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CONTENTS

A note on the diophantine equation $x^{p-1} - 1 = py^q$

Le Maohua *
Presented by J. Friedlander, F.R.S.C.

Abstract. Let p, q be primes with $p > 2$. In this note we prove that if $p \equiv 1 \pmod 4$ and $p > 4 \cdot 10^{176}$, then the equation $x^{p-1} - 1 = py^q$ has no positive integer solution (p, x, y, q).

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ denote the sets of integers, positive integers and rational numbers respectively. Let p, q be primes with $p > 2$. In [4] and [5], Osada and Terai discussed the solvability of the equation

$$x^{p-1} - 1 = py^q, \quad x, y \in \mathbb{N}. \tag{1}$$

In this note we prove the following result:

Theorem. *If $p \equiv 1 \pmod 4$ and $p > 4 \cdot 10^{176}$, then (1) has no solution (p, x, y, q).*

The proof of the Theorem depends on the following lemmas.

Lemma 1 ([3]). *The equation*

$$X^2 + 1 = 2Y^n, \quad X, Y, n \in \mathbb{N}, \quad Y > 1, \quad n > 2$$

has no solution (X, Y, n).

Let α be an algebraic number with the minimal polynomial

$$a_0 z^d + \cdots + a_d = a_0 \prod_{i=1}^{d} (z - \delta_i \alpha), \; a_0 > 0,$$

where $\delta_1 \alpha, \ldots, \delta_d \alpha$ are conjugates of α. Then

$$h(\alpha) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \max (1, |\delta_i \alpha|) \right)$$

is called the logarithmic absolute height of α.

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Lemma 2. *Let α_1, α_2 be real algebraic numbers with $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, and let D denote the degree of $\mathbb{Q}(\alpha_1, \alpha_2)$. Let $b_1, b_2 \in \mathbb{N}$, and let $b = b_1/Dh(\alpha_2) + b_2/Dh(\alpha_1)$. For any T with $T \geq 1$, if $0.52 + \log b \geq T$ and $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 \neq 0$, then*

$$\log|\Lambda| \;\geq\; 70\left(1 + \frac{0.1137}{T}\right)^2 D^4 h(\alpha_1) h(\alpha_2)(0.52 + \log b)^2.$$

Proof. Let $B = \log(5c_4/c_1) + \log b$, $K = [c_1 D^3 B h(\alpha_1) h(\alpha_2)]$, $L = [c_2 D B]$, $R_1 = [c_3 D^{\frac{3}{2}} B^{\frac{1}{2}} h(\alpha_2)] + 1$, $S_1 = [c_3 D^{\frac{3}{2}} B^{\frac{1}{2}} h(\alpha_1)] + 1$, $R_2 = [c_4 D^2 B h(\alpha_2)]$, $S_2 = [c_4 D^2 B h(\alpha_1)]$, $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, where c_1, c_2, c_3, c_4 are positive numbers. Notice that $(u - 1/T)v < [uv] \leq uv$ for any real numbers u, v with $u \geq 0$ and $v \geq T$. By the proof of [2, Theorems 1 and 3], if $B \geq T$,

$$\text{(2)} \qquad \sqrt{c_1} \;=\; \frac{\rho+1}{(\log\rho)^{\frac{3}{2}}} + \sqrt{\frac{(\rho+1)^2}{(\log\rho)^3} + \frac{\rho+1}{T\log\rho}},\; c_2 > \frac{2}{\log\rho},$$

$$c_3 = \max(\sqrt{c_1}, \sqrt{c_2}),\; c_4 = \sqrt{2c_1c_2} + \frac{1}{T},$$

for any ρ with $\rho > 1$, then

$$\text{(3)} \qquad \log|\Lambda| \;\geq\; -(c_1c_2\log\rho + 1)D^4 h(\alpha_1)h(\alpha_2)B^2.$$

Setting $\rho = 5.803$. We may choose c_1, c_2, c_3, c_4 which make (2) hold and

$$\text{(4)} \qquad c_1c_2\log\rho + 1 \;<\; 70\left(1 + \frac{0.1137}{T}\right)^2,\; B < 0.52 + \log b.$$

Substituting (4) into (3), the lemma is proved.

Lemma 3 ([1]). *Let $f(X,Y) \in \mathbb{Z}[X,Y]$ be a binary form of degree d which is irreducible over $\mathbb{Q}$. If $d \geq 3$, then all solutions (X,Y) of the equation*

$$f(X,Y) \;=\; \pm 1, \quad X, Y \in \mathbb{Z}$$

satisfy

$$\log\max(|X|, |Y|) \;\leq\; 10^{150} d^6 \log H,$$

where H is the maximum absolute value of coefficients of $f(X,Y)$.

Proof of the Theorem. Let (p, x, y, q) be a solution of (1). By [4], (1) has only solution $(p, x, y, q) = (5, 3, 4, 2)$ with $q = 2$, and by [5], (1) has only solution $(p, x, y, q) = (7, 2, 3, 2)$ with $p > 3$ and $2|x$. We may assume that $q > 2$ and $2 \nmid x$ for $p \equiv 1 \pmod 4$.

If $p \equiv 1 (\mathrm{mod}\, 4)$, then we have

$$x^{\frac{p-1}{2}}+1 = \begin{cases} 2y_1^q, \\ 2py_1^q, \end{cases} \quad x^{\frac{p-1}{2}}-1 = \begin{cases} 2^{q-1}py_2^q, \\ 2^{q-1}y_2^q, \end{cases}$$

where $y_1, y_2 \in \mathbb{N}$ such that $2y_1y_2 = y$. By Lemma 1, we have $x^{(p-1)/2}+1 \neq 2y_1^q$, and hence

$$x^{\frac{p-1}{2}}+1 = 2py_1^q, \quad x^{\frac{p-1}{4}}+1 = 2y_{21}^q, \quad x^{\frac{p-1}{4}}-1 = 2^{q-2}y_{22}^q$$

for $p \equiv 1 \ (\mathrm{mod}\, 8)$, where $y_{21}, y_{22} \in \mathbb{N}$ such that $y_{21}y_{22} = y_2$. It is impossible by Lemma 1. Therefore (1) has no solution for $p \equiv 1 \ (\mathrm{mod}\, 8)$.

By the above analysis, if $p \equiv 5 (\mathrm{mod}\, 8)$, then

$$(5) \qquad x^{\frac{p-1}{2}}+1 = 2py_1^q, \quad x^{\frac{p-1}{4}}+1 = \begin{cases} 2y_{21}^q, \\ 2^{q-2}y_{22}^q, \end{cases} \quad x^{\frac{p-1}{4}}-1 = \begin{cases} 2^{q-2}y_{22}^q, \\ 2y_{21}^q, \end{cases}$$

where $y_1, y_{21}, y_{22} \in \mathbb{N}$ such that $2y_1y_{21}y_{22} = y$. We get from (5) that

$$(6) \qquad y_{21}^q = 2^{q-3}y_{22}^q + \delta, \quad \delta \in \{-1, 1\}.$$

From (6), we have

$$(7) \qquad \left| q \log \frac{y_{21}}{2y_{22}} - 3 \log 2 \right| = \frac{2}{y_{21}^q + 2^{q-3}y_{22}^q} \sum_{k=0}^{\infty} \frac{(y_{21}^q + 2^{q-3}y_{22}^q)^{-2k}}{2k+1} < \frac{1}{2^{q-5}y_{22}^q}.$$

Since $h(y_{21}/2y_{22}) = \log 2y_{22}$ and $h(2) = \log 2$, by Lemma 2, if $q \geq 4000$, then we may choose $T = 9$ and hence

$$(8) \qquad \left| q \log \frac{y_{21}}{2y_{22}} - 3 \log 2 \right| > \exp\left(-71.8 (\log 2y_{22})(\log 2)(0.52 + \log b)^2\right),$$

where $b = q/\log 2 + 3/\log 2y_{22}$. The combination of (7) and (8) yields

$$\log 32 + 49.21 \left(0.52 + \log\left(\frac{q}{\log 2} + \frac{3}{\log 2y_{22}}\right)\right)^2 > q,$$

whence we deduce that

$$(9) \qquad q < 4200.$$

On the other hand, by Lemma 3, we get from (6) that

$$(10) \qquad \log 2y_{22} = \log \max(y_{21}, 2y_{22}) < 10^{150} q^6 \log 64.$$

Substitute (9) into (10), we obtain $\log 2y_{22} < 2 \cdot 10^{172}$. Further, by (9), we find from (5) that $p < 4 \cdot 10^{176}$. The theorem is proved.

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Department of Mathematics
Hunan Normal University
P.O. Box 410081
Changsha, Hunan
P.R. China

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COINCIDENCES OF COMPOSITES OF ADMISSIBLE U.S.C. MAPS AND APPLICATIONS

by

Sehie Park

Presented by P.A. Fillmore, F.R.S.C.

ABSTRACT. A composite of compact acyclic maps from a convex subset of a locally convex Hausdorff topological vector space into itself has a fixed point. This also holds for a very large class of u.s.c. multifunctions. Generalizations of this new result and their applications to the KKM theory are obtained.

1. Introduction and preliminaries

In this paper, we obtain general coincidence theorems for a very large class of upper semicontinuous multifunctions. From these, we obtain a new fixed point theorem and an extended form of the classical Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem. Consequently, our new results extend many of known theorems in [B, BD2. BDG1,2, GL, H, Hi, L1,2, P1,2. S] and others.

A *multifunction* $F : X \to 2^Y$ is a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-1}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For topological spaces X and Y, F is *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, $F^{-1}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$ is closed in X; and *compact* if $F(X) = \bigcup\{Fx : x \in X\}$ is contained in a compact subset of Y. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets [L1]. Such convex hull is called a *polytope*. Let D be a subset of X and $\langle D\rangle$ the set of all nonempty finite subsets of D. A multifunction $G : D \to 2^X$ is called a *KKM map* if $\operatorname{co} N \subset G(N)$ for each $N \in \langle D\rangle$.

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The *KKM theory* is the study of KKM maps and their applications. For the literature, see [P2,3].

Given a class $\mathbf{X}$ of multifunctions. we denote

$$\mathbf{X}(X,Y) = \{T : X \to 2^Y \mid T \in \mathbf{X}\};$$

$$\mathbf{X}_c = \{T = T_m T_{m-1} \cdots T_1 \mid T_i \in \mathbf{X}\}.$$

Let X be a convex space and Y a topological space. We define

$T \in \Phi(X,Y) \iff T^{-1}y$ is convex for each $y \in Y$ and $\{\operatorname{Int} Tx\}_{x \in X}$ covers Y.

$T \in \mathbb{M}(X,Y) \iff T^{-1}|_K$ has a continuous selection $s : K \to X$ for every nonempty compact subset K of Y such that $s(K) \subset P$ for some polytope P of X.

For topological spaces X and Y, we define

$f \in \mathbb{C}(X,Y) \iff f$ is a (single-valued) continuous function.

$T \in \mathbb{K}(X,Y) \iff T$ is a *Kakutani map*; that is, Y is a convex space and T is u.s.c. with nonempty compact convex values.

$T \in \mathbb{V}(X,Y) \iff T$ is an *acyclic map*; that is, T is u.s.c. with compact acyclic values.

Motivated by [BD2], we introduce an abstract class of multifunctions as follows:

$T \in \mathfrak{A}(X,Y) \iff T$ is an *admissible map*; that is, (i) $\mathfrak{A} \supset \mathbb{C}$; (ii) each $F \in \mathfrak{A}_c$ is u.s.c. and nonempty compact-valued; and (iii) for any polytope P, each $F \in \mathfrak{A}_c(P,P)$ has a fixed point.

Note that $\mathbb{C}$, $\mathbb{K}$, and $\mathbb{V}$ are admissible classes. See [S, P1]. Moreover, the class of approachable maps in topological vector spaces is admissible. See [BD2]. For other examples of admissible classes, see [P3].

2. Coincidences of compact composites of admissible maps

We begin with the following coincidence theorem:

Theorem 1. *Let X be a convex space, Y a topological space, $F \in \mathfrak{A}_c(X,Y)$, and $G \in \mathbb{M}_c(X,Y)$. If F is compact, then F and G have a coincidence point $x_0 \in X$; that is, $Fx_0 \cap Gx_0 \neq \emptyset$.*

Proof. Since F is compact, there exists a compact set K such that $F(X) \subset K \subset Y$. Since $G \in \mathbb{M}_c(X,Y) = \mathbb{M}(X,Y)$ by [B, Lemma 2.1], $G^{-1}|_K$ has a continuous selection $s : K \to P$, where P is a polytope in X. Then $(sF)|_P \in \mathfrak{A}_c(P,P)$ has a fixed point $x_0 \in P$. Since $x_0 \in (sF)x_0$, we have $Fx_0 \cap Gx_0 \supset Fx_0 \cap s^{-1}(x_0) \neq \emptyset$. This completes our proof.

Particular forms of Theorem 1 are appeared in Ben-El-Mechaiekh *et al.* [BDG1,2, B], Ha [H], and Granas and Liu [GL].

Theorem 2. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c(X,Y)$ compact, and $G \in \Phi(X, \overline{F(X)})$. Then F and G have a coincidence point $x_0 \in X$.*

Proof. We show that $G \in \mathbb{M}(X, \overline{F(X)})$. In fact, since $\{\operatorname{Int} Gx : x \in X\}$ covers any compact subset K of $\overline{F(X)}$, there exists an $N = \{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that $\{\operatorname{Int} Gx : x \in N\}$ covers K. Let $\{\lambda_i\}_{i=1}^n$ be the partition of unity subordinated to this cover and $P = \operatorname{co} N \subset X$. Define $f : K \to P$ by

$$fy = \sum_{i=1}^{n} \lambda_i(y)x_i = \sum_{i \in N_y} \lambda_i(y)x_i \quad \text{for } y \in K,$$

where

$$i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in \operatorname{Int} Gx_i.$$

Then $x_i \in G^{-1}y$ for $i \in N_y$. Clearly f is continuous and $fy \in \operatorname{co}\{x_i : i \in N_y\} \subset G^{-1}y$ since $G^{-1}y$ is convex for each $y \in \overline{F(X)}$. Therefore, Theorem 2 follows from Theorem 1.

From Theorem 2, we have the following:

Theorem 3. *Let X and C be nonempty convex subsets of a locally convex Hausdorff topological vector space E, and $F \in \mathfrak{A}_c(X, X + C)$ a compact multifunction. Suppose that one of the following conditions holds:*

(i) *X is closed and C is compact.*

(ii) *X is compact and C is closed.*

(iii) *$C = \{0\}$.*

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Proof. Let V be an open convex neighborhood of the origin 0 in E, and Y a compact set such that $F(X) \subset Y \subset X + C$. Define $G : X \to 2^Y$ by $Gx = (x + C + V) \cap Y$ for $x \in X$. Then each Gx is open in Y and $G^{-1}y = (y - C - V) \cap X$ is convex for each $y \in Y$. Moreover, since $Y \subset X + C$, for every $y \in Y$, there exists an $x \in X$ such that $y \in x + C + V$; it follows that $\{Gx : x \in X\}$ covers Y. This shows $G \in \Phi(X, \overline{F(X)})$. Therefore, by Theorem 2, there exist $x_V \in X$ and $y_V \in Y$ such that $y_V \in Fx_V \cap Gx_V$; that is, $y_V - x_V \in C + V$. In other words, we obtain the assertion:

($*$) for each neighborhood V of 0 in E,

$$(F - i)(X) \cap (C + V) \neq \emptyset,$$

where $i : X \to E$ is the inclusion. Now we consider Cases (i)-(iii).

Case (i). Since X is closed so is $(F-i)(X)$. Since C is compact and E is regular, (*) implies $(F-i)(X)\cap C\neq\emptyset$; that is, there exists an $\hat{x}\in X$ such that $F\hat{x}\cap(\hat{x}+C)\neq\emptyset$.

Case (ii). Since $(F-i)(X)$ is compact and C is closed, the same conclusion follows as in Case (i).

Case (iii). For each neighborhood V of 0 in E, there exist $x_V, y_V\in X$ such that $y_V\in Fx_V$ and $y_V-x_V\in V$. Since $F(X)$ is relatively compact, we may assume that y_V converges to some $\hat{x}$. Then x_V also converges to $\hat{x}$. Since the graph of F is closed in $X\times\overline{F(X)}$, we have $\hat{x}\in F\hat{x}$.

This completes our proof.

