

CONTENTS

Q. CHEN	
Existence of weak solutions to the nonstationary thermistor problem	1
S.C. CHANG	
μ -uniqueness of $\ell \otimes e$	6
M.Ya. ANTIMIROV, A.A. KOLYSHKIN and R. VAILLANCOURT	
Analytical solution to the direct unsymmetric eddy current testing problem for a symmetric flaw	9
E. GÖRLICH and A.P. ROHS	
Asymptotic minimal projection constants for Lebesgue spaces	15
M. DOSSA	
Problème de Cauchy sur un conoïde caractéristique pour des systèmes quasi-linéaires hyperboliques	17
R.J. ASHTON and J.H.H. CHALK	
On the representation of integers by indefinite diagonal quadratic forms	23
A. KAROUI and R. VAILLANCOURT	
Smooth biorthogonal wavelet bases	25
M. RØRDAM	
A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$	31
E.J. BEGGS	
Imbedding algebra bundles in trivial bundles	37
G. GONG	
Approximation by dimension drop C^* -algebras and classification	40
G.A. ELLIOTT, G. GONG, H. LIN and C. PASNICU	
Homomorphisms, homotopies, and approximations by circle algebras	45
Mailing addresses	51

EXISTENCE OF WEAK SOLUTIONS TO THE NONSTATIONARY THERMISTOR PROBLEM

Qihong Chen

(Suzhou Institute of Urban Construction and Environmental Protection,
Suzhou 215008, China)

Presented by G.F.D. Duff, F.R.S.C.

Abstract

In this paper, we establish the existence of weak solutions for a degenerate nonlinear elliptic - parabolic system, which models the nonstationary heat conduction in an electric conductor with Joule heating as the only source.

Key words. nonstationary thermistor, degenerate, quadratic growth, weak solution, existence

AMS (MOS) subject classification. 35R35

1 Introduction

The thermistor is an electrical device which can be used as a current surge regulator. The basis of its operation is the temperature-dependent electrical conductivity. Let u be the absolute temperature and φ the electric potential in a solid conductor represented by a bounded domain Ω , then their evolution is governed by the following nonlinear system:

$$\nabla \cdot (\sigma(u) \nabla \varphi) = 0 \quad \text{in } Q_T \equiv \Omega \times (0, T) \quad (1.1)$$

$$u_t - \nabla \cdot (k(u) \nabla u) = \sigma(u) |\nabla \varphi|^2 \quad \text{in } Q_T \quad (1.2)$$

$$\varphi = \eta \quad u = h \quad \text{on } S_T \equiv \partial \Omega \times (0, T) \quad (1.3)$$

$$u(x, 0) = u_0(x) \quad x \in \Omega \quad (1.4)$$

where $\sigma(u)$ and $k(u)$ are respectively electrical and thermal conductivities which are supposed to be given positive functions of the temperature. When $\sigma(u)$ decreases markedly in a narrow temperature range,

the device is called a thermistor. In this case, equation (1.1) may become degenerate.

The problem of the electrical heating of a conductor has a long history (see [1]) and has been reconsidered recently. Various results have been given for the steady state case. (See [2], [3], [4] and the references therein.) Also the nonstationary problem has been investigated by several authors. (See [5], [6], [7] etc.) In papers [5]-[7], it is assumed that

$$0 < \sigma_* < \sigma(u) < \sigma^* \quad \forall u \in \mathbb{R} \quad (1.5)$$

where σ_* and σ^* are two positive constants. This condition is very crucial to the existence and uniqueness results. However, it excludes the physically important case of the metallic conduction in which

$$\sigma(u) \sim O(1/u) \quad \text{as } u \rightarrow \infty$$

In the present paper, instead of (1.5), we make the following assumptions on the electrical conductivity

$$\sigma \in C(\mathbb{R}) \quad 0 < \sigma(u) < \sigma^* \quad \forall u \in \mathbb{R} \quad (1.6)$$

$$0 < \sigma_* < u \sigma(u) \quad \forall u > m > 0 \quad (1.7)$$

where $m = \inf \{h, u_*\} > 0$. We note explicitly, by (1.6), equation (1.1) is not uniformly elliptic.

2 Weak formulation

Let Ω be a bounded domain of \mathbb{R}^N with a smooth boundary $\partial\Omega$. (The cases of practical interest are for $N < 3$, but our results hold in all space dimensions.) Suppose all the initial-boundary values η , h and u_* to be the traces of smooth functions. For simplicity, we assume $k(u) \equiv 1$. Then (1.2) becomes

$$u_* - \Delta u = \sigma(u) |\nabla \varphi|^2 \quad \text{in } Q_T \quad (2.1)$$

Let us consider the problem (1.1), (2.1), (1.3) and (1.4), which will be denoted by (P). We say that a pair $\{\varphi, u\}$ is a weak solution of (P), if

$$\begin{aligned} \varphi - \eta &\in \dot{W}_2^{1,0}(Q_T) \\ \int_{Q_T} \sigma(u) \nabla \varphi \cdot \nabla \psi &= 0 \quad \forall \psi \in \dot{W}_2^{1,0}(Q_T) \end{aligned} \quad (2.2)$$

$$\begin{aligned} u - h &\in \dot{V}_2(Q_T) \\ - \int_{Q_T} u v_t + \int_{Q_T} \nabla u \cdot \nabla v &= \int_{Q_T} \sigma(u) |\nabla \varphi|^2 v + \int_{\Omega} u_0 v(x, 0) \\ &\text{for all } v \in \dot{W}_2^{1,1}(Q_T) \cap L^\infty(Q_T) \text{ such that } v(x, T) = 0 \end{aligned} \quad (2.3)$$

where the notations for the functional spaces are those of the book [8].

The strategy we used to deal with the quadratic growth is to rewrite $\sigma |\nabla \varphi|^2$ as $\nabla \cdot (\varphi \sigma \nabla \varphi)$. More precisely, we have

$$\begin{aligned} \text{Lemma 2.1} \quad &\text{If } \{\varphi, u\} \text{ is a weak solution of (P), then} \\ &\int_{Q_T} \sigma(u) |\nabla \varphi|^2 v = - \int_{Q_T} \varphi \sigma(u) \nabla \varphi \cdot \nabla v \\ &\quad \forall v \in \dot{W}_2^{1,0}(Q_T) \cap L^\infty(Q_T) \end{aligned} \quad (2.4)$$

Proof. (2.4) is obtained by choosing $\psi = \varphi v$ as a test function in (2.2).

Clearly, any weak solution of (P) satisfies (2.2) and

$$u - h \in \dot{V}_2(Q_T)$$

$$-\int_{Q_T} u v_t + \int_{Q_T} \nabla u \cdot \nabla v = -\int_{Q_T} \varphi \sigma(u) \nabla \varphi \cdot \nabla v + \int_{\Omega} u_0 v(x, 0) \quad (2.5)$$

for all $v \in \dot{W}_2^{1,1}(Q_T) \cap L^\infty(Q_T)$ such that $v(x, T) = 0$

and vice versa. Thus, to avoid the difficulty due to the quadratic term, we can take (2.2) and (2.5) as an equivalent expression of the weak solution.

The main result of this paper is the following existence theorem.

Theorem 2.2 Suppose (1.6) and (1.7) are satisfied. Then there exists a weak solution to the thermistor problem (P).

3 Proof of the main theorem

Let us introduce a family of functions $\sigma_n(u)$ which approximates $\sigma(u)$ as $n \rightarrow \infty$, each uniformly positive:

$$\sigma_n \in C(\mathbb{R}) \quad 1/n < \sigma_n(u) < 2\sigma \quad \forall u \in \mathbb{R} \quad (3.1)$$

$$\sigma(u) < \sigma_n(u) \quad \forall u > m > 0 \quad (3.2)$$

$$\sigma_n(u) \rightarrow \sigma(u) \text{ as } n \rightarrow \infty, \text{ uniformly in } u \text{ in any bounded intervals} \quad (3.3)$$

By arguments analogous to those employed in [7], there exists at least one C^∞ solution $\{\varphi_n, u_n\}$ to the approximating problem (P_n) :

$$\begin{aligned} \varphi_n - \eta \in \dot{W}_2^{1,0}(Q_T) \\ \int_{Q_T} \sigma_n(u_n) \nabla \varphi_n \cdot \nabla \psi = 0 \quad \forall \psi \in \dot{W}_2^{1,0}(Q_T) \end{aligned} \quad (3.4)$$

$$u_n - h \in \dot{V}_2(Q_T)$$

$$-\int_{Q_T} u_n v_t + \int_{Q_T} \nabla u_n \cdot \nabla v = -\int_{Q_T} \varphi_n \sigma_n(u_n) \nabla \varphi_n \cdot \nabla v + \int_{\Omega} u_0 v(x, 0) \quad (3.5)$$

for all $v \in \dot{W}_2^{1,1}(Q_T) \cap L^\infty(Q_T)$ such that $v(x, T) = 0$

where equation (3.4) is uniformly elliptic by (3.1).

The following lemma gives the basic uniform estimates.

Lemma 3.1 Let $\{\varphi_n, u_n\}$ be any solution of the problem (P_n) . Then we have

$$u_n(x, t) > m > 0 \quad \text{for any } (x, t) \in Q_T \quad (3.6)$$

$$\|\varphi_n\|_{L^\infty(Q_T)} < C \quad (3.7)$$

$$\int_{Q_T} |\nabla u_n|^2 < C \quad (3.8)$$

$$\int_{Q_T} |\nabla \varphi_n|^2 < C \quad (3.9)$$

where C is a constant independent of n .

Proof. (3.6) and (3.7) are consequences of the weak maximum principle.

Denote by w the solution of the following problem

$$\begin{aligned} w_t - \Delta w &= 0 && \text{in } Q_T \\ w &= h && \text{on } \bar{S}_T \\ w(x, 0) &= u_0 && x \in \Omega \end{aligned}$$

Then, $u_n - w$ solves

$$(u_n - w)_t - \Delta(u_n - w) = \sigma_n(u_n) |\nabla \varphi_n|^p$$

Multiplying by $(u_n - w)$ and integrating over Q_T , we obtain

$$1/2 \int_{\Omega} (u_n - w)^2 |_{t=0} + \int_{Q_T} |\nabla(u_n - w)|^2 = \int_{Q_T} \sigma_n(u_n) |\nabla \varphi_n|^p (u_n - w)$$

Applying Lemma 2.1 to the term on the right-hand side, we have

$$1/2 \int_{\Omega} (u_n - w)^2 |_{t=0} + \int_{Q_T} |\nabla(u_n - w)|^2 \leq \int_{Q_T} \varphi_n \sigma_n(u_n) \nabla \varphi_n \cdot \nabla(u_n - w)$$

Discarding the first term on the left-hand side and using Cauchy inequality, we get

$$\int_{Q_T} |\nabla(u_n - w)|^2 \leq C \int_{Q_T} \sigma_n(u_n) |\nabla \varphi_n|^p$$

where C is independent of n . Putting $\psi = \varphi_n - \eta$ in (3.4), we easily find

$$\int_{Q_T} \sigma_n(u_n) |\nabla \varphi_n|^p \leq 2\sigma^* \int_{Q_T} |\nabla \eta|^2 \quad (3.10)$$

Thus (3.8) follows.

To prove (3.9), we choose $\psi = (\varphi_n - \eta)u_n$ in (3.4). In a similar way as above, it can be deduced that

$$\int_{Q_T} u_n \sigma_n(u_n) |\nabla \varphi_n|^p < C \left(\int_{Q_T} \sigma_n(u_n) |\nabla \varphi_n|^p + \int_{Q_T} |\nabla u_n|^2 \right) \quad (3.11)$$

where C is also independent of n . From (1.7), (3.2) and (3.6), we get

$$u_n \sigma_n(u_n) > u_n \sigma(u_n) > \sigma_n > 0 \quad (3.12)$$

Substituting (3.8), (3.10) and (3.12) into (3.11), we arrive at (3.9).

To show compactness, we invoke a known proposition from the theory of real functions. (See [8], pp72.)

Lemma 3.2 Let $\{f_n\}$ be a given sequence of functions. If $\|f_n\|_p \leq C$ and $\|f_n\|_{p, Q'} \leq \mu$ (mes Q') for any measurable subset Q' of Q , where $\mu(\tau)$ is a continuous function of $\tau > 0$ that is equal to zero for $\tau = 0$, then it is possible to extract a subsequence from $\{f_n\}$ that is strongly convergent in $L^p(Q)$.

In the light of Lemma 3.2 and the estimates (3.8) and (3.9), we can draw a conclusion as follows:

Lemma 3.3 For a given p , $1 < p < 2$, it is possible to extract two subsequences from $\{\varphi_n\}$ and $\{u_n\}$ respectively, which are strongly convergent in $L^p(Q_T)$.

In view of the above lemmas, there exist two subsequences, still denoted by $\{\varphi_n\}$ and $\{u_n\}$, and two functions

$$\varphi \in W_2^{1,0}(Q_T) \cap L^\infty(Q_T) \quad u \in W_2^{1,0}(Q_T)$$

such that

$$\varphi_n \rightarrow \varphi \text{ weakly-}^* \text{ in } L^\infty(Q_T), \text{ weakly in } W_2^{1,0}(Q_T) \text{ and a. e. in } Q_T \quad (3.13)$$

$$u_n \rightarrow u \text{ weakly in } W_2^{1,0}(Q_T) \text{ and a. e. in } Q_T \quad (3.14)$$

Now, we are in a position to verify that (φ, u) is a weak solution of (P) by passing to the limit in (3.4) and (3.5). This can be reduced to the following lemma:

Lemma 3.4 For an arbitrary $\xi \in [L^p(Q_T)]^N$

$$\int_{Q_T} \varphi_n \sigma_n(u_n) \nabla \varphi_n \cdot \xi \rightarrow \int_{Q_T} \varphi \sigma(u) \nabla \varphi \cdot \xi \quad (3.15)$$

$$\int_{Q_T} \sigma_n(u_n) \nabla \varphi_n \cdot \xi \rightarrow \int_{Q_T} \sigma(u) \nabla \varphi \cdot \xi \quad (3.16)$$

as $n \rightarrow \infty$.

Proof. From (3.3), (3.13) and (3.14), we have

$$\varphi_n \sigma_n(u_n) \rightarrow \varphi \sigma(u) \quad \text{a.e. in } Q_T$$

which implies that, by the Lebesgue dominated convergence theorem,

$$\varphi_n \sigma_n(u_n) \xi \rightarrow \varphi \sigma(u) \xi \quad \text{strongly in } [L^p(Q_T)]^N \quad (3.17)$$

The strong convergence (3.17), together with the weak convergence (3.13), leads to (3.15). The proof of (3.16) is the same.

The proof of the existence theorem is completed.

Remark. In a similar way, with a slight modification, the existence theorem for the degenerate thermistor problem with mixed boundary conditions can be proved.

Acknowledgement. This research is supported by the National Natural Science Foundation of China.

References

- [1] Diesselhorst, H., Ueber das Probleme eines elektrisch erwarmten Leiters, Ann. Physics, 1(1900), 312-325.
- [2] Cimatti, G. and Prodi, G., Existence results for a nonlinear elliptic system modelling a temperature dependent electrical resistor, Ann. Mat. Pura Appl., 152(1988), 227-236.
- [3] Hong Xie and Allegretto, W., C^{α} solution of a class of nonlinear degenerate elliptic system arising in the thermistor problem, SIAM J. Math. Anal., 22(1991), 1491-1499.
- [4] Chen, X., Existence and regularity of solutions of a nonlinear degenerate elliptic system arising from a thermistor problem, to appear.
- [5] Cimatti, G., Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor, Ann. Mat. Pura Appl., 162(1992), 33-42.
- [6] Chipot, M. and Cimatti, G., A uniqueness result for the thermistor problem, Euro. Jnl. of Applied Mathematics, 2(1991), 97-103.
- [7] Yuan, G. and Liu, Z., Existence and uniqueness of C^{α} -solution for the thermistor problem with mixed boundary value, to appear.
- [8] Ladyzenskaja, O., Solonnikov, V. and Uralceva, N., Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, A.M.S., (1968).

μ -uniqueness of $\ell \oplus e$

Shao-Chien Chang

Presented by G.F.D. Duff, F.R.S.C.

Abstract

The problem of μ -uniqueness for FK-spaces was studied by Beekmann and Chang in the late 80s. The question whether the space $\ell \oplus e$ is μ -unique was left open. In this note we shall provide, in a sense, a partial solution to this problem.

Let A be an infinite matrix with complex entries. We let the summability domain of A be

$$c_A := \{x \in \omega \mid Ax \in c\},$$

where ω and c are spaces of all complex sequences and convergent sequences respectively.

It is known (see [9]) that for each continuous functional $f \in c'_A$ we can express f as

$$f(x) = \mu \lim_A x + t(Ax) + sx, \quad x \in c_A, \quad \mu \in \mathbb{C}, \quad t \in \ell \text{ and } s \in c'_A. \quad (0.1)$$

For each given f , the question whether the coefficient μ is unique or the μ -uniqueness of A was studied by Beekmann, Beekmann-Chang, Chang, Macphail-Wilansky and others since the 60s.

It has been shown in [4] that if c_A does not contain all finite sequences, i.e. $\varphi \notin c_A$, then some of those distinguished subsets are no longer invariant. eg. for some given matrices A and B with $c_A = c_B$, we may have $L_A \neq L_B$. Hence we shall confine our discussion to those FK-spaces containing the set of all finite sequences.

For a given matrix A with $\varphi \subset c_A$, and an FK-space $X \supset \varphi$, we define

$$\mu_A^\perp := \{f \in c'_A \mid \mu = 0 \text{ in (0.1)}\},$$

$\mu_X^\perp := \{f \in X' \mid \forall \mu\text{-unique matrix } D \text{ with } \varphi \subset c_D \subset X, \text{ we have } f \mid c_D \in \mu_D^\perp\}$

and

$\mu_X^\perp = \{f \in X' \mid \exists \text{ a matrix } D \text{ and a } g \in \mu_D^\perp \text{ with } c_D \supset X \text{ and } g \mid_X = f\}$.

It is easy to see that $\mu_X^\perp \subseteq \mu_X^\perp$. Whether the equality holds is an open question.

Definition. An FK-space X is said to be μ -unique, if \exists an $f \in X'$ such that $\mu_D(f \mid c_D) \neq 0$ for some μ -unique matrix D with $\varphi \subset c_D \subset X$.

It is shown in (5) that,

Theorem 1. X is μ -unique, if and only if $X' \neq \mu_X^\perp$.

We let $X = \ell \oplus e$, where $e = (1, 1, 1, \dots)$. It is an BK-space under the norm

$$\|x\| = \sum_k |x_k - \lim x| + |x|, \quad \forall x \in X.$$

Now we observe that $X' = m \oplus C$, in here m is the Banach space of all bounded sequences and C is the set of all complex numbers. Hence for any $f \in X'$, we can express f as

$$f(x) = \sum_k \alpha_k (x_k - \lim x) + \alpha \lim x, \quad x \in X$$

where $(\alpha_k) \in m$ and $\alpha \in C$. As we shall see that the second term in the above expression is a member of μ_X^\perp . In view of theorem 1, this constitute a partial solution of the question whether $\ell \oplus e$ is μ -unique. Since $\mu_X^\perp \subset \mu_X^\perp$, so it is sufficient to show that the functional $x \mapsto \lim x \in \mu_X^\perp$.

