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# CHARACTERIZATIONS OF $\varphi$-SYMMETRIC SPACES IN TERMS OF THE CANONICAL CONNECTION

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

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Presented by E. Bierstone, F.R.S.C.

## 1. Introduction.

Let $(M, g)$ be a Riemannian manifold and $R$ the curvature tensor of the Levi-Civita connection $\nabla$. Define the Jacobi operator $K_v(\cdot) = R(\cdot, v)v$ for each $v \in SM$, the unit tangent bundle of $M$. In Riemannian geometry, the Jacobi operators play important roles and, in particular, give us much informations about symmetric spaces(cf. Besse [1], Chavel [6]). But in a naturally reductive Riemannian homogeneous space the Jacobi operator of a canonical connection is nicer, since the curvature tensor of $\nabla$ is, in general, not parallel(cf. Ziller [12]).

On the other hand, a globally Sasakian $\varphi$-symmetric space is a typical example of a naturally reductive homogeneous space, and has nice geometric properties.

Let $M$ be a Sasakian manifold with structure tensors $(\varphi, \xi, \eta, g)$. We denote by $\tilde{R}$ the curvature tensor of a canonical connection, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(\varphi X, Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X$$

for any vector fields $X, Y$. Then the purpose of this paper is to prove the following.

MAIN THEOREM. *Let $M$ be a $(2n+1)$-dimensional connected, simply-connected, complete Sasakian manifold with structure tensors $(\varphi, \xi, \eta, g)$. Then the following conditions are equivalent.*

(I) *$M$ is $\varphi$-symmetric.*

(II) $(\tilde{\nabla}_X \tilde{R})(\cdot, X)X = 0$ *for any vector field $X$ orthogonal to $\xi$.*

(III) *The operator $\tilde{K}_{\gamma'}$ along any $\varphi$-geodesic $\gamma$ is parallel with respect to $\tilde{\nabla}$.*

## 2. Sasakian $\varphi$-symmetric spaces.

Let $M$ be a $(2n+1)$-dimensional Sasakian manifold with structure tensors $(\varphi, \xi, \eta, g)$ (cf. Blair [2], Takahashi [9]):

$$\begin{aligned}
&\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \\
(2.1)\qquad &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad d\eta(X, Y) = g(\varphi X, Y), \\
&(\nabla_X \varphi)(Y) = \eta(Y)X - g(X, Y)\xi, \quad \nabla_X \xi = \varphi X
\end{aligned}$$

for any vector fields $X, Y$, where $\nabla$ is the Levi-Civita connection for $g$. The curvature tensor

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

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on a Sasakian manifold satisfies the following

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi \tag{2.2}$$

for any vector fields $X, Y, Z$.

A geodesic $\gamma$ on a Sasakian manifold is said to be a $\varphi$-geodesic if $\eta(\gamma') = 0$. From (2.1) it is easy to see that a geodesic which is initially orthogonal to $\xi$ remains orthogonal to $\xi$. A local diffeomorphism $s_m$ of $M, m \in M$, is said to be a $\varphi$-geodesic symmetry if its domain $U$ is such that, for every $\varphi$-geodesic $\gamma$ such that $\gamma(0)$ lies in the intersection of $U$ with the integral curve of $\xi$ through $m$,

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all $s$ with $\gamma(\pm s) \in U$, $s$ being the arc length. Setting $S = -I + 2\eta \otimes \xi$, we have

$$s_m = exp_m \circ S_m \circ exp_m^{-1}, \tag{2.3}$$

since $\xi$ is a Killing vector field(see Blair-Vanhecke [3]).

In [8], Takahashi introduced the notion of a locally $\varphi$-symmetric space by requiring that

$$\varphi^2(\nabla_V R)(X,Y)Z = 0$$

for any vector fields $V, X, Y, Z$, orthogonal to $\xi$. Moreover, he defined a globally $\varphi$-symmetric space by requiring that any $\varphi$-geodesic symmetry of a locally $\varphi$-symmetric space to be extendible to a global automorphism of $M$ and that the Killing vector field $\xi$ generates a global one-parameter subgroup of isometries.

Here we collect three results from Takahashi [8] and Blair-Vanhecke [3],[4],[5].

THEOREM 1. *A necessary and sufficient condition for a Sasakian manifold to be locally $\varphi$-symmetric is that it admits a $\varphi$-geodesic symmetry at every point, which is a local automorphism of the structure $(\varphi, \xi, \eta, g)$.*

THEOREM 2. *A Sasakian manifold $M$ is locally $\varphi$-symmetric if and only if, for any vector fields $V, X, Y, Z$,*

$$\begin{aligned}(\nabla_V R)(X,Y)Z = \; & [g(X,V)g(\varphi Y,Z) - g(Y,V)g(\varphi X,Z) + g(\varphi R(X,Y)V,Z)]\xi \\ & + \eta(X)[-g(\varphi V,Z)Y + g(Y,Z)\varphi V + R(Y,\varphi V)Z] \\ & + \eta(Y)[g(\varphi V,Z)X - g(X,Z)\varphi V - R(X,\varphi V)Z] \\ & + \eta(Z)[g(Y,V)\varphi X - g(X,V)\varphi Y - \varphi R(X,Y)V].\end{aligned}$$

THEOREM 3. *A Sasakian manifold $M$ is locally $\varphi$-symmetric if and only if*

$$g((\nabla_X R)(X,\varphi X)X, \varphi X) = 0$$

*for any vector field $X$, orthogonal to $\xi$.*

## 3. The canonical connection.

Let $M$ be a Sasakian manifold with structure tensors $(\varphi, \xi, \eta, g)$. Let $T$ be a tensor field of type (1,2) defined by

$$T(X,Y) = g(\varphi X,Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X \tag{3.1}$$

for any vector fields $X, Y$. We now define a linear connection

$$\tilde{\nabla}_X Y = \nabla_X Y + T(X, Y). \tag{3.2}$$

for any vector fields $X, Y$. This linear connection is called the canonical connection. By direct calculations, we see that

$$\tilde{\nabla}\varphi = 0, \quad \tilde{\nabla}\xi = 0, \quad \tilde{\nabla}\eta = 0, \quad \tilde{\nabla}g = 0 \text{ and } \tilde{\nabla}T = 0. \tag{3.3}$$

Remark that $\tilde{\nabla}$ has the same geodesics as $\nabla$.

Let $\tilde{R}$ and $R$ be the curvature tensors of $\tilde{\nabla}$ and $\nabla$, respectively. Then we have

$$\begin{aligned} \tilde{R}(X,Y)Z = \; & R(X,Y)Z + \eta(Z)[\eta(X)Y - \eta(Y)X] \\ & + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(\varphi X, Y)\varphi Z \\ & + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi. \end{aligned} \tag{3.4}$$

Hence by Theorem 2 we have

$$\begin{aligned} (\tilde{\nabla}_V \tilde{R})(X,Y)Z = \; & (\nabla_V R)(X,Y)Z \\ & + T(V, R(X,Y)Z) - R(T(V,X),Y)Z \\ & - R(X, T(V,Y))Z - R(X,Y)T(V,Z). \end{aligned} \tag{3.5}$$

Thus, we have the following result(see Takahashi [9]).

THEOREM 4. *A necessary and sufficient condition for a Sasakian manifold to be locally $\varphi$-symmetric is that*

$$\tilde{\nabla}\tilde{R} = 0, \tag{3.6}$$

*where $\tilde{\nabla}$ is the canonical connection defined by* (3.2).

In particular, a locally $\varphi$-symmetric space is locally homogeneous (see Takahashi [9]). Moreover, since T(X,X) = 0, it follows that in this case, all local geodesic symmetries are volume-preserving up to sign(cf. Watanabe [11]). Finally, the same condition $T(X,X) = 0$ implies that a simply-connected, complete, locally $\varphi$-symmetric space is a naturally reductive homogeneous space (cf. Tricerri-Vanhecke [10]).

## 4. Jacobi fields.

Let $m$ be a point of a Sasakian manifold $M$ and let $\gamma$ be a geodesic, parametrized by arc length $s$, through $m = \gamma(0)$. A Jacobi field along $\gamma$ is a vector field $X$ satisfying the equation

$$\nabla_{\gamma'}\nabla_{\gamma'}X + R(X,\gamma')\gamma' = 0. \tag{4.1}$$

In order to prove the Main Theorem, we write down (4.1) in terms of the connection $\tilde{\nabla}$ of a Sasakian manifold as follows:

$$\tilde{\nabla}_{\gamma'}\tilde{\nabla}_{\gamma'}X - 2T(\gamma', \tilde{\nabla}_{\gamma'}X) + \tilde{R}(X,\gamma')\gamma' = 0, \tag{4.2}$$

where the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is given by $\tilde{T} = 2T$.

Suppose that $(\tilde{\nabla}_{\gamma'}\tilde{R})(\cdot,\gamma')\gamma' = 0$ along a geodesic $\gamma$. We take an orthonormal basis $\{e_1, e_2, \ldots, e_{2n+1}\}$ such that $\gamma'(0) = e_1$, and we denote by $\{\tilde{E}_i, i = 1, \ldots, 2n+1\}$ the orthonormal frame field along $\gamma$, obtained by $\bar\nabla$-parallel translation of the $e_i$ along $\gamma$. Then, from (3.3) and the assumption $(\bar\nabla_{\gamma'}\bar R)(\cdot,\gamma')\gamma' = 0$, we have

$$g(\bar E_a, \bar R(\bar E_b,\gamma')\gamma') = constant, \quad g(\bar E_a, T(\bar E_b,\gamma')) = constant. \tag{4.3}$$

In particular, we choose an orthonormal frame field $\{\bar E_1, \bar E_2, \dots, \bar E_{2n+1}\}$ such that $e_1 = \gamma'(0)$, $e_2 = \bar E_2(0) = (\xi_m - \alpha\gamma'(0))/\sqrt{1-\alpha^2}$ and

$$\bar K_{\gamma'}(\bar E_a) = \bar R(\bar E_a,\gamma')\gamma' = \lambda_a \bar E_a$$

for $a = 3, \dots, 2n+1$, where $\alpha = \eta(\gamma') = constant$ and $\lambda_a = constant$. Thus, if we denote the Jacobi field along $\gamma$ by

$$X = f_1\bar E_1 + f_2\bar E_2 + f_3\bar E_3 + \dots + f_{2n+1}\bar E_{2n+1},$$

then the Jacobi equation (4.2) reduces to the following.

$$(f_1)'' - 2\alpha\Sigma_{a=3}^{2n+1}(f_a)'g(\varphi\gamma'(0),e_a) = 0,$$

$$(f_2)'' - 2\frac{(1+\alpha-\alpha^2)}{\sqrt{1-\alpha^2}}\Sigma_{a=3}^{2n+1}g(\varphi\gamma'(0),e_a)(f_a)' = 0, \tag{4.5}$$

$$(f_a)'' - 2\{\alpha\Sigma_{c=3}^{2n+1}g(e_a,\varphi e_c)(f_c)' - \frac{g(\varphi\gamma'(0),e_a)}{\sqrt{1-\alpha^2}}(f_2)'\} + \lambda_a f_a = 0$$

for a=3, ... , 2n+1.

## 5. Proof of Main Theorem.

By Theorems 2 and 3 and (3.5) we see that (I) is equivalent to (II). Moreover, since $(\bar\nabla_{\gamma'}\bar R)(Y,\gamma')\gamma' = (\bar\nabla_{\gamma'}\bar K_{\gamma'})(Y)$ for any vector field $Y$ along each geodesic $\gamma$, it is trivial that (II) implies (III).

We shall prove that (III) implies (I). So, we may assume that $(\bar\nabla_{\gamma'}\bar R)(\cdot,\gamma')\gamma' = 0$ along any $\varphi$-geodesic $\gamma$. From a result of Blair-Vanhecke [3] it is sufficient to prove that the $\varphi$-geodesic symmetry $s_m$ is a (local) isometry. First, from a result of Blair-Vanhecke [5] it follows that for sufficient small $s > 0$,

$$(s_m)_{*\gamma(s)}\xi = \xi_{\gamma(-s)}. \tag{5.1}$$

We now obtain Jacobi fields $X$ along each $\varphi$-geodesic $\gamma$. For this purpose, we have the following.

LEMMA. *If $\bar K_{\gamma'}$ along a $\varphi$-geodesic $\gamma$ is parallel with respect to $\bar\nabla$, then $\varphi\gamma'$ is an eigenvector of $\bar K_{\gamma'}$.*

PROOF. It is sufficient to prove that $\varphi\gamma'(0)$ is an eigenvector of $\bar K_{\gamma'(0)}$. Assume that $\varphi\gamma'(0)$ is not an eigenvector of $\bar K_{\gamma'(0)}$. If we take an eigenvector $e \neq \mu\varphi\gamma'(0), \mu(\neq 0) \in \mathbb{R}$, orthogonal to both $\xi_m$ and $\gamma'(0)$, then $e$ is also orthogonal to $\varphi\gamma'(0)$. In fact, we denote $E$ by the $\nabla$-parallel vector field along $\gamma$ such that $E(0) = e$. Then, from the assumption $(\bar\nabla_{\gamma'}\bar R)(\cdot,\gamma')\gamma' = 0$, we have

$$\begin{aligned} 0 &= g(\xi,(\bar\nabla_{\gamma'}\bar R)(E,\gamma')\gamma')(0) \\ &= g(\xi,(\nabla_{\gamma'}R)(E,\gamma')\gamma')(0) - [-g(\varphi E,\gamma') + g(\varphi R(E,\gamma')\gamma',\gamma')](0) \\ &= g(\xi,\nabla_{\gamma'}(R(E,\gamma')\gamma'))(0) - [g(\varphi\gamma',E) - g(R(E,\gamma')\gamma',\varphi\gamma')](0) \\ &= -g(e,\varphi\gamma'(0)), \end{aligned}$$

taking account of (3.5) and (2.2). Therefore, we can now choose an orthonormal basis $\{e_1 = \gamma'(0), e_2 = \xi_m, e_3, \ldots, e_{2n+1}\}$ of $T_m(M)$ such that $\bar{K}_{\gamma'(0)}e_a = \lambda_a e_a (a = 1, \ldots, 2n+1)$. This is a contradiction, since $\varphi\gamma'(0)$ is orthogonal to all $\gamma'(0), \xi_m, e_3, \ldots, e_{2n+1}$.

Since $\varphi\gamma'$ is an eigenvector of $\bar{K}_{\gamma'}$, we can choose an orthonormal basis $\{e_1 = \gamma'(0), e_2 = \xi_m, e_3 = \varphi\gamma'(0), e_4, \ldots, e_{2n+1}\}$ such that $\bar{K}_{\gamma'(0)}e_a = \lambda_a e_a$ for $a = 1, \ldots, 2n+1$, where $\lambda_2 = 0$. So, by (3.4) we have

$$R(\varphi\gamma', \gamma')\gamma' = (\lambda_3 + 1)\varphi\gamma', \tag{5.2}$$

Moreover, it follows from (4.5) that along the $\varphi$-geodesic $\gamma$,

$$f_2'' - 2f_3' = 0, \quad f_3'' + 2f_2' + \lambda_3 f_3 = 0, \tag{5.3}$$

and

$$(f_a)'' + \lambda_a f_a = 0 \tag{5.4}$$

for a=4, ... , 2n+1. Since $\xi$ is an isotropic Jacobi field along $\gamma$ (see Ziller [12]) with initial condition $(\xi(0) = \xi_m, \bar{\nabla}_{\gamma'(0)}\xi = 0)$ and parallel with respect to $\bar{\nabla}$, by (5.3) we have

$$f_2 = 1, \quad \lambda_3 = 0,$$

from which (2.1) and (5.2) imply that $\varphi\gamma'$ is a Jacobi field along $\gamma$. By (2.3) we have

$$(s_m)_{*\gamma(s)}\varphi\gamma' = \varphi\gamma'(-s).$$

Any vector field $X$ along $\gamma$ orthogonal to $\gamma', \xi$ and $\varphi\gamma'$ satisfies the initial condition $(\bar{\nabla}_{\gamma'(0)}X = \nabla_{\gamma'(0)}X$. Then, by the same arguments as a locally symmetric space(cf. Chavel [6], Osserman-Sarnak [8]), we can see from (5.1) and (5.2) that

$$\|s_{m*}Y\| = \|Y\|$$

for any vector $Y \in T_{\gamma(s)}(M)$. This implies that $s_m$ is an isometry.

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# Large Deviations: Upper Bound on Metric Space and Application

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**Abstract:** Let $X$ be a space with two metrics $d_1$ and $d_2$. The topology of $(X, d_1)$ is weaker than the topology of $(X, d_2)$. Some general results about large deviation upper bound on space $(X, d_2)$ is obtained under assumptions which are related to the condition that the complement of a $d_2$-ball has an exponential decay rate and the condition of the usual "exponential tightness". The result we get is for a class of $\mathcal{B}$-measurable, $d_2$-closed subsets of $X$. $\mathcal{B}$ is a $\sigma$-algebra on $X$ contained in the Borel $\sigma$-algebra of $(X, d_2)$. Applying this result and some properties of the vector-valued $L^p$ space, we obtain some large deviation upper bounds for a sequence of probability measures on the space of measure-valued paths.

## 1. Introduction

Let $X$ be a topological space, $\{a_N\}_{N=1,\dots}$ be a sequence of positive numbers which goes to infinity as $N$ goes to infinity. Let $\{P_N\}_{N=1,\dots}$ be a sequence of probability measures on $(X, \mathcal{L})$, where $\mathcal{L}$ is a $\sigma$-algebra of $X$ containing all open subset of $X$. One says that $\{P_N\}_{N=1,\dots}$ obeys a large deviation principle in $X$, with speed $\{a_N\}_{N=1,\dots}$ and rate function $I(\cdot) : X \to [0, +\infty]$ if

i). for every open subset $G$ of $X$

$$\liminf_{N\to\infty} a_N^{-1} \log P_N(G) \geq -\inf_{x\in G} I(x). \tag{1.1}$$

ii). for every closed subset $F$ of $X$

$$\limsup_{N\to\infty} a_N^{-1} \log P_N(F) \leq -\inf_{x\in F} I(x). \tag{1.2}$$

iii). the level sets $\{x \in X : I(x) \leq s\}$ are compact for all $s \geq 0$.

The study of large deviations of the infinite dimensional systems has been very active in recent years. For finite dimensional spaces the following condition of "exponential tightness"

$$\forall \eta > 0,\ \exists \text{a compact subset } K_\eta \text{ of } X \ni$$
$$\limsup_{N\to\infty} \frac{1}{a_N} \log P_N(K_\eta^c) \leq -\eta. \tag{1.3}$$

plays a key role in proving the upper bound. Under this condition, in order to obtain the upper bound it suffices to show that (1.2) is satisfied for any compact subset of $X$. Since in finite dimensional spaces a set is compact if and only if it is closed and bounded, (1.3) is equivalent to the following condition

$$\forall \eta > 0,\ \exists \text{a bounded closed subset } B_\eta \text{ of } X \ni$$
$$\limsup_{N\to\infty} \frac{1}{a_N} \log P_N(B_\eta^c) \leq -\eta. \tag{1.4}$$

In infinite dimensional space compact sets and closed bounded sets are not equivalent in general. Thus we have two candidates for the condition to get the large deviation upper bound in infinite dimensional space: one is the "exponential tightness" and the other is (1.4). Recently, P.Baldi [1] studied the large deviation principle for a sequence of probability measures on an arbitrary normed vector spaces. Under the assumption of "exponential tightness", he obtained a large deviation principle using the same method as that in finite dimensional case. C.Léonard [5] suggested that when dealing with the case where $X$ is the strong dual space of some separable normed vector space one could use the condition (1.4) instead of the "exponential tightness". This is not only a technically simpler condition but also a weaker condition in the case of infinite dimension.