Particular forms of Theorem 3 were due to Lassonde [L1], Park [P2], Ben-El-Mechaiekh and Deguire [BD2], Simons [S], Himmelberg [Hi], and many others. For the literature, see [P3].

3. Non-compact case and the KKM theorem

Theorem 2 can be extended to the non-compact case as follows:

Theorem 4. *Let X be a convex space, D a nonempty subset of X, Y a Hausdorff space, $S: D\to 2^Y$, $T: X\to 2^Y$ multifunctions, and $F\in\mathfrak{A}_c(X,Y)$. Suppose that*

(1) *for each $x\in D$, $Sx\subset Tx$ and Sx is compactly open;*

(2) *for each $y\in F(X)$, $T^{-1}y$ is convex;*

(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)}\cap K\subset S(D)$; and*

(4) *for each $N\in\langle D\rangle$, there exists a compact convex subset L_N of X containing N such that $F(L_N)\backslash K\subset S(L_N\cap D)$.*

Then F and T have a coincidence point.

Proof. Since $\overline{F(X)}\cap K$ is compact, by (1) and (3), there exists an $N\in\langle D\rangle$ such that $\overline{F(X)}\cap K\subset S(N)$. Let L_N be the set in (4). We claim that $F(L_N)\subset S(L_N\cap D)$. Note that

$$F(L_N)\cap K\subset\overline{F(X)}\cap K\subset S(N)\subset S(L_N\cap D).$$

On the other hand, $F(L_N)\backslash K\subset S(L_N\cap D)$ by (4). Hence, we have $F(L_N)\subset S(L_N\cap D)$.

Note that $F(L_N)$ is compact since it is the image of the compact set L_N under the composite F of compact-valued u.s.c. multifunctions. Now we apply Theorem 2 with $F|_{L_N}$, $T|_{L_N}$, L_N and $F(L_N)$ replacing F, G, X and Y, respectively. Note that $T|_{L_N}$ has convex fibers $T^{-1}y\cap L_N$ for each $y\in F(L_N)$ by (2), and $S(L_N\cap D)$ covers $F(L_N)$; that is, $\{\operatorname{Int}Tx : x\in L_N\cap D\}$ covers $\overline{F(L_N)}=F(L_N)$, where $\operatorname{Int}Tx$ denotes the interior of Tx in $F(L_N)$. Hence, all of the requirements are satisfied, and thus $F|_{L_N}$ and $T|_{L_N}$ have a coincidence point $x_0\in L_N$. This completes our proof.

Theorem 4 is equivalent to the following generalization of the KKM theorem:

Theorem 5. *Let X be a convex space, D a nonempty subset of X, Y a Hausdorff space, and $F \in \mathfrak{A}_c(X, Y)$. Let $G : D \to 2^Y$ be a multifunction such that*

(5) *for each $x \in D$, Gx is compactly closed in Y;*

(6) *for any $N \in \langle D \rangle$, $F(\mathrm{co}\, N) \subset G(N)$; and*

(7) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof of Theorem 5 using Theorem 4. Let $S : D \to 2^Y$ and $T : X \to 2^Y$ be defined by $Sx = Y \backslash Gx$ for $x \in D$ and $T^{-1}y = \mathrm{co}\, S^{-1}y$ for $y \in Y$. Then (1) and (2) hold by (5) and the definitions of S and T. Suppose that $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} = \emptyset$; that is, $\overline{F(X)} \cap K \subset S(D)$. Then (3) holds. Note that (4) and (7) are equivalent. Therefore, by Theorem 4, F and T have a coincidence point $x_0 \in X$. For $y \in Fx_0 \cap Tx_0$, we have $x_0 \in T^{-1}y = \mathrm{co}\, S^{-1}y$ and hence, there exists an $N \in \langle S^{-1}y \rangle \subset \langle D \rangle$ such that $x_0 \in \mathrm{co}\, N$. Since $y \in Fx_0 \subset F(\mathrm{co}\, N) \subset G(N) = Y \backslash \bigcap\{Sx : x \in N\}$ by (6), $y \notin Sx$ for some $x \in N$; that is, $x \notin S^{-1}y$ and $x \in N$, which is a contradiction. This completes our proof.

Proof of Theorem 4 using Theorem 5. Let $Gx = Y \backslash Sx$ for $x \in D$. Then (5) and (7) follow from (1) and (4), resp. Moreover, from (3), we have

$$\overline{F(X)} \cap K \subset S(D) = \bigcup_{x \in D} (Y \backslash Gx) = Y \backslash \bigcap_{x \in D} Gx,$$

contrary to the conclusion of Theorem 5. Therefore, F and G do not satisfy (6) and hence, there exist an $N \in \langle D \rangle$ and a $y \in F(\mathrm{co}\, N) \backslash G(N)$; that is, $y \notin Gx$ or $y \in Sx$ for all $x \in N$. This implies $y \in Tx$ or $x \in T^{-1}y$ for all $x \in N$. Since $y \in F(\mathrm{co}\, N) \subset F(X)$, $T^{-1}y$ is convex by (2). Therefore, $\mathrm{co}\, N \subset T^{-1}y$. Note that $y \in Fx_0$ for some $x_0 \in \mathrm{co}\, N \subset T^{-1}y$, and hence $y \in Fx_0 \cap Tx_0$. This completes our proof.

Theorems 4 and 5 include very large number of known results and have many applications in the fixed point theory and the KKM theory. For the details, see [P3].

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SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA

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SPECTRAL RIGIDITY OF HOPF SURFACES

by Domenico PERRONE[1]

Presented by P.C. Greiner, F.R.S.C.

Summary. We characterize Hopf surfaces in the class of compact generalized Hopf manifolds by the spectrum of the Laplacian.

1. Introduction

Let (M,g) be a compact (connected and C^∞) Riemannian n-manifold with metric tensor g. We denote by $\mathrm{Spec}(M,g) = \{0 = \lambda_0 \leq \lambda_1 \leq \ldots\ldots \uparrow +\infty\}$ the spectrum of the Laplacian Δ acting on functions, where each eigenvalue λ is written a number of times equal to its multiplicity.

The following problem :

(1.1) " when Spec(M,g) determines the geometry of (M,g)? "

is very interesting, it has been studied by many people and there are many of papers devoted to this broad question.

Bedford and Suwa [1] determined the spectrum of the Laplacian Δ of a Hopf manifold M_α and given some isospectral results in the class of the Hopf manifolds. More precisely they proved that : if M_α and $M_{\alpha'}$ are two Hopf manifolds, $\dim_{\mathbb{C}} M = n$, with $\alpha = e^{-2\pi(b-ia)}$ and $b\sqrt{2n+1} > 1/2$ or $b\sqrt{2n+1} > |a|$, then $\mathrm{Spec}(M_\alpha) = \mathrm{Spec}(M_{\alpha'})$ implies that $M_{\alpha'}$ is isometric (as a Riemannian manifold) to M_α.

In order to study the general question (1.1), a useful tool is the following formula of Minakshisundaram

$$(1.2) \qquad \sum_{k=0}^{\infty} \exp(-\lambda_k t) = (4\pi t)^{-n/2} \sum_{i=0}^{m} a_i t^i + O(t^{m+1-n/2})$$

where the coefficients a_0, a_1 and a_2 were calculated by Mckean and Singer [6] and Berger [2], and a_3 was calculated by Sakai [7].

In this paper we study spectral rigidity of a *Hopf surface*, namely a Hopf manifold of real dimension 4, which in the sequel we denote by H_α. Our results are the following.

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THEOREM 1. *Let (M,g) be a compact Riemannian manifold with Euler-Poincare' characteristic $\chi(M) \leq 0$. If $Spec(M,g) = Spec(H_\alpha)$, then (M,g) is locally isometric to H_α.*

THEOREM 2. *Let (M,J,g) be a compact g.H.m. (i.e. generalized Hopf manifold). If $Spec(M,g) = Spec(H_\alpha)$, then (M,g) is locally isometric to H_α.*

THEOREM 3. *If $\alpha = e^{-2\pi(b-ia)}$ with $b2\sqrt{5}>1$, the only compact g.H.m. (M,J,g) which have $Spec(M,g) = Spec(H_\alpha)$ are the Hopf manifold $H_{\bar{\alpha}}$ and the Hermitian manifold $N_\alpha = V/G$, where $V=(\mathbb{C}^2-\{0\},(\Sigma dz_i d\bar{z}_i)/\|z\|^2)$ and G is the infinite group generated by the biholomorphic isometry*

$$f_\alpha = e^{-2\pi b}\begin{bmatrix} e^{-ia'} & 0 \\ 0 & e^{ia'} \end{bmatrix}$$

with $a' = 2\pi\,|\,a\,|$; $H_{\bar{\alpha}}$ and N_α are both globally isometric, as Riemannian manifolds, to H_α.

Remark. Tsukada [8] obtains isospectral results on Hopf manifolds in the class of compact Hermitian manifolds, of complex dimension ≥ 6, using the spectrum of two real Laplacians and three complex Laplacians.

2. Hopf manifolds and generalized Hopf manifolds

We define Hopf manifolds following [1]. Let V be $\mathbb{C}^n-\{0\}$ with $n \geq 2$, and let α be a complex number with $0<|\alpha|<1$. The analytic automorphism $g_\alpha(z_1,z_2,\ldots,z_n) = (\alpha z_1,\alpha z_2,\ldots,\alpha z_n)$ of V, generates an infinite cyclic group G_α acting on V freely and properly discontinuously. Thus the quotient $M_\alpha = V/G_\alpha$ is an n-dimensional complex manifold, which is called a *Hopf manifold*. H_α does not admit a Kaehler metric, but the hermitian metric $g_0 = (\Sigma dz_i d\bar{z}_i)/\|z\|^2$ on V is G_α-invariant and hence induces a Hermitian metric on M_α. In the sequel we consider M_α with this metric as a Riemannian manifold. It is known that, if α is real, M_α is globally isometric to $S^1 x S^{n-1}$; in general M_α is diffeomorphic and locally isometric to $S^1 x S^{n-1}$.

Let (M,g,J) be a Hermitian manifol with metric g, complex structure J fundamental 2-form Ω and of complex dimension $n\geq 2$. Recall that by [9], (M,g,J) is called a *generalized Hopf manifold (shortly g.H.m.)* if it is *locally conformally Kaehler (shortly l.c.K.)* and its Lee form $\omega=(1/(2n-1))\delta\Omega\cdot J$ is parallel, i.e. $\nabla\omega=0$ $(\omega\neq 0)$, where ∇ is the the

Levi-Civita connection of g. Of course Hopf manifolds are examples of g.H.m. A l.c.K. manifold whose l.c.K. structure consists of flat Kaehler metrics, is called *locally conformally Kähler (shortly l.c.K_0) manifold.*

In the sequel, for a Riemannian manifold (M,g), by $R=(R_{ijkh})$, $\rho=(\rho_{ij})$, τ and $W=(W_{ijkh})$, we denote the curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal tensor. Recall that

$$W_{ijkh} = R_{ijkh} - \frac{1}{m-2}(\rho_{jh}g_{ik} + \rho_{ik}g_{jh} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih}) + \frac{\tau}{(m-1)(m-2)}(g_{ik}g_{jh} - g_{ih}g_{jk}) \tag{2.1}$$

where m=dimM. For an tensor field T on (M,g), $\|T\|$ denotes the norm of T with respect to g. Then, we have

$$\|W\|^2 = \|R\|^2 - \frac{4}{m-2}\|\rho\|^2 + \frac{2}{(m-1)(m-2)}\tau^2$$

We note that a Hermitian manifold (M,J,g) is a *l.c.K_0* manifold iff it is l.c.K. and locally conformally flat (i.e. the tensor W=0). Finally a *l.c.K.* manifold is called a *P_0K-manifold* if it is *l.c.K_0*. and its Lee form is parallel, in other words if it is a conformally flat *g.H.m.*.

3. Proof of Theorem 1

Let (M,g) be a compact Riemannian manifold with $\chi(M)\leq 0$ and H_α a Hopf surface. Assume Spec(M,g)=Spec(H_α). Then, by formula (1.2), we get that dimM=dimH_α=4 and $a_k=a'_k$ for any k, where the a'_k are the corresponding coefficients for H_α. The first four coefficients a_k are given (see [2], [6], [7]) by

$$a_0 = \mathrm{vol}(M,g), \quad a_1 = \frac{1}{6}\int_M \tau dv, \quad a_2 = \frac{1}{360}\int_M \{2\|R\|^2 - 2\|\rho\|^2 + 5\tau^2\}dv,$$

$$a_3 = \frac{1}{6!}\int_M \{-\frac{1}{9}\|\nabla R\|^2 - \frac{26}{63}\|\nabla\rho\|^2 - \frac{142}{63}\|\nabla\tau\|^2 + \frac{5}{9}\tau^3 - \frac{2}{3}\tau\|\rho\|^2 + \frac{2}{3}\tau\|R\|^2 + \frac{8}{21}A - \frac{8}{63}B + \frac{20}{63}C - \frac{4}{7}D\}dv,$$

where
$$A = \sum R^{ij}{}_{kh} R^{kh}{}_{pq} R^{pq}{}_{ij}, \qquad B = \sum \rho_{ij} R^{i}{}_{abc} R^{jabc},$$
$$C = \sum \rho^{ij}\rho^{kh} R_{ikjh}, \qquad D = \sum \rho^{i}{}_{j}\,\rho^{j}{}_{k}\,\rho^{k}{}_{i}.$$

Moreover the Euler-Poincaré characteristic $\chi(M)$ can be expressed by (cf.[2] p.225)

(3.1) $$\chi(M) = (1/32\pi^2)\int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\}dv.$$

Since H_α is conformally flat, $a_2 = a_2'$ implies

(3.2) $$\int_{H_\alpha}\{2\|\rho'\|^2 + (13/3)\tau'^2\}dv' = \int_M \{2\|W\|^2 + 2\|\rho\|^2 + (13/3)\tau^2\}dv.$$

Moreover, $a_0 = a_0'$, $a_1 = a_1'$, $\tau' =$ const. and the Schwarz inequality imply

(3.3) $$\int_{H_\alpha}\tau'^2 dv' \le \int_M \tau^2 dv,$$

where the equality holds iff $\tau =$ const. $= \tau'$. Now, since H_α is conformally flat with $\chi(H_\alpha)=0$, using (3.1), (3.2) and (3.3), we get

$$32\pi^2\chi(M) = 32\pi^2\{\chi(M) - \chi(H_\alpha)\} = \int_M \{\|W\|^2 - 2\|\rho\|^2 + (2/3)\tau^2\}dv +$$

$$\int_{H_\alpha}\{2\|\rho'\|^2 - (2/3)\tau'^2\}dv' = \int_M 3\|W\|^2 + 5\{\int_M \tau^2 dv - \int_{H_\alpha}\tau'^2 dv'\} \ge 0.$$

Therefore, $\chi(M) \le 0$ implies that M is conformally flat with $\chi(M) = 0$,

(3.4) $$\tau = \text{const.} = \tau' \quad \text{and} \quad \int_M \|\rho\|^2 dv = \int_{H_\alpha}\|\rho'\|^2 dv'.$$

On the other hand, for any conformally flat manifold of dimension 4 and with $\tau =$ const., the following formula holds (see [3])

(3.5) $$\frac{1}{2}\Delta\|\rho\|^2 = 2D - \frac{7}{6}\tau\|\rho\|^2 + \frac{1}{6}\tau^3 + \|\nabla\rho\|^2.$$

Integrating this formula, we have

(3.6) $$6\int_M \|\nabla\rho\|^2 dv = \int_M (7\tau\|\rho\|^2 - 12D - \tau^3)dv.$$

For H_α, since $\|\rho'\|^2 =$ const and $\|\nabla\rho'\|^2 = 0$, from (3.5) we get

(3.7) $$12D' = 7\tau'\|\rho'\|^2 - \tau'^3$$

By (3.4), (3.6) and (3.7), we obtain

(3.8) $$\int_M \|\nabla\rho\|^2 dv = 2\{\int_{H_\alpha} D'dv' - \int_M D dv\}.$$

Since M is conformally flat, by (2.1), A, B and C are given by

$$\tau\|\rho\|^2 - \frac{2}{9}\tau^3, \quad \frac{5}{6}\tau\|\rho\|^2 - \frac{1}{6}\tau^3 \text{ and } \frac{7}{6}\tau\|\rho\|^2 - \frac{1}{6}\tau^3 - D.$$

Moreover, we have $\|\nabla R\|^2 = 2\|\nabla\rho\|^2 - (1/3)\|\nabla\tau\|^2$ and $\|R\|^2 = 2\|\rho\|^2 - (1/3)\tau^2$; the corresponding identies hold for H_α. So $a_3 = a_3'$ and (3.4) imply

(3.9) $$5\int_M \|\nabla\rho\|^2 dv + 7\int_M D dv = 5\int_{H_\alpha}\|\nabla\rho'\|^2 dv' + 7\int_{H_\alpha} D'dv'.$$

Moreover H_α has Ricci tensor parallel, therefore from (3.8) and (3.9) we get that $\nabla\rho = 0$. If (M,g) is irreducible, it is an Einstein space and hence (since W=0) of constant sectional curvature; then $\chi(M)=0$ and (3.1) imply that (M,g) is flat. This is a contradiction because $\tau = \tau' \neq 0$. So (M,g) is reducible and hence it has as its simply connected Riemannian covering one of the following spaces (cf. [5]) :

$$R \times S^3(c),\ R^4,\ R \times H^3(-c),\ S^2(c) \times H^2(-c).$$

Then, $\tau = \tau'$ implies that (M,g) is locally isometric to $S^1 \times S^3$ and thus locally isometric to H_α.

