\alpha}_n(s,t) - \Gamma_n(s,t)\right|^p \Big/ q(t)dsdt = o_P(1).$$

(c) *We have, as $n \to \infty$,*

$$\int_0^1 \int_0^1 \left|\hat{\alpha}_n(s,t)\right|^p \Big/ q(t)dsdt \xrightarrow{\mathcal{D}} \int_0^1 \int_0^1 \left|\Gamma(s,t)\right|^p \Big/ q(t)dsdt,$$

where $\{\Gamma(s,t);\ 0 \leq s,t \leq 1\}$ is a Gaussian process as defined in (2.2).

(d) *With the sequence of stochastic processes $\Gamma_n(\cdot,\cdot)$ as in* (b) *we have, as $n \to \infty$*

$$\int_0^1 \int_0^1 \left|\hat{\beta}_n(s,t) - \Gamma_n(s,t)\right|^p \Big/ q(t)dsdt = o_P(1).$$

(e) *We have, as $n \to \infty$*

$$\int_0^1 \int_0^1 \left|\hat{\beta}_n(s,t)\right|^p \Big/ q(t)dsdt \xrightarrow{\mathcal{D}} \int_0^1 \int_0^1 \left|\Gamma(s,t)\right|^p \Big/ q(t)dsdt,$$

where $\{\Gamma(s,t);\ 0 \le s,t \le 1\}$ is a Gaussian process as defined in (2.2).

3. An application to the change-point problem. As mentioned already in the Introduction, in order to propose a statistic for the change-point problem, we put $s = F(s)$ and consider $\hat{\alpha}_n(F(x),t) = \hat{\hat{\alpha}}_n(x,t)$.

Theorem 3.1. *Let q be a positive function on $(0,1)$ and F be a continuous distribution function. Then, with $1 \le p < \infty$, the following three statements are equivalent.*

(a) *We have*

$$\int_0^1 (t(1-t))^{p/2}/q(t)dt < \infty \quad \textit{and} \quad \int_{-\infty}^{\infty} (F(x)(1-F(x)))^{p/2}dx < \infty.$$

(b) *One can construct a Kiefer process $\{K(F(x),t); x \in \mathbb{R},\ 0 \le t \le 1\}$ such that with the sequence of stochastic processes Γ_n*

$$\begin{aligned}\{\Gamma_n(F(x),t); x \in \mathbb{R}, 0 \le t \le 1\} &= \{n^{-1/2}(K(F(x),t) - tK(F(x),1)); x \in \mathbb{R}, 0 \le t \le 1\} \\ &\stackrel{\mathcal{D}}{=} \left\{\Gamma(F(x),t); x \in \mathbb{R}, 0 \le t \le 1\right\} \textit{ for each } n \ge 1,\end{aligned}$$

we have, as $n \to \infty$,

$$\int_0^1 \int_{-\infty}^{\infty} |\hat{\hat{\alpha}}_n(x,t) - \Gamma_n(F(x),t)|^p/q(t)dxdt = o_P(1).$$

(c) *We have, as $n \to \infty$*

$$\int_0^1 \int_{-\infty}^{\infty} |\hat{\hat{\alpha}}_n(x,t)|^p/q(t)dxdt \xrightarrow{\mathcal{D}} \int_0^1 \int_{-\infty}^{\infty} |\Gamma(F(x),t)|^p/q(t)dxdt,$$

where $\{\Gamma(s,t);\ 0 \le s,t \le 1\}$ is a Guassian process as defined in (2.2).

For example, assuming (cf. Hoeffding, 1973; also Csörgő, Csörgő, Horváth, 1986)

$$\int_{-\infty}^{\infty} x^2(\log(1+x))^{1+\delta}dF(x) < \infty, \textit{ with any } \delta > 0,$$

we have

$$\int_0^1 \int_{-\infty}^{\infty} |\widehat{\widehat{\alpha}}_n(x,t)|/(t(1-t))dxdt \xrightarrow{\mathcal{D}} \int_0^1 \int_{-\infty}^{\infty} |\Gamma(F(x),t)|/(t(1-t))dxdt.$$

Consequently, in this weighted L_1-convergence in distribution, we obtain a most natural statistic (computable) for the change-point problem as formulated in (1.4) (cf. also the discussion of this problem in the Introduction).

