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## Markov uniqueness and its applications to martingale problems, stochastic differential equations and stochastic quantization

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Presented by Donald A. Dawson, F.R.S.C. .

**Abstract.** We prove that for a class of positive finite measures  $\mu$  on a locally convex topological vector space  $E$  "Markov uniqueness" holds for the corresponding Dirichlet operators  $\Delta_\infty + \beta \cdot \nabla$  on  $L^2(E; \mu)$ . Here  $\beta$  is the logarithmic derivative of  $\mu$  associated with a rigging  $E' \subset H \subset E$  for some Hilbert space  $H$ . Markov uniqueness means that there exists exactly one self-adjoint operator  $L$  generating a sub-Markovian semigroup on  $L^2(E; \mu)$  and  $L$  is on smooth cylinder functions of the above type. As a consequence we prove that we have Markov uniqueness for all positive finite measures  $\mu$  on  $E$  admitting an integration by parts formula provided  $H$  is chosen properly. We also discuss applications to show uniqueness of the corresponding martingale problem under symmetry assumptions and of the stochastic quantization of infinite volume quantum fields.

Before we state our results we first present the finite dimensional case solved in [RZ 92a,b] (see also [R 91]) and recall our (infinite dimensional) framework. Let  $dx$  denote Lebesgue measure on  $\mathbb{R}^d$  and let  $\varphi$  be locally in  $H^{1,2}(\mathbb{R}^d; dx)$  (= (real) Sobolev space of order 1 in  $L^2(\mathbb{R}^d; dx)$ ). Consider the operator  $S_\varphi$  on  $L^2(\mathbb{R}^d; \varphi^2 dx)$  defined by

$$S_\varphi := \Delta + 2\varphi^{-1}(\nabla\varphi, \nabla \cdot)_{\mathbb{R}^d}$$

$$D(S_\varphi) := C_0^\infty(\mathbb{R}^d)$$

where  $C_0^\infty(\mathbb{R}^d)$  denotes the set of all differentiable functions on  $\mathbb{R}^d$  with compact support and  $\Delta$  is the Laplacian on  $\mathbb{R}^d$ . Note that  $\varphi \in H_{loc}^{1,2}(\mathbb{R}^d; dx)$  is necessary and sufficient so that  $C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d; \varphi^2 \cdot dx)$  and so that  $S_\varphi$  maps  $C_0^\infty(\mathbb{R}^d)$  into  $L^2(\mathbb{R}^d; \varphi^2 dx)$ . The following was proved in [RZ 92b].

**Theorem 1.** *There exists exactly one negative definite self-adjoint operator  $L_\varphi$  on  $L^2(\mathbb{R}^d; \varphi^2 dx)$  which extends  $(S_\varphi, C_0^\infty(\mathbb{R}^d))$ , such that  $T_t := e^{tL_\varphi}$ ,  $t > 0$ , is sub-Markovian (that is  $0 \leq u \leq 1$  implies  $0 \leq T_t u \leq 1$  for all  $u \in L^2(\mathbb{R}^d; \varphi^2 \cdot dx)$ ,  $t > 0$ ).*

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The uniqueness in Theorem 1 is referred to as *Markov* or *Dirichlet uniqueness*, since a negative definite self-adjoint operator  $L$  on  $L^2(\mathbb{R}^d; \varphi^2 dx)$  which generates a sub-Markovian semigroup is called a *Dirichlet operator* (cf. [MR 92, Chap.I, Sect. 4]). The Dirichlet form  $(\mathcal{E}_\varphi, D(\mathcal{E}_\varphi))$  corresponding to  $L_\varphi$  in Theorem 1 is the closure on  $L^2(\mathbb{R}^d; \varphi^2 \cdot dx)$  of

$$(1) \quad \mathcal{E}_\varphi(u, v) = \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d} \varphi^2 dx; \quad u, v \in C_0^\infty(\mathbb{R}^d).$$

(cf. [F 80], [MR 92, Chap.I, Sect. 4]). It provides a means to formulate and study the Markov uniqueness problem in infinite dimensions.

Let  $E$  be a separable real Banach space with dual  $E'$  and dualization  $E' \langle \cdot, \cdot \rangle_E$ . Let  $\mathcal{B}(E)$  denote its Borel  $\sigma$ -algebra and let  $\mu$  be a finite positive measure on  $(E, \mathcal{B}(E))$  with  $\text{supp}[\mu] = E$ . We emphasize, however, that as in [RZ 92 a,b], [ARZ 92] all results below carry over to more general locally convex topological vector spaces  $E$  with possibly  $\text{supp}[\mu] \neq E$ . Define for  $K \subset E'$  the linear space

$$\mathcal{FC}_b^\infty(K) := \{f(l_1, \dots, l_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \dots, l_m \in K\}.$$

Set  $\mathcal{FC}_b^\infty := \mathcal{FC}_b^\infty(E')$ . Compared with the finite dimensional case we have that  $E, \mu, \mathcal{FC}_b^\infty(K)$  replace  $\mathbb{R}^d, dx, C_0^\infty(\mathbb{R}^d)$  respectively. To define the analogue of (1) we need a gradient  $\nabla$ . To this end fix  $u = f(l_1, \dots, l_m) \in \mathcal{FC}_b^\infty, z \in E$  and define for  $k \in E$

$$(2) \quad \frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)|_{s=0} = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(l_1(z), \dots, l_m(z)) E' \langle l_i, k \rangle_E$$

Furthermore, we assume that we are given a "tangent space"  $H$  to  $E$  at each point, i.e.,  $H$  is a separable real Hilbert space such that  $H \subset E$  continuously and densely. Thus, identifying  $H$  with its dual  $H'$  we have

$$(3) \quad E' \subset H \subset E \text{ continuously and densely.}$$

We can define  $\nabla u(z)$  to be the unique element in  $H$  such that

$$\langle \nabla u(z), h \rangle_H = \frac{\partial u}{\partial h}(z) \quad \text{for all } h \in H (\subset E).$$

Now it is possible to define a positive definite symmetric bilinear form (henceforth briefly called *form*) on (real)  $L^2(E; \mu)$  by

$$(4) \quad \mathcal{E}_{\mu, H}^0(u, v) := \int_E \langle \nabla u, \nabla v \rangle_H d\mu; \quad u, v \in \mathcal{FC}_b^\infty.$$

which is densely defined, since  $\mathcal{FC}_b^\infty$  is dense in  $L^2(E; \mu)$ . An element  $k \in E$  is called *well- $\mu$ -admissible* if there exists  $\beta_k \in L^2(E; \mu)$  such that

$$\int \frac{\partial u}{\partial k} d\mu = - \int u \beta_k d\mu \quad \text{for all } u \in \mathcal{FC}_b^\infty.$$

Assume:

- (5) There exists a dense linear subspace  $K$  of  $E'$  ( $\subset H \subset E$ ) consisting of well- $\mu$ -admissible elements in  $E$ .

(5) corresponds to the assumption  $\varphi \in H_{loc}^{1,2}(\mathbb{R}^d; dx)$  in the finite dimensional case (cf. [R 90, Subsection 4a]). Then it is easy to see that the form  $(\mathcal{E}_{\mu,H}^0, \mathcal{FC}_b^\infty)$  defined in (4) is closable on  $L^2(E; \mu)$  and that its closure  $(\mathcal{E}_{\mu,H}^0, D(\mathcal{E}_{\mu,H}^0))$  is a Dirichlet form on  $L^2(E; \mu)$  (cf. [AR 90], [MR 92, Chap.II, Subsection 3b]). We also denote the closure of  $\nabla$  with domain  $D(\mathcal{E}_{\mu,H}^0)$  by  $\nabla$ . Let  $L_{\mu,H}^0$  with domain  $D(L_{\mu,H}^0)$  be its generator (cf. [MR 92, Chap.I]).  $(L_{\mu,H}^0, D(L_{\mu,H}^0))$  is a *Dirichlet operator*, i.e.,  $(e^{tL_{\mu,H}^0})_{t \geq 0}$  is sub-Markovian. It is immediate that if  $u = f(l_1, \dots, l_m) \in \mathcal{FC}_b^\infty(K)$  and  $K_0 \subset K$  is an orthonormal basis of  $H$  having  $l_1, \dots, l_m$  in its linear span, then  $u \in D(L_{\mu,H}^0)$  and

$$(6) \quad L_{\mu,H}^0 u = \sum_{k \in K_0} \left( \frac{\partial}{\partial k} \left( \frac{\partial u}{\partial k} \right) + \beta_k \frac{\partial u}{\partial k} \right).$$

Note that the right hand side is only a finite sum and defines a linear operator  $S_{\mu,H,K}$  on  $L^2(E; \mu)$  with dense domain  $\mathcal{FC}_b^\infty(K)$ . Now we can extend the notion of "Markov uniqueness" to this infinite dimensional setting.

**Definition 2.** We say that *Markov uniqueness* holds for  $(S_{\mu,H,K}, \mathcal{FC}_b^\infty(K))$  if  $(L_{\mu,H}^0, D(L_{\mu,H}^0))$  is the only Dirichlet operator on  $L^2(E; \mu)$  extending  $(S_{\mu,H,K}, \mathcal{FC}_b^\infty(K))$ .

Let for a sub- $\sigma$ -algebra  $\mathcal{A}$  of  $\mathcal{B}$ ,  $E_\mu[\cdot | \mathcal{A}]$  denote conditional expectation w.r.t.  $\mu$  given  $\mathcal{A}$ . Now we can formulate our main results.

**Theorem 3.** *Suppose that there exist  $e_n \subset K$ ,  $n \in \mathbb{N}$ , forming an orthonormal basis in  $H$  such that for  $\mathcal{B}_N := \sigma\{E'(\langle e_n, \cdot \rangle_E) | n \leq N\}$*

$$(7) \quad \sup_{N \in \mathbb{N}} \sum_{n=1}^N \|\beta_{e_n} - E_\mu[\beta_{e_n} | \mathcal{B}_N]\|_{L^2(E; \mu)}^2 < \infty.$$

*Then Markov uniqueness holds for  $(S_{\mu,H,K}, \mathcal{FC}_b^\infty(K))$ .*

(7) is a weakened form of a condition in [T 87] where an analogue of Theorem 3 is proved under much stronger smoothness assumptions on  $\mu$  (cf. [T 87, Theorem 3]).