Theorem 2. $x \mapsto \lim x \in \mu_X^\perp$.

ANALYTICAL SOLUTION TO THE DIRECT UNSYMMETRIC EDDY CURRENT TESTING PROBLEM FOR A SYMMETRIC FLAW

M. YA. ANTIMIROV, A. A. KOLYSHKIN AND RÉMI VAILLANCOURT

Presented by K.B. Ranger, F.R.S.C.

ABSTRACT. An analytical solution is obtained by a perturbation method for the change of impedance in a double conductor line due to eddy currents induced by an infinitely long horizontal cylindrical flaw with symmetric cross-section with respect to the vertical axis in the case the conductivities of the flaw and of the surrounding material do not differ by much. Numerical results for flaws of circular and elliptic cross-sections are presented.

RÉSUMÉ. On obtient une solution analytique, par la méthode des perturbations, pour le changement d'impédance causé par les courants de Foucault induits dans un double fil conducteur par une longue faille horizontale cylindrique de section symétrique par rapport à l'axe vertical et contenue dans le demi-espace inférieur si la conductivité de la faille diffère de peu de celle du demi-espace conducteur environnant. On présente des résultats numériques pour des failles de section circulaire et elliptique.

Subject-classification: AMS(MOS): 35K05, 65R10.

Keywords: Eddy current NDE, perturbation method, Fourier transform techniques

1. Introduction.

A central problem in eddy current testing is to determine, from the output signal of an eddy current probe, the values of the parameters of a flaw contained in a conducting medium. This is a complicated inverse problem which has given rise to many methods of solution. We shall restrict ourselves to the direct problem.

We consider an infinitely long horizontal cylindrical flaw of cross-section symmetric with respect to the vertical axis, thus generalizing the results obtained in [1] for a flaw of rectangular cross-section. We use an idealized eddy current probe in the form of a double conductor line situated above a conducting halfspace containing the flaw. A probe, in the form of a long rectangular frame, the ratio of the sides ranging from 10 to 20, is often used in practical eddy current testing [2]. It was shown in [3] that probes with width-to-length ratio equal to 1:4 or smaller can be modelled by a double conductor line. With a double line, one can solve analytically problems in which the probe is not in a symmetric position with respect to the flaw, because in this case the vector potential has only one

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under grants OGP0007916, EQP0122258 and EQP0139636, and the Centre de recherches mathématiques of the Université de Montréal.

nonzero component, provided the flaw is an infinitely long cylinder. In the particular case of circular and elliptical cylindrical flaws, we determine in detail the influence of the shape of the flaw on the output signal.

2. Formulation of the problem. Consider a horizontal double conductor line in free upper half-space, $R_0 = \{z \geq 0\}$, at height h above conducting lower half-space, $R_1 = \{z \leq 0\}$. An infinitely long horizontal cylindrical flaw, R_2 , parallel to the y -axis and symmetric with respect to the z -axis, is contained in R_1 (see Fig. 1).

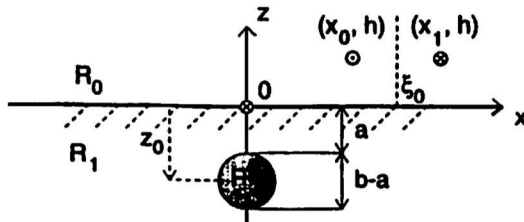


FIGURE 1. Double conductor line above conducting halfspace containing an infinitely long horizontal flaw of cylindrical cross-section symmetric with respect to the z -axis (cross-section pictured as a disk in figure).

Let $f(x, z) = 0$ describe the cross-section, R_2 , of the flaw in the xz -plane, assumed, for simplicity, to be convex. Let a and b denote the vertical distances, respectively, of the upper and lower parts of the flaw measured from the plane $z = 0$.

Let the xz -coordinates of the wires be (x_0, h) and (x_1, h) , respectively, with alternating current $\pm I \exp j\omega t$. Here I is the amplitude, ω the frequency, and $j = \sqrt{-1}$.

The vector potential A in an isotropic conducting nonferromagnetic medium satisfies

$$\Delta A = \mu_0 \sigma \frac{\partial A}{\partial t} - \mu_0 \mathbf{i}^e, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad (1)$$

where σ is the electrical conductivity of the medium, μ_0 is the magnetic constant and \mathbf{i}^e is the density of the external current.

3. Mathematical analysis. Due to symmetry, A has only one nonzero component,

$$A = A(x, z) e^{j\omega t} \mathbf{e}_y, \quad (2)$$

where \mathbf{e}_y is the unit vector in the y -direction. Using (1) and (2) we obtain the following system of equations for the amplitude of A in each of the regions R_0 , R_1 and R_2 :

$$\Delta A_0 = -\mu_0 I \delta(x - x_0) \delta(z - h) + \mu_0 I \delta(x - x_1) \delta(z - h), \quad (3)$$

$$\Delta A_1 + k_1^2 A_1 = 0, \quad \Delta A_2 + k_2^2 A_2 = 0, \quad (4)$$

where $k_i^2 = -j\omega\sigma_i\mu_0$, and σ_i is the conductivity of the i -th medium ($i = 1, 2$). The boundary conditions are

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}\Big|_{z=0} = \frac{\partial A_1}{\partial z}\Big|_{z=0}, \quad A_1|_L = A_2|_L, \quad \frac{\partial A_1}{\partial \mathbf{n}}\Big|_L = \frac{\partial A_2}{\partial \mathbf{n}}\Big|_L, \quad (5)$$

where L is the boundary of the region R_2 , and \mathbf{n} is the unit outer normal to R_2 . Moreover,

$$A_0, A_1, A_2, \frac{\partial A_0}{\partial x}, \frac{\partial A_1}{\partial x}, \frac{\partial A_2}{\partial x} \rightarrow 0, \quad \text{as } \sqrt{x^2 + z^2} \rightarrow \infty. \quad (6)$$

If the parameter $\varepsilon = 1 - \sigma_2/\sigma_1$ is small, then k_1 and k_2 are related by

$$k_2^2 = k_1^2(1 - \varepsilon). \quad (7)$$

We seek the solution to (3)-(6) in power series in ε ,

$$A_0 = A_0^0 + \varepsilon A_0^1 + \dots, \quad A_1 = A_1^0 + \varepsilon A_1^1 + \dots, \quad A_2 = A_2^0 + \varepsilon A_2^1 + \dots \quad (8)$$

In the case $\varepsilon = 0$ corresponding to a flawless conducting halfspace, we have

$$A_1^0 = A_2^0 = \frac{\mu_0 I}{\pi} \int_0^\infty \frac{e^{q_1 z - \xi h}}{\xi + q_1} [\cos \xi(x - x_0) - \cos \xi(x - x_1)] d\xi, \quad (9)$$

where $q_1 = \sqrt{\xi^2 - k_1^2}$. Solution (9) is found, for example, in [4], p. 66.

Substituting (7) and (8) into (3)-(6) and equating the coefficients of the first power in ε , we obtain

$$\Delta A_0^1 = 0, \quad (x, z) \in R_0. \quad (10)$$

$$\Delta A_1^1 + k_1^2 A_1^1 = \begin{cases} 0, & (x, z) \in R_1 \setminus R_2 \\ k_1^2 A_1^0, & (x, z) \in R_2, \end{cases} \quad (11)$$

where, now, A_1^1 denotes the vector potential in the whole region $z < 0$. The boundary conditions on L in (5) are satisfied automatically (see [1]). Equations (10) and (11) are solved with the boundary conditions (5) at $z = 0$ and (6) at infinity.

Expressing A_i^0 as the sum of its even and odd parts, and using (9), we obtain

$$A_{1\text{even}}^0(x, z) = \frac{\mu_0 I}{\pi} \int_0^\infty \frac{e^{-\xi h + q_1 z}}{\xi + q_1} (\cos \xi x_0 - \cos \xi x_1) \cos \xi x d\xi, \quad (12)$$

$$A_{1\text{odd}}^0(x, z) = \frac{\mu_0 I}{\pi} \int_0^\infty \frac{e^{-\xi h + q_1 z}}{\xi + q_1} (\sin \xi x_0 - \sin \xi x_1) \sin \xi x d\xi. \quad (13)$$

Taking the Fourier cosine transform of A_i^1 ,

$$\tilde{A}_i^1(\lambda, z) = \int_0^\infty A_{i\text{even}}^1(x, z) \cos \lambda x dx, \quad i = 0, 1, \quad (14)$$

to determine $A_{0\text{even}}^1$ and $A_{1\text{even}}^1$, we obtain, from (10), (11), (5) at $z = 0$, and (6), the following problem, with $q = \sqrt{\lambda^2 - k_1^2}$,

$$\frac{d^2 \bar{A}_0^1}{dz^2} - \lambda^2 \bar{A}_0^1 = 0, \quad \frac{d^2 \bar{A}_1^1}{dz^2} - q^2 \bar{A}_1^1 = \begin{cases} 0, & z \notin [-b, -a], \\ \bar{G}_1, & z \in [-b, -a], \end{cases} \quad (15)$$

where, with $x = \psi(z) > 0$ describing the right-hand part of R_2 (see Fig. 1),

$$\bar{G}_1(z, \lambda) = \frac{k_1^2 \mu_0 I}{2\pi} \int_0^\infty \frac{e^{-\xi h + q_1 z}}{\xi + q_1} (\cos \xi x_0 - \cos \xi x_1) \left[\frac{\sin(\lambda - \xi)\psi(z)}{\lambda - \xi} + \frac{\sin(\lambda + \xi)\psi(z)}{\lambda + \xi} \right] d\xi.$$

The boundary conditions are

$$\bar{A}_0^1|_{z=0} = \bar{A}_1^1|_{z=0}, \quad \left. \frac{d\bar{A}_0^1}{dz} \right|_{z=0} = \left. \frac{d\bar{A}_1^1}{dz} \right|_{z=0}. \quad (16)$$

The bounded solution, as $z \rightarrow \infty$, to the first equation in (15) is $\bar{A}_0^1 = C_1 e^{-\lambda z}$, and the solution to the second equation, found by the method of variation of parameters, is

$$\bar{A}_1^1 = \begin{cases} C_2 e^{qz} + C_3 e^{-qz}, & -a < z < 0, \\ C_4 e^{qz} + C_5 e^{-qz} + \frac{1}{q} \int_{-b}^z \bar{G}_1(\eta, \lambda) \sinh[q(z - \eta)] d\eta, & -b < z < -a, \\ C_6 e^{qz}, & z < -b. \end{cases}$$

We use conditions (16) and the fact that the vector potential and its first derivative are continuous at $z = -a$ and $z = -b$ to determine the arbitrary constants C_1, \dots, C_6 . Then, applying the inverse Fourier cosine transform,

$$A_{1\text{even}}^1(x, z) = \frac{2}{\pi} \int_0^\infty \bar{A}_1^1(\lambda, z) \cos \lambda x d\lambda,$$

we obtain the solution $A_{0\text{even}}^1$ in the form

$$A_{0\text{even}}^1(x, z) = \frac{k_1^2 \mu_0 I}{\pi^2} \int_0^\infty \frac{e^{-\lambda z}}{\lambda + q} \cos \lambda x d\lambda \int_{-b}^{-a} e^{(q_1 + q)\eta} d\eta \int_0^\infty e^{-\xi h} (\cos \xi x_1 - \cos \xi x_0) \\ \times \left[\frac{\sin(\lambda - \xi)\psi(\eta)}{\lambda - \xi} + \frac{\sin(\lambda + \xi)\psi(\eta)}{\lambda + \xi} \right] \frac{1}{\xi + q_1} d\xi. \quad (17)$$

Similarly, by using the direct and inverse Fourier sine transforms, we obtain

$$A_{0\text{odd}}^1(x, z) = \frac{k_1^2 \mu_0 I}{\pi^2} \int_0^\infty \frac{e^{-\lambda z}}{\lambda + q} \sin \lambda x d\lambda \int_{-b}^{-a} e^{(q_1 + q)\eta} d\eta \int_0^\infty e^{-\xi h} (\sin \xi x_1 - \sin \xi x_0) \\ \times \left[\frac{\sin(\lambda - \xi)\psi(\eta)}{\lambda - \xi} - \frac{\sin(\lambda + \xi)\psi(\eta)}{\lambda + \xi} \right] \frac{1}{\xi + q_1} d\xi. \quad (18)$$

Finally, adding (17) and (18), we have the solution

$$A_0^1(x, z) = \frac{k_1^2 \mu_0 I}{\pi^2} \int_0^\infty \frac{e^{-\lambda z}}{\lambda + q} d\lambda \int_{-b}^{-a} e^{(q_1 + q)\eta} d\eta \int_0^\infty e^{-\xi h} \left\{ \cos \lambda x (\cos \xi x_1 - \cos \xi x_0) \right. \\ \times \left[\frac{\sin(\lambda - \xi)\psi(\eta)}{\lambda - \xi} + \frac{\sin(\lambda + \xi)\psi(\eta)}{\lambda + \xi} \right] + \sin \lambda x (\sin \xi x_1 - \sin \xi x_0) \\ \left. \times \left[\frac{\sin(\lambda - \xi)\psi(\eta)}{\lambda - \xi} - \frac{\sin(\lambda + \xi)\psi(\eta)}{\lambda + \xi} \right] \frac{1}{\xi + q_1} \right\} d\xi. \quad (19)$$

5. Numerical results and conclusion. The change of impedance, per unit wire length, due to an infinitely long cylindrical flaw is computed by the formula

$$Z_{\text{ind}} = \varepsilon \frac{j\omega}{I} \oint_l A_0^1(x, z) dl, \quad (20)$$

where l is the path of integration along unit length of lines. Let $d = x_1 - x_0$ be the distance between the wires. Using the dimensionless quantities

$$\bar{h} = \alpha = \frac{h}{d}, \quad \beta = d\sqrt{\omega\sigma_1\mu_0}, \quad \bar{a} = \frac{a}{d}, \quad \bar{b} = \frac{b}{d}, \quad \bar{x} = \frac{x_0}{d}, \quad \bar{x}_1 = \frac{x_1}{d},$$

and omitting the tildes, formula (20) becomes

$$Z_{\text{ind}} = \varepsilon \frac{\omega\mu_0}{\pi^2} Z_0, \quad (21)$$

where, with $x = \psi(z) > 0$, as above, describing the right-hand part of the flaw,

$$\begin{aligned} Z_0 = & \beta(b-a) \int_0^\infty \frac{e^{-h\beta\xi}}{\xi + \sqrt{\xi^2 + j}} d\xi \int_0^\infty \frac{e^{-h\beta\eta}}{\eta + \sqrt{\eta^2 + j}} d\eta \\ & \times \int_0^1 d\zeta \exp\left(-\beta[a + \zeta(b-a)]\left[\sqrt{\xi^2 + j} + \sqrt{\eta^2 + j}\right]\right) \\ & \times \left\{ (\cos \beta\xi x_0 - \cos \beta\xi x_1)(\cos \beta\eta x_1 - \cos \beta\eta x_0) \left[\frac{\sin \beta(\xi - \eta)\psi(\zeta)}{\xi - \eta} + \frac{\sin \beta(\xi + \eta)\psi(\zeta)}{\xi + \eta} \right] \right. \\ & \left. + (\sin \beta\xi x_0 - \sin \beta\xi x_1)(\sin \beta\eta x_1 - \sin \beta\eta x_0) \left[\frac{\sin \beta(\xi - \eta)\psi(\zeta)}{\xi - \eta} - \frac{\sin \beta(\xi + \eta)\psi(\zeta)}{\xi + \eta} \right] \right\}. \end{aligned} \quad (22)$$

We now consider in greater detail flaws with circular and elliptical cross-section.

5.1. Long circular flaw. Consider a long circular flaw given by $(z + z_0)^2 + x^2 = R^2$ (see Fig. 1), or, in terms of the dimensionless variable $\zeta = (z - a)/(b - a)$,

$$\xi = \psi(\zeta) = \pm(b-a)\sqrt{\zeta(1-\zeta)}. \quad (23)$$

Computational results obtained by formulae (22) and (23), with the help of the IMSL routine DMLIN, for $b = 0.3$ and $b = 1.1$ are shown in Fig. 2, where $\xi_0 = x_0 + 0.5$.

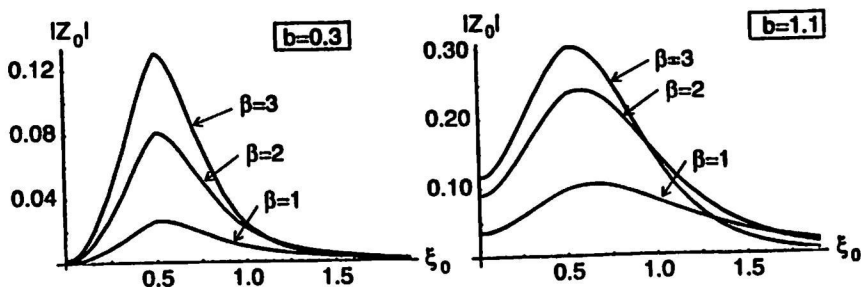


FIGURE 2. Absolute impedance change, $|Z_0|$, against ξ_0 for $\beta = 1, 2, 3$, $\alpha = 0.05$, $a = 0.05$, and $b = 0.3$ (left) and $b = 1.1$ (right) for circular flaw.

As seen in Fig. 2, the maximum of $|Z_0|$ increases with β and its abscissa, ξ_0 , hardly depends on the vertical size, b , of the flaw. Curves are smoother for larger values of b .

5.2. Long elliptic flaw. Consider a long elliptic flaw given, in the $\xi\zeta$ -plane, by

$$\xi = \pm\gamma(b-a)\sqrt{\zeta(1-\zeta)}, \quad (24)$$

where $\gamma = d/[(b-a)/2]$ is the ratio of the semi-axes of the ellipse, in the x - and z -directions, respectively (see Fig. 1). Computational results for $\gamma = 2.0$ and $\gamma = 0.5$ are shown in Fig. 3.

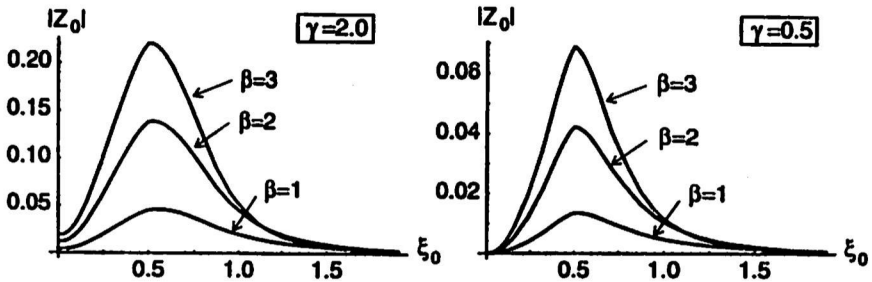


FIGURE 3. Absolute impedance change, $|Z_0|$, against ξ_0 for $\beta = 1, 2, 3$, $\alpha = 0.05$, $a = 0.05$, $b = 0.3$, $\gamma = 2.0$ (left) and $\gamma = 0.5$ (right) for elliptic flaw.