4. Proof of Theorems 2,3.

Proof of Theorem 2. By Corollary 3.3 of [9] we have the vanishing of the Euler-Poincaré characteristic of a compact *g.H.m.*. So, Theorem 2 is a consequence of the Theorem 1.

Proof of Theorem 3. Let (M,g,J) be a compact g.H.m. with Spec(M,g) = Spec(H_α). By Theorem 1 we have that (M,g,J) is a 4-dimensional compact P_0K-manifold with constant scalar curvature $\tau=6$. Now, we show that the Lee form ω of M has $\|\omega\|=1$. Let ξ the Lee vector field, namely $\omega=g(\xi,\cdot)$. From

$$(d\|\omega\|^2)(X) = X(\|\omega\|^2) = Xg(\xi,\xi) = 2\omega(\nabla_X\xi),$$

and

$$X(\|\omega\|^2) = X\omega(\xi) = (\nabla_X\omega)(\xi) + \omega(\nabla_X\xi),$$

we get

$$d\|\omega\|^2 = 2(\nabla\omega)(\xi) = 0$$

and hence $\|\omega\|$ = constant. Moreover

$$\delta\omega = -\operatorname{Tr}\nabla\omega = 0.$$

So, using the classical relation between the scalar curvature of two conformally related Riemannian metrics (cf. for example [4]), we obtain that the scalar curvature of (M,g,J) is given by

$$\tau = 6\|\omega\|^2.$$

Since $\tau=6$, we get $\|\omega\|^2=1$.

Now, the conditions of prop.4.5 of [8] apply and thus M is $S \times R/\Gamma$, where S is a 3-dimensional compact Sasakian manifold of constant sectional curvature 1 and Γ is an infinite cyclic group. Moreover, we have assumed $b = (-2\pi)^{-1}\operatorname{Re}\log\alpha > 1/2\sqrt{5}$. Then, by the proof of lemma 6.4 of [8], we obtain that (M,g,J) is biholomorphically isometric to a 4-dimensional Hermitian manifold $M_{c,d} = (\mathbb{C}^2-\{0\},(\Sigma dz_i d\bar{z}_i)/\|z\|^2)/G$, where G is the infinite group generated by the biholomorphic isometry $f_{c,d}$ defined by

$f_{c,d}(z_1,z_2) = e^{-2\pi d}(e^{2\pi i c'}z_1, e^{2\pi i c''}z_2)$ with $d>0$ and $-\frac{1}{2}\leq c'\leq c''\leq \frac{1}{2}$.

Finally, $b>1/2\sqrt{5}$ and $\mathrm{Spec}(M_{c,d}) = \mathrm{Spec}(M,g) = \mathrm{Spec}(H_\alpha)$ imply (cf. lemma 5.4 of [8]) that $d=b$ and $c'^2 = a^2 = c''^2$. Thus, we can have the following cases :

1°) $c' = c'' = a$; 2°) $c' = c'' = -a$; 3°) $c' = -|a|$, $c'' = |a|$.

In the case 1°), $M_{c,d}$ is biholomorphically isometric to H_α. In the case 2°), the diffeomorphism of $\mathbb{C}^2-\{0\}$ defined by $(z_1,z_2)\longmapsto(\bar{z}_1,\bar{z}_2)$ induces an isometry between $M_{c,d} = H_{\bar{\alpha}}$ and H_α. In the case 3°), if $a\geq 0$ (resp. $a\leq 0$), the diffeomorphism of $\mathbb{C}^2-\{0\}$ defined by

$$(z_1,z_2)\longmapsto(\bar{z}_1,z_2) \quad (\text{resp. } (z_1,z_2)\longmapsto(z_1,\bar{z}_2))$$

induces an isometry between $M_{c,d}$ and H_α.

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Dipartimento di Matematica
Universita' degli Studi di Lecce
73100 Lecce, Italy.

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OPERATOR REPRESENTATIONS FOR SIMILARITY SOLUTIONS OF 2+1-D EVOLUTION EQUATIONS

Oleg I. Bogoyavlenskij[1] William F. Shadwick[1,2]

Presented by J.E. Marsden, F.R.S.C.

Abstract. A Lax representation for similarity solutions of the KP equation is found. Three families of similarity solutions of a 2 + 1-dimensional equation with breaking solitons are determined and operator representations for them are obtained.

I. It is well known that one of the major advances in the study of integrable partial differential equations was the discovery that they could be written in what is now called a Lax representation. This formulation, in terms of commutators of differential operators provides insights into properties of the equations that are far from obvious in the normal coordinate representation. This formulation provides immediate access to the associated scattering problem, the infinity of conserved quantities and the action-angle variables in which the integrability of the system is manifest. It is clear that the Lax representation provides a highly efficient means of encoding key properties of the differential equation. It remains a problem to provide means other than trial and error for establishing that such a formulation either does or does not exist for a given equation.

Another important theme in the study of integrable partial differential equations is the equations satisfied by the similarity solutions [1-8] (which the equations always possess by virtue of their scale invariance). In the case of 1 + 1-dimensional evolution equations, the similarity solutions satisfy ordinary differential equations with the Painleve property and it is conjectured that it is necessary for complete integrability of the full equations [1]. In reference [4] Flaschka showed that the equation satisfied by the similarity solutions for the Korteweg-de Vries equation has its own operator representation

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(1) $$L + [L, A] = 0,$$

where the operators L and A are determined by the formulae

(2) $$L = -\partial_z^2 + v, \quad A = 6\partial_z^3 - \frac{9}{2}(v\partial_z + \partial_z v) - \frac{1}{2}z\partial_z, \quad z = \frac{x}{t^{1/3}}.$$

In this paper we show that similarity solutions of some 2 + 1-dimensional integrable nonlinear equations admit operator representations, which differ from (1).

The main method of this work consists of the following. The considered 2 + 1-D evolution equations have certain operator representations. We change variables t, x, y in the corresponding operators to the variables t, z, τ, where z and τ are the similarity variables. The nontrivial fact is that for the similarity solutions the variable t can be excluded from these operator representations. After that the validity of the derived operator representations for the similarity solutions of the considered 2+1-D evolution equations can be checked by a direct verification.

II. The KP equation

(3) $$(u_t - 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy}$$

has the known Lax representation

(4) $$[L_0, \quad \partial_t + A_0] = 0$$

with operators L_0 and A_0 of the form

(5) $$L_0 = -\partial_x^2 + \sigma\partial_y + u, \quad A_0 = 4\partial_x^3 - 3(u\partial_x + \partial_x u) + w.$$

Here the functions u and w are connected by the relation $3\sigma u_y = -w_x$.

Similarity solutions of the KP equation (3) have the form

(6) $$u = \frac{1}{t^{2/3}}v(z, \tau), \quad w = \frac{1}{t}w_0(z, \tau), \quad \text{where } z = \frac{x}{t^{1/3}}, \quad \text{and } \tau = \frac{y}{t^{2/3}}.$$

For solutions of this form the KP equation is reduced to the equation [3,6]

(7) $$\left(\frac{2}{3}v + \frac{1}{3}zv_z + \frac{2}{3}\tau v_\tau + 6vv_z - v_{zzz}\right)_z = 3\sigma^2 v_{\tau\tau}.$$

Proposition 1 *Equation (7), describing similarity solutions (6) of the KP equation, admits an equivalent Lax representation of the form*

$$[L_1, \quad \sigma\partial_\tau + A_1] = 0 \tag{8}$$

with linear operators

$$\begin{aligned} L_1 &= 4\partial_z^3 - \frac{2\tau}{3\sigma}\partial_z^2 - 3(v\partial_z + \partial_z v) - \frac{1}{3}z\partial_z + \frac{2\tau}{3\sigma}v + w_0, \\ A_1 &= -\partial_z^2 + v. \end{aligned} \tag{9}$$

The proof consists of a direct verification.

III. Let us consider the following 2+1-dimensional equation having breaking solitons [8,9]

$$u_{tx} = 4u_x u_{xy} + 2u_y u_{xx} - u_{xxxy}. \tag{10}$$

Equation (10) has three families of similarity solutions

$$u(t,x,y) = t^\alpha v(z,\tau), \quad z = xt^\alpha, \quad \tau = yt^{-1-2\alpha}, \tag{11}$$

$$u(t,x,y) = e^t v(z,\tau), \quad z = xe^t, \quad \tau = ye^{-2t}, \tag{12}$$

$$u(t,x,y) = f(t,y)V(\zeta), \quad \zeta = f(t,y)x. \tag{13}$$

Solutions (11) depend on an arbitrary parameter α. Solutions (12) can be obtained from (11) as a limiting case as $\alpha \longrightarrow \pm\infty$. Solutions (13) imply that the function $f(t,y)$ is an arbitrary solution of the following equation, equivalent to the invicid Burgers equation:

$$f_t = cf^2 f_y, \quad c = \text{const}. \tag{14}$$

All functions (13) have breaking graphs in view of equation (14).

Equation (10) for similarity solutions (11) reduces to the form

$$2\alpha v_z + \alpha z v_{zz} - (1+2\alpha)\tau v_{z\tau} = 4v_z v_{z\tau} + 2v_\tau v_{zz} - v_{zzz\tau}. \tag{15}$$

Equation (10) for similarity solutions (12) reduces to the equation

$$2v_z + zv_{zz} - 2\tau v_{z\tau} = 4v_z v_{z\tau} + 2v_\tau v_{zz} - v_{zzz\tau}. \tag{16}$$

Equation (10) for similarity solutions (13) reduces to the equation

(17) $$f_t(V+\zeta V')' = f^2 f_y(4V'(\zeta V)'' + 2V''(\zeta V)' - (\zeta V)''''),$$

where $V' = dV/d\zeta$. This equation implies that the function $f(t,y)$ satisfies the partial differential equation (14) and the function $V(\zeta)$ satisfies the ordinary differential equation

(18) $$(\zeta V)'''' + c(\zeta V)'' = 4V'(\zeta V)'' + 2V''(\zeta V)'.$$

Equation (18) can be reduced (by scale transformations) to two cases when $c = 1$ and $c = 0$ (the last case describes the stationary similarity solutions of equation (10)).

Equation (18), after multiplying by $(\zeta V)'$ and integrating takes the form

(19) $$v'''v' - \frac{1}{2}v''^2 + \frac{c}{2}v'^2 = 2(\frac{v}{\zeta})'v'^2 + d,$$

where $v = \zeta V$, and d is a constant of integration. The order of equation (19) can be reduced for the special case $c = 0$, $d = 0$. After substituting

(20) $$v'(\zeta) = \frac{1}{\zeta}w(z), \quad z = v + \frac{3}{4}$$

equation (19) for $c = d = 0$ takes the form

(21) $$w'' + \frac{1}{2w}w'^2 - \frac{2}{w}w' + \frac{2z}{w} - 2 = 0,$$

where $w' = dw/dz$. It is natural to conjecture that equation (21) possesses the Painleve property.

Proposition 2 *Equation (15) has an equivalent operator representation*

(22) $$2\alpha L - 2(LL_\tau + L_\tau L) - (1+2\alpha)\tau L_\tau = [L, A],$$

where

$$L = -\partial_z^2 + v_z, \quad A = \alpha z\partial_z - v_\tau\partial_z - \partial_z v_\tau.$$

Equation (16) is equivalent to the operator equation

(23) $$2L - 2(LL_\tau + L_\tau L) - 2\tau L_\tau = [L, A],$$

where

$$L = -\partial_z^2 + v_z, \quad A = z\partial_z - v_\tau\partial_z - \partial_z v_\tau.$$

Equation (18) has an equivalent operator representation

(24) $$L^2 - \frac{c}{4}L + [L, A] = 0,$$

where

(25) $$L = -\partial_\zeta^2 + V',$$

$$\begin{aligned} 8A &= c\zeta\partial_\zeta - 2\{\zeta\partial_\zeta, L\} - \{V + \zeta V', \partial_\zeta\} \\ &= 4\zeta\partial_\zeta^3 + 4\partial_\zeta^2 + (c\zeta - 6\zeta V' - 2V)\partial_\zeta - 3\zeta V'' - 2V' \end{aligned}$$

and the usual notation for the anticommutator is used: $\{x, y\} = xy + yx$.

The proof consists again of a direct verification.

We note that the operator equation (24) differs in an essential way from equation (1).

These derived operator representations lead to a new type of behaviour of the eigenvalues of the corresponding Schrodinger operator $L = -\partial_z^2 + v_z$, under the assumption that $|v_z(z,t)| \longrightarrow 0$ as $z \longrightarrow \pm\infty$. For example, for the operator equation (22) the eigenvalues $f_k(t)$ of the operator L satisfy the equation

(26) $$f_{k\tau} = \frac{2\alpha f_k}{(1+2\alpha)\tau + 4f_k}.$$

This equation is equivalent to the linear system

(27) $$\dot{f}_k = 2\alpha f_k, \quad \dot{\tau} = (1+2\alpha)\tau + 4f_k.$$

Integrating (27), we obtain the following first integral

(28) $$F = (\tau + 4f_k)f_k^{-1-1/2\alpha}.$$

Thus, all trajectories of (26) are determined by the equation

(29) $$\tau = f_k(F f_k^{1/2\alpha} - 4).$$

The function $f_k(\tau)$ (29) for $F > 0$ is multivalued in some domain of the variable τ.

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Department of Mathematics, Queen's University, Kingston, K7L 3N6.

The Fields Institute for Research in Mathematical Sciences and Pure Mathematics Department, University of Waterloo, Ontario, N2L 3G1.

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EXISTENCE OF RIEMANN INVARIANTS AND HAMILTONIAN STRUCTURES

O. I. Bogoyavlenskij

Presented by J.E. Marsden, F.R.S.C.

Abstract. A necessary and sufficient condition is found for the existence of Riemann invariants for a hyperbolic system of équations. The necessary criteria are pointed out for the existence of a Hamiltonian and bi-Hamiltonian structures for a nonlinear system of first order. The results are formulated in terms of an intrinsic geometry, connected with the Nijenhuis and Haantjes tensors of an operator tensor field.

I. Riemann invariants of a nonlinear hyperbolic system of equations in one space dimension,

$$u_t^i = \sum_{j=1}^{n} A_j^i(u) u_x^j, \tag{1}$$

are coordinates $v^1(u), \ldots, v^n(u)$, in which this system has the diagonal form

$$v_t^i = \lambda_i(v^1, \ldots, v^n) v_x^i. \tag{2}$$

Existence of Riemann invariants for $n = 2$ has been proven by Riemann in his work [1]. Nonexistence of such invariants for a general system (1) for $n \geq 3$ is well known [2-4].

II. Let $u, v \in T_p(M^n)$ be two tangent vectors at a point $p \in M^n$ in an n-manifold M^n, and let $\tilde{u}$ and $\tilde{v}$ be any smooth vector fields, extending the vectors u and v. The Nijenhuis tensor $N(u,v)$, determined by a given (1,1) tensor field $A : TM^n \to TM^n$, has the form [5]

$$N(u,v) = A^2[\tilde{u}, \tilde{v}] + [A\tilde{u}, A\tilde{v}] - A([A\tilde{u}, \tilde{v}] + [\tilde{u}, A\tilde{v}]) \tag{3}$$

and does not depend on the prolongations $\tilde{u}$, $\tilde{v}$. Haantjes tensor $H(u,v)$ is [6]

$$H(u,v) = A^2 N(u,v) + N(Au, Av) - A(N(Au,v) + N(u, Av)). \tag{4}$$

For our applications we choose A to be defined by equation (1) and choose $M^n = R^n$.