**Corollary 4.** *Consider the situation of Theorem 3; i.e., in particular (7) holds. Let  $\varphi \in D(\mathcal{E}_{\mu,H}^0)$  such that  $\varphi\beta_k \in L^2(E; \mu)$  for all  $k \in K$ . Then Markov uniqueness holds for  $(S_{\mu,H,K} + 2\varphi^{-1}\langle \nabla\varphi, \nabla\cdot \rangle_H, \mathcal{FC}_b^\infty(K))$  (on  $L^2(E; \varphi^2\mu)$ ).*

**Corollary 5.** *There exists a separable real Hilbert space  $\tilde{H} \subset H$  and a linear subspace  $\tilde{K} \subset K$  such that the image of  $\tilde{K}$  under the imbedding  $\tilde{K} \subset E' \subset \tilde{H}' \equiv \tilde{H} \subset E$  consists of well- $\mu$ -admissible elements in  $E$  and Markov uniqueness holds for the corresponding operator  $(S_{\mu,H,\tilde{K}}, \mathcal{FC}_b^\infty(\tilde{K}))$ .*

Now we turn to the probabilistic consequences. By [Sch 90] there exists a diffusion process  $M_{\mu,H} = (\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), (P_z)_{z \in E})$  associated with the Dirichlet form  $(\mathcal{E}_{\mu,H}^0, D(\mathcal{E}_{\mu,H}^0))$ , i.e., for all  $u \in L^2(E; \mu)$ ,  $t > 0$ ,

$$\int u(X_t) dP_z = T_t u(z) \quad \text{for } \mu\text{-a.e. } z \in E$$

where  $T_t u := \exp(tL_{H,\mu}^0)u$ .  $M_{\mu,H}$  is easily seen to be conservative (i.e., has infinite lifetime) and to have  $\mu$  as an invariant measure.

**Definition 6.** Let  $D \subset L^2(E; \mu)$  such that each  $u \in D$  has a  $\mu$ -version  $\tilde{u}$  which is continuous. Let  $A$  be a linear operator on  $L^2(E; \mu)$  having  $D$  in its domain. We say that a  $\mu$ -symmetric right process  $M = (\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), (P_z)_{z \in E})$  with state space  $E$  solves the martingale problem for  $(A, D)$  if for every  $u \in D$

$$(8) \quad \tilde{u}(X_t) - \tilde{u}(X_0) - \int_0^t Au(X_s) ds, \quad t \geq 0,$$

is an  $(\mathcal{F}_t)_{t \geq 0}$ -martingale under  $P_\mu := \int P_z \mu(dz)$ .

[MR 92, Chap.IV, Theorem 6.4] then implies.

**Theorem 7.** *Suppose Markov uniqueness holds for  $(S_{\mu,H,K}, \mathcal{FC}_b^\infty(K))$ . Then there exists (up to  $\mu$ -equivalence) exactly one  $\mu$ -symmetric right process  $M$  with state space  $E$  which solves the martingale problem for  $(S_{\mu,H,K}, \mathcal{FC}_b^\infty(K))$ .  $M$  is, in fact, the conservative diffusion process  $M_{\mu,H}$  introduced above. This holds, in particular, in the situations described in Theorem 3 and Corollaries 4, 5.*

In the situation described before Theorem 3 it was proved in [AR 91, cf. Theorem 6.6] that if  $\int_{E'}(k, z)_E^2 \mu(dz) < \infty$  for all  $k \in K$  and if there exists a Brownian semigroup on  $E$  over  $H$  (i.e. with covariance given by  $2(\cdot, \cdot)_H$ ) then  $M_{\mu, H}$  satisfies the stochastic equation

$$(9) \quad X_t = z + W_t + N_t, \quad t \geq 0, P_z\text{-a.s.}$$

for  $\mathcal{E}_{\mu, H}^0$ -quasi-every  $z \in E$  (cf. [MR 92, Chap.III, Sect.2] for the notion of  $\mathcal{E}_{\mu, H}^0$ -quasi-everywhere). Here  $(W_t)_{t \geq 0}$  is an  $(\mathcal{F}_t)$ -Brownian motion on  $E$  with covariance  $2(\cdot, \cdot)_H$  starting at the origin under  $P_z$  and  $(N_t)_{t \geq 0}$  is a continuous  $E$ -valued  $(\mathcal{F}_t)$ -adapted process such that for every  $k \in K$ ,  $E'(k, N_t)_E = \int_0^t \beta_k(X_s) ds$ ,  $t \geq 0$ ,  $P_z$ -a.s. for  $\mathcal{E}_{\mu, H}^0$ -quasi-every  $z \in E$ . By using Itô's formula (in finite dimensions!) it follows by Theorem 7 that  $M_{\mu, H}$  is (up to  $\mu$ -equivalence) the only  $\mu$ -symmetric right process satisfying (9) if Markov uniqueness holds for  $(S_{\mu, H, K}, \mathcal{F}C_b^\infty(K))$ .

Corollary 5 obviously applies, in particular, to the examples studied in detail in [AR 91, Subsection 7 II]. There  $\mu$  was an (*infinite volume*) *space time* resp. *time zero quantum field* with  $E$  a Banach space of tempered Schwartz distributions on  $\mathbb{R}^d$ ,  $H := L^2(\mathbb{R}^d; dx)$  and  $K := C_0^\infty(\mathbb{R}^d)$  with  $d = 2$  resp.  $d = 1$ . We emphasize, however, that it seems unlikely that (7) holds for these  $E, H, \mu$ . Therefore, to have uniqueness we change the Hilbert space  $H$  into  $\tilde{H}$ , and  $K$  into  $\tilde{K}$  as in Corollary 5. Nevertheless, we have thus constructed a solution  $M_{\mu, \tilde{H}}$  of (9) (with  $\tilde{H}, \tilde{K}$ ) which is within the class of  $\mu$ -symmetric right processes (up to  $\mu$ -equivalence) unique, is conservative, and has  $\mu$  as an invariant measure.  $M_{\mu, \tilde{H}}$  is sometimes called the *stochastic quantization* of  $\mu$ .

Detailed proofs of the above results can be found in [ARZ 92]. All results of this paper have been announced in [ARZ 92], at the "Conference on Measure-valued Processes, Stochastic Partial Differential Equations and Interacting Systems", Montreal, 11-16 October, 1992, and the meeting "Stochastische Analysis", Oberwolfach, 26-31 October, 1992.

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**Extremal Measures For q-Hermite Polynomials When  $q > 1$**

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**Abstract.** We characterize the solutions of the indeterminate moment problem associated with the continuous q-Hermite polynomials when  $q > 1$ . The extremal measures are found explicitly. As a byproduct we evaluate several q beta integrals and their discrete analogs. A system of biorthogonal rational functions is also introduced.

**1. Main Results.** The continuous q-Hermite polynomials  $\{H_n(x|q)\}$  of L. J. Rogers are orthogonal on  $[-1, 1]$  when  $|q| < 1$ , [3], [4]. To accommodate the case  $q > 1$ , Askey [3] introduced the polynomials  $\{h_n(x|q)\}$ ,  $h_n(x|q) := i^{-n} H_n(ix|1/q)$ , which satisfy the three term recurrence relation

$$(1.1) \quad h_{n+1}(x|q) = 2xh_n(x|q) - q^{-n}(1 - q^n)h_{n-1}(x|q), \quad n > 0, \quad 0 < q < 1.$$

The initial conditions are  $h_0(x|q) = 1, h_1(x|q) = 2x$ . The moment problem associated with (1.1) is indeterminate; that is, there are infinitely many probability measures with respect to which these q-Hermite polynomials are orthogonal. There is a one to one correspondence between solutions of an indeterminate moment problem  $\{d\psi(x; \sigma)\}$  and functions  $\sigma(z)$  analytic in the open upper half plane which map  $0 < \text{Im } z$  into  $\text{Im } \sigma(z) \leq 0$ , [11]. Furthermore [11] there exist four entire functions  $A(z), B(z), C(z), D(z)$  with real and simple zeros such that the Stieltjes transform of  $d\psi(x, \sigma)$  is given by

$$(1.2) \quad \frac{A(z) - \sigma(z)C(z)}{B(z) - \sigma(z)D(z)} = \int_{-\infty}^{\infty} \frac{d\psi(t; \sigma)}{z - t},$$

where  $\sigma$  is as above. The entire functions also satisfy

$$(1.3) \quad A(z)D(z) - B(z)C(z) = 1.$$

The spectral theory of orthogonal polynomials is essentially the spectral theory of symmetric second order difference operators or infinite symmetric Jacobi matrices, [12]. The Jacobi matrix associated with (1.1) is

$$J = (a_{mn}), \quad a_{m,m+1} = a_{m+1,m} = \frac{1}{2}\sqrt{q^{-m-1} - 1}, \quad \text{otherwise } a_{m,n} = 0, m, n \geq 0.$$

The operator  $J$  is unbounded and closed on  $l^2$  but does not have a unique self-adjoint extension. The extremal measures are precisely the spectral measures corresponding to orthogonal spectral

resolutions of the identity. The operator  $J$  is a Schrodinger operator and our analysis seems to be the first complete analysis of an unbounded Schrodinger operator not possessing a unique self-adjoint extension, [12].

The purpose of this note is to announce some of the results of our forthcoming paper [9]. In this section we state some of our main results and in Section 2 we include some of their consequences.

**Theorem 1.** *In the case of the moment problem associated with the polynomials  $\{h_n(x|q)\}$  the functions  $A(x)$ ,  $B(x)$ ,  $C(x)$ ,  $D(x)$  are given by*

$$(1.4) \quad A(x) = \frac{4xq(q^2; q^2)_\infty}{(1-q)(q; q^2)_\infty} {}_2\phi_1(qe^{2\xi}, qe^{-2\xi}; q^2, q^2), \quad C(x) = {}_2\phi_1(e^{-2\xi}, e^{2\xi}; q; q^2, q^2),$$

$$(1.5) \quad B(x) = -(q; q^2)_\infty^{-2} \prod_{n=0}^{\infty} [1 - (4x^2 + 2)q^{2n+1} + q^{4n+2}] = -(q; q^2)_\infty^{-2} (qe^{2\xi}, qe^{-2\xi}; q^2)_\infty,$$

$$(1.6) \quad D(x) = \frac{x}{(q; q)_\infty} \prod_{n=0}^{\infty} [1 - (4x^2 + 2)q^{2n+2} + q^{4n+4}] = \frac{x}{(q; q)_\infty} (q^2 e^{2\xi}, q^2 e^{-2\xi}; q^2)_\infty,$$

where  $x = \sinh \xi$ .

In (1.4)-(1.6) we employed the notation in [8] for  $q$ -shifted factorials and basic hypergeometric functions. The evaluation of the entire functions  $A(x)$ ,  $B(x)$ ,  $C(x)$ ,  $D(x)$  in Theorem 1 involves identifying the polynomials  $\{h_n(x|q)\}$  of odd and even degrees as multiples of Al-Salam-Chihara polynomials [6] of argument  $x^2$ . We also use Darboux's asymptotic method. An independent calculation gave expressions to base  $\sqrt{q}$ . Thus we also obtained some quartic transformations which are not stated here.