Again, the abscissa of the maximum for $|Z_0|$ is not influenced by the remaining parameters of the problem but its height depends on the ratio γ .

Conclusion. A maximum of $|Z_0|$ indicates the presence of a flaw in a conducting medium; but it is computationally difficult to determine any parameter of the flaw (depth, horizontal size etc.) by using only the position of the maximum along the ξ_0 -axis.

REFERENCES

1. M. Ya. Antimirov, A. A. Kolyshkin and R. Vaillancourt, *Eddy current nondestructive testing by a perturbation method*, J. Nondestr. Eval. 10 (1991), no. 1, 31-37.
2. Yu. M. Shkarlet, *Eddy current nondestructive testing by a perturbation method*, The Soviet J. of Nondestructive Testing 10 (1974), no. 1, 5-11.
3. N. N. Zatepin and V. M. Tatarnikov, *Theory of a superposed transducer*, The Soviet J. of Nondestructive Testing 18 (1982), no. 2, 81-87.
4. R. L. Stoll, *The analysis of eddy currents*, CRM Monograph Series, Clarendon Press, Oxford, 1974.

DEPARTMENT OF APPLIED MATHEMATICS, RIGA TECHNICAL UNIVERSITY, RIGA, LATVIA, LV1010

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO, CANADA K1N 6N5
E-mail address: rxvsg@acadvm1.uottawa.ca

Received January 20, 1994

Asymptotic minimal projection constants for Lebesgue spaces

Ernst Görlich Alfred P. Rohs

Presented by G.D.F. Duff, F.R.S.C.

Abstract: For the spaces $X = L_{w_i}^1(-1, 1), 1 \leq i \leq 4$, with weights $w_1(x) = 1, w_2(x) = (1-x^2)^{-1/2}, w_3(x) = (1+x)^{-1/2}$, and $w_4(x) = (1-x)^{-1/2}$ the asymptotic behaviour of the relative minimal projection constants $p(\mathcal{P}_n, X)$ with respect to the set \mathcal{P}_n of algebraic polynomials of degree n is announced to be $\lim_{n \rightarrow \infty} p(\mathcal{P}_n, X) / \log n = 4/\pi^2$. An outline of proof is given.

Given a normed linear space X and a subspace Y , the number

$$p(Y, X) = \inf\{\|P\|; P \text{ is a continuous linear projection of } X \text{ onto } Y\}$$

is called the relative projection constant of Y in X , see e.g. Cheney [1] for a survey of related problems and results.

Recently Petras [3] has proved that in case $X = C[-1, 1]$ the precise asymptotic behaviour is given by

$$\lim_{n \rightarrow \infty} \frac{p(\mathcal{P}_n, X)}{\log n} = \frac{4}{\pi^2} \quad (1)$$

The purpose of the present note is to announce the following.

Theorem 1 Equation (1) is true if X is one of the spaces $X_i = L_{w_i}^1(-1, 1), 1 \leq i \leq 4$, with norm $\|f\|_{X_i} = \int_{-1}^1 |f(x)| w_i(x) dx$, where $w_1(x) = 1, w_2(x) = (1-x^2)^{-1/2}, w_3(x) = (1+x)^{-1/2}$, and $w_4(x) = (1-x)^{-1/2}$.

We outline the proof for $i = 1$. Following [2], Prop. 2, we first map the space X_1 onto X_2 by means of an isometric isomorphism T , where

$$T(f; x) = f(x) \cdot \frac{w_1(x)}{w_2(x)} \quad .$$

The set \mathcal{P}_n is mapped by T onto $\text{span}\{q_k; 0 \leq k \leq n\}$, where $q_k(x) = \sqrt{2/\pi} \sin((k+1) \arccos x)$ and the principal value of \arccos is taken. Then the operator $Q_n = TP_nT^{-1}$ is a (minimal) projection from X_2 onto $\text{span}\{q_k; 0 \leq k \leq n\}$ iff P_n is a (minimal) projection from X_1 onto \mathcal{P}_n . Second, the standard isomorphism M of X_2 onto the space $\overline{L_{1/\pi}^1}$ of even, 2π -periodic, integrable functions with norm $\|g\|_{\overline{L_{1/\pi}^1}} = 2 \int_0^\pi |g(x)| dx$ is applied, which is defined by

$$M(f; x) = \frac{1}{2} f(\cos x) \quad .$$

1991 Mathematics Subject Classification: Primary 41A10; Secondary 42A10, 33C45.

Key words and Phrases: Minimal projection, Berman-Marcinkiewicz equation, product formula, generalized translation, Jacobi polynomials.

Now $H_n = MQ_n M^{-1}$ is a (minimal) projection from $\widetilde{L}_{2\pi}^1$ onto $\text{span}\{h_k; 0 \leq k \leq n\}$, where $h_k(x) = \sin((k+1)x) \text{sign}(\sin x)$. The functions h_k form a fundamental set in $\widetilde{L}_{2\pi}^1$, so that they can substitute the trigonometric functions in the proof of the Berman-Marcinkiewicz equation ([1], p. 63) to the effect that

$$\frac{1}{\pi} \int_0^\pi f(s) \{D_{n+1}(x-s) - D_{n+1}(x+s)\} ds = \frac{1}{\pi} \int_0^\pi [T_\lambda^* H_n T_\lambda^* f](x) d\lambda \quad (2)$$

is obtained, where $D_n(x) = 1 + 2 \sum_{k=1}^n \cos kx$ and $T_\lambda^*(f; x) = [T_\lambda + T_{-\lambda}](f; |x|)$, $T_\lambda(f; x) = f(x + \lambda) \text{sign}(\sin(x + \lambda))$. As compared with [2], (2.10) (or rather its trigonometric equivalent obtained via M), equation (2) uses a different translation operator which, however, has a discrete support like the Chebychev translation, and satisfies the product formula

$$T_\lambda^*(h_k; x) = 2h_k(x) \cos((k+1)\lambda) \quad (3)$$

for $k \in \mathbb{N} \cup \{0\}$, $\lambda \in [0, \pi]$, and $x \in \mathbb{R}$. In fact, [2], (2.17) is incorrect and has to be replaced by (3).

The next step is to represent H_n as a linear combination of functionals in $(\widetilde{L}_{2\pi}^1)^*$, which in turn have a Riesz-type representation in terms of certain functions $G_{k,n} \in \widetilde{L}_{2\pi}^\infty$, the space of even $L_{2\pi}^\infty$ -functions, and to show that the operator norm is

$$\|H_n\|_{[\widetilde{L}_{2\pi}^1]} = \text{esssup}_{u \in [0, \pi]} \left| \int_0^\pi \sum_{k=0}^n \overline{G_{k,n}(u)} h_k(x) \right| dx$$

Replacing the f in (2) by a test function $g_{\varepsilon,n}(s) = \text{sign}(D_{n+1}(\pi/2 - s))$ for $s \in [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$ and $g_{\varepsilon,n}(s) = 0$ elsewhere in $[0, \pi]$, and adapting a technique of estimation from [3], p. 210, one obtains for $n > 44$, say:

$$\begin{aligned} & \frac{1}{2\pi} \|D_{n+1}\|_{L_{2\pi}^1} - \frac{4}{\pi^2} \log \frac{4}{\varepsilon} - \frac{4}{(2n+3)\pi \sin \frac{\varepsilon}{2}} - \frac{\varepsilon}{\pi \cos \frac{\varepsilon}{2}} \\ & \leq \frac{1}{2\pi} \int_0^\pi g_{\varepsilon,n}(s) \left\{ D_{n+1} \left(\frac{\pi}{2} - s \right) - D_{n+1} \left(\frac{\pi}{2} + s \right) \right\} ds \leq \left(1 + \frac{4\varepsilon}{\pi} \right) \|H_n\|_{[\widetilde{L}_{2\pi}^1]} \end{aligned}$$

Here $1/(2\pi) \|D_{n+1}\|_{L_{2\pi}^1}$ behaves like $4/\pi^2 \log n + \mathcal{O}(1)$ as $n \rightarrow \infty$, so that, choosing $\varepsilon = \pi/(4 \log n)$ and letting $n \rightarrow \infty$, the assertion follows.

In the cases $i = 3, 4$, the proof is similar, and for $i = 2$ it is much simpler.

Full details will appear elsewhere.

References

- [1] CHENEY, E. W. : Projection operators in approximation theory, in: Studies in functional analysis, *MAA Stud. Math.* 21 (R. G. Bartle, ed.) (1980), 50–80.
- [2] GÖRLICH, E. and MARKET, C. : A lower bound for projection operators on $L^1(-1, 1)$, *Ark. Mat.* 24 (1986), 81–92.
- [3] PETRAS, K. : On the minimal norms of polynomial projections, *J. Approx. Theory* 62 (1990), 206–212.

Address: Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany.

PROBLEME DE CAUCHY SUR UN CONOÏDE CARACTERISTIQUE
 POUR DES SYSTEMES QUASI-LINEAIRES HYPERBOLIQUES

Marcel DOSSA

Presented by G.F.D. Duff, F.R.S.C.

Résumé. De nouveaux résultats d'existence et de régularité sont établis pour une classe de systèmes quasi-linéaires hyperboliques du second ordre, les données initiales étant portées par un demi-conoïde caractéristique.

CAUCHY PROBLEM ON A CHARACTERISTIC CONOID FOR QUASI-LINEAR HYPERBOLIC SYSTEMS

Abstract. New existence and smoothness results are obtained for a class of quasi-linear hyperbolic systems of second order, with initial data on a characteristic conoid.

I. INTRODUCTION

On considère dans le domaine Y intérieur à un demi-conoïde \mathcal{C} de \mathbb{R}^{n+1} , de sommet O et d'équation $x^0 = S(x^1, \dots, x^n)$, le problème de Cauchy pour un système de N équations (E_r) à N fonctions inconnues (w_r) de $n+1$ variables (x^α) de la forme :

$$(1) : \begin{cases} \Lambda^{\lambda\mu}(x^\alpha, w_s) D_{\lambda\mu} w_r + f_r(x^\alpha, w_s, D_\nu w_s) = 0 \\ r, s = 1, 2, \dots, N; \alpha, \lambda, \mu, \nu = 0, 1, \dots, n; \\ D_\nu = \frac{\partial}{\partial x^\nu}, D_{\lambda\mu} = \frac{\partial^2}{\partial x^\lambda \partial x^\mu} \end{cases}$$

de type x^0 hyperbolique avec $\Lambda^{00} > 0$ et la forme quadratique $\Lambda^{ij} x_i x_j$ définie négative ($i, j = 1, \dots, n$), avec des conditions initiales :

$$(2) : \begin{cases} w_r = \tilde{w}_r \text{ sur } \mathcal{C}, \tilde{w}_r \text{ étant définie dans un voisinage } U \text{ de } O \text{ dans } \mathbb{R}^{n+1}, \\ \mathcal{C} \text{ étant un demi-conoïde de sommet } O, \text{ caractéristique pour} \\ \text{l'opérateur différentiel } \Lambda^{\lambda\mu}(x^\alpha, \tilde{w}_s(x^\alpha)) D_{\lambda\mu}. \end{cases}$$

Dans [1], pour $n = 3$ et sous des hypothèses de différentiabilité sur les données $(\Lambda^{\lambda\mu}, f_r, \tilde{w}_r)$ de classe respective C^{3t+2} , C^{3t} , C^{3t+2} avec

$t \geq 6$ et moyennant la condition de compatibilité suivante, notée (0_t) : « les \tilde{w}_r vérifient au point 0 les équations (E_r) et toutes les équations dérivées jusqu'à l'ordre $t-2$ », F. CAGNAC résoud le problème (1), (2) dans $\Pi^{t+1}(\tilde{Y})$, \tilde{Y} étant un voisinage assez petit de 0 dans Y .

Dans [5], A. D. RENDALL conjecture l'inutilité de la condition (0_t) .

La présente note propose une nouvelle méthode de solution qui conduit à des résultats dont la nouveauté, par rapport aux travaux antérieurs ([1], [5]), réside dans la petitesse de l'ordre de différentiabilité minimum exigé des données et dans l'absence de toute condition de compatibilité portant sur les valeurs des données au sommet du cône. On démontre ainsi la conjecture de RENDALL.

II NOTATIONS GEOMETRIQUES ET ESPACES FONCTIONNELS UTILISES

$\forall t \in]0, +\infty[$, on note :

$$Y_t = \left\{ (x^0, \dots, x^n) \in Y; S(x^1, \dots, x^n) \leq x^0 \leq t \right\}$$

$$G_t = \left\{ (x^0, \dots, x^n) \in Y; x^0 = t \right\}$$

$C^\infty(Y_t)$ = espace des fonctions de classe C^∞ sur Y_t .

$F^p(Y_t)$ = l'espace de Banach défini par la norme :

$$\|X\|_{F^p(Y_t)} = \text{Esssup}_{\tau \in]0, t[} \tau^{-n/2} \|X\|_{H^p(G_\tau, Y)}$$

avec, p étant un entier naturel :

$$\|X\|_{H^p(G_\tau, Y)}^2 = \sum_{k=0}^p \int_{G_\tau} |D^k X|^2 dx^1 \dots dx^n$$

où :

$$|D^k X|^2 = \sum_{|\alpha| \leq k} |D^\alpha X|^2, \quad D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^0)^{\alpha_0} \dots (\partial x^n)^{\alpha_n}}, \quad \text{les } D^\alpha X \text{ étant des}$$

fonctions qui sont des dérivées au sens des distributions de X .

III. HYPOTHESES ET ENONCES DES RESULTATS

Soit $p \in \mathbb{N} \cup \{+\infty\}$.

Hypothèse α_p .

Les $A^{\lambda\mu}(x^\alpha, w_\beta)$ sont définies et de classe C^p dans $U \times W$, où U est

un ouvert de \mathbb{R}^{n+1} contenant 0, W est un ouvert de \mathbb{R}^n . Dans $U \times W$, les $A^{\lambda\mu}(x^\alpha, w_\beta)$ définissent une forme quadratique définie de signature

+ - ... - avec $A^{00} > 0$, $A^{ij} x_i x_j$ définie < 0 ($i, j = 1, \dots, n$).

Il existe un point $(a_r) \in W$ tel que $A^{\lambda\mu}(0, a_r) = \eta^{\lambda\mu}$ (métrique de Minkowski)

Hypothèse β_p

Les fonctions $f_r(x^\alpha, w_\beta, w_{\beta\nu})$ sont définies et de classe C^p dans $U \times W \times W'$ où W' est un ouvert de $\mathbb{R}^{(n+1)n}$.

Hypothèse γ_p

Les \tilde{w}_r sont des applications de U dans W de classe C^p et telles que :

$$\tilde{w}_r(0) = a_r, \text{ le point } (a_{r\lambda}) = (D_\lambda \tilde{w}_r(0)) \in W'.$$

Hypothèse δ

$$S(x^1, \dots, x^n) = s \text{ avec } s = \sqrt{\sum_{i=1}^n (x^i)^2}.$$

Théorème 1.

Si $n \geq 2$, si les $A^{\lambda\mu}$, f_r , \tilde{w}_r vérifient les hypothèses α_{2t+2} , β_{2t} , γ_{2t+2} , δ , avec $t > n/2 + 1$, alors il existe un réel $T_0 > 0$ tel que le problème (1), (2) a une solution unique (w_r) telle que :

- les w_r sont définies dans Y_{T_0} et $w_r \in F^{t+1}(Y_{T_0})$.
- sur \mathcal{E} : $w_r = \tilde{w}_r$ et les $D^\lambda w_r$, $|\lambda| \leq t$, sont bornées.

Théorème 2

Si $n \geq 2$, si les $A^{\lambda\mu}$, f_r , \tilde{w}_r vérifient les hypothèses α_∞ , β_∞ , γ_∞ , alors il existe un réel $T_0 > 0$ tel que le problème (1), (2) possède une solution unique $(w_r) \in C^\infty(Y_{T_0})$ dans le domaine Y_{T_0} .

IV ESQUISSES DES PREUVES DES THEOREMES 1 ET 2

Preuve du théorème 1

On montre d'abord l'inutilité de la condition (0_r) en construisant une nouvelle donnée initiale (\tilde{w}_r) de classe C^{2t+2} telle que $\tilde{w}_r/\mathcal{E} = \tilde{w}_r/\mathcal{E}$ et (\tilde{w}_r) vérifie la condition (0_{2t+2}) . On en déduit d'après [1] :

- (3) : $\left\{ \begin{array}{l} \text{la détermination unique des fonctions } \psi_r^{(l)} \ (1 \leq l \leq t), \text{ égales} \\ \text{aux restrictions au conoïde } \mathcal{G} \text{ des dérivées } D_0^l w_r \text{ des fonctions} \\ \text{inconnues } w_r, \\ \bullet \text{ le "polynôme de Taylor" à l'ordre } t \text{ en } 0 \text{ de la fonction incon-} \\ \text{nue } w_r \text{ est égal à } w_r^0 = \text{polynôme de Taylor à l'ordre } t \text{ en } 0 \text{ de} \\ \bar{w}_r. \end{array} \right.$

On pose ensuite $w_r = v_r + \omega_r$ et on transforme le problème (1), (2) en un problème à données nulles, d'inconnues (v_r) , qu'on peut résoudre dans $F^{t+1}(Y_T)$, pour T assez petit, à l'aide d'une variante de la théorie de Leray^[4]. Cela impose aux (ω_r) de vérifier vu (3) :

$$(4) : \omega_r \in F^{t+2}(Y) \text{ et } D_0^l \omega_r / \mathcal{G} = \psi_r^{(l)}, \quad 0 \leq l \leq t.$$

On choisit les ω_r sous la forme $\omega_r = \omega_r^0 + \omega_r^1$, les ω_r^1 étant construites comme solution dans $F^{t+2}(Y)$ du problème :

$$(5) : P_{c_1 \dots c_m} \omega_r^1 = 0 \text{ dans } Y \text{ et } D_0^l \omega_r^1 / \mathcal{G} = \sigma_r^{(l)} = \psi_r^{(l)} \cdot D_0^l \omega_r^0 / \mathcal{G}, \quad 0 \leq l \leq t.$$

où : si $t = 2m - 1$, $P_{c_1 \dots c_m} = \prod_{i=1}^m (c_i D_0^2 - \Delta)$; si $t = 2m$, $P_{c_1 \dots c_m} = (\prod_{i=1}^m (c_i D_0^2 - \Delta)) \cdot D_0$
avec $\Delta = \sum_{i=1}^n D_i^2$, les c_k étant des réels tels que $1 < c_1 < \dots < c_m$.

Revenons maintenant au problème de la construction d'une nouvelle donnée initiale \bar{w}_r vérifiant (α_{2t+2}) . Il se résoud au moyen des propositions suivantes qui découlent d'une combinaison du théorème 5-4-2 de [3] et des résultats de [1].

Notations préliminaires

Soit $p \in \mathbb{N}$, $p \geq 2$. Soit des données $(A^{\lambda\mu}, f_r, \bar{w}_r)$ vérifiant $(\alpha_p, \beta_{p-2}, \gamma_p)$. On note $A_{(p)}^{\lambda\mu}$, $f_{r,(p)}$, ${}^{(p)}\bar{w}_r$ les polynômes de Taylor aux points respectifs $(0, \bar{w}_r(0))$, $(0, \bar{w}_r(0))$, $(D_v \bar{w}_r(0))$, 0 , aux ordres respectifs $p, p-2, p$ des fonction respectives $A^{\lambda\mu}, f_r, \bar{w}_r$.