In this article we will discuss the large deviation upper bound on any metric space. Two abstract conditions are given and their relations with (1.3) and (1.4) are also discussed. Roughly speaking our result shows that assumption (1.1) can be weakened to get a good upper bound while (1.4) may not sufficient to get even a weaker upper bound.

The abstract results will be given in section 2. The necessity of exponential tightness is shown on a non-Polish space for obtaining large deviation principle. In section 3 we first discuss some basic properties of the vector-valued $L^p$ space which can be found in [3]. Then the result is used to obtain the large deviation upper bound for the measure-valued Markov process.

The new contributions of this article are to show the appropriate formulation of the large deviation problem on non-separable metric spaces and to obtain a very general upper bound result on metric spaces.

Proofs, as well as further details, will appear in a forthcoming paper by the author.

**2. Upper Bound.** Let $X$ be any set, $d_1, d_2$ be two metrics on $X$. Let $\mathcal{B}_1$ be the Borel $\sigma$-algebra of $(X, d_1)$ with respect to which for every $x_0 \in X$ the function $f_2(x) = d_2(x, x_0)$ is measurable. We assume

(a). The topology of $(X, d_1)$ is weaker than the topology of $(X, d_2)$.

(b). Any bounded closed ball in $(X, d_2)$ is compact in $(X, d_1)$.

Let $X_0 \subset X$, $\mathcal{C}$ be a $\sigma$-algebra on $X_0$ containing $\mathcal{B}_1 \cap X_0$ but contained in the Borel $\sigma$-algebra of $(X_0, d_2)$. Let $\mathcal{B} = \{A \subset X; A \cap X_0 \in \mathcal{C}, A \text{ is in the Borel } \sigma\text{-algebra of the space } (X, d_2)\}$. Then $\mathcal{B}$ contains $\mathcal{B}_1$ and is contained in the Borel $\sigma$-algebra of space $(X, d_2)$. Let $\{P_N\}_{N\geq 1}$ be a sequence of probability measures on $(X_0, \mathcal{C})$, $Z$ be the set of all bounded continuous functions on $(X, d_2)$ whose restriction to $X_0$ are $\mathcal{C}$-measurable, $(Z^*, \sigma\{Z^*, Z\})$ be the dual space of $Z$ with *weak** topology $\sigma\{Z^*, Z\}$ and $(Z^*, \|\cdot\|^*)$ be the dual space of $Z$ with strong topology $\|\cdot\|^* = \sup\{|\langle\cdot, g\rangle|; g \in Z \text{ and } \|g\|_\infty = \sup_{x\in X}|g(x)| \leq 1\}$. By definition we can see that $Z = C_{\cdot}(X, d_2, \mathcal{B})$, the set of all $\mathcal{B}$-measurable, bounded continuous functions on $(X, d_2)$ and for each $N \geq 1$ we can extend $P_N$ to $(X, \mathcal{B})$ by letting $P_N(X \setminus X_0) = 0$. If we introduce a $\sigma$-algebra $\tilde{\mathcal{B}}$ on $Z^*$ as

$$\tilde{\mathcal{B}} = \{B \subset Z^*; \{x \in X : \delta_x \in B\} \in \mathcal{B}\}$$

Let the map $\Theta$ be defined by

$$\Theta : (X, \mathcal{B}) \longrightarrow (Z^*, \tilde{\mathcal{B}}), \quad x \mapsto \delta_x \in Z^*.$$

Let $Q_N(\cdot) = P_N \circ \Theta^{-1}(\cdot)$. $\forall f \in Z$, $z^* \in Z^*$, we define

$$H_N(f) = \frac{1}{N} \log \int_{Z^*} \exp(N\langle z^*, f\rangle) Q_N(dz^*),$$

$$H(f) = \limsup_{N\to\infty} H_N(f), \quad H^*(z^*) = \sup_{f\in Z}\{\langle z^*, f\rangle - H(f)\}.$$

**Lemma 2.1** *For any strong bounded, $\tilde{\mathcal{B}}$-measurable and weak* closed subset $A$ of $Z^*$, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_N(A) \le - \inf_{z^*\in A} H^*(z^*). \tag{2.1}$$

The next lemma gives a sufficient condition for an element of $Z^*$ to be a probability measure on $(X, \mathcal{B})$.

**Lemma 2.2** *If $H^*(z^*) < \infty$ and for any decreasing sequence of functions $\{f_n\} \subset Z$ converging to zero pointwise we have $H(f_n) \to 0$, then $z^*$ is a probability measure on $(X, \mathcal{B})$.*

Here are the main results:

**Theorem 2.3** *Let $(X, d_1, d_2)$ satisfy the assumptions (a) and (b). We have*
(i). *Under the assumption that*

$$\text{For every sequence } \{f_n\} \subset Z,\ f_n \downarrow 0, \text{ we have } H(f_n) \to 0, \tag{2.2}$$

*then for any $\mathcal{B}$-measurable, closed subset $A$ of $(X, d_2)$ we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(A) \le - \inf_{x\in A} L(x). \tag{2.3}$$

(ii). *If*

$$\text{For every sequence} \{f_n\} \subset C_b(X, d_2, \mathcal{B}_1),\ f_n \downarrow 0, \text{ we have } H(f_n) \to 0, \tag{2.4}$$

*then for any $\mathcal{B}$-measurable, closed subset $A$ of $(X, d_2)$ we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(A) \le - \inf_{x\in A} \tilde{L}(x). \tag{2.5}$$

*where $L(x) = H^*(\Theta(x))$, $\tilde{L}(x) = \sup_{f\in C_b(X,d_2,\mathcal{B}_1)}\{f(x) - H(f)\}$. Obviously*

$$\tilde{L}(x) \le L(x),\ \forall x \in X.$$

**Corollary 2.4** *Let $\bar{X}_0$ be the closure of $X_0$ in the space $(X, d_2)$. Then we have*
(i). *If $\bar{X}_0 \cap \{x \in X; L(x) < \infty\} \subset X_0$ and (2.2) is true, then for any $\mathcal{C}$-measurable, closed subset $A$ of $(X_0, d_2)$ we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(A) \le - \inf_{x\in A} L(x). \tag{2.6}$$

(ii). *If $\bar{X}_0 \cap \{x \in X; \tilde{L}(x) < \infty\} \subset X_0$ and (2.2) is true, then for any $\mathcal{C}$-measurable, closed subset $A$ of $(X_0, d_2)$ we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(A) \le - \inf_{x\in A} \tilde{L}(x). \tag{2.7}$$

The following two results discuss the relation of the above results with (1.3) and (1.4).

**Theorem 2.5** *If the condition*

$$\forall a > 0 \quad \exists \text{ compact subset } K_a \subset (X, d_2), \ni$$
$$\limsup_{N\to\infty} \frac{1}{N} \log P_N\{x : x \notin K_a\} \le -a. \tag{2.8}$$

*is satisfied, then $\forall \{f_n\} \subset Z$, $f_n \downarrow 0$ pointwise, we have $H(f_n) \to 0$.*

**Theorem 2.6** *If the condition*

$$\forall a > 0,\ \exists b \geq 0,\ x_0 \in X,\ \ni\ \limsup_{N\to\infty} \frac{1}{N} \log P_N\{x; d_2(x, x_0) > b\} \leq -a. \tag{2.9}$$

*is satisfied, then* $\forall \{f_n\} \subset C_b(X, d_1)$, $f_n \downarrow 0$ *pointwise, we have* $H(f_n) \to 0$.

**Remarks.** 1. The exponential tightness is crucial in the passage from a weak to a full large deviation principle. But it is not a necessary condition in general.

2. If $X$ is a locally compact topological space or a Polish space, then it is necessary in order to get a full large deviation principle (cf. exercises 1.2.19 and 4.1.10 in [2]).

In chapter 7 of Feng [4] the following result is proved: Let $X$ be a Polish space. $X_n$, $n = 1, \cdots$, is a increasing sequence of closed subset of $X$. Let $X_\infty = \cup_{n=1}^{\infty} X_n$ be equipped with the inductive topology. $X_\infty$ is not Polish in general. We have

**Theorem 2.7** *If* $\{P_N\}_{N\geq 1}$ *is a sequence of probability measures on space* $X_\infty$ *satisfying a large deviation upper bound with a good rate function* $I$ *and for any* $s > 0$ *there exists an* $n_0$ *such that for any* $n \geq n_0$

$$\limsup_{N\to\infty} N^{-1} \log P_N((X_n)^c) \leq -s \tag{2.10}$$

*then* $\{P_N\}_{N\geq 1}$ *is exponential tight on* $X_\infty$.

## 3. Application.

Let $E = \{0, 1, \cdots\}$ be equipped with the discrete topology, $\varphi$ be a strictly positive function on $E$ satisfying: $\lim_{x\to\infty} \frac{\varphi(x)}{x} = +\infty$ and

$$\mathcal{X} = \{g; g \text{ is a real valued function on } E,\ |||g|||_\varphi < \infty \text{ and } \lim_{|x|\to\infty} \frac{g(x)}{\varphi(x)} \text{ exists}\}$$

where

$$|||g|||_\varphi = \sup_{x\in E} \frac{|g(x)|}{\varphi(x)}.$$

Then $(\mathcal{X}, |||\cdot|||_\varphi)$ is a separable normed vector space.

$(\mathcal{X}^*, |||\cdot|||_\varphi^*)$ is the strong dual space of $(\mathcal{X}, |||\cdot|||_\varphi)$. Where

$$\forall \nu \in \mathcal{X}^*, |||\nu|||_\varphi^* = \sup\{|\langle \nu, g\rangle|; g \in \mathcal{X}, |||g|||_\varphi \leq 1\}$$

Let

$$\begin{aligned} L^1_{\mathcal{X}} &= \{F : [0, T] \to \mathcal{X}, F \text{ is measurable and } |F|_1 = \int_0^T |||F(t)|||_\varphi dt < \infty\}; \\ C_{\mathcal{X}} &= \{F : [0, T] \to \mathcal{X}; F \text{ is continuous}\}. \end{aligned}$$

Then $C_{\mathcal{X}}$ is a separable dense subset of space $L^1_{\mathcal{X}}$. Let

$$\Lambda_{\mathcal{X}^*} = \{\nu(\cdot) : [0, T] \to \mathcal{X}^*, \ni t \mapsto \langle \nu(t), g\rangle \text{ is measurable for any } g \in \mathcal{X} \text{ and } |\nu(\cdot)|_\infty < \infty\}.$$

where $|\nu(\cdot)|_\infty = \text{ess sup}_{0\leq t\leq T} |||\nu(t)|||_\varphi^*$. Then we have

**Theorem 3.8** *The strong dual of $L^1_X$ can be identified as $\Lambda_{X^*}$.*

Since $C_X$ is separable, we can introduce one more topology on $\Lambda_{X^*}$ as follow. Let $C$ be a countable dense subset of $C_{X^*}$. $\forall\mu(\cdot),\ \nu(\cdot)\in\Lambda_{X^*}$ we define

$$\gamma(\mu(\cdot),\nu(\cdot))=\sum_{F_n\in C}\frac{1}{2^n}|\langle\mu(\cdot),F_n\rangle-\langle\nu(\cdot),F_n\rangle|\wedge 1.$$

which is a metric on $\Lambda_{X^*}$. It is easy to see that the topology of space $(\Lambda_{X^*},\gamma)$ is weaker than the *weak** topology on $\Lambda_{X^*}$ and $(\Lambda_{X^*},\gamma)$ is separable. Let $B_1$ denote the Borel $\sigma$-algebra of $(\Lambda_{X^*},\gamma)$.

Let $M_1(E)$ denote the set of all probability measures on $E$ with the usual weak topology. For $m\geq 1$, we define $M^w_{\varphi,m}(E)=\{u\in M_1(E);\langle u,\varphi\rangle\leq m\}$ endowed with the subspace topology of $M_1(E)$. Define $\mathcal{D}_m=D([0,T],M^w_{\varphi,m}(E))$, $\mathcal{D}_\infty=\cup_{m\geq 1}\mathcal{D}_m$. Let $(\mathcal{D}_m,d_{sko})$ denote the space $\mathcal{D}_m$ endowed with the Skorohod topology, $d_{sko}$ the Skorohod metric. $\mathcal{D}^{ind}_\infty$ denotes the space $\mathcal{D}_\infty$ endowed with the "inductive topology" of $(\mathcal{D}_m,d_{sko})$, i.e., $V$ is open in $\mathcal{D}^{ind}_\infty$ if and only if for each $m\geq 1$, $V\cap\mathcal{D}_m$ is open in $\mathcal{D}_m$. Let $\mathcal{D}^{sko}_\infty$ be the space $\mathcal{D}_\infty$ with the Skorohod topology, $C_{ind}$, $C_{sko}$ denote the Borel $\sigma$-algebra of $\mathcal{D}^{ind}_\infty$ and $\mathcal{D}^{sko}_\infty$ respectively. Let $C_0=B_1\cap\mathcal{D}_\infty$. Then we have

**Lemma 3.9** $\mathcal{D}_\infty\subset\Lambda_{X^*}$.

**Lemma 3.10** $C_{ind}=C_{sko}\supset C_0$.

We will use $C$ to denote $C_{sko}$ in the sequel. Now let $D([0,T],E^{\otimes N})$ be equipped with the Skorohod topology, $\{P_N\}$ be a sequence of probability measures on $D([0,T],E^{\otimes N})$. $\forall N\geq 1$, let $\mathcal{P}_N$ be defined as

$$\mathcal{P}_N=P_N\circ\varepsilon^{-1}_{x(N)}.$$

$$\varepsilon_{x(N)}:(x_i^{(N)})_{1\leq i\leq N}\in D([0,T],E^{\otimes N})\longrightarrow(t\to\frac{1}{N}\sum_{i=1}^N\delta_{x_i^{(N)}(t)})\in\mathcal{D}_\infty.$$

Then $\{\mathcal{P}_N\}_{N\geq 1}$ is a sequence of probability measures on $(\mathcal{D}_\infty,C_{sko})$. If we introduce the $\sigma$-algebra $B$ on $\Lambda_{X^*}$ to be the family of all subset $B\subset\Lambda_{X^*}$ such that $B$ is a Borel measurable subset of space $(X,d_2)$ and $B\cap\mathcal{D}^{ind}_\infty\in C$. Applying corollary 2.4 to $X_0=\mathcal{D}^{ind}_\infty$, $X=\Lambda_{X^*}$, $d_1=\gamma$, $d_2=|\cdot|_\infty$, $C$, $B_1$ and $B$ we have

**Theorem 3.11** *Assume that there exists a function $h(x)$ with $\lim_{x\to\infty}h(x)=\infty$ and $\beta>0$ such that*

$$\limsup_{N\to\infty}\frac{1}{N}\log E^{P_N}\{\exp[\beta\sum_{i=1}^N\int_0^T h(x_i^{(N)}(t))dt]\}<\infty. \tag{3.1}$$

*Then we have that*

(i). *If (2.2) is satisfied, then for any $C$-measurable, closed subset $A$ of $\mathcal{D}^{ind}_\infty$ we have*

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathcal{P}_N(A)\leq-\inf_{x\in A}L(x). \tag{3.2}$$

(ii). *If (2.4) is satisfied, then for any $C$-measurable, closed subset $A$ of $\mathcal{D}^{ind}_\infty$ we have*

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathcal{P}_N(A)\leq-\inf_{x\in A}\tilde{L}(x). \tag{3.3}$$

*where*

$$L(\mu) = \sup\{F(\mu) - H(F); F \in \mathcal{F}\}, \mu \in \Lambda_{\mathcal{X}^*} \tag{3.4}$$

$$\tilde{L}(\mu) = \sup\{F(\mu) - H(F); F \in \tilde{\mathcal{F}}\}, \mu \in \Lambda_{\mathcal{X}^*} \tag{3.5}$$

$$H(F) = \limsup_{N \to \infty} \frac{1}{N} \log E^{P_N}\{\exp[NF(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^{(N)}(\cdot)})]\}. \tag{3.6}$$

*where $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are the sets of all bounded, $|\cdot|_\infty$-continuous functions which are $\mathcal{B}$ and $\mathcal{B}_1$ measurable respectively.*

This result is the result of the following lemmas.

**Lemma 3.12** *The closure $\overline{\mathcal{D}}_\infty$ of $\mathcal{D}_\infty$ in the space $(\Lambda_{\mathcal{X}^*}, |\cdot|_\infty)$ is a subset of $\mathcal{D}$.*

**Lemma 3.13** *For any $\mu(\cdot) \notin \overline{\mathcal{D}}_\infty$, $L(\mu(\cdot)) = \tilde{L}(\mu(\cdot)) = \infty$.*

**Lemma 3.14** *Let $u$ be any element of $\mathcal{X}^*$. If the following three conditions are satisfied:*

(a). $\forall f \in C_b(E)$, $f \geq 0$ *we have* $\langle u, f\rangle > 0$;

(b). $\langle u, 1\rangle = 1$;

(c). $\lim_{n \to \infty} \langle u, \chi_{E_n}\rangle = 0$;

*where $E_n = \{n+1, \cdots\}$. Then $u \in M_\varphi(E)$.*

**Lemma 3.15** *Assume that the sequence $\{\mathcal{P}_N\}_{N=1,\dots}$ satisfies condition* (3.1). *Then $\{\mu(\cdot) \in \Lambda_{\mathcal{X}^*}; \tilde{L}(\mu(\cdot)) < \infty\} \cap \bar{\mathcal{D}}_\infty \subset \mathcal{D}_\infty$.*

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# MORPHISMES GAUCHES ET MONADES ADJOINTS DANS UNE 2-CATÉGORIE

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**Résumé.** On étend ici les résultats de $[B_1]$, auquel on devra se référer constamment, en introduisant la notion de morphisme gauche, tirée de $[M_1]$, entre monades à gauche, ou m.a.g., et co-m.a.g. dans une 2-catégorie quelconque $\mathbb{C}$, ce qui permet d'obtenir en corollaire et de préciser les résultats de $[M_2]$ ainsi que la notion de monades adjoints ("adjoint triples") de [B - W] et [E - M]. On trouvera tous les détails dans $[B_2]$.

1. À la notion "habituelle" de morphisme entre m.a.g. (on omet $\mu . T\eta = \iota_T$) on ajoute

**Définition.** Un morphisme gauche du co-m.a.g. $T'_* = (T', \mu', \eta')$ sur $\underline{C}'$ au m.a.g. $T = (T, \mu, \eta)$ sur $\underline{C}$ dans $\mathbb{C}$ est un triplet ordonné $(T'_*, (X, \phi), T)$, noté $(X, \phi) : T'_* \longrightarrow T$ ou simplement $(X, \phi)$, où $X : \underline{C}' \longrightarrow \underline{C}$ et $\phi : TXT' \longrightarrow X$ vérifient dans $\mathbb{C}$ :

(1) $\phi . \eta XT' = X\eta'$ (2) $\phi . T\phi T' . T^2 X\mu' = \phi . \mu XT'$ (3) $\phi . TX\eta' = \phi$

où $\nu' = T'\eta' . \mu'$ et dualement, i.e. en passant à $\mathbb{C}_*$ obtenue en inversant la structure verticale de $\mathbb{C}$ pour $(X, \phi) : T' \longrightarrow T_*$, i.e. $\phi : X \longrightarrow TXT'$ etc.

