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QUOTIENTS OF MOMENT FUNCTIONS.

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Abstract: A formula is found giving the quotient of two moment functions as a moment function.

1. Introduction. I have been trying to prove that L(s), the maximum eigenvalue of the integral equation

(1) $$L(s)\phi(x,s) = \int_0^1 \phi(y,s)\,(1-sxy)^{-1}\,dy$$

is a moment function of the parameter <u>s</u> in the sense that

(2) $$L(s) = \int_0^1 m(t)(1-st)^{-1}dt$$

with the weight distribution, m(t), non-negative. This stems from an earlier paper, (1) which considered an integral equation, differing from equation (1) above only in having the integration from -1 to 1. This paper contained the history of the problem, and presented a number of conjectures based on computer data. These conjectures are equally plausible for equation (1).

If the solution $\phi(x,s)$ is normalized so that $\phi(0,s)=1$ for all <u>s</u>, by putting s=0 in equation (1) we obtain

(3) $$L(s) = \int_0^1 \phi(y,s)\,dy.$$

If we can express $\phi(y,s)$ as a moment function with weight w(y,t), and if it is legitimate to reverse the order of integration, we shall be able to express L(s) as a moment function.

By a method involving iteration, a sequence of functions, $\phi_n(x,s)$, can be found that increasingly approximate to a solution of equation (1). I have been able to express the earliest members, at any rate, of this sequence as moment functions. To obtain normalized functions it is necessary to consider $\phi_n(x,s)/\phi_n(0,s)$. It is then desirable to express this quotient of moment functions as a single moment function.

2. Theorem. If, for summable functions $W(t)$ and $w(t)$ we have

$$F(z) = \int_0^1 W(t)\ (z-t)^{-1}\ dt \tag{4}$$

$$f(z) = \int_0^1 w(t)\ (z-t)^{-1}\ dt \tag{5}$$

$$\theta(z) = F(z)/[zf(z)] \tag{6}$$

then

$$\theta(z) = I(z) \tag{7}$$

where

$$I(z) = \int_0^1 W^*(t)/(z-t)\ dt \tag{8}$$

with

$$W^*(t) = \frac{W(t)q(t) - w(t)p(t)}{t(q(t)^2 + \pi^2 w(t)^2)}, \tag{9}$$

where $p(z)$ and $q(z)$ are given by the right-hand sides of (4) and (5) respectively, treated as principal integrals.

The following conditions must be satisfied;-

(a) the function $W^*(t)/(z-t)$ is summable,

(b) neither $\theta(z)$ nor $I(z)$ has an <u>unbranched</u> singularity in the closed interval of reals, [0,1].

3. Standard results. The functions defined by equations (4) and (5) are analytic in the plane with a cut from 0 to 1. If a real value, z, in (0,1) is reached by continuation from above, $F(z)$ takes the value $p(z) - i\pi W(z)$; approached from below $F(z)$ takes the value $p(z) + i\pi W(z)$. When $F(z)$ is known, $W(z)$ can be found by considering the difference of the values above and below. (2). Similarly $f(z)$ has imaginary parts involving $w(t)$, and $q(z)$ as a real part.

4.) Weight formula. If $\theta(z)$ can be expressed as a moment function with a mass distribution in $(0,1)$, the considerations in section (3) show that $-2\pi i\, W^*(z)$ must equal $\theta_+(z) - \theta_-(z)$, where $\theta_+(z)$ represents the value reached by continuation, from above, to z on the real axis between 0 and 1, and $\theta_-(z)$ represents the value reached by continuation from below. Then

$$(10) \qquad \theta_+(z) = F_+(z)/[zf_+(z)] = \frac{p(z) - i\pi W(z)}{z[q(z) - i\pi w(z)]} .$$

$\theta_-(z)$ is given by a similar expression, with $-i$ replaced by $+i$. Taking the difference of the two expressions, simplifying and dividing by $-2\pi i$, we find

$$(11) \qquad W^*(z) = \frac{W(z)q(z) - w(z)p(z)}{z[q(z)^2 + \pi^2 w(z)^2]} .$$

(5.) Proof.

4.) We have shown that <u>if</u> $\theta(z)$ can be obtained as a moment function of the type shown in equation (8), then the weight must be given by $W^*(t)$. It remains to investigate whether such a representation is possible.

$F(z)$, $f(z)$ and $I(z)$ are analytic in the plane cut from 0 to 1. In this region $\theta(z)$ could have a singularity only where $f(z)=0$, which is impossible, as the following argument shows.

Let A,B,T,Z be the points representing $0,1,t,z$ respectively. The line TZ represents the vector $z-t$, and it lies between AZ and

BZ. Hence all the vectors z-t have directions lying within an angle less than 180°. The direction of the inverse, $1/(z-t)$, is the reflection in the real axis of that of z-t, so all these inverse directions also lie within an angle less than 180°, which also holds for the directions of $w(t)/(z-t)$, as $w(z)$ is real and non-negative. Hence $\int_0^1 w(t)\ (z-t)^{-1}\ dt$ could only be zero if $w(t)$ were identically zero or, more strictly,zero almost everywhere.

Hence in the cut plane $\theta(z)$ and $I(z)$ are analytic and tend to zero as z tends to oo.

Let their difference be given by $D(z)$, where

(12) $D(z) = \theta(z) - I(z)$.

As $\theta(z)$ and $I(z)$ experience the same discontinuity in crossing the cut, $D(z)$ is unbranched. As $D(z)$ has no singularity in the cut plane and, by condition (b), no singularity in the interval (0,1), $D(z)$ must be analytic in the finite plane and it tends to zero as z tends to oo. Hence $D(z)$ is constant and its behaviour at infinity shows that the constant is zero. The theorem is established.

(6.) Comments. The need for condition (a) is shown by the example $W(t)=1$, $w(t)=t$. This leads to a weight $W^*(t)$ that resembles C/t with C constant near $t=0$. This gives a divergent integral. As $W^*(t)$ is the only possible weight, and as $W^*(t)$ does not define any function, this means that the quotient in question cannot be represented by a moment function of the type specified.

The need for condition (b) is shown by the example

(13) $W(t) = \sqrt{[(1-t)/t]}/[\pi(1+t)]$, $w(t) = (1/\pi)/\sqrt{[(1-t)t]}$.

Then $f(z) = 1/\sqrt{[(z-1)z]}$ so $f(z)$ is purely imaginary in (0,1) and $q(z)=0$. This, as can be seen from equation (11), leads to negative values for $W^*(t)$, which is clearly impossible, as both $F(z)$ and $f(z)$ are positive for $z > 1$. The explanation is that

(14) $\theta(z) = [1/(z+1)]\sqrt{[2(z-1)/z]} - 2/(z+1) + 1/z$.

$I(z)$ gives the first two terms of this, but as $1/z$ has no discontinuity over (0,1), it makes no contribution to the weight and in fact could not be given by any moment function that did not employ a delta function or a Stieltjes integral with a jump at $t=0$. If condition (b) were dropped, the most we would be able to assert would be that $I(z)$ differs from $\theta(z)$ by an unbranched function.

We have excluded unbranched singularities but our argument does not imply that $\theta(z)$ must be free of infinities in the real interval from 0 to 1. For instance, if $W(t)=t$ and $w(t)=1$ we have

$$(15)\quad \theta(z) = 1 - \frac{1}{z(\ln z - \ln(z-1)} = I(z).$$

Here there are singularities resembling $-1/(z \ln z)$ in both $\theta(z)$ and $I(z)$, but these cancel out in the difference $D(z)$.

7.) Corollary.

If $W(t)/w(t)$ is an increasing function, then $W^*(t)$ is non-negative.

Proof. Equations (4) and (5) show for the numerator of $W^*(z)$ in equation (9) that

$$(16)\quad W(z)q(z) - w(z)p(z) = \int_0^1 [W(z)w(t)-w(z)W(t)]/(z-t)\,dt$$

If $z>t$ we have $W(z)/w(z)>W(t)/w(t)$ so $W(z)w(t)-w(z)W(t)>0$, so both the numerator and the denominator of the integrand in equation (4) are positive. Similarly, if $z<t$, they are both negative. Either way, the integrand is positive. In equation (3), the denominator is automatically positive.

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A CUMULATIVE HAZARD PROCESS FOR THE COX MODEL

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Abstract: We present asymptotic results for the cumulative hazard process based on the Cox model. We also obtain a weak approximation for this process through the bootstrap. A distribution-free goodness-of-fit test is given as an application.

1. Introduction. For analyzing failure time data, Cox (1972) has introduced a conditional hazard model which takes the form

$$\lambda(t|z) = \lambda_0(t)\exp(\beta z), \tag{1.1}$$

where z is a p dimensional covariate vector, β is a vector of p parameters, and $\lambda_0(t)$ is an arbitrary baseline hazard function. This model implies that the covariates contribute exponentially to the hazard function. Due to the fact that $\lambda_0(t)$ is arbitrary, the Cox model is very flexible for many applications.

Let X be the failure time and C be the censoring time. We assume that X and C are conditionally independent given the covariate Z which has density and finite range (a, b). We observe the values of $T_i = \min(X_i, C_i)$, Z_i and of $\delta_i = I(X_i \le C_i)$, in the form of an independent sample of size n.

Define $F(t,z) = P(T_i \le t, Z_i \le z, \delta_i = 1)$ and $H(t,z) = P(T_i > t, Z_i \le z)$, and denote their empirical counterparts by F_n and H_n respectively. From the definition of hazard function and (1.1), we have

$$\Lambda_0(t) = \Lambda(t,z), \tag{1.2}$$

where $\Lambda_0(t) = \int_0^t \lambda_0(s)ds$ and

$$\Lambda(t,z) = \int_0^t \left[1 \Big/ \int_a^z \exp(\beta u) d_s H(s,u) \right] d_s F(s,z). \tag{1.3}$$

The regression parameter β will be estimated by $\hat{\beta}$ which maximizes the partial likelihood (Cox, 1975)

$$\prod_{i\in D}\left\{\exp(\beta Z_i)\Big/\sum_{j\in R(t_i)}\exp(\beta Z_j)\right\},$$

where D denotes the set of indices corresponding to individuals who died and $R(t_i)$ is the risk set at time t_i. Then, we define the cumulative hazard process

$$\eta_n(t,z)=\sqrt{n}\big(\Lambda_n(t,z)-\Lambda(t,z)\big), \tag{1.4}$$

where Λ_n has the form (1.3) with F, H and β replaced by F_n, H_n and $\hat{\beta}$ respectively.

The asymptotic properties of, and a weak approximation for the process (1.4), as well as an application of these results, are presented in the following sections. For proofs, we refer to a forthcoming paper by the author (see Yuen (1993), [11]).

2. Asymptotic results for the cumulative hazard process. From now on, we shall assume that F and H are continuous and that the survival study is terminated at time T_0 with $P(T\geq T_0)>0$.

Theorem 2.1. Two asymptotic results for $\hat{\beta}$ are

$$|\hat{\beta}-\beta|=O\big(n^{-1/2}(\log\log n)^{1/2}\big)\quad \text{a.s.}, \qquad \text{and} \tag{2.1}$$

$$|\sqrt{n}(\hat{\beta}-\beta)-G|=O\big(n^{-1/(4p+2)}\log n\big)\quad \text{a.s.}, \tag{2.2}$$

where $G\sim N(0,\theta)$.

Remark 2.1. The covariance matrix θ is given in Tsiatis (1981), where he proved (2.1) without the rate of convergence and obtained the weak version of (2.2).

Approximation of empirical processes as well as some related results (see Borisov (1982), Theorem 3, Burke (1988), Lemma 4.4, and Kiefer (1961), Theorem 2) are utilized in establishing the next theorem. The idea is to transform η_n of (1.4) into a bunch of error terms vanishing in the limit and into a number of stochastic integrals converging to a Gaussian process almost surely. Consider the process

$$\xi(t,z) = \sum_{i=1}^{3} \xi_i(t,z), \tag{2.3}$$

where

$$\xi_1(t,z) = \int_0^t (1/A(s,z))d_sB^F(s,z)\,, \qquad A(s,z) = \int_a^z \exp(\beta u)d_uH(s,u)$$
$$\xi_2(t,z) = \int_0^t (-B^A(s,z)/A^2(s,z))d_sB^F(s,z)\,, \quad B^A(s,z) = \int_a^z \exp(\beta u)d_uB^H(s,u)$$
$$\xi_3(t,z) = \int_0^t [(D(s,z)\theta G)/A^2(s,z)]d_sB^F(s,z)\,, \quad D(s,z) = \int_a^z u\exp(\beta u)d_uH(s,u).$$

The random vector G can be expressed in terms of a sum of six stochastic integrals consisting of two mean-zero Gaussian processes, B^F and B^H. That ξ is Gaussian follows from the fact that B^F and B^H are jointly Gaussian with covariance

$$E\big(B^H(t,z)B^H(s,y)\big) = H(s\vee t, y\wedge z) - H(t,z)H(s,y)$$
$$E\big(B^F(t,z)B^F(s,y)\big) = F(s\wedge t, y\wedge z) - F(t,z)F(s,y)$$
$$E\big(B^F(t,z)B^H(s,y)\big) = F(t, y\wedge z) - F(s\wedge t, y\wedge z) - F(t,z)H(s,y).$$

Theorem 2.2. For $t \le T_0 < \infty$, the cumulative hazard process $\eta_n(t,z)$ converges to a sequence of Gaussian processes ξ_n such that

$$\sup_{(t,z)\in R^{p+1}} |\eta_n(t,z) - \xi_n(t,z)| = O(n^{-1/(4p+2)}\log n) \quad \text{a.s.}, \tag{2.4}$$

and ξ of (2.3) and ξ_n have the same distribution for each n.

Due to the complexity of the covariance matrix of ξ_n and its dependence on the underlying distribution, we seek a bootstrapped version of the cumulative hazard process.