The identity (1.3) is an instance of the nonterminating  $q$  analog of the Chu-Vandermonde sum [8, (II.23)]. The generating function

$$(1.7) \quad \sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/2}}{(q)_n} h_n(x|q) = (-t(x + \sqrt{x^2 + 1}), t(\sqrt{x^2 + 1} - x); q)_\infty,$$

proved to be very useful in our analysis. Using Rogers's formula for the linearization of products of two continuous  $q$ -Hermite polynomials [5] and (1.7) one can prove Theorem 2.

**Theorem 2.** *Let  $\{h_n(x|q)\}$  be orthogonal with respect to a normalized probability measure  $d\psi(x)$ .*

Then

$$(1.8) \quad \int_{-\infty}^{\infty} \prod_{j=1}^4 (-t_j(x + \sqrt{x^2 + 1}), t_j(\sqrt{x^2 + 1} - x); q)_\infty d\psi(x) \\ = \left[ \prod_{1 \leq j < k \leq 4} (-t_j t_k / q; q)_\infty \right] / (t_1 t_2 t_3 t_4 q^{-3}; q)_\infty.$$

provided that the integral exists.

In the process of proving (1.8) we also derived the Poisson kernel

$$(1.9) \quad \sum_{n=0}^{\infty} h_n(\sinh \xi | q) h_n(\sinh \eta | q) \frac{q^{n(n+1)/2}}{(q; q)_n} R^n = (-qRe^{\xi+\eta}, -qRe^{-\xi-\eta}, qRe^{\xi-\eta}, qRe^{\eta-\xi}; q)_{\infty} / (qR^2; q)_{\infty}.$$

This is not surprising since in the case of continuous q-Hermite polynomials when  $q < 1$  the Poisson kernel is equivalent to the linearization formula of L. J. Rogers. The interested reader may consult the excellent monograph [1] for applications of Rogers's work and for references to Rogers's original papers, see also [5].

R. Askey [3] proved that the polynomials  $\{h_n(x|q)\}$  are orthogonal on  $(-\infty, \infty)$  with respect to the normalized weight function

$$(1.10) \quad \frac{1}{(-\ln q) \cosh \xi (-qe^{2\xi}, -qe^{-2\xi}; q)_{\infty}}, \quad x = \sinh \xi.$$

This fact when combined with Theorem 2 gives the following result of Askey [2].

**Corollary 3.** *We have*

$$(1.11) \quad \int_{-\infty}^{\infty} \frac{\prod_{j=1}^4 (-t_j e^{\xi}, t_j e^{-\xi}; q)_{\infty}}{(-\ln q) (-qe^{2\xi}, -qe^{-2\xi}; q)_{\infty}} d\xi = \left[ \prod_{1 \leq j < k \leq 4} (-t_j t_k / q; q)_{\infty} \right] / (t_1 t_2 t_3 t_4 q^{-3}; q)_{\infty}.$$

Note that the orthogonality of the polynomials  $\{h_n(x|q)\}$  is equivalent to the special case  $t_3 = t_4 = 0$  of (1.8). One can think of Theorem 2 as a way of introducing new parameters in the spectral measure of the q-Hermite polynomials and the result is the evaluation of more general q beta integrals, [2], [3], [8].

Let

$$(1.12) \quad \varphi_n(\sinh \xi; t_1, t_2, t_3, t_4) = {}_4\phi_3 \left( \begin{matrix} q^{-n}, -t_1 t_2 q^{n-2}, -t_1 t_3 / q, -t_1 t_4 / q \\ -t_1 e^{\xi}, t_1 e^{-\xi}, t_1 t_2 t_3 t_4 q^{-3} \end{matrix} \middle| q, q \right).$$

**Theorem 4.** *The rational functions  $\varphi_n$  of (1.12) satisfy the biorthogonality relation*

$$(1.13) \quad \int_{-\infty}^{\infty} \varphi_m(x; t_1, t_2, t_3, t_4) \varphi_n(x; t_2, t_1, t_3, t_4) w(x) d\psi(x) = g_n \delta_{m,n},$$

where the  $h_n$ 's are orthogonal with respect to  $d\psi$ , and  $w(x)$  is given by

$$(1.14) \quad w(x) = \prod_{j=1}^4 (-t_j(x + \sqrt{x^2 + 1}), t_j(\sqrt{x^2 + 1} - x); q)_{\infty}$$

and  $g_n$  is

$$(1.15) \quad g_n = \frac{1 + t_1 t_2 q^{2n-1} \frac{(t_1 t_2 t_3 t_4 q^{-3})^n (-t_1 t_2 q^{n-1}; q)_\infty}{(t_1 t_2 t_3 t_4 q^{-3}; q)_n}}{1 + t_1 t_2 q^{n-1} \frac{(-t_1 t_3/q, -t_1 t_4/q, -t_2 t_3/q, -t_2 t_4/q, -t_3 t_4/q; q)_\infty}{(t_1 t_2 t_3 t_4 q^{-3}; q)_\infty}}$$

We also obtained another orthogonality relation for the functions  $\{\varphi_n\}$  associated with the strong asymptotics of  $\{\varphi_n\}$ . After we discovered Theorem 4, M. Rahman kindly pointed out to us that he discovered the special case of Theorem 4 when  $\frac{d\psi(x)}{dx}$  is Askey's weight function (1.10), [10].

**2. Applications.** The extremal measures  $\{d\psi(x; \sigma)\}$  of an indeterminate moment problem are precisely those measures which correspond to real constant functions  $\sigma$ . Here  $\sigma$  is allowed to take the values  $\pm\infty$ . The polynomials orthogonal with respect to  $d\psi(x; \sigma)$  are complete in  $L_2(d\psi(x; \sigma))$  if and only if  $d\psi(x; \sigma)$  is an extremal measure, [11]. For general indeterminate moment problems the entire functions  $A(z)$ ,  $B(z)$ ,  $C(z)$ ,  $D(z)$  have real and simple zeros and are uniform limits of polynomials.

**Lemma 5.** *Let*

$$(2.1) \quad f(z) := -\frac{(q; q^2)_\infty B(z)}{(q^2; q^2)_\infty D(z)} = \frac{(qe^{2\xi}, qe^{-2\xi}; q^2)_\infty}{z(q^2 e^{2\xi}, q^2 e^{-2\xi}; q^2)_\infty}, \quad z = \sinh \xi.$$

*Then the function  $f(z)$  is a meromorphic function that maps the open upper half plane into the open lower half plane. Furthermore  $f(\sqrt{z})$  is analytic in the open upper half plane and maps it into the open lower half plane.*

Lemma 5 follows from the Mittag-Leffler expansion

$$(2.2) \quad f(z) = \frac{(q; q^2)_\infty^2}{(q^2; q^2)_\infty^2} \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \frac{q^n + q^{-n}}{2} \left[ \frac{1}{z - (q^{-n} - q^n)/2} + \frac{1}{z + (q^{-n} - q^n)/2} \right] \right\}.$$

It is clear from (2.1) and (2.2) that  $B(x)/D(x)$  is increasing on any open interval whose end points are consecutive zeros of  $D(z)$ . For  $\sigma \in (-\infty, \infty)$  define  $\eta = \eta(\sigma)$  as the unique solution of

$$(2.3) \quad \sigma = B(\sinh \eta)/D(\sinh \eta), \quad 0 < \sinh \eta < (q^{-1} - q)/2.$$

We define  $\eta(\pm\infty)$  by

$$\eta(-\infty) = 0, \quad \eta(\infty) = (q^{-1} - q)/2.$$

It is more convenient to use the parameter  $a$ ,

$$a := e^{-\eta}, \quad q \leq a \leq 1.$$

**Theorem 6.** Define doubly infinite sequences  $\{x_n(a) : n = 0, \pm 1, \dots\}$  and  $\{m_n(a) : n = 0, \pm 1, \dots\}$  by

$$(2.4) \quad x_n(a) = \frac{1}{2}(q^{-n}a^{-1} - aq^n), \quad n = 0, \pm 1, \pm 2, \dots,$$

$$(2.5) \quad m_n(a) = a^{4n}q^{n(2n-1)}(1 + a^2q^{2n})/(-a^2, -q/a^2, q; q)_{\infty}, \quad n = 0, \pm 1, \dots$$

Then the extremal measures of the moment problem associated with the  $q$ -Hermite polynomials with  $q > 1$  are supported on  $\{x_n(a) : n = 0, \pm 1, \dots\}$  and the mass concentrated at  $x_n(a)$  is  $m_n(a)$ .

Our proof of Theorem 6 uses product formulas for theta functions. The positive measures  $\{d\psi(x; \sigma)\}$  in (1.2) are normalized by  $\int_{-\infty}^{\infty} d\psi(x; \sigma) = 1$ . This is equivalent to the Jacobi triple product identity, [8, (II.28)]

$$(2.6) \quad \sum_{-\infty}^{\infty} z^n p^{n^2} = (p^2, -pz, -p/z; p^2)_{\infty}.$$

In addition to the parameterization (2.3), Lemma 5 and (2.2) are used to introduce suitable functions  $\sigma(z)$  in (1.2). This process lead to curious explicit evaluations of sums and integrals. The details are in [9].

**Corollary 7** (Bailey's  ${}_6\psi_6$  sum). We have

$$(2.7) \quad {}_6\psi_6 \left( \begin{matrix} qa^{1/2}, -qa^{1/2}, b, c, d, e \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right) = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde)_{\infty}}.$$

**Proof.** Apply (1.8) with  $d\psi$  being any extremal measure of Theorem 6.

It is worth mentioning that that neither (2.6) nor (2.7) were used in our analysis and they indeed are consequences of our general analytic results. The only summation theorem used is the  $q$  analog of the Pfaff-Saalschütz theorem [8, (II.12)] which was used to establish (1.13). In fact the aforementioned  $q$  analog of the Pfaff-Saalschütz theorem follows from (2.7). The Stieltjes transforms of most cases of the Al-Salam-Chihara polynomials for  $q > 1$  have been identified in [7] but the extremal measures were not identified as explicitly as in the present work. The Al-Salam-Chihara polynomials are more general than the  $h_n$ 's of this work but, as was noted in [7], the results of [7] do not cover the case of continuous  $q$ -Hermite polynomials.