Theorem 1 *A hyperbolic system of equations (1) possesses n Riemann invariants if and only if the Haantjes tensor $H(u,v)$(4) vanishes.*

Proof. By definition of hyperbolicity the (1,1) tensor field $A^i_j(p)$, $p \in M^n$ for a hyperbolic system (1) has real eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$ and a diagonal Jordan normal form [2]. Let $r^i_1(p), \ldots, r^i_n(p)$ and $\ell^1_i(p), \ldots, \ell^n_i(p)$ be vector fields associated with right (contravariant) and left (covariant) eigenvectors for the operator A:

$$Ar_k = \lambda_k r_k, \qquad \ell^k A = \lambda_k \ell^k, \tag{5}$$

i.e. with the eigenvectors of the operator A and its dual.
If $\lambda_j \neq \lambda_k$ then $(\ell^j, r_k) = 0$. Suppose at first that all eigenvalues $\lambda_i(p)$ are distinct and consider fields $S_k(p)$ of the hyperplanes, generated by the eigenvectors $r_1(p), \ldots, r_{k-1}(p), r_{k+1}(p), \ldots, r_n(p)$. For the Haantjes tensor one has [5,6]

$$H(r_i, r_j) = (A - \lambda_i)^2 (A - \lambda_j)^2 [r_i, r_j]. \tag{6}$$

Therefore if $H(u,v) \equiv 0$ then $[r_i, r_j] = a_{ij} r_i + b_{ij} r_j$ and hence the fields of the hyperplanes $S_k(p)$ are integrable. Let $v^k(u^1, \ldots, u^n)$ be a smooth function, constant on the integral submanifolds, tangent to the field $S_k(p)$. Then $dv^k(r_j) = 0$, $j \neq k$ and hence $(dv^k)A = \lambda_k (dv^k)$. Using this equality, we obtain

$$v^k_t = \sum_{i=1}^n \frac{\partial v^k}{\partial u^i} u^i_t = \sum_{i,j=1}^n \frac{\partial v^k}{\partial u^i} A^i_j u^j_x = \lambda_k \sum_{j=1}^n \frac{\partial v^k}{\partial u^j} u^j_x = \lambda_k v^k_x. \tag{7}$$

Therefore in the new coordinates $v^1, \ldots, v^n$ system (1) has a diagonal form (2), thus $v^1, \ldots, v^n$ are Riemann invariants.

If an eigenvalue λ_k has multiplicity n_k then we consider the field S_{n-n_k} of planes generated by $n - n_k$ other eigenvectors r_j, and choose n_k independent functions $v^k_1, \ldots, v^k_k$, constant on the corresponding integral submanifolds. Obviously $(dv^k_i)A = \lambda_k (dv^k_i)$, and the equality (7) follows.

Therefore the condition $H(u,v) \equiv 0$ is sufficient for the existence of Riemann invariants. Its necessity follows immediately from the formula (6) for a diagonal system (2).

Theorem 2 *A non-hyperbolic system (1) with operator tensor field $A^i_j(p)$ having distinct (complex) eigenvalues has the following properties:*

1. The system is transformed locally into a block-diagonal form with blocks of sizes 2×2 and 1×1 if and only if the Haantjes tensor $H(u,v)$(4) vanishes.

2. The system is split into noninteracting subsystems of two or one equations if and only if the Nijenhuis tensor $N(u,v)$(3) vanishes.

3. The system is reduced locally to two diagonal blocks of sizes $k \times k$ and $(n-k) \times (n-k)$ if and only if the tangent bundle $T(M^n)$ splits as $L_1 + L_2$, where $\dim L_1 = k$, $\dim L_2 = n-k$ and $H(L_i, L_i) \subset L_i$, $A(L_i) \subset L_i$.

4. The system is split into two noninteracting subsystems of k and $n-k$ equations if and only if $T(M^n)$ splits as $L_1 + L_2$, where $N(L_i, L_i) \subset L_i$ and $A(L_i) \subset L_i$.

III.

Theorem 3 *The Haantjes tensor $H(u,v)$ has the following properties:*

1. The Haantjes tensor is a gauge and scale invariant: Under a transformation of the (1,1) tensor field $A(p) \longrightarrow f(p)A(p) + g(p)1$ (where $f(p)$ and $g(p)$ are arbitrary functions, and 1 is the identity tensor field) the Haantjes tensor is transformed as $H(p) \longrightarrow f^4(p)H(p)$.

2. For $n = 3$ the Jacobi identity is satisfied for any operator tensor field $A^i_j(p)$:

$$H(H(u,v),w) + H(H(v,w),u) + H(H(w,u),v) = 0.$$

Therefore for $n = 3$ the Haantjes tensor $H(u,v)$ defines a structure of Lie algebra in each tangent space $T_p(M^3)$.

IV. For any vector $u \in T_p(M^n)$ we define linear operators N_u, M_u and H_u

$$N_u(v) = N(u,v), \qquad M_u = N_{Au} - AN_u, \qquad H_u(v) = H(u,v).$$

The relation $H_u = [M_u, A]$ follows from (4).

We define three symmetric bilinear forms on $T_p(M^n)$:

$$(u,v)_N = \mathrm{Tr}(N_u N_v), \qquad (u,v)_M = \mathrm{Tr}(M_u M_v), \qquad (u,v)_H = \mathrm{Tr}(H_u H_v).$$

On each tangent space $T_p(M^n)$ we define three polynomials

$$P_N(u) = \det(N_u - 1), \qquad P_M(u) = \det(M_u - 1), \qquad P_H(u) = \det(H_u - 1).$$

Zero levels of these polynomials define three algebraic varieties V_N, V_M and V_H, embedded into the tangent space $T_p(M^n)$. Therefore in the whole we get three fibrations of the algebraic varieties V_N, V_M and V_H, embedded into the tangent bundle $T(M^n)$.

We determine the following (3,1) tensors

$$J_N(u,v) = N_{N(u,v)} - [N_u, N_v], \qquad J_H(u,v) = H_{H(u,v)} - [H_u, H_v],$$

which we call the Jacobi tensors. It is obvious that

$$J_H(u,v)(w) = H(H(u,v),w) + H(H(v,w),u) + H(H(w,u),v),$$

Hence we get that the condition $J_H(u,v) \equiv 0$ means that the Jacobi identity is fulfilled for the Haantjes tensor $H(u,v)$. Therefore the Haantjes tensor determines a deformation of the structures of Lie algebras in the tangent bundle $T(M^n)$ if and only if the Jacobi tensor $J_H(u,v) \equiv 0$.

V. In [7] it is shown that system (1) is Hamiltonian with non-degenerate Poisson brackets if and only if it has the form

$$u^i_t = A^i_j(u) u^j_x, \qquad A^i_j(u) = g^{i\alpha}(u) \nabla_\alpha \nabla_j f(u), \tag{8}$$

where the covariant derivatives ∇_α are determined by a non-degenerate metric $g_{ij}(u)$, having zero Riemann curvature tensor.

Theorem 4 *If a system (1) has a non-degenerate Hamiltonian structure then the following necessary conditions are satisfied:*

1. For $n = 3$, the Lie algebra determined by the Haantjes tensor in each tangent space $T_p(M^3)$ is simple or commutative.

2. For $n = 4$, the Haantjes tensor $H(u,v)$ defines a deformation of the structures of Lie algebras in the tangent bundle $T(M^4)$.

3. The polynomial $P_H(u) = \det(H_u - 1)$ is even: $P_H(-u) = P_H(u)$.

4. The fibration of algebraic manifolds $V_H \subset T(M^n)$ is invariant under the reflection $u \longrightarrow -u$ in the tangent bundle $T(M^n)$.

5. The eigenvalues of the operators H_u, $[M_u, M_v]$, $[H_u, H_v]$, $[M_u, A^k]$, $H_u M_v + M_v H_u$ are symmetric with respect to zero.

6. The differential 2-forms from the two countable families

$$\alpha_k(u,v) = \mathrm{Tr}(A^k[M_u, M_v]), \qquad \beta_k(u,v) = \mathrm{Tr}(A^k[H_u, H_v])$$

are all equal zero.

7. For $n = 3$ the following alternative concerning the metric $(u,v)_H$ on $T(M^3)$ is true: a) the metric is nondegenerate and conformally flat, b) the metric is zero. In the last case the Hamiltonian system has Riemann invariants and is integrable by the generalized hodograph method.

The proof of these necessary criteria is based on the following Lemma.

Lemma 1 *If a system (1) has a non-degenerate Hamiltonian structure, then the operators M_u and A are symmetric with respect to the bilinear form $(v,w) = g_{\alpha\beta}v^\alpha w^\beta$ and the operators $H_u = [M_u, A]$ are skew-symmetric.*

This statement has the tensor character, so it is enough to prove the Lemma 1 in one system of coordinates $u^1,\dots,u^n$. We choose the coordinates $u^1,\dots,u^n$, in which the flat metric $g^{ij}(u)$ is constant and diagonal, $g^{ij}(u) = q^i\delta^i_j$. In these coordinates the Hamiltonian system (8) and tensors $M(u,v)$ and $H(u,v)$ have simple form and the proof of the Lemma 1 follows from the analysis of the exact formulae for the operators M_u, A and H_u.

VI. The Haantjes tensor $H(u,v)$ (or $M(u,v) = M_u v$) is called reducible, if there exists a subspace $L \subset T_p(M^n)$ that is invariant under operators H_u (or M_u), for all $u \in T_p(M^n)$.

Theorem 5 *If a system (1) has two linear independent non-degenerate Hamiltonian structures with metrics g_{1ij} and g_{2ij}, then the following necessary conditions are satisfied:*

1. The tensors $H(u,v)$ and $M(u,v)$ are reducible. There exists a non-trivial subspace $L \subset T_p(M^n)$ invariant under operators H_u, M_v and A for all $u, v \in T_p(M^n)$.

2. The polynomials $P_M(u)$ and $P_H(u)$ are reducible.

3. If two Hamiltonian structures are in general position, or if the operator $B = g_1 g_2^{-1}$ has distinct eigenvalues at some points, then the Haantjes tensor $H(u,v) \equiv 0$ and all operators M_u, M_v and A commute with each other. The system (1) in this case possesses Riemann invariants and is integrable by the generalized hodograph method.

The proof of these necessary criteria follows from the next Lemma 2. The generalized hodograph method was developed in the work [8].

Lemma 2 *If a system (1) has two non-degenerate Hamiltonian structures, then all operators H_u, M_u and A commute with the operator $B = g_1 g_2^{-1}$.*

Lemma 2 is a direct consequence of the basic Lemma 1.

The effectiveness of the methods of this work has been checked for many concrete systems of equations (1), for example, equations of gas dynamics, equations of magnetohydrodynamics, the Benney equations, matrix equations (1) of Lax type and others, which are in progress now, and will be reported elsewhere.

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Department of Mathematics and Statistics
Queen's University, Kingston, K7L 3N6, Canada.

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On Sub-Riemannian Metrics and $\tilde{SL}_2$-geometry

Zhong Ge *

Presented by. E. Bierstone, F.R.S.C.

Abstract

We use sub-Riemannian metrics to give a differential-geometrical description of those compact 3-manifolds which admit $\tilde{SL}_2\mathbb{R}$-geometry, one of the eight three-dimensional geometries studied by Thurston [19]. Then we show that if a compact manifold M admits a $sl(2,\mathbb{R})$-form satisfying the Maurer-Cartan equation up to a small error, then M is a $\tilde{SL}_2\mathbb{R}$-manifold.

A **sub-Riemannian metric** (sR-metric for short) on a differentiable 3-manifold M is a metric $g : H \times H \to \mathbb{R}$ on a contact distribution $H \subset TM$ (H is contact if $H + [H, H] = TM$). Associated to this metric is the **sub-Riemannian distance**, or, Carnot-Caratheodory distance on M, defined by

$$d(x_0, x_1) = inf \ (\int_0^1 g(\dot{\gamma}, \dot{\gamma})dt)^{1/2}$$

where γ runs over horizontal curves (i.e. tangent to H almost everywhere) joining x_0 to x_1. This kind of metrics has appeared in hyperbolic geometry, control theory, 2-nd order sub-elliptic differential equations, probability, and is closely related to CR geometry.

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Recently, Bryant-Hsu, and independently, Wilkens have computed the local invariants of a sR-metric g, which we describe briefly. The structure equation is $d\omega_1 = A\omega_2 \wedge \omega_3 + B\omega_1 \wedge \omega_3$, $d\omega_2 = D\omega_1 \wedge \omega_3 + E\omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$, where ω_1, ω_2 is an orthonormal coframe for this metric, ω_3 is such that H is defined by the equation $\omega_3 = 0$. $A + D$ is always an invariant, which we will call the torsion. If $A + D \neq 0$, then A, B, D, E are four invariants, and any other invariant can be derived from them. If $A + D \equiv 0$, then the sR metric is called type-I.

It turns out that there is a very interesting relation between sR-metrics and three-dimensional geometry. The simplest example of this is the following result

Theorem 1 *A closed manifold M admits a locally homogeneous type-I metric if and only if it admits either S^3, or Nil, or $\tilde{SL}_2\mathbb{R}$-geometry.*

In this note we will announce some results which give a differential-geometrical description of those compact 3-manifolds which admit $\tilde{SL}_2\mathbb{R}$-geometry (which we will call $\tilde{SL}_2\mathbb{R}$-manifolds), one of the eight three-dimensional geometries studied by Thurston [19]. Recall that Thurston [19] used group-theoretic methods to classify three-dimensional locally homogeneous Riemannian geometry, and among the eight classes of compact metrizable manifolds, those admitting either S^3, or $E(3)$, or Nil, or H^3-geometry have differential-geometric descriptions (due to Hamilton, Gromov, Min-oo and Ruh[15] respectively), but so far there has been no differential-geometric description of $\tilde{SL}_2\mathbb{R}$-manifolds.

The geometric invariant that we will use is

$$K = e_2(B) - e_1(E) - B^2 - E^2 + \frac{A - D}{2}$$

which will be called the curvature of g. This invariant is well defined even at those points where the structure equation breaks down. For the canonical sR-metrics on S^3, Nil, $\tilde{SL}_2\mathbb{R}$, respectively we have $K = 1, K = 0, K = -1$. If g is type-I, then K is the invariant of Bryant-Hsu-Wilkens. In the general case we are led to this invariant through the computation of the Jacobi equation.

We say e_3 is an ϵ-almost Killing field if $m_1 < \epsilon, m_2 < \epsilon^{3/2}, m_3 < \epsilon^2$, where $m_1 = max\ |A+D|$, $m_2 = max\ (|grad(A+D)| + 2|A+D|(|B|+|E|))$, $m_3 = max\ |e_3(A+D)| + |A^2 - D^2|/2$.

Let $L(\epsilon, \delta_0, d, r)$ be the set of sR-manifolds M with the following properties: (1) e_3 is an ϵ-Killing field; (2) $-1 < K < -\delta_0 < 0$; (3) $diam(M) < d$; (4) there exists an embedding $u : S^1 \to M$ onto an orbit of v and

$$r = max_{t_1,t_2} \quad d(u(t_1), u(t_2)).$$

The following result gives a differential-geometric description of $\tilde{SL}_2R$-manifolds.

Theorem 2 *There is a function $\epsilon_0 = \epsilon(\delta_0, d)$ (does not depend on r_0) such that if $M \in L(\epsilon_0, \delta_0, d, r)$, then M is a $\tilde{SL}_2R$-manifold.*

We would like to remark that in Theorem 2, e_3 does not need to have a closed orbit, as we can slightly perturb e_3 so that it has a closed orbit.

Recall that topologically a compact $\tilde{SL}_2R$-manifold is a Seifert fibre space of non-zero Euler number over a hyperbolic orbifold. The following result gives an estimate of the Euler number of M in terms of r.