3. Weak approximation of the bootstrapped cumulative hazard process. Let $\{T_i^*, Z_i^*, \delta_i^*\}_{i=1}^m$ be a bootstrapped sample of size m drawn from our random sample. Analogously to $\hat{\beta}$, we obtain the bootstrapped maximum partial likelihood estimator β^*. Define the bootstrapped cumulative hazard process $\eta_{mn}(t,z) = \sqrt{m}(\Lambda_{mn}(t,z) - \Lambda_n(t,z))$, where Λ_{mn} is defined similarly to Λ_n by using $F_{mn}(t,z) = (1/m)\sum_{i=1}^{m} I(T_i^* \le t, Z_i^* \le Z, \delta_i^* = 1)$, $H_{mn}(t,z) = (1/m)\sum_{i=1}^{m} I(T_i^* > t, Z_i^* \le Z)$, and β^* instead. With the conditions

$$0 < \liminf(m/n) \le \limsup(m/n) < \infty \quad \text{and} \quad m/\log n \to \infty \tag{3.1}$$

as $m \wedge n \to \infty$, we have

Theorem 3.1. If conditions (3.1) hold, then $|\beta^* - \beta| \to 0$ almost surely and $\sqrt{m}(\beta^* - \beta) \to G$ in probability, where G is defined in Theorem 2.1.

Theorem 3.2. If conditions (3.1) hold, then we can define a sequence of Gaussian processes ξ_{mn} such that

$$\sup_{(t,z)\in R^{p+1}} |\eta_{mn}(t,z) - \xi_{mn}(t,z)| = o_p(1) \tag{3.2}$$

as $m \wedge n \to \infty$. Furthermore, for each m and n, ξ_{mn} and ξ of (2.3) have the same distribution.

The proofs of Theorems 3.1 and 3.2 involve the bootstrapped empirical process, the bootstrapped version of strong law of large number by Athreya (1983), and that of Glivenko Cantelli theorem by Gaenssler (1986) (see Yuen (1993), [10] and [11]).

4. Goodness-of-fit tests for the Cox model. Here we explore the possibility of using strong approximation of empirical fields for building goodness-of-fit test statistics in the present context.

Our null hypothesis is specified in (1.2), which means that the value of the cumulative hazard function should not depend on z for any t if the Cox model is correct. The equality (1.2) also gives rise to a statistic, namely

$$V_n(t,z) = \eta_n(t,z) - \eta_n(t,b) = \sqrt{n}\big(\Lambda_n(t,z) - \Lambda_n(t,b)\big).$$

Similarly, we define $V(t,z) = \xi(t,z) - \xi(t,b)$ and $V_{mn}(t,z) = \eta_{mn}(t,z) - \eta_{mn}(t,b)$. We now generate N bootstrapped cumulative hazard processes $\{\eta_{jmn}\}_{j=1}^{N}$ from independent samples of size m, drawn from the original data. Finally, a Cramér-von Mises-type procedure can be established by considering

$$\phi(V) = \int_0^{T_0} \int_a^b V^2(t,z) d_t d_z H(t,z). \tag{4.1}$$

The empirical and bootstrapped versions of (4.1) are denoted by $\phi_n(V_n) = \int_0^{T_0} \int_a^b V_n^2(t,z)$ $d_t d_z H_n(t,z)$, and $\phi_n(V_{jmn}) = \int_0^{T_0} \int_a^b V_{jmn}^2(t,z) d_t d_z H_n(t,z)$, for $j = 1, 2, \cdots, N$, respectively. Using (2.4), (3.2), and ideas presented in Csörgő et. al. (1986), we can show that $\phi_n(V_n)$ and $\phi_n(V_{jmn})$ converge in distribution to $\phi(V)$. Anderson-Darling and Kolmogorov-Smirnov type statistics can be built along the same lines.

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AN ADAPTED RUNGE–KUTTA PAIR FOR STATE-DEPENDENT DELAY DIFFERENTIAL EQUATIONS

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Presented by T.E. Hull, F.R.S.C.

ABSTRACT. An adaptated Runge–Kutta–Verner (5, 6) formula pair is used to construct a numerical method to solve state-dependent delay differential equations with nonvanishing lag. The derivative jump discontinuities of the solution, which are assumed to be isolated, are located by a fifth-degree interpolation polynomial. The solution at the delay is approximated by a three-point Hermite polynomial. A Fortran program, called SYSDEL, is available from the authors.

RÉSUMÉ. On emploie une paire de formules du type Runge–Kutta–Verner (5, 6) adaptée pour la résolution numérique d'équations différentielles avec retard. On localise les sauts, supposés isolés, dans les dérivées de la solution au moyen d'un polynôme d'interpolation du 5ème degré et on interpole la solution au retard au moyen d'un polynôme d'Hermite à trois points. Les auteurs fourniront le programme Fortran SYSDEL sur demande.

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Keywords: delay differential equation, adapted Runge–Kutta pair, switching function

1. Introduction. Consider the d-dimensional system of state-dependent delay differential equations (DDEs) for $\mathbf{y} = ({}^1y, {}^2y, \dots, {}^dy)^T : [\bar{a}, b] \to \mathbb{R}^d$ (left superscripts denote vector components),

$$\mathbf{y}'(t) = \mathbf{f}\Big(t, \mathbf{y}(t), \mathbf{y}\big(\boldsymbol{\alpha}(t, \mathbf{y})\big)\Big) \quad \text{for } t \in [a, b], \qquad \mathbf{y}(t) = \boldsymbol{\phi}(t) \quad \text{for } t \in [\bar{a}, a], \tag{1.1}$$

with sufficiently smooth functions $\mathbf{f} : [a, b] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{\phi} : [\bar{a}, a] \to \mathbb{R}^d$ and $\boldsymbol{\alpha} : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$, where

$$\bar{a} = \min_{\substack{a \le t \le b \\ l = 1, \dots, d}} \big\{{}^l\alpha(t, \mathbf{y}(t))\big\} < a, \tag{1.2}$$

and the following simplifying notation for vector systems is used:

$$\mathbf{y}(\boldsymbol{\alpha}(t, \mathbf{y})) := \big[{}^1y({}^1\alpha(t, \mathbf{y})), \dots, {}^dy({}^d\alpha(t, \mathbf{y}))\big]^T. \tag{1.3}$$

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The terms ${}^{l}\alpha(t, y(t))$ and $t - {}^{l}\alpha t, y(t))$ are called, respectively, the delays and the lags. We consider only nonvanishing lags: ${}^{l}\alpha(t, y(t)) < t$, for all $t \in [a, b]$. A general procedure for solving vanishing-lag DDEs has been proposed recently [3].

Schemes for the numerical solution of ordinary differential equations (ODEs) have been adapted to solve the delay system (1.1). In [4] and in this work, the results of [5] are used to characterize the derivatives jump discontinuities. In section 2, a new method for the location of these discontinuities is given. The proposed method is as follows:

The Runge–Kutta–Verner (5,6) pair with a variable stepsize strategy and a three-point Hermite interpolation polynomial are used for the integration of the DDEs. Newton's backward divided-difference interpolation polynomial of degree 5 is used to extrapolate the switching function and the bisection method is used to locate the root of the extrapolated polynomial to machine precision (16 digits).

Numerical experiments were run, with different values of the tolerance, in double precision Fortran conforming to the IEEE floating point standard on an AMDAHL 4370 and an IBM 3090 at the University of Ottawa. The Fortran code, called SYSDEL, is available upon resquest from the authors.

2. Derivative Jump Discontinuities. For simplicity, we consider only single-lag scalar DDEs. But, the results of this section are valid for systems with multiple lags.

Definition 1. A number ξ is called a *derivative jump discontinuity* of the solution $y(t)$ of (1.1) if y or some derivative of y has a jump discontinuity at $t = \xi$.

The set of derivative jump discontinuities which is propagated from the initial jump point, generally $t = a$, is called the set of *primary discontinuities* of $y(t)$. This set is characterized [6] as the set of zeros of the nonlinear *switching function*,

$$g(t) = \alpha(t, y(t)) - Z, \tag{2.1}$$

where Z is a previous jump. The jump discontinuities are assumed to be sufficiently isolated to permit an accurate appoximation to the switching function. To obtain a high order numerical scheme for DDEs, it is shown in [5] that the jump discontinuities have to be located to sufficient accuracy. Since generally the exact derivative jump discontinuities, ξ_ν, are not known beforehand, a numerical approximation to these ξ_ν's is needed.

2.1. A numerical method for the location of jump discontinuities. The next zero of the extrapolated switching function, $g(t)$, is determined by Newton's backward divided-difference interpolation polynomial. The resulting discontinuity becomes the next grid point.

The Switching Function Algorithm. If t_{n+1} is the most recent grid point, y_{n+1} is a numerical approximation to $y(t_{n+1})$ and T_h is a numerical approximation to a previous exact derivative jump discontinuity, and the derivative jump discontinuities are sufficiently far apart, then:

(1) If $[\alpha(t_n, y_n) - T_h] \times [\alpha(t_{n+1}, y_{n+1}) - T_h] > 0$, proceed to the next integration step.

(2) If $[\alpha(t_n, y_n)-T_h]\times[\alpha(t_{n+1}, y_{n+1})-T_h] \leq 0$, locate the derivative jump discontinuity in $[t_n, t_{n+1}]$:

(α) Construct Newton's backward interpolation polynomial $g_h(t)$ from $q+1$ previous values of the discrete switching function

$$g_h(t_\nu) = \alpha(t_\nu, y_\nu) - T_h, \qquad \nu = n-q, \dots, n.$$

(β) Use the bisection method to find the zero Z_h of this polynomial in the extrapolated interval (t_n, t_{n+1}) and take Z_h as an approximation to the exact root Z of $\alpha(t, y(t)) - T = 0$.

(γ) Take $t_{n+1} = Z_h$ as the next grid point. □

The following theorem addresses the important question of the accuracy of the above method. For simplicity we assume a constant stepsize h; however, the theorem is still valid for a system and with a variable stepsize h_ν.

Theorem 1. *Consider the DDE:*

$$y'(t) = f\Big(t, y(t), y\big(\alpha(t, y(t))\big)\Big) \quad \text{if } t \in [a, b], \qquad y(t) = \phi(t) \quad \text{if } t \in [\bar{a}, a]. \tag{2.2}$$

Consider also the continuous and discrete switching functions,

$$g(t) = \alpha(t, y(t)) - T, \qquad g_h(t_\nu) = \alpha(t_\nu, y_\nu) - T_h,$$

where y_ν is a numerical approximation to the solution $y(t_\nu)$ of the DDE, and T and T_h are the exact and the approximate derivative jump discontinuities of $y(t)$ and y_ν, respectively. Assume that the function $\alpha(t, y)$ is Lipschitzian with respect to y with Lipschitz constant M_α and $g(t)$ has a zero in (t_n, t_{n+1}). Then, by extrapolating Newton's backward interpolation polynomial $P_{g_h}(t)$ for $g_h(t_\nu)$, one can approximate the zero Z of $g(t)$ in (t_n, t_{n+1}) by the zero Z_P of $P_{g_h}(t)$ to the order $O\left(h^{\min\{p/r, p/s\}}\right)$ provided

(1) *the degree of the interpolation polynomial is at least $(p-1)$,*
(2) $| T - T_h | = O(h^p)$,
(3) *the global integration method is of order at least p,*
(4) *the zeros, Z_P^* and Z_P, nearest to Z, of P_g and P_{g_h} are of multiplicity r and s respectively, where P_g is Newton's backward polynomial interpolating g at the points $\{t_{n-p+1}, \dots, t_n\}$,*
(5) *the divided difference $g[t_n, \dots, t_{n-p+1}, Z]$ is bounded.*

2.2. Numerical results on the switching functions. Numerical examples will illustrate the accuracy of the switching function method in locating jump discontinuities.

Example 1. The first three jumps in the solution to the DDE:

$$y'(t) = \frac{1}{t} y(t)\, y(\ln y(t)) \quad \text{for } t \geq 1, \qquad y(t) = 1 \quad \text{for } t \in [0, 1]. \tag{2.3}$$

occur at $e_1 = e \approx R_1$, $e_2 = e^2 \approx R_2$, and $e_3 = \exp(3 - \exp(1 - e)) \approx R_3$, where R_i are the numerical approximations to e_i obtained by the switching function method. Columns 2, 3 and 4 of Table 1 lists the absolute error made in locating these discontinuities.

Table 1. Absolute error in derivative jump discontinuities for Examples 1 and 2.

TOL	$\lvert R_1 - e\rvert$	$\lvert R_2 - e^2\rvert$	$\lvert R_3 - e_3\rvert$	$\lvert S_1 - \xi_1\rvert$	$\lvert S_2 - \xi_2\rvert$
1E-05	$1.10E-07$	$2.72E-07$	$4.32E-05$	$1.49E-07$	$2.99E-06$
1E-07	$5.08E-10$	$1.23E-08$	$1.14E-07$	$3.11E-09$	$3.79E-09$
1E-09	$1.26E-10$	$2.13E-10$	$7.29E-10$	$1.00E-10$	$1.00E-09$
1E-12	$3.77E-15$	$9.01E-14$	$8.52E-13$	$1.68E-13$	$1.29E-14$

Example 2. Since the exact solution to the DDE:

$$y'(t) = \frac{t-1}{t} y(t) y(t - \ln(t) - 1) \quad \text{for } t \geq 1, \qquad y(t) = 1 \quad \text{for } t \in [0, 1]. \tag{2.4}$$

is not known, we compare the reference values of the first two jumps [1]:

$$\xi_1 = 3.146\,193\,220\,620\,582\,585\,2 \approx S_1, \qquad \xi_2 = 5.925\,449\,824\,508\,246\,492\,6 \approx S_2, \tag{2.5}$$

with our numerical approximations, S_1 and S_2, in columns 5 and 6 of Table 1.