Si $(X, \phi)$ et $(Y, \psi)$ sont de tels morphismes de $T'_*$ à $T$ alors un 2-morphisme $\sigma : (X, \phi) \longrightarrow (Y, \psi)$ est un triplet ordonné $((X, \phi), \sigma, (Y, \psi))$ où $\sigma : X \longrightarrow Y$ vérifie (4) $\sigma\phi = \psi . T\sigma T'$. Dans le cas dual ci-haut, on aura $(4)_*$ $T\sigma T' . \phi = \psi\sigma$ où $\sigma : X \longrightarrow Y$.

Ces divers types de morphismes forment une 2-catégorie $\underline{\mathbb{M}}_g(\mathbb{C})$ avec morphisme identité $(I, \nu) : T \longrightarrow T$, $\nu = \mu . T\eta$ et dualement, laquelle contient pleinement (i.e. comme sous 2-catégorie pleine) celles des m.a.g. $\mathbb{M}_{\cdot 2}(\mathbb{C})$ et des co-mag $(\mathbb{M}_{\cdot 2}(\mathbb{C}_*))_*$ de [B₁]. On les compose en recollant bout à bout les "cylindres" évidents. On aura par exemple:

$$T_*^{-} \underset{(Y',\psi')}{\overset{(X',\phi')}{\rightrightarrows}} {\scriptstyle\downarrow\sigma'}\ T' \underset{(Y,\psi)}{\overset{(X,\phi)}{\rightrightarrows}} {\scriptstyle\downarrow\sigma}\ T = T'' \qquad \underset{(YY',\,Y\psi'.\,\psi Y'T'')}{\overset{(XX',\,X\phi'.\,\phi X'T'')}{\rightrightarrows}} {\scriptstyle\downarrow\sigma * \sigma'} \qquad T$$

2. Soit maintenant dans $\mathbb{C}$ les quasi-adjonctions: $\underline{B} \xrightarrow{U} \underline{C} \xrightarrow{F} \underline{B}$, $\eta : I \longrightarrow UF$, $\varepsilon : FU \longrightarrow I$ où (5) $U\varepsilon.\eta U = \iota_U$, noté $(\eta,\varepsilon) : F \dashv U$ et dualement: $\underline{B}' \xrightarrow{U'} \underline{C}' \xrightarrow{F'} \underline{B}'$, $\eta' : UF' \longrightarrow I$, $\varepsilon' : I \longrightarrow F'U'$ où $(5)_*$ $\eta'U'.U'\varepsilon' = \iota_{U'}$ noté $(\varepsilon',\eta')$: $U' \overset{*}{\dashv} F'$. De la même source on tire:

**Définition.** Un morphisme gauche du second au premier est un triplet ordonné $(U' \overset{*}{\dashv} F'$, $(X, Z)$, $F \dashv U)$, noté $(X, Z) : U' \overset{*}{\dashv} F' \longrightarrow F \dashv U$ ou simplement $(X, Z)$, où $X : \underline{C}' \longrightarrow \underline{C}$ et $Z : \underline{B}' \longrightarrow \underline{B}$ vérifient (6) $UZ = XU'$.

Si de même $(Y, W) : U' \overset{*}{\dashv} F' \longrightarrow F \dashv U$ alors un 2-morphisme gauche du premier au second est un triplet ordonné $((X, Z), (\alpha, \beta), (Y, W))$, noté $(\alpha, \beta) : (X, Z) \longrightarrow (Y, W)$ et parfois simplement $(\alpha, \beta)$, où $\alpha : X \longrightarrow Y$ et $\beta : Z \longrightarrow W$ vérifient (7) $U\beta = \alpha U'$.

Si on compose de façon habituelle ces morphismes gauches avec ceux entre quasi-adjoints et dualement, on obtient une 2-catégorie $\mathbb{A}_g(\mathbb{C})$ contenant pleinement $\mathbb{A}_{\cdot 2}(\mathbb{C})$ et $(\mathbb{A}_{\cdot 2}(\mathbb{C}_*))_*$ de [B₁].

3. Le 2-foncteur $\mathbb{D}_g : \mathbb{C} \longrightarrow \mathbb{M}_g(\mathbb{C})$ qui à $\underline{C}$ associe le (co)-monade trivial (à gauche) $I_{\underline{C}} = (I_{\underline{C}}, \iota, \iota)$ possède un 2-adjoint à droite $\mathbb{L}_g$ ssi c'est vrai pour $\mathbb{D}_{\cdot 2}$ : $\mathbb{C} \longrightarrow (\mathbb{M}_{\cdot 2}(\mathbb{C}_*))_*$ définis comme ci-haut ([B₁]). Comme corollaire du Théorème d'Eilenberg-Moore-Maranda (E-M-M) pour m.a.g. (Théorème 2 (iii) de [B₁]) on obtient:

<u>Théorème "E-M-M-gauche".</u> Soit $\mathbb{D}_g \dashv \mathbb{L}_g$, $T$ un m.a.g. sur $\underline{C}$ engendré par $(\eta, \varepsilon^T)$ : $F^T \dashrightarrow U^T$ avec $\mu = U^T \varepsilon^T F^T$, $\eta : I \longrightarrow U^T F^T = T$ et $T'_*$ un co-m.a.g. sur $\underline{C}'$ engendré par $(\varepsilon', \eta') : U' \overset{*}{\dashv} F'$ avec $\mu' = U' \varepsilon' F'$ et $\eta' : T' = U'F' \longrightarrow I$. Alors aux morphismes $(X, \phi) \xrightarrow{\sigma} (Y, \psi)$ du co-m.a.g. $T'_*$ au m.a.g. $T$ correspond un morphisme $\widetilde{X} : \underline{B}' \longrightarrow \underline{C}^T$ de $\mathbb{C}$ unique tel que (7) $U^T \widetilde{X} = XU'$ et (8) $\phi = X\eta' \,.\, U^T \varepsilon^T \widetilde{X} F'$, de même pour $\widetilde{Y}$, et un 2-morphisme $\widetilde{\sigma} : \widetilde{X} \longrightarrow \widetilde{Y}$ de $\mathbb{C}$ unique tel que (9) $U^T \widetilde{\sigma} = \sigma U'$. Et dualement.

**Démonstration.** Il suffit d'appliquer E-M-M à la composition des morphismes ci-haut avec $(U', U'\varepsilon') : I_{\underline{B}'} \longrightarrow T'_*$. Q.E.D.

Dans le cas concret de CAT on obtient (10) $\widetilde{X}(C') = (XU'(B'), (\phi U' \,.\, TXU'\varepsilon')_{B'})$

4. Grâce à ce théorème on obtient un 2-foncteur $\mathbb{R}_g : \mathbb{M}_g(C) \longrightarrow \mathbb{A}_g(C)$ qui prolonge $\mathbb{R}_{\cdot 2} : \mathbb{M}_{\cdot 2}(\mathbb{C}) \longrightarrow \mathbb{A}_{\cdot 2}(\mathbb{C})$ et $\mathbb{R}^0_{\cdot 2} : \mathbb{M}_{\cdot 2}(\mathbb{C}_*))_* \longrightarrow \mathbb{A}_{\cdot 2}(\mathbb{C}_*))_*$ de [$B_1$] et à $(X, \phi)$ ci-haut associe $(X, \widetilde{X})$, avec un 2-adjoint à gauche $\mathbb{S}_g$ : ce dernier prolonge lui aussi $\mathbb{S}_{\cdot 2}$ et $\mathbb{S}^0_{\cdot 2}$ et fait correspondre au morphisme gauche $(X, Z)$ le morphisme $(X, X\eta' \,.\, U\varepsilon ZF')$ : $T'_* \longrightarrow T$, ces deux derniers engendrés de façon habituelle et dualement. De plus, $\mathbb{S}_g \,.\, \mathbb{R}_g$ est l'identité sur $\mathbb{M}g$.

L'unité de cette 2-adjonction est alors donnée par $(I, K^T)$ où $K^T : \underline{B} \longrightarrow \underline{C}^T$ provient du théorème "classique" d'Eilenberg-Moore pour m.a.g. (Théorème 25 de [$B_2$]). On remarque encore une fois que $\mathbb{S}_g(I, I) = (I, \nu) : T \longrightarrow T$ qui fournit la co-unité de cette 2-adjonction si on choisit de définir $\mathbb{S}_g$ comme ci-haut (c'est l'approche retenue dans [$B_1$]). Dans ce cas, l'existence de $\mathbb{R}_g$ entraîne $\mathbb{D}_g \dashv \mathbb{L}_g$ car il suffit de composer $\mathbb{S}_{\cdot 2}$ avec le 2-foncteur trivial de $\mathbb{C}$ dans $\mathbb{A}_{\cdot 2}(\mathbb{C})$ ayant comme 2-adjoint à droite celui qui de $(\eta, \varepsilon) : F \dashrightarrow U$ ne conserve que $\underline{C}$ et dualement. Dans une 2-catégorie, on a donc équivalence entre le Théorème E-M-M, $\mathbb{D}_g \dashv \mathbb{L}_g$ et $\mathbb{S}_g \dashv \mathbb{R}_g$.

5. On obtient les mêmes équivalences pour les 2-catégories $\mathbb{M}(\mathbb{C})$ des (co)-monades et $\mathbb{A}(\mathbb{C})$ des adjonctions, incluant les morphismes gauches, car elles sont contenues pleinement dans $\mathbb{M}_g(\mathbb{C})$ et $\mathbb{A}_g(\mathbb{C})$ respectivement, d'où $\mathbb{D}_g \dashv \mathbb{L}_g$ entraîne $\mathbb{D} \dashv \mathbb{L}$ . Le Théorème " E-M-M gauche" constitue alors dans le cas de CAT, avec $\widetilde{X}$ donné par (10), le théorème 6 de $[M_1]$.

Pour les (co)-monades (resp. à gauche) de degrés supérieurs de $[\mathbb{M}_2]$ on procède "mutatis mutandis" comme dans $[B_1]$ pour obtenir $\mathbb{M}_s(\mathbb{C})$ , (resp. $\mathbb{M}_{sg}(\mathbb{C})$) . Ainsi, si $\mathbb{T} = (T, \mu, \eta)$ est un monade (resp. à gauche) de degré $n \geq 2$ sur $\mathbb{C}$ dans $\mathbb{C}$ avec $\mu : T^n \longrightarrow T$ , $\eta : I \longrightarrow T^{n-1}$ et $\mathbb{T}' = (T', \mu', \eta')$ un co-monade (resp. à gauche) de degré $n' \leq -2$ avec $\mu' : T' \longrightarrow T'^{-n'}$ , $\eta' : T'^{-n'-1} \longrightarrow I$ , alors $(X, \phi) : \mathbb{T}'_* \longrightarrow \mathbb{T}$ est un morphisme de $\mathbb{M}_s(\mathbb{C})$ (resp. $\mathbb{M}_{sg}(\mathbb{C})$) ssi $(X, \phi) : \overline{\mathbb{T}}'_* \longrightarrow \overline{\mathbb{T}}$ est dans $\mathbb{M}_g(\mathbb{C})$ où $\overline{\mathbb{T}} = (T^{n-1}, \mu T^{n-2}, \eta)$ et $\overline{\mathbb{T}}' = (T'^{-n'-1}, \mu' T'^{-n'-2}, \eta')$ , de même pour les 2-morphismes et dualement.

Le morphisme identité sur $\mathbb{T}$ , de degré $n$ , dans $\mathbb{M}_s(\mathbb{C})$ (resp. $\mathbb{M}_{sg}(\mathbb{C})$) sera encore $(I, \nu)$ avec $\nu = \mu T^{n-2} . T^{n-1}\eta$ et dualement, avec les 2-équivalences $\mathbb{M}_g(\mathbb{C}) \cong \mathbb{M}_{sg}(\mathbb{C})$ et les 2-inclusions pleines évidentes, d'où ici aussi $\mathbb{D}_g \dashv \mathbb{L}_g$ entraîne $\mathbb{D}_{sg} \dashv \mathbb{L}_{sg}$ et $\mathbb{D}_s \dashv \mathbb{L}_s$ . On applique le théorème "E-M-M gauche" pour (co)-m.a.g. à $\overline{\mathbb{T}}$ et $\overline{\mathbb{T}}'$ avec $\overline{F}' = F' T'^{-n'-2}$ comme dans $[B_1]$ pour obtenir les "versions n-aires". La conjonction du Théorème E-M-M et de son corollaire "gauche" ci-haut pour $\mathbb{M}_s(\mathrm{CAT})$ constitue, avec les versions duales, le Théorème 2 de $[M_2]$ . On fait de même pour $\mathbb{A}_s(\mathbb{C})$ et $\mathbb{A}_{sg}(\mathbb{C})$ . La 2-adjonction $\mathbb{S}_s \dashv \mathbb{L}_s$ est mentionée pour CAT dans $[\mathbb{M}_2]$ bien qu'il n'est aucunement fait mention du fait que $\mathbb{S}_s(I, I) = (I, \mu T^{n-2} . T^{n-1}\eta)$ .

6. Quant au cas minimal, ou "Kleisli", il suffit de passer de $\mathbb{C}$ à $\mathbb{C}^*$ obtenue en dualisant la structure horizontale de $\mathbb{C}$ et de considérer $\mathbb{M}_d(\mathbb{C}) = (\mathbb{M}_g(\mathbb{C}^*))^*$ laquelle contient pleinement celle des monades à droite (on omet $\mu . \eta T = \iota_T$ cette fois) $(\mathbb{M}_{\cdot 2}(\mathbb{C}^*))^*$ et des co-monades à droite $(\mathbb{M}_{\cdot 2}(\mathbb{C}^*_*))^*_*$ . On a alors $\mathbb{L}_d \dashv \mathbb{D}_d : \mathbb{C} \longrightarrow \mathbb{M}_d(\mathbb{C})$ et $\mathbb{R}_d \dashv \mathbb{S}_d : \mathbb{A}_d(\mathbb{C}) = (\mathbb{A}_g(\mathbb{C}^*))^* \longrightarrow \mathbb{M}_d(\mathbb{C})$. On obtient ainsi les résultats correspondants pour (co)-monades de degrés supérieurs et dans le cas particulier de CAT, le Théorème 1 de $[\mathbb{M}_2]$.

7. Soit $T'_* = (T', \mu', \eta')$ un co-monade et $T = (T, \mu, \eta)$ un monade sur $\underline{C}$ dans CAT. Selon [E-M] on a une adjonction monadique ("adjoint triples"), notée $\alpha : T \dashv T'_*$, ssi $\alpha$ : $\underline{C}(T(-), -) \cong \underline{C}(-, T'(-))$ vérifie pour tout $C$ et $C'$ de $\underline{C}$:

$(\eta \dashv \eta') : \underline{C}(C, \eta'_C) . \alpha_{C,C'} = \underline{C}(\eta_{C,C'})$

$(\mu \dashv \mu') : \alpha_{C,T'(C')} . \alpha_{T'(C),C'} . \underline{C}(\mu_C, C') = \underline{C}(C, \mu'_{C'}) . \alpha_{C,C'}$

Si $\bar{\eta} : I \longrightarrow T'T$ et $\bar{\varepsilon} : TT' \longrightarrow I$ sont respectivement l'unité et la counité de cette adjonction $\alpha : T \dashv T'$, on notera aussi $(\bar{\eta}, \bar{\varepsilon}) : T'_* \dashv T$ l'adjonction monadique. Ainsi, une double adjonction $F \dashv U \dashv R$ dans $\mathbb{C}$ engendre une telle adjonction monadique.

<u>Proposition</u>: $(\bar{\eta}, \bar{\varepsilon}) : T \dashv T'_*$ ssi $(I_{\underline{C}}, \bar{\varepsilon}) : T'_* \longrightarrow T \in \mathbb{M}$ (CAT) avec inverse $(I_{\underline{C}}, \bar{\eta})$.

<u>Démonstration</u>. $\bar{\varepsilon}$ vérifie (1) $\bar{\varepsilon} . \eta T' = \eta'$ ssi $\eta \dashv \eta'$ et (2) $\bar{\varepsilon} . \mu T' = \bar{\varepsilon} . T \bar{\varepsilon} T' . T^2 \mu'$ et dualement pour $(I_{\underline{C}}, \bar{\eta})$. Il suffit alors de composer les deux. Q.E.D.

On peut donc définir une adjonction monadique dans une 2-catégorie quelconque $\mathbb{C}$ comme un isomorphisme de $\mathbb{M}(\mathbb{C})$ de la forme $(I_{\underline{C}}, \phi) : T'_* \longrightarrow T$.

D'où $(\bar{\eta}, \bar{\varepsilon}) : T \dashv T'_*$ sera envoyé par $\mathbb{R} : \mathbb{M}(\mathbb{C}) \longrightarrow \mathbb{A}(\mathbb{C})$ dans un isomorphisme $\tilde{I}_{\underline{C}}$ : $\underline{C}^{T'} \longrightarrow \underline{C}^T$ "au-dessus de $\underline{C}$", à condition que $\mathbb{D} : \mathbb{C} \longrightarrow \mathbb{M}(\mathbb{C})$ possède un 2-adjoint à gauche $\mathbb{L}$. Dans le cas particulier de CAT, $\tilde{I}_{\underline{C}}$ est défini pour toute $T'$-coalgèbre $(A, a)$ par

$\tilde{I}_{\underline{C}}(A, a) = (A, T(A) \xrightarrow{T(a)} TT'(A) \xrightarrow{\bar{\varepsilon}_A} A)$. La réciproque est toujours vraie grâce à $\mathbb{S}$. De plus, si $\mathbf{T} = (T, \mu, \eta)$ est un monade sur $\underline{C}$ dans $\mathbb{C}$ quelconque avec une adjonction $(\bar{\eta}, \bar{\varepsilon}) : T \dashv T'$, on obtient alors un co-monade $\mathbf{T}' = (T', \mu', \eta')$ où $\eta'$ est donné par (1) ci-haut et $\mu' = T'^2\, \bar{\varepsilon}\, .\, T'^2 \mu\, T'\, .\, T'\, \bar{\eta}\, TT'\, .\, \bar{\eta}\, T'$, équivalent à (2) dans la Proposition précédente, d'où un isomorphisme $(I_{\underline{C}}, \bar{\varepsilon}) : \mathbf{T}'_{\bullet} \longrightarrow \mathbf{T}$ de $\mathbb{M}(\mathbb{C})$, i.e. $(\bar{\eta}, \bar{\varepsilon}) : \mathbf{T} \dashv \mathbf{T}'_{\bullet}$. On obtient ainsi pour CAT la Proposition 3.3 de [E - M], équivalente au Théorème 5, page 137, de [B-W]. À noter que ces derniers ont omis les conditions $\eta \dashv \eta'$ et $\mu \dashv \mu'$ dans leur définition mais comme ils ne s'intéressent qu'au cas d'une double adjonction cela ne fait aucune différence pour la suite.