Theorem 3 *Let $M \in L(\epsilon_0, \delta_0, d, r)$, where ϵ_0 is as in Theorem 2. There are two functions $\mu_1 = \mu_1(\delta_0, d) > 0$, $\mu_2 = \mu_2(\delta_0, d) > 0$ such that if u satisfies the following additional condition*

There is no $[\gamma] \in \pi_1(M)$ and $n > 1$ such that $[u(S^1)] = [\gamma]^n$ (A)

then the Euler number of the Seifert fibre space M satisfies

$$\frac{\mu_1}{r_0} > e(M) > \frac{\mu_2}{r_0}. \tag{1}$$

Theorem 2 has interesting applications to both semi-Riemannian geometry and CR geometry. In the case of semi-Riemannian geometry, Kulkarni and Raymond [14] proved that if M is a semi-Riemannian manifold of signature $(1,1,-1)$ with constant curvatures 1 and a time-like Killing field, then M is a $\tilde{SL}_2\mathbb{R}$-manifold. We generalize their result to the case that there is a time-like almost Killing field and the curvatures are $1/(3-\delta)$ pinched ($\delta > 0$). The same idea also applies to strongly pseudo-convex CR-manifolds with a transversal infinitesimal almost CR-automorphisms.

In the third application, we study manifolds with a $sl(2,\mathbb{R})$-form satisfying the Maurer-Cartan equation up to a small error, and prove that such a manifold must be a $\tilde{SL}_2\mathbb{R}$-manifold (compare Min-oo and Ruh[15], [16]). First let us recall the formulation of the problem in a broader context below.

Let h be a Lie algebra, $\theta : TM \to h$ a parallelization of M (that is, a Cartan connection of type h), i.e. $\theta_x : T_xM \to h$ is a vector space isomorphism of T_xM with h for all $x \in M$. The curvature of θ is the h-valued 2-form $\Omega = d\theta + [\theta, \theta]$. Note that if $\Omega = 0$, then M can be written as G/Γ, where G is a Lie group with Lie algebra h, Γ a discrete subgroup of G, and $\Omega = 0$ becomes the Maurer-Cartan equation $d\theta + [\theta, \theta] = 0$. In view of this fact, a natural question is

If Ω is small enough, does that imply that M is of the form G/Γ? (Q)

Min-oo and Ruh[15] first proved that this is the case for compact sem-simple Lie groups. In [16] they further extended their method, and observed that the main obstruction to their method is $H^2(G/\Gamma, ad) \neq 0$. Indeed, for abelian Lie algebras, the answer to (Q) is negative, as there are nilpotent groups which are almost flat.

We will give an affirmative answer to (Q) for the case $h = sl(2,\mathbb{R})$, even though in general $H^2(SL_2\mathbb{R}/\Gamma, ad) \neq 0$. In the following let $h = sl(2,\mathbb{R})$.

To measure the norm of Ω, we fix a Cartan decomposition $h = k \oplus m$, where k is spanned by Y, m by X_1, X_2, satisfying $[X_1, X_2] = Y, [Y, X_1] = -X_2, [Y, X_2] = X_1$. Fix a metric $< \cdot, \cdot >$ on h by insisting that X_1, X_2, Y are orthonormal. This metric induces a Riemannian metric on M with Levi-Civita connection D and diameter d.

Instead of C^0-norm, we will use C^1-norm. We define

$$\|\Omega\|_1 \quad = \quad max\{ \ \|\Omega(x,y)\|, \ \|D\Omega(x,y)\|, \ \ x, y \in TM, \|x\| = \|y\| = 1\}.$$

Theorem 4 *There is a function $\epsilon = \epsilon(d)$ depending only on the diameter d, such that if $\|\Omega\|_1 < \epsilon$, then M is a $\tilde{S}L_2\mathbb{R}$-manifold.*

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MATHEMATICALLY MODELING A SIGNAL SIMULATOR FOR THE CALIBRATION OF EDDY CURRENT PROBES

E. N. DERUN, A. A. KOLYSHKIN and R. VAILLANCOURT

Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. By simultaneously using an excitation coil and an electrically conducting plane, a simulator, designed to calibrate eddy current probes, can emulate signals which correspond to spot welds with different flaws. The presence of a conducting plane allows a wide variation of the amplitude and phase of the excitation coil voltage during the calibration of a probe. A mathematical model for such a simulator is derived and used to study the variation of the impedance change upon the parameters of the probe. Numerical computation shows that the impedance change depends in an important way on the frequency of the excitation coil current.

RÉSUMÉ. L'ajout d'un plan conducteur à un simulateur à bobine excitatrice permet de reproduire des signaux qui correspondent à diverses soudures à arc. Cet ajout relâche de beaucoup la précision requise sur l'amplitude et la phase du voltage lors du calibrage de sondes à courants de Foucault . On développe un modèle mathématique pour étudier la variation du changement d'impédance comme fonction des paramètres de la sonde. Les calculs illustrent la forte influence de la fréquence de la bobine excitatrice sur ce changement.

Subject-classification: AMS(MOS): 35Q60, 65N35.
Keywords: electromagnetic equations, signal simulator, eddy current probes.

1. Introduction.

Signal simulators are used extensively [7] for the calibration of eddy current probes. In the quality control of spot welds, the probe calibration is an especially delicate problem. It could be achieved by means of different spot weld patterns from a kit of pre-tested welds according to diameter and depth of the cast cores. But it would be technically difficult to produce spot welds with prescribed parameter values and even more difficult to reproduce welding patterns.

To overcome some of these difficulties, a signal simulator equipped with an excitation coil and an electrically conducting plane [1] and [2], as shown schematically in Fig. 1(a), presents experimental evidence of superior performance as shown in Fig. 2.

The simultaneous use of an excitation coil and an electrically conducting plane permits a simulation of signals corresponding to spot welds with different flaws. The plane produces

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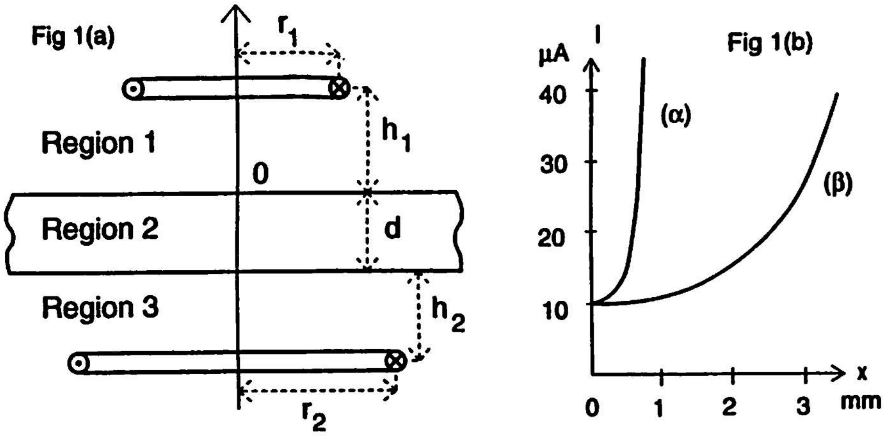

FIGURE 1. (a) Schematic disposition of simulator elements: probe coil in region 1, conducting plate (region 2) and excitation coil in region 3. (b) Experimental dependence of current I, in μA, of eddy current probe on radial shift x mm away from center of excitation coil; (α) for simulator with excitation coil but without electrically conducting plane; (β) for simulator with excitation coil and electrically conducting plane.

the simulation base signal from weld details while the coil produces the signal only from the spot weld flaw. Since the coil signal is weaker than the plane signal, then, for a small change in the total simulation signal, one need only regulate the amplitude and phase of the voltage of the excitation coil within wide limits. Therefore the usual stringent accuracy imposed on the setting and maintaining of the amplitude and phase of the excitation coil voltage can be considerably relaxed.

In section 2, we derive, from the potential of the linearized model, an analytical expression for the impedance change, Z, in the sensor coil and use this expression, in section 3, to study numerically the dependance of Z upon the parameters of the problem; finally, we draw a qualitative result on the dependance of Z upon the frequency of the current in the excitation coil.

2. Mathematical analysis.

To determine the change in impedance in the upper coil for a probe with a conducting plane, we shall superpose the vector potentials in the upper half-space due to the eddy currents induced in this plane by the current in the upper (resp. lower) coil in the absence of the lower (resp. upper) coil.

Let the sensor coil, of radius r_1, be situated (in region 1) at distance h_1 above, and parallel to, a horizontal electrically conducting plane (region 2) of thickness d with electrical conductivity σ. The excitation coil, of radius r_2, is situated (in region 3) coaxially with the sensor coil, at distance h_2 below, and parallel to, the plane (see Fig. 1(a)).

Suppose the current distributions in the upper and lower coils are, respectively,

$$\dot{I}_1 = I_1 e^{j\omega_1 t}, \qquad \dot{I}_2 = I_2 e^{j\omega_2 t}, \tag{1}$$

where ω_1, ω_2 and I_1, I_2 are the frequencies and amplitudes, respectively. Let (r, φ, z) be the system of cylindrical polar coordinates where the z-axis, directed upwards, coincides with the axes of the coils and the origin is located on the upper surface of the plane.

First, we find the vector potential in the upper half-space in the case the upper coil is present and the lower coil is absent. The mathematical formulation of this first linearized problem, as derived from Maxwell's equations, is

$$\Delta A_1 = -\mu_0 I_1 \delta(z - h_1)\,\delta(r - r_1), \tag{2}$$

$$\Delta A_2 + k_1^2 A_2 = 0, \tag{3}$$

$$\Delta A_3 = 0, \tag{4}$$

$$A_1\Big|_{z=0} = A_2\Big|_{z=0}, \qquad \frac{\partial A_1}{\partial z}\Big|_{z=0} = \frac{\partial A_2}{\partial z}\Big|_{z=0} \tag{5}$$

$$A_2\Big|_{z=-d} = A_3\Big|_{z=-d}, \qquad \frac{\partial A_2}{\partial z}\Big|_{z=-d} = \frac{\partial A_3}{\partial z}\Big|_{z=-d}, \tag{6}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}, \qquad k_1^2 = -j\omega_1\sigma\mu_0,$$

$\delta(x)$ is the Dirac delta function, and $A_i(r, z)$ is the vector potential in the ith region shown in Fig. 1(a) $(1 : z > 0, \quad 2 : -d < z < 0, \quad 3 : z < -d)$.

Applying the Hankel transform

$$\widetilde{A}_i(\lambda, z) = \int_0^\infty A_i(r, z)\, r\, J_1(\lambda r)\, dr, \quad i = 1, 2, 3,$$

to problem (2)–(6), we obtain

$$\frac{d^2\widetilde{A}_1}{dz^2} - \lambda^2\widetilde{A}_1 = -\mu_0 I_1 \delta(z - h_1)\, r_1 J_1(\lambda r_1), \tag{7}$$

$$\frac{d^2\widetilde{A}_2}{dz^2} - q_1^2\widetilde{A}_2 = 0, \tag{8}$$

$$\frac{d^2\widetilde{A}_3}{dz^2} - \lambda^2\widetilde{A}_3 = 0, \tag{9}$$

$$\widetilde{A}_1\Big|_{z=0} = \widetilde{A}_2\Big|_{z=0}, \qquad \frac{d\widetilde{A}_1}{dz}\Big|_{z=0} = \frac{d\widetilde{A}_2}{dz}\Big|_{z=0}, \tag{10}$$

$$\widetilde{A}_2\Big|_{z=-d} = \widetilde{A}_3\Big|_{z=-d}, \qquad \frac{d\widetilde{A}_2}{dz}\Big|_{z=-d} = \frac{d\widetilde{A}_3}{dz}\Big|_{z=-d}, \tag{11}$$

where $q_1 = \sqrt{\lambda^2 - k_1^2}$.

Solving problem (7)-(11) and using the inverse Hankel transform

$$A_i(r,z) = \int_0^\infty \widetilde{A}_i(\lambda, z)\,\lambda J_1(\lambda r)\,d\lambda, \qquad i = 1,2,3,$$

we find the following formula for the vector potential in the upper half-space ($z > 0$) due to eddy currents induced in the plane by the current from the upper coil:

$$A^{(1)}(r,z) = \int_0^\infty C(\lambda)\,J_1(\lambda r)\,e^{-\lambda(z+h_1)}\,d\lambda, \tag{12}$$

where

$$C(\lambda) = \frac{\mu_0}{2} I_1 r_1 J_1(\lambda r_1) \frac{\lambda\left[(\lambda - q_1) - (\lambda + q_1)e^{2q_1 d}\right] + q_1\left[(\lambda + q_1)e^{2q_1 d} + (\lambda - q_1)\right]}{\lambda\left[(\lambda - q_1) - (\lambda + q_1)e^{2q_1 d}\right] - q_1\left[(\lambda + q_1)e^{2q_1 d} + (\lambda - q_1)\right]}.$$

Solution (12) is found in [5] and is the limiting case, as $d \to \infty$, of formulae found in [6] and, later, in [3] for a similar problem, where the upper coil is situated above a conducting half-space. A more general problem, for a multi-layer medium, is solved in [4].

Second, we find the vector potential in region 1 if the lower coil is present and the upper one is absent. The mathematical formulation of this second linearized problem is

$$\Delta A_1 = 0, \tag{13}$$

$$\Delta A_2 + k_2^2 A_2 = 0, \tag{14}$$

$$\Delta A_3 = -\mu_0 I_2\,\delta(z + h_2 + d)\,\delta(r - r_2), \tag{15}$$

$$A_1\Big|_{z=0} = A_2\Big|_{z=0}, \qquad \frac{\partial A_1}{\partial z}\Big|_{z=0} = \frac{\partial A_2}{\partial z}\Big|_{z=0}, \tag{16}$$

$$A_2\Big|_{z=-d} = A_3\Big|_{z=-d}, \qquad \frac{\partial A_2}{\partial z}\Big|_{z=-d} = \frac{\partial A_3}{\partial z}\Big|_{z=-d}, \tag{17}$$

where $k_2^2 = -j\,\omega_2\,\sigma\,\mu_0$. Solving problem (13)-(17) by the technique used for problem (2)-(6), we obtain the vector potential in the upper half-space ($z > 0$), due to eddy currents induced in the conducting plane by the current from the lower coil, in the form

$$A^{(2)}(r,z) = \int_0^\infty D(\lambda)\,J_1(\lambda r)\,e^{-\lambda(z+h_2)}\,d\lambda, \tag{18}$$

where

$$D(\lambda) = \frac{2\lambda q_2 \mu_0 I_2 r_2 J_1(\lambda r_2)}{(\lambda + q_2)^2 e^{q_2 d} - (\lambda - q_2)^2 e^{-q_2 d}}, \qquad q_2 = \sqrt{\lambda^2 - k_2^2}.$$

The impedance change in the upper coil may be found by means of the formula

$$Z = -\frac{E}{I_1}, \qquad \text{where} \quad E = -j\omega \int_L \left[A^{(1)}(r,z) + A^{(2)}(r,z)\right] dl, \tag{19}$$

L is the contour of the upper coil and, for the sake of simplicity, we restrict ourselves to the case $\omega_1 = \omega_2 = \omega$. If we introduce the dimensionless variables

$$\alpha = \frac{2h_1}{r_1}, \quad \beta = r_1\sqrt{\omega\sigma\mu_0}, \quad \gamma = \frac{2d}{r_1}, \quad I = \frac{I_2}{I_1}, \quad R = \frac{r_2}{r_1}, \quad H = \frac{h_2}{h_1},$$

we obtain, from (12), (18) and (19), the following formula for the change in impedance:

$$Z = \omega \pi \mu_0 I_1 r_1 \left[j\beta \int_0^\infty F(y) J_1^2(\beta y) e^{-\alpha\beta y} dy + 4j\beta I R \int_0^\infty G(y) J_1(\beta y) J_1(\beta R y) e^{-\alpha\beta y(1+H)/2} dy \right], \tag{20}$$

where

$$F(y) = \left\{ y \left[\left(y - \sqrt{y^2+j}\right) e^{-\gamma\beta\sqrt{y^2+j}} - \left(y + \sqrt{y^2+j}\right) \right] + \sqrt{y^2+j} \left[y + \sqrt{y^2+j} + \left(y - \sqrt{y^2+j}\right) e^{-\gamma\beta\sqrt{y^2+j}} \right] \right\} \Big/ \left\{ y \left[\left(y - \sqrt{y^2+j}\right) e^{-\gamma\beta\sqrt{y^2+j}} - \left(y + \sqrt{y^2+j}\right) \right] - \sqrt{y^2+j} \left[y + \sqrt{y^2+j} + \left(y - \sqrt{y^2+j}\right) e^{-\gamma\beta\sqrt{y^2+j}} \right] \right\},$$

and

$$G(y) = \frac{y\sqrt{y^2+j}\, e^{-\gamma\beta\sqrt{y^2+j}/2}}{\left(y + \sqrt{y^2+j}\right)^2 - \left(y - \sqrt{y^2+j}\right)^2 e^{-\gamma\beta\sqrt{y^2+j}}}.$$

3. Numerical results.

Since formula (20) contains several parameters, we investigated the influence of each one of γ, α, H and R on Z while the others were kept fixed [2]. Examples are shown in Fig. 2.