3. Adapted Runge–Kutta Formulae for DDEs. In using an (explicit) Runge–Kutta formula to solve the d-dimensional DDE (1.1), the lth component ${}^l y({}^l\alpha(t, \mathbf{y}(t))$ of the solution at the delay is approximated by the lth component of the Hermite osculating vector polynomial, $\mathbf{Q}_q^h(t)$, through the numerical values ${}^l y_{\nu-q+1}, \dots, {}^l y_\nu$ and ${}^l y'_{\nu-q+1}, \dots, {}^l y'_\nu$ at the q previous times $t_{\nu-q+1}, \dots, t_\nu$. We note that ${}^l\alpha(t_\nu, \mathbf{y})$ lies in $[t_{\nu-q+1}, t_\nu]$ and the values $\mathbf{y}'_{\nu-q+1}, \dots, \mathbf{y}'_\nu$ are provided by the adapted RK scheme.

In our error analysis below, we shall need a similar polynomial, denoted by $\mathbf{Q}_q(t)$, through the exact values $\mathbf{y}(t_{\nu-k+1}), \dots, \mathbf{y}(t_\nu)$ and $\mathbf{y}'(t_{\nu-k+1}), \dots, \mathbf{y}'(t_\nu)$.

An adapted r-stage Runge–Kutta formula for DDEs is as follows:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\Psi\left(t_n, \mathbf{y}_n, \mathbf{Q}_q^h(\alpha(t_n, \mathbf{y}_n)), h\right) \tag{3.1}$$

where the vector increment function is

$$\Psi\left(t_n, \mathbf{y}_n, \mathbf{Q}_q^h(\alpha(t_n, \mathbf{y}_n)), h\right) = \sum_{i=1}^{r} c_i \mathbf{k}_i,$$

and the lth component of $\mathbf{k}_i$ ($l = 1, \dots, d$) is

$$ {}^l k_i = {}^l f\left(t_n + \lambda_i h, \mathbf{y}_n + h\sum_{j=1}^{i-1}\beta_{ij}\mathbf{k}_j, \mathbf{Q}_q^h\Big(\alpha\Big(t_n + \lambda_i h, \mathbf{y}_n + h\sum_{j=1}^{i-1}\beta_{ij}\mathbf{k}_j\Big)\Big)\right). \tag{3.2}$$

Again, for simplicity, we restrict ourselves to the scalar case. According to Lemma 2 below, we take $Q_q^h = Q_3^h$ to be a three-point Hermite interpolation polynomial of degree five since we use a Runge–Kutta formula pair of order $(5, 6)$ in the extrapolation mode.

The study of the convergence of an adapted Runge–Kutta scheme is complicated by the interpolation process. We start by quoting a lemma on the Lipschitz property of Ψ.

Lemma 1. *Assume that*

(1) *$f(t, y, z)$ is Lipschitzian with respect to its second and third arguments, with Lipschitz constants M_2 and M_3, respectively,*
(2) *$\alpha(t, y)$ is Lipschitzian with respect to y, with Lipschitz constant M_α,*
(3) *$Q_q^h(\alpha(t, y_n))$ is Lipschitzian with respect to α and the interpolated values y_ν, $\nu = n - q + 1, \dots, n$, with Lipschitz constants M_Q and M respectively.*

Then there exists a constant L_Ψ such that

$$\left\|\Psi\big(t, y_n, Q_q^h(\alpha(t, y_n)), h\big) - \Psi\big(t, \bar{y}_n, Q_q^h(\alpha(t, \bar{y}_n)), h\big)\right\| \le L_\Psi \max_{\nu \le n} \|y_\nu - \bar{y}_\nu\|.$$

3.1. Local truncation error. We consider the modified local problem on $[t_n, t_{n+1}]$:

$$\bar{y}'(t) = f\Big(t, \bar{y}(t), Q_q\big(\alpha(t, \bar{y}(t))\big)\Big), \qquad \bar{y}(t_n) = y(t_n). \tag{3.3}$$

For a solution of (3.3) obtained by an adapted pth-order RK method, we have:

Proposition 2. *The local truncation error in the numerical solution $\bar{y}_{n+1}$ of (3.3) obtained by an adapted pth-order Runge–Kutta method is of the order $(p + 1)$ provided $y(t) \in C^p[t_n, t_{n+1}]$ and the set of interpolation points is fixed.*

For a local solution of (1.1) obtained by an adapted pth-order RK method, we have:

Lemma 2. *Consider an adapted RK method of order p to solve (1.1). If*

(1) *the solution $y(t)$ of (1.1) is Lipschitzian with Lipschitz contant L_y,*
(2) *$\left\|Q_q\big(\alpha(t, \bar{y}(t))\big) - y\big(\alpha(t, \bar{y}(t))\big)\right\| \le Lh^q$, where $L > 0$ is a constant and $t \in [t_n, t_{n+1}]$,*
(3) *the conditions of Proposition 2 and Lemma 1 hold,*

then, for some constant $M > 0$, the local error in y_{n+1} satisfies the bound:

$$\|y(t_{n+1}) - y_{n+1}\| \le M h^{\min\{p,q\}+1}.$$

3.2. Global truncation error. The following theorem gives us a bound for the global error in the numerical solution obtained by our numerical method.

Theorem 2. *Let $y(t)$ and y_ν denote the exact and the numerical solution, respectively, of problem (1.1). If y_ν is obtained by a stable numerical method of order p, and local order $p+1$, for ordinary differential equations together with an interpolation scheme, then the global error $\|y(t_\nu) - y_\nu\|$ over the interval $[a,b]$ is of the order p.*

4. Numerical Results. The accuracy and cost of our scheme, at final time t_f, are listed in Table 2, for Examples 3 and 4 below, where:

TOL: tolerance for the maximum norm of the error estimate.

NFE(i): number of function evaluations, $i = 3, 4$.

MRE(i): maximum relative error in solution at t_f, $i = 3, 4$.

Example 3. Consider the delay DDE [2] with exact solution:

$$y'(t) = y(t-1), \quad t \in [0,15], \qquad y(t) = 1, \quad t \in [-1,0], \tag{4.1}$$

$$y(t) = \sum_{i=0}^{k} \frac{(t-i+1)^i}{i!}, \qquad t \in [k-1,k]. \tag{4.2}$$

The numerical results at $t_f = 15$ for different values of the tolerance are given in columns 2 and 3 of Table 2.

Table 2. Numerical results at $t_f = 15$ and ξ_2, resp. for Examples 3 and 4.

TOL	NFE(3)	MRE(3)	NFE(4)	MRE(4)
10^{-4}	741	3.51E-06	190	4.19E-05
10^{-6}	1079	1.64E-07	295	2.56E-06
10^{-8}	1471	3.81E-09	561	1.28E-08
10^{-10}	1926	3.51E-10	1030	1.00E-10
10^{-12}	2962	4.69E-12	2941	1.66E-13

Example 4. The numerical results for the DDE of Example 2 at the second jump, $t_f = \xi_2$, as in (2.13), are listed in columns 4 and 5 of Table 2 for different tolerance values.

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Un critère élémentaire pour l'équation de Catalan

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Résumé .— Nous montrons que l'équation de Catalan $x^p - y^q = 1$ n'a pas de solution non triviale lorsque certaines congruences sont vérifiées pour un nombre premier convenable de la forme $hpq + 1$.

Nous considérons l'équation de Catalan

$$x^p - y^q = 1$$

où p and q sont des nombres premiers impairs. Si cette équation possède une solution non triviale (*i.e.* avec $|x|$, $|y| > 1$) alors Cassels, [Ca], a démontré que

$$(1) \qquad x - 1 = u^q/p, \qquad \frac{x^p - 1}{x - 1} = p\,u'^q, \qquad y + 1 = v^p/q, \qquad \frac{y^q + 1}{y + 1} = q\,v'^p,$$

où u, u', v et v' sont des entiers rationnels.

Soit ℓ un nombre premier de la forme $\ell = hpq + 1$, alors :

- Si $\ell | u$, on a $\ell | y$ et (1) implique $q^{hq} \equiv 1 \pmod{\ell}$.
- Si $\ell | v$, on a $\ell | x$ et (1) implique $p^{hp} \equiv 1 \pmod{\ell}$.
- Reste le cas où ℓ ne divise pas uv. Soit alors g une racine primitive modulo ℓ, les relations (1) entrainent

$$x \equiv 1 + ag^{jq} \pmod{\ell}, \qquad y \equiv -1 + bg^{kp} \pmod{\ell},$$

où

$$ap \equiv 1 \pmod{\ell}, \quad bq \equiv 1 \pmod{\ell}, \quad j \in \{0, 1, \ldots, hp - 1\}, \quad k \in \{0, 1, \ldots, hq - 1\},$$

avec

$$(1 + ag^{jq})^p + (1 - bg^{kp})^q \equiv 1 \pmod{\ell}.$$

Remarquons que cette dernière relation implique

$$\exists j \in \{0, 1, \ldots, hp - 1\}, \quad \left((1 + ag^{jq})^p - 1\right)^{hp} \equiv 1 \pmod{\ell}.$$

En conclusion, nous avons démontré le résultat suivant.

Critère 1. — *Soient p et q des nombres premiers impairs et ℓ un nombre premier de la forme $\ell = hpq + 1$. Soient a et b des entiers tels que $ap \equiv 1 \pmod{\ell}$, $bq \equiv 1 \pmod{\ell}$. Alors, si on a simultanément*

$$\text{(i)} \qquad q^{hq} \not\equiv 1 \pmod{\ell}, \qquad p^{hp} \not\equiv 1 \pmod{\ell},$$

et, pour tout $j \in \{0, 1, \ldots, hp - 1\}$ et tout $k \in \{0, 1, \ldots, hq - 1\}$

$$\text{(ii)} \qquad (1 + ag^{jq})^p + (1 - bg^{kp})^q \not\equiv 1 \pmod{\ell},$$

l'équation de Catalan n'a que des solutions triviales. Cette conclusion est encore vraie si on remplace la condition (ii) précédente par la condition plus faible

$$\text{(ii')} \qquad \forall j \in \{0, 1, \ldots, hp - 1\}, \quad \left((1 + ag^{jq})^p - 1\right)^{hp} \not\equiv 1 \pmod{\ell}.$$

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* Cette recherche a été effectuée lors d'un séjour à l'Université de Thessalonique en juin 1993.

Exemples : Ce critère permet d'exclure des couples (p,q) pour lesquels les critères antérieurs échouaient. C'est en particulier le cas pour les couples $(83,4871)$ et $(193,4877)$: le critère ci-dessus s'applique en choisissant respectivement $\ell = 16980307$ et $\ell = 30120353$, avec les conditions (i) et (ii').

Rappelons le critère démontré en [M], qui contient les deux critères d'Inkeri publiés en [I1] et [I2].

Critère 2. — *Soient p et q des nombres premiers impairs. Posons $p-1 = d\ell$, où ℓ est impair et d une puissance de deux, $\zeta = e^{2i\pi/p}$. Soient g une racine primitive modulo p, $m = g^d \bmod p$, $K = \mathbb{Q}(\xi)$ où $\xi = \zeta + \zeta^m + \cdots + \zeta^{m^{\ell-1}}$. Désignons par h_K le nombre de classes de K. Alors, l'équation de Catalan n'a pas de solution non triviale quand on a à la fois*

$$q \nmid h_K \qquad \text{et} \qquad p^{q-1} \not\equiv 1 \pmod{q^2}.$$

Dans les exemples précédents, le Critère 2 ne s'applique pas pour les raisons suivantes :

$$83^{4870} \equiv 1 \pmod{4871^2} \qquad \text{et} \qquad 4871^{82} \equiv 1 \pmod{83^2}$$

et

$$193^{4877} \equiv 1 \pmod{4877^2} \qquad \text{et} \qquad 193 | h_{4877}.$$

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Effective Lower Bounds for the Fractional Parts of Powers of a Dense Set of Rationals

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Presented by D.W. Boyd, F.R.S.C.

Abstract

The author applies Padé approximation techniques à la F. Beukers to deduce lower bounds for fractional parts of powers of certain rational numbers. In particular, one finds "strong" effective bounds for an infinite class of rationals and can use this to construct a dense set of rationals satisfying "weak" effective bounds.

1 Introduction

In 1957, Mahler [6] showed that if p and q are relatively prime integers with $p > q \geq 2$ and $\epsilon > 0$ is given, then there exists a k_0 such that if $k \geq k_0$, then

$$\|(p/q)^k\| > e^{-\epsilon k} \tag{1}$$

where $\|x\|$ denotes the distance from x to the nearest integer. This implies, amongst other things, that the number $g(k)$ in Waring's problem satisfies

$$g(k) = 2^k + \left[(3/2)^k\right] - 2$$

for all but finitely many k (see e.g. Hardy and Wright [5]). Unfortunately, the result is ineffective — that is, it is not possible to determine the value of k_0 from the proof. By way of effective bounds, Baker and Coates [1] used the theory of linear forms in logarithms to construct for each p/q an ϵ with $0 < \epsilon < 1$ such that p/q satisfies

$$\|(p/q)^k\| > q^{-\epsilon k} \tag{2}$$

for all $k \geq k_0(p,q)$, where the latter constant is effectively computable. While the technique is applicable in a very general setting, the values of ϵ are extremely close to 1 (for $p = 3$, $q = 2$, one finds perhaps $\epsilon \sim 1 - 10^{-30}$). In this paper, we will use Padé approximation to a particular hypergeometric function to obtain stronger bounds in a more restricted setting. In particular, we note that the techniques of Beukers [4] may be used to construct, for any given $\epsilon > 0$, a dense set (in the interval $(1, \infty)$) of rationals p/q for which a bound of the form (2) is obtained.

2 Basic Notation and Introductory Lemmas

For convenience, we introduce, in analogy to the binomial coefficients,

$$\left[\!\!\left[\begin{matrix} x \\ y \end{matrix} \right]\!\!\right] = \frac{x^x}{y^y (x-y)^{x-y}} \quad \text{for real } x \geq y \geq 0$$

where we adopt the convention $\left[\!\left[\begin{matrix} x \\ 0 \end{matrix} \right]\!\right] = \left[\!\left[\begin{matrix} x \\ x \end{matrix} \right]\!\right] = 1$.