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# A Remark on Harmonic Maps in three dimensional Minkowski Space

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**Abstract.** The purpose of this paper is to extend Sideris's result of existence of harmonic maps from four dimensional Minkowski to the three dimensional Minkowski space. Moreover, the maps are still a small perturbation of geodesics.

## 1. Main Problem

Shatah [SH] has constructed global weak solutions of the nonlinear sigma model in $H^{1,2}$ for arbitrary dimension. Moreover, for the case of maps from Minkowski space $R^{3+1}$ to sphere $S^3$ he has found an example of smooth symmetic initial data for which the corresponding solution develops singularities of the second order derivatives in finite time, showing that the large initial data may contribute to the formation of singularities. To avoid this situation, Sideris [SI] proved the existence of a global smooth solution from $R^{3+1}$ into an $n$ dimensional complete Riemannian manifold $N$, as a perturbation of a special geodesic class of harmonic maps. Geometrically, the solution lies in a small tubular region of a given geodesic on $N$ instead of a small neighborhood of a point on $N$ (therefore, the initial data is partly small).

Formally, a harmonic map from the Minkowski space $R^{2+1}$ into an $n$ dimensional complete Riemannian manifold $(N, h)$ is described as a critical point $\xi : R^{2+1} \mapsto N$ of the energy integral

$$E(\xi) = \int_{R^3} g_{\alpha\beta} h_{ij}(\xi) \frac{\partial \xi^i}{\partial x_\alpha} \frac{\partial \xi^j}{\partial x_\beta} dx \tag{1.1}$$

where $(g_{\alpha\beta}) = \text{diag}(-1,1,1)$ is the Lorentzian metric on $R^{2+1}$. The corresponding Euler-Lagrange equation for $\xi$ takes the following form

$$-\Delta_g \xi^i + \gamma^i_{jk}(\xi)(\partial\xi^j, \partial\xi^k)_g = 0 \tag{1.2}$$

with $(\gamma^i_{jk}(\xi))$ are Christoffel symbols of $h$ on $N$ and

$$\begin{gathered}(\partial\xi^j, \partial\xi^k)_g = g^{\alpha\beta}\partial_\alpha\xi^j\partial_\beta\xi^k =: Q(\xi^j,\xi^k)\\ \Delta_g = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_\alpha}(\sqrt{|g|}g^{\alpha\beta}\frac{\partial}{\partial x_\beta}) = \partial_1^2 + \partial_2^2 - \partial_t^2 = -\Box\end{gathered} \tag{1.3}$$

Here we used the standard summation convention of repeated upper and lower values. Following [SI], we write the equation (1.2) in a convenient coordinate chart $(u^1,..,u^n)$ (Fermi's chart) defined on a tubular neighborhood of a given geodesic $\Upsilon(u^1) \in N$, with $u^1 \in R^1$ representing arc length on $\Upsilon$. In this coordinate system, geodesics are straight lines, thus the Christoffel symbols vanish along $\Upsilon$ ,so,

$$\gamma^i_{jk}(u^1,0,..0) = 0, \qquad \text{for all } 1 \le i,j,k \le 2 \tag{1.4}$$

We are searching for a global smooth solution $u = (u^1,..,u^n)$ to the Cauchy problem

$$\begin{cases}\Box u^i + \gamma^i_{jk}(u)Q(u^j,u^k) = 0, & (t,x) \in R^{2+1}\\ u^i = f^i(x), \quad \partial_t u^i = g^i(x), & t=0,\ x \in R^2\end{cases} \qquad i = 1,..,n \tag{1.5}$$

in a small neighborhood of $\Upsilon$, ie. $\tilde{u} = (u^2,..,u^n)$ is small.

In the remainder of this section, we introduce notations and functional spaces related to solutions of the problem (1.5). Let $x = (x_0,x_1,x_2) = (x_\alpha)$ represent points in $R^{2+1}$ with $x_0 = t$ acting as time variable and $r = \sqrt{x_1^2 + x_2^2}$. We first consider the following vector fields on $R^{2+1}$ (see [KL1,2])

$$\begin{gathered}\partial_\alpha = \frac{d}{dx_\alpha}, \quad \alpha = 0,1,2 \qquad \Omega_{12} = x_1\partial_2 - x_2\partial_1,\\ L_i = t\partial_i + x_i\partial_t, \quad i = 1,2 \qquad L_0 = t\partial_t + x_1\partial_1 + x_2\partial_2\end{gathered} \tag{1.6}$$

These are Killing or conformal Killing vector fields and satisfy some commutation relations which can be outlined by

$$[\Gamma,\ \Gamma] = \Gamma, \quad [\partial,\ \Gamma] = \partial \tag{1.7}$$

where $\Gamma = \{\partial, \Omega_{12}, L_i, L_0\}$ and the $\Gamma$ ( or $\partial$ ) on the right hand side stands for a linear combination of $\Gamma'$s (or $\partial'$s). Moreover, by the conformal invariance of the wave equation, we know

$$[\Gamma, \ \Box] = c\Box \tag{1.8}$$

for some $c$ depending on $\Gamma$ ( $c = -2$ if $\Gamma = L_0$ and $c = 0$ otherwise). We denote vector function $u = (u^1, u^2, ..u^n)$ by $u = (u^1, \tilde{u})$, i.e., $\tilde{u} = (u^2, ..u^n)$, and for any norm $|\cdot|$, we set

$$|\tilde{u}| = \sum_2^n |u^i|, \qquad |u| = |u^1| + |\tilde{u}|.$$

It is a standard procedure to use $\Gamma$ to define the generalized Sobolev norm (see [KL1,2])

$$\begin{aligned} &\|u(t)\|_m^2 = \|u(t)\|_{\Gamma,m,2}^2 = \sum_{|\alpha|\le m} \|\Gamma^\alpha u(t,)\|_{L^2(R^2)}^2 \\ &|u(t)| = |u(t)|_{\Gamma,m,\infty} = \sum_{|\alpha|\le m} |\Gamma^\alpha u(t,)|_{L^\infty(R^2)} \\ &\|u(t)\| = \|u(t)\|_{L^2(R^2)} \end{aligned} \tag{1.9}$$

where

$$\Gamma^\alpha = \Gamma^{\alpha_1}\Gamma^{\alpha_2}..\Gamma^{\alpha_7}, \quad \alpha = (\alpha_1, .., \alpha_7) \in (N^+)^7, \quad |\alpha| = \sum_1^7 \alpha_i$$

We use $c > 0$ to represent some generic constants which may depend on some other constants and functions.

## 2. Main Result and A Priori Estimate

**Theorem.** *Let $m \ge 4$ be an integer, $f^i, g^i \in C_0^\infty(R^2)(i = 1, ..n)$ and*

$$A = 4C_0(\|f^1\|_{m+1} + \|g^1\|_m) \tag{2.1}$$

*with some absolute constant $C_0 > 1$. Then, there exist sufficiently small constants $\epsilon, \delta >$ 0, depending only on $m, N$ and $A$, such that if*

$$\|\tilde{f}\|_{m+1} + \|\tilde{g}\|_m \le \delta \tag{2.2}$$

*the Cauchy problem (1.5) has an unique global smooth solution* $u(t,x) = (u^1(t,x), .. u^n(t,x))$ *satisfying*

$$\|\partial u^1(t)\|_m \le A, \quad \|\partial \tilde{u}(t)\|_m \le \epsilon, \quad \|u(t,x)\|_{m+1} \le A(1+t)^\alpha, \tag{2.3}$$

*where* $0 < \alpha < 1/4$ *is some small number.*

Since (1.5) is semilinear wave equation, we can prove existence of its local solution (in time) by using the standard procedure(see, for example, [SI]). The proof of the theorem can be carried out in a standard procedure, for examplesee [KL1], if we can derive certain a priori estimate results. Before doing this, we define the folowing set of functions: Let integer $m \ge 4$. For $T > 0, \epsilon > 0, 0 < \alpha < 1/4$ and $A$ defined in (2.1), let

$$\begin{aligned} X_{A,\epsilon,T} = \{ u(t,x) = (u^1, \tilde{u}) | \ & \|\partial u^1(t)\|_m \le A, \\ & \|\partial \tilde{u}(t)\|_m \le \epsilon, \|u(t)\|_{m+1} \le A(1+t)^\alpha, 0 \le t \le T \} \end{aligned}$$

**Lemma 2.1.** *(A Priori Estimate). There exist sufficiently small positive constants* $\epsilon, \delta$, *depending only on* $m$ *and* $A$ *such that if*

$$\|\tilde{u}(0)\|_{m+1} = \|\tilde{f}\|_{m+1} + \|\tilde{g}\|_m \le \delta$$

*and* $u \in X_{A,\epsilon,T}$ *is a solution of (1.5) over* $[0,T)$ *for some* $T > 0$, *then*

$$u \in X_{A/2,\epsilon/2,T}.$$

The proof of lemma 2.1 is based on the following lemmas 2.2-2.4.

**Lemma 2.2.** *Let* $n \ge 2$ *and* $u \in C^\infty(R \times R^n)$ *satisfying* $u = 0$ *if* $|t| - r \le -\beta$ *for some* $\beta > 0$. *Then*

$$\begin{gathered} \left\|\frac{u(t)}{1+||t|-r|}\right\| \le c\|\partial_s u(t)\|, \\ |u(t,x)| \le c(1+||t|-r|)^{-\frac{1}{2}}(1+|t|+r)^{-\frac{n-1}{2}}\|u(t)\|_{[\frac{n}{2}]+1}, \quad t > 0 \\ |u(t,x)| \le c(1+||t|-r|)^{\frac{1}{2}}(1+|t|+r)^{-\frac{n-1}{2}}\|\partial u(t)\|_{[(n+1)/2]} \end{gathered}$$

By the property of finite speed of propagation of solutions of wave equation, we know that this lemma can apply to the solution of problem (1.5). We notice that the quadratic function $Q(u,v)$ satisfies the 'Null Condition', which was proposd and used by Klainerman (see [KL2]) when $n=3$ and has the following major property:

**Lemma 2.3.** *The quadratic form $Q(u,v)$ satisfies the 'Null Condition', ie., for any integer $k \geq 0$*

$$|\Gamma^k Q(u,v)| \leq \frac{c_k}{(r+|t|)} \sum_{i+j\leq k} (|\partial\Gamma^i u||\Gamma^{j+1}v| + |\partial\Gamma^i v||\Gamma^{j+1}u|)$$

*where $|\Gamma u|$ is the absolute value of vector valued function $\Gamma u$.*

**Lemma 2.4.** *Let $m>0$ be an integer and $u=(u^1,..u^n)=(u^1,\tilde{u})$ be a smooth map from $R^{2+1}$ to $R^n$ such that*

$$|\partial u(t)|_{[\frac{m}{2}]} \leq M, \quad |\tilde{u}(t)|_{[\frac{m}{2}]} \leq 1$$

*for all $t \geq 0$. Let also $\gamma^i_{jk}$ be smooth functions on $R^n$ satisfying (1.4). Then, for all $(t,x) \in R^{2+1}$,*

$$|\Gamma^m \gamma^i_{jk}(u)(t,x)| \leq c\Big(\sum_{|\alpha|\leq m} |\Gamma^\alpha \tilde{u}(t,x)| + \sum_{|\alpha|\leq m} |\Gamma^\alpha u^1(t,x)| \sum_{|\alpha|\leq[\frac{m}{2}]} |\Gamma^\alpha \tilde{u}(t,x)|\Big)$$

*where $c>0$ is a constant depending only on $m, M, A$ and $N$.*

The proof of this result is similar to that of lemma 2 of [SI].

The derivation of (2.3) will employ the invariance property (1.8) and conserved quantities of the wave equation In fact, time translation of $R^2$ is generated by $\partial_t$ and the conservation quantity associated with it is the conservation of energy:

$$\|\partial u(t)\| = \|\partial u(0)\|$$

for the solution of homogenous wave equation. Moreover, the motions generated by the operator

$$K_1 = (1 + t^2 + |x|^2)\partial_t + 2tx^i\partial_i + t$$

is inversions together with appropriate scalings and will leave the quadratic quantity

$$K(u)(t) = \int_{R^3} (\frac{1}{2}(1 + t^2 + |x|^2)|\partial u|^2 + 2tx^i\partial_t u\partial_i u + tu\partial_t u - \frac{1}{2}u^2)$$

invariant. This can be seen from the identity:

$$\frac{d}{dt}K(u)(t) = \int_{R^3} (K_1 u)\Box u = 0$$

for solutions of the homogenous wave equation. It can be easily shown that $K(u)(t)$ is equivalent to $||u(t)||_1^2$:

$$C_0^{-1}||u(t)||_{m+1}^2 \le \sum_{|\alpha|\le m} K(\Gamma^\alpha u)(t) \le C_0||u(t)||_{m+1}^2 \tag{2.4}$$

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# Fourier coefficients of modular forms over arithmetic progressions. I

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**Abstract** Let $f$ be a modular form for $\Gamma_0(N)$ of integral weight $r$. For any integers $\ell$ and $k$ with $k \geq 1$, let $A_f(x;\ell,k) = \sum_{n \leq x;\, n \equiv \ell \bmod k} a_n$ where $a_n$ is the $n^{\text{th}}$ Fourier coefficient of $f$. If $f$ is not a cusp form, then $A_f(x;\ell,k)$ has an asymptotic expansion as $x \to \infty$ which holds uniformly in $\ell$ and $k$ in the range $k \leq x^{2r/(2r+1)}$ if $(k, \ell N) = 1$. If $f$ is a cusp form, then $A_f(x;\ell,k) = O(x^{r/2-1/6} d(k) \log 4kx)$ holds uniformly in $\ell$ and $k$ as $x \to \infty$ provided $k \leq x^{\frac{2}{3}}$ and $(k, \ell N) = 1$. where the implied constant in the O-symbol is independent of $\ell, k$ and $x$, and $d(k)$ is the ordinary divisor function.

**1. Introduction.** Let $V$ be the space of holomorphic functions $f$ on the upper half-plane $\mathsf{H}$ which are holomorphic at $i\infty$ in the sense that $f$ has a Fourier expansion of the form ($z \in \mathsf{H}$)

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z} \quad \text{with} \quad a_n = O(n^c) \quad \text{as} \quad n \to \infty \tag{1}$$

where $c \geq 0$ is a constant depending only on $f$; we call $a_n$ the $n^{\text{th}}$ Fourier coefficient of $f$. For each $x \geq 1$ and $f \in V$, define

$$A_f(x;\ell,k) = \sum_{\substack{n \leq x \\ n \equiv \ell \bmod k}} a_n \tag{2}$$

where $\ell$ and $k$ are integers with $k \geq 1$. A simple calculation shows that (1) implies that $A_f(x;\ell,k) \ll k^{-1}x^{c+1} + x^c$ holds uniformly in $\ell$ and $k$ as $x \to \infty$, where $\ll$ is the Vinogradov notation, i.e., $f \ll g$ iff $f = O(g)$. (We remark that throughout this paper, the constants inherent in all $O$-terms are independent of the parameters $\ell, k, q, y$ and $x$; these

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remarks apply equally well to the Vinogradov notation $\ll$.) In this paper, we will impose additional restrictions on $f$ to ensure that more precise information can be obtained about $A_f(x;\ell,k)$ as $x \to \infty$. Indeed, we will show that if $f$ is a modular form for $\Gamma_0(N)$ (the congruence subgroup of $SL_2(\mathbf{Z})$ of upper triangular matrices mod $N$) of integral weight $r \geq 3$, then there exists a simple arithmetic function $H_f(\ell,k)$, depending only on $\ell$, $k$ and $f$, such that

$$A_f(x;\ell,k) \;=\; H_f(\ell,k)x^r + O((\ell,k)^{1/(2r+1)}x^{r-1+1/(2r+1)}d(k)) \tag{3}$$

holds uniformly in $\ell$ and $k$ as $x \to \infty$ provided that $k(\ell,k)^{1/(2r+1)} \;\leq\; x^{2r/(2r+1)}$ and $(k,N)=1$, where $(\ell,k)$ denotes the greatest common divisor of $\ell$ and $k$, and $d(k)$ is the ordinary divisor function.

On the other hand, if $f$ is a cusp form, then $H_f(\ell,k)$ is identically zero so that (3) provides only an upper bound for $A_f(x;\ell,k)$; moreover, Rankin [13] in 1940 proved that $A_f(x) \ll x^{r/2-1/10}$ for cusp forms, a much stronger result than (3) provides; here $A_f(x) = A_f(x;1,1)$. Recently, Rankin remarked that (cf. [14], p. 133) this result can be improved to $A_f(x) \ll x^{r/2-1/6+\epsilon}$ by an argument of Walfisz [18] together with the Ramanujan-Petersson conjecture which was established in 1974 by Deligne (cf. [3], [4]). In this paper, we will show that Theorem 1 of Smith [17], together with Deligne's result, implies that

$$A_f(x;\ell,k) \;\ll\; (\ell,k)^{1/3}d(k)x^{r/2-1/6}\log 4kx \tag{4}$$

holds uniformly in $\ell$ and $k$ as $x \to \infty$ provided that $k(\ell,k)^{1/3} \leq x^{2/3}$ and $(k,N)=1$.

**2. Some Preliminaries.** Since $G = GL_2^+(\mathbf{R})$ acts on $\mathbf{H}$ by bilinear action, i.e. $\sigma z = \frac{az+b}{cz+d}$ for $\sigma = \binom{ab}{cd} \in G$ and $z = x+iy \in \mathbf{H}$, then for each positive integer $r$, we define an $r$-action of $G$ on $V$ by $(f\|[\sigma]_r)(z) \;=\; (\det\sigma)^{r/2}j(\sigma,z)^{-r}f(\sigma z)$ where $j(\sigma,z) = cz+d$. A simple calculation shows that $(f\|[\sigma]_r)\|[\tau]_r \;=\; f\|[\sigma\tau]_r$ for all $\sigma,\tau \in G$. For each $\lambda > 0$, define a Mellin transform on $V$ by

$$M_\lambda f(s) \;=\; \int_0^\infty f^\times(i\lambda^{-\frac{1}{2}}y)y^{s-1}\,dy \tag{5}$$

where $f^\times(z) = f(z) - f(i\infty)$ (note that $f(i\infty)$ is just the $O^{\text{th}}$ Fourier coefficient of $f$). Clearly, (1) implies that $f^\times(z) = O(y^{-c-1})$ uniformly in $x$ as $y \to 0^+$ so that the integral in (5) converges absolutely and locally uniformly in the half-plane $Re(s) > c+1$, i.e., $M_\lambda f$ is holomorphic in this half-plane. A straightforward calculation shows that for $Re(s) > c+1$ (cf. Ogg [11], 1-5)

$$M_\lambda f(s) \;=\; \left(\frac{2\pi}{\sqrt{\lambda}}\right)^{-s}\Gamma(s)L_f(s) \tag{6}$$

where $L_f$ is the Dirichlet series associated with $f$ defined by

$$L_f(s) = \sum_{n\geq 1} a_n n^{-s}. \tag{7}$$

We observe that by (1), the series in (7) converges absolutely and locally uniformly in the half-plane $Re(s) > c+1$ and so represents a holomorphic function there. By the following well-known result of Hecke [7], we can provide a meromorphic continuation of $L_f$ to the entire complex plane for a class of $f \in V$ (cf. Ogg [11], Theorem 1).