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## SUMMARY OF SPECTRAL INVARIANCE RESULTS

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*Presented by G.A. Elliott, F.R.S.C.*

**Abstract.** The author's recent results on spectral invariant dense subalgebras of  $C^*$ -algebras associated with dynamical systems are summarized. If  $G$  is a compactly generated polynomial growth Type R Lie group, and the action of  $G$  on  $S(M)$  (Schwartz functions on a locally compact  $G$ -space  $M$ ) is tempered in a certain sense, then there is a natural smooth crossed product  $S(G \times M)$  which is dense and spectral invariant in the  $C^*$ -crossed product  $C^*(G \times M)$ .

The theory of differential geometry on a  $C^*$ -algebra (or noncommutative space) Connes [3] requires the use of "differentiable structures" for these noncommutative spaces, or some sort of algebra of "differentiable functions" on the noncommutative space. Such algebras of functions have usually been provided by a dense subalgebra of smooth functions  $A$  for which the  $K$ -theory  $K_*(A)$  is the same as the  $K$ -theory of the  $C^*$ -algebra  $K_*(B)$  (see for example Blackadar-Cuntz [1], and the recent works of J. Bost, G. Elliott, T. Natsume, R. Nest, R. Ji, P. Jollissaint, V. Nistor and many others).

One goal of both of the papers Schweitzer [10] [11] was to construct such dense subalgebras of smooth functions in the case that  $B$  is a  $C^*$ -crossed product  $C^*(G \times M)$  associated with a dynamical system, or more specifically with an action of a Lie group  $G$  (not necessarily connected) on a locally compact space  $M$ . In these papers, we realize this goal by constructing smooth crossed products  $S(G \times M)$  of Schwartz functions on  $G \times M$ , which are spectral invariant in the  $C^*$ -crossed product. Spectral invariant means that the spectrum of every element of  $S(G \times M)$  is the same in  $S(G \times M)$  and  $C^*(G \times M)$ . In the language of Palmer

[5], this is the same as saying that  $\mathcal{S}(G \times M)$  is a spectral subalgebra of  $C^*(G \times M)$ . By Schweitzer [8], Lemma 1.2, Corollary 2.3, and Connes [2], VI.3, spectral invariant subalgebras have the same  $K$ -theory as the  $C^*$ -algebra itself, so these smooth crossed products  $\mathcal{S}(G \times M)$  provide us with the "noncommutative differentiable structures" we are looking for.

I will begin by describing the results obtained in [11]. The idea in that paper is to employ the following theorem, which is interesting in its own right. It gives a condition for a dense Fréchet subalgebra  $A$  to be a spectral invariant subalgebra of  $B$  when certain subrepresentations of topologically irreducible representations of  $A$  extend appropriately to  $B$ . This is in contrast to the situation in [8], Theorem 1.4, Corollary 1.5, which says that algebraically irreducible representations extend iff the subalgebra is spectral invariant.

If  $E$  is an  $A$ -module, we say that  $E$  is algebraically cyclic iff there exists an  $e \in E$  such that the algebraic span  $Ae$  is equal to  $E$ .

**Theorem 1** ([11], Theorem 1.4). *Let  $A$  be a dense  $m$ -convex Fréchet subalgebra of a  $C^*$ -algebra  $B$  with continuous inclusion map  $A \hookrightarrow B$ . Assume that every algebraically cyclic subrepresentation of every topologically irreducible representation of  $A$  on a Banach space is contained in a  $*$ -representation of  $B$  on a Hilbert space. Then  $A$  is spectral invariant in  $B$ .*

The smooth subalgebras  $\mathcal{S}(G \times M)$  are shown to be  $m$ -convex Fréchet algebras in Schweitzer [9], §3, and their topologically irreducible representations are relatively accessible when the  $C^*$ -crossed product is CCR. Hence Theorem 1 gives many new interesting cases of spectral invariant smooth crossed products  $\mathcal{S}(G \times M)$ . For example, results are obtained when  $G$  is a closed subgroup of a connected, simply connected nilpotent Lie group with certain restrictions on the isotropy subgroups (that they be CCR for one), and when the action of  $G$  on  $M$  has closed orbits. (See the examples in [11], §18, §2, §16-17.)

A simple illustrative example is given by  $\mathbb{Z}$  acting by translation on  $\mathbb{R}$ . The Schwartz functions  $\mathcal{S}(\mathbb{Z} \times \mathbb{R})$  with convolution multiplication provide a dense subalgebra of smooth functions of the  $C^*$ -crossed product  $C^*(\mathbb{Z} \times \mathbb{R})$ . Any topologically irreducible representation of  $\mathcal{S}(\mathbb{Z} \times \mathbb{R})$  must factor through an orbit to a representation of the convolution algebra  $\mathcal{S}(\mathbb{Z} \times \mathbb{Z})$  [11], Theorem 14.1. The latter algebra is a smooth version of the compact operators,

whose representation theory is quite nice Du Cloux [4], Corollary 3.5 or [11], Theorem 15.1, Example 2.5. Theorem 1 may then be applied to obtain the spectral invariance of  $S(\mathbb{Z} \times \mathbb{R})$  in  $C^*(\mathbb{Z} \times \mathbb{R})$ .

As one might speculate, when the  $C^*$ -crossed product is not CCR (or at least when it is not GCR), the representation theory of the dense subalgebra becomes quite complicated as does the representation theory of the  $C^*$ -algebra. For example, the dense subalgebra  $A_\theta^\circ$ , given by the canonical action of  $\mathbb{T}^2$  on the irrational rotation  $C^*$ -algebra  $A_\theta$ , is spectral invariant but does not satisfy the hypothesis of Theorem 1. That is, there exists certain "bad" topologically irreducible representations of  $A_\theta^\circ$ , which have algebraically cyclic subrepresentations which do not extend to  $A_\theta$  [11], Example 7.1.

In order to get results in the non-CCR case, a new method is needed to replace Theorem 1. Such a method, or methods, is introduced in [10], which I shall now describe.

We begin by trying to show that the smooth crossed product  $S(G, A)$  is spectral invariant in  $L^1(G, B)$  instead of in the  $C^*$ -crossed product  $C^*(G, B)$ . Let  $\| \cdot \|_0$  be the norm on  $B$ , and let  $\{\| \cdot \|_n\}_{n=0}^\infty$  be a family of increasing submultiplicative norms giving the topology of  $A$ . In the paper Blackadar-Cuntz [1], the condition  $\| ab \|_n \leq C \sum_{i+j=n} \| a \|_i \| b \|_j$  for all  $a, b$  in  $A$ , is used to show that  $A$  is a spectral invariant subalgebra of  $B$ . The commutative Fréchet algebra  $S(M)$  of Schwartz functions on  $M$  satisfies this condition in  $C_0(M)$  [10], §2. Moreover, for some very nice actions of  $G$  on  $A$  (isometric on each norm), one can show that if the norms on  $A$  satisfy the condition in  $B$ , then the norms on the smooth crossed product  $S(G, A)$  satisfy the condition in  $L^1(G, B)$ . The following more general condition introduced in [10] does the same thing without requiring an isometric action.

We say that  $A$  is strongly spectral invariant in  $B$  if

$$(*) \quad (\exists C > 0)(\forall m)(\exists D_m > 0)(\exists p_m \geq m)(\forall n)(\forall a_1, \dots, a_n \in A) \\ \left\{ \| a_1 \dots a_n \|_m \leq D_m C^n \sum_{k_1 + \dots + k_n \leq p_m} \| a_1 \|_{k_1} \dots \| a_n \|_{k_n} \right\}.$$

Notice that in the summand of (\*), at most  $p_m$  of the natural numbers  $k_j$  are nonzero, regardless of  $n$ . The idea behind showing that strong spectral invariance implies spectral

invariance is given by setting  $a_1 = \dots a_n = a$  in (\*). We have

$$\begin{aligned} \|a^n\|_m &\leq DC^n \sum_{k_1 + \dots + k_n \leq p} \|a\|_{k_1} \dots \|a\|_{k_n} \\ &\leq K^n \|a\|_0^{n-p} \|a\|_p^p \end{aligned}$$

where  $p$  is fixed as  $n$  runs. It follows that the series  $(1 - a)^{-1} = 1 + a + a^2 + \dots$  converges absolutely in the norm  $\|\cdot\|_m$  when  $\|a\|_0$  is sufficiently small. So  $1 - a$  is invertible in the completion of  $A$  in  $\|\cdot\|_m$  when  $a$  is sufficiently close to 0 in  $B$ . The rest of the argument is in Theorem 1.17 of [10].

There are also examples of spectral invariant dense subalgebras which are not strongly spectral invariant [10], Example 1.13. The following theorem and corollary illustrates the usefulness of the concept of strong spectral invariance.

We say that a Lie group  $G$  (not necessarily connected) is compactly generated if  $G$  has an open relatively compact neighborhood  $U$  of the identity which satisfies  $\bigcup_{n=0}^{\infty} U^n = G$  and  $U^{-1} = U$ . We call  $\tau(g) = \min\{n \mid g \in U^n\}$  the word gauge on  $G$ . (The smooth crossed product  $S(G, A)$  is then defined to be the set of  $G$ -differentiable  $\tau$ -rapidly vanishing functions from  $G$  to  $A$ .) We say that the action of  $G$  on  $A$  is  $\tau$ -tempered if for every  $m$ ,  $\|\alpha_g(a)\|_m$  is bounded by a polynomial in  $\tau(g)$  times  $\|a\|_n$  for some  $n$ . Finally, we say that  $G$  is Type R if all the eigenvalues of  $Ad_g$  lie on the unit circle for each  $g \in G$ .

**Theorem 2.** *If  $A$  is strongly spectral invariant in  $B$  and  $G$  is a compactly generated Type R Lie group, and the action of  $G$  on  $A$  is  $\tau$ -tempered, then the smooth crossed product  $S(G, A)$  is strongly spectral invariant in  $L^1(G, B)$ .*

**Corollary 3.** *For compactly generated Type R Lie groups, for which the action of  $G$  on  $S(M)$  is  $\tau$ -tempered, the smooth crossed product  $S(G \times M)$  is strongly spectral invariant in  $L^1(G, C_0(M))$ .*

(Note that  $S(G \times M)$  is shorthand for  $S(G, S(M))$ .) It is the subadditivity of  $\tau$  and the strong spectral invariance of  $A$  in  $B$  that play the essential role in the proof of Theorem 2 and Corollary 3. The hypotheses that  $G$  be Type R and that the action is  $\tau$ -tempered are not used in the proof, but they are necessary to assure the existence of the smooth crossed product

$S(G \times M)$ , and to assure that  $S(G \times M)$  is a Fréchet  $*$ -algebra. There are a wide variety of Type R Lie groups (see below or [9], §1.4, [10]), and also many examples of  $\tau$ -tempered actions of such groups  $G$  on  $S(M)$  [9], §5, [10], Examples 6.26-7, 7.20, [11].