On note : $A_{(\infty)}^{\lambda\mu} = A^{\lambda\mu}, f_{r,(\infty)} = f_r$, ${}^{(\infty)}\bar{w}_r = \bar{w}_r$, ${}^{(p)}Y = \text{intérieur de } {}^{(p)}\mathcal{G}$
avec ${}^{(p)}\mathcal{G} = \text{demi-conoïde de sommet } 0, \text{ orienté vers les } x^0 \geq 0,$
caractéristique par rapport à $A_{(p)}^{\lambda\mu}(x^\alpha, {}^{(p)}\bar{w}_r(x^\alpha)) D_{\lambda\mu}$ dans un domaine D de \mathbb{R}^{n+1} où cet opérateur est hyperbolique.

Alors on a :

Proposition 1 (cf [2]).

Soit $p \in \mathbb{N} \cup \{+\infty\}$, $p \geq 2$. soit des données $(A^{\lambda\mu}, f_r, \bar{w}_r)$ vérifiant

$(\alpha_p, \beta_{p-2}, \gamma_p)$. Alors il existe deux suites (décroissantes) $(\bar{D}_k)_{k \in \mathbb{N}}$, $(D_k)_{k \in \mathbb{N}}$ de voisinages de 0 dans U et une suite $({}^{(p)}\omega_{r,k})_{k \in \mathbb{N}}$ de fonctions de $C^\infty(D_k)$ telles que :

$$\bullet \bar{D}_0 = D_0 = D, \quad {}^{(p)}\omega_{r,0} = {}^{(p)}\bar{\omega}_r,$$

• pour $k \geq 1$: \bar{D}_k est un voisinage de 0 dans D_{k-1} , dans lequel la métrique $(\bar{A}_{(p),k}^{\lambda\mu})$ est hyperbolique; D_k est un voisinage de 0 dans \bar{D}_k , qui est de plus un domaine causal de \bar{D}_k pour la métrique $(\bar{A}_{(p),k}^{\lambda\mu})$;

$({}^{(p)}\omega_{r,k})$ est un élément de $C^\infty(D_k)$, solution du problème de Cauchy linéaire caractéristique (cf [3]) :

$$\left\{ \begin{array}{l} \bar{A}_{(p),k}^{\lambda\mu} \cdot D_{\lambda\mu}^{(p)} \omega_{r,k} + \bar{f}_{r,(p),k} = \text{dans } {}^{(p)}Y \cap D_k \\ {}^{(p)}\omega_{r,k} = {}^{(p)}\omega_{r,k-1} = {}^{(p)}\bar{\omega}_r \text{ sur } {}^{(p)}\mathcal{E} \cap D_k \end{array} \right.$$

$$\text{avec : } \bar{A}_{(p),k}^{\lambda\mu}(x^\alpha) = A_{(p)}^{\lambda\mu}(x^\alpha, {}^{(p)}\omega_{s,k-1}(x^\alpha));$$

$$\bar{f}_{r,(p),k}(x^\alpha) = f_{r,(p)}(x^\alpha, {}^{(p)}\omega_{s,k-1}(x^\alpha), D_v^{(p)}\omega_{s,k-1}(x^\alpha))$$

Proposition 2 (cf [2]). Hypothèses et notations identiques à celles de la proposition 1.

I) Pour tout entier k , $1 \leq k < p$, les polynômes de Taylor à l'ordre k en 0 des fonctions $({}^{(p)}\omega_{r,k})$ et $({}^{(p)}\omega_{r,k-1})$ coïncident.

II) Soient k un entier $< p$ et $({}^{(p)}\bar{\omega}_{r,k})$ le polynôme de Taylor à l'ordre $k+1$ en 0 de la fonction $({}^{(p)}\omega_{r,k})$. Alors :

• $({}^{(p)}\bar{\omega}_{r,k})$ vérifie la condition (0_{k+1}) et il existe une fonction (\bar{w}_r) de classe C^{k+1} telle que le polynôme de Taylor à l'ordre $k+1$ en 0, de \bar{w}_r est égal à $({}^{(p)}\bar{\omega}_{r,k})$ et $\bar{w}_r = \bar{w}_r$ sur \mathcal{E} .

Remarque : Vu la proposition 2, on a : $\omega_r^0 = ({}^{(2+2)}\bar{\omega}_{r,2+1})$.

Preuve du théorème 2

Elle est identique à celle du théorème 1, sauf la construction de $\omega_r = \omega_r^0 + \omega_r^1$. Ici, vu la proposition 2, les ω_r^0 et ω_r^1 doivent vérifier :

$$\left\{ \begin{array}{l} D^\beta \omega_r^0(0) = D^\beta ({}^{(p)}\bar{\omega}_{r,k})(0) \quad \forall k \in \mathbb{N}, \quad \forall \beta \in \mathbb{N}^{n+1}, |\beta| = k \\ D^\beta \omega_r^0/\mathcal{E} = \sigma_r^{(k)} = \phi_r^{(k)} \cdot D_0^k \omega_r^0/\mathcal{E} \quad \forall k \in \mathbb{N} \end{array} \right.$$

Leur construction se fait au moyen de certaines variantes du théorème classique de Borel.

Remarques

- 1) les détails des démonstrations des théorèmes 1 et 2 se trouvent dans [2].
- 2) Les théorèmes 1 et 2 peuvent s'appliquer aux Equations de Yang-Mills-Higgs en jauge de Lorentz, aux Equations d'Einstein du vide en coordonnées harmoniques, au système conforme régulier des Equations d'Einstein

BIBLIOGRAPHIE

- [1] : F. CAGNAC, Annali di Matematica Pura ed Applicata (IV), Vol CXXIX, PP 13-41, (1980)
- [2] : M. DOSSA, Thèse de Doctorat d'Etat, Université de Yaoundé, 1992.
- [3] : A.G. FRIEDLANDER, The wave equation on a curved space-time. Cambridge University press, 1975
- [4] : J. LERAY, Hyperbolic Differential Equations, Princeton, I.A.S, 1952
- [5] : A. D. RENDALL, Proc. Roy. Soc. London Ser. A 427 (1990), n°1872, 221-239
- [6] : H. MÜLLER ZUM HAGEN, Ann. Inst. H. Poincaré Phys. Théo. 53 (1990), n° 2, 159-216
- [7] : A. D. RENDALL, nonlinear hyperbolic equations and field theory, 154-163 Pitman Res. Notes Math-Ser., 253, Longman Sci. Techn., Harlow, 1992.

Adresse de l'auteur : Faculté des Sciences, B.P. 812 Yaoundé (Cameroun).

Received February 11, 1994

ON THE REPRESENTATION OF INTEGERS BY INDEFINITE DIAGONAL QUADRATIC FORMS

J.H.H. Chalk, F.R.S.C. and R.J. Ashton

In Memory of D.B. DeLury (obit. 21 Nov. 1993) and T. Estermann (obit. 29 Nov. 1991).

Abstract

"A brief review of the developments of the Circle-method of Hardy-Littlewood adapted for diagonal quadratic forms."

Let $a = (a_1, a_2, a_3, a_4)$ denote non-zero integers, not all of the same sign. Let $v(n)$ denote the number of solutions of the diophantine equation

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = c, \quad (c \in \mathbb{Z}/\{0\}) \quad (1)$$

in positive integers $x = (x_1, x_2, x_3, x_4)$, subject to a restraint of the form

$$|a_i x_i^2| \leq n \quad (1 \leq i \leq 4), \quad (2)$$

for some $n > 0$. (From classical theory of representations it is known that if c is actually represented by the form in (1), then it has an infinity of such representations of c ; moreover, it represents an integer c if, and only if, (1) is solvable p -adically for every prime p .) The restraint (2) provides a measure of the 'size' of a solution of (1). Estermann [3] adapted the Circle-method to obtain an asymptotic formula for $v(n)$ as $n \rightarrow \infty$:

$$\left| v(n) - K \Delta^{-\frac{1}{2}} \mathfrak{S}(a, c)n \right| < C_2 n^{\frac{1}{2} + \epsilon}, \quad \text{for } n > C_1, \quad (3)$$

where

- (i) $\Delta = |a_1 a_2 a_3 a_4|$, K is a known absolute constant,
- (ii) C_1 and C_2 are constants depending at most on a , c and ϵ ,
- (iii) $\mathfrak{S}(a, c)$ is the so-called Singular Series (embodying a p -adic component for every prime p). (cf. [3] §3, 3.5).

He remarked that there is a different and perhaps more interesting problem if (2) is replaced by

$$\sum_{1 \leq i \leq 4} |a_i x_i^2| \leq n \quad (4)$$

particularly for the case when the quadratic form on the left of (1) has signature zero. However, this was already done by D.B. DeLury [2] albeit with a weaker estimate for the error term on the right of (3). Later, K.S. Williams [4] obtained explicit bounds in terms of Δ and $|c|$ for $\mathfrak{S}(a, c)$ and for the constants C_1, C_2 and from this he was able to assert the existence of a 'small' solution of (1); in fact a solution $x = (x_1, x_2, x_3, x_4)$ satisfying

$$\max_{1 \leq i \leq 4} |x_i| \ll \Delta^{\alpha+\epsilon} |c|^{\beta+\epsilon} \quad (5)$$

with

$$\alpha = 27 + \frac{27}{17} \sqrt{102} = 43.04, \quad \beta = \frac{14}{17} \sqrt{102} = 22.31... \quad (6)$$

Recently, this work was improved by R.J. Ashton [1], who introduced the Poisson Summation formula in place of Estermann's use of the Euler Summation method. As a direct consequence, (5) may now be replaced by

$$\max_{1 \leq i \leq 4} |x_i^2| \ll \Delta^{\frac{3}{2}+\epsilon} |c|^{\frac{1}{2}+\epsilon}. \quad (7)$$

(Note: The implied constant in the Vinogradov symbol ' \ll ' depends at most on c .)

References

1. R.J. Ashton, "On Indefinite Diagonal Quaternary Quadratic", Ph.D thesis, University of London, 1993.
2. D.B. DeLury, "On the Representation of Numbers by the Indefinite Form $ax^2 + by^2 + cz^2 + dt^2$ ", University of Toronto Studies (Math. Series 5) 1938.
3. T. Estermann, "A New Application of the Hardy-Littlewood-Kloosterman Method", Proc. London Math. Soc., (3) 12 (1962), 425-444.
4. K.S. Williams, "An Application of the Hardy-Littlewood Method", Ph.D thesis, University of Toronto, 1965.

Department of Mathematics,
Imperial College,
180 Queen's Gate,
London SW7 2BZ,
England.

Received February 25, 1994

2. Biorthogonal Wavelet Bases.

2.1. *Preliminaries.* Two scaling functions, ϕ and $\tilde{\phi}$ in $L^2(\mathbb{R})$, satisfying

$$\phi(x) = 2 \sum_{n \in \mathbb{Z}} \alpha_n \phi(2x - n), \quad \tilde{\phi}(x) = 2 \sum_{n \in \mathbb{Z}} \tilde{\alpha}_n \tilde{\phi}(2x - n), \quad (2.1)$$

are said to be *dual scaling functions* if

$$(\phi(\cdot - j), \tilde{\phi}(\cdot - k)) := \int_{-\infty}^{\infty} \phi(x - j) \overline{\tilde{\phi}(x - k)} dx = \delta_{j,k}, \quad j, k \in \mathbb{Z}. \quad (2.2)$$

By (2.1) and [4], pp. 72-73, it is seen that the pair of functions

$$\psi(x) = 2 \sum_{n \in \mathbb{Z}} (-1)^n \overline{\tilde{\alpha}_{1-n}} \phi(2x - n), \quad \tilde{\psi}(x) = 2 \sum_{n \in \mathbb{Z}} (-1)^n \overline{\alpha_{1-n}} \tilde{\phi}(2x - n), \quad (2.3)$$

are (in general, non-orthonormal) dual wavelets associated with the scaling functions $\phi(x)$, $\tilde{\phi}(x)$. Using notation (1.1) for ϕ_{jk} , $\tilde{\phi}_{jk}$, ψ_{jk} and $\tilde{\psi}_{jk}$, we have, for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} (f, \phi_{jk}) \tilde{\phi}_{jk}(x) + \sum_{\substack{n \in \mathbb{Z} \\ n < j}} \sum_{k \in \mathbb{Z}} (f, \psi_{nk}) \tilde{\psi}_{nk}(x) \\ &= \sum_{k \in \mathbb{Z}} (f, \tilde{\phi}_{jk}) \phi_{jk}(x) + \sum_{\substack{n \in \mathbb{Z} \\ n < j}} \sum_{k \in \mathbb{Z}} (f, \tilde{\psi}_{nk}) \psi_{nk}(x). \end{aligned} \quad (2.4)$$

Taking the Fourier transform of $\phi(x/2)$ and $\tilde{\phi}(x/2)$ as given by (2.1), we see that there exists a pair of 2π -periodic functions, $m_0(\xi)$ and $\tilde{m}_0(\xi)$,

$$m_0(\xi) = \sum_{n \in \mathbb{Z}} \alpha_n e^{in\xi}, \quad \tilde{m}_0(\xi) = \sum_{n \in \mathbb{Z}} \tilde{\alpha}_n e^{in\xi}, \quad (2.5)$$

such that

$$\hat{\phi}(2\xi) = m_0(\xi) \hat{\phi}(\xi), \quad \hat{\tilde{\phi}}(2\xi) = \tilde{m}_0(\xi) \hat{\tilde{\phi}}(\xi). \quad (2.6)$$

2.2. *Conditions for the existence of biorthogonal wavelet bases.* We now assume that the coefficients, α_n and $\tilde{\alpha}_n$, of m_0 and \tilde{m}_0 , respectively, as defined in (2.1), are real, satisfy the symmetry relations $\alpha_{-n} = \alpha_n$ and $\tilde{\alpha}_{-n} = \tilde{\alpha}_n$, and are finite in number. The last assumption is equivalent to the compact support property of the constructed wavelets.

Using (2.6), one can see [4] that

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi). \quad (2.7)$$

To have regular biorthogonal wavelet bases, $m_0(\xi)$ and $\tilde{m}_0(\xi)$ have to satisfy a set of conditions given in [3]. Among these, by (2.2) we have

$$m_0(\xi) \overline{\tilde{m}_0(\xi)} + m_0(\xi + \pi) \overline{\tilde{m}_0(\xi + \pi)} = 1, \quad \forall \xi \in [0, \pi]. \quad (2.8)$$

Hence the coefficients α_n and $\tilde{\alpha}_n$ of the two-scale difference equations (2.1) satisfy

$$\sum \alpha_n \tilde{\alpha}_n = \frac{1}{2}; \tag{2.9}$$

moreover, by (2.7), ϕ and $\tilde{\phi}$ are in $L^2(\mathbb{R})$ only if

$$m_0(0) = 1, \quad \tilde{m}_0(0) = 1, \tag{2.10}$$

and continuous only if

$$m_0(\pi) = 0, \quad \tilde{m}_0(\pi) = 0. \tag{2.11}$$

Conditions (2.8) and (2.10) imply neither the biorthogonality of the scaling functions nor that these are in $L^2(\mathbb{R})$. A positive answer is provided in [3] by the following proposition.

Proposition. Assume that $m_0(\xi)$ and $\tilde{m}_0(\xi)$ can be factored in the form:

$$m_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^L f(\xi), \quad \tilde{m}_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^{\tilde{L}} \tilde{f}(\xi), \tag{2.12}$$

respectively, where $L, \tilde{L} \geq 1$, and suppose, that for some $k, \tilde{k} > 0$,

$$B_k = \sup_{\xi} |f(\xi)f(2\xi)\dots f(2^{k-1}\xi)|^{1/k} < 2^{L-1/2}, \tag{2.13}$$

$$\tilde{B}_{\tilde{k}} = \sup_{\xi} |\tilde{f}(\xi)\tilde{f}(2\xi)\dots \tilde{f}(2^{\tilde{k}-1}\xi)|^{1/\tilde{k}} < 2^{\tilde{L}-1/2}. \tag{2.14}$$

Then $\phi, \tilde{\phi} \in L^2(\mathbb{R})$ and $(\phi_{0,0}, \tilde{\phi}_{0,n}) = \delta_{0n}$.

If (2.13) and (2.14) are satisfied, then there exist [3] two positive numbers, $\epsilon, \tilde{\epsilon} > 0$, and a positive constant, $c > 0$, such that

$$|\hat{\phi}(\xi)| < c(1 + |\xi|)^{-L-\epsilon+\log(B_k)/\log(2)}, \quad |\hat{\tilde{\phi}}(\xi)| < c(1 + |\xi|)^{-\tilde{L}-\tilde{\epsilon}+\log(\tilde{B}_{\tilde{k}})/\log(2)}. \tag{2.15}$$

By Theorem 3.8 in [3], if (2.8) and (2.15) are satisfied, then the dual wavelets constructed from the scaling functions $\phi(x)$ and $\tilde{\phi}(x)$ generate two biorthogonal wavelet bases, and (2.4) holds. Thus, the construction of a biorthogonal wavelet basis reduces to the easier construction of 2π -periodic functions (2.5) satisfying (2.8), (2.10), (2.13) and (2.14).

3. Numerical Construction of Biorthogonal Wavelets.

3.1. The construction of $m_0(\xi)$. By using a result from [5], one can easily see that a 2π -periodic function, $m_0(\xi)$, which satisfies the first parts of (2.10), (2.11) and (2.12), respectively, and (2.13) for some $k > 0$, is a candidate for generating a biorthogonal wavelet basis. Hence, let $n_0 \geq 1$ be a positive integer and consider the function

$$m_{n_0}(\xi) = \sum_{n=-n_0}^{n_0} \alpha_n e^{in\xi}, \tag{3.1}$$

which generates a symmetric filter since $m_{n_0}(-\xi) = m_{n_0}(\xi)$ follows from $\alpha_n = \alpha_{-n}$, for $1 \leq n \leq n_0$.

From the first parts of (2.10) and (2.11) we derive the pair of linear equations:

$$\alpha_0 + 2 \sum_{n=1}^{n_0} \alpha_n = 1, \quad \alpha_0 + 2 \sum_{n=1}^{n_0} (-1)^n \alpha_n = 0, \quad (3.2)$$

in the $n_0 + 1$ unknowns α_j , $j = 0, 1, \dots, n_0$. By fixing $n_0 - 1$ of these, one obtains a unique solution to system (3.2).

In general, (2.13) is verified numerically by approximating the upper bound of B_k . This problem simplifies considerably if, instead of estimating B_k , one estimates the maximum of the absolute value of a piecewise polynomial which approximates $|f(\xi)f(2\xi) \cdots f(2^{k-1}\xi)|$, as is done in the following theorem, proved in [6].