The primary object of our approximation, $H(A, B, z)$, is given by

$$z^B H(A,B,z) = (1-z)^{A+B} - \sum_{r=0}^{B-1} \binom{A+B}{r} (-z)^r.$$

where z is a real variable and A and B are nonnegative integers.

We fix $A = bcm$ and $B = (a-b)cm$, where a, b, c and m are positive integers with $a \geq b$. If n is a positive integer with $n < bcm$, then we can find polynomials in $\mathbb{Z}[z]$, $P_n(z)$, $Q_n(z)$ and $E_n(z)$ such that

$$P_n(z) - H(bcm, (a-b)cm, z) Q_n(z) = (-1)^{(a-b)cm+n} z^{2n+1} E_n(z). \tag{3}$$

These represent the (n,n)-diagonal Padé approximants to $H(bcm, (a-b)cm, z)$ and satisfy, defining $\Theta = \frac{(bcm+n)!}{((a-b)cm+n)!(bcm-n-1)!n!}$

Lemma 2.1 *a)* $Q_n(z) = \Theta \cdot \int_0^1 t^{bcm-n-1}(1-t)^{(a-b)cm+n}(1-t+zt)^n\,dt$

$$= \sum_{r=0}^{n} \binom{(a-b)cm+2n-r}{(a-b)cm+n} \binom{bcm-n-1+r}{r} z^r$$

b) $E_n(z) = \Theta \cdot \int_0^1 t^n(1-t)^{(a-b)cm+n}(1-zt)^{bcm-n-1}\,dt$

$$= \sum_{r=0}^{bcm-n-1} \binom{n+r}{r} \binom{bcm+n}{(a-b)cm+2n+r+1} (-z)^r$$

Proof: See Beukers [4]. ∎

We will take $n = dm$ or $dm - 1$ for d a positive integer, $d < bc$. At least one of these values will assure the nonvanishing of the form $P_n(z) - rQ_n(z)$ (where r and z are nonzero reals) which is necessary for the theorems that follow (see, for example, Beukers [4]). Also, if we let $z = -1/p^a$ where p is a positive integer with $p \geq 2$, then the study of the function $H(A, B, z)$ is motivated by

Lemma 2.2 $\left\| \left(\frac{p^a+1}{p^b} \right)^{acm} \right\| = \left\| H\left(bcm, (a-b)cm, -1/p^a\right) \right\|.$

Proof: Immediate from the definition of $H(A, B, z)$ which differs from the binomial by a fixed integer for our choice of z. ∎

3 Some Upper and Lower Bounds

The integral representations of the approximant $Q_n(z)$ and the error term $E_n(z)$ from Lemma 2.1 enable one to readily deduce upper bounds for these quantities (or even asymptotics if one is so inclined). If we write $s = c/d$ and set $C(s) = \left(1 + \frac{1}{p^a}\right)^{bs-1}$, then we have

Lemma 3.1 *If* $\left[\!\left[\begin{matrix} as+1 \\ bs-1 \end{matrix}\right]\!\right] \cdot C(s) < p^a$ *then*

a) $|Q_n(-1/p^a)| \ll \left(\left[\!\left[\begin{matrix} (a-b)s+2 \\ (a-b)s+1 \end{matrix}\right]\!\right]\right)^{dm}$

b) $|E_n(-1/p^a)| \ll \left(C(s) \cdot \left[\!\left[\begin{matrix} as+1 \\ bs-1 \end{matrix}\right]\!\right]\right)^{dm}$

Proof: The result follows from Bennett [3]. We note here only that the implied constants are independent of m. ∎

Arguing as in Beukers [4], then, we may conclude that

Theorem 3.2 *If* $\left[\!\left[\begin{matrix} as+1 \\ bs-1 \end{matrix}\right]\!\right] \cdot C(s) < p^a$*, then there is an effectively computable constant* $k_0 = k_0(a, b, s, p)$ *such that for all* $k \geq k_0$*, we have*

$$\left\| \left(\frac{p^a+1}{p^b}\right)^k \right\| > \left(p^{1/s} \left[\!\left[\begin{matrix} (a-b)s+2 \\ (a-b)s+1 \end{matrix}\right]\!\right]^{1/(as)}\right)^{-k}.$$

Proof: Again, see Bennett [3]. ∎

In particular, if we take $b = a - 1$ and $s \sim \log a$, then we obtain

$$\lim_{a \to \infty} \left(p^{1/s} \left[\!\left[\begin{matrix} s+2 \\ s+1 \end{matrix}\right]\!\right]^{1/(as)}\right) = 1$$

and hence

Corollary 3.3 *If* $\epsilon > 0$ *is given, then there is a computable constant* $a_0 = a_0(p, \epsilon)$ *such that if* $a \geq a_0$ *we may find an effective* $k_0 = k_0(a, p, \epsilon)$ *with*

$$\left\| \left(\frac{p^a+1}{p^{a-1}}\right)^k \right\| > e^{-\epsilon k}$$

for all $k \geq k_0$.

In general, one finds bounds of the above form via these techniques whenever (in the notation of Theorem 3.2) either $b/a \to 0$ or $b/a \to 1$ (see Bennett [2] for the case $a = b = 1$). Both of these situations coincide with bounding kth powers of rationals r for which the ratio between the integral and fractional parts of r is extremely large.

4 Construction of Our Dense Set

We are now in a position to present our main result.

Theorem 4.1 *Let $\epsilon > 0$ be given. Then there is a constructible set of rationals S_ϵ, dense in the interval $(1,\infty)$, such that each element p/q of S_ϵ satisfies*

$$\|(p/q)^k\| > q^{-\epsilon k}$$

for all $k \geq k_0(p,q,\epsilon)$ where the constant is effectively computable.

Proof: Take $p/q > 1$ to be any rational and both $\epsilon > 0$, $\delta > 0$ be arbitrary. We construct a rational p_1/q_1 in a deleted δ-neighbourhood of p/q for which the above bound obtains effectively. From Corollary 3.3, we find an a_0 with

$$\left\|\left(\frac{p^a+1}{p^{a-1}}\right)^k\right\| > q^{-\epsilon k} \tag{4}$$

for $a \geq a_0$ and all $k \geq k_1 = k_1(a,p,\epsilon)$. Let

$$a_1 = \max\left\{a_0, \frac{\ln q}{\epsilon \ln p} + 1, -\frac{\ln(q\delta)}{\ln p} + 2\right\}.$$

Then there is an effectively computable constant $k_0 = k_0(p,q,\epsilon,\delta)$ such that $k \geq k_0$ implies (4) with a replaced by a_1. It follows that

$$\left\|\left(\frac{p^{a_1}+1}{qp^{a_1-1}}\right)^k\right\| > (q^{1+\epsilon})^{-k}$$

for all such k. We take $p_1 = p^{a_1}+1$ and $q_1 = qp^{a_1-1}$ and note that $a_1 \geq \frac{\ln q}{\epsilon \ln p} + 1$ implies that

$$\|(p_1/q_1)^k\| > q_1^{-\epsilon k}$$

for $k \geq k_0$. Also, since $a_1 \geq -\frac{\ln(q\delta)}{\ln p} + 2$, we have $q/(qp^{a_1-1}) < \delta$ and thus, from $p_1/q_1 = p/q + 1/(qp^{a_1-1})$, that p_1/q_1, lies in a deleted δ-neighbourhood of p/q, as required. Since the choices of ϵ and δ were arbitrary, the result obtains. ∎

5 Concluding Remarks

While a detailed analysis of the greatest common factor $G(n,s)$ present in the coefficients of $Q_n(z)$ is not needed for the above result, one can in fact show that

$$\lim_{n\to\infty} \sqrt[n]{G(n,s)} = \frac{a^{a/(2b(a-b))}}{b^{b/(2a(a-b))}(a+b)^{(a+b)/(2ab)}}\sqrt{2\pi/e^{\gamma}} + o(1)$$

as a function of s (provided $a > b \geq 1$). Despite this, it appears that constructing a dense set of rationals effectively satisfying (1) for small $\epsilon > 0$ is unfeasible by this approach.

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THE PROBABILITY OF MAXIMAL RANK FOR A MATRIX OVER A FINITE COMMUTATIVE RING

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Presented by P. Ribenboim, F.R.S.C.

Abstract. We consider what sense can be made of the limit as m, n approach infinity of the probability that an $m \times n$ matrix over a given finite commutative ring have maximal rank.

In [3, Theorem 2], the first two authors found the probability that a square matrix over a finite commutative ring with identity be invertible. In particular, this recovered the observation of Waterhouse [7, p. 104] that $(1 - q^{-1}) \cdots (1 - q^{-n})$ is the probability that an $n \times n$ matrix over the finite field $\mathbf{F}_q$ be of maximal rank, n. It was shown in [3, Proposition 5] that for fixed q, the limit of this probability, as $n \to \infty$, is a number $L(q) = \prod_{1 \leq k} (1 - q^{-k})$ which is strictly between 0 and 1 and is asymptotic to $e^{-1/q}$. Hence, for "large" square matices over "large" finite fields $\mathbf{F}_q$, the probability of maximal rank is approximately $e^{-1/q}$. This paper addresses similar questions for rectangular matrices.

Our main concern is the probability that a "large" $m \times n$ matrix over $\mathbf{F}_q$ be of maximal rank. (Since a matrix has the same rank as its transpose, there is no loss of generality in supposing that $m \geq n$.) The final two results address extensions to matrices over finite rings. Our results depend on a formula, recalled in Proposition 1, for the number of

$m \times n$ matrices over $\mathbf{F}_q$ of rank r. This formula was stated in case q is prime by Landsberg [6]; since then, a number of proofs of the formula have been given for arbitrary prime-power q (cf. [4, Theorem 2]).

All rings discussed below are assumed to be finite and have an identity. Adapting notation from [3], we let $M(R, m, n)$ denote the set of $m \times n$ matrices over R; $M(R, m, n, r)$ denote the subset of $M(R, m, n)$ consisting of its rank r matrices; $N(R, m, n, r) = |M(R, m, n, r)|$; and $a(R, m, n, r) = N(R, m, n, r)/|R|^{mn}$, the probability that a random $m \times n$ matrix over R have rank r. (As in [3], we assume each element of R is equally likely at each entry of a matrix.) Recall our *riding hypothesis* that $m \geq n > 0$. We also *assume henceforth* that the integer r satisfies $0 \leq r \leq n$, and that a product indexed by the empty set is 1.

<u>PROPOSITION 1.</u> (cf. Landsberg [6]) $N(\mathbf{F}_q, m, n, r) =$

$$(\textstyle\prod_{0\leq i\leq r-1} (q^m - q^i))(\prod_{0\leq i\leq r-1} (q^n - q^i))/(\prod_{0\leq i\leq r-1} (q^r - q^i)).$$

Corollary 2 generalizes the motivating result of Waterhouse (and the field case of [3, Theorem 2]) by determining the probability that an $m \times n$ matrix over a finite field have maximal rank.

<u>COROLLARY 2.</u> Let $m \geq n$ be positive integers. Then the probability that an $m \times n$ matrix over $\mathbf{F}_q$ have (maximal) rank $\min(m, n) = n$ is

$$a(\mathbf{F}_q, m, n, n) = \textstyle\prod_{0\leq i\leq n-1} (1 - q^{-(m-i)}).$$

Next, we need a result that says, essentially, that $N(\mathbf{F}_q, m, n, r)$ is an $(m + n - r)r$ - degree polynomial in q with integer coefficients.

<u>THEOREM 3.</u> Let m, n and r be as above. Define polynomials g

and h in $\mathbb{Z}[X]$ by $g = \prod_{0 \le i \le r-1} (X^{m-i} - 1)(X^{n-i} - 1)X^i$ and $h = \prod_{0 \le i \le r-1} (X^{r-i} - 1)$. Then $f = f_r = g/h \in \mathbb{Z}[X]$ is such that $N(F_q, m, n, r) = f(q)$ for each (prime-power) q. Moreover, for fixed m and n, $\deg(f_r) = (m + n - r)r$ is maximized only at $r = \min(m,n) = n$.

In the spirit of [2] and [3], we show next that "maximal rank is typical." As usual, take $m \ge n$; fix q for the moment. Then the rank of an $m \times n$ matrix over F_q is a discrete random variable X taking on the values $0, 1, \cdots, \min(m, n) = n$. If p is the associated probability function, then $p(r) = a(F_q, m, n, r) = N(F_q, m, n, r)/q^{mn} = f_r(q)/q^{mn}$. The expected rank is then $E_q = E(X) = \Sigma_{0 \le r \le n} \; rp(r)$. Applying limit theorems, Theorem 3, and Corollary 2, we infer the following result.

COROLLARY 4. Let m and n be positive integers. Then, as $q \to \infty$, the expected rank of an $m \times n$ matrix over F_q has limiting value $\min(m, n)$; moreover, $\lim_{q \to \infty} a(F_q, m, n, \min(m, n)) = 1$.

We next state our main results on the limit of the probability of maximal rank as matrices grow "large." Theorem 5 (b), (c) should be contrasted with the contexts of [5, Propositions 1.2 and 1.1], inasmuch as our limiting processes allow m, n, and the rank in question all to approach infinity. The proof of Theorem 5 is obtained by combining limit theorems, Corollary 2, and the bounds for L(q) in [3, Proposition 5].

THEOREM 5. (a) Fix a positive integer k and a prime-power q. Then, as $n \to \infty$, the probability that an $(n + k) \times n$ matrix over F_q be of maximal rank has limiting value $\lim_{n \to \infty} a(F_q, n + k, n, n)$. This limit exists, it equals $L(q)/[(1 - q^{-1})\cdots(1 - q^{-k})]$, and it is strictly between 0 and 1. Moreover, this limit is bounded below by

$$[\exp((-2q^3 + q^2 + 2q - 2)/2(q^2 - 1)(q - 1)^2)]/[(1 - q^{-1})\cdots(1 - q^{-k})]$$

and is bounded above by

$$[\exp((-2q - 3)/2(q^2 - 1))]/[(1 - q^{-1})\cdots(1 - q^{-k})].$$

(b) Fix an integer $k \geq 2$ and a prime-power q. Then, as $n \to \infty$, the probability that an $nk \times n$ matrix over $\mathbf{F}_q$ be of maximal rank has limiting value $\lim_{n\to\infty} a(\mathbf{F}_q, nk, n, n) = 1$.