It follows from the results of the numerical modeling and experimental investigation that, for the regulation of the simulator signal, one has to vary not only the amplitude and phase of the current in the excitation coil, but also the frequency.

A qualitative comparison can be drawn between our theoretical and experimental results since the linear theoretical model obtained in this paper allows one to study the problem in the first approximation. For example, this model explains the stong influence of the frequency of the current in the excitation coil, a fact that was unclear to us before.

Conclusion.

It is known experimentaly that the addition of an electrically conducting plane to a signal simulator reduces the necessity of accurately setting and maintaining the amplitude and phase of the voltage. A linearized mathematical model is derived for the simulator and the dependance of the impedance change on the current frequency is studied numerically.

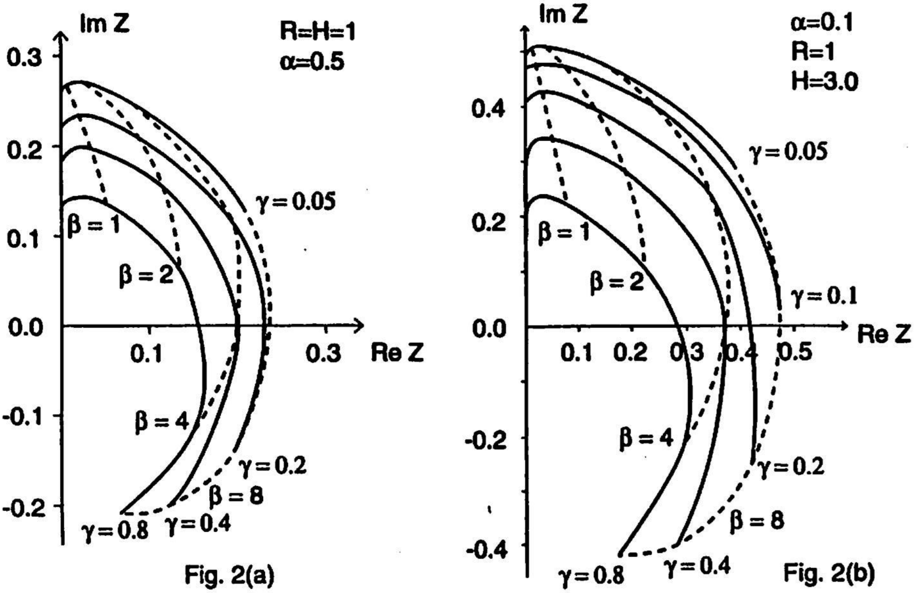

FIGURE 2. Change in impedance, Z as a function of: (a) dimensionless distance, α, between upper coil and plane for $R = H = 1$ and $\alpha = 0.5$; (b): ratio $H = h_2/h_1$ of distances between plane and lower and upper coils for $\alpha = 0.1$, $R = 1$ and $H = 3$.

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DEPT. OF AUTOMATED CONTROL SYSTEMS, RIGA TECHNICAL UNIVERSITY, RIGA, LATVIA, LV 1010

DEPT. OF APPLIED MATHEMATICS, RIGA TECHNICAL UNIVERSITY, RIGA, LATVIA, LV 1010
E-mail address: podvis@bella.riga.lv

DEPT. OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO, CANADA K1N 6N5
E-mail address: rxvsg@acadvm1.uottawa.ca

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Curvature of the Ellipse and Dynamical Consequences

John Faulkner

Presented by H.S.M. Coxeter, F.R.S.C.

Abstract: A simple theorem on the curvature of the ellipse yields a 2–line derivation of the inverse-square law of gravitation.

§1. Introduction: A Dynamically Useful Geometrical Theorem.

Using only elementary, non-calculus mathematics, I have rediscovered a simple but useful geometrical theorem [see (1.1) below] that permits a brief and apparently novel derivation of the inverse-square law of attraction from elliptical motion and Kepler's (focal) areal law. It has emerged that theorem (1.1) is closely related to (3.5), a result discovered long ago [1, 2] and quoted in an encyclopaedic review [3]. Indeed, theorem (1.1) itself appears in another guise in [1], but it was neither recognized by subsequent generations of geometers [e.g. 4 - 6], nor used directly in [1] for the simple purpose illustrated in §2 below, or (as far as I can tell) by later generations of dynamicists. The theorem possesses intrinsic elegance, relates to developments in an heroic age and, through its absence from Newton's work, casts light on his currently debated modes of thought (though this point is necessarily more speculative). For all these reasons, it deserves to be known more widely.

a) **The Theorem.** *Let ρ be the radius of curvature at a point P on the ellipse, ϕ the angle between the tangent at P and a focal ray or radius vector, and l the semi-latus rectum. Then:*

$$\rho \sin^3 \phi = l. \tag{1.1}$$

The theorem became evident upon noticing that an analytically obtained expression for ρ contained the cube of a factor already found in $\sin \phi$ in an earlier but differently directed geometrical investigation. The theorem's intriguingly simple final mathematical form soon led to a short, more geometrical derivation using only first order differences (i.e., no second differentiation), in the most basic spirit of the Principia [7]. Since alternatively derived expressions for ρ and $\sin \phi$ are already known separately in several forms [see 4–6], I shall merely sketch my different initial approaches here. These approaches are conditioned by my long-standing wish to provide the shortest possible, self-contained non-calculus and therefore more accessible proofs of Newton's most celebrated propositions. The details of this, and a 3–line proof of the external attraction of a spherical shell, meeting the challenge by Littlewood (whose own ingenious "short proof" [8] is flawed by an unfortunate misidentification), will be published elsewhere.

b) Finding sin ϕ: Let F_1 and F_2, separated by $2ae$, be the foci of the ellipse, $r_1(\equiv r)$ and r_2 the two focal radii. Apply the cosine formula to triangle F_1PF_2, noting that $\angle F_1PF_2$ is $\pi-2\phi$. Expressing $\cos 2\phi$ in terms of $\sin\phi$ and using $r_1+r_2=2a$, $b^2=a^2(1-e^2)$, quickly yields:

$$\sin\phi=\frac{b}{(r_1r_2)^{1/2}}=\frac{b}{[r(2a-r)]^{1/2}}. \qquad (1.2)$$

This "shape" relationship for the ellipse has its own important dynamical implications. Defining the "perpendicular component" of the total velocity v by $v_\perp = v\sin\phi$, it leads on immediately to the familiar *vis viva* equation of dynamics:

$$v^2=\frac{r^2v_\perp^2}{l}\left(\frac{2}{r}-\frac{1}{a}\right), \qquad (1.3)$$

where $l=b^2/a=a(1-e^2)$. By appealing to Kepler's 2nd and 3rd laws (note how $rv_\perp \equiv h$ appears here!), those willing to difference $1/r$ explicitly or wishing, despite all exhortation, to make a (minimal) use of formal calculus, can indeed derive "Universal Gravitation" from equation (1.3); but that approach is not our focus in this paper.

c) Finding the Radius of Curvature, ρ: Start with the Cartesian equation $(x/a)^2+(y/b)^2=1$. Move the origin to P (x_p, y_p), deduce the slope of the normal from the local linear approximation, rotate to (outward) normal and tangential (ξ, η) axes, and check that the two lowest order terms satisfy an equation of type $\eta^2+2\nu\xi=0$, i.e. the local form for a circle of radius ν passing through P orthogonally to the normal. Deduce ρ operationally as half the ratio of coefficients of the ξ and η^2 terms. Eliminating y_p (via the Cartesian equation) and transforming x_p (via $x_p=ae+r\cos\theta$ and the polar equation $r+er\cos\theta=l$) leads to a final simplification:

$$\rho=a^2b^2\left(\frac{x_p^2}{a^4}+\frac{y_p^2}{b^4}\right)^{3/2}=\frac{1}{ab}[r(2a-r)]^{3/2}=\frac{(r_1r_2)^{3/2}}{ab}. \qquad (1.4)$$

From equations (1.2) and (1.4) we have, immediately:

$$\rho\sin^3\phi=b^2/a=l. \qquad (1.5)$$

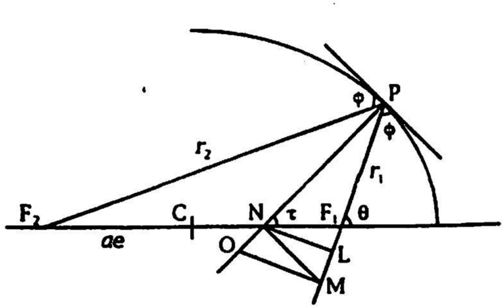

Figure 1. Geometry discussed in §§1 - 3. Note: PL = l and PO = ρ.

§2. Dynamical Corollary: Inverse Square Law of Gravitation.

Theorem (1.1) permits a 2–line dynamical derivation of the inverse-square law of gravitation. Let an elliptically orbiting particle, of mass m, satisfy Kepler's focal areal law so that, by Newton's beautiful geometrical argument, the implied force of attraction, F, is focus-directed. Then the normal inward component of that force satisfies

$$\frac{F}{m}\sin\phi = \frac{v^2}{\rho} = \frac{h^2}{\rho r^2 \sin^2\phi} , \tag{2.1}$$

where the latter expression, with $h(= rv_\perp)$ the angular momentum per unit mass, itself follows from the areal law.

Using theorem (1.1), we have, immediately, the corollary:

$$\frac{F}{m} = \frac{h^2}{(\rho\sin^3\phi)r^2} = \frac{h^2}{lr^2} . \tag{2.2}$$

§3. Additional Corollaries.

a) **Geometrical.** The analytical approach used in obtaining ρ, above, implies several other results with interesting consequences. Among these, a particularly fruitful one involving τ, the angle between the major axis and the normal at P, is:

$$\sin\tau = \frac{ay_p}{b(r_1r_2)^{1/2}} . \tag{3.1}$$

Therefore, from equations (1.2) and (1.4),

$$\rho\sin^2\phi\sin\tau = y_p . \tag{3.2}$$

Thus, if the normal at P intercepts the major axis in N,

$$PN = \rho\sin^2\phi \tag{3.3}$$

(implying NO = $\rho\cos^2\phi$, where O is the center of curvature), and, by (1.1),

$$PN\sin\phi = l, \tag{3.4}$$

$$\rho = PN^3/l^2 . \tag{3.5}$$

Geometrically, $\rho\sin^2\phi$ is a double projection (from the normal onto a focal ray and back again), so that N is a pivotal point in a number of interesting relationships and constructions. Differently ordered, the results derived here are part of, or arise in, the geometrical proof mentioned in §1 (which also turns out to be a more efficiently organized and self-contained equivalent of that in [1]). Note that (3.4), (3.5), and (1.1) are valid for all conics, of course

b) **Dynamical.** What of elliptical motion satisfying a central areal law? Let ω be the angle between a central radius vector CP, of length R, and the normal, PN. Analogously to (1.1) we find the less appealing but relevant relationship:

$$\rho R^3 \cos^3 \omega = a^2 b^2 . \tag{3.6}$$

The normal force equation has the same form as before (i.e. equation [2.1]), with R and $\cos \omega$ replacing r and $\sin \phi$. Thus, in this case,

$$\frac{F}{m} = \frac{h^2}{(\rho \cos^3 \omega) R^2} = \frac{h^2}{a^2 b^2} R \,, \tag{3.7}$$

the linear force law well-known (though not so deduced before, I believe) for centered attractive elliptical motion.

§4. Historical perspectives.

Newton recognized the importance of the ellipse's curvature (via the phrase "a tangent circle of equall crookednesse") in his Waste Book of 1664, and found relationship (3.5) from a fluxional argument (unpublished in his lifetime) by 1671 [9]. However, he never used it, nor even the general concept of instantaneous normal acceleration, directly, in the Principia (despite at least one confident assertion [10] to the contrary). Apparently he did not discover the geometrical projection equivalent of (3.4) until his never published "preliminary ameliorations" [9] to the first edition in the early 1690's. Without (3.4), needed to establish theorem (1.1) by this route, ρ remains in an inductively unhelpful and inapplicable form.

My colleague M. Nauenberg has nevertheless made the fascinating suggestion [11] that Newton had used the general concepts of curvature and normal acceleration as computational aids prior to recognizing (at once ruefully yet gleefully?) the more fundamental significance of Kepler's 2nd law during correspondence with Hooke in 1679/80, and that a vestigial trace of this survives in his otherwise long-puzzling fascination with the $1/r^3$ force-law. If Newton did use normal curvature directly early on, it becomes even more imperative to address a question that already presents itself: why did Newton not convert his elliptical curvature formula (3.5) into a more directly useful form and employ it as such?

I speculate that there may have been another important philosophical reason for discarding his long-held normal curvature concept (and tool?) in favor of Hooke's more physically appealing idea of deviation induced from continued straight-line motion by averted (rather than normal) forces directed towards a material point: sensitive to the accusation that "action at a distance" might itself seem to have something occult about it, he had no wish to compound the potential expository problem by seeming to appeal to instantaneous circular motion around an ever-moving, non-material, mathematical point. Certainly his initial chordal (Prop. VI, Cors. III, IV) and subsequent astounding "comparison" theorems (Prop. VII, Cors. II, III), so beautifully discussed in [12], all involve forces towards explicitly "immovable" or given force-centers, and derive from averted (in contrast to normally-directed) measures of curvature. (Newton's so-called "chord of curvature" formulation arises as a special example of the averted case for a general curve, applied to the osculating circle, and not from a de-projection of normal circular acceleration.) In fact Newton's emphasis on immovable force-centers begins with Prop. I Theorem I, and continues unabated through the celebrated Prop. XI and well beyond.

On the other hand, Keill [1] not only independently discovered (3.5), but, quoting Milnes for (3.4) who may in turn have obtained it from Newton via Gregory post-1690 [9], effectively deduced (1.1) (so that it should properly be called Keill's theorem). However, instead of applying it directly as in §2, he combined it instead with yet another transformed measure of force in a slightly inefficient detour. Perhaps because of that, and rather poor figures (certainly in the abridged English translation), theorem (1.1) not only failed to appear in later geometrical treatises, but also made no seeming impact on dynamical works.

Theorem (1.1), at once complete but of the starkest simplicity, has almost been broached a number of times recently [e.g. 9, 10, 12–14]; yet when its adumbrations are examined, one finds that neither the theorem itself nor the argument embodied in equations (2.1) and (2.2) have been given in their most complete and transparently appealing form.

Finally, I note that countless generations of published calculus-employing dynamicists, starting from a postulated force-law

$$F/m = \mu/r^2, \tag{4.1}$$

have reversed the process, integrating the equations of motion to produce the orbit and somewhat tediously concluding that the angular momentum per unit mass, h, and the semi-latus rectum, l, satisfy:

$$h^2 = \mu l. \tag{4.2}$$

Any of them, reflecting on the significance of this result in terms of the required normal force component, could have realized that this elementary piece of dynamics implied the geometrical necessity of theorem (1.1); but apparently either none did, or if they did, thought it worth noting. That the relation (4.2) holds necessarily is constructively obvious from equation (2.2), which seems to be the shortest and clearest derivation of that fact.

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Lick Observatory, UC Santa Cruz
Santa Cruz, CA 95064 USA

RELEVEMENT DE MODULES DE DRINFELD EN CARACTERISTIQUE ZERO

Yves HELLEGOUARCH

François RECHER

Presented by P. Ribenboim, F.R.S.C.

Abstract - In this paper, we show that Drinfeld modules over $\mathbb{F}_p[t]$ possess "canonical" liftings as "generalized" modules in characteristic zero.

Suggérée par une conversation avec O.A. Laudal, notre méthode de relèvement utilise le foncteur de Witt.