Theorem 1. *Suppose the 2π -periodic function,*

$$m_0(\xi) = \sum_{j=-n}^N \alpha_j e^{ij\xi}, \quad \alpha_j \in \mathbb{R}, \quad -n \leq j \leq N, \quad (3.3)$$

satisfies (2.10) and (2.12). If we write $F_k(\xi) = f(\xi)f(2\xi) \cdots f(2^{k-1}\xi)$, then, for all $\epsilon > 0$ and $k > 0$, there exist a positive integer r and a finite partition of $[0, 2\pi]$, say $(I_i)_{i \in I}$, such that

$$\left| \sup_{\xi} |F_k(\xi)|^{1/k} - \sup_{\xi} |P_{F_k}(\xi)|^{1/2k} \right| < \epsilon,$$

where, for each i , $P_{F_k}(\xi)$ is equal to a polynomial of degree r if $\xi \in I_i$, and 0 otherwise.

Remark 1. In practice, $P_{F_k}(\xi)$ is approximated on I_i by the constant value $\bar{F}_k(\xi_i)F_k(\xi_i)$, where ξ_i is the midpoint of the interval I_i . If $M = N - L + n$, $K = (1 + 2^{k-1})2^{k-2}$ and h is the step size of the partition, then it follows from the method of proof of Theorem 1 that the error made in approximating $\sup_{\xi} |F_k(\xi)|^{1/k}$ is bounded by

$$\text{const} \left\{ \left[1 + \frac{1}{\sup_{\xi} |F_k(\xi)|^2} \sum_{j=-KM}^{KM} |\beta_j| h \right]^{1/2k} - 1 \right\}. \quad (3.4)$$

3.2. The construction of $\tilde{m}_0(\xi)$. The dual, $\tilde{m}_0(\xi)$, to $m_0(\xi)$ follows from identity (2.8). Since $m_0(0) = 1$ and $m_0(\pi) = 0$, then necessarily $\tilde{m}_0(0) = 1$.

If we require at least continuity of the wavelet $\tilde{\psi}(x)$, then by an argument given in [5], $\tilde{m}_0(\xi)$ also has to satisfy the condition $\tilde{m}_0(\pi) = 0$.

With a notation for N_0 which anticipates the result, one easily sees that if the following two functions:

$$m_{n_0}(\xi) = \sum_{n=-n_0}^{n_0} \alpha_n e^{in\xi}, \quad \tilde{m}_{N_0}(\xi) = \sum_{n=-N_0}^{N_0} \beta_n e^{in\xi}, \quad N_0 = n_0 + (2k + 1), \quad (3.5)$$

satisfy (2.8), then the integer $N_0 + n_0$ is odd.

If $m_0(\xi)$ and $\tilde{m}_0(\xi)$ are as in (3.5) for some integer k , then by conditions (2.8), (2.10) and (2.11), one obtains the following linear system of equations in β_j :

$$\begin{aligned} \sum_{i+j=0} \alpha_i \beta_j &= 1, & \sum_{i+j=2n} \alpha_i \beta_j &= 0, & 1 \leq n \leq \frac{n_0 + N_0 - 1}{2}, \\ \beta_0 + 2 \sum_{j=1}^{N_0} (-1)^j \beta_j &= 0. \end{aligned} \tag{3.6}$$

The numerical method mentioned in Remark 1 is then used to decide whether or not $\tilde{m}_0(\xi)$ generates a biorthogonal wavelet basis.

By the following theorem, proved in [6], our method produces an infinite family of dual filters all of the same length.

Theorem 2. Consider a symmetric 2π -periodic function,

$$m_{n_0}(\xi) = \sum_{j=-n_0}^{n_0} \alpha_j e^{ij\xi}, \quad \alpha_j \in \mathbb{R}, \quad \alpha_{-j} = \alpha_j, \quad 1 \leq j \leq n_0. \tag{3.7}$$

Assume that, for some $N_0 > n_0$, there exists a dual real trigonometric function

$$\tilde{m}_{N_0}(\xi) = \sum_{j=-N_0}^{N_0} \beta_j e^{ij\xi}, \quad \beta_j \in \mathbb{R}, \quad \beta_{-j} = \beta_j, \quad 1 \leq j \leq N_0, \tag{3.8}$$

such that $m_{n_0}(\xi)$ and $\tilde{m}_{N_0}(\xi)$ satisfy condition (2.8) and $\tilde{m}_{N_0}(\xi)$ factors in the form (2.12) with $\tilde{L} = 2$, where $\tilde{f}(\xi)$ is a trigonometric function satisfying

$$\sup_{\xi} |\tilde{f}(\xi) \tilde{f}(2\xi) \cdots \tilde{f}(2^{k-1}\xi)|^{1/k} \leq 2^{(3/2) - \epsilon}, \tag{3.9}$$

for some positive integer $k \geq 1$ and $\epsilon > 0$. Then, for all $N = N_0 + 2l$, $l \in \mathbb{N}$, there exists a one-parameter family of trigonometric functions of length $2N + 1$, dual to $m_{n_0}(\xi)$, with parametric coefficients $\beta_j(\mu)$.

4. Numerical Results. The above technique has been used [6] to construct filters of length five, seven and nine, respectively, with dual filters in parametric form. Let $m_N(\xi)$ and $\tilde{m}_N(\xi)$ denote the dual trigonometric functions that generate a set of biorthogonal wavelet bases, where N is the number of vanishing moments [3] of the corresponding wavelets. Since the coefficients of $\tilde{m}_N(\xi)$ are given in parametric form, an approximation to the range, $[l_N, L_N]$, of the parameter μ , has been computed for which condition (2.14) is satisfied by $\tilde{m}_N(\xi)$.

The decay associated with $\hat{\phi}(\xi)$ is the largest number $\epsilon > 0$ such that

$$\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|(1 + |\xi|)^{\epsilon} d\xi < C < \infty. \tag{3.10}$$

In this case, $\phi(x)$ and $\psi(x)$ are at least of class $C^{\epsilon-1}$.

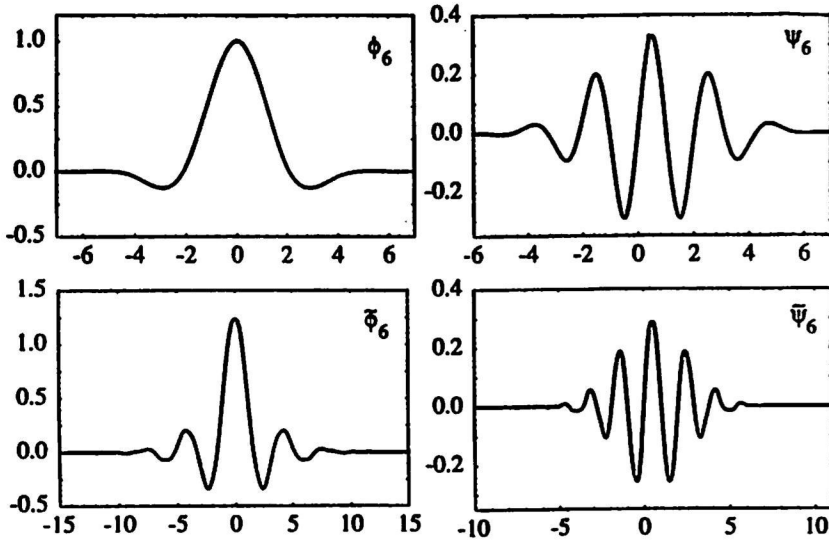


FIGURE 1. Dual scaling functions, $\phi_6(x)$, $\tilde{\phi}_6(x)$, and corresponding wavelets, $\psi_6(x)$, $\tilde{\psi}_6(x)$, at $\mu = 2.5$.

The coefficients α_n of $m_N(\xi)$ and $\beta_n(\mu)$ of $\tilde{m}_N(\xi)$ for $l_N \leq \mu \leq L_N$, and the decays, ϵ_N and $\tilde{\epsilon}_N$, associated with ϕ_N and $\tilde{\phi}_N$ are given in [6] for $N = 2, 4$ and 6 .

Eight iterations of the constructive cascade algorithm given in [7], pp. 202–205, produce a good approximation to the graphs of the scaling functions and the corresponding wavelets. Figure 1 presents the graphs of $\phi_6(x)$, $\tilde{\phi}_6(x)$, $\psi_6(x)$ and $\tilde{\psi}_6(x)$, at $\mu = 2.5$.

REFERENCES

1. P. P. Vaidyanathan, *Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having the perfect reconstruction property*, IEEE Trans. on Acoustics, Speech and Signal Processing ASSP-35 (1987), no. 4, 476–492.
2. D. E. Dudgeon and R. M. Mersereau, *Multidimensional digital signal processing*, Prentice-Hall, Englewood Cliffs, NJ, 1984.
3. A. Cohen, I. Daubechies, and J.-C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math. 45 (1992), no. 5, 485–560.
4. Y. Meyer, *Wavelets and operators*; transl. from the French by D. H. Salinger, Cambridge Studies in Advanced Mathematics, vol. 37, Cambridge University Press, Cambridge, 1992.
5. I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. 41 (1988), no. 7, 909–996.
6. A. Karoui and R. Vaillancourt, *Families of biorthogonal wavelets*, Computers and Mathematics with Applications (to appear).
7. I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia, PA, 1992.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ON, CANADA K1N 6N5
E-mail address: rxvsg@acadvm1.uottawa.ca

A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$

Mikael Rørdam

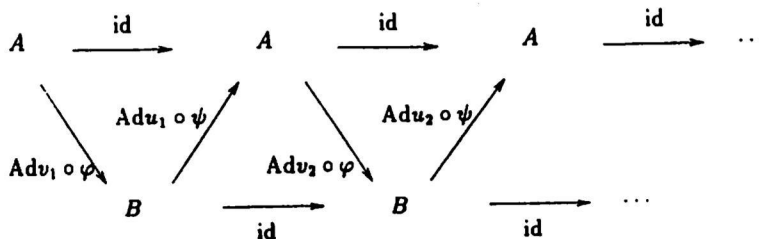
Presented by G.A. Elliott, F.R.S.C.

The purpose of this note is to present a direct and relatively self-contained proof of the theorem of George Elliott that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is isomorphic to \mathcal{O}_2 . This theorem will also be included in [2], where it will follow from a much more general result. The present proof differs from the proof given in [2], in that it gives a characterization of all separable unital C^* -algebras A for which A is isomorphic to $A \otimes \mathcal{O}_2$. Most of the ideas of the proof given here are due to Elliott and are based on conversations, but there are some new ideas and short cuts, most notably in the proof of Lemma 1.

We need the following three facts for the proof. Call two $*$ -homomorphisms $\varphi, \psi: A \rightarrow B$ between separable C^* -algebras approximately unitarily equivalent, written $\varphi \approx \psi$, if there is a sequence $\{u_k\}$ of unitaries in B (or in B with a unit adjoined) such that $Adv_k \circ \varphi \rightarrow \psi$ pointwise. The first result we need follows from a version of Elliott's approximate intertwining argument.

Proposition A (Elliott [4]). *If A and B are two separable C^* -algebras and there are $*$ -homomorphisms $\varphi: A \rightarrow B$ and $\psi: B \rightarrow A$ such that $\psi \circ \varphi \approx id_A$ and $\varphi \circ \psi \approx id_B$, then A and B are isomorphic.*

Sketch of the proof. Find sequences $\{u_k\}$ and $\{v_k\}$ of unitaries in A , respectively B , making each triangle in the diagram



commute well enough on appropriate finite subsets of A and B so that the sequences $\{Adv_k \circ \varphi\}$ and $\{Adv_k \circ \psi\}$ of $*$ -homomorphisms are pointwise Cauchy. The limits of these Cauchy sequences will then be $*$ -isomorphisms (each inverse to the other). \square

The map λ referred to below is the endomorphism on \mathcal{O}_n given by $\lambda(a) = \sum_{j=1}^n s_j a s_j^*$, where s_1, \dots, s_n are the canonical generators of \mathcal{O}_n , that satisfy $s_i^* s_i = 1$ and $\sum s_i s_i^* = 1$.

Theorem B ([6] Theorem 3.6). (i) Every pair of unital *-homomorphisms from \mathcal{O}_2 into any simple, unital, purely infinite C^* -algebra are approximately unitarily equivalent.

(ii) For every even n and for every unitary u in \mathcal{O}_n there is a sequence $\{v_k\}$ of unitaries in \mathcal{O}_n such that $\lambda(v_k^*)v_k \rightarrow u$.

Lemma C $\mathcal{O}_2 \otimes \mathcal{O}_2$ is simple and purely infinite.

Proof. Simplicity of $\mathcal{O}_2 \otimes \mathcal{O}_2$ follows from the fact that \mathcal{O}_2 is simple ([3]) together with the fact that the minimal tensor product of simple C^* -algebras is again simple ([7], Chap. IV, Corollary 4.21). Clearly, $\mathcal{O}_2 \otimes \mathcal{O}_2$ is infinite. From [1], Corollary 1.3, it follows that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes M_\infty$. It now follows from [5], Theorem 6.8 (essentially a reformulation of results of Blackadar and Kumjian and of Cuntz) that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is purely infinite. \square

We now turn to the proof of Elliott's theorem and begin this with the following three lemmas:

Lemma 1. For every even n there is an asymptotically central sequence $\{\varphi_k\}$ of unital endomorphisms of \mathcal{O}_n (i.e. $\{\varphi_k\}$ satisfies $[\varphi_k(a), b] \rightarrow 0$ as $k \rightarrow \infty$ for all $a, b \in \mathcal{O}_n$).

Proof. The element

$$u = \sum_{i,j=1}^n s_i s_j s_i^* s_j^* (= \sum_{j=1}^n \lambda(s_j) s_j^*)$$

is a unitary in \mathcal{O}_n . Hence, by Theorem B, there is a sequence $\{v_k\}$ of unitaries in \mathcal{O}_n such that $\lambda(v_k^*)v_k \rightarrow u$. Let $\varphi_k : \mathcal{O}_n \rightarrow \mathcal{O}_n$ be the *-endomorphism given by $\varphi_k(s_j) = v_k s_j$. Then

$$\left\| \sum_{j=1}^n \lambda(v_k s_j) s_j^* - v_k \right\| = \|\lambda(v_k)u - v_k\| \rightarrow 0,$$

whence $\|\lambda(v_k s_j) - v_k s_j\| \rightarrow 0$. Since $\lambda(v_k s_j) s_i = s_i v_k s_j$, it follows that $\{\varphi_k\}$ is asymptotically central. \square

Before proving the next lemma, which produces an exact embedding of $\mathcal{O}_2 \otimes \mathcal{O}_2$ into \mathcal{O}_2 from approximate ones, we need the following observation. Suppose that p is a polynomial in eight non-commuting variables and that the element x obtained by evaluating p at the eight elements $s_i \otimes 1, s_i^* \otimes 1, 1 \otimes s_i, 1 \otimes s_i^*, i = 1, 2$, in $\mathcal{O}_2 \otimes \mathcal{O}_2$ satisfies $\|x\| < \epsilon$. Let $\varphi_1^{(n)}, \varphi_2^{(n)}$ be two sequences of embeddings of \mathcal{O}_2 into a C^* -algebra A , and let x_n denote p evaluated at $\varphi_1^{(n)}(s_i), \varphi_1^{(n)}(s_i^*), \varphi_2^{(n)}(s_i), \varphi_2^{(n)}(s_i^*), i = 1, 2$. Assume that $\|[\varphi_1^{(n)}(s_i), \varphi_2^{(n)}(s_j)]\| \rightarrow 0$ for all $i, j \in \{1, 2\}$. Then $\|x_n\| < \epsilon$ for all sufficiently large n . To see this, observe first that we can define embeddings $\bar{\varphi}_1, \bar{\varphi}_2 : \mathcal{O}_2 \rightarrow \ell^\infty(A)/c_0(A)$ by setting $\bar{\varphi}_i(s_j) = \{\varphi_i^{(n)}(s_j)\} + c_0(A)$, and that $\bar{\varphi}_1(\mathcal{O}_2)$ commutes with $\bar{\varphi}_2(\mathcal{O}_2)$. We therefore get an embedding $\bar{\varphi} : \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \ell^\infty(A)/c_0(A)$ given by $\bar{\varphi}(a \otimes 1) = \bar{\varphi}_1(a)$ and $\bar{\varphi}(1 \otimes a) = \bar{\varphi}_2(a), a \in \mathcal{O}_2$. Now $\bar{x} = \{x_n\} + c_0(A)$ is equal to $\bar{\varphi}(x)$, and so $\|\bar{x}\| < \epsilon$. Since $\|\bar{x}\| = \limsup \|x_n\|$ the assertion follows.

Lemma 2. *Let A be a simple, unital, purely infinite C^* -algebra. For every finite subset F of A and every $\varepsilon > 0$ there is a finite subset F' of A and a $\delta > 0$ such that if $\varphi_1, \varphi_2 : \mathcal{O}_2 \rightarrow A$ are unital $*$ -homomorphisms such that $\|[\varphi_1(s_i), \varphi_2(s_j)]\| < \delta$ and $\|[\varphi_k(s_i), a]\| < \delta$ for all $i, j, k \in \{1, 2\}$ and $a \in F'$, then there is a unitary v in A such that*

$$\|vav^* - a\| < \varepsilon, \quad \|v\varphi_1(s_j)v^* - \varphi_2(s_j)\| < \varepsilon$$

for all $a \in F$ and $j = 1, 2$.

Proof. It suffices to show that if $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$ are asymptotically central sequences of embeddings of \mathcal{O}_2 into A such that $[\varphi_1^{(n)}(s_i), \varphi_2^{(n)}(s_j)] \rightarrow 0$, then the conclusion of the lemma holds for the pair $\varphi_1^{(n)}, \varphi_2^{(n)}$ for n sufficiently large.

By Theorem B there is a unitary v in $\mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\|Adv(s_j \otimes 1) - 1 \otimes s_j\| < \varepsilon/5$$

for $j = 1, 2$. There is an element z in the $*$ -algebra generated by $s_j \otimes 1$ and $1 \otimes s_j$ such that $\|v - z\| < \varepsilon/5$. Note that

$$\|z(s_j \otimes 1) - (1 \otimes s_j)z\| < 3\varepsilon/5.$$

Express z as a polynomial in the eight non-commuting variables $s_j \otimes 1, s_j^* \otimes 1, 1 \otimes s_j, 1 \otimes s_j^*, j = 1, 2$, and denote by z_n the evaluation of that polynomial at $\varphi_1^{(n)}(s_j), \varphi_1^{(n)}(s_j^*), \varphi_2^{(n)}(s_j), \varphi_2^{(n)}(s_j^*)$. By the remarks preceding the statement of the lemma we have that

$$\|z_n \varphi_1^{(n)}(s_j) - \varphi_2^{(n)}(s_j) z_n\| < 3\varepsilon/5$$

for sufficiently large n . Also, $\|z_n - 1\| < \varepsilon/5$ for large n , and so there are unitaries v_n in A with $\|v_n - z_n\| < \varepsilon/5$. We have

$$\|v_n \varphi_1^{(n)}(s_j) - \varphi_2^{(n)}(s_j) v_n\| < \varepsilon$$

for large n . Finally, $[z_n, a] \rightarrow 0$ for all $a \in A$, and so $\|v_n a v_n^* - a\| < 2\varepsilon/5$ for all a in any finite set F when n is large. \square

Lemma 3. *Suppose that A is a unital, separable C^* -algebra and that there is an asymptotically central inclusion of \mathcal{O}_2 into A (i.e. there is a sequence of unital $*$ -homomorphisms $\varphi_n : \mathcal{O}_2 \rightarrow A$ such that $\|[\varphi_n(b), a]\| \rightarrow 0$ for all $a \in A$ and $b \in \mathcal{O}_2$). It follows that there is a unital embedding $\psi : A \otimes \mathcal{O}_2 \rightarrow A$ such that $\psi \circ \varphi \approx \text{id}_A$, where $\varphi : A \rightarrow A \otimes \mathcal{O}_2$ is given by $\varphi(a) = a \otimes 1$.*

Proof. Choose an increasing sequence $\{F_n\}$ of finite subsets of A such that $\bigcup_{n=1}^{\infty} F_n$ is dense in A , and find finite subsets $F'_n \subseteq A$ and $\delta_n > 0$ such that Lemma 2 holds with $\epsilon_n = 2^{-n}$. We may assume that the sequence $\{F'_n\}$ is increasing and that $\{\delta_n\}$ decreases to zero. By assumption there is a sequence $\{\varphi_n\}$ of unital embeddings of \mathcal{O}_2 into A such that

$$\|[\varphi_n(s_i), \varphi_{n+1}(s_j)]\| < \delta_n, \quad \|[a, \varphi_n(s_j)]\| < \delta_n,$$

for all $a \in F'_n$ and $i, j \in \{1, 2\}$.