**Remark.** We clarify what the  $\tau$ -tempered assumption can mean in practice. Let  $G$  be the integers  $\mathbb{Z}$ , and let  $G$  act on  $\mathbb{R}$  via  $\alpha_n(r) = e^{-nr}$ . The word gauge  $\tau$  is equivalent in an appropriate sense to the absolute value function  $\tau(n) = |n|$ . If we take  $S(M) = C_0(\mathbb{R})$ , or  $S(M) = C_0^\infty(\mathbb{R})$ , then  $\alpha$  is an isometric action of  $\mathbb{Z}$  on  $S(M)$ , meaning that  $\alpha$  leaves each seminorm invariant, and so  $\alpha$  is  $\tau$ -tempered. However, if we take  $S(M)$  to be the standard Schwartz functions  $S(\mathbb{R})$ , then for a fixed  $\varphi \in S(M)$ ,  $\|\alpha_n(\varphi)\|_m$  will in general grow exponentially in  $n$  as  $n \rightarrow +\infty$ , so  $\alpha$  is no longer  $\tau$ -tempered. So the  $\tau$ -temperedness condition does place a restriction on what  $S(M)$  can be for a given action. If the action of  $\mathbb{Z}$  on  $\mathbb{R}$  were by translation  $\alpha_n(r) = r + n$ , then the action of  $\mathbb{Z}$  on  $S(\mathbb{R})$  would be  $\tau$ -tempered. For general  $M$  and  $G$  as in Corollary 3, and regardless of the action, one can always get a  $\tau$ -tempered action by taking  $S(M) = C_0(M)$ , or  $S(M) = C_0^\infty(M)$ , where the superscript  $\infty$  means " $G$ -differentiable".

Note that Corollary 3 makes no assumption about the crossed product being CCR. No restrictions on the action of  $G$  or the isotropy subgroups are needed. However, we are left with the question of whether  $S(G \times M)$  is spectral invariant in the  $C^*$ -crossed product  $C^*(G \times M)$ , and not just  $L^1(G, C_0(M))$ . To take care of this we generalize a result of Pytlik [7] which says that if  $G$  has polynomial growth, then the rapidly vanishing  $L^1$ -functions on  $G$  form a symmetric Fréchet  $*$ -algebra, which consequently is spectral invariant in  $C^*(G)$ . In particular, we show in §7 of [10] that the rapidly vanishing  $L^1$ -functions from  $G$  to  $B$  is spectral invariant in the  $C^*$ -crossed product  $C^*(G, B)$  when  $G$  has polynomial growth. Since these rapidly vanishing  $L^1$ -functions are also spectral invariant in  $L^1(G, B)$ , and since they contain the smooth crossed product, we are able to conclude that the smooth crossed product is spectral invariant in the  $C^*$ -crossed product when  $G$  has polynomial growth. Our main result is then:

**Corollary 4.** For compactly generated polynomial growth Type R Lie groups  $G$ , and  $\tau$ -

tempered actions of  $G$  on  $S(M)$ , the smooth crossed product  $S(G \times M)$  is spectral invariant in the  $C^*$ -crossed product  $C^*(G \times M)$ .

Examples of such groups are given by finitely generated polynomial growth discrete groups, compact or connected nilpotent Lie groups, the group of Euclidean motions on the plane, any motion group, or any closed subgroup of one of these. Numerous examples of smooth crossed products which are spectral invariant because of Corollary 4 can be found in [10], Examples 2.6-7, 6.26-7, 7.20, [11], [9], §5.

We remark that in [6], methods are given to show that  $S(G \times M) \hookrightarrow C^*(G \times M)$  is an isomorphism on  $K$ -theory without using spectral invariance, whenever  $G$  is a closed subgroup of a connected, simply connected nilpotent Lie group, and the action of  $G$  on  $S(M)$  is  $\tau$ -tempered [6], Example 3.2.

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# Some Results on Hyperbolic Harmonic Maps

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**Abstract.** We show that certain harmonic maps from Lorentzian manifolds can be transformed into problems of finding global solutions for nonlinear wave equations in Minkowski space-times. The resulting problem can be solved by a method [GA] similar to Klainerman's general results [KL].

## 1. The Problem

Let  $(M, g)$  be a four dimensional Lorentzian manifold with metric  $g$  in local coordinates  $(x^0, x^1, \dots, x^3)$  and  $(N, h)$  be an  $n$  dimensional complete Riemannian manifold with metric  $h$  in local coordinates  $(u^1, \dots, u^n)$ . For a  $C^1$  map  $u: M \rightarrow N$ , recall that the energy of  $u$  is defined as the intrinsic Dirichlet integral:

$$(1) \quad E(u) = \int_M e(u) dM$$

with energy density:

$$(2) \quad e(u) = g^{ab} h_{ij} \partial_a u^i \partial_b u^j$$

Here  $(g^{ab}) = (g_{ab})^{-1}$ . We say that  $u$  is harmonic if  $E(u)$  is stationary at  $u$  (see [2]), ie,  $u$  is a critical point of  $E(u)$  which gives rise to the nonlinear equations(see [2])

$$(3) \quad -\Delta_g u^i + \gamma_{jk}^i(u)(\nabla u^j, \nabla u^k)_g = 0, \quad i = 1, 2, \dots, n$$

Here  $\nabla$  is the covariant derivative on  $M$ ,

$$\Delta_g = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^a} (\sqrt{|g|} g^{ab} \frac{\partial}{\partial x^b}), \quad |g| = |\det(g_{ab})|$$

is the Laplace -Beltrami operator on  $M$ ,  $(\gamma_{jk}^i(u))$  are Christoffel symbols for  $N$  and  $(\cdot)_g$  is the induced pointwise norm on  $TM$ , namely, for any  $x \in M, u, v \in (TM)_x, (u, v)_g = g^{ab}u_a v_b$ . Since  $M$  is Lorentzian, (3) is a system of  $n$  weakly-coupled nonlinear wave equations on  $M$  with at least quadratic nonlinearity near  $\nabla u = 0$ . Therefore, the existence of harmonic maps from  $M$  to  $N$  is equivalent to that of solutions of (3).

The Riemannian case of this problem has been extensively studied (see [ES]). The work on this problem with Lorentzian source manifold  $M$  started a few years ago mostly on Minkowski space  $M^{m+1}$ , the case which has various applications in relativity and gauge field theory.  $M^{m+1}$  is topological space  $R \times R^+ \times S^{m-1}$  ( $R^+ = [0, \infty)$ ) equipped with Lorentzian metric  $\eta: -dt^2 + dr^2 + r^2 d\theta^2$  in spherical coordinates  $(t, r, \theta)$ . The equation corresponding to (3) for  $M = M^{m+1}$  is

$$(4) \quad \square_{\eta} u^i = (\partial_i^2 - \partial_1^2 - \dots - \partial_m^2)u^i + \gamma_{jk}^i(u)(\partial u^j, \partial u^k)_{\eta} = 0 \quad i = 1, 2, \dots, n$$

In the case of a so called  $\sigma$ -model, ie,  $M = M^{m+1}$ ,  $N = S^n$ , J.Shatah [SH] showed the existence of globally weak solutions in  $H^{1,2}$  for (4) with initial data of finite energy and the solution may not lie in a small region. For  $m = n = 3$ , he also constructed smooth symmetric initial data for which the corresponding solution develops a singularity in a finite time. For  $M = M^4$  and  $N$  is some complete Riemannian manifold, T.Sideris [SI] proved the existence of a unique global classical solution in time for smooth and small initial data. Geometrically, his solution concentrates in a small neighbourhood of a given geodesic on  $N$  instead of a small neighbourhood of a point (therefore, the initial data is partly small), as a perturbation of a special class of harmonic maps: geodesics. His method is based on the Fermi chart constructed along the geodesic and the resulting nonlinear system has higher order nonlinearity with respect only to some components of  $u$ , which permits the solution to lie over a tubular region.

We consider here the case that  $M = R^+ \times H^3$  and  $g = S^2(\tau)\nu$  where  $\nu$  is a canonical

Lorentzian metric on  $M$  given by

$$(5) \quad ds^2 = -d\tau^2 + d\sigma^2,$$

and  $d\sigma^2$  is the line element on three dimensional hyperbolic space  $H^3$  given by

$$(6) \quad d\sigma^2 = d\rho^2 + \sinh^2 \rho d\omega^2, \quad d\omega^2 = (d\phi^2 + \cos^2 \phi d\psi^2)$$

in spherical coordinates  $(\rho, \phi, \psi)$ .  $S(\tau)$  is a positive function of the time variable  $\tau \in R^+$ . Given a geodesic  $\Upsilon \in N$ , following [SI], we may construct a Fermi chart in a neighborhood of  $\Upsilon$  such that the parameter  $u^1$  along  $\Upsilon$  may vary freely and the Christoffel symbols of  $N$  satisfy

$$(7) \quad \gamma_{jk}^i(u^1, 0, \dots, 0) = 0, \quad u^1 \in (-\infty, \infty), \quad 1 \leq i, j, k \leq n$$

We prescribe the initial data for (3) at  $\tau = 0$

$$(8) \quad \begin{aligned} u(0, \sigma) &= (f^1(\sigma), \delta f(\sigma)) = f_\delta(\sigma), \\ (\partial_\tau u)(0, \sigma) &= (g^1(\sigma), \delta g(\sigma)) = g_\delta(\sigma) \end{aligned} \quad \sigma \in H^3$$

with  $\delta > 0$  a small parameter. Here  $f, g$  are smooth functions with compact supports, i.e., there is  $\sigma_0 > 0$  such that

$$f_\delta(\sigma) = g_\delta(\sigma) = 0, \quad |\sigma| \geq \sigma_0.$$

Thus, (3) can be written as

$$(9) \quad \frac{1}{S^2(\tau)} \partial_\tau^2 u + \frac{2\dot{S}(\tau)}{S^3(\tau)} \partial_\tau u - \frac{1}{S^2(\tau)} \Delta_{H^3} u + \gamma_{jk}(u) (\nabla u^j, \nabla u^k)_g = 0,$$

where  $\dot{S}(\tau) = \frac{d}{d\tau} S(\tau)$ ,  $\gamma_{jk}(u) = (\gamma_{jk}^i(u))$  and  $\Delta_{H^3}$  is the Laplace-Beltrami operator of  $H^3$ .

Now we define a class of  $S(\tau)$  for which we may solve problem (3),(8).