(c) Given any limit process such that $m \geq n$ and $n \to \infty$, then 1 is the limit of the probability that an $m \times n$ matrix over $\mathbf{F}_q$ be of maximal rank if and only if the limit process entails that $m - n \to \infty$.

REMARK 6. (a) Theorem 5(a) can be interpreted as saying that over "large" finite fields $\mathbf{F}_q$ and for each $k \geq 1$, the probability that a "large" $(n + k) \times n$ matrix be of maximal rank n is asymptotic to $e^{-1/q}/(1 - q^{-1})\cdots(1 - q^{-k})$. By the convention regarding empty products, we can now view the result on square matrices in [3, Proposition 5] as a degenerate case of Theorem 5(a); indeed, we need only to formally put $k = 0$ in Theorem 5(a). For this reason, one may consider an $(n + k) \times n$ matrix as being "nearly square" (when k is fixed and $n \to \infty$). As Theorem 5(b) shows, this is the wrong intuition for an $nk \times n$ matrix (when $k > 1$). In fact, the result in [3, Proposition 5] cannot be viewed as a degenerate case of Theorem 5(b) because the limit in question is 1 whenever $k > 1$, but is $L(q)$ (which is unequal to 1) when $k = 1$.

(b) Since the limit in Theorem 5(a) depends on k (with q fixed, of course), the naive question suggested in the abstract has no answer. In other words, the probability of maximal rank for an $m \times n$ matrix over a given $\mathbf{F}_q$ does not approach a limit, if one stipulates only that the limit process permits both m and n to approach infinity. If one's expectations are drawn from the case of square matrices, as in [3], this is, arguably, the biggest surprise of the present work. Those expectations, which were perhaps reinforced by Corollary 4, were partially realized in Theorem 5(b), with the essential point being identified in Theorem 5(c).

Passing from finite fields to arbitrary finite commutative rings, we present analogues of [3, Theorem 2 and Proposition 5].

PROPOSITION 7. (a) Let (R, M) be a finite commutative local ring, with residue field $K = R/M$. Then an $m \times n$ matrix over R is just as likely to have rank r as is an $m \times n$ matrix over K; that is, $a(R, m, n, r) = a(K, m, n, r)$.

(b) Let $A_1, \cdots, A_s$ be $m \times n$ matrices over commutative rings $R_1, \cdots, R_s$, respectively. View $A = (A_1, \cdots, A_s)$ as a matrix over $R = \prod R_i$ by means of the canonical isomorphism of abelian groups $M(R, m, n) \to \prod M(R_i, m, n)$. Then:

(1) If $m = n$, then $\det(A) = \prod \det(A_i)$.

(2) A has maximal rank, namely $\min(m,n)$, if and only if each A_i has maximal rank.

The definition of "rank" used here is the standard one over commutative rings (cf. [1]). If one were to adopt instead the definition "the maximal r for which the matrix has an invertible $r \times r$ submatrix," then one would lose the "if" assertion in Proposition 7 (b)(2) and the "$=$" symbols in Corollary 8 would become "$\leq$".

COROLLARY 8. Suppose $R = \prod R_i$, a product of finitely many finite commutative rings. If $m \geq n$, then $N(R, m, n, n) = \prod N(R_i, m, n, n)$ and $a(R, m, n, n) = \prod a(R_i, m, n, n)$, and hence $\lim_{n\to\infty} a(R, m, n, n) = \prod \lim_{n\to\infty} a(R_i, m, n, n)$. If the above is the canonical decomposition with each R_i local, this value can then be determined by appeals to Proposition 7(a) and Theorem 5.

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A NOTE ON THE LAW OF ITERATED LOGARITHM FOR ABEL SUMS

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Presented by Miklós Csörgő F.R. S. C.

Abstract: We give a short proof of the main result in Bovier and Picco (1993) and discuss two applications of invariance principles to Abel sums.

Let $\{\epsilon_i, 0 \leq i < \infty\}$ be a sequence of independent identically distributed random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = 1$. The Abel sums of $\{\epsilon_i, 0 \leq i < \infty\}$ are defined by

$$A(x) = \begin{cases} \sum_{0 \leq i < \infty} \left(1 - \frac{1}{x}\right)^i \epsilon_i, & \text{if } \quad 1 < x < \infty \\ \epsilon_0, & \text{if } \quad 0 \leq x \leq 1. \end{cases}$$

The main result in Bovier and Picco (1993) can be formulated in the following way:

Theorem 1.1 *The cluster set of $(x \log\log x)^{-1/2} A(x)$, when x converges to infinity, is the interval $[-1, 1]$ with probability one.*

Proof: Lai (1974a) proved that

$$\limsup_{x\to\infty} (x \log\log x)^{-1/2} A(x) = 1 \quad a.s. \tag{1.1}$$

and

$$\liminf_{x\to\infty}(x\log\log x)^{-1/2}A(x) = -1 \quad a.s. \tag{1.2}$$

Since $\{A(x), 2 \le x < \infty\}$ is continuous with probability one, (1.1) and (1.2) imply immediately Theorem 1.1.

Combining Strassen's (1964) invariance principle with the properties of Gaussian processes it is easy to get a functional version of Theorem 1.1. Let $\mathcal{H}$ be the reproducing kernel Hilbert space generated by $\left\{\frac{2ts}{t+s}, 0 \le t, s \le 1\right\}$. Let K denote the unit ball of $\mathcal{H}$, i.e.

$$K = \{h \in \mathcal{H} : \|h\| \le 1\},$$

where $\|\cdot\|$ is the norm of $\mathcal{H}$.

Theorem 1.2 *The set of the limit points of* $\{(n\log\log n)^{-1/2}A(nt), 0 \le t \le 1\}$ *in* $C[0,1]$ *is* K *with probability one as* $n \to \infty$.

Proof: Let $S(u) = \sum_{0\le i\le u} \epsilon_i$. By Strassen (1964) we can define a sequence of i.i.d. standard normal r.v.'s $\{\xi_i, 0 \le i < \infty\}$ such that

$$|S(u) - W(u)| \overset{a.s.}{=} o((u\log\log u)^{1/2}) \quad \text{as} \quad u \to \infty. \tag{1.3}$$

Thus for all $\delta > 0$ we can find a r.v. $\eta = \eta(\delta)$ such that

$$|S(u) - W(u)| \le \delta(u\log\log u)^{1/2}, \quad \text{if} \quad u \ge \eta. \tag{1.4}$$

Integration by parts and (1.4) give

$$\begin{aligned}&\left|\int_0^\infty \exp(u\log(1-1/x))d(S(u)-W(u))\right|\\ &\le |\epsilon_0 - \xi_0| + |\log(1-1/x)|\int_0^\eta |S(u)-W(u)|du\end{aligned}$$

$$+\delta|\log(1-1/x)|\int_n^\infty (u\log\log u)^{1/2}\exp(u\log(1-1/x))du.$$

and therefore we have

$$|\int_0^\infty \exp(u\log(1-1/x))d(S(u)-W(u))| \overset{a.s.}{=} o((x\log\log x)^{1/2}), \qquad (1.5)$$

as $x \to \infty$. Let

$$B(x) = \begin{cases} \sum_{0\le i<\infty} \left(1-\frac{1}{x}\right)^i \xi_i, & \text{if} \quad 1 < x < \infty \\ \xi_0, & \text{if} \quad 0 \le x \le 1. \end{cases}$$

Now (1.5) yields that

$$\sup_{0\le t\le 1} |A(nt) - B(nt)| \overset{a.s.}{=} o((n\log\log n)^{1/2}). \qquad (1.6)$$

Since $\{B(x), 0 \le x < \infty\}$ is a Gaussian process, the method in Oodaira (1972, 1973), Lai (1974b) and Mangano (1976) gives that the set of the limit points of $\{(n\log\log n)^{-1/2}B(nt), 0 \le t \le 1\}$ in $C[0,1]$ is K with probability one. By (1.6) this also completes the proof of Theorem 1.2.

Remark: We can drop the condition that $\{\epsilon_i, 0 \le i < \infty\}$ are i.i.d.r.v.'s. If we can find a constant $\sigma > 0$ and a sequence of i.i.d. standard normal r.v.'s $\{\xi_i, 0 \le i < \infty\}$ such that

$$\left|\sum_{0\le i\le k} \epsilon_i - \sigma \sum_{0\le i\le k} \xi_i\right| \overset{a.s.}{=} o((k\log\log k)^{1/2}), \quad \text{as} \quad k \to \infty,$$

then the set of the limit points of $\{(\sigma^2 n\log\log n)^{-1/2}A(nt), 0 \le t \le 1\}$ in $C[0,1]$ is K with probability one, as $n \to \infty$.

Invariance principle also gives the weak convergence of $(\frac{2}{n})^{1/2}A(nt)$.

Theorem 1.3 *As $n \to \infty$, we have*

$$\left(\frac{2}{n}\right)^{1/2} A(nt) \xrightarrow{C[0,1]} \Gamma(t),$$

where $\{\Gamma(t), 0 \le t \le 1\}$ *is a Gaussian process with* $E\Gamma(t) = 0$ *and* $E\Gamma(t)\Gamma(s) = 2ts/(t+s)$.

Proof: By Major (1976) we can find a sequence of independent normal r.v.'s $\{\eta_i, 0 \le i < \infty\}$ with $E\eta_i = 0$ and $E\eta_i^2 = \sigma_i^2$ such that $\sigma_k^2 \to \sigma^2$ $(k \to \infty)$ and

$$\left|\sum_{0\le i\le k} \epsilon_i - \sum_{0\le i\le k} \eta_i\right| \overset{a.s.}{=} o(k^{1/2}), \quad \text{as} \quad k \to \infty. \tag{1.7}$$

Replacing (1.3) with (1.7), we get similarity to (1.6) that

$$\sup_{0\le t\le 1} |A(nt) - B_n^*(nt)| \overset{a.s.}{=} o(n^{1/2}),$$

where

$$B^*(x) = \begin{cases} \sum_{0\le i<\infty} \left(1 - \frac{1}{x}\right)^i \eta_i, & \text{if} \quad 1 < x < \infty \\ \eta_0, & \text{if} \quad 0 \le x \le 1 \end{cases}$$

Since $\{B^*(x), 0 \le x < \infty\}$ is a Gaussian process, elementary arguments give that

$$\left(\frac{2}{n}\right)^{1/2} B^*(nt) \xrightarrow{C[0,1]} \Gamma(t).$$

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FREE EXACT COMPLETIONS OF WEAK-LEX CATEGORIES

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Abstract. In this work we introduce the notion of flat functor from an arbitrary category to an exact category. By using a duality theorem on κ-accessible categories with products and small categories with weak κ-limits, we show that there exists the free exact completion of a small category with weak κ-limits; here κ is an infinite regular cardinal. In case of $\kappa = \aleph_0$, this gives A. Carboni and R. C. Magno's result on the free exact completion of a small category with finite limits (see [4]).

Introduction

Let $\mathbf{C}$ be a small lex (=finitely complete) category. It is well-known that there is a free-exact completion of $\mathbf{C}$ (see [12], [4], [5], [10], [7]). That is, there are an exact category $\mathbf{D}$ and a lex functor $F : \mathbf{C} \to \mathbf{D}$ such that F has the following universal property: for any exact category $\mathbf{B}$, the functor

$$\begin{aligned} F^* : Reg(\mathbf{D}, \mathbf{B}) &\to Lex(\mathbf{C}, \mathbf{B}) \\ M &\mapsto M \circ F \end{aligned}$$

induced by F is an equivalence of categories; here $Reg(\mathbf{D}, \mathbf{B})$ is the category of regular functors from $\mathbf{D}$ to $\mathbf{B}$. The fundamental construction of A. Carboni and R. C. Magno gives an explicit description of this completion by adding as new objects the equivalence relations in $\mathbf{C}$ and as new arrows the suitable classes of compatible maps (see [4]). Recently, M. Makkai has shown that the free exact completion of $\mathbf{C}$ is equivalent to $\prod Filt(Lex(\mathbf{C}, \mathbf{Set}), \mathbf{Set})$, the category of functors from $Lex(\mathbf{C}, \mathbf{Set})$ to $\mathbf{Set}$ that preserve products and filtered colimits (see [7] and [10]); where $\mathbf{Set}$ is the category of small sets. Since $\mathbf{C}$ is a small category with finite limits, $Lex(\mathbf{C}, \mathbf{Set})$ is locally finitely presentable, thus, it is finitely accessible with products.

Our goal is to generalize the above mentioned result to a small category with weak finite limits, more precisely, to a small category with weak κ-limits (weak limits of diagrams whose domain category is of size less than κ), for any infinite regular cardinal κ.

Inspired by a result in [7] (Proposition 5.5), we observe that for any κ-accessible category $\mathbf{A}$ with products, the opposite category $\mathbf{A}_\kappa^{op}$ of $\mathbf{A}_\kappa$ (whose objects are κ-presentable in $\mathbf{A}$) has weak κ-limits. We prove the following duality theorem on κ-accessible categories with products: the κ-accessible categories with products are exactly the categories of the form κ-$Flat(\mathbf{C})$, the category of the κ-flat functors from $\mathbf{C}$ into $\mathbf{Set}$, where $\mathbf{C}$ is a small category with weak κ-limits. We recall from [8] and [11] that for any small category $\mathbf{C}$, a functor $F : \mathbf{C} \to \mathbf{Set}$ is κ-flat iff F is a κ-filtered colimit of representable functors. We denote κ-$Flat(\mathbf{C})$ by $\mathbf{C}^*$. Thus, $\mathbf{C}^*$ is κ-accessible with products. Let $\mathbf{C}^{*+}$ be the category $\prod Filt_\kappa(\mathbf{C}^*, \mathbf{Set})$ of functors from $\mathbf{C}^*$ to $\mathbf{Set}$ that preserve products and κ-filtered colimits. Then $\mathbf{C}^{*+}$ is small κ-Barr-exact (see [10] and [7]).