**Lemma.** *For any $f, g \in V$, the following are equivalent:*
*(A) $f = i^r g\|[H_\lambda]_r$ with $H_\lambda = \begin{pmatrix} 0 & -1 \\ \lambda & 0 \end{pmatrix} \in G$;*
*(B) $M_\lambda f$ can be meromorphically continued to the entire complex plane such that $M_\lambda f(s) + \frac{f(i\infty)}{s} + \frac{g(i\infty)}{r-s}$ is an entire function which is bounded in every vertical strip and satisfies the functional equation $M_\lambda f(s) = M_\lambda g(r-s)$.*

Consequently, for any $f \in V$ satisfying condition (A) for some $g \in V$, Theorem 4.1 of Chandrasekharan and Narasimhan [2] gives a result like (3) with $\ell = k = 1$. In order to attempt to establish (3) for arbitrary $\ell$ and $k$ using similar ideas, we first replace the congruence condition in (2) by a character sum. For any integers $p$ and $q$ with $q \geq 1$, we say that

$$\omega = e_q(p) \tag{8}$$

is an *additive character* mod $q$, where $\omega(n) = e_q(p)(n) = e^{2\pi i np/q}$; in case $(p,q) = 1$, we say that $\omega$ is *primitive* with *conductor* $q$. By a well-known result in number theory (cf. Hardy and Wright [6], p. 234, equation (16.2.3)), it follows that (2) can be rewritten as

$$A_f(x;\ell,k) = \frac{1}{k}\sum_{q|k}\ \sideset{}{^*}\sum_{\omega \bmod q} \omega(-\ell)A_f(x,\omega) \tag{9}$$

where the inner sum in (9) is taken over all primitive additive characters mod $q$, and $A_f(x,\omega) = \sum_{n\leq x} a_n\omega(n)$.

For any $f \in V$ with a Fourier expansion given by (1), and for any additive character $\omega$ mod $q$, we define

$$f_\omega(z) = \sum_{n\geq 0} a_n\omega(n)e^{2\pi inz}, \tag{10}$$

which clearly is equivalent to

$$f_\omega(z) = f(z+p/q) \tag{11}$$

if $\omega$ is given by (8). Thus, (6) and (10) imply that $M_\lambda f_\omega(s) = \left(\frac{2\pi}{\sqrt{\lambda}}\right)^{-s}\Gamma(s)L_f(s,\omega)$

where

$$L_f(s,\omega) = \sum_{n\geq 1} a_n\omega(n)n^{-s}. \tag{12}$$

If it were possible to prove that for each $\omega$, $f_\omega$ satisfies (A) for some "nice" $g^\omega \in V$, then a result like (3) would in principle follow by Theorem 1 of Smith [17]. Since we don't know how to construct such a $g^\omega$ in general, we shall restrict our attention in this paper to the subspace $M(N,r,\varepsilon)$ of $V$ introduced by Hecke consisting of modular forms $f$ for the congruence group $\Gamma_0(N)$ of weight $r \geq 1$ with a Dirichlet character $\varepsilon \bmod N$. For the convenience of the reader, we recall that $f \in M(N,r,\varepsilon)$ means that in addition to $f \in V$, $f|[\sigma]_r = \varepsilon(d)f$ for all $\sigma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(N)$ and $f$ is holomorphic at each cusp of $\Gamma_0(N)$. Furthermore, let $S(N,r,\varepsilon)$ denote the subspace of cusp forms of $M(N,r,\varepsilon)$, i.e., $f \in S(N,r,\varepsilon)$ if and only if $f$ vanishes at all the cusps of $\Gamma_0(N)$. We shall also assume that $\varepsilon(-1) = (-1)^r$ since otherwise, $M(N,r,\varepsilon) = \{0\}$.

For any $N \geq 1$, let $X_N$ denote the collection of primitive additive characters whose conductors are relatively prime to $N$. For any integers $p$ and $q$ with $q \geq 1$, let $\bar{p}$ be an integer satisfying the congruence $p\bar{p} \equiv 1 \bmod q$. For each $\omega = e_q(p) \in X_N$, define $\omega' = e_q(-\overline{pN})$.

**3. The Functional Equation for $L_f(s,\omega)$.** For any $N \geq 1$, the map $\sigma \mapsto H_N\sigma H_N^{-1}$ defines an automorphism of $\Gamma_0(N)$ with $H_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Consequently, the map $f \mapsto f|[H_N]_r$ defines a linear isomorphism of $M(N,r,\varepsilon)$ onto $M(N,r,\bar{\varepsilon})$. In particular, for each $f \in M(N,r,\varepsilon)$, there exists a unique $\tilde{f} \in M(N,r,\bar{\varepsilon})$ such that

$$f = i^r\tilde{f}|[H_N]_r. \tag{13}$$

**Theorem 1.** *For each $f \in M(N,r,\varepsilon)$, then*

$$f_\omega = \varepsilon(q)i^r\tilde{f}_{\omega'}|[H_{q^2N}]_r \tag{14}$$

*for all $\omega \in X_N$, where $q$ is the conductor of $\omega$.*

**Proof.** For each primitive $\omega = e_q(p) \in X_N$, we associate a pair of non-singular integral matrices defined by

$$\alpha = \alpha_\omega = \begin{pmatrix} q & p \\ 0 & q \end{pmatrix} \tag{15}$$

and

$$\rho = \rho_\omega = \begin{pmatrix} q & -p \\ cN & d \end{pmatrix} \tag{16}$$

where $c$ and $d$ are integers defined by

$$cpN + dq = 1. \tag{17}$$

From (11), (15) and (16), together with the fact that $f \in M(N, r, \varepsilon)$, it follows that

$$f_\omega = f\|[\alpha]_r = \varepsilon(q) f\|[\rho\alpha]_r. \tag{18}$$

Using the functional equation of $f$ given by (13), we see that (18) can be rewritten as

$$f_\omega = \varepsilon(q) i^r \bar{f}\|[H_N \rho\alpha]_r. \tag{19}$$

A matrix calculation shows that $H_N\rho\alpha = q^{-1}\alpha' H_{q^2N}$ where $\alpha' = \begin{pmatrix} q & -c \\ 0 & q \end{pmatrix}$. Since the action of scalar matrices is trivial on $V$, we see that (14) follows immediately from (19) where $\omega'$ is the primitive character mod $q$ associated with the matrix $\alpha'$, i.e., $\omega' = e_q(-c) = e_q(-\overline{pN})$ in view of (17), which completes the proof of Theorem 1.

By Lemma 1, we immediately have the following consequence of Theorem 1.

**Theorem 2.** *For each $f \in M(N, r, \varepsilon)$ and $\omega \in X_N$ with conductor $q$, $M_{q^2N} f_\omega$ can be meromorphically continued to the entire complex plane. Furthermore, $M_{q^2N} f_\omega(s) + \frac{f(i\infty)}{s} + \frac{\varepsilon(q)\bar{f}(i\infty)}{r-s}$ is an entire function which is bounded in every vertical strip and satisfies the functional equation $M_{q^2N} f_\omega(s) = \varepsilon(q) M_{q^2N} f_{\omega'}(r-s)$. Moreover, $L_f(s,\omega)$ is holomorphic everywhere except possibly for a simple pole (which exists iff $f(i\infty) \neq 0$) at $s = r$ with residue*

$$\operatorname*{res}_{s=r} L_f(s,\omega) = \varepsilon(q) q^{-r} \operatorname*{res}_{s=r} L_f(s).$$

**Remarks.** For each Dirichlet character $\chi$ mod $q$, define $L_f(s,\chi)$ as in (12). Furthermore, let $\chi(\omega) = \chi(p)$ for each additive character $\omega = e_q(p)$ mod $q$. From the identity

$$W(\bar{\chi}) L_f(s,\chi) = \sum_{\omega \bmod q}{}^{*} \bar{\chi}(\omega) L_f(s,\omega), \quad (f \in M(N, r, \varepsilon)) \tag{20}$$

where $W(\chi)$ is the Gaussian sum defined by $W(\chi) = \sum^{*}_{\omega \bmod q} \chi(\omega)\omega(1)$, we can trivially deduce a theorem of Weil [20] for $L_f(s,\chi)$ analogous to Theorem 2 given above. Moreover, in [20], Weil establishes a criterion for $f \in V$ to be in $M(N, r, \varepsilon)$ in terms of certain analytic properties of the Dirichlet series $L_f(s,\chi)$. We remark that a similar criterion can be obtained using additive characters; such a result is implicit in Razar [15]. Indeed, these results are quite easy to obtain and provide a transparent proof of Weil's criterion with multiplicative characters in view of (20). We shall leave the details to the interested reader.

**4. The Order of $A_f(x;\ell,k)$.** In order to efficiently determine the order of $A_f(x;\ell,k)$ as $x \to \infty$ (uniformly in $\ell$ and $k$ in some sense) for $f \in M(N, r, \varepsilon)$, we use Smith's [17] adaptation of Theorem 4.1 of Chandrasekharan and Narasimhan [2] which enables one to determine the average order of a class of arithmetic functions when averaged over certain arithmetic progressions. In view of Theorem 2 given above, Theorem 1 of [17] implies the existence of a positive integer $\rho$ and numbers $0 < \xi_1 < \xi < \rho$ such that for arbitrary integers $\ell$ and $k$ with $k \geq 1$ and $(k, N) = 1$, then

$$A_f(x;\ell,k) = Q_f(x;\ell,k) + \xi y Q_f'(x+\xi_1 y;\ell,k) + O(\sum_{\substack{x<n\leq x+\rho y\\ n\equiv \ell \bmod k}} |a_n|) + \tag{21}$$
$$+ O(\frac{1}{k}\sum_{q|k}\lambda(q)^r \max_{1\leq n\leq q}|K(n\overline{N},\ell;q)|\ F(xy^{-2}\lambda(q)^{-2})x^{-\frac{1}{2}}y^{r+\frac{1}{2}})$$

for all $x \geq 1$ and $0 < y \leq x$ with $\lambda(q) = \frac{2\pi}{q\sqrt{N}}$, where

$$Q_f(x;\ell,k) = \frac{1}{2\pi i}\int_C \frac{x^s}{s} L_f(s;\ell,k)ds \tag{22}$$

and

$$L_f(s;\ell,k) = \sum_{\substack{n\geq 1\\ n\equiv \ell \bmod k}} a_n n^{-s}, \tag{23}$$

$C$ being a cycle enclosing all the singularities of the integrand in (22). Furthermore, $K(n\overline{N},\ell;q)$ is the Kloosterman sum defined by

$$K(n\overline{N},\ell;q) = \sum_{\omega \bmod q}{}^{*}\ \omega(-\ell)\omega'(n)$$

and $F$ is any continuous function satisfying, certain technical conditions automatically satisfied in the special conditions considered in this paper, such that

$$\sum_{n\leq x}|b_n| \ll F(x) \tag{24}$$

for $x \geq 1$ where $\{b_n\}$ are the Fourier coefficients of $\bar{f}$ (cf. (13)). Finally, the dash on $Q_f'$ in (21) denotes differentiation with respect to $x$. By Petersson [11] (or Lehner [9], p. 298), we know that

$$a_n, b_n = O(n^{r-1}) \quad \text{as} \quad n\to\infty \tag{25}$$

for $r \geq 3$. Consequently, we may take $F$ above to be $F(x) = x^r$ for $x \geq 1$. From the estimate of Hooley [8] for the Kloosterman sum $|K(n\overline{N},\ell;q)| \leq q^{\frac{1}{2}}(\ell,q)^{\frac{1}{2}}d(q)$ (which incidentally depends on the estimate of Weil [19] for $q$ a prime), it follows that the second $O$-term in (21) is

$$\ll \frac{1}{k}\sum_{q|k} q^{r+\frac{1}{2}}(\ell,q)^{\frac{1}{2}}d(q)(xy^{-1})^{r-\frac{1}{2}} \quad \ll (\ell,k)^{\frac{1}{2}}d(k)(kxy^{-1})^{r-\frac{1}{2}}.$$

Also, (25) implies that the first $O$-term in (21) is

$$\ll x^{r-1}\sum_{\substack{x<n\leq x+\rho y\\ n\equiv \ell \bmod k}} 1 \ll x^{r-1}(\frac{y}{k}+1) \tag{26}$$

since $\sum_{A<n\leq A+B} 1 \leq B+1$ for $B \geq 0$. (Continued in Part II.)

# Fourier coefficients of modular forms over arithmetic progressions. II.

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Presented by J.B. Friedlander, F.R.S.C.

**Abstract.** This paper completes the proof of the main theorem stated in Part I and gives an estimate for cusp forms.

1. To estimate $A_f(x, \ell, k)$ in [21] of Part I we must determine $Q_f(x; \ell, k)$. Using the same idea that led to the decomposition in (9), we immediately obtain the identity (cf. (12) and (23))

$$L_f(s; \ell, k) = \frac{1}{k} \sum_{q|k} \sum_{\omega \bmod q}{}^{*} \omega(-\ell), L_f(s, \omega), \tag{27}$$

from which we may conclude that $L_f(s; \ell, k)$ is holomorphic everywhere except possibly for a simple pole at $s = r$ by Theorem 2. Hence, by (22), (27) and the last part of Theorem 2, we have

$$Q_f(x; \ell, k) = \Phi_f(\ell, k) \operatorname*{res}_{s=r} L_f(s) x^r + L(O; \ell, k) \tag{28}$$

where

$$\Phi_f(\ell, k) = \frac{1}{k} \sum_{q|k} \varepsilon(q) q^{-r} c_q(\ell) \tag{29}$$

with

$$c_q(\ell) = \sum_{\omega \bmod q}{}^{*} \omega(-\ell),$$

the Ramanujan sum. By (27), together with the functional equation of $L_f(s, \omega)$ given in Theorem 2, we have

$$L_f(O; \ell, k) \ll \frac{d(k)}{k}, \tag{30}$$

and since $|c_q(\ell)| < q$, (29) implies that $\Phi_f(\ell, k) \ll \frac{1}{k}$ for $r \geq 3$. Substituting these results into (21) gives

$$A_f(x; \ell, k) - \Phi_f(\ell, k) \operatorname*{res}_{s=r} L_f(s) x^r \ll k^{-1} y x^{r-1} + x^{r-1} + (\ell, k)^{\frac{1}{2}} d(k) (kxy^{-1})^{r-\frac{1}{2}}.$$

By taking $y = k[(\ell, k)x]^{1/(2r+1)}$, we obtain

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† Obiit 1983. This paper written in 1979 is now published in two parts by permission of Mrs. Karin Smith

**Theorem 3.** *If $f \in M(N,r,\varepsilon)$ with $r \geq 3$, then*

$$A_f(x;\ell,k) = \Phi_f(\ell,k) \operatorname*{res}_{s=r} L_f(s)x^r + O(\ell,k)^{1/(2r+1)}d(k)x^{r-1+1/(2r+1)})$$

*holds uniformly in $\ell, k$ and $x \geq 1$ provided $(k,N) = 1$ and $k(\ell,k)^{1/(2r+1)} \leq x^{2r/(2r+1)}$. Furthermore, $\Phi_f(\ell,k)$ is defined by (29) which if $(\ell,k) = 1$ takes the form*

$$\Phi_f(\ell,k) = \frac{1}{k}\prod_{p|k}(1-\varepsilon(p)p^{-r}) \qquad (p = \text{ prime}).$$

We observe that a similar result holds for $r = 1$ and $r = 2$. Indeed, by Petersson [11], (25) still holds for $r = 1$, while for $r = 2$, (25) must be replaced by $a_n, b_n = O(n\log n)$ as $n \to \infty$, from which the corresponding theorem may easily be obtained. We omit the details.

As a simple application of Theorem 3, we may deduce a special case of a result obtained by Smith [17] using different methods. Indeed, let $Q$ be a positive definite integral quadratic form of level $N$ in $2r$ variables. If $f = \theta_q$ denotes the theta function of $Q$, then $f \in M(N,r,\varepsilon)$ where $\varepsilon(q) = \left(\frac{(-1)^r D}{q}\right)$ is the Kronecker symbol and $D$ denotes the discriminant of $Q$. Furthermore, $f$ satisfies (13) with $\bar{f} = D^{-\frac{1}{2}}N^{r/2}\theta_{Q^*}$ where $Q^*$ denotes the adjoint form of $Q$. (For details, see Schoeneberg [16] or Ogg [11]). Since the $n^{\text{th}}$ Fourier coefficient of $f$ is $r_Q)(n)$, the number of representations of $n$ by the form $Q$, Theorem 3 implies the following

**Corollary.** *Using the above notation (with $r \geq 3$ then*

$$\sum_{\substack{n \leq x \\ n \equiv \ell \bmod k}} r_Q(n) = N_Q(\ell,k)x^r + O((\ell,k)^{1/(2r+1)}d(k)x^{r-1+1/(2r+1)})$$

*holds uniformly in $\ell, k$ and $x \geq 1$ provided $(k,N) = 1$ and $k(\ell,k)^{1/(2r+1)} \leq x^{2r/(2r+1)}$, where $N_Q(\ell,k)$ is an arithmetic function depending only on $\ell, k$ and $Q$ (cf. Smith [17] for an arithmetic interpretation of $N_Q(\ell,k)$).*

The remainder of this paper will be devoted to establishing (4) for cusp forms $f$ in $S(N,r,\varepsilon)$. Recall that in this situation, $L_f(s)$ is holomorphic at $s = r$ so that by Theorem 3, we only obtain an upper bound for $A_f(x;\ell,k)$ which we know is extremely poor (cf. (4)). Before we can establish (4), we shall require some additional preparation.

In [1], Atkin and Lehner construct a basis for the (finite dimensional) vector space $S(N,r,1)$ consisting of newforms of level $N$ together with oldforms which are induced from newforms of lower levels (cf. [1], Theorem 5); a similar result has been obtained for $S(N,r,\varepsilon)$ by Li [10] (cf. p. 294) where $\varepsilon$ is any Dirichlet character mod $N$. Plainly, to establish (4), it suffices to assume that $f$ is a normalized newform of level $N$ (normalized means $a_1 = 1$; see Theorem 2 of [10]). One of the fundamental properties of a newform of level $N$ is that it is an eigenform for all the Hecke operators $T_p$ with $p \nmid N$ while for $p|N$, it is an eigenform for the Atkin and Lehner operator $U_p$, where $p$ is a prime. From these facts, we may derive the following properties of the Fourier coefficients of a normalized newform $f \in S(N,r,\varepsilon)$ (cf. Atkin and Lehner [1], Theorem 3, and Li [10], Theorem 3).

**Lemma 2.** *Suppose $f \in S(N,r,\varepsilon)$ is a normalized newform. Then the Fourier coefficients $\{a(n)\}$ of $f$ satisfy the following properties* [2]

(i) If $p$ is a prime not dividing $N$, then

$$\varepsilon(p)p^{r-1}a(n/p) + a(np) = a(p)a(n) \quad \text{for all } n \geq 1.$$

(ii) If $p$ is a prime dividing $N$, then

$$a(np) = a(n)a(p) \quad \text{for all } n \geq 1 \qquad \text{and} \quad |a(p)| \leq p^{(r-1)/2}.$$

(In (i), we define $a(x) = 0$ if $x$ is not an integer.)