I. DESCRIPTION DE CERTAINS ANNEAUX DE WITT.

Soit $\overline{\mathbb{F}_p}$ une clôture algébrique du corps $\mathbb{F}_p$; dans ce qui suit, nous utiliserons les notations :

$$R_0 = \mathbb{F}_p((t^{-1})) \ , \ S_0 = \overline{\mathbb{F}_p} \otimes R_0 \ , \ R_\infty = \mathbb{F}_p((t^{-p^{-\infty}})) = \varinjlim \mathbb{F}_p((t^{-p^{-i}})),$$
$$S_\infty = \overline{\mathbb{F}_p} \otimes R_\infty \ , \ \mathbb{Z}_p((\theta^{-p^{-\infty}})) = \{x \in \mathbb{Q}_p((\theta^{-p^{-\infty}})), \text{coefficients dans } \mathbb{Z}_p\}.$$
$$\text{où} \quad \mathbb{Q}_p((\theta^{-p^{-\infty}})) = \varinjlim \mathbb{Q}_p((\theta^{-p^{-i}}))$$

Le but de ce premier paragraphe est de donner une description précise de $W(R_0)$ (Th. 2).

Rappelons ([1] p.45, Th. 3) que pour tout corps parfait k de caractéristique p, $W(k)$ est un anneau de valuation discrète absolument non ramifié dont k est le corps résiduel. De plus, il existe une application $f : k \mapsto W(k)$, multiplicative, telle que tout élément de $W(k)$ s'écrive d'une manière et d'une seule sous la forme :

$$\sum_{i=0}^{\infty} f(\alpha_i)p^i$$

pour une certaine suite $\alpha_0, \alpha_1, \ldots$ d'éléments de k. Puisque f est multiplicative, nous avons, dans $W(k)$ et pour tout $i \in \mathbb{N}$:

$$\begin{aligned} f(t_1 + \ldots + t_n) &= f(t_1^{p^{-i}} + \ldots + t_n^{p^{-i}})^{p^i} \\ &\equiv (\theta_1^{p^{-i}} + \ldots + \theta_n^{p^{-i}})^{p^i} \quad mod.\ p^{i+1}W(k) \end{aligned}$$

où l'on pose $\theta_h = f(t_h)$.

La formule multinômiale donne :

$$f(t_1 + \ldots + t_n) \equiv \sum_{h_1+\ldots+h_n=p^i} \frac{p^i!}{h_1!\ldots h_n!} \theta_1^{\frac{h_1}{p^i}} \ldots \theta_n^{\frac{h_n}{p^i}} \; mod\, p^{i+1}.$$

LEMME. Soient $r_1, \ldots, r_n$ des éléments de $\mathbb{Q}_+$ dont le dénominateur est une puissance de p et qui sont tels que : $r_1 + \ldots + r_n = 1$. Alors il existe un nombre $(r_1, \ldots, r_n) \in \mathbb{Z}_p$, bien déterniné, tel que si $r_1 = \frac{h_1}{p^i}, \ldots, r_n = \frac{h_n}{p^i}$ (fractions non nécessairement irréductibles) on ait :

$$(r_1, \ldots, r_n) \equiv \frac{p^i!}{h_1!\ldots h_n!} \; mod \; p^{i+1} \quad \text{pour tous les choix possibles de } i \in \mathbb{N}.$$

THEOREME 1. Soit k un corps parfait de caractéristique p et soient $t_1, \ldots, t_n \in k$, on pose $\theta_h = f(t_h)$ pour $1 \leq h \leq n$. Alors nous avons:

1) $f(t_1 + \ldots + t_n) = \sum_{\substack{r_1 + \ldots + r_n = 1 \\ r_1, \ldots, r_n \in \mathbb{Z}[\frac{1}{p}]}} (r_1, \ldots, r_n) \theta_1^{r_1} \ldots \theta_n^{r_n}.$

2) Pour tout $i \in \mathbb{N}$:

$$f(t_1 + \ldots + t_n) \equiv \sum_{\substack{r_1+\ldots+r_n=1 \\ p^i r_1 \in \mathbb{N}, \ldots, p^i r_n \in \mathbb{N}}} (r_1, \ldots, r_n) \theta_1^{r_1} \ldots \theta_n^{r_n} \qquad mod \quad p^{i+1}.$$

3) Si $t_1, \ldots, t_n$ sont des indéterminées sur $\mathbb{F}_p$ et si $F(t_1, \ldots, t_n)$ est une forme homogène en $t_1, \ldots, t_n$ de degré d, alors $f[F(t_1, \ldots, t_n)]$ est une somme homogène de monômes $\theta_1^{s_1} \ldots \theta_n^{s_n}$, avec $(s_1, \ldots, s_n) \in \left(\mathbb{Z}[\frac{1}{p}]\right)^n$ de degré d.

<u>Preuve</u>. 1) et 2) proviennent du lemme. 3) résulte de 1) par la spécialisation :

$$t_1 \mapsto t, \ldots, t_n \mapsto t,$$

où t est une indéterminée sur $\mathbb{F}_p$. □

THEOREME 2. Avec les notations précédentes nous avons :

$$W(R_0) = \mathbb{Z}_p((\theta^{-1})) + p\mathbb{Z}((\theta^{-1/p})) + \ldots \subset W(R_\infty).$$

<u>Preuve</u>. On se ramène à l'étude de $W(\mathbb{F}_p[[t^{-1}]])$ et, en remplaçant t par $1/t$, on voit qu'il suffit de montrer que :

$$W(\mathbb{F}_p[[t]]) = \mathbb{Z}_p[[\theta]] + p\mathbb{Z}_p[[\theta^{1/p}]] + \ldots.$$

Le théorème 1 prouve l'inclusion du premier membre dans le second. Pour montrer l'inclusion réciproque, on se ramène à la démonstration de l'inclusion de $\mathbb{Z}_p[[\theta]]$ dans $W(\mathbb{F}_p[[t]])$.

Soit $P_i(\theta) \in \mathbb{Z}_p[\theta]$ pour $i = 1$ ou 2, son vecteur de Witt s'écrit :

$$(a_0^{(i)}(t), a_1^{(i)}(t), \ldots, a_n^{(i)}(t), \ldots) \quad \text{où } a_n^{(i)}(t) \in \mathbb{F}_p[t] \quad \text{pour } n \in \mathbb{N}.$$

On sait que l'homomorphisme canonique $h : \mathbb{F}_p[t] \to \mathbb{F}_p[t]/_{(t^m)}$ induit un homomorphisme $W(h) : W(\mathbb{F}_p[t]) \to W(\mathbb{F}_p[t]/_{(t^m)})$ et que $\theta^m \in Ker W(h)$.

On voit donc que si $P_1(\theta) \equiv P_2(\theta) \ mod \ \theta^m$ on a : $W(h)(P_1(\theta)) = W(h)(P_2(\theta))$. Donc les composantes $a_n^{(1)}(t)$ et $a_n^{(2)}(t)$ des vecteurs de Witt de $P_1(\theta)$ et $P_2(\theta)$ sont congrues modulo t^n. Passant à la limite projective, on voit que : $\mathbb{Z}_p[[\theta]] \subset W(\mathbb{F}_p[[t]])$.
Le reste est simple (voir [2] p.497). □

COROLLAIRE. Soit σ le Frobenius de R_0 et soit $\tau = W(\sigma)$.
Alors si

$$z = \sum_n a_n^{(0)}\theta^{-n} + \sum_n a_n^{(1)}\theta^{-n/p} + \ldots \in W(R_0)$$

on a

$$\tau(z) = \sum_n a_n^{(0)}\theta^{-np} + \sum_n a_n^{(1)}\theta^{-n} + \ldots.$$

Preuve. Il suffit d'écrire le vecteur de Witt de z et de remarquer que :

$$\tau(\theta) = \tau\left[f(t)\right] = f(\sigma(t)) = f(t^p) = \theta^p. \ \square$$

Remarque : Si l'on veut prolonger τ à $W(S_\infty)$ on posera :

$$\tau(z) = \sum_n \tau(a_n^{(0)})\theta^{-np} + \sum_n \tau(a_n^{(1)})\theta^{-n} + \ldots.$$

II. RELEVEMENT D'UN MODULE DE DRINFELD.

Soit φ un module de Drinfeld : $\mathbb{F}_p[T] \xrightarrow{\varphi} R_\infty\{\sigma\}$. On sait que φ (qui est $\mathbb{F}_p$-linéaire) est défini par l'image de T :

$$\varphi(T) = t\sigma^0 + a_1(t)\sigma + \ldots + a_r(t)\sigma^r \quad \text{avec } a_i(t) \in R_\infty.$$

A ce module est attaché un T motif pur M qui est le $R_\infty\{\sigma\}$ $[T]$- module, libre et de rang 1 sur $R_\infty\{\sigma\}$, défini par le générateur m et la relation fondamentale (voir [3]) :

(2) $$Tm = tm + a_1(t)\sigma(m) + \cdots + a_r(t)\sigma^r(m)$$

On a donc :

$$M \simeq R_\infty\{\sigma\}[T] \Big/ R_\infty\{\sigma\}[T]\,(t-\varphi(T)).$$

Posons $t_\nu = t^{p^{-\nu}}$, nous aurons encore :

$$Tm = (t_\nu)^{p^\nu} m + [a_1(t_\nu)]^{p^\nu}\sigma(m) + \cdots + [a_r(t_\nu)]^{p^\nu}\sigma^r(m) \tag{3}$$

Le rélèvement en caractéristique 0 de ce T-motif sera un "Θ - motif pur N" qui (par définition) est le $W(R_\infty)\{\tau\}$ $[\Theta]$- module, libre et de rang 1 sur $W(R_\infty)$ $\{\tau\}$, défini par le générateur n et la relation fondamentale :

$$\Theta n = \theta n + A_r(\theta)\tau(n) + \cdots + A_r(\theta)\tau^r(n) \tag{4}$$

On désigne par π la réduction modulo p : $W(R_\infty) \to R_\infty$, et on suppose de plus que :

$$\pi(A_i(\theta)) = a_i(t) \quad \text{pour} \quad i = 1, \cdots, r \tag{5}$$

et

$$\Theta\; n = (\theta_\nu)^{p^\nu} n + A_1(\theta_\nu)^{p^\nu}\tau(n) + \cdots + A_r(\theta_\nu)^{p^\nu}\tau^r(n) \tag{6}$$

pour tout $\nu \in \mathbb{N}$ et avec $\theta_\nu := \theta^{p^{-\nu}} = f(t^{p^{-\nu}})$.

Les relations (4), (5), (6) entraînent que :

$$A_i(\theta) = f\;[a_i(t)] \quad \text{pour} \quad i = 1, \cdots, r.$$

A ce Θ - motif N est attaché un module de Drinfeld généralisé ψ :

$$\mathbb{Z}_p\;[\Theta] \xrightarrow{\psi} W(S_\infty)\;\{\tau\}$$

qui est un homomorphisme de $\mathbb{Z}_p$- algèbres déterminé par l'image de Θ :

$$\psi(\Theta) = \theta\tau^0 + f[a_1(t)]\tau + \cdots + f[a_r(t)]\tau^r \tag{7}$$

<u>Remarque</u> : Cette définition s'étend de même aux modules de Drinfeld généralisés d'Alain Thiery $\mathbb{F}_p[t] \xrightarrow{\phi} \mathbb{F}_p((t^{-1}))\{\{\sigma\}\}$, voir [4] Ch.2.

L'homomorphisme ψ est un module de Drinfeld "généralisé" au sens de [5] p.131. Il lui est donc associé une exponentielle exp_ψ par la construction de [5] p.132 .

THEOREME 3. L'exponentielle $exp_\psi(\tau)$ est dans $\mathbb{Z}_p((\theta^{-1}))\{\{\tau\}\}$.
1) Si $exp_\psi(\tau) = \tau^0 + c_1\tau + c_2\tau^2 + \ldots$ alors, par réduction modulo p, exp_ψ devient exp_ϕ.
2) L'exponentielle de ψ induit une application $\mathbb{Z}_p$-linéaire $W(R_\infty) \to W(R_\infty)$.
3) Le diagramme suivant est commutatif :

$$\begin{array}{ccc} W(R_\infty) & \xrightarrow{exp_\psi} & W(R_\infty) \\ \downarrow \pi & & \downarrow \pi \\ R_\infty & \xrightarrow{exp_\phi} & R_\infty \end{array}$$

Preuve. 1) D'après [5] p.132, on pose :

$$exp_\phi = \sum_k b_k\sigma^k = \sigma^0 + b_1\sigma + b_2\sigma^2 + \ldots$$

La relation dans l'anneau de Ore :

$$exp_\phi.t\sigma^0 = \phi(t).exp_\phi$$

fournit les équations :

$$\begin{cases} [1]_\sigma\, b_1 = a_1 \\ [2]_\sigma\, b_2 = a_2 + a_1\, b_1^\sigma \\ \ldots \end{cases} \qquad \text{où l'on pose } [n]_\sigma = \sigma^n(t) - t.$$

De même, on obtient pour exp_ψ :

$$\begin{cases} [1]_\tau\, c_1 = f(a_1) \\ [2]_\tau\, c_2 = f(a_2) + f(a_1)\, c_1^\sigma \\ \ldots \end{cases} \qquad \text{où l'on pose } [n]_\tau = \tau^n(\theta) - \theta.$$

Ces équations donnent le résultat cherché (remarquons que $[1]_\tau^{-1}, [2]_\tau^{-1}, \ldots$ sont bien dans $\mathbb{Z}_p[[\theta^{-1}]]$).

2) Puisque τ est $\mathbb{Z}_p$-linéaire, on a :

$$(\sum^n b_k\tau^k)(z_0(\theta^{-1}) + pz_1(\theta^{-1/p}) + \ldots) =$$

$$(\sum_0^n b_k\tau^k)(z_0(\theta^{-1})) + p(\sum_0^n b_k\tau^k)(z_1(\theta^{-1/p})) + \ldots$$

Le Théorème 1 permet d'étudier le degré en θ des c_i comme dans [5] p.133. Alors, chaque terme tend vers une limite respectivement dans $\mathbb{Z}_p((\theta^{-1}))$, $p\mathbb{Z}_p((\theta^{-1/p}))$,... donc la limite existe dans $W(R_\infty)$ d'après le Théorème 2.

3) La dernière partie est évidente. □

Remarques : 1) Si l'on fait agir exp_ψ sur $W(S_\infty)$ conformément à la convention de la fin du premier paragraphe, le diagramme étendu que l'on obtient reste commutatif et on constate que la construction ci-dessus commute avec l'action de $Gal(\overline{\mathbf{F}_p}/\mathbf{F}_p)$.

2) On peut de même définir le logarithme.

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Département de Mathématiques
Université de Caen
14032 Caen Cedex

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PSEUDO-SPECTRAL ANALYSIS OF STABLE NONISOTHERMAL CIRCULAR COUETTE FLOW

A. A. KOLYSHKIN AND RÉMI VAILLANCOURT

Presented by. K.B. Ranger, F.R.S.C.

ABSTRACT. The stability of a circular Couette flow with internal heat generation is studied in the annular region between an inner rotating cylinder and a fixed outer one, with same axis. The investigation is carried out for wide ranges of the Prandtl number and radius ratio R, for the axisymmetric mode ($n = 0$) and the first two asymmetric modes ($n = 1, 2$), respectively.

RÉSUMÉ. On étudie la stabilité d'un écoulement de Couette circulaire dans la région bornée par deux cylindres coaxiaux. Le cylindre interne tourne à vitesse constante et le cylindre externe ne bouge pas. La recherche porte sur un grand éventail du nombre de Prandtl et du rapport R des rayons pour le mode axisymétrique ($n = 0$) et les deux premiers modes asymétriques ($n = 1, 2$).

Subject-classification: AMS(MOS): 65N35, 35B35, 65F15, 70K20.
Keywords: Pseudo-spectral method, hydrodynamic stability

1. Introduction.

An unsteady circular Couette flow was studied experimentally by Cooper, Jankowski, Neitzel and Squire [1] and measurements were performed by means of a laser-Doppler velocimeter. It was found that the laser beam initiated instability in the flow of silicone oil (which has a high Prandtl number, approximately 115). Thus, internal heating has a significant influence on the measurements and has to be taken into account.

The stability characteristics of a convective motion with internal heat generation in an annulus are shown [2] to depend upon the gap between the cylinders and the Prandtl number.

The influence of an axial convective motion caused by internal heat sources on the stability of a circular Couette flow between two rotating cylinders is studied in [3]. The numerical investigation was carried out for wide ranges of the Prandtl number and radius ratio. Both the axisymmetric and asymmetric modes with mode numbers $n = 0$, and $n = 1$ and 2, respectively, are also studied. The mathematical problem is described in this report and the results are summarized.

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2. Formulation of the problem. Consider an infinitely long annulus with inner and outer radii, $R_1 < R_2$, filled with a viscous incompressible fluid. The inner cylinder rotates with constant angular velocity ω_1 while the outer one remains at rest. Internal heat sources with constant volume density Q are distributed within the fluid. Both cylinders are maintained at constant and equal temperature.