From the point of view of the present work, the most interesting concept is that of κ-flat functor from $\mathbf{C}$ to any κ-Barr-exact category. Notice that the original definition of κ-flat functor deals with small categories and $\mathbf{Set}$, and κ-$Flat(\mathbf{C})$ is the κ-filtered colimits completion of a small category $\mathbf{C}$ (see [8] and [11]). Given an arbitrary category $\mathbf{C}$ and a κ-Barr-exact category $\mathbf{B}$, we define that a functor $F : \mathbf{C} \to \mathbf{B}$ is *κ-flat* iff for any κ-diagram $G : I \to \mathbf{C}$, there is a cone $(f_i : D \to D_i)_{i \in I}$ on G such that $F(f_i) = p_i \circ k$ for all $i \in I$ and some regular epimorphism $k : F(D) \to limF \circ G$; here the p_i are limit projections, i.e., every κ-limit of a diagram whose objects are images of F can be recovered by some image of F. We denote by κ-$Flat(\mathbf{C}, \mathbf{B})$ the full subcategory of the functor category $(\mathbf{C}, \mathbf{B})$ whose objects are κ-flat.

Before describing the main theorems of this paper, we make a few remarks on κ-flat functors. Firstly, if $\mathbf{C}$ is small, and if $\mathbf{B}$ is replaced by $\mathbf{Set}$, then the above mentioned definition of κ-flat functor coincides with the original one. Secondly, in case $\mathbf{C}$ is small with κ-limits, a functor between $\mathbf{C}$ and $\mathbf{B}$ preserves κ-limits iff it is κ-flat.

1 Main results

Theorem 1.1 *For any small category $\mathbf{C}$ with weak κ-limits, the evaluation functor $e_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}^{*+}$ has the following properties:*

(i) $e_{\mathbf{C}}$ is κ-flat, full and faithful;

(ii) for any $C \in \mathbf{C}$, $e_{\mathbf{C}}(C)$ is (regular) projective in $\mathbf{C}^{*+}$, i.e., given a regular epi $p : M \to N$ in $\mathbf{C}^{*+}$, every morphism $f : e_{\mathbf{C}}(C) \to N$ factors through p;

(iii) for each $F \in \mathbf{C}^{*+}$, *there are* $C \in \mathbf{C}$ *and a regular epimorphism* $\eta : e_{\mathbf{C}}(C) \to F$ *in* $\mathbf{C}^{*+}$.

Theorem 1.2 *(i) The functor* $e_{\mathbf{C}}$ *of Theorem 1.1 has the universal property of a free* κ*-Barr-exact completion of* $\mathbf{C}$*:* $\mathbf{C}^{*+}$ *is* κ*-Barr-exact, and for any* κ*-Barr-exact category* $\mathbf{B}$*, the functor*

$$\Sigma : \kappa\text{-}Reg(\mathbf{C}^{*+}, \mathbf{B}) \to \kappa\text{-}Flat(\mathbf{C}, \mathbf{B})$$
$$M \mapsto M \circ e_{\mathbf{C}}$$

induced by $e_{\mathbf{C}}$ *is an equivalence of categories.*

(ii) The quasi-inverse of the equivalence Σ *of (i) takes a* κ*-flat functor* $F : \mathbf{C} \to \mathbf{B}$ *to its left Kan extension* $F!$ *along* $e_{\mathbf{C}}$.

Remark 1.3 *In order to prove Theorem 1.2, we only use the properties of* $e_{\mathbf{C}}$ *of Theorem 1.1. Therefore, for a* κ*-Barr-exact category* $\mathbf{D}$*, we have that, if there is a functor* $F : \mathbf{C} \to \mathbf{D}$ *such that the properties (i), (ii) and (iii) of Theorem 1.1 are satisfied by* F*, then* $\mathbf{D}$ *is a* κ*-Barr-exact completion of* $\mathbf{C}$*. Consequently, these are necessary and sufficient conditions describing the free* κ*-Barr-exact completion of* $\mathbf{C}$.

2 Sketch of proofs

The proof of Theorem 1.2 consists of three parts: (1) the functor Σ of Theorem 1.2 is full and faithful; (2) every κ-flat functor $F : \mathbf{C} \to \mathbf{B}$ has a left Kan extension $F!$ along $e_{\mathbf{C}}$; (3)$F!$ is κ-regular.

The proof of part (1) is quite similar to the proof of Proposition 5.8 in [7] by using the properties of Theorem 1.1.

For the existence of $F!$, from the dual of Theorem X.3.1. in [9], it suffices to show that the composite $F \circ P : e_{\mathbf{C}}/C' \to \mathbf{C} \to \mathbf{B}$ has a colimit in $\mathbf{B}$ for each $C' \in \mathbf{C}^{*+}$, where P is the projection $< C, e_{\mathbf{C}}(C) \to C' > \mapsto C$. Since $C' \in \mathbf{C}^{*+}$, according to Theorem 1.1, we can take a regular epimorphism $t : e_{\mathbf{C}}(A) \to C'$ with A in $\mathbf{C}$, and let

$$D \underset{v'}{\overset{u'}{\rightrightarrows}} e_{\mathbf{C}}(A)$$

be the kernel pair of t; then t is the coequalizer of (u', v'). Let $d : e_{\mathbf{C}}(S) \to D$ be a regular epimorphism, then t is a coequalizer of the morphisms $(u' \circ d, v' \circ d)$. Denote $u' \circ d$ by u, and $v' \circ d$ by v. Consider the category $\mathbf{E}$ as follows.

$$t \underset{v}{\overset{u}{\rightrightarrows}} t \circ u$$

Let $i : \mathbf{E} \to e_{\mathbf{C}}/C'$ be the inclusion. We have

Lemma 2.1 *i is final.*

Since i is final, according to Theorem IX.3.1 in [9], to prove that $F!$ exists, we only need to show that the pair of morphisms $(F(u), F(v))$ has a coequalizer in $\mathbf{B}$.

Let (p, q) be the product projections of $F(e_{\mathbf{C}}(A)) \prod F(e_{\mathbf{C}}(A))$, and let $\alpha : F(e_{\mathbf{C}}(S)) \to F(e_{\mathbf{C}}(A)) \prod F(e_{\mathbf{C}}(A))$ be the unique morphism so that $F(u) = p \circ \alpha$ and $F(v) = q \circ \alpha$. Since $\mathbf{B}$ is κ-Barr-exact, α has a factorization $\alpha = y \circ x$ with $y : Q \to F(e_{\mathbf{C}}(A)) \prod F(e_{\mathbf{C}}(A))$ mono and $x : F(e_{\mathbf{C}}(S)) \to Q$ regular epi, for some $Q \in \mathbf{B}$. We can prove the following lemma.

Lemma 2.2 *y is an equivalence relation on $F(e_{\mathbf{C}}(A))$.*

Since $\mathbf{B}$ is κ-Barr-exact, every equivalence relation is effective, so $(p \circ y, q \circ y)$ has a coequalizer from Lemma 2.2. Since $F(u) = p \circ y \circ x$ and $F(v) = q \circ y \circ x$, it follows that $(F(u), F(v))$ has a coequalizer as x is regular epi. This completes the proof of the existence of $F!$.

$F!$ is κ-regular. That $F!$ preserves regular epimorphisms follows from the proof of the existence of $F!$. In order to prove that $F!$ is κ-regular, we only need to show that $F!$ preserves κ-limits.

Without loss of generality, we may assume that $\mathbf{B}$ is small, since both of $\mathbf{C}$ and $\mathbf{C}^{*+}$ are essentially small. For any κ-regular functor $M : \mathbf{B} \to \mathbf{Set}$, that $M \circ F!$ is κ-regular follows from $M \circ F$ κ-flat (see Proposition 5.8 in [7]). For any $B \in \mathbf{B}$, we have that $\mathbf{B}(B, -) \circ F!$ preserves κ-limits. Indeed, $\mathbf{B}(B, -)$ is an equalizer of a pair between κ-regular functors from $\mathbf{B}$ into $\mathbf{Set}$ (see [3] and [10]), and limits are computed pointwise in $L_\kappa(\mathbf{B}, \mathbf{Set})$, the category of functors from $\mathbf{B}$ to $\mathbf{Set}$ that preserve κ-limits. We conclude that $\mathbf{B}(B, -) \circ F!$ preserves κ-limits from that $M \circ F!$ preserves κ-limits. Thus, $F!$ preserves κ-limits.

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Algebraic aspects of the Jones basic construction

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Abstract

We define finite separable extensions of algebras, show that they admit the basic construction of V. Jones' index theory, and give examples. We symmetrize finite separability to an equivalence relation on categories and rings, then study properties invariant under equivalence.

1 Finite separable extensions

V.F.R. Jones' index theory of type II_1 von Neumann algebra subfactors led to new representations of the braid group and subsequently to the solving of old problems in knot theory. In assessing these achievements, it was observed that much of this theory could be made purely algebraic, using in a novel way the finite dimensional Hecke algebras and Bratteli diagrams for multimatrix inclusions [2]. Another algebraic direction to Jones' theory was indicated by M. Pimsner and S. Popa in [7] in which they show using an "orthonormal basis" that a factor M is a finitely generated projective module over a finite index subfactor N. In this note we will follow this other and quite different algebraic direction by building on the observation by D. Kastler and the author [4] that $N \subseteq M$ is a separable extension of algebras. We define finite separable extensions of algebras, and a certain useful generalization, f.s.-equivalence. This concept possesses features of Jones' theory that have certain representation-theoretic implications; for example, a question implicit in [7] about what properties are shared by subalgebra N and algebra M can be answered in the setting of finite separable extensions. As the name would indicate, an example of this otherwise noncommutative theory is finite separable field extension, a building block of Galois theory.

Let k be a commutative ring denoting by k° its group of units. Let S be a subalgebra of a faithful k-algebra A such that $1_A \in S$. We identify k with $k1_A$. Consider the natural modules and bimodules formed from $S \subseteq A$ and the tensor product over S. Let μ denote the multiplication map $A \otimes_S A \to A$, an A-A bimodule morphism.

Definition 1.1 A is called a finite separable extension of S if there exists an element $f \in A \otimes_S A$, an S-S bimodule homomorphism $E : A \to S$, and $\tau \in k^\circ$ such that

1. $af = fa$ $(\forall a \in A)$ *and* $\mu(f) = 1$;
2. $E(1) = 1$;
3. $\mu(1 \otimes_S E)f = \mu(E \otimes_S 1)f = \tau$.

An element f satisfying (1) is a separating element and its existence alone defines a separable extension of rings, a theory developed by Sugano [8]. The existence of f is equivalent to μ being a split epimorphism of A-bimodules, which is in turn equivalent to the vanishing of relative Hochschild cohomology groups, $H^n(A, S; -) = 0$ for $n > 0$. The map $E : A \to S$ satisfying (2) is a conditional expectation as in operator theory, and its existence for a subalgebra $S \subseteq A$ is equivalent to requiring the subalgebra be a direct summand in the natural bimodule ${}_SA_S$: then A is called a split extension of S. Conditional expectations and separating elements are not unique, but condition (3) demands the existence of a conditional expectation $E : A \to S$ and separating element in $A \otimes_S A$

$$f = \tau \sum_{i=1}^{n} x_i \otimes_S y_i$$

such that $\tau \in k^\circ$ and $\sum_{i=1}^n E(x_i)y_i = \sum_{i=1}^n x_i E(y_i) = 1$. We will say that E and f are compatible in case they satisfy condition (3). Now fix the notations $S \subseteq A$, f, E, x_i, y_i, μ, and τ for the rest of this section.

Lemma 1.1 For every $a \in A$, we have $\sum_{i=1}^n E(ax_i)y_i = \sum_{i=1}^n x_i E(y_i a) = a$.

Corollary 1.1 E is a nondegenerate S-valued bilinear form on A such that $\{E(-x_i)\}_{i=1}^n$ is a dual basis of $\{y_i\}_{i=1}^n$ for the projective module ${}_SA$ and $\{E(y_i-)\}_{i=1}^n$ is a dual basis of $\{x_i\}_{i=1}^n$ for A_S.

The Basic Construction A_1. Define A_1 as the k-algebra $A \otimes_S A$ where multiplication is given by

$$(a_0 \otimes_S a_1)(a_2 \otimes_S a_3) = a_0 E(a_1 a_2) \otimes_S a_3,$$

Note that A_1 has unity element $1 = \sum_{i=1}^n x_i \otimes_S y_i$. A is contained in A_1 via the monomorphism $a \longmapsto a1$. Define index of S in A, $[A : S]_E = \tau^{-1}$, an invertible element in k. This definition is independent of f since $\mu(1) = \tau^{-1}$.

Proposition 1.1 A_1 is a finite separable extension of A with index $[A : S]_E$.

Proof. A separating element $f = \sum_{i=1}^n x_i \otimes_S 1 \otimes_S y_i$ and a conditional expectation $E_1 = \tau\mu : A_1 \to A$ are compatible. □

Theorem 1.1 If A_{i+1} is defined inductively for i = 1,2,3,... as the basic construction of the finite separable extension $A_{i-1} \subseteq A_i$, and the ground ring k possesses a solution q to the quadratic equation $q^2\tau = q - 1$, then there exist nontrivial homomorphisms of the braid groups B_n into the group of units of A_{n-1}.