By Lemma 2 there are unitaries v_n in A such that

$$\|\text{Adv}_n(a) - a\| < 2^{-n}, \quad \|(\text{Adv}_n \circ \varphi_{n+1})(s_j) - \varphi_n(s_j)\| < 2^{-n}, \quad (a \in F_n).$$

Set $\psi_n = \text{Ad}(v_1 v_2 \cdots v_{n-1})$ and $\psi'_n = \text{Ad}(v_1 v_2 \cdots v_{n-1}) \circ \varphi_n$. Then

$$\begin{aligned} \|\psi_{n+1}(a) - \psi_n(a)\| &= \|\text{Adv}_n(a) - a\| < 2^{-n}, \quad (a \in F_n), \\ \|\psi'_{n+1}(s_j) - \psi'_n(s_j)\| &= \|(\text{Adv}_n \circ \varphi_{n+1})(s_j) - \varphi_n(s_j)\| < 2^{-n}. \end{aligned}$$

The sequences $\{\psi_n\}$ and $\{\psi'_n\}$ are therefore pointwise convergent to *-homomorphisms $\psi_0 : A \rightarrow A$, respectively $\psi'_0 : \mathcal{O}_2 \rightarrow A$, and $\psi_0(A)$ commutes with $\psi'_0(\mathcal{O}_2)$. It follows that ψ_0 and ψ'_0 extend to an embedding $\psi : A \otimes \mathcal{O}_2 \rightarrow A$ given by $\psi(a \otimes 1) = \psi_0(a)$ and $\psi(1 \otimes b) = \psi'_0(b)$ where $a \in A$ and $b \in \mathcal{O}_2$. Since $\psi \circ \varphi = \psi_0$ and $\psi_0 \approx \text{id}_A$ by construction, the last assertion follows. \square

Theorem (Elliott). *Let A be a separable unital C^* -algebra. Then $A \otimes \mathcal{O}_2$ is isomorphic to A if and only if there is an asymptotically central inclusion of \mathcal{O}_2 into A (see Lemma 3).*

Proof. The "only if" part follows from Lemma 1. To prove the "if" part, let $\varphi : A \rightarrow A \otimes \mathcal{O}_2$ denote the inclusion $\varphi(a) = a \otimes 1$, and use Lemma 3 to find an embedding $\psi : A \otimes \mathcal{O}_2 \rightarrow A$ such that $\psi \circ \varphi \approx \text{id}_A$. By Proposition A it now suffices to show that $\varphi \circ \psi \approx \text{id}_{A \otimes \mathcal{O}_2}$. This is proved in two steps.

Set

$$B = C^*((\varphi \circ \psi)(1 \otimes \mathcal{O}_2), 1 \otimes \mathcal{O}_2) \subseteq A \otimes \mathcal{O}_2,$$

and observe that B is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_2$ (and hence is simple and purely infinite by Lemma C). Also, B commutes with $(\varphi \circ \psi)(A \otimes 1)$. Define $\lambda, \mu : \mathcal{O}_2 \rightarrow B$ by $\lambda(a) = (\varphi \circ \psi)(1 \otimes a)$ and $\mu(a) = 1 \otimes a$. By Theorem B there is a sequence $\{v_n\}$ of unitaries in B such that $\text{Adv}_n \circ \lambda \rightarrow \mu$.

The two *-homomorphisms $\varphi, \varphi \circ \psi \circ \varphi : A \rightarrow A \otimes 1$ are approximately unitarily equivalent. We can therefore find a sequence $\{w_n\}$ of unitaries in $A \otimes 1$ such that $\text{Ad}_{w_n} \circ \varphi \circ \psi \circ \varphi \rightarrow \varphi$.

Let us now prove that $\text{Ad}(w_n v_n) \circ \varphi \circ \psi \rightarrow \text{id}_{\mathcal{A} \otimes \mathcal{O}_2}$. Observe that $(\text{Ad} v_n \circ \varphi \circ \psi)(a \otimes 1) = (\varphi \circ \psi)(a \otimes 1)$, because $v_n \in B$. Hence,

$$(\text{Ad}(w_n v_n) \circ \varphi \circ \psi)(a \otimes 1) = (\text{Ad} w_n \circ \varphi \circ \psi)(a \otimes 1) \rightarrow a \otimes 1.$$

Also,

$$\begin{aligned} \|(\text{Ad}(w_n v_n) \circ \varphi \circ \psi)(1 \otimes a) - 1 \otimes a\| &= \|\text{Ad} w_n((\text{Ad} v_n \circ \varphi \circ \psi)(1 \otimes a) - 1 \otimes a)\| \\ &= \|(\text{Ad} v_n \circ \varphi \circ \psi)(1 \otimes a) - 1 \otimes a\| \rightarrow 0. \end{aligned}$$

This proves the assertion. In particular, $\varphi \circ \psi \approx \text{id}_{\mathcal{A} \otimes \mathcal{O}_2}$, as desired. \square

Corollary (Elliott). *The C^* -algebras \mathcal{O}_2 and $\mathcal{O}_2 \otimes \mathcal{O}_2$ are isomorphic.*

Proof. Combine the theorem with Lemma 1. \square

It also follows that the infinite tensor product $\otimes_1^\infty \mathcal{O}_2$ is isomorphic to \mathcal{O}_2 . Indeed, $\otimes_1^\infty \mathcal{O}_2$ is the inductive limit of the sequence

$$\mathcal{O}_2 \rightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \dots,$$

and each of these C^* -algebras is isomorphic to \mathcal{O}_2 by Elliott's theorem. It follows from [6], Theorem 7.2, that every inductive limit of a sequence

$$\mathcal{O}_2 \rightarrow \mathcal{O}_2 \rightarrow \mathcal{O}_2 \rightarrow \dots$$

with unital maps is isomorphic to \mathcal{O}_2 .

References

- [1] O. Bratteli, A. Kishimoto, M. Rørdam and E. Størmer: The crossed product of a UHF-algebra by a shift, *Ergod. Th. & Dynam. Sys.* 13 (1993), 615–626.
- [2] O. Bratteli, G. Elliott, D. Evans and A. Kishimoto: On the classification of C^* -algebras of real rank zero, III: The infinite case. In preparation.
- [3] J. Cuntz: Simple C^* -algebras generated by isometries, *Comm. Math. Phys.* 57 (1977), 173–185.

- [4] G. Elliott: On the classification of C^* -algebras of real rank zero, *J. Reine angew. Math.* 443 (1993), 179–219.
- [5] M. Rørdam: On the structure of simple C^* -algebras tensored with a UHF-algebra, *J. Funct. Anal.* 100 (1991), 1–17.
- [6] M. Rørdam: Classification of inductive limits of Cuntz algebras, *J. reine angew. Math.* 440 (1993), 175–200.
- [7] M. Takesaki: *Theory of operator algebras I*, Springer-Verlag, New York/Heidelberg/Berlin, 1979.

Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark. email: mikael@imada.ou.dk

Received March 7, 1994

IMBEDDING ALGEBRA BUNDLES IN TRIVIAL BUNDLES

BY EDWIN J. BEGGS,
UNIVERSITY COLLEGE, SWANSEA,
WALES SA2 8PP.

Presented by G.A. Elliott, F.R.S.C.

We take the algebra of sections of a locally trivial bundle of algebras over a normal paracompact topological space X . This paper shows that the suspension of such an algebra of sections imbeds in a trivial algebra bundle over X .

INTRODUCTION

The purpose of this paper is to examine the question of whether a locally trivial bundle of algebras over a topological space can be imbedded in a trivial bundle algebra over the space. We will not be able to answer this question, but in terms of algebraic topology, we can answer the next best question: The suspension of the bundle of algebras can be imbedded in a trivial bundle algebra.

If we think of the case of vector space bundles over a topological space, we can imbed such a bundle in a trivial bundle by using a partition of unity. This relies on the fact that multiplication by a scalar is a vector space linear map, and that we can continuously deform the identity to the zero map by such linear maps. In other words, vector spaces are contractible by vector space maps. However not all algebras are contractible by algebra maps, so we cannot do this for algebras in general. However in the case where the fibre algebras are contractible, the answer to the original question is yes.

For a general algebra bundle, we can take the suspension, and this does imbed in a trivial bundle. To do this we map the suspension to the cone algebra, and then use the contractible algebra result.

I have been informed by S. Wassermann [3] that the result is false if 'locally trivial' is omitted. The counter example is constructed in [1, Ex 4.4 (2)], where a continuous bundle algebra A is constructed so that $A \otimes C_0(\mathbb{R})$ is not a sub-bundle of a trivial (or locally trivial) algebra.

I would like to thank G. Elliott for several conversations.

SECTION 1 THE CONTRACTIBLE FIBER CASE

Let X be a normal paracompact topological space. A locally trivial algebra bundle \mathcal{A} over X is given by an open cover U_α ($\alpha \in \mathcal{I}$) of X , and topologised algebra fibers A_α . For every α and β with $U_\alpha \cap U_\beta \neq \emptyset$, there is an algebra isomorphism $\theta_{\alpha\beta} : A_\alpha \rightarrow A_\beta$. Then

$$\mathcal{A} = \left\{ (s_\alpha) \in \prod_\alpha C(U_\alpha, A_\alpha) \mid \theta_{\alpha\beta} s_\alpha = s_\beta \in C(U_\alpha \cap U_\beta, A_\beta) \right\}.$$

We will give the sets of continuous functions the compact open topology. The algebra operations on \mathcal{A} are pointwise addition, pointwise multiplication, and (when defined) pointwise adjoint.

Let us now suppose that all the fibers are contractible, by a homotopy $h_\alpha : A_\alpha \times [0, 1] \rightarrow A_\alpha$, with $h_\alpha(a, 1) = a$ (the identity) and $h_\alpha(a, 0) = 0$. Since X is paracompact, we shall assume that the cover U_α is locally finite. Then take another open cover V_α , with $\overline{V_\alpha} \subset U_\alpha$, and functions $\theta_\alpha : X \rightarrow [0, 1]$, with the support of θ_α contained in U_α , and with θ_α being identically 1 on V_α . Then we define the map

$$F : \mathcal{A} \rightarrow \prod_\alpha C(X, A_\alpha) \quad \text{by} \quad F((s_\alpha)) = \left(x \mapsto \prod_\alpha h_\alpha(s_\alpha(x), \theta_\alpha(x)) \right).$$

Each map $x \mapsto h_\alpha(s_\alpha(x), \theta_\alpha(x))$ can be considered as a continuous function on all of X by defining it to be zero outside U_α . Since each map $a \mapsto h_\alpha(a, t)$ is an algebra homomorphism, the map F is an algebra homomorphism. In addition F is a product of continuous functions, and so is continuous. Also F is 1-1, because

for each x there is an α so that $h_\alpha(s_\alpha(x), \theta_\alpha(x)) = h_\alpha(s_\alpha(x), 1) = s_\alpha(x)$. We have succeeded in imbedding the algebra \mathcal{A} into $\prod_\alpha C(X, A_\alpha)$, which is isomorphic to the trivial bundle algebra $C(X, \prod_\alpha A_\alpha)$.

In the C^* algebra case, we require X to be compact, in which case we can take a finite cover. Then the construction gives a finite direct sum of C^* algebras as the fiber.

SECTION 2 THE SUSPENSION

The suspension SA of an algebra A is defined to be

$$SA = \left\{ s : [0, 1] \rightarrow A \mid s(0) = s(1) = 0 \right\}.$$

The suspension is imbedded in the cone CA of A , which is defined as

$$CA = \left\{ s : [0, 1] \rightarrow A \mid s(1) = 0 \right\}.$$

Further the cone is contractible, by using the homotopy $h(s, t)(x) = 0$ for $x \geq t$ and $h(s, t)(x) = s(1 - t + x)$ for $x \leq t$.

Now let us return to our locally trivial algebra bundle \mathcal{A} over X . The suspension SA is also a locally trivial algebra bundle over X . Its fibers are SA_α , and its transition functions are the suspensions of the $\theta_{\alpha\beta}$. There is also the cone algebra CA , which has fibers CA_α . The suspension SA naturally embeds in CA . Then by the result of section 1, CA imbeds in a trivial algebra $\prod_\alpha C(X, CA_\alpha)$. The net result is that the suspension SA embeds in a trivial algebra bundle.

REFERENCES

- (1) E. Kirchberg & S. Wassermann, 'Operations on continuous bundles of C^* algebras', Preprint, Humboldt University, Berlin & University of Glasgow (1993).
- (2) S. Wassermann, 'Operations on continuous fields of C^* algebras', Lecture, University College, Swansea (18th Feb 1993).
- (3) S. Wassermann, private communication (1994).

Approximation by dimension drop C^* -algebras and classification

Guihua Gong

presented by George Elliott, F.R.S.C.

Abstract

In this note, we would like to announce the following theorem and several related results. If a simple C^* -algebra A of real rank zero is an inductive limit of $\bigoplus_{i=1}^n M_{[n,i]}(C(X_{n,i}))$, where the $X_{n,i}$'s are finite CW complexes with uniformly bounded dimension and with $H^{2l}(X_{n,i})$ torsion free (for $l \geq 2$), then A can be written as an inductive limit of $\bigoplus_{i=1}^n M_{[n,i]}(C(Y_{n,i}))$ with $\dim(Y_{n,i}) \leq 3$. Hence it can be classified by its K-theory. The result is proved by using approximation by dimension drop C^* -algebras.

G. Elliott has initiated a program with the ambitious goal to classify all "amenable" C^* -algebras by invariants like K-theory, tracial data, spectrum, etc.

In this note, we will only consider the class of C^* -algebras of inductive limits of $\bigoplus_{i=1}^n M_{[n,i]}(C(X_{n,i}))$, where the $X_{n,i}$'s are finite CW complexes. This class of C^* -algebras has been intensively studied by many authors.

We would like to announce the following three theorems. (Theorems 1 and 2 are in [G1], Theorem 3 is in [G2].) We will suppose that $A = \lim_{\rightarrow} \bigoplus_{i=1}^n M_{[n,i]}(C(X_{n,i}))$ is of real rank zero in all of the following theorems.

The main result of this note has been presented in AMS meeting, College station, Texas, Oct. 1993

Theorem 1. *If $H^l(X_{n,i})$ is torsion free for every $X_{n,i}$ and $\sup_{n,i} \dim(X_{n,i}) < +\infty$, then A can be written as an inductive limit of direct sums of matrix algebras over $C(S^1)$.*

Theorem 2. *Let $M_{\mathbb{Q}}$ denote the UHF algebra with $K_0(M_{\mathbb{Q}}) = \mathbb{Q}$. Then $A \otimes M_{\mathbb{Q}}$ can be written as an inductive limit of direct sums of matrix algebras over $C(S^1)$.*

Theorem 3. *If A is simple with the slow dimension growth condition*

$$\lim_{n \rightarrow \infty} \frac{\dim X_{n,i}}{[n, i]} = 0$$

and $H^{2l}(X_{n,i})$ are torsion free for $l \geq 2$, then A can be written as an inductive limit of direct sums of matrix algebras over $C(Y_{n,i})$ with $Y_{n,i}$ being 3-dimensional finite CW complexes.

Theorem 1 and Theorem 2 can be proved by the method which was used in the proof of Theorem 5.28 of [EG]. In the proof, we use the following result (which was proved by factoring through $C(S^n)$ and using [EGLP]): any homomorphism from $C(X)$ to $M_k(C(X))$ (with $H^*(X)$ torsion free) is homotopic to a homomorphism whose image of any given set $F \subset C(X)$ can be approximated by matrix algebras over $C(S^1)$ arbitrarily well in a certain sense for k large enough. (Larger k is needed to get better approximation.)

As pointed out in Remark 5.36 of [EG], to deal with a space X with torsion in $H^{2l+1}(X)$, one needs to use the dimension drop C^* -algebras (introduced by G. Elliott) instead of the circle algebras.

Definition 1. *The basic dimension drop C^* -algebra D_k is the subalgebra of $M_k(C[0, 1])$ consisting of $f \in M_k(C[0, 1])$ with $f(0)$ and $f(1)$ scalars. A*

dimension drop C^* -algebra is a direct sum of matrix algebras over basic dimension drop C^* -algebras.

It is well known that $K_0(D_k) = \mathbb{Z}$ and $K_1(D_k) = \mathbb{Z}_k$.

Definition 2. Let $\phi : A \rightarrow B$ be a homomorphism, $\epsilon > 0$ and $F \subset A$ be a finite set. We say that ϕ is approximately contained in a dimension drop C^* -algebra on F within ϵ , if there is a subalgebra B_1 of B which is a dimension drop C^* -algebra (or a quotient of a dimension drop C^* -algebra) such that

$$\text{dist}(\phi(f), B_1) < \epsilon$$

for any $f \in F$.

Since we allow a quotient of a dimension drop C^* -algebra to be used in the definition, one knows that if $\phi : A \rightarrow B$ is approximately contained in a dimension drop C^* -algebra on F within ϵ , then $\psi \circ \phi : A \rightarrow C$ is approximately contained in a circle algebra on F to within ϵ for an arbitrary homomorphism $\psi : B \rightarrow C$.

The following theorem will play the role of Theorem 2 of [EGLP] (or Theorem 1.7.2 of [EG]) in the proof of our Theorem 3.

Theorem 4. Suppose that X is a finite CW complex with $\tilde{H}^{\text{even}}(X) = 0$ and $H^{\text{odd}}(X) = H^{2l+1}(X) = \mathbb{Z}_k$ (i.e., the reduced cohomology group is a torsion group and is supported in only one dimension $2l + 1$). For any $\epsilon > 0$, and any finite set $F \subset C(X)$, there exists an N such that if $n \geq N$, then any unital homomorphism $\phi : C(X) \rightarrow M_n(C(X))$ is homotopic to a unital homomorphism $\psi : C(X) \rightarrow M_n(C(X))$ with ψ approximately contained in a dimension drop C^* -algebra on F to within ϵ .