**Definition.**  $(M, g)$  with  $g = S^2(\tau)\nu$  is strongly conformally flat if  $S(\tau) > 0$  and

$$(10) \quad \left| \frac{d^i}{d\tau^i} (\dot{S}(\tau) - S(\tau)) \right| \leq c_i \exp(-2\tau) \quad i = 0, 1, 2, \dots$$

This condition means  $S(\tau)$  is sufficiently close to  $\exp \tau$  as  $\tau \rightarrow +\infty$ . In particular, (10) implies

$$(11) \quad \left| \frac{d^i}{d\tau^i} \left( \frac{\dot{S}(\tau)}{S(\tau)} - 1 \right) \right| \leq c \exp(-3\tau), \quad 0 < c' \leq \left| \frac{\exp(\tau)}{S(\tau)} \right| \leq c'' \quad i = 0, 1, 2, \dots$$

**Theorem.** Let  $(M, g)$  be strongly conformally flat and  $\Upsilon \subset N$  be any geodesic. Then we can construct a (Fermi) chart in a neighborhood of  $\Upsilon$  with parameter  $u^1$  along  $\Upsilon$  such that (7) is satisfied. In this coordinate, for any given smooth initial data of (8), there is  $\delta_1 > 0$  such that if  $0 \leq \delta \leq \delta_1$ , there will be a unique smooth harmonic map  $u : M \rightarrow N$  satisfying initial conditions (8). The range of this map is in a tubular neighborhood of  $\Upsilon$ .

**Remark.** This result can be easily extended to higher spatial dimension  $n > 3$ .

To prove the theorem we transform the Cauchy problem (3),(8) to a corresponding problem in Minkowski space  $M^4$  by a conformal mapping and study the global existence of smooth solutions for the transformed problem.

## 2. A Conformal Transformation

By  $g = S^2(\tau)\nu$ , we derive

$$(12) \quad (\nabla u^j, \nabla u^k)_g = g^{ab} \partial_a u^j \partial_b u^k = S^{-2}(\tau) (\nu^{ab} \partial_a u^j \partial_b u^k) = S^{-2}(\tau) (\nabla u^j, \nabla u^k)_\nu$$

Then (9), we know  $v = S(\tau)u$  satisfies

$$(13) \quad \partial_\tau^2 v - \frac{\dot{S}(\tau)}{S(\tau)} v - \Delta_{\mathbb{H}^n} v = \bar{F}(v, \nabla v)$$

with

$$\bar{F}(v, \nabla v) = -S(\tau) \gamma_{jk} \left( \frac{v}{S} \right) \left( \nabla \left( \frac{v^j}{S} \right), \nabla \left( \frac{v^k}{S} \right) \right)_\nu$$

If  $\tilde{S}(\tau) = S(\tau)$ , i.e.,

$$(14) \quad S(\tau) = c_1 \exp(\tau) + c_2 \exp(-\tau)$$

for constants  $c_1, c_2$ , then (13) is a nonlinear automorphic wave equation on  $H^3$ . If  $S(\tau) > 0$  and  $S(\tau) \rightarrow +\infty$  as  $\tau \rightarrow +\infty$ , then  $c_1 > 0, c_2 \geq 0$ ;  $\exp(\tau), \cosh(\tau), \sinh(\tau)$  are special cases of (14).

We define a mapping  $\Phi$  from  $M_+^{3+1} = \{(t, r, \theta) \in M^{3+1} | t^2 - r^2 \geq 1\}$  to  $M$  by

$$\Phi(t, r, \theta) = \left( \frac{1}{2} \ln(t^2 - r^2), \cosh^{-1} \frac{t}{\sqrt{t^2 - r^2}}, \theta \right) = (\tau, \rho, \omega),$$

It can be shown (see [GA]) that  $\Phi$  is bijective and conformal, i.e.,  $\Phi^* \nu = \gamma^2 \eta$  with conformal factor  $\gamma^2 = (t^2 - r^2)^{-1}$ . Furthermore, if  $v$  is the solution of (13), then  $u = \gamma \nu \circ \Phi$  satisfies the Cauchy problem

$$(15) \quad \begin{cases} \square u - \kappa(\rho)u = F\left(\frac{\rho}{S(\ln \rho)}, u, \partial u\right) & (\rho, \sigma) \in M^4, \rho > 1 \\ u = (u_0^1, \delta \bar{u}_0(\sigma)), L_0 u = (u_1^1, \delta \bar{u}_1(\sigma)), & \rho = 1, \end{cases}$$

where  $\rho = \sqrt{t^2 - r^2}$  and  $\kappa, F$  are defined by

$$(16) \quad \begin{aligned} \kappa(\lambda) &= \Phi^*(\exp(-2\tau)\left(\frac{\tilde{S}}{S} - 1\right)) = \lambda^{-2} \left(\frac{\tilde{S}(\ln \lambda)}{S(\ln \lambda)} - 1\right), \\ F(\bar{\lambda}, u, \partial u) &= \Phi^*(\exp(-3\tau)\bar{F}(v, \nabla v)) = \bar{\lambda}^{-1} \gamma_N(\bar{\lambda}u)(\partial(\bar{\lambda}u), \partial(\bar{\lambda}u))_\eta, \\ \lambda &= \sqrt{t^2 - r^2}, \quad \bar{\lambda} = \frac{\lambda}{S(\ln \lambda)} \end{aligned}$$

Also

$$(17) \quad \begin{aligned} (u_0^1, \delta \bar{u}_0) &= f_\delta(\Phi(\sigma)), \\ (u_1^1, \delta \bar{u}_1) &= g_\delta(\Phi(\sigma)). \end{aligned} \quad \sigma \in H^3$$

is defined on the hypersurface  $\rho = \sqrt{t^2 - r^2} = 1$ .

Then we can prove the main Theorem by proving a corresponding result, i.e., existence of global and smooth solutions. We can use a standard method to show the existence of

global and smooth solutions for (15)(see [KL]) under the assumptions transformed by  $\Phi$  from those in the main Theorem. But, we have to point out some of the differences when following the procedure of proof. First, (15) is not a standard Cauchy problem in the sense that the initial surface is a hyperboloid. Secondly, nonlinear term  $F$  contains  $t, x$  explicitly which is not dealt with in standard proofs of the global solutions. The solution to overcome these difficulties is to use appropriate derivatives for the norm of the space to which solution  $u$  belongs, such that we can derive some a priori estimates for  $u$  without invoking the difficulties due to the presence of  $t, x$  in the nonlinear term. Details of this procedure and the proof of existence of global and smooth solutions of (15) are contained in [GA].

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# SUR LE SPECTRE DE $CV_0(X)$

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## Abstract

*We give a necessary and sufficient condition for  $CV_0(X)$  to be a locally convex algebra or a locally convex algebra with continuous product. We then show that in most of the important cases, The spectrum of  $CV_0(X)$  is homeomorphic to  $X$ .*

En utilisant un théorème d'approximation basé sur la théorie des modules, Prolla [4] montre que si  $X$  est un complètement régulier séparé et  $V$  une famille de Nachbin sur  $X$  telle que  $CV_0(X)$  est une algèbre localement convexe à produit continu et essentielle, (i. e., pour tout  $x \in X$ , il existe  $f \in CV_0(X)$  telle que  $f(x) \neq 0$ ), alors le spectre de  $CV_0(X)$  est homéomorphe à  $X$ . Dans cette note, nous considérons des familles de Nachbin plus générales et montrons que dans la plupart des cas, le spectre de  $CV_0(X)$  est encore homéomorphe à  $X$ . La technique que nous utilisons ici diffère de celle de Prolla.

**Notation.** Soit  $X$  un complètement régulier séparé. On appelle poids sur  $X$  toute fonction positive semi-continue supérieurement sur  $X$ . Une famille  $V$  de poids est dite de Nachbin si:

- 1)  $\forall x \in X, \exists v \in V: v(x) \neq 0,$
- 2)  $\forall v_1, v_2 \in V, \forall \lambda > 0, \exists v \in V: \max(\lambda v_1, \lambda v_2) \leq v.$

On notera par  $CV_0(X)$  l'espace vectoriel des fonctions  $f$  continues sur  $X$  et telles que  $fv$  s'annule à l'infini, pour tout  $v \in V$ . On le munit de la topologie  $\tau_v$ , des semi-normes  $P_v$ 's, où  $P_v(f) := \|vf\|_\infty = \sup\{|f(t)|v(t), t \in X\}, v \in V$ .

Une algèbre localement convexe  $E$  (a. l. c.) est toute algèbre munie d'une topologie localement convexe pour laquelle le produit de  $E$  est séparément continu. Le spectre d'une a. l. c.  $E$  est l'ensemble  $M(E)$  de tous ses caractères (non nuls) continus. On munit  $M(E)$  de la topologie faible induite par  $\sigma(E', E)$ ,  $E'$  étant le dual topologique de  $E$ .

Dans toute la suite, on suppose en plus que  $CV_0(X)$  est essentielle.

**1. Proposition.** On a: 1)  $CV_0(X)$  est une algèbre localement convexe si, et seulement si,  $V |g| \leq V, \forall g \in CV_0(X)$  i.e., pour tout  $v \in V$ , il existe  $v' \in V$  tel que  $v |g| \leq v'$ .  
2)  $CV_0(X)$  est une a.l.c à produit continu si, et seulement si,  $V \leq VV$ .

**Preuve.** 1) Si  $CV_0(X)$  est une a. l. c., alors le produit par  $g$  est continu pour tout  $g \in CV_0(X)$ . Donc pour tout  $v \in V$ , il existe  $v' \in V$  tel que:  $P_v(fg) \leq P_{v'}(f) \quad f \in CV_0(X)$ . On va donc montrer que  $v |g| \leq v'$ . Soient  $x_0 \in X$  et  $U_n := \{x \in X : v'(x) < v'(x_0) + 1/n\}$ . Alors  $U_n$  est un voisinage ouvert de  $x_0$ . Soit  $f_n$  un élément de  $CV_0(X)$  dont le support,  $\text{supp } f_n$ , est contenu dans  $U_n$  et telle que  $0 \leq f_n \leq 1, f_n(x_0) = 1$ . Une telle fonction existe toujours d'après ([3], lemme 2, p. 69). On a donc  $P_v(f_n g) \leq P_{v'}(f_n)$ . En particulier  $v(x_0) f_n(x_0) |g(x_0)| \leq \sup\{v'(t) f_n(t), t \in U_n\}$ . Donc  $v(x_0) |g(x_0)| \leq v'(x_0) + 1/n$ . Comme  $n$  est quelconque, on aura  $v(x_0) |g(x_0)| \leq v'(x_0)$ . La réciproque est évidente.