Thus, (ii) implies that

$$|a(p^m)| \leq p^{m(r-1)/2} \qquad (m \geq 1) \tag{31}$$

for all primes $p$ dividing $N$. The analogue of (31) for primes $p$ not dividing $N$ (with $m = 1$) is the Ramanujan-Petersson conjecture, which asserts that

$$|a(p)| \leq 2p^{(r-1)/2}. \quad (p \nmid N) \tag{32}$$

This conjecture has been established by Deligne for $r \geq 2$ (cf. [3], [4]) and by Deligne and Serre for $r = 1$ (cf. [5]), the proof of (32) depending upon Deligne's proof of the Riemann Hypothesis associated with the Weil conjectures. In order to obtain the analogue of (31) for prime powers with $p$ not dividing $N$, we first observe that Lemma 2(ii) implies that $x_m = a(p^m)$ satisfies

$$x_{m+2} - a(p)x_{m+1} + \varepsilon(p)p^{r-1}x_m = 0 \tag{33}$$

[2] For convenience, we shall sometimes write $a(n)$ instead of $a_n$.

for all $m \geq 0$. Clearly, the characteristic polynomial of the second order linear recursive equation in (33) is $x^2 - a(p)X + \varepsilon(p)p^{r-1}$. Since the Ramanujan-Petersson conjecture is equivalent to the characteristic roots of

$$X^2 - \eta_p a(p)X + p^{r-1} \tag{34}$$

being conjugate where $\eta_p^2 = \overline{\varepsilon(p)}$ with $\eta_p a(p)$ real (cf. Ogg [11], IV-29), it follows that the solutions of (33) are given by

$$x_m \;=\; p^{m(r-1)/2}\eta_p^m \frac{\sin(m+1)\theta_p}{\sin\theta_p} \tag{35}$$

where $p^{(r-1)/2}e^{i\theta_p}$ is a root of (34) with $\theta_p$ real. Thus, (35) implies that

$$|a(p^m)| \;\leq\; (m+1)p^{m(r-1)/2} \tag{36}$$

for all $m \geq 1$. Since the Fourier coefficients of $f$ are multiplicative, (31) and (36) imply that

$$|a(n)| \;\leq\; n^{(r-1)/2}d(n) \quad \text{for all} \quad n \geq 1.$$

In order to derive (4) from (21), we must apply (37) twice, once for each of the $O$-terms in (21). Therefore, the first $O$-term in (21) is

$$\ll\; x^{(r-1)/2} \sum_{\substack{x<n\leq x+\rho y \\ n\equiv \ell \bmod k}} d(n). \tag{38}$$

Unfortunately, we are unable to use the simple minded approach to estimate the sum in (38) that we used in (26) to establish Theorem 3. To deal with the sum in (38), we need another lemma. We note that a similar result has been obtained by Hooley [8], though his error term is somewhat different from ours.

**Lemma 3.** *For any $x \geq 1$,*

$$\sum_{\substack{n\leq x \\ n\equiv \ell \bmod k}} d(n) \;=\; H(\ell,k)x\log x + K(\ell,k)x + O(\ell,k)^{\frac{1}{2}}x^{\frac{1}{2}}d(k)\log 4kx)$$

*holds uniformly in $\ell$ and $k$ provided $k(\ell,k)^{\frac{1}{2}} \leq x^{\frac{2}{3}}$, where $H(\ell,k)$ and $K(\ell,k)$ are certain arithmetic functions (for precise definitions, see [8]) which satisfy*

$$H(\ell,k) \ll \frac{d(k)}{k} \quad \text{and} \quad K(\ell,k) \ll \frac{d(k)}{k}\log 4k.$$

**Proof.** For $Re(s) > 1$, define

$$D(s) \;=\; \sum_{n\geq 1} d(n)n^{-s}.$$

For any primitive additive character $\omega \bmod q$, $D(s,\omega)$ has a meromorphic continuation into the entire complex plane and satisfies the functional equation (cf. Hooley [8], Lemma 1)

$$D(s,\omega) = \frac{\Delta(1-s;q)}{\Delta(s;q)}D(1-s,\omega_+) + \frac{\Delta(2-s;q)}{\Delta(1+s;q)}D(1-s,\omega_-)$$

where $\omega_\pm = \frac{1}{2}(\omega' \pm \bar{\omega}')$ (with $N=1$) and $\Delta(s;q) = (\pi/q)^{-s}\Gamma(s/2)^2$. Hence, we may apply (21) in this case with $r=1$ (cf. Smith [17]; also see remark 3 in section 2). The remainder of the proof is now omitted as it closely parallels the above proof of Theorem 3 (cf. Hooley [8] for details regarding the main terms).

We can now establish (4) for any $r \geq 1$.

**Theorem 4.** *If $f \in S(N,r,\varepsilon)$, then*

$$A_f(x;\ell,k) \ll (\ell,k)^{\frac{1}{3}} d(k) x^{r/2-\frac{1}{6}} \log 4kx$$

*holds uniformly in $\ell, k$ and $x \geq 1$ for $(k,N)=1$ provided $k(\ell,k)^{\frac{1}{5}} \leq x^{\frac{2}{5}}$.*

**Proof.** Since $f$ is a cusp form, $L_f(s)$ is holomorphic at $s=r$ so that by (28) and (30), we have $Q_f(x;\ell,k) \ll d(k)/k$. By (38) and Lemma 3, we find that the first $O$-term in (21) is $\ll k^{-1}yx^{(r-1)/2}d(k)\log 4kx$. Finally, we must estimate the second $O$-term in (21), i.e., we must obtain an upper bound for the sum in (24). Since $S(N,r,\varepsilon)$ has a basis consisting of newforms of level $N$ together with oldforms which are induced from newforms of lower levels, it follows by (37) that the Fourier coefficients $\{b_n\}$ of $\tilde{f}$ satisfy $b_n \ll n^{(r-1)/2}d(n)$ from which we immediately obtain $\sum_{n\leq x} b_n \ll x^{(r+1)/2}\log x$. By taking $F(x) = x^{(r+1)/2}\log x$, the second $O$-term in (21) is

$$\ll [k(\ell,k)]^{\frac{1}{2}} d(k) x^{r/2} y^{-\frac{1}{2}} \log(xy^{-2}k^2).$$

Combining these results with (21) completes the proof of Theorem 4 if we choose $y = k[(\ell,k)x]^{\frac{1}{3}}$.

As a final remark, we note that Walfisz [18] proved that (this can also be derived from Theorem 3.2 of Chandrasekharan and Narasimhan [2])

$$A_f(x) = \Omega(x^{r/2-\frac{1}{4}}).$$

Consequently, Theorem 4 is quite close to the best possible result, at least for $\ell = k = 1$. Perhaps some of the special techniques in analytic number theory can be brought to bear on the problem of either further reducing the exponent on $x$ in Theorem 4 or of increasing the range of uniformity in $k$. Progress in either problem would have interesting applications.

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# Remarks on the preceding paper by R.A. Smith

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The paper [8] was written in 1979 and since that time many fundamental developments have taken place. The two main results of [8] are striking for various reasons. The uniform error term for the summatory function of Fourier coefficients of cusp forms in Theorem 4 is not surprising, and certainly it was known to the experts that one can expect uniformity for $k \leq x^{\frac{2}{3}}$ for all automorphic forms of $GL_2(\mathbb{A}_Q)$. However, that one can expect a larger range of uniformity for non-cusp forms, and that this range increases with the weight of the modular form comes as a surprise. Indeed, we have been led to believe that in the spirit of the Langlands philosophy normalization should not affect any result otherwise obtained by not normalizing. Theorem 3 of [8] is a counterexample to such a belief.

Since 1979, we know more about $L$-functions attached to symmetric powers of automorphic representations of $GL_2(\mathbb{A}_Q)$. In fact, from the work of Shahidi [7] and Rankin [6] we know that

$$A_f(x) \ll x^{\frac{k}{2}-\frac{1}{6}}(\log x)^{-\delta+\epsilon}$$

with $\delta = (8-3\sqrt{6})/10 = 0.065153...$ and that in the notation of [8],

$$\sum_{n\leq x} |b_n| \ll \frac{x^{(r+1)/2}}{(\log x)^{\delta}}.$$

If this result is used, we can increase the range of validity of Theorem 4 in [8] to

$$k(\ell,k)^{\frac{1}{2}} \leq x^{\frac{2}{3}}(\log x)^{\frac{2}{3}+\delta}.$$

The exponent '$\frac{2}{3}$' has come to play a prominent role in analytic number theory. In this connection, there is a result of Fouvry [2]: if $d(n)$ denotes the usual divisor function, for any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that

$$\sum_{\substack{(q,a)=1 \\ q\leq x^{1-\epsilon} \\ q\notin[x^{\frac{2}{3}-\epsilon}, x^{\frac{2}{3}+\epsilon}]}} \Big| \sum_{\substack{n\leq x \\ n\equiv a (\bmod q)}} d(n) - \frac{1}{Q(q)} \sum_{\substack{n\leq x \\ (n/q)=1}} d(n)\Big| \ll_{\epsilon} x\exp(-c(\log x)^{\frac{1}{3}}) \qquad (1)$$

uniformly for $0 < |a| \leq \exp(c(\log x)^{\frac{1}{3}})$. The $\varepsilon$ appearing in the above estimate can perhaps be replaced by a power of a logarithm. By Smith [8], we know the result above is valid for $q \leq x^{\frac{2}{3}-\epsilon}$ because $d(n)$ can (almost) be identified with the Fourier coefficient of an Eisenstein series of weight 2. Moreover, the above result, without the constraint $q \notin [x^{\frac{2}{3}-\epsilon}, x^{\frac{2}{3}+\epsilon}]$, represents a result of Elliott-Halberstam type, and is a major conjecture in analytic number theory with serious implications. In fact, if $d(n)$ is replaced by $\Lambda(n)$ which is the classical von Mangoldt function, then the analogue of (1) without the constraint on $q$ is the famous conjecture of Elliott and Halberstam. Only recently, Bombieri, Friedlander and Iwaniec [1] managed to make progress towards the Elliott-Halberstam conjecture by extending the range of the celebrated Bombieri-Vinogradov Theorem slightly beyond the exponent $\frac{1}{2}$. In a paper of Fouvry [3], it is shown that $d(n)$ and the higher divisor functions $d_k(n)$ are not unrelated to $\Lambda(n)$ and that uniform results on

$$\sum_{\substack{n\leq x\\ n\equiv a \pmod q}} d_k(n)$$

and certain allied sums implies a version of the Elliott-Halberstam conjecture which goes beyond the exponent $\frac{1}{2}$. In this context, the results of Friedlander and Iwaniec [4,5] represent major advances towards the conjecture.

These remarks are intended to amplify the results of [8] within the modern context. It is likely that the methods of [8] are applicable for coefficients of Dirichlet series attached to automorphic representation of higher $GL_n(\mathbb{A}_Q)$.

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# EXISTENCE D'UN POINT D'EQUILIBRE FORT

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*Presented by P. Ribenboim, F.R.S.C.*

Dans le présent travail on obtient des conditions suffisantes pour le point d'équilibre fort et on précise son existence dans une stratégie mixte.

## I. Conditions suffisantes

Considérons le jeu sans alliance à deux joueurs.

$$\langle X_1, X_2, \{ f_i(x_1, x_2) \}\ i=1,2 \rangle \qquad (1)$$

où $X_1 \in \operatorname{comp} \mathbf{R}^{m_1}$ l'ensemble des stratégies $x_1$ du premier joueur, la stratégie du second joueur $x_2 \in X_2 \in \operatorname{comp} \mathbf{R}^{m_2}$; $f_i\ (x_1, x_2)$ est la fonction de gain. On suppose que $f_i\ (x_1, x_2)$ est continue sur $X_1 \times X_2$.

La situation $x^* = (x_1^*, x_2^*) \in X_1 \times X_2$ du jeu(1) d'équilibre fort d'après([1, p.96] et[3]) nous allons définir par les inégalités

$$f_1(x_1^*, x_2) \leq f_1(x_1^*, x_2^*), \quad \forall x_2 \in X_2$$
$$f_2(x_1, x_2^*) \leq f_2(x_1^*, x_2^*), \quad \forall x_1 \in X_1$$

Soit la fonction

$$\varphi(x,y) = f_1(x_1, y_2) + f_2(y_1, x_2) - f_1(x_1, x_2) - f_2(x_1, x_2) \quad (3)$$

où $y = (y_1, y_2) \in X_1 \times X_2$, et le jeu de deus personnes à somme nulle $\langle X, Y, \varphi(x, y) \rangle$ avec $X = Y = X_1 \times X_2$. (4)

**Affirmation** Si $(x^*, y^*)$ représente un point-selle du jeu (4) s'est-à-dire

$$\varphi(x^*, y) \leq \varphi(x^*, y^*) \leq \varphi(x, y^*), \forall x \in X, y \in Y \qquad (5)$$

alore $x^* = (x_1^*, x_2^*)$ est le point d'équilibre fort.

**Démonstration**

Si dans (5) nous posons $x = y^*$ alors de (5) et (3) on obtient

$$f_1(x_1^*, y_2) + f_2(y_1, x_2^*) - f_1(x_1^*, x_2^*) - f_2(x_1^*, x_2^*) \leq 0 \qquad (6)$$

quelque soit $y_2 \in X_2$ et $y_1 \in X_1$

Si dans (6) nous posons $y_1 = x_1^*$, et $y_2 = x_2^*$ nous obtenons respectivement les inégalités (2)

<u>II.Existence</u>. L'Extension mixte du jeu (1) se present par

$$< \{\mu_1\}, \{\mu_2\}, F(\mu_1, \mu_2) >$$

où $\{\mu_i\}$ est l'ensemble des mesures de probabilité sur le compacts $X_i$, et l'espérance mathématique de la fonction de gain du i-ème joueur est :

$$F_i(\mu_1, \mu_2) = \int_{X_1 \times X_2} f_i(x_1, x_2)\, \mu_1(dx_1)\, \mu_2(dx_2) \qquad (i=1,2)$$

La situation $\mu^* = (\mu^*_1, \mu^*_2) \in \{\mu_1\} \times \{\mu_2\}$ du jeu (7) d'équilibre fort est définie par les inégalités

$$F_1(\mu^*_1, \mu_2) \le F_1(\mu^*_1, \mu^*_2) \quad \forall \mu_1 \in \{\mu_1\}$$

$$F_2(\mu_1, \mu^*_2) \le F_2(\mu^*_1, \mu^*_2) \quad \forall \mu_2 \in \{\mu_2\} \qquad (8)$$

**Théorème** Dans le jeu (7) il existe une situation d'équilibre fort.

La démonstration se divise en deux étapes. Tout d'abord analogiquement à ce qu'on a démontré dans l'affirmation nous obtenons que la première composante $\mu^* = (\mu_1^*, \mu_2^*)$ du point-selle $(\mu^*, \nu^*)$ de la fonction $\psi(\mu, \nu) = F_1(\mu_1, \nu_2) + F_2(\nu_1, \mu_2) - F_1(\mu_1, \mu_2) - F_2(\mu_1, \mu_2)$ verifie l'inégalité (8)

En deuxième étape à l'aide du théorème [2, p. 353] on obtient l'existence du point-selle de la fonction.

<u>Conclusion</u> Il y a des exemples où l'équilibre fort dans les stratégies pures du jeu (1) n'existe pas. D'après le théorème concidéré dans les stratégies mixtes, la situation d'équilibre fort existe. A l'aide de l'affirmation considérée, on peut obténir parfois cette situation en forme explicite.

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# UNE CARACTERISATION DES CARACTERES DANS LES ALGEBRES NORMEES NON COMPLETES

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**ABSTRACT** : We give a characterization of multiplicative linear functionals in normed Q-algebras. Some consequences are obtained.

**INTRODUCTION** · Dans [2], J.P.Kahane et W.Żelazko ont montré que si A est une algèbre de Banach complexe commutative et unitaire, alors toute forme linéaire f vérifiant la propriété :

(1) $f(x) \in Spx$, pour tout $x \in A$

est en fait un caractère. Ceci a été généralisé par W.Żelazko au cas non commutatif [4]. Ce dernier a aussi montré que le résultat reste vrai dans les algèbres localement multiplicativement convexes (a.l.m.c) complètes si on suppose la continuité de la forme liénaire f [5].

Le but de ce travail est d'étendre ce résultat aux algèbres normées non complètes qui sont des Q-algèbres i.e dont l'ensemble des éléments inversibles est oouvert. la démonstration est basée sur le fait que dans les algèbres de Banach la fonction multivoque $x \longmapsto Spx$ est semi-continue superieurement [1]. On obtient alors que, plus généralement, le résultat est vrai dans toute algèbre complexe commutative vérifiant les deux conditions suivantes :

(i) Le spectre de tout élément est borné.

(ii) Pour tout élément x, $Spx = \{\chi(x)/\chi \in M^*\}$ où $M^*$ désigne l'ensemble des caractères non nuls.

Enfin, signalons que ce résultat s'étend à des algèbres topologiques à savoir les algèbres multiplicativement convexes (a.l.m.c.) ou même les algèbres localement A-convexes (a.l.A-convexes) qui sont des Q-algèbres.

Dans tout ce qui suit les algèbres considérées sont complexes et unitaires.

Nous nous intéressons d'abord aux algèbres normées.

**THEOREME 1** Soient $(A, \|\ .\ \|)$ une algèbre normée qui est une Q-algèbre et $f$ une forme linéaire sur A vérifiant :

(1) $f(x) \in Sp\,x$, pour tout $x \in A$

Alors $f$ est un caractère de A.

**PREUVE** : Soit B la complétée de A. Puisque (1) est vérifiée et A est une Q-algèbre, $f$ est nécessairement continue. Donc $f$ se prolonge en une forme linéaire continue $\tilde{f}$ sur B. Nous allons montrer que $\tilde{f}(x) \in Sp_B x$ pour tout $x \in B$. Soit $x \in B$. Il existe une suite $(x_n)_{n\geq 1}$ d'éléments de A qui converge vers x dans B. Comme $\tilde{f}$ est continue, $(f(x_n))_n$ converge vers $\tilde{f}(x)$ dans $\mathbb{C}$. Soit un voisinage fermé V de 0 dans $\mathbb{C}$. Il existe un voisinage ouvert U de 0 contenu dans V. L'application multivoque définie sur B qui à tout élément de B fait correspondre son spectre est semi-continue supérieurement [1]. Donc pour l'ouvert $U + Sp_B x$, il existe $\alpha > 0$ tel que :

$$\|y - x\| < \alpha \Rightarrow Sp_B y \subset U + Sp_B x.$$

Et comme la suite $(x_n)_n$ converge vers x, il existe un entier $N \in \mathbb{N}$ tel que :

$$Sp_B x_n \subset U + Sp_B x, \text{ pour tout } n \geq N$$

Mais, d'après [3], A est pleine dans sa complétée, donc $Sp_A x = Sp_B x$, pour tout $x \in A$. En particulier $Sp_A x_n = Sp_B x_n$, pour tout n. Or par hypothèse nous avons $f(x_n) \in Sp_A x_n$, pour tout n. D'où $f(x_n) \in U + Sp_B x \subset V + Sp_B x$, pour tout $n \geq N$. En passant à la limite, on obtient $\tilde{f}(x) \in \overline{V + Sp_B x} = V + Sp_B x$, car V est fermé et $Sp_B x$ est compact. Comme le voisinage V est arbitraire, $\tilde{f}(x) \in Sp_B x$. Le théorème découle du résultat de W.Żelazko ([4]).