The above theorem is proved by factoring a certain homomorphism through $D_k \otimes M_{n_1}(C(S^{2l}))$. Notice that a homomorphism from $C(S^{2l})$ to $M_{n_2}(C(S^{2l}))$ (n_2 large enough) is homotopic to a homomorphism which was approximately contained in a finite dimensional C^* -algebra.

Using the above theorem, one can prove the following.

Theorem 5. *If A is as in Theorem 3, then A can be written as an inductive limit of direct sums of dimension drop C^* -algebras and matrix algebras over 2-dimensional finite CW complexes.*

Theorem 3 follows from the next theorem, and Theorem 5.8 and 4.20 of [EG].

Theorem 6. *Suppose that A is simple and of real rank zero. If A can be written as an inductive limit of direct sums of dimension drop C^* -algebras and matrix algebras over 2-dimensional finite CW complexes, then A can be classified by its graded ordered K -group with dimension range.*

This theorem is proved by combining the techniques in [EG] and [Ell].

To remove the restriction of $H^{2l}(X)$ torsion free for $l \geq 2$, one needs to prove a certain analogue of Theorem 4. However one can not use $D_k \otimes C(S^1)$ to replace D_k , for some technical reasons. One may use $C(T_{1l,k})$, where $T_{1l,k}$ are the spaces in [EG] with $H^2(T_{1l,k}) = \mathbb{Z}_k$, and $H^1(T_{1l,k}) = 0$. There are still some difficulties remaining to be overcome for this case.

Acknowledgement This work was done while the author was at Queen's University. He would like to thank Professor J. Mingo for his support.

References

- [E11] Elliott, G. A. The classification of C^* -algebras of real rank zero, J. Reine Angew. Math. 443 (1993), 179-217
- [EG] Elliott, G.A. and Gong, G. On the classification of C^* -algebras of real rank zero, II, preprint.
- [EGLP] Elliott, G., Gong, G., Lin, H. and Pasnicu, C. Homomorphisms, homotopies, and approximations by circle algebras, preprint.
- [G1] Gong, G., On inductive limits of matrix algebras over higher dimensional spaces, I, preprint
- [G2] Gong, G., On inductive limits of matrix algebras over higher dimensional spaces, II, preprint.

Department of Mathematics, University of Toronto, Toronto, Ontario,
M5S 1A1, Canada

Department of Mathematics, University of Puerto Rico, Rio Piedras,
PR 00931, USA

Received March 20, 1994

Homomorphisms, homotopies, and approximations by circle algebras

George A. Elliott, F.R.S.C., Guihua Gong, Huaxin Lin, and Cornel Pasnicu

Abstract

Suppose that X is a product of spheres and $F \subseteq C(X)$ is a finite subset. In this note, we shall prove that any unital homomorphism $\phi : C(X) \rightarrow M_l(C(Y))$ is homotopic to a unital homomorphism $\phi' : C(X) \rightarrow M_l(C(Y))$ such that $\phi'(F)$ is approximately contained in a sub C^* -algebra of $M_l(C(Y))$ of circle type (i.e., a direct sum of matrix algebras over $C(S^1)$ or a quotient of $C(S^1)$), provided that $l/(\dim Y + 1)$ is large enough. This result plays an important role in the classification theory of C^* -algebras of real rank zero. We shall present a classification result for simple real rank zero inductive limits of direct sums of matrix algebras over $C(S^2)$ and $C(T^2)$.

In [EG], the authors constructed the first example of a homomorphism from $C(T^2)$ to $M_k(C(T^2))$, nontrivial on $H^2(T^2) (\subseteq K_*(C(T^2)))$ and such that the images of the generators of $C(T^2)$ under the map are approximately contained in a circle algebra to within an arbitrarily small given tolerance. Using this example and the results in [P] and [P1], they proved that several classes of inductive limits of direct sums of matrix algebras over the 2-torus (or the n -torus) can be written as inductive limits of direct sums of matrix algebras over circles. The class of C^* -algebras of real rank zero which can be expressed as inductive limits of direct sums of matrix algebras over circles is classified in [EII].

In this note, the above mentioned example will serve as a model. Using this model and the results in [DN], we will prove that any homomorphism from $C(T^2)$ to $M_k(C(T^2))$ (k large enough) is homotopic to a homomorphism which is approximately contained in a circle algebra on the generators of $C(T^2)$ to within an arbitrarily small given tolerance (see definition below). Actually, we will prove the theorem not only for T^2 , but also for general tori T^n , spheres S^n , and products of spheres $S^{n_1} \times S^{n_2} \times \dots \times S^{n_k}$. Our result is a bridge from [EG] to the general classification of real rank zero inductive limits of matrix algebras over $C(X)$, with X being arbitrary finite CW complexes with torsion free cohomology groups (see [EG1], [G2-3]). For the case of space X being with torsion in $H^{odd}(X)$, we need the result saying that certain homomorphisms between $C(X)$ and $M_k(C(Y))$ (for k large enough) are approximately contained in a dimension drop C^* -algebra which was proved by the second author (see [G1-2]). A classification result for X being the 2-sphere or the 2-torus will also be presented in this note.

Definition 1. Let $\phi : A \rightarrow B$ be a homomorphism, $\epsilon > 0$ and $F \subseteq A$ a finite set. We shall say that ϕ is approximately contained in a circle algebra on F , to within ϵ , if there is a subalgebra B_1 of B which is isomorphic to a finite direct sum of matrix algebras over $C(S^1)$ (or quotients of $C(S^1)$) such that $\text{dist}(\phi(f), B_1) < \epsilon$ for any $f \in F$.

Since we allow quotients of $C(S^1)$ to be used in the definition, we have that if $\phi : A \rightarrow B$ is approximately contained in a circle algebra on F to within ϵ , then $\psi \circ \phi : A \rightarrow C$ is approximately contained in a circle algebra on F to within ϵ for an arbitrary homomorphism $\psi : B \rightarrow C$.

In this paper, all spaces X, Y are either a sphere S^n or a product of spheres $S^{n_1} \times S^{n_2} \times \dots \times S^{n_k}$. The main purpose of this paper is to prove the following theorem, and certain generalizations of it.

Theorem 2. For any $\epsilon > 0$, any finite set $F \subseteq C(X)$, and a fixed space Y , there exists an N such that if $k \geq N$, then any unital homomorphism $\phi : C(X) \rightarrow M_k(C(Y))$ is homotopic to a unital homomorphism $\psi : C(X) \rightarrow M_k(C(Y))$ with ψ approximately contained in a circle algebra on F to within ϵ .

Let $\text{ch} : K^*(S^n) \rightarrow H^*(S^n, \mathbb{Q})$ denote the Chern map. It is well known (see Theorem 6.1 of [Bo]) that

$$\text{image}(\text{ch}) \subseteq H^*(S^n, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \subseteq \mathbb{Q} \oplus \mathbb{Q} = H^*(S^n, \mathbb{Q}).$$

Furthermore, the map ch induces an isomorphism $K^*(S^n) \rightarrow H^*(S^n, \mathbb{Z})$ (We still denote it by ch). If $X = S^{n_1} \times S^{n_2} \times \dots \times S^{n_k}$, by using the Kunnetth formula both for K -theory and cohomology, it is easy to see that the Chern map induces an isomorphism $ch: K^*(X) \rightarrow H^*(X, \mathbb{Z})$. For convenience, we shall always identify $K_0(M_k(C(X))) = K_0(C(X)) = K^0(X)$ with $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus \dots$ and $K_1(M_k(C(X))) = K_1(C(X)) = K^1(X)$ with $H^1(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z}) \oplus \dots$. To avoid confusion we shall use $KH^n(X)$ to denote $H^n(X, \mathbb{Z})$ to indicate that we identify $H^n(X, \mathbb{Z})$ with a subgroup of $K^*(X)$. In this way, $\psi: M_{k_1}(C(X)) \rightarrow M_{k_2}(C(Y))$ induces the group homomorphisms $\psi_*: KH^{n_1}(X) \rightarrow KH^{n_2}(Y)$, where n_1 and n_2 need not be the same, but $n_1 - n_2$ must be even.

The following result is a consequence of Lemma 3.20 of [EG].

Lemma 3. *Let $X = Y = T^n$. For any $\epsilon > 0$ and any finite set $F \subseteq C(X)$, there exists an N such that when $k \geq N$, there is a unital homomorphism $\psi: C(X) \rightarrow M_k(C(Y))$ with the following properties:*

- (i) ψ is approximately contained in a circle algebra on F to within ϵ ;
- (ii) $\psi_*: KH^n(T^n) = \mathbb{Z} \rightarrow KH^n(T^n) = \mathbb{Z}$ is the identity.

Lemma 4. *Theorem 2 is true for $X = S^n, Y = S^n$.*

Proof. By Theorem 3.4.5 and Theorem 6.4.4 of [DN], there is a unital homomorphism

$$\phi_1: C(T^n) \rightarrow M_l(C(S^n))$$

with $\phi_{1*}: KH^n(T^n) = \mathbb{Z} \rightarrow KH^n(S^n) = \mathbb{Z}$ equal to the identity map when l is large enough (for example, $l \geq 2n$). Let $\phi_2: C(S^n) \rightarrow C(T^n)$ denote the homomorphism induced by the canonical continuous map $\alpha: T^n \rightarrow S^n$, i.e. α is the map obtained by identifying the closed subset $(\{pt\} \times T^{n-1}) \vee (T^1 \times \{pt\} \times T^{n-2}) \vee (T^2 \times \{pt\} \times T^{n-3}) \vee \dots \vee (T^{n-1} \times \{pt\})$ of T^n to a single point, where $\{pt\}$ is the set consisting of a single point in $T^1 = S^1$. It is well known that $\phi_{2*}: KH^n(S^n) = \mathbb{Z} \rightarrow KH^n(T^n) = \mathbb{Z}$ is the identity map. For any finite set $F \subseteq C(S^n)$ and $\epsilon > 0$, consider $\phi_2(F) \subseteq C(T^n)$ and $\epsilon > 0$ in Lemma 3; one can find an N_1 and a $\phi_3: C(T^n) \rightarrow M_{N_1}(C(T^n))$ as in Lemma 3. We choose our $N = N_1 l$. For any $k \geq N$, we may take $\psi: C(S^n) \rightarrow M_k(C(S^n))$ to be the direct sum of $(\phi_1 \otimes id_{N_1}) \circ \phi_3 \circ \phi_2$ and certain point evaluations, where id_{N_1} is the identity map from $M_{N_1}(C)$ to itself.

Lemma 5. *Let $X = S^{n_1}, Y = S^{n_2}$, with $0 < n_1 \leq n_2$ and $n_2 - n_1$ even. For any $\epsilon > 0$ and any finite set $F \subseteq C(S^{n_1})$, there exists an N such that when $k \geq N$, there is a unital homomorphism $\psi: C(S^{n_1}) \rightarrow M_k(C(S^{n_2}))$ with the following properties:*

- (i) ψ is approximately contained in a circle algebra on F to within ϵ ;
- (ii) $\psi_*: KH^{n_1}(S^{n_1}) = \mathbb{Z} \rightarrow KH^{n_2}(S^{n_2}) = \mathbb{Z}$ is the identity map.

Proof. There is a unital homomorphism $\phi: C(S^{n_1}) \rightarrow M_l(C(S^{n_2}))$ such that $\phi_*: KH^{n_1}(S^{n_1}) \rightarrow KH^{n_2}(S^{n_2})$ is the identity map when l is large enough. (This follows from the fact that $F_{k+2}K_*(A) \subseteq F_k K_*(A)$ in [EL], or see Theorem 3.5.5 of [DN].) Applying Lemma 4 to $C(S^{n_2})$, it is easy to see that we can choose $N = l \cdot N_1$, where N_1 is the N in Lemma 4 for $\phi(F)$ and ϵ (of course the map ψ can be chosen to be the composition of ϕ with the map in Lemma 4).

Remark 6. For any group homomorphism α from \mathbb{Z} to \mathbb{Z} , one may require $\psi_* = \alpha$ instead of $\psi_* = \text{identity}$ in (ii) of Lemmas 3, 4 and 5, without changing the integer N . This can be done by combining the homomorphism in the lemmas with the homomorphism from $M_k(C(Y))$ to itself induced by a degree $\alpha(1)$ map from Y to Y .

Let $X = S^{n_1} \times S^{n_2} \times \dots \times S^{n_k}$. Then $K^*(X) = \mathbb{Z}^{2^k}$ and $KH^n(X) = \mathbb{Z}^{\tau(n)}$, where $\tau(n)$ is the cardinal number of the set

$$\{(n_{i_1}, n_{i_2}, \dots, n_{i_j}) | n_{i_1} + n_{i_2} + \dots + n_{i_j} = n, (i_1, i_2, \dots, i_j) \subseteq \{1, 2, \dots, k\}\}.$$

More precisely, $KH^n(X)$ is the direct sum of several standard \mathbb{Z} -summands and each standard \mathbb{Z} -summand is the pull-back of the top dimensional cohomology $KH^n(S^{n_{i_1}} \times S^{n_{i_2}} \times \dots \times S^{n_{i_j}})$ via the canonical projection $\pi_{(i_1, i_2, \dots, i_j)}: S^{n_1} \times S^{n_2} \times \dots \times S^{n_k} \rightarrow S^{n_{i_1}} \times S^{n_{i_2}} \times \dots \times S^{n_{i_j}}$. Let $\alpha: S^{n_{i_1}} \times S^{n_{i_2}} \times \dots \times S^{n_{i_j}} \rightarrow S^n$ be the canonical degree 1 map between the two manifolds, i.e., the map obtained by identifying the closed subset

$$(\{pt\} \times S^{n_2} \times \dots \times S^{n_k}) \vee (S^{n_1} \times \{pt\} \times \dots \times S^{n_k}) \vee \dots \vee (S^{n_1} \times S^{n_2} \times \dots \times \{pt\})$$

to a single point. Then α induces an isomorphism from $KH(S^n) \rightarrow KH^n(S^{n_{i_1}} \times S^{n_{i_2}} \times \dots \times S^{n_{i_j}})$. Hence $\alpha \circ \pi_{(i_1, i_2, \dots, i_j)}: X \rightarrow S^n$ induces a homomorphism which takes $KH^n(S^n) = \mathbb{Z}$ to that

standard \mathbb{Z} -summand of $KH^n(X)$ (i.e., the \mathbb{Z} -summand corresponding to the subset (i_1, i_2, \dots, i_j) of $(1, 2, \dots, k)$). Therefore we have proved that, for each standard \mathbb{Z} -summand of $KH^n(X)$, there is a homomorphism $C(S^n)$ to $C(X)$ taking $KH^n(S^n) = \mathbb{Z}$ to that \mathbb{Z} -summand.

On the other hand, by using Theorem 3.5.5 of [DN], for a \mathbb{Z} -summand of $KH^n(X)$ ($n > 0$) (denote it by $G = \mathbb{Z} \subseteq KH^n(X)$), there is a unital homomorphism $\phi : C(X) \rightarrow M_k(C(S^n))$ such that

(i) $\phi_* : G = \mathbb{Z} \rightarrow KH^n(X) = \mathbb{Z}$ is the identity, and $\phi_* : KH^0(X) = \mathbb{Z} \rightarrow KH^0(S^n) = \mathbb{Z}$ is the map of multiplying by k which is guaranteed by ϕ being a unital homomorphism,

(ii) ϕ_* maps all the other \mathbb{Z} -summands of $K_*(C(X))$ into zero.

Combining the above argument and Lemma 5, we have proved

Lemma 7. *Let G_1 be a \mathbb{Z} -summand of $KH^{n_1}(X)$ and G_2 , a \mathbb{Z} -summand of $KH^{n_2}(Y)$, where $n_2 \geq n_1 > 0$ and $n_2 - n_1$ is even. For any $\epsilon > 0$ and a finite set $F \subseteq C(X)$, there exists an N such that when $k \geq N$, there is a unital homomorphism $\psi : C(X) \rightarrow M_k(C(Y))$ with the following properties:*

(i) ψ is approximately contained in a circle algebra on F to within ϵ ;

(ii) ψ_* maps G_1 to G_2 identically, and maps all the other \mathbb{Z} -summands of $K_*(C(X))$ to zero except $KH^0(C(X)) = \mathbb{Z}$.

As in Remark 6, we can also require ϕ_* from G_1 to G_2 to be any given group homomorphism.

From Lemma 7, and Theorem 3.5.5 of [DN], we know that when k is large enough, for any unital homomorphism $\phi : C(X) \rightarrow M_k(C(Y))$ there is a unital homomorphism $\psi : C(X) \rightarrow M_k(C(Y))$ such that

(i) ψ is approximately contained in a circle algebra on F to within ϵ ,

(ii) $\psi_* = \phi_*$ on K -theory.

By Theorem 3.5.5 and Remark 6.4.5 of [DN], it follows from (ii) that ψ and ϕ are homotopic.

This proves Theorem 2.

The following is a generalization of Theorem 2.

Theorem 8. *For any $\epsilon > 0$, any finite set $F \subseteq C(X)$, and a fixed space Y , there exists an N such that if $P < P'$ are two projections of $M_k(C(Y))$ with $\text{rank}(P) \geq N$, then for any unital homomorphism $\phi : C(X) \rightarrow P'M_k(C(Y))P'$, there is a unital homomorphism $\psi : C(X) \rightarrow PM_k(C(Y))P$ which is approximately contained in a circle algebra on F to within ϵ , such that ϕ is homotopic to $\psi \oplus \psi'$, where $\psi' : C(X) \rightarrow (P' - P)M_k(C(Y))(P' - P)$ is defined by $\psi'(f) = f(x_0)(P' - P)$, i.e., $\psi'(f)(y) = (P' - P)(y)f(x_0)(P' - P)(y)$. Sometimes, we say that ψ' is a homomorphism defined by point evaluation.*

Proof. P contains a trivial projection $Q \in M_k(C(Y))$ with $\text{rank}(Q) = \text{rank}(P) - \dim(Y)$. So when $\text{rank}(P)$ is large enough, one can construct a unital homomorphism $\psi : C(X) \rightarrow QM_k(C(Y))Q = M_{\text{rank}(Q)}(C(Y))$ such that $\psi_* = \phi_*$ on all the $KH^n(X)$ except $KH^0(X)$ and ψ_1 is approximately contained in a circle algebra on F to within ϵ . Then set $\psi = \psi_1 \oplus \psi_2$ with $(\psi_2 f)(y) = (P - Q)(y)f(x_0)(P - Q)(y)$ for a fixed $x_0 \in X$. Hence $(\psi \oplus \psi')_* = \phi_*$ on all the K -theory.

In order to prove that ϕ is homotopic to $\psi \oplus \psi'$, we need to generalize Corollary 6.4.4 of [DN] to the following assertion: If Y is a finite CW complex of dimension n and $P \in M_k(C(Y))$ is a projection, then the natural map

$$[C_0(X), PM_k(C_0(Y))P] \rightarrow kk(Y, X)$$

is a bijection when $\text{rank}(P) \geq 3n + 3$.

To prove this assertion, by the original corollary, one only needs to prove that

$$[C_0(X), QM_k(C_0(Y))Q] = [C_0(X), M_{\text{rank}(Q)}(C_0(Y))] \rightarrow [C_0(X), PM_k(C_0(Y))P]$$

is a bijection, where Q is a trivial subprojection of P of rank equal to $\text{rank}(P) - n$.