2) Maintenant si  $V \leq VV$ , alors le produit de  $CV_0(X)$  est continu. Réciproquement si  $CV_0(X)$  est à produit continu, on a:

$$\forall v \in V, \exists v' \in V : P_v(fg) \leq P_{v'}(f)P_{v'}(g); \quad f, g \in CV_0(X).$$

Montrons alors que  $v \leq v'v'$ . Soient  $x_0 \in X, U_n := \{x \in X : v'(x) < v'(x_0) + 1/n\}$  et  $f_n$  une fonction comme ci-dessus. Alors on a:  $P_v(f_n^2) \leq (P_{v'}(f_n))^2$ . Donc  $v(x_0) \leq (v'(x_0) + 1/n)^2$ . Comme  $n$  est quelconque, on a  $v(x_0) \leq v'(x_0)v'(x_0)$ .

**Exemple:** Il existe des familles de Nachbin  $V$  telles que  $CV_0(X)$  est une a. l. c. mais pas à produit continu. Considérer, par exemple, le cas où  $X = \mathbb{R}$  et  $V = \{\lambda v, \lambda > 0\}$ , où:

$$v(x) = \begin{cases} x, & \text{si } x > 0 \\ 1 & \text{sinon.} \end{cases}$$

Maintenant on suppose que  $CV_0(X)$  est une a. l. c. et soient  $x \in X$  et  $\delta_x$  l'évaluation en  $x$  définie sur  $CV_0(X)$ . Si  $v(x) \neq 0$ , alors on a:  $|\delta_x(f)| = |f(x)| \leq \frac{1}{v(x)} P_v(f)$ . Donc  $\delta_x$  est un caractère continu de  $CV_0(X)$ . De plus, l'application  $\delta: \delta(x) = \delta_x$  est injective. En effet si  $x, y \in X$  avec  $x \neq y$ , alors il existe  $g \in C(X)$  tel que:  $0 \leq g \leq 1, g(x) = 1$  et  $g(y) = 0$ . On prend alors  $f \in CV_0(X)$  tel que  $f(x) \neq 0$  et  $h = fg$ . Alors  $h \in CV_0(X)$  et  $h(x) \neq h(y)$  i.e.,  $\delta_x \neq \delta_y$ <sup>1</sup>. On peut donc considérer  $X$  comme un sous ensemble du spectre  $\Delta$  de  $CV_0(X)$ . Comme  $X$  est complètement régulier séparé et  $CV_0(X)$  essentielle,

<sup>1</sup> Prolla a prouvé la continuité de  $\delta_x$  et l'injection de  $\delta$  dans un cadre plus général (cf. [4], p. 729).

cette inclusion est même topologique. La question est donc:  $Y'$  a-t-il égalité? ou encore: Est ce que  $\delta$  est surjective? La réponse est donnée par

**2. Théorème.** *L'application  $\delta$  est surjective dans les cas suivants:*

- 1)  $X$  est localement compact (en particulier si  $V \subset C(X)$ ).
- 2)  $CV_0(X)$  contient la fonction constante 1.

**Preuve.** Dans le cas 1), Nachbin a montré que l'algèbre  $\mathcal{K}(X)$  des fonctions continues à support compact est dense dans  $CV_0(X)$  (cf. [3], p. 64). Par suite ces deux algèbres ont (algébriquement) le même spectre. Mais la topologie  $\tau_w$  est moins fine sur  $\mathcal{K}(X)$  que la topologie  $\eta$  limite inductive localement convexe des  $\mathcal{K}_K(X)$ 's, algèbres des fonctions continues sur  $X$  et à support dans  $K$ . Le spectre de  $(\mathcal{K}(X), \eta)$  étant égal à  $X$ , il en est de même de  $\Delta$ . Dans le cas 2),  $C_b(X)$  est dense dans  $CV_0(X)$  (cf. [3], proposition 5, p. 66). Donc  $(C_b(X), \tau_w)$  et  $CV_0(X)$  ont le même spectre. Mais  $\tau_w$  est moins fine sur  $C_b(X)$  que la topologie de la convergence uniforme. Donc  $\Delta$  est contenu dans  $\beta X$ . Supposons que pour un point  $x \in \beta X \setminus X$ ,  $\delta_x$  est continu sur  $(C_b(X), \tau_w)$ . Alors il existe  $v \in V$  tel que:  $|\tilde{f}(x)| \leq P_v(f)$ ,  $f \in C_b(X)$  et  $\tilde{f}$  est le prolongement à  $\beta X$  de  $f$ . Comme  $1 \in CV_0(X)$ ,  $v$  s'annule à l'infini. Donc la fonction  $\tilde{v}$  définie par:  $\tilde{v} = v$  sur  $X$  et  $\tilde{v} = 0$  hors de  $X$  est semi-continue supérieurement sur  $\beta X$  (cf [2]) et l'on a encore:  $|\tilde{f}(x)| \leq P_{\tilde{v}}(\tilde{f})$ ,  $f \in C_b(X)$ . Maintenant, si  $O$  est l'ouvert  $\{t \in \beta X : \tilde{v}(t) < 1/2\}$  et si  $g \in C_b(X) = C(\beta X)$  est telle que  $0 \leq g \leq 1$ ,  $g(x) = 1$  et  $\text{supp } g \subset O$ , alors on a:  $|g(x)| \leq P_{\tilde{v}}(g)$ . Ceci donne que  $1 \leq 1/2$ . ce qui est absurde. Par suite  $\Delta$  est homéomorphe à  $X$ .

Une conséquence de 2) du théorème ci-dessus est que pour tout complètement régulier séparé  $X$ , on a:

$$M(C_{\mathcal{P}}(X)) = \bigcup \{ \bar{B}^{vX}, B \in \mathcal{P} \},$$

où  $\mathcal{P}$  est une famille quelconque de parties bornantes de  $vX$  comme dans [5].

Remarquons qu'on peut supposer, sans perdre de généralité, que chaque  $B \in \mathcal{P}$  est fermé dans  $X$  de telle sorte que sa fonction caractéristique soit semi-continue supérieurement.

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## On homogeneous complex manifolds having more than two ends

BRUCE GILLIGAN

*Presented by J.S. Halperin, F.R.S.C.*

**ABSTRACT.** Suppose  $G$  is a connected Lie group and  $H$  is a closed subgroup such that  $G/H$  has an invariant complex structure and more than two ends. Then there exist holomorphic fibrations  $G/H \rightarrow G/I \rightarrow G/P$  with  $I/H$  a compact complex manifold,  $P/I$  of the form  $S/\Gamma$ , where  $\Gamma$  is a Zariski dense discrete subgroup of some semi-simple complex Lie group  $S$  and  $S/\Gamma$  has the same number of ends as  $G/H$ , and  $G/P$  a homogeneous rational manifold. We also note that if  $\pi_1(G/H)$  is solvable, then  $G/H$  has at most two ends.

1. **INTRODUCTION.** In this paper we consider complex manifolds  $X$  which are homogeneous under the action of a connected Lie group  $G$  and thus can be written in the form  $X = G/H$ . It has been proved, see [12] and [11], that if  $H$  has a finite number of connected components, then  $G/H$  fibers as a vector bundle over the minimal orbit of a maximal compact subgroup of  $G$  in  $G/H$ . In particular, this implies  $G/H$  has at most two ends in the sense of Freudenthal [5]. Thus a necessary condition in order that  $G/H$  have more than two ends is that  $H$  have an infinite number of connected components. Some examples of discrete subgroups of  $SL(2, \mathbb{C})$  whose coset spaces do have more than two ends were given by L. Bianchi [4]. And it was proved in [7] that for every integer  $k > 2$  there exists a discrete subgroup  $\Gamma_k \subset SL(2, \mathbb{C})$  such that  $SL(2, \mathbb{C})/\Gamma_k$  has  $k$  ends. We do not know of any other types of examples. The purpose of this short note is to make an observation about homogeneous complex manifolds which have more than two ends which essentially explains why this is so. We show that a homogeneous complex manifold  $G/H$  has more than two ends exactly if there are closed subgroups  $I$  and  $P$  of  $G$  with  $H \subset I \subset P \subset G$  such that  $G/P$  is a homogeneous rational manifold,  $I/H$  is a compact complex manifold and  $P/I = S/\Gamma$ , where  $\Gamma$  is a Zariski dense discrete subgroup of some semi-simple complex Lie group  $S$ , with  $S/\Gamma$  having the same number of ends as  $G/H$  has. Incidentally many questions remain; e.g., must the group  $S$  be simple?

2. **SOME TECHNICAL TOOLS.** We begin with a result of D. N. Akhiezer [1] which describes the structure of homogeneous spaces of algebraic groups with two ends.

**Theorem** Suppose  $G$  is a linear algebraic group over  $\mathbb{C}$  and  $H$  is an algebraic subgroup such that  $G/H$  has two ends. Then there exist a parabolic subgroup  $P$  in  $G$  containing  $H$  and a nontrivial character  $\phi: P \rightarrow \mathbb{C}^*$  with  $H = \ker \phi$ . Thus there is a homogeneous fibration  $G/H \xrightarrow{C^*} G/P$ , where  $G/P$  is a homogeneous rational manifold, and, in fact,  $P = N_G(H^\circ)$ .

One would like to fiber the space in question. The following allows control over the ends of locally trivial fiber bundles; for proofs see [1] or [6, Lemma 2]. For a connected manifold  $X$  we let  $e(X)$  denote the number of ends of  $X$ .

**Lemma** (*The fibration lemma*) Suppose  $X \xrightarrow{F} B$  is a locally trivial fiber bundle, where the fiber  $F$  is connected. If  $e(X) > 1$ , then one of the following holds:

- a)  $F$  is compact and  $e(B) = e(X)$
- b)  $B$  is compact and  $e(F) \geq e(X)$ . Moreover, if  $B$  is simply connected, then  $e(F) = e(X)$ .

We now make an observation about orbits in projective space which have at least two ends. This gives another proof of [10, Th. 10] for complex groups. But we find it instructive in that one sees how [8, Prop. 2] can play a role. It also underlines the relationship between the normalizer fibrations relative to  $G$  and  $G'$ , if  $G'$  is transitive. In this case note that  $P = N_G(H^\circ)$ , à la Tits [14].

**Proposition** Suppose  $G/H$  is an orbit in some  $\mathbb{P}_N$  of a connected complex Lie group  $G$  with  $e(G/H) \geq 2$ . Then  $G/H$  has two ends and there exists a parabolic subgroup  $P$  in  $G$  containing  $H$  such that  $P/H = \mathbb{C}^*$ .