Nous allons maintenant étendre le résultat précédent à une classe plus vaste d'algèbres.

Rappelons que dans une algèbre A, le rayon spectral est défini par :

$$\rho_A(x) = \sup\{|\lambda|, \lambda \in Sp_A x\}, \quad x \in A.$$

**PROPOSITION 1** : Soit A une algèbre telle que :

(i) pour tout $x \in A$, $Sp_A x$ est borné

(ii) Pour tout x de A, $Sp_A x = \{\chi(x) / \chi \in M^*(A)\}$.

Si f est une forme linéaire sur A vérifiant :

(i) $f(x) \in Sp_A x$, pour tout $x \in A$

Alors f est un caractere.

**PREUVE** : Soit $\rho_A$ le rayon spectral , Par (ii) on a :

$$\rho_A(x) = \sup\{|\chi(x)|, \chi \in M^*(A)\}$$

et

RadA= $\{x \in A \,/\, \rho_A(x) = 0\}$ ; où RadA est le radical de Jacobson de A. Considérons $B = A \,/\, RadA$. On munit l'algèbre B de la norme définie par :

$$\|\bar{x}\|_1 = \rho_A(x), \quad x \in A$$

C'est une norme d'algèbre. Et comme $Sp_A x = Sp_B \bar{x}$, on a $\rho_A(x) = \rho_B(\bar{x})$ pour tout x de A. D'où $\|\bar{x}\|_1 = \rho_B(x)$, pour tout $x \in A$. D'après [3], $(B, \|\ .\ \|_1)$ est une Q-algèbre . On considère l'application définie sur B par :

$$\bar{f}(\bar{x}) = f(x), \text{pour tout } x \in A .$$

$\bar{f}$ est bien définie et c'est une forme linéaire sur B vérifiant :

$$\bar{f}(\bar{x}) \in Sp_B x \text{ , pour tout } x \text{ de } A.$$

Par le théorème précédent , $\bar{f}$ est un caractère sur B et aussi f est un caractère de A.

Comme conséquence du résultat précédent nous avons le :

**COROLLAIRE 1** : Soit A une a.l.m.c. commutative qui est une Q-algèbre. Si f est une forme linéaire sur A vérifiant :

$$f(x) \in Sp_A x, \text{ pour tout } x \in A \text{ ;}$$

alors f est un caractère de A.

**PREUVE** . Par la proposition précédente, il suffit de montrer que (ii) est vérifiée. Soient $x \in A$ et $\lambda \in Sp_A x$. Il existe un idéal maximal $M$ tel que $\lambda e - x \in M$ ; où e est l'unité de A. Comme A est une Q-algèbre, l'idéal M est fermé. D'après le théorème de Gelfand-Mazur, on obtient que M est de codimension 1. Donc il existe un caractère $\chi \in M^*(A)$ tel que $M = \operatorname{Ker} \chi$ . D'où $\lambda = \chi(x)$.

**REMARQUES** :

1) D'après ce qui précède le résultat est vrai dans toute algèbre commutative topologique (resp.bornologique) qui est une Q-algèbre (resp, Q-algèbre boronologique) et qui appartient à une classe d'algèbres vérifiant le théorème de Gelfand-Mazur.

2) Si A est une algèbre vérifiant :

$$Sp_A x = \{\chi(x) / \chi \in M^*(A)\}, \text{ pour tout } x \in A \text{ ;}$$

alors on a l'équivalence suivante : $Sp_A x$ est borné pour tout x si, et seulement si, $Sp_A x$ est compact pour tout x. En effet, dans la preuve de la proposition1 l'algèbre B est une algèbre normée qui est une Q-algèbre. Donc $Sp_B \bar{x}$ est compact pour tout $x \in A$. Or $Sp_A x = Sp_B \bar{x}$, pour tout $x \in A$. D'où le résultat .

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# Large Deviations for the Empirical Process of Mean Field Particle System with Unbounded Jumps

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**Abstract:** An N-particle system with mean field interaction is considered. Large deviations for the empirical process as N goes to infinity are obtained under conditions which are satisfied by many interesting models including the first and the second Schlögl models. This result is obtained without using the large deviation result for the related empirical distribution because of the difficulty in using the contraction principle

**1. Introduction** Let $E = \{0, 1, \dots\}$, $E^{\otimes N}$ be the $N$-fold product of $E$ both equipped with the discrete topology. $x^{(N)}(t) = (x_1^{(N)}(t), \dots, x_N^{(N)}(t))$ on $E^{\otimes N}$ is a Markov process generated by

$$\Omega^{(N)}\psi(x^{(N)}) = \sum_{k=1}^{N} Q^{(k)}_{\varepsilon_{x^{(N)}}}\psi(x^{(N)}) \tag{1.1}$$

where $\varepsilon_{x^{(N)}} = \frac{1}{N}\sum_{k=1}^{N}\delta_{x_k^{(N)}}$, $x^{(N)} = (x_1^{(N)}, \dots, x_N^{(N)}) \in E^{\otimes N}$. For any $\nu \in M_1(E)$ (the space of all probability measures on $E$ with weak topology), $Q_\nu$ is defined by

$$Q_\nu f(x) = \sum_{y \in E} q_{x,y}(f(y) - f(x)) + \|\nu\|(f(x+1) - f(x)) \tag{1.2}$$

$Q = (q_{x,y})$ is a $Q$-matrix. $\|\nu\|$ is the first moment of $\nu$. $Q_\nu^{(k)}$ is used instead of $Q_\nu$ when it acts on the $k$-th variable of $\psi \in C_b(E^{\otimes N})$, $f \in C_b(E)$. Where $C_b(E)$ and $C_b(E^{\otimes N})$ denote the set of bounded continuous functions on $E$ and $E^{\otimes N}$ respectively.

This is an $N$-particle system with mean field interaction. The jump rates $q_{x,y}$ can be unbounded. Consider the empirical process $\varepsilon_{x^{(N)}(t)}, t \geq 0$ of the N-particle system

$$\varepsilon_{x^{(N)}(t)} = \frac{1}{N}\sum_{k=1}^{N}\delta_{x_k^{(N)}(t)} \tag{1.3}$$

This is a measure-valued process on $D([0,T], M_1(E))$, the space of all right continuous functions $\omega : [0,T] \to M_1(E)$ which have left limits at each $t \in (0,T]$ and are left continuous at $T$ with the Skorohod metric $d$ on it. Where $\delta_x$ denotes the Dirac measure with unit mass at $x$. Assume. $N \geq 1$, that $(x_1^{(N)}(0), \ldots, x_N^{(N)}(0))$ are independent and identically distributed with common distribution $u \in M_1(E)$. We prove that $\varepsilon_{x^{(N)}(\cdot)}$ converges in distribution to the solution of the following nonlinear master equation

$$\frac{du(t)(\cdot)}{dt} = \sum_{y \neq \cdot} \{u(t)(y)q_{y,\cdot} - u(t)(\cdot)q_{\cdot,y}\} + \|u(t)\|\{u(t)(\cdot - 1) - u(t)(\cdot)\}. \tag{1.4}$$

with initial distribution $u \in M_1(E)$.

The nonlinear master equation was proposed by G.Nicolis and I.Prigogine [6] as a mean-field model of a chemical reaction with spatial diffusion. S.Feng and X.Zheng [3] established the existence and uniqueness of the solution to the martingale problem associated with the nonlinear master equation and proved the existence of at least three equilibrium states for the second Schlögl model (see [7]). It is very natural to study the multiple equilibra transitions and the metastability for the nonlinear master equations. As a first step we discuss the problem of large deviations of the empirical process $\varepsilon_{x^{(N)}(t)}$, which is the main topic of this article.

Several recent papers have discussed similar kind of large deviation problem for varies models. Among them see for instance [1], [2], [5] and [8]. Feng [4] discussed the large deviation problem of the empirical distribution for the same model as we discussed here. But because of the difficulty in using the contraction principle the result here is proved separately.

Proofs, as well as further details, will appear in a forthcoming paper by the author.

**2. The Main Result.** We introduce a new metric $r$ on $M_1(E)$ by $\forall u, v \in M_1(E)$, $r(u,v) = \sum_{n=0}^{\infty} 2^{-n}(|u(n) - v(n)|)$. $M_1^v(E)$ denotes the space $M_1(E)$ endowed with the vague topology. For any fixed $T > 0$, let $\mathcal{D} = D([0,T], M_1(E))$.

Let $\varphi$ be defined as $\varphi : E \to R, x \to 1 + x \log\log(x+2)$. For each $m \geq 1$, define $\mathcal{D}_m = \{\mu(\cdot) \in \mathcal{D}; \sup_{0 \leq t \leq T} \langle \nu(t), \varphi \rangle \leq m\}$ equipped with the Skorohod topology. $\mathcal{D}_\infty = \bigcup_{m \geq 1} \mathcal{D}_m$ is equipped with the "inductive topology". By definition, a set $V$ is open in $\mathcal{D}_\infty$ if and only if $V \cap \mathcal{D}_m$ is open in $\mathcal{D}_m$ for each $m \geq 1$. Another useful property is that a function is continuous on $\mathcal{D}_\infty$ if and only if it is sequential continuous. $\forall \mu(\cdot), \nu(\cdot) \in \mathcal{D}_\infty$ let $r_{0T}(\mu(\cdot), \nu(\cdot)) = \sup_{0 \leq t \leq T} r(\mu(t), \nu(t))$. Then $r_{0T}$ induces the uniform topology on $\mathcal{D}_\infty$. From the definition of $r$ and the right continuity we can prove that for any fixed $\nu_0(\cdot) \in \mathcal{D}_\infty$, $F(\mu(\cdot)) = r_{0T}(\mu(\cdot), \nu_0(\cdot))$ is measurable with respect to the Borel $\sigma$-algebra of space $\mathcal{D}_\infty$ with the "inductive topology". This will be used in proving the upper bound.

Let $Q = (q_{x,y})_{x,y \in E}$ be a totally stable conservative $Q$-matrix satisfying:

$$\inf_{x \in E}\{q_{x,x+1}\} > 0; \tag{2.1}$$

$$\exists\, \Lambda > 0, \text{ such that } q_{x,y} = 0 \text{ for } |x - y| \geq \Lambda; \tag{2.2}$$

$$\exists\, \lambda > 0 \text{ such that } \forall x \in E,\ \sum_{y \in E} q_{x,y}(y - x) \leq \lambda x, \tag{2.3}$$

$$\sum_{x,y} q_{y\in E}(\varphi(y)-\varphi(x)) \leq \lambda\varphi(x).\ x \in E;$$

(2.4) $\exists\, 0 < c < \infty$, such that $\forall x, y \in E. y > x$, $\sum_{z\neq 0}(q_{y,y+z} - q_{x,x+z})z$

$$+2\sum_{z=1}^{\infty}[(q_{y,x-z} - q_{x,2x-y-z}) \vee 0 + (q_{x,y+z} - q_{y,2y-x+z}) \vee 0]z \leq c(y-x);$$

(2.5) $\forall\, l \geq 0$, $\exists\, \lambda(l) > 0$, $\ni$

$$\sup_{x\in E} \max\{\sum_{y\in E} q_{x,y}(e^{y-x}-1) + (e-1)x,$$

$$\sum_{y\in E} q_{x,y}(e^{\varphi(y)-\varphi(x)}-1) + (e^{\varphi(x+1)-\varphi(x)}-1)l\} \leq \lambda(l).$$

**Remark.** Conditions (2.3) and (2.4) guarantee that the corresponding martingale problems are well-posed. Conditions (2.1),(2.2) and (2.5) are needed only for the large deviation result. Note also that (2.3) can be implied by (2.5). These conditions are satisfied the first and the second Schlögl models (cf. [7]).

**Theorem 2.1** *Under assumptions (2.3) and (2.4) we have*

(a). *The time-inhomogeneous martingale problem for $Q_{u(t)}$ with initial distribution $u$ is well-posed. The solution is denoted by $P_{u(\cdot),u}$.*

(b). *For each $N \geq 1$, the martingale problem for $\Omega^{(N)}$ on $D([0,T], E^{\otimes N})$ with initial distribution $u^{\otimes N}$ is well-posed. This solution is denoted by $P_u^{(N)}$.*

Let $X$ be a Hausdorff topological space. $\{P_N\} \subset M_1(X)$. $\{a_N\}$ is a sequence of positive numbers tending to $\infty$. $I$ is a function from $X$ to $[0,\infty)$. Usually $(X, P_N, a_N)$ is said to be a large deviation system with action functional $I$ if

i). for every open subset $G$ of $X$

$$\liminf_{N\to\infty} a_N^{-1} \log P_N(G) \geq -\inf_{x\in G} I(x). \tag{2.6}$$

ii). for every closed subset $F$ of $X$

$$\limsup_{N\to\infty} a_N^{-1} \log P_N(F) \leq -\inf_{x\in F} I(x). \tag{2.7}$$

iii). the level sets $\{x \in X : I(x) \leq s\}$ are compact for all $s \geq 0$.

Note: The infimum is replaced by $+\infty$ if the set is empty.

Let $\tilde{\mathcal{P}}_u^{(N)}(\cdot) = P_u^{(N)} \circ \varepsilon_{x^{(N)}(\cdot)}^{-1}$. $C_0^{1,0}([0,T] \times E)$ denotes the set of all continuous functions on $[0,T] \times E$ with compact support and first order continuous derivative with respect to $t$.

$\forall u, \nu \in M_1(E)$, define

$$S_u^{\tilde{\mu}(\cdot)}(\mu(\cdot)) = \sup\{\langle \mu(0), g\rangle - \log\langle u, e^g\rangle; g \in C_b(E)\}$$

$$+ \sup_{f\in C_0^{1,0}([0,T]\times E)} \{\langle \mu(T), f(T)\rangle - \langle \mu(0), f(0)\rangle - \int_0^T \langle \mu(s), \frac{\partial f(s)}{\partial s} + e^{-f(s)} Q_{\bar\mu(s)} e^{f(s)}\rangle ds\}$$

$$S_u(\mu(\cdot)) = S_u^{\mu(\cdot)}(\mu(\cdot)).$$

We assume in the following that $u$ satisfies $\eta = \int_E e^{\varphi(x)} u(dx) < \infty$. The main result of this article is:

**Theorem 2.2** *Under assumptions* (2.1)-(2.5), $(\mathcal{D}_\infty, \hat{\mathcal{P}}_u^{(N)}, N)$ *is a large deviation system with action functional* $S_u(\cdot)$.

The following lemma will be frequently used in the sequel.

**Lemma 2.3** *Assume conditions* (2.1)-(2.5) *are satisfied. Then for any* $r > 0$, *there exists an* $R_0 > 0$ *such that for all* $R \geq R_0$ *and* $N \geq 1$, *we have*

$$\mathcal{P}_u^{(N)}\{\mathcal{D}_\infty \backslash \mathcal{D}_R\} \leq \exp(-Nr). \tag{2.8}$$

**3. Lower Bound and Compactness of Level Sets.** The main result of this section is

**Lemma 3.4** *For every* $\bar\mu(\cdot) \in \mathcal{D}_\infty$ *and any open neighborhood* $V$ *of* $\bar\mu(\cdot)$, *we have*

$$\liminf_{N\to\infty} \frac{1}{N} \log \hat{\mathcal{P}}_u^{(N)}(V) \geq -S_u(\bar\mu(\cdot)). \tag{3.1}$$

This result is obtained by a series of lemmas.

For $\bar\mu(\cdot) \in \mathcal{D}_\infty$, let $P_{\bar\mu(\cdot),(s,x)}$ be the unique solution of the martingale problem for $\{Q_{\bar\mu(t)}; t \in [0,T]\}$ with initial distribution $\delta_x$ at time $s$. For $0 \leq s < t \leq T, u, \nu \in M_\varphi(E), f \in C_b(E)$, we define

$$l_{\bar\mu(\cdot)}^{s,t} : M_1(E) \times M_1(E) \times C_b(E) \longrightarrow R^1,$$

$$l_{\bar\mu(\cdot)}^{s,t}(u, \nu; f) = \langle \nu, f(x(t))\rangle - \langle u, \log E^{P_{\bar\mu(\cdot),(s,x)}}[e^{f(x(t))}]\rangle.$$

For $0 = t_0 < t_1 < \cdots < t_M = T$, we denote by $\pi^{(N)}(t_0, t_1, \cdots, t_M)$ the joint distribution of $(\varepsilon_{x^{(N)}(t_0)}, \cdots, \varepsilon_{x^{(N)}(t_M)})$ on $M_1(E)^{\otimes(M+1)}$ under $P_{\bar\mu(\cdot),u}^{(N)}$ (the $N$-fold independent product of $P_{\bar\mu(\cdot),u}$), $p(t_0, t_1, \cdots, t_M)$ the joint distribution of $(x(t_0), \cdots, x(t_M))$ on $E^{\otimes(M+1)}$ under $P_{\bar\mu(\cdot),u}$. Then we have

**Lemma 3.5** *If* $M_1(E)$ *is endowed with the weak topology, then for any* $M \geq 1, 0 = t_0 < t_1 < \cdots < t_M = T, (M_1(E)^{\otimes(M+1)}, \pi^{(N)}(t_0, t_1, \cdots, t_M), N)$ *is a large deviation system with action functional* $L_{\bar\mu(\cdot),u}^{t_0,\cdots,t_M}$ *defined by*

$$\begin{aligned} L_{\bar\mu(\cdot),u}^{t_0,\cdots,t_M} &= \sup_{g\in C_b(E)} l_u(\mu_0; g) + \sum_{k=0}^{M-1} \sup_{f\in C_b(E)} l_{\bar\mu(\cdot)}^{t_k,t_{k+1}}(\mu_k, \mu_{k+1}; f) \\ &= \sup_{g\in C_b(E)} l_u(\mu_0; g) + \sum_{k=0}^{M-1} \sup_{f\in C_k(E)} l_{\bar\mu(\cdot)}^{t_k,t_{k+1}}(\mu_k, \mu_{k+1}; f). \end{aligned} \tag{3.2}$$

*The second equality holds because* $C_k(E)$ *is pointwise dense in* $C_b(E)$.