For each fixed point $y \in Y$,

$$\text{Hom}(C_0(X), P(y)M_k P(y)) = \text{Hom}(C_0(X), M_{\text{rank}(P)}).$$

Therefore, $\text{Hom}(C_0(X), PM_k(C_0(Y))P)$ can be regarded as the space of cross sections ($= 0$ at the base point $y_0 \in Y$) of a fibre bundle over Y with fibre equal to $\text{Hom}(C_0(X), M_{\text{rank}(P)}) = F_0^{\text{rank}(P)}(X)$

(see page 276 of [DN] for $F_0^r(X)$). Then $Hom(C_0(X), QM_k(C_0(Y))Q)$ corresponds to the cross section of a sub fibre bundle of the above bundle over Y with fibre $F_0^{rank(Q)}(X)$. It was proved in [DN] that $(F_0^{rank(P)}(X), F_0^{rank(Q)}(X))$ is an $n + 1$ connected CW complex pair. Since $dim(Y) = n$, it is routine to prove that any cross section of the total fibre bundle is homotopic to a cross section of the sub fibre bundle (e.g., using Lemma 3.1 of Chapter II of [Wh]), construct the homotopy lifting cell by cell). Therefore,

$$[C_0(X), QM_k(C_0(Y))Q] \rightarrow [C_0(X), PM_k(C_0(Y))P]$$

is a surjection. Since the composition

$$[C_0(X), QM_k(C_0(Y))Q] \rightarrow [C_0(X), PM_k(C_0(Y))P] \rightarrow [C_0(X), M_k(C_0(Y))]$$

is a bijection, we have proved the assertion. Theorem 8 follows from the assertion as before.

The following theorem will be used to study inductive limit systems with the dimension of the space unbounded (but with the slow dimension growth property; see [EG1]).

Theorem 9. For any $\epsilon > 0$ and any finite set $F \subseteq C(X)$, there exists an N such that for Y an arbitrary sphere or product of spheres, if $P' > P \in M_k(C(Y))$ with $rank(P) \geq [3dim(Y) + 3]N$, then for any unital homomorphism $\phi : C(X) \rightarrow P'M_k(C(Y))P'$, there exists a unital homomorphism $\psi : C(X) \rightarrow PM_k(C(Y))P$ with ψ approximately contained in a circle algebra on F to within ϵ , such that ϕ is homotopic to $\psi \oplus \psi'$, where $\psi'(f) = f(x_0)(P' - P)$ for a fixed $x_0 \in X$.

Proof. Applying Lemma 7 for both spaces to be X , one can find an N and a unital homomorphism $\psi_1 : C(X) \rightarrow M_N(C(X))$ with the following properties:

- (i) ψ_1 is approximately contained in a circle algebra on F to within ϵ ;
- (ii) $\psi_{1*} : KH^n(X) \rightarrow KH^n(X)$ are identity except on $KH^0(X)$.

By the argument in the proof of Theorem 8, we can find a unital homomorphism $\psi_2 : C(X) \rightarrow M_{2dim(Y)+2}(C(Y))$ such that $\psi_{2*} = \phi_*$ on $KH^n(X)$ except on $KH^0(X)$. It should be noticed that $(\psi_2 \otimes id_N)_* = \psi_{2*} : KH^n(X) \rightarrow KH^n(Y)$. One can embed $M_{2dim(Y)+2}(C(Y))$ into $PM_k(C(Y))P$ (denote the map by β). Define

$$\psi = \beta \circ (\psi_2 \otimes id_N) \circ \psi_1 \oplus \psi_3,$$

where $\psi_3(f)(y) = (P - \beta(1))(y)f(x_0)(P - \beta(1))(y)$ for a fixed $x_0 \in X$. Then $(\psi \oplus \psi')_* = \phi_*$ on all the K-theory $KH^n(X)$ (it is true on $KH^0(X)$ since both maps are unital). Hence $\psi \oplus \psi'$ is homotopic to ϕ .

Corollary 10. For any $\epsilon > 0$ and any finite set $F \subseteq M_n(C(X))$, there exists an N such that for Y an arbitrary sphere or product of spheres, if $P' > P \in M_k(C(Y))$ with $rank(P) \geq (4dim(Y) + 3)N$, then for any unital homomorphism $\phi : M_n(C(X)) \rightarrow P'M_k(C(Y))P'$, there is a unital homomorphism $\psi : M_n(C(X)) \rightarrow QM_k(C(Y))Q$ with $Q \leq P$ and ψ approximately contained in a circle algebra on F to within ϵ such that ϕ is homotopic to $u(\psi \oplus \psi')u^*$ for some unitary $u \in P'M_k(C(Y))P'$, where $\psi'(f) = f(x_0)(P' - Q)$ for a fixed $x_0 \in X$.

Proof. Let $F_1 \subseteq C(X)$ be a finite set of generators of $C(X)$. Then each element in $M_n(C(X))$ can be approximated arbitrarily well by a polynomial in elements of F_1 with the coefficients in $M_n(C)$. Therefore for any $\epsilon > 0$, there is a $\delta > 0$, such that if $\phi : C(X) \rightarrow A$ is approximately contained in a circle algebra on F_1 to within δ , then $\phi \otimes 1_n : M_n(C(X)) \rightarrow M_n(A)$ is approximately contained in a circle algebra on F to within ϵ . Applying Theorem 9 to F_1 and δ , find an N_1 satisfying the conclusion of Theorem 9. Set $N = nN_1$. Let $\phi : M_n(C(X)) \rightarrow P'M_k(C(Y))P'$ be any unital homomorphism and $P' > P$ with $rank(P) \geq (4dim(Y) + 3)N$. There is a trivial subprojection Q of P with $rank(Q) = (3dim(Y) + 3)N$.

Let $p_1 = \phi(e_{11})$, where e_{ij} are the matrix units. And let $\phi_1 = \phi|_{e_{11}M_n(C(X))e_{11}} : C(X) \rightarrow p_1M_k(C(Y))p_1$. Then one may identify $P'M_k(C(Y))P'$ with $M_n(p_1C(Y)p_1)$, and ϕ with $\phi_1 \otimes 1_n$. Since $rank(p_1) = \frac{rank(P)}{n} \geq (4dimY + 3)N_1$, there is a trivial subprojection q_1 of p_1 with $rank(q_1) = (3dimY + 3)N_1$. Then by applying Theorem 9 to ϕ_1 and $p_1 > q_1$, one gets $\psi_1 : C(X) \rightarrow q_1M_k(C(Y))q_1$ and ψ'_1 satisfying the conclusion of Theorem 9 for F_1 and δ . Hence $\psi_1 \otimes 1_n$ and $\psi'_1 \otimes 1_n$ satisfy the conclusion of the Corollary for F and ϵ but with respect to projections $P' (= p_1 \otimes 1_n) > \tilde{Q} (= q_1 \otimes 1_n)$.

However, \tilde{Q} is unitarily equivalent to Q and $P' - \tilde{Q}$ is unitarily equivalent to $P' - Q$. This completes the proof.

Theorem 11. *Let A be the inductive limit C^* -algebra of the system (A_i, ϕ_{ij})*

$$A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A$$

where each A_i is a finite direct sum of $M_{[i,k]}C(X_{i,k})$, with each $X_{i,k}$ a product of finitely many spheres. Suppose that A is simple, unital, and of real rank zero. Furthermore, suppose that the inductive limit system satisfies the slow dimension growth condition, i.e.,

$$\lim_{i \rightarrow \infty} \max_k \frac{\dim(X_{i,k})}{[i,k]} = 0.$$

Then for any finite subset F of any A_i , any $\epsilon > 0$, and any projection $P \in A$, there are a projection $Q < P$ and two unital homomorphisms $\psi : A_i \rightarrow PAP$ and $\psi' : A_i \rightarrow (1 - Q)A(1 - Q)$ such that

- (i) ψ' has finite dimensional image,
- (ii) ψ is approximately contained in a circle algebra on F to within ϵ ,
- (iii) there is a unitary $u \in A$, such that $u(\psi \oplus \psi')u^*$ is homotopic to the canonical homomorphism $\phi_i : A_i \rightarrow A$

Proof. It should be noticed that if the theorem is true for a subprojection of P in place of P , then it is true for P . Furthermore, if the theorem is true for a projection q in place of P with $[q] < [P]$ in $K_0(A)$, then the theorem is true for P , since a unitary conjugacy is allowed.

Let $A_i = \bigoplus_{k=1}^j M_{[i,k]}C(X_{i,k})$ and set $\phi_i^k = \phi_i|_{M_{[i,k]}C(X_{i,k})}$, the restriction of $\phi_i : A_i \rightarrow A$ to the k^{th} -summand. Let $p_k (1 \leq k \leq j)$ denote the image of $1 \in M_{[i,k]}C(X_{i,k})$ under ϕ_i . Then $p_1 + p_2 + \cdots + p_j = 1 \in A$. If for some k , $p_k = 0$, then $\phi_i^k : M_{[i,k]}C(X_{i,k}) \rightarrow A$ is the zero map. So we may simply drop this copy from A_i . Without loss of generality, we may assume that $p_k \neq 0$ for all $1 \leq j \leq k$.

Since A is simple and of real rank zero (also suppose it is not $M_n(\mathbb{C})$), A has the following properties:

- (1) For any $q \in A$, and integer $K > 0$, there is a projection $0 \neq q_1 < q$ such that $K[q_1] < [q]$.
- (2) For any two nonzero projections q_1, q_2 , there is an integer K such that $K[q_1] > [q_2]$.

By using these two properties, it is routine to prove that there are projections q_1, q_2, \dots, q_j with the properties $q_k < p_k$, and $[q_1 \oplus q_2 \oplus \cdots \oplus q_j] < [p]$. Furthermore, we can choose q_k in A_m for m large enough. Let $\pi_k : A_i \rightarrow M_{[i,k]}C(X_{i,k})$ denote the canonical projection, and set $F_k = \pi_k(F) \subseteq M_{[i,k]}C(X_{i,k})$. If the theorem is true for each $\phi_i^k = M_{[i,k]}C(X_{i,k})$, $p_k A p_k$, finite set $F_k \subseteq M_{[i,k]}C(X_{i,k})$, $\frac{\epsilon}{j} > 0$, and the projection $q_k \in p_k A p_k$, then the theorem is true for $\phi_i : A_i \rightarrow A$, the finite set $F \subseteq A_i$, $\epsilon > 0$, and the projection $q_1 \oplus q_2 \oplus \cdots \oplus q_j$. In other words, the theorem follows.

Let us prove the theorem for ϕ_i^1 and projection q_1 . First, there is a $M > 0$ such that $M[q_1] > 1$ in A_m . For the space $X_{i,1}$, a finite set $F_1 \subseteq M_{[i,1]}C(X_{i,1})$ and $\frac{\epsilon}{j} > 0$, there is an $N > 0$ satisfying the conclusion of Corollary 10. Since $\lim_{l \rightarrow \infty} \max_k \frac{\dim(X_{l,k})}{[l,k]} = 0$, and $[l,k] \rightarrow \infty$ as $l \rightarrow \infty$ (see [GL]), there is an $l > m (> i)$ such that $\max_k \frac{4\dim(X_{l,k})+3}{[l,k]} < \frac{1}{NM}$. Let $\tilde{\phi}_k$ denote the partial map of $\phi_{i,l} : A_l \rightarrow A_l$ corresponding to the summands $M_{[i,1]}C(X_{i,1})$ and $M_{[l,k]}C(X_{l,k})$, i.e., the cut-down of the map $\phi_{i,l}|_{M_{[i,1]}C(X_{i,1})}$ to the k^{th} summand $M_{[l,k]}C(X_{l,k})$ of A_l . Let p_1^k and q_1^k denote the cut-down of p_1 and q_1 into $M_{[l,k]}C(X_{l,k})$ respectively. Then $p_1^k > q_1^k$, and ϕ_k is a unital homomorphism from $M_{[i,1]}C(X_{i,1})$ to $p_1^k M_{[l,k]}C(X_{l,k}) p_1^k$. Since $M[q_1] > [1]$ in A_l and $M_{[l,k]}C(X_{l,k})$ is a direct summand of A_l , $M[q_1^k] > [1]$ in $M_{[l,k]}C(X_{l,k})$. Hence $\text{rank}(q_1^k) > \frac{1}{NM}[l,k] \geq 4(\dim X_{l,k} + 3)N$. Applying Corollary 10 to $\tilde{\phi}_k$ and the projection q_1^k , one can find $\psi_1^k : M_{[i,1]}C(X_{i,1}) \rightarrow (q_1^k)' M_{[l,k]}C(X_{l,k}) (q_1^k)'$ (with $(q_1^k)' < q_1^k$) and $(\psi_1^k)'' : M_{[i,1]}C(X_{i,1}) \rightarrow (p_1^k - (q_1^k)') M_{[l,k]}C(X_{l,k}) (p_1^k - (q_1^k)')$ with ψ_1^k approximately contained in a circle algebra on F_1 to within $\frac{\epsilon}{j}$ and $(\psi_1^k)''$ is as in Corollary 10 and therefore with finite dimensional range. Furthermore, $\tilde{\phi}_k$ is homotopic to $(\psi_1^k) \oplus (\psi_1^k)''$. Set $\psi_1 = \bigoplus \psi_1^k$ and $\psi_1' = \bigoplus (\psi_1^k)''$; then ψ_1 and ψ_1' are the homomorphisms we need for

$$\phi_i^1 : M_{[i,1]}C(X_{i,1}) \rightarrow A_l \rightarrow A,$$

with respect to ξ and the projection q_1 . This completes the proof of the theorem.

The following theorem was stated in an early version of [EGLP].

Theorem 12. *Let $A = \lim_{\leftarrow} (A_n, \phi_n)$ be a simple C^* -algebra of real rank zero, where*

$$A_n \cong \sum_{i=1}^{m(n)} \oplus C(X_n^{(i)}) \otimes M_{n(i)},$$

where each $X_n^{(i)}$ is homeomorphic to T^2 or S^2 . Suppose that $K_0(A)$ has finite rank. Then A is isomorphic to an inductive limit of finite direct sums of $C(S^1) \otimes M_k$. In particular, these algebras are uniquely determined by their graded order-unit groups $K_*(A)$ (up to isomorphism).

In the proof of the above theorem in the earlier version of [EGLP], we used the following lemma, the proof of which was however missing.

Lemma 13. *Let A be a separable simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ of finite rank. Suppose that X is homeomorphic to the space T^2 or S^2 and $\phi : C(X) \rightarrow A$ is a monomorphism which is homotopy trivial. Then for any $\epsilon > 0$ and any finite subset $g_1, g_2, \dots, g_k \in C(X)$ there are mutually orthogonal projections $p_1, p_2, \dots, p_N \in A$ and $\xi_1, \xi_2, \dots, \xi_N \in X$ such that*

$$\|\phi(g_s) - \sum_{i=1}^N g_s(\xi_i) p_i\| < \epsilon,$$

$s = 1, 2, \dots, k$.

The above lemma has been proved by Lin[L] and Bratteli-Elliott-Evans-Kishimoto independently.

To prove the above Theorem 12, one needs to use Lemma 3.8, Theorem 3.10 and Lemma 4.4 of [EGLP] together with the above Theorem 11 and Lemma 13. We omit the details.

Acknowledgement. The second author would like to thank Professor J. Mingo for his support.

References

- [Bo] Bott, R. Lectures on $K(X)$, W.A. Benjamin Inc., New York, 1969.
- [DN] Dadarlat, M. and Nemethi, A. Shape theory and (connective) K -theory, *J. Operator Theory* 23 (1990), 207-291.
- [Eil] Elliott, G. A. The classification of C^* -algebras of real rank zero, *J. Reine Angew. Math.* 443 (1993), 179-217.
- [EG] Elliott, G. A. and Gong, G. On inductive limits of matrix algebras over two-tori, preprint.
- [EG1] Elliott, G.A. and Gong, G. On the classification of C^* -algebras of real rank zero, II, preprint.
- [EGLP] Elliott, G., Gong, G., Lin, H. and Pasnicu, C. Abelian C^* -subalgebras of C^* -algebras of real rank zero and inductive limit C^* -algebras, preprint.
- [EL] Exel, R. and Loring, T. A. Extending cellular cohomology to C^* -algebras, *Trans. Amer. Math. Soc.* 329 (1992), 141-160.
- [G1] Gong, G. Approximations by dimension drop C^* -algebras and classification, preprint.
- [G2] Gong, G. On inductive limits of matrix algebras over higher dimensional spaces, I and II, preprint
- [G3] Gong, G. On inductive limits of matrix algebras over higher dimensional spaces, III, in preparation.
- [L] Lin, H. Homomorphisms from $C(X)$ into C^* -algebras, preprint.
- [P] Pasnicu, C. On certain inductive limit C^* -algebras, *Indiana University Math. J.* 35 (1986), 269-288.
- [P1] Pasnicu, C. Homomorphisms compatible with covering maps of the two-torus, *C. R. Math. Rep. Acad. Sci. Canada*, 14 (1992), 143-147.
- [Wh] Whitehead, G. W. *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.

Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 1A1, Canada
 Department of Mathematics, University of Puerto Rico, Rio Piedras, PR 00931, USA
 Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14214, USA
 Department of Mathematics, University of Puerto Rico, Rio Piedras, PR 00931, USA

Mailing Addresses

- | | |
|------------------------|---|
| M.Y. Antimirov | Department of Applied Mathematics
Riga Technical University
Riga, Latvia, LV 1010 |
| R.J. Ashton | Department of Mathematics
Imperial College, 180 Queen's Gate
London, SW7 2BZ, England |
| E.J. Beggs | University College of Swansea
Swansea, Wales SA2 8PP, U.K. |
| J.H.H. Chalk | Department of Mathematics
Imperial College, 180 Queen's Gate
London, SW7 2BZ, England |
| S.C. Chang | Department of Mathematics
Brock University
St. Catharines, Ontario Canada L2S 3A1 |
| Q. Chen | Suzhou Institute of Urban Construction
and Environmental Protection
Suzhou 215008, P.R. China |
| M. Dossa | Faculté des Sciences,
B.P. 812 Yaounde, Cameroun |
| G.A. Elliott | Department of Mathematics
University of Toronto
Toronto, Ontario, Canada M5S 1A1 |
| G. Gong | Department of Mathematics
University of Puerto Rico
Rio Pedras, PR 00931, USA |
| E. Görlich | Lehrstuhl A für Mathematik
RWTH Aachen
D52056 Aachen, Germany |
| K. Karoui | Department of Mathematics
University of Ottawa
Ottawa, Ontario, Canada K1N 6N5 |
| A.A. Kolyshkin | Department of Applied Mathematics
Riga Technical University
Riga, Latvia, LV 1010 |
| H. Lin | Department of Mathematics
SUNY at Buffalo
Buffalo, New York 14214, USA |
| C. Pasnicu | Department of Mathematics
University of Puerto Rico
Rio Pedras, PR 00931, USA |
| A.P. Rohs | Lehrstuhl A für Mathematik
RWTH Aachen
D52056 Aachen, Germany |
| M. Rørdam | Department of Mathematics & Computer Science
Odense University, Campusvej 55
Odense M, Denmark |
| R. Vaillancourt | Department of Mathematics
University of Ottawa
Ottawa, Ontario, Canada K1N 6N5 |