We introduce cylindrical polar coordinates (r, φ, z), with the origin on the axis of the cylinders and the z-axis directed upwards.

Dimensionless quantities are chosen as measures, respectively, of length, $h = (R_2 - R_1)/2$; time, $t = h^2/\nu$; temperature, $T = Qh^2/(2\kappa\rho c_p)$; and pressure, $p = g\beta Qh^3/(2\kappa c_p)$.

Different scales are used to measure the components of the velocity vector, i.e., $\omega_1 R_1$ for the azimuthal component and $u_0 = g\beta Qh^4/(2\nu\kappa\rho c_p)$ for the radial and vertical components. Here ρ is the density, ν the kinematic viscosity, β the coefficient of thermal expansion, κ the thermal conductivity, g the acceleration due to gravity and c_p the constant heat capacity. We introduce the following notation: $\mathrm{Gr} = g\beta Qh^5/(2\nu^2\rho\kappa c_p)$ is the Grashof number, $\mathrm{Pr} = \nu/\kappa$ the Prandtl number; $S = \omega_1 R_1/u_0$ the parameter arising from the different scalings of the velocity components; lastly $R = R_1/R_2$, $r_1 = 2R/(1-R)$, and $r_2 = 2/(1-R)$.

If we assume weak thermal convection, we can use the Oberbeck–Boussinesq approximation (see [4] and the references therein) of the Navier–Stokes equations in the form

$$\frac{\partial v_r}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial v_r}{\partial r} + S\frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi} + v_z\frac{\partial v_r}{\partial z} - S^2\frac{v_\varphi^2}{r}\right) = -\frac{\partial p}{\partial r} + \nabla^2 v_r - \frac{v_r}{r^2} - S\frac{2}{r_2}\frac{\partial v_\varphi}{\partial \varphi}, \quad (1)$$

$$S\frac{\partial v_\varphi}{\partial t} + \mathrm{Gr}\, S\left(v_r \frac{\partial v_\varphi}{\partial r} + S\frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + v_z\frac{\partial v_\varphi}{\partial z} + \frac{v_r v_\varphi}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + S\,\nabla^2 v_\varphi - S\frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}, \quad (2)$$

$$\frac{\partial v_z}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial v_z}{\partial r} + S\frac{v_\varphi}{r}\frac{\partial v_z}{\partial \varphi} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \nabla^2 v_z + T, \quad (3)$$

$$\frac{\partial T}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial T}{\partial r} + S\frac{v_\varphi}{r}\frac{\partial T}{\partial \varphi} + v_z\frac{\partial T}{\partial z}\right) = \frac{1}{\mathrm{Pr}}\nabla^2 T + \frac{2}{\mathrm{Pr}}, \quad (4)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + S\frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0, \quad (5)$$

where

$$\mathbf{v} = (v_r, v_\varphi, v_z), \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

System (1)–(5) has a steady axisymmetric solution of the following form

$$v_r = 0, \quad v_\varphi = V_0(r), \quad v_z = W_0(r), \quad T = T_0(r), \quad p_0(r,z) = p_{01}(z) + p_{02}(r). \quad (6)$$

We assume that the fluid flux through any cross-section of the channel is equal to zero,

$$\int_{r_1}^{r_2} r\, W_0(r)\, dr = 0. \quad (7)$$

Substitution of (6) into (1)–(5) leads to the following system of ordinary differential equations:

$$\mathrm{Gr}\, S^2 \frac{V_0^2}{r} = \frac{dp_{02}}{dr}, \qquad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) V_0 - \frac{V_0}{r^2} = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) W_0 + T_0 = \frac{dp_{01}}{dz}, \qquad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) T_0 = -2.$$

The solution to this system which satisfies (7) and the following boundary conditions

$$V_0|_{r=r_1} = 1, \quad V_0|_{r=r_2} = 0, \quad W_0|_{r=r_i} = 0, \quad T_0|_{r=r_i} = 0, \quad i = 1, 2,$$

has the form

$$V_0(r) = \frac{r_1 r_2^2}{(r_2^2 - r_1^2) r} - \frac{r_1 r}{r_2^2 - r_1^2}, \tag{8}$$

$$W_0(r) = A \ln\frac{r}{r_1} + \frac{A \ln R}{r_2^2 - r_1^2}(r^2 - r_1^2) + B(r^2 - r_1^2) + \frac{r_2^2 - r_1^2}{8 \ln R}(r^2 \ln r - r_1^2 \ln r_1) + \frac{(r^2 - r_1^2)(r^2 - r_2^2)}{32}, \tag{9}$$

$$T_0(r) = -\frac{r_2^2 - r_1^2}{2 \ln R} \ln\frac{r}{r_1} - \frac{r^2 - r_1^2}{2}, \tag{10}$$

where

$$A = -\frac{(r_2^2 - r_1^2)[12 r_1^2 r_2^2 + 3(r_2^4 - r_1^4)/\ln R + (r_2^2 - r_1^2)^2]}{96[r_2^2 - r_1^2 + (r_1^2 + r_2^2)\ln R]}, \qquad B = -\frac{r_2^2 \ln r_2 - r_1^2 \ln r_1}{8 \ln R}.$$

The formula for the pressure p_0 is omitted since it is not used in the sequel.

3. Method of solution. The method of normal perturbations is used to investigate the stability of flow (6). Consider a perturbed motion

$$\begin{bmatrix} v_r \\ v_\varphi \\ v_z \\ T \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ V_0(r) \\ W_0(r) \\ T_0(r) \\ p_0(r,z) \end{bmatrix} + \begin{bmatrix} u(r) \\ v(r) \\ w(r) \\ \theta(r) \\ q(r) \end{bmatrix} e^{-\lambda t + ikz + in\varphi} \tag{11}$$

where $u(r)$, $v(r)$, $w(r)$, $\theta(r)$ and $q(r)$ are the complex amplitudes of the normal perturbations, k is the wave number, $\lambda = \alpha + i\gamma$ is a complex eigenvalue; the mode number $n = 0$ corresponds to axisymmetric (toroidal) perturbations while $n \neq 0$ corresponds to asymmetric (spiral) ones.

Substituting (11) into (1)–(5), and linearizing the equations in the neighborhood of the base flow $\mathbf{v}_0 = V_0(r)\mathbf{e}_\varphi + W_0(r)\mathbf{e}_z$, T_0, p_0, we obtain the following system of ordinary differential equations

$$-\lambda u + \mathrm{Gr}\left(S\frac{V_0}{r} i n u + W_0 i k u - S^2 \frac{2}{r} v V_0\right) = -\frac{dq}{dr} + Lu - \frac{u}{r^2} - i n S \frac{2}{r^2} v, \tag{12}$$

$$-\lambda v+\mathrm{Gr}\left(u\frac{dV_0}{dr}+S\frac{V_0}{r}\,i\,n\,v+W_0\,i\,k\,v+\frac{u}{r}\,V_0\right)=-\frac{1}{S}\frac{i\,n\,q}{r}+Lv-\frac{v}{r^2}+\frac{i\,n}{S}\frac{2}{r^2}\,u, \quad (13)$$

$$-\lambda w+\mathrm{Gr}\left(u\,W_0'+S\frac{V_0}{r}\,i\,n\,w+W_0\,i\,k\,w\right)=-i\,k\,q+Lw+\theta, \quad (14)$$

$$-\lambda\,\mathrm{Pr}\,\theta+\mathrm{Gr}\,\mathrm{Pr}\left(u\,T_0'+S\frac{V_0}{r}\,i\,n\,\theta+W_0\,i\,k\,\theta\right)=L\theta, \quad (15)$$

$$u'+\frac{u}{r}+S\frac{i\,n\,v}{r}+i\,k\,w=0, \quad (16)$$

where $L := \frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}-k^2$ and the boundary conditions are

$$u|_{r=r_i}=0, \quad v|_{r=r_i}=0, \quad w|_{r=r_i}=0, \quad \theta|_{r=r_i}=0, \qquad i=1,2. \quad (17)$$

The eigenvalues, $\lambda_m=\alpha_m+i\gamma_m$, $m=1,2,\dots$, of the boundary value problem (12)–(17) determine the stability of flow (6), (8)–(10). The flow is stable if $\alpha_m>0$ for all m, and unstable if $\alpha_m<0$ for at least one value of m.

The system (12)–(17) depends upon many parameters, namely Gr, Pr, S, R, k and n. Usually, Pr, S, R and n are kept fixed, and the values Gr $=$ Gr(k) are determined as a function of k in the case only one eigenvalue has zero real part, while the others have positive real part. Then the critical Grashof numbers are found by setting Gr $=$ $\min_k$ Gr(k). This procedure is repeated for other sets of values of Pr, S, R and n.

A collocation method based on Chebyshev polynomials is used to obtain the numerical solution to (12)–(17). To reduce the size of the generalized eigenvalue problem

$$(A-\lambda B)\psi=0, \quad (18)$$

obtained by discretizing system (12)–(17), the amplitudes $w(r)$ and $q(r)$ are eliminated from (12)–(16). The resulting system is bulky and is not presented here. This elimination increases the order of the differential equation for $u(r)$ by two so that two extra boundary conditions are needed; these can be found, from (16) and (17), in the form

$$u'|_{r=r_i}=0, \qquad i=1,2. \quad (19)$$

Therefore, we seek the solution to (12)–(17) and (19) (after eliminating $w(r)$ and $q(r)$) in terms of the coordinate $x=r-(1+R)/(1-R)$, in the form

$$u(x)=\sum_{j=1}^{m}a_jS_{mj}(x), \quad v(x)=\sum_{j=1}^{m}b_jq_{mj}(x), \quad \theta(x)=\sum_{j=1}^{m}c_jq_{mj}(x), \quad (20)$$

where a_j, b_j and c_j are arbitrary (complex) constants, and the fundamental interpolation polynomials

$$q_{mj}(x)=(1-x^2)\frac{T_m(x)}{(x-x_j)T_m'(x_j)}, \quad S_{mj}(x)=(1-x^2)^2\frac{T_m(x)}{(x-x_j)T_m'(x_j)}, \quad (21)$$

satisfy the boundary conditions (17) and (19). Hence, the functions $u(x)$, $v(x)$, and $\theta(x)$ automatically satisfy these boundary conditions. Here $T_m(x)$ denotes Chebyshev's polynomial of the first kind of degree m whose zeros are $x_j = (2j-1)\pi/(2m)$, $j = 1, 2, \dots, m$.

Solutions of the form (20) are more convenient than those obtained by traditional collocation methods [5] for two reasons: first, in the present case the matrix B in (18) is not singular, and, second, the use of base functions which satisfy the given boundary conditions considerably reduces [6] the condition number of the matrices in this spectral method. This numerical method has been described in some detail by the authors [7, 8].

A check on the algorithm and code, and a comparison with results from other authors [9, 10] were made in the following two particular cases:

1) isothermal Couette flow, and

2) pure convection with internal heat generation without rotation.

In the first case, the computed critical Taylor numbers for different values of R, from 0.1 to 0.95, differ by at most 0.08% from the values found in Walowit *et al.* [8].

In the second case, the computed critical Grashof numbers for $R = 0.95$ differ by at most 0.3% from the values given by Gershuni *et al.*[22]. It thus becomes clear that for $R = 0.95$ the curvature of the layer is negligible and the computational results agree with those of the case of a plane vertical layer.

5. Numerical results and conclusion. Numerical investigation [3] was carried out for a wide range of radius ratio and Prandtl number for the axisymmetric ($n = 0$) mode and the first two asymmetric ($n = 1, 2$) modes. For small to large gaps between the cylinders ($R = 0.95$, 0.7, and 0.4) axisymmetric disturbances alone lead to instability for all Prandtl numbers in the interval $1 \leq \mathrm{Pr} \leq 100$. An increase in the Prandtl number as well as in the Taylor number has a destabilizing effect on the flow—the critical Grashof number decreases. The pertubations propagate downstream, and the phase speed of the perturbations increases with the growth of the Prandtl number. For very large gaps ($R = 0.1$), there exist regions of stabilization of the Taylor vortices by internal heat generation—for example, at $\mathrm{Pr} = 5$ the critical Grashof number is a monotone increasing function of the Taylor number, Ta. The perturbations take the form of upstream or downstream spiral vortices ($n = 1$), the direction of which depending upon the ratio of the parameters of the problem. The region of stabilization of Taylor vortices for large Prandtl numbers at $R = 0.1$ corresponds to asymmetric mode ($n = 1$) and to small Taylor numbers; the size of the region decreases with the growth of the Prandtl number. Several bifurcations occur in the case $\mathrm{Pr} = 1$: (a) at $(\mathrm{Gr}, \mathrm{Ta}) = (884, 18)$ there is a transition from a long-wave spiral mode ($n = 1$) to an upstream spiral mode ($n = 1$) with a different wave number, (b) at $(\mathrm{Gr}, \mathrm{Ta}) = (903, 43)$ the transition is to the axisymmetric mode, and (c) at $(\mathrm{Gr}, \mathrm{Ta}) = (887, 177)$ the transition is again to the spiral mode with $n = 1$. Typical situations are shown in Fig. 1(a) and 1(b).

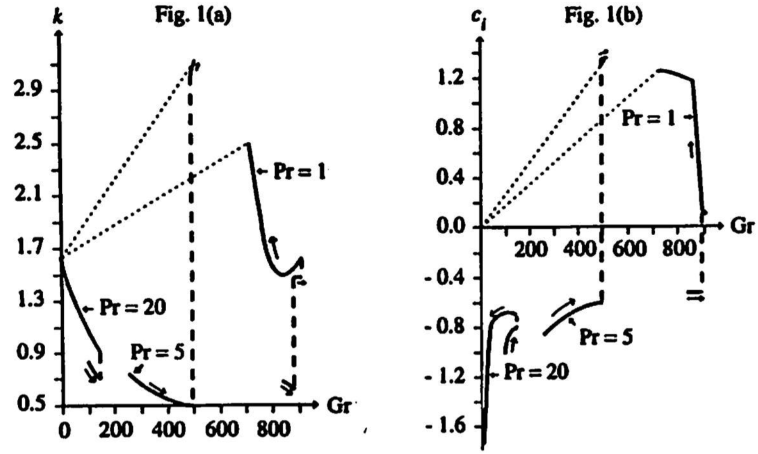

FIGURE 1. As Ta increase to 18, Fig. 1(a) shows a jump from $(\mathrm{Gr}, k) = (884, 0.62)$ to $(\mathrm{Gr}, k) = (884, 1.49)$. There, not only the size of the cells changes, but also the direction of the perturbation propagation—in the region $\mathrm{Ta} > 18$ they spread upstream—and the phase speed changes from $c_i = -0.53$ to $c_i = 0.11$ as shown in Fig. 1(b). The next jump in the (Gr, k)- and (Gr, c_i)-planes is related to the transition from the axisymmetric branch to the asymmetric $(n = 1)$ one.

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DEPARTMENT OF APPLIED MATHEMATICS, RIGA TECHNICAL UNIVERSITY, RIGA, LATVIA, LV1010

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO, CANADA K1N 6N5

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Mailing Addresses

O.I. Bogoyavlenskij	Department of Mathematics and Statistics Queen's University Kingston, Ontario K7L 3N6 Canada
E.N. Derun	Department of Automated Control Systems Riga Technical University Riga, Latvia LV 1010
J. Faulkner	Lick Observatory University of California, Santa Cruz Santa Cruz, CA 95064 U.S.A.
Z. Ge	Fields Institute 185 Columbia Street West Waterloo, Ontario N2L 5Z5 Canada
Y. Hellegouarch	Département de Mathématiques Université de Caen F-14032 Caen Cedex, France
A.A. Kolyshkin	Department of Applied Mathematics Riga Technical University Riga, Latvia LV 1010
M. Le	Department of Mathematics Hunan Normal University P.O. Box 410081 Changsha, Hunan, P.R. China
S. Park	Department of Mathematics Seoul National University Seoul 151-742, Korea
D. Perrone	Dipartimento di Matematica Universitá degli Studi di Lecce I-73100 Lecce, Italy
F. Recher	Département de Mathématiques Université de Caen F-14032 Caen Cedex, France
W.F. Shadwick	Fields Institute 185 Columbia Street West Waterloo, Ontario N2L 5Z5 Canada
R Vaillancourt	Department of Mathematics University of Ottawa Ottawa, Ontario K1N 6N5 Canada