Let $\hat{\mathcal{P}}^{(N)}_{\bar{\mu}(\cdot),u}(\cdot) = P^{(N)}_{\bar{\mu}(\cdot),u} \circ \epsilon^{-1}_{x^{(N)}(\cdot)}$. We have

**Lemma 3.6** *For every $\gamma > 0, \bar{\mu}(\cdot) \in \mathcal{D}_\infty \cap C([0,T], M_1(E))$ and open neighborhood $V$ of $\bar{\mu}(\cdot)$ in $\mathcal{D}_\infty$ with the inductive topology, there exist finitely many $0 = t_0 < t_1 < \cdots < t_M = T$ and open neighborhood $\bar{V}_i$ of $\bar{\mu}(t_i), i = 0, \cdots, M-1$ in $M_1(E)$ with vague topology such that*

$$\limsup_{N\longrightarrow\infty} \frac{1}{N} \log P^{(N)}_{\bar{\mu}(\cdot),u}(\{\epsilon_\cdot \notin V\} \cap \bigcap_{i=0}^{M-1} \{\epsilon_\cdot(t_i) \in \bar{V}_i\}) \leq -\gamma. \tag{3.3}$$

**Lemma 3.7** *If $\mu(\cdot) \in \mathcal{D}_\infty, S^{\bar{\mu}(\cdot)}_u(\mu(\cdot)) < \infty$, then there exists an integrable function $h_{0,T}$ such that*

$$\langle \mu(t), g\rangle - \langle \mu(s), g\rangle = \int_s^t h_{0,T}(r)dr + \int_s^t \langle \mu(r), Q_{\bar{\mu}(r)} g(r)\rangle dr. \tag{3.4}$$

**Lemma 3.8** *For every $\hat{\mu}(\cdot) \in \mathcal{D}_\infty$ and open neighborhood $V$ of $\hat{\mu}(\cdot)$ in $\mathcal{D}_\infty$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{N} \log \hat{\mathcal{P}}^{(N)}_{\bar{\mu}(\cdot),u}(V) \geq -\tilde{S}^{\bar{\mu}(\cdot)}_u(\hat{\mu}), \tag{3.5}$$

$$\tilde{S}^{\bar{\mu}(\cdot)}_u(\mu(\cdot)) = \sup_{0=t_0<\cdots<t_M, M\geq 1} L^{t_0,\cdots,t_M}_{\bar{\mu}(\cdot),u}(\mu(t_0), \cdots, \mu(t_M)) \tag{3.6}$$

**Lemma 3.9** *For every $\bar{\mu}(\cdot), \mu(\cdot) \in \mathcal{D}_\infty$ we have*

$$\tilde{S}^{\bar{\mu}(\cdot)}_u(\mu(\cdot)) \leq S^{\bar{\mu}(\cdot)}_u(\mu(\cdot)). \tag{3.7}$$

The following lemma shows the compactness of level sets.

**Lemma 3.10** *Let us assume (2.1)-(2.5). Then for any $\gamma > 0$, the level set $\Phi_u(\gamma) = \{\mu(\cdot) \in \mathcal{D}_\infty; S_u(\mu(\cdot)) \leq \gamma\}$ is compact in $\mathcal{D}_\infty$.*

## 4. Upper Bound.

The main result of this section is

**Theorem 4.11** (Upper bound on $\mathcal{D}_\infty$) Under assumptions (2.1)-(2.5), for any closed subset $A$ of $\mathcal{D}_\infty$ we have

$$\limsup_{N\longrightarrow\infty} \frac{1}{N} \log \hat{\mathcal{P}}^{(N)}_u(A) \leq -\inf_{\mu\in A} S_u(\mu). \tag{4.1}$$

We prove this theorem by the following three lemmas.

**Lemma 4.12** *For any $\bar{\mu}(\cdot) \in \mathcal{D}_\infty$ and $I < S_u(\bar{\mu}(\cdot))$ there exists $\delta' > 0$ and an integer $N_1$ such that $\forall N \geq N_1$,*

$$\hat{\mathcal{P}}^{(N)}_u\{r_{0T}(\mu(\cdot), \bar{\mu}(\cdot)) < \delta'\} \leq \exp[-NI]. \tag{4.2}$$

**Lemma 4.13** *For all $a \geq 0$, there exists an $R \geq 1$ and a compact subset $K_a \subset \mathcal{D}_R \cap C([0,T], M_1(E))$ such that the following is true: $\forall \delta > 0$, $\exists N_2 \geq 1$ such that $\forall N \geq N_2$*

$$\hat{\mathcal{P}}^{(N)}_u\{r_{0T}(\mu(\cdot), K_a) \geq \delta\} \leq \exp[-Na]. \tag{4.3}$$

**Lemma 4.14** *Assume* (2.1)-(2.5) *are satisfied.* $\forall\gamma > 0, \varepsilon > 0, s \geq 0$, *there exists an integer* $N_0$ *such that* $\forall N \geq N_0$,

$$\hat{\mathcal{P}}_u^{(N)}\{r_{0T}(\mu(\cdot), \Phi_u(s)) \geq \varepsilon\} \leq \exp[-N(s-\gamma)]. \tag{4.4}$$

*where* $r_{0T}(\mu(\cdot), \Phi_u(s)) = \inf_{\nu(\cdot)\in\Phi_u(s)} \sup_{0\leq t\leq T} r(\mu(t), \nu(t))$.

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# TWO THEOREMS ON REPRESENTATIONS OF $q$-DEFORMED CENTERLESS VIRASORO ALGEBRAS

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**Abstract** *A new $q$-deformed centerless Virasoro algebra $U_q(W)$ is presented and two theorems on representations of $q$-deformed centerless Virasoro algebras are obtained.*

## §1. The classification of a class of $CZ_q$-modules

T.L.Curtright and C.K.Zachos introduced a $q$-deformed centerless algebra $CZ_q$ in [C-Z]. The associative algebra $CZ_q$ can be described as follows:

Generators: $\{ z_m \mid m \in \mathbf{Z} \}$,

Relations: $q^{m-n} z_m z_n - q^{n-m} z_n z_m = [m-n] z_{m+n}$ for $m, n \in \mathbf{Z}$,

where $q \in \mathbf{C}$ with $q^2 \neq 0, 1$, $[m] := \dfrac{q^m - q^{-m}}{q - q^{-1}}$ for $m \in \mathbf{Z}$.

**Definition** A $U_q(Vir)$-module $V$ is called a **Harish–Chandra module** if

$(i)$ $\ell_0$ acts semisimply on $V$,

$(ii)$ all eigenspaces of $\ell_0$ are finite-dimensional,

$(iii)$ $\dfrac{1}{q-q^{-1}} \notin spec(\ell_0)$.

The eigenspaces of $\ell_0$ is called the **weight spaces** of $V$.

We have found the following $CZ_q$-modules

$$\begin{cases} V_{ab} := \oplus_{k \in \mathbf{Z}} \mathbf{C} v_k, \\ z_n(v_k) := \dfrac{1 + aq^{2n+2k} + bq^{2k}}{q - q^{-1}} v_{n+k}; \end{cases}$$

$$\begin{cases} \mathcal{A}_\alpha^{\pm} := \oplus_{k \in \mathbf{Z}} \mathbf{C} v_k, \\ \ell_n(v_k) := -[n+k+1] q^{n+k+1} v_{n+k}, & \text{for } k \neq -1, \\ \ell_n(v_k) := -([n]q^n + [n][n \pm 1]\alpha q^{2n-1}) v_{n-1}, & \text{for } k = -1; \end{cases}$$

$$\begin{cases} \mathcal{B}_\alpha^\pm := \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, & \\ \ell_n(v_k) := -[k]q^k v_{n+k}, & \text{for } n+k\neq 0, \\ \ell_n(v_k) := ([n]q^{-n} + [n][n\pm 1]\alpha q^{-2n})v_0, & \text{for } n+k=0, \end{cases}$$

where $n, k \in \mathbf{Z}$ and $a, b, \alpha \in \mathbf{C}$.

**Theorem 1.** *Let $q$ be not a root of unity. If $V$ is an indecomposable Harish-Chandra $CZ_q$-module with one-dimensional weight spaces, then there exist some $a, b, \alpha \in \mathbf{C}$ such that $V$ is isomorphic to one of the $U_q(Vir)$-modules : $V_{ab}$, $\mathcal{A}_\alpha^\pm$, $\mathcal{B}_\alpha^\pm$.*

## §2. A new $q$-deformed centerless Virasoro algebra $U_q(W)$

Let $A$ be an algebra and $\sigma \in Aut(A)$. A linear map $D : A \to A$ is called a **$\sigma$-derivation (or a skew-derivation)** if

$$D(ab) = D(a)b + \sigma(a)D(b) \qquad \text{for all } a, b \in A.$$

The set of all $\sigma$-derivations of $A$ is denoted by $Der_\sigma A$.

If $H$ is a Hopf algebra with co-mutiplication $\Delta : H \to H \otimes H$, given by $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$, then an algebra $A$ is an **$H$-module algebra** if $A$ is an $H$-module, and $H$ "measures" $A$; that is, $h \cdot 1 = \varepsilon(h)1$ and

$$h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b) \qquad \text{for } a, b \in A,$$

where $\varepsilon$ is the counit.

Now we present a new $q$-deformed centerless Virasoro algebra $U_q(W)$ for $q^4 \neq 0, 1$. The associative algebra $U_q(W)$ is defined as follows:

Generators: $\{E_m, K, K^{-1} \mid m \in \mathbf{Z}\}$,

Relations: $KE_mK^{-1} = q^{2m}E_m, \quad E_0 = \dfrac{K - K^{-1}}{q^2 - q^{-2}},$

$$E_mE_n - E_nE_m = \frac{[n-m]E_{n+m}(q^{n+m}K + q^{-n-m}K^{-1})}{[2]},$$

where $m, n \in \mathbf{Z}$.

One of advantages of the algebra $U_q(W)$ is that the subalgebra $U_{q,m}(W)$ generated by $\{E_m, E_{-m}, K, K^{-1}\}$ is isomorphic to $U_{q^m}(s\ell(2))$ for each $m \in \mathbb{Z}_{\geq 1}$, where $U_q(s\ell(2))$ is the quantum group introduced by P.P.Kulish and N.Reshetikhin [K-R]. Hence, all those subalgebras $U_{q,m}(W)$ are Hopf algebra for $m \in \mathbb{Z}_{\geq 1}$.

As an application of this behavior of the algebra $U_q(W)$, we can completely describe $U_q(W)$-module actions on the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$ in the following case:

**Theorem 2.** *Let $q$ be not a root of unity and $\varphi : U_q(W) \to End(\mathbb{C}[t,t^{-1}])$ a representation. If $\mathbb{C}[t,t^{-1}]$ is a $U_{q,m}(W)$-module algebra for $m \in \mathbb{Z}_{\geq 1}$, then $\varphi(K^2) \in Aut(\mathbb{C}[t,t^{-1}])$, and there exist $a, b \in \mathbb{C} \setminus \{0\}$, $u \in \mathbb{Z} \setminus \{0\}$ such that*

$$\varphi(K) : t^m \mapsto s(m)a^m t^m, \qquad \varphi(E_m) = \frac{a^2-1}{q^2-q^{-2}} b^m t^{mu+1} \partial \varphi(K^{-1}) \quad \text{for } m \in \mathbb{Z},$$

*where $\partial \in Der_{\varphi(K^2)}(\mathbb{C}[t,t^{-1}])$ with $\partial(t) = 1$ and $s : \mathbb{Z} \to \{\pm 1\}$ is a map satisfying $q^{2m} = s(n)s(mu+n)a^{mu}$ for $m, n \in \mathbb{Z}$.*

**Remark.** $\varphi(K) \in Aut(\mathbb{C}[t,t^{-1}])$ if and only if $s : \mathbb{Z} \to \{\pm 1\}$ is a group homormophism.

Details of the proofs of these results will appear elsewhere.

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# SOLUTIONS OF SOME FUNCTIONAL INEQUALITIES CONNECTED WITH CONVEX FUNCTIONS

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*Presented by J. Aczel, F.R.S.C.*

*Abstract.* With each $T > 0$ we associate the class of functions satisfying the following functional inequality which, for $T = 1$, reduces to the definition of convex functions:

$$f(tx + (T-t)y) \leq tf(x) + (T-t)f(y), \quad x, y \in J; \ t \in [0, T],$$

where $J = (0, +\infty)$ or $J = [0, +\infty)$. Making use of a sandwich type theorem we establish some relationships between these classes. It turns out that these properties depend essentially upon $J$.

## Introduction

Let $T$ be a fixed positive real number. Denote by $J$ either $[0, +\infty)$ or $(0, \infty)$. In a recent paper [2] the following *sandwich type* result was proved:

**Theorem 0.** *A function $f : J \to \mathbb{R}$ satisfies the inequality*

$$\text{(T)} \qquad f(tx + (T-t)y) \leq tf(x) + (T-t)f(y), \quad x, y \in J; \ t \in [0, T],$$

*if and only if there exists a convex function $g : J \to \mathbb{R}$ such that*

$$T^{-1}g(Tx) \leq f(x) \leq g(x), \quad x \in J.$$

This theorem not only completely characterizes the solutions of the inequality $(T)$ but gives a simple method of construction of solutions of the inequality $(T)$ as well. For instance, taking $T \in (0,1)$, and $g(x) = x^2$, $x \in [0, +\infty)$, we infer that every function $f : [0, +\infty) \to \mathbb{R}$ such that

$$Tx^2 \leq f(x) \leq x^2, \quad x \in [0, +\infty),$$

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is a solution of inequality $(T)$. Similarly, taking $T > 1$ and $g(x) = x^{-1}$, $x \in (0, +\infty)$, we see that every function $f : (0, +\infty) \to \mathbb{R}$ such that

$$T^{-2}x^{-1} \leq f(x) \leq x^{-1}, \quad x \in (0, +\infty),$$

satisfies $(T)$ (cf. [2]). Clearly, for $T = 1$ the set of solutions of the inequality $(T)$ coincides with the class of convex functions.

The aim of this note is to describe how the class of all solutions of the inequality $(T)$ depends on $T$. It turns out that the behaviour of these classes in the case $J = (0, +\infty)$ is essentially different from that in the case $J = [0, +\infty)$.

## 1. The case $J = (0, +\infty)$

Given a positive real number $T$, we denote by $W(T)$ the class of all solutions of $(T)$ on the interval $J = (0, +\infty)$:

$$W(T) := \{f : (0, +\infty) \to \mathbb{R} \mid f \quad \text{satisfies } (T)\}.$$

The main result of this paragraph reads as follows.

**Theorem 1.** *If $0 < T_1 < T_2 < 1$ then $W(T_2) \subset W(T_1)$. If $1 < T_1 < T_2$ then $W(T_1) \subset W(T_2)$.*

In the proof we need the following

**Lemma.** *If a function $g : (0, +\infty) \to \mathbb{R}$ is convex and there exists a constant $T \in (0, 1)$ (resp. $T \in (1, +\infty)$) such that*

$$g(Tx) \leq Tg(x), \quad x \in (0, +\infty), \tag{1}$$

*then the function $x \to g(x)/x$ is nonde creasing (resp. nonincreasing).*

*Proof.* Since $g$ is convex, the function $x \to g(x)/x$ is either monotonic or there exists a point $s \in (0, +\infty)$ such that it is nonincreasing on $(0, s)$ and nondecreasing on $(s, +\infty)$ (cf. [1. Lemma]). If (1) holds with $T \in (0, 1)$, then for each fixed $x_0 \in (0, +\infty)$ we have

$$\frac{g(T^n x_0)}{T^n x_0} \leq \frac{g(x_0)}{x_0}, \quad n \in \mathbb{N}, \tag{2}$$

and the sequence $(T^n x_0)$ tends to zero. Hence $g(x)/x$ cannot be decreasing on any interval $(0,s) \subset (0,+\infty)$. Therefore this function is nondecreasing on $(0,+\infty)$. In the second case, if (1) holds with $T \in (1,+\infty)$, then for every $x_0 \in (0,+\infty)$ we have also (2), but now the sequence $(T^n x_0)$ tends to $+\infty$. This implies that the function $g(x)/x$ cannot be increasing on any interval $(s,+\infty) \subset (0,+\infty)$. So $g(x)/x$ is nonincreasing on $(0,+\infty)$.

*Proof of Theorem 1.* Take $0 < T_1 < T_2 < 1$, and $f \in W(T_2)$. By the Theorem 0, there exists a convex function $g : (0,+\infty) \to \mathbb{R}$ such that

$$T_2^{-1} g(T_2 x) \le f(x) \le g(x), \quad x \in (0,+\infty). \tag{3}$$

Hence $g$ satisfies (1) with $T_2 \in (0,1)$ and, in view of the Lemma, $g(x)/x$ is nondecreasing. In particular,

$$g(T_1 x)/T_1 x \le g(T_2 x)/T_2 x, \quad x \in (0,+\infty).$$

Combining this with (3) we get

$$T_1^{-1} g(T_1 x) \le f(x) \le g(x), \quad x \in (0,+\infty),$$

which, in view of the Theorem, implies that $f \in W(T_1)$. The proof in the remaining case is analogous.

*Remark 1.* The class $W(1)$ (of all the convex functions on $(0,+\infty)$) is not included in any $W(T)$ for $T \neq 1$. In particular, positive constant functions do not belong to $W(T)$ with $T \in (0,1)$, and negative constant functions do not belong to $W(T)$ with $T \in (1,+\infty)$.

## 2. The case $J = [0,+\infty)$

Now, given a positive real number $T$, put

$$W^0(T) := \{f : [0,+\infty) \to \mathbb{R} \mid \ f \ \text{ satisfies } (T) \ \text{ and } f(0) = 0\}$$

We have the following

**Theorem 2.** *If* $0 < T_1 < T_2$, *then* $W^0(T_2) \subset W^0(T_1)$. *Moreover, for every* $T > 1$,

$$W^0(T) = \{f : [0,+\infty) \to \mathbb{R} \mid \ f(x) = f(1)x, \ x \in [0,+\infty)\}.$$

*Proof.* Take $0 < T_1 < T_2$, and $f \in W^0(T_2)$. By the Theorem 0, there exists a convex function $g : [0, +\infty) \to \mathbb{R}$ such that

$$T_2^{-1} g(T_2 x) \le f(x) \le g(x), \quad x \in [0, +\infty). \tag{4}$$

Since $f(0) = 0$, putting here $x = 0$ we get $g(0) = 0$. It follows that the function $x \to g(x)/x$ is nondecreasing on $(0, +\infty)$ (cf. [4, Theorem 11B ] or [3, Theorem 2. §3, chapter VII]). Therefore

$$g(T_1 x)/T_1 x \le g(T_2 x)/T_2 x, \quad x \in (0, +\infty).$$

Hence, using (4), we get

$$T_1^{-1} g(T_1 x) \le f(x) \le g(x), \quad x \in [0, +\infty),$$

which, in view of the Theorem 0, implies that $f \in W^0(T_1)$.

Now assume that $T > 1$ and fix an $f \in W^0(T)$. By the Theorem 0, there is a convex function $g : [0, +\infty) \to \mathbb{R}$ such that

$$T^{-1} g(Tx) \le f(x) \le g(x), \quad x \in (0, +\infty). \tag{5}$$

Hence

$$g(Tx)/Tx \le g(x)/x, \quad x \in (0, +\infty).$$

On the other hand, the function $x \to g(x)/x$ is nondecresing and, consequently,

$$g(x)/x \le g(Tx)/Tx \quad x \in (0, +\infty).$$

Thus $g(Tx)/Tx = g(x)/x$ for $x \in (0, +\infty)$, and hence

$$g(T^n x)/T^n x = g(x)/x, \quad x \in (0, +\infty); \ n \in \mathbb{N}. \tag{6}$$

The monotonicity of the function $g(x)/x$ implies that the following limit exists (it may be infinite)

$$c := \lim_{x \to +\infty} (g(x)/x)$$

It follows that, for every $x \in (0, +\infty)$,

$$\lim_{n \to +\infty} g(T^n x)/T^n x = c.$$

On the other hand, in view of (6), for all $x \in (0, +\infty)$

$$\lim_{n \to +\infty} g(T^n x)/T^n x = g(x)/x.$$

Thus $g(x) = cx$, $x \in (0, +\infty)$. By (5) we infer that also $f(x) = cx$ for all $x \in [0, +\infty)$, which was to be shown.

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