Proof. Let $e_i = 1 \otimes_S 1$ be the idempotent in A_{i+1} formed from the unity element $1 \in A_i$. The family of idempotents $\{e_i\}_{i=1}^{\infty}$ in the tower of algebras $S \subseteq A \subseteq A_1 \subseteq \ldots$ satisfies the braid-like relations: $e_i e_{i+1} e_i = \tau e_i$, $e_{i+1} e_i e_{i+1} = \tau e_{i+1}$, and $e_i e_j = e_j e_i \quad (|i-j| > 1)$. Map the Artin generators σ_i of B_n to $w_i = q e_i - 1$, which are units of A_{n-1} and satisfy the Artin relations. □

Proposition 1.2 A_1 is Morita equivalent to S.

Proof. One notes the ring isomorphism $A_1 \to End\, A_S$ given by $\sum_{i=1}^{m} a_i e_1 b_i \longmapsto \sum_{i=1}^{m} \lambda_{a_i} E \lambda_{b_i}$ where $\lambda_x(y) = xy$ is the left multiplication map and e_1 is the idempotent defined above. A_S is a generator module since $E(1) = 1$. □

Proposition 1.3 If B is a finite separable extension of A, which in turn is a finite separable extension of S, with conditional expectations E_1 and E_2, resp., then B is a finite separable extension of S with index satisfying Lagrange's condition,

$$[B : S]_{E_2 \circ E_1} = [B : A]_{E_1} [A : S]_{E_2}.$$

It follows as a corollary that finite separable extension is closed under tensor product with index behaving multiplicatively.

2 Examples

1. **Subfactors.** Let N be a subfactor of M with Jones index $n \leq [M : N] < n + 1$. If $\{m_j\}_{j=1}^{n+1}$ is the Pimsner- Popa orthonormal basis with respect to the trace-preserving conditional expectation $E : M \to N$, a separating element is then given by

$$\frac{1}{[M : N]} \sum_{j=1}^{n+1} m_j \otimes m_j^*$$

(observed in [4] and [9] independently) compatible with E (cf. [3]).

2. **Finite Separable Extensions of Fields** F_2/F_1 with characteristic coprime to the degree n. Let α be a primitive element, $F_2 = F_1(\alpha)$, with minimal polynomial $p(x) = x^n - \sum_{i=0}^{n-1} c_i x^i$. Let $E = \frac{1}{n} trace : F_2 \to F_1$, the normalized trace, a nondegenerate bilinear form on the F_1-vector space F_2 with dual bases $\{\alpha^i\}_{i=0}^{n-1}$ and $\{\frac{\sum_{j=0}^{i} c_j \alpha^j}{p'(\alpha)\alpha^{i+1}}\}_{i=0}^{n-1}$. Then a compatible separating element is given by

$$f = \sum_{i=0}^{n-1} \alpha^i \otimes_{F_1} \frac{\sum_{j=0}^{i} c_j \alpha^j}{p'(\alpha)\alpha^{i+1}}$$

3. **Crossed product algebras.** Let H be a subgroup of G with finite index $[G : H] \in k^{\circ}$, B a k-algebra with action $\alpha : G \to Aut(B)$. Then $A = B \times_{\alpha} G$ is a finite

separable extension of $S = B \times_\alpha H$. For if $\{g_i\}_{i=1}^n$ is a left transversal of H in G, then

$$f = \frac{1}{[G:H]} \sum_{i=1}^n g_i \otimes_S g_i^{-1}$$

is a separating element compatible with the natural projection

$$\pi_H : B \times_\alpha G \to B \times_\alpha H;$$

whence $\tau = \frac{1}{[G:H]}$. Note that group algebras, and specifically those generated by Sylow p-subgroups of finite groups over characteristic p are included in this example.

Remark 2.1 Galois extensions of commutative rings [1], multimatrix extensions $M_{n_1}(S) \times \cdots \times M_{n_t}(S)$, and separable algebras over a local or global field are also finite separable extensions.

3 Structure theory

A category $\mathcal{C}$ we say is f.s.-equivalent to a category $\mathcal{D}$ if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ that are adjoint functors in either order with split epic counits: i.e., there exist adjunctions $\langle F, G, \eta, \epsilon \rangle$ and $\langle G, F, \eta', \epsilon' \rangle$ such that $\epsilon : FG \dot{\to} 1$ and $\epsilon' : GF \dot{\to} 1$ are split epis on every object. For example, an additive category $\mathcal{C}$ is f.s.-equivalent to the finite product category $\mathcal{C} \times \cdots \times \mathcal{C}$. When $\mathcal{C}$ and $\mathcal{D}$ are module categories A-**Mod** and S-**Mod**, the following useful definition of an equivalence relation among rings is obtained.

Definition 3.1 Rings A and S are f.s.-equivalent if there exist bimodules ${}_AP_S$ and ${}_SQ_A$, elements $\sum_{i=1}^n p_i \otimes q_i \in P \otimes_S Q$ and $\sum_{j=1}^m q_j' \otimes_A p_j' \in Q \otimes_A P$, and split epimorphisms of bimodules

$$\mu : P \otimes_S Q \to A$$

$$\epsilon : Q \otimes_A P \to S$$

such that (i) $\sum_{i=1}^n p_i \epsilon(q_i \otimes p) = p$; (ii) $\sum_{i=1}^n \epsilon(q \otimes p_i) q_i = q$; (iii) $\sum_{j=1}^m q_j' \mu(p_j' \otimes q) = q$; and (iv) $\sum_{j=1}^m \mu(p \otimes q_j') p_j' = p$.

Examples.

1. Suppose $S \subseteq A$ is a finite separable extension. Then A and S are f.s.-equivalent: take $P = A$, $Q = A$, $\epsilon = E$, $\sum p_i \otimes q_i = \sum_{i=1}^n x_i \otimes_S y_i$, and $\sum q_j' \otimes p_j' = 1 \otimes 1$. Saying that the functors restricting and inducing modules between S-**Mod** and A-**Mod** are adjoints in either order says that finite separable extensions enjoy symmetric Frobenius reciprocity relations like that of finite group representations. [1]

[1] A finite separable extension is a left or right quasi-Frobenius extension, but not conversely: take for example a group algebra of a p-group over a field of characteristic p.

2. If A and S are Morita equivalent, it is well-known that there are bimodules P and Q, isomorphisms μ and ϵ satisfying an associativity condition, so that $\sum p_i \otimes q_i = \mu^{-1}(1)$ and $\sum q'_j \otimes p'_j = \epsilon^{-1}(1)$ satisfy (i)-(iv). Hence, A and S are f.s.-equivalent.

Remark 3.1 P and Q are progenerators from right and left. $P \otimes_S Q$ and $Q \otimes_A P$ have unital algebra structures like in the basic construction, and these algebras are Morita equivalent to S and A respectively. Symmetric indices may be defined in the center though not in the group of units.

Proposition 3.1 If A is f.s.-equivalent to S (notation above), then $Tor_n^A(M, N)$ is isomorphic to a direct summand of $Tor_n^S(M \otimes_A P, Q \otimes_A N)$ for each pair of A-modules M and N and integer $n \geq 0$. A corresponding statement is true for S-modules as well as Ext functors.

Proof. It is clear that $M \otimes_A N$ is isomorphic to a direct summand of $M \otimes_A P \otimes_S Q \otimes_A N$ via a natural split epic transformation derived from μ: this result extends to the higher derived functors of $\otimes$. □

The next theorem follows directly from the last proposition and is closely related to Serre's extension theorem since the cohomological dimension of a group G over a field k equals the global dimension $D(-)$ of the group ring $k[G]$. The theorem is valid for left, right and weak global dimension.

Theorem 3.1 If A and S are f.s.-equivalent rings, then $D(A) = D(S)$.

The next theorem is closely related to Villamayor's and Wallace's theorems about radicals of group algebras in characteristic p. If $f = \tau \sum_{i=1}^n x_i \otimes_S y_i$ is the relevant separating element then by a normalizing basis $\{x_i\}$ for A_S we mean that A is free over S with basis $\{x_i\}_{i=1}^n$ and that for every $i = 1, \ldots, n$ there exists σ_i an automorphism of S such that $sx_i = x_i\sigma_i(s)$.

Theorem 3.2 If A is a finite separable extension of S with normalizing basis $\{x_i\}_{i=1}^n$ for A_S, then

$$J(A) \cap S = J(S).$$

Proof. It is generally true for split extensions that $J(A) \cap S \subseteq J(S)$. Hence $J(A_1) \cap A \subseteq J(A)$. Then $J(A_1) \cap S \subseteq J(A) \cap S \subseteq J(S)$. Under the identifications $A_1 \cong End(A_S) \cong M_n(S)$, s in S maps to the matrix $\text{Diag}(\sigma_1(s), \ldots, \sigma_n(s))$; hence $J(M_n(S)) = M_n(J(S))$, so that $J(S) \subseteq J(A_1) \cap S$. □

Theorem 3.3 (D.G. Higman) A finite group G has finite representation type over a field k of characteristic p iff each Sylow p-subgroup H is cyclic.

Proof. By the Krull-Schmidt theorem and example 2.3, k[G] has a finite list of indecomposables iff k[H] has. The theorem follows from the representation types of the p-groups [6]. □

Higman's theorem and theorem 3.1 lead naturally to the question of what other properties are shared by two f.s.-equivalent rings. Certain nice classes of modules like the projective modules are closed under passing to a direct summand and under change of rings by tensoring with bimodules finitely generated and projective on left and right. Now certain homological properties of rings like "von Neumann regular" or "left perfect" are characterized by the coincidence of nice classes of modules, like "all modules = flat modules" or "flat left modules = projective left modules". It is not hard to make precise the assertion that homological properties are shared by f.s.-equivalent rings (cf. [5]).

The next theorem is a direct consequence of Sugano's theorem in [8], where local ring refers to the noncommutative-theoretic notion. X might be thought of as a primitive element for the special finite separable extension considered.

Theorem 3.4 Suppose that A is a finite separable extension of a local ring A_0 such that $A = A_0 \oplus A_1$ is a Z_2-graded algebra. Then A is isomorphic to a quadratic factor ring of a skew polynomial ring over A_0, i.e., there exists an automorphism σ of A_0 and a unit a such that $A \cong A_0[X;\sigma]/(X^2 - a)$.

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Large Deviations for I.I.D. Random Sums when Cramér's Condition is Fulfilled only on a Finite Interval

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Abstract. We derive the exact asymptotics of the probabilities of large deviations and of densities of sums S_n of i.i.d.r.v.'s as well as the expectations of smooth functions of S_n/n in the case in which the Cramér's condition is fulfilled only on a finite interval. The probabilistic interpretation of our results differs from that relevant to the famous Cramér's theorem.

1. Introduction. Let $\{X_n, n \geq 1\}$ be i.i.d. r. v.'s with common distribution function $F(\cdot)$ having the following asymptotics on the right-hand tail:

$$1 - F(x) \sim C_a \cdot \exp(-z_0 \cdot x) \cdot x^{-a} \tag{1.1}$$

as $x \to \infty$, where $z_0 > 0$, $C_a > 0$, and $a > 2$. Hereinafter, we refer to such tails as *exponential-power tails.*

It is obvious that (1.1) implies that the famous Cramér's condition of finiteness of exponential moments is fulfilled only on a finite interval. Namely,

$\psi(z) := \mathbb{E}\exp(z \cdot X_i) < \infty$ for $z \in [0, z_0]$, whereas $\psi(z) = \infty$ for $z > z_0$.

Let us point out that we do not make any assumption on the behavior of $F(\cdot)$ on the left-hand tail.

In this work we reveal that under fulfilment of (1.1) the exact asymptotics of the probabilities of large deviations of $S_n := X_1 + \ldots + X_n$ as well as the densities of S_n, $p_{S_n}(\cdot)$, are not given by the famous Cramér's formula. Naturally, it is true only in a certain range of large deviations (see Theorems 2.1 - 2.2). These results are related to the fact that under fulfilment of (1.1) the rate function $H(u) := \sup_z [zu - V(z)]$ (where $V(z) := \log \psi(z)$) is not strongly downward convex. We also present the probabilistic interpretation of this phenomenon in terms of conditioned limit theorems for the random step-function $\eta_n(t) := S_{[nt]}/n$, where $t \in [0,1]$ and $S_0 := 0$ (see Theorem 2.3), and then establish that this phenomenon is in fact linked with the generalized concept of *action functional* relevant to a wide class of possibly discontinuous functions (see Proposition 2.1). Note that the just mentioned generalized concept of action functional was first introduced in the ramarkable paper by Lynch and Sethuraman (1987) in the context of families of processes with independent increments (treated in their work as *probability measures* on [0,1]) and then developed in the recent paper by Mogulskii (1993) in the context of families of processes with independent increments taking values in the space $\mathbb{D}[0,1]$ endowed with the Skorokhod or the uniform topology.

In addition, we also derive the exact asymptotics of expectations of $F(S_n/n)$ for a certain class of smooth functions F (see Theorem 2.4). Note that this result is of the same spirit as the results obtained by Dubrovskii (1976) (see also their version in Chapter 5 of the monograph by Wentzell (1990)).

It is interesting to note that distributions having exponential moments only on finite intervals or not strongly downward convex rate functions (functionals) arise as in the classical theory of branching processes, as well as in the modern theory of measure-valued processes. They also arise in various models of the Actuarial Science (cf., e.g. Klüppelberg (1989)). Thus, Chover, Ney and Wainger (1973) considered the subcritical age-dependent process whose particle lifetime distribution $G(\cdot)$ belongs to a certain class $\mathfrak{G}(d)$ (which is closely related to the class of distributions satisfying (1.1)). Here $d := \max\left\{\rho\colon \int_0^\infty e^{\rho\cdot t}\cdot dG(t) < \infty\right\}$. Let m denote the average number of children born at each birth epoch. In particular, Chover, Ney and Wainger (1973) established that if a subcritical branching process starts with one particle at the initial moment, $G(\cdot) \in \{\mathfrak{G}(d), d > 1\}$ and $d\cdot m < 1$ (this means that the Malthusian parameter does not exist) then the total number of descendants of the initial particle alive at time t conditioned on non-extinction converges to a nondegenerate limit as $t \to \infty$ (see Theorem 4 therein). However, the generation number converges to a finite limit as $t \to \infty$ in a certain sense, which suggests that the class $\{\mathfrak{G}(d), d > 1\}$ plays a borderline role for branching processes (cf. Section 5 of that paper for more details). Apart from this, it was established in the recent work by Fleischmann and Kaj (1992) that for a certain family of measure-valued processes the rate functional (which plays the role analogous to the rate function $H(\cdot)$ and is related to the action functional) is not strongly downward convex and even not continuous as a rule (see Theorem 5.2.1 and Example 5.2.4 therein). This justifies our interest to the class of distributions satisfying (1.1).