Similar applicable statistics-type results can be formulated also in terms of the other stochastic processes considered in this paper.

Acknowledgements. The author wishes to acknowledge her gratitude to Miklós Csörgő for stimulating discussions. Research supported by an NSERC Canada grant at Carleton University, Ottawa.

References

[1] Csörgő, M., Csörgő, S. and Horváth, L. (1986). An Asymptotic Theory for Empirical Reliability and Concentration Processes. *Lecture Notes in Statistics* 33. Springer-Verlag, Berlin.

[2] Deshayes, J. and Picard, D. (1986). Off-line statistical analysis of change-point models using non parametric and likelihood methods, Lecture Notes in Control and Information Sciences, M. Thoma and A. Wyner, eds., 77: Detection of Abrupt Changes in Signals and Dynamical Systems, M. Basseville and A. Benveniste, eds. Springer-Verlag, Berlin 103-168.

[3] Hoeffding, W. (1973). On the centering of a simple linear rank statistic. *Ann. Statist.* 1, 54-66.

[4] Pardzhanadze, A.M. and Khmaladze, E.V. (1986). On the asymptotic theory of statistics of sequential ranks. *Theory Probab. Appl.* 31, 669-682.

[5] Picard, D. (1985). Testing and estimating change-points in time series. *Adv. Appl. Prob.* 17, 126-135.

[6] Szyszkowicz, B. (1991). Empirical type processes and contiguity. *C.R. Math. Rep. Acad. Sci. Canada XIII*, No. 4 161-166.

[7] Szyszkowicz, B. (1993). L_p-approximations of weighted empirical type processes. In *Tech. Rep. Ser. Lab. Res. Stat. Prob.* No. 231 Carleton U. - U. Ottawa.

[8] Szyszkowicz, B. (1994). Weak convergence of weighted empirical type processes under contiguous and changepoint alternatives. *Stochastic Process. Appl.* 50, 281-313.

Department of Mathematics & Statistics
Carleton University
Ottawa, Ontario K1S 5B6
Canada

Received October 6, 1994

Mailing Addresses

I. Ágoston	Mathematical Institute Hungarian Academy of Sciences P.O. Box 127, H-1364 Budapest, Hungary
M. Akkar	Université de Bordeaux I U.F.R. de Mathématiques ed d'informatique 351 Cours de la Liberation F33405 Talence Cedex, France
A. Borbély	Department of Mathematics Kuwait University P.O. Box 5969 Safat 13060 Kuwait
V. Dlab	Department of Mathematics & Statistics Carleton University Ottawa, Ontario K1S 5B6, Canada
G. Fournier	Departement de mathématiques et d'informatique Université de Sherbrooke Sherbrooke, Québec, J1K 2R1, Canada
H. Hu	Department of Mathematics & Statistics York University North York, Ontario, M3J 1P3, Canada
P. Kannappan	Department of Pure Mathematics University of Waterloo Waterloo. Ontario. N2L 3G1, Canada
E. Lukács	Department of Mathematics Faculty of Transport Engineering Technical University of Budapest H1111 Budapest, Hungary
V. Nistor	Department of Mathematics Harvard University 1 Oxford Street, Cambridge, MA 02138, U.S.A.
L. Oubbi	Departement de Mathématiques Ecole Normale Superieure Takkadoum B.P. 5118. 10,000 Rabat, Morocco
M. Oudadess	Departement de Mathématiques Ecole Normale Superieure Takkadoum B.P. 5118, 10,000 Rabat, Morocco
B. Szyszkowicz	Department of Mathematics & Statistics Carleton University Ottawa, Ontario K1S 5B6, Canada
K. Trimeche	Department of Mathematics Faculty of Sciences of Tunis. 1060 Tunis, Tunisia
D. Violette	Departement de mathématiques Université de Moncton Moncton, N.B., E1A 3E9, Canada
E.V. Vlasov	Department of Mathematics and Statistics Queen's University Kingston, Ontario, K7L 3N6. Canada