To complete the Introduction, let us give some auxiliary notation. Let z(u) denote the unique root of the equation $V'(z) = u$. Let $u_0 := \sup\{u\colon$ such that z(u) exists$\}$. Then it is easily seen that uder fulfilment of (1.1),

$$u_0 = V'(z_0-) < \infty. \tag{1.2}$$

Recall that under fulfilment of (1.1) the rate function $H(\cdot)$ is not strongly downward convex. Namely, $H(u) = uz(u) - V(z(u))$ for $u \le u_0$ and $H(u) = H(u_0) + z_0(u - u_0)$ for $u > u_0$.

Now, let us introduce the family of the probability measures $\mathbb{P}^z$ (with $z \in [0,z_0]$) on the space of infinite sequences $\Omega := \{\omega\colon (x_1, \ldots, x_n, \ldots)\colon X_i(\omega) = x_i\}$ as follows:

Let μ^0 be the probability measure on $\mathbb{R}^1$ generated by the distribution function F, and define the new probability measure on $\mathbb{R}^1$, μ^z, by the equation: $\mu^z(du) := e^{zu}\cdot\mu(du)/\psi(z)$. Set $\mathbb{P}^z := \mu^z \otimes \ldots \otimes \mu^z \otimes \ldots$ It is easily seen that although the probability measures $\mathbb{P}^0$ and $\mathbb{P}^z$ are mutually singular, but they are absolutely continuous on the σ-algebras generated by any finite set of $(X_1, \ldots, X_n)$ such that the Radon-Nykodim derivatives are as follows:

$$(1.3)\quad \frac{d\mathbb{P}^z}{d\mathbb{P}^0}\Bigg|_{\sigma(X_1,\ldots,X_n)} = \frac{\exp(z(X_1+\ldots+X_n))}{\psi(z)^n}\ ;\ \frac{d\mathbb{P}^0}{d\mathbb{P}^z}\Bigg|_{\sigma(X_1,\ldots,X_n)} = \frac{\exp(-z(X_1+\ldots+X_n))}{\psi(z)^{-n}}.$$

Note that the above transformation of measures goes back to Cramér (1938) and is often called *Cramér's transformation*. Let us point out that the expectation and the variance of r.v.'s X_n with respect to $\mathbb{P}^{z(u)}$ are as follows: $\mathbb{E}_{z(u)}X_i = V'(z(u)) = u$, and $\mathrm{Var}_{z(u)}X_i = V''(z(u))$. In particular, under fulfilment of (1.1) $\mathbb{E}_{z_0}X_i = u_0$, and $\mathrm{Var}_{z_0}X_i = V''(z_0-)$.

2. Results. Note that under fulfilment of (1.1) the exact asymptotics of $\mathbb{P}\{S_n > nu\}$ as $n \to \infty$ for $u \in (\mathbb{E}X_i, u_0]$ is expressed in terms of the famous Cramér's formula (cf., e.g., [2]). The idea of the proof consists in employing the Cramér's transformation of measures (see (1.3)) with the subsequent use of the various refinements of CLT for studying the asymptotic behavior of $\mathbb{P}^{z(u)}\{S_n > nu\}$ $(= \mathbb{P}^{z(u)}\{S_n - \mathbb{E}_{z(u)}S_n > 0\})$. Below we derive the exact asymptotics of $\mathbb{P}\{S_n > y\}$ as $n \to \infty$, $y - nu_0 \to \infty$. Note that in contrast to the Cramér's case, we cannot reduce our problem to the consideration of the probabilities of *normal deviations* of S_n with respect to any transformed measure $\mathbb{P}^z$. This follows from (1.2), since we cannot make $\mathbb{E}_zX_i$ greater than u_0 $(< y/n)$. However, we can still consider the distribution of S_n with respect to the "rightmost" measure $\mathbb{P}^{z_0}$ reducing our problem to studying the probabilities of *large deviations* of S_n:

$$(2.1)\qquad \mathbb{P}^{z_0}\{S_n - \mathbb{E}_{z_0}S_n > y - nu_0\} \qquad \text{as } n \to \infty,\ y - n\cdot u_0 \to \infty.$$

It turns out that under fulfilment of (1.1) the asymptotics of the right-hand tail of the distribution of X_i with respect to $\mathbb{P}^{z_0}$ is of *power type*, which in turn allows us to apply the known results on the asymptotic behavior of the *probabilities of large deviations of* S_n *in the case of power tails* to get the asymptotics of (2.1). This is clarified by the following (purely analytical) lemma.

Lemma 2.1. Let (1.1) be fulfilled. Then (a) the distribution function $F(\cdot)$ is not lattice. (b) Let $F^{z_0}(\cdot)$ denote the common d.f. of $\{X_n, n \geq 1\}$ with respect to μ^{z_0}. Then

$$\text{i)}\qquad 1 - F^{z_0}(y) \sim \frac{z_0\cdot c_a}{\psi(z_0)\cdot(a-1)}\cdot y^{-(a-1)}$$

as $y \to \infty$.

ii) $$|F^{z_0}(y+x) - F^{z_0}(y)| \sim \frac{z_0 \cdot c_a}{\psi(z_0)} \cdot |x| \cdot y^{-a}$$

as $y \to \infty$, where $0 < Const \le |x| = o(y)$.

It should be emphasized that the index of the tail $\alpha = a - 1 \in (1,\infty)$.

Now, we proceed with

Theorem 2.1 (compare to Vinogradov (1983), Nagaev (1985, 1989)). Let (1.1) be fulfilled with $a \in (2,3) \cup (3,\infty)$. Let $\kappa > 0$ be any fixed real. Then

$$\mathbb{P}\{S_n > y\} \sim n \cdot \mathbb{P}\{X_1 > y - (n-1)\cdot u_0\} \cdot \exp(-(n-1)\cdot H(u_0)) \tag{2.2}$$

as $n \to \infty$, $(y - nu_0)/n^{1/(a-1)} \to \infty$ if $a \in (2,3)$ and $y - nu_0 \ge Const \cdot n^{1/2+\kappa}$ if $a \in (3,\infty)$.

Proof of Theorem 2.1 involves the Cramér's transformation with the subsequent application of Lemma 2.1 and the *interval limit theorems on large deviations* in the case of *power tails* (derived in [13] for $\alpha \in (1,2)$ and in [9] for $\alpha \in (2,\infty)$). □

Remark 2.1. (a) Note that in the case of *fixed* n Chover, Ney and Wainger (1973) (see (2.9) therein) obtained a representation similar to (2.2) which was in fact the key tool in deriving their conditioned limit theorems for branching processes.

(b) Note that if we are interested in results on the asymptotic behavior of the probabilities of large deviations only up to logarithmic equivalence (often called *rough* theorems) then $\log \mathbb{P}\{S_n > y\} \sim -n \cdot H(y/n)$ as $n \to \infty$ in the full range of large deviations of y (see, e.g., Vinogradov (1985)).

The local analog of (2.2) is also true.

Theorem 2.2. Let i.i.d. r.v.'s $\{X_n, n \ge 1\}$ have the common bounded density $p(\cdot)$ such that

$$p(x) \sim c_a \cdot \exp(-z_0 \cdot x) \cdot x^{-a} \tag{2.3}$$

as $x \to \infty$, where $z_0 > 0$, $c_a > 0$, and $a > 2$. Let $\kappa > 0$ be any fixed real. Then the density of S_n, $p_{S_n}(\cdot)$, has the following asymptotics on the right-hand tail:

$$p_{S_n}(y) \sim n \cdot p(y - (n-1)\cdot u_0) \cdot \exp(-(n-1)\cdot H(u_0)) \quad \text{as } n \to \infty \text{ with } y - n \cdot u_0 \ge Const \cdot n^{1+\kappa}.$$

Remark 2.2. Note that (2.3) implies (1.1) with the same a and $C_a = c_a/z_0$.

It is well known that if Cramér's condition is fulfilled on the whole positive semi-axis then the main part of the probability of the large deviation, $\mathbb{P}\{S_n > y\}$, is generated by small and approximately equal individual summands. In contrast, under fulfilment of (1.1) this is true only in the range of large deviations in which the exact asymptotics of $\mathbb{P}\{S_n > y\}$ is given by the Cramér's formula (i.e. for $y \le n \cdot u_0$). Thus, for $u \in (\mathbb{E}X_1, u_0]$ the conditioned random step-function $(\eta_n(\cdot) \mid \eta_n(1) > u)$ converges in $\mathbb{D}[0,1]$ (endowed with the Skorokhod topology) to *the smooth non-random function* $f_u(t) := ut$. However, the corresponding result for $u > u_0$ is quite opposite. Namely, the maximal term is approximately equal to $n(u - u_0)$, whereas the (appropriately normalized) rest of the sum is closed to a Brownian bridge, $w_0(\cdot)$.

Theorem 2.3. Let (2.3) be fulfilled. Let $t \in [0,1]$ and $u > u_0$ be fixed. Then (a) Let $\theta_u(t) := u_0 t + (u-u_0)\cdot I_{\{\chi \le \tau\}}$, where I_A is the indicator of the set A, and the random variable χ is uniformly distributed in [0,1]. Then

$$(\eta_n(\cdot) \mid \eta_n(1) > u) \xrightarrow{\mathbb{D}[0,1]} \theta_u(\cdot) \qquad \text{as } n \to \infty.$$

(b) Let $X_1(n) := \max(X_1, \ldots, X_n)$; $\tau_n := \min(i \le n: X_i = X_1(n))$, $\sigma := (\psi''(z_0-)/\psi(z_0) - (\psi'(z_0-)/\psi(z_0))^2)^{1/2}$, and $\zeta_n(t) := n^{1/2}\cdot(\eta_n(t) - tu_0 - X_1(n)/n)/\sigma$. Then

$$(\zeta_n(\cdot) \mid \eta_n(1) > u) \xrightarrow{\mathbb{D}[0,1]} w_0(\cdot) \qquad \text{as } n \to \infty.$$

Proof of Theorem 2.3. Proof of (a) involves the Cramér's transformation with the subsequent application of the *rough* limit theorem on large deviations (up to logarithmic equivalence) from Vinogradov (1985) as well as Theorem 2' of Godovan'chuk (1981) (which describes the asymptotic behavior of $\eta_n(\cdot)$ as $n \to \infty$ with respect to the "rightmost" transformed measure, $\mathbb{P}^{z_0}$). Proof of (b) follows from (a) and the theorem of Poleshchuk (1990). □

It is well known that the rough asymptotics of $\mathbb{P}(\eta_n(\cdot) \in A)$ as $n \to \infty$ for a certain class of sets $A \subset \mathbb{D}[0,1]$ such that $dist(A,\mathbf{0}) > 0$ is expressed in terms of the action functional, $I(\cdot)$ (here $\mathbf{0}$ is the function on [0,1] identically equal to 0). Namely,

$$\log \mathbb{P}(\eta_n(\cdot) \in A) \sim -n\cdot \inf_{f\in A} I(f) \qquad \text{as } n \to \infty. \tag{2.4}$$

Let us assume for simplicity that (2.3) is fulfilled and r.v.'s $\{X_n, n \ge 1\}$ are non-negative. In particular, it means that the trajectories of $\eta_n(\cdot)$ are monotonically increasing functions on [0,1], equal to 0 if $t = 0$. Under these conditions, Theorem 5.1 of Lynch and Sethuraman (1987) implies that

$I(f) = \int_0^1 H(\dot f)\cdot dt + z_0\cdot\left(f(1) - f(0) - \int_0^1 \dot f\cdot dt \right)$ for monotonically increasing f: $f(0)=0$;

$I(f) = +\infty$ otherwise.

Then, in view of (2.4), the typical behavior of the trajectories of $\eta_n(\cdot)$ is determined by the extremal (or extremals) at which $I(\cdot)$ attains minimum. Set

$$A_u := \{f \in \mathbb{D}[0,1]: f(1) > u\}.$$

Then the following proposition demonstrates the difference between the cases $u \le u_0$ and $u > u_0$ (compare to Theorem 2.3.a).

Proposition 2.1. (a) If $u \le u_0$ then $\inf_{f\in A_u} I(f)$ is attained at the unique function $f_u(t) = ut$. (b) If $u > u_0$ then $\inf_{f\in A_u} I(f)$ is attained as at the function $f_u(t)$ as well as at the functions having the first derivative equal to u_0 almost everywhere with finite (or countable) number of positive jumps whose sum is equal to $u - u_0$.

The asymptotics of expectations of functionals of S_n/n also deserves being studied.

Theorem 2.4. Let (1.1) be fulfilled. Let $F \in \mathbb{C}^2(\mathbb{R}^1_+)$ such that the function $z_0 v - F(v)$ attains minimum at $v = u_1 > u_0$ such that $F''(u_1) < 0$. Then

$$\mathbb{E}\exp(n\cdot F(S_n/n)) \sim \exp(n\cdot(F(u_1) - H(u_1)) \cdot \left(2\pi/|F''(u_1)| \right)^{1/2} \cdot \psi(z_0)^{-1} \cdot \frac{z_0 \cdot c_a}{a-1} \cdot u_0^{-a} \cdot n^{3/2-a}$$

as $n \to \infty$.

Proof of Theorem 2.4 is carried out by a combination of the Laplace method and the arguments used for the proof of Theorem 2.1 □

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