**PROOF:** Since  $G$  is represented as a linear group acting on  $\mathbb{P}_N$ , its commutator subgroup  $G'$  has closed orbits in  $G/H$ . Thus one has the homogeneous fibration  $G/H \rightarrow G/G'H$ . The base of this fibration is a Stein abelian Lie group, e.g. see [9, p. 168], and thus is isomorphic to  $\mathbb{C}^k \times (\mathbb{C}^*)^l$ . Now since  $G'$  is connected, so are its orbits. If  $\dim G/G'H > 0$ , it follows from the fibration lemma that  $G/G'H$  has at least two ends. But  $G/G'H$  is homeomorphic to  $\mathbb{R}^{2k+l} \times (S^1)^l$  and so has exactly two ends. Hence  $k = 0$ ,  $l = 1$ , and  $G/G'H = \mathbb{C}^*$ . The fibration lemma also implies that the fiber is compact and hence is a homogeneous rational manifold. By [10, Lemma 6, p. 75] the fibration  $G/H \rightarrow G/PH$  realizes  $G/H$  as a homogeneous  $\mathbb{C}^*$ -bundle over a homogeneous rational manifold. Otherwise,  $G/G'H$  is a point and  $G'$  is transitive on  $G/H$ . In this case  $N_{G'}((G' \cap H)^\circ)$  is parabolic in  $G'$  and  $N_{G'}((G' \cap H)^\circ)/G' \cap H = \mathbb{C}^*$ , by the theorem of D. N. Akhiezer [1] quoted above. We must show that this  $\mathbb{C}^*$ -fibration is  $G$ -equivariant.

Consider the normalizer fibrations of  $G/H$  relative to the two groups  $G$  and  $G'$ . We claim that if  $G/H$  is an orbit in some projective space with more than one end and  $G'$  is transitive on  $G/H$ , then the  $G$  and  $G'$  normalizer fibrations of  $G/H$  coincide! As noted above the fibration  $\pi : G'/G' \cap H \rightarrow G'/A$  has fiber  $C^*$  and base a homogeneous rational manifold, where  $A := N_{G'}((G' \cap H)^\circ)$ . Note that  $H \subset N_G(A)$  and thus the fibers of the map  $\pi$  are  $G$ -invariant. This implies that one has the diagram

$$\begin{array}{ccc} G/H & = & G'/G' \cap H \\ \downarrow & & \downarrow C^* \\ G/P & = & G'/A \end{array}$$

where  $P$  is the isotropy of the  $G$ -action on  $G'/A$  and so is parabolic in  $G$ . Since  $P$  is connected and  $P/H = C^*$ , it follows that  $P \subset N_G(H^\circ)$ . We claim that  $P = N_G(H^\circ)$ . First we show that  $\dim N_G(H^\circ) = \dim P$ . Let  $L := N_G(H^\circ)/H^\circ$  and  $\Gamma := H/H^\circ$ . Because  $e(G/H) \geq 2$  and the fibration  $G/H \rightarrow G/N_G(H^\circ)$  has a rational manifold as base,  $e(L/\Gamma) \geq 2$ . Clearly  $L/\Gamma \subset G/H$  and so is Kähler. It then follows from [8, Prop. 2] that  $L^\circ$  is solvable and  $L/\Gamma$  is either a torus bundle over  $C^*$  or a Cousin group with two ends. But  $L/\Gamma$  lies in  $G/H$  and is the orbit of a linear group, i.e.,  $N_G(H^\circ)$ , in some projective space. This implies  $L/\Gamma = C^*$  and so  $\dim N_G(H^\circ) = \dim H + 1 = \dim P$ . Finally the fact that  $P$  is parabolic in  $G$  implies that  $P = N_G(H^\circ)$  and thus the two normalizer fibrations coincide. ††

**3. STRUCTURE OF  $G/H$  WITH  $e > 2$ .** We now look at the structure of complex homogeneous spaces which have more than two ends. It is known that semi-simple complex Lie groups possess many interesting discrete subgroups. The next result says that this is essentially the only way that one can have a homogeneous complex manifold with more than two ends. For, one has the locally trivial holomorphic fibrations  $G/H \xrightarrow{I/H} G/I \xrightarrow{P/I} G/P$ , where the ends of  $G/H$  are "displayed" by the fiber  $P/I$  which can be expressed as a coset space of a semi-simple complex Lie group modulo a Zariski dense discrete subgroup.

**Theorem** *Suppose  $G$  is a connected Lie group and  $H$  is a closed subgroup such that  $X := G/H$  has a  $G$ -invariant complex structure and  $e(X) > 2$ . Then there exist closed subgroups  $I$  and  $P$  of  $G$  with  $H \subset I \subset P \subset G$  such that*

- 1)  $G/P$  is a homogeneous rational manifold,
- 2)  $I/H$  is a compact complex parallelizable manifold,
- 3)  $P/I = S/\Gamma$ , where  $\Gamma$  is a Zariski dense discrete subgroup of some complex semi-simple Lie group  $S$ , with  $e(S/\Gamma) = e(G/H) > 2$ ,
- 4) The fibrations  $G/H \rightarrow G/I \rightarrow G/P$  are locally trivial holomorphic fiber bundles.

*Conversely, any  $G/H$  which fibers in such a fashion satisfies  $e(G/H) = e(S/\Gamma)$ .*

**PROOF:** If  $G$  is a complex Lie group, let  $N := N_G(H^\circ)$  be the normalizer of  $H^\circ$  in  $G$  and let  $\tilde{N}$  consist of those connected components of  $N$  which meet  $H$ . Then the fibration  $G/H \rightarrow G/\tilde{N}$  has connected fiber. We claim that  $G/\tilde{N}$  is compact and thus is a homogeneous rational manifold. Note that  $G/\tilde{N} \rightarrow G/N$  is a covering and so  $G'$  has closed orbits in  $G/\tilde{N}$ , since it has closed orbits in  $G/N$ . As in the proposition consider the fibration  $G/\tilde{N} \rightarrow G/G'\tilde{N}$ . If  $\dim G/G'\tilde{N} > 0$ , then  $G/G'\tilde{N}$  has at most two ends which implies  $G/\tilde{N}$  has at most two ends. And if  $G'$  is transitive on  $G/\tilde{N}$  and thus on  $G/N$ , then  $G' \cap N$  is algebraic in  $G'$ . Hence the covering  $G/\tilde{N} \rightarrow G/N$  is finite and  $G/\tilde{N}$  again has at most two ends. Since  $e(G/H) > 2$ , the fibration lemma implies  $G/\tilde{N}$  is compact and thus  $\tilde{N} = N$  and  $G/N$  is rational. Let  $P$  be a parabolic subgroup of minimal dimension contained in  $N$  and containing  $H$ . Then  $H^\circ$  is normal in  $P$  and the fibration lemma implies  $e(P/H) = e(G/H) > 2$ .

Now if  $G$  is real, let  $G/H \rightarrow G/U$  be the  $\mathfrak{g}$ -anticanonical fibration of  $G/H$ . It is known that  $U \subset N$ , the group  $U/H^\circ$  is a complex Lie group, and the  $\mathfrak{g}$ -anticanonical fibration is a locally trivial holomorphic fiber bundle, see [10, Cor. 5, p. 64]. Also from [10, Th. 10, p. 78] because  $e(G/H) > 2$ , it follows that  $G/U$  is rational and  $e(U/H) = e(G/H) > 2$ . Choose  $P$  to be a closed subgroup of minimal dimension contained in  $U$  and containing  $H$  such that the corresponding group  $P/H^\circ$  is complex and is parabolic in  $U/H^\circ$ . Again  $H^\circ$  is normal in  $P$  and  $e(P/H) = e(G/H) > 2$ .

In either of the above situations we let  $L := P/H^\circ$  and  $\Lambda := H/H^\circ$ . Any solv-manifold fibers as a vector bundle over a compact solv-manifold (see [3] and [13]) and thus a solv-manifold has at most two ends. Hence the group  $L$  cannot be solvable. If  $L$  is semi-simple, then the situation is handled by the last paragraph of the proof. Otherwise, by [6, Th. 2] there exists a fibration  $L/\Lambda \rightarrow L/J$ , where  $J$  is a proper closed complex subgroup of  $L$  which contains both  $\Lambda$  and the radical  $R_L$  of  $L$ . Assume  $J$  has minimal dimension with these properties. Let  $\tilde{J}$  consist of those connected components of  $J$  which meet  $\Lambda$ . Since  $\tilde{J}/\Lambda$  is connected, it follows by the fibration lemma that either the fiber  $\tilde{J}/\Lambda$  of the fibration  $L/\Lambda \rightarrow L/\tilde{J}$  is compact or the base  $L/\tilde{J}$  is compact.

Assume that the latter holds. Then  $e(\tilde{J}/\Lambda) \geq e(L/\Lambda) > 2$ . As above,  $\tilde{J}$  cannot be solvable. And since  $\tilde{J} \supset R_L$ , it is clear that  $\tilde{J}$  is not semi-simple. Again there exists a proper closed complex subgroup  $J_1$  of  $\tilde{J}$  which contains  $\Lambda$  and the radical  $R_{J_1}$ . (The argument in [6, Th. 2] is for connected Lie groups. We showed in [2, Prop. 3] how to apply this method to groups which are not connected.) But  $J_1 \supset R_{J_1} \supset R_L$  and  $\dim J_1 < \dim J$ . This contradicts the minimality of  $J$ . So  $L/\tilde{J}$  cannot be compact.

Hence  $\tilde{J}/\Lambda$  is compact and  $e(L/\tilde{J}) = e(L/\Lambda) > 2$  with  $\tilde{J} \supset R_L$ . Let  $Q := L/R_L$ . Note that  $Q$  is transitive on  $L/\tilde{J}$  and let  $M$  be the isotropy for this action, i.e.,

$L/\bar{J} = Q/M$ . Suppose  $A$  is any algebraic subgroup of  $Q$  containing  $M$ . We claim  $A = Q$ . Since  $e(Q/M) > 2$  and  $Q/A$ , as a quotient of algebraic groups, has at most two ends, it follows as in the first part of the proof that  $Q/A$  is compact and thus is a homogeneous rational manifold. Let  $P \xrightarrow{\phi} P/H^\circ =: L \xrightarrow{\psi} L/R_L =: Q$  be the natural maps. Then  $\phi^{-1}(\psi^{-1}(A))$  is parabolic in  $G$ , contains  $H$  and is contained in  $P$ . Since  $P$  was chosen to be a minimal parabolic subgroup containing  $H$ , it follows that  $A = Q$ . Now we apply this to  $N_Q(M^\circ)$  which is algebraic in  $Q$ . Thus  $N_Q(M^\circ) = Q$  and hence  $M^\circ$  is normal in  $Q$ . Then the quotient group  $S := Q/M^\circ$  is semi-simple and  $\Gamma := M/M^\circ$  is a discrete subgroup of  $S$ . Note that the Zariski closure  $\Gamma^Z$  of  $\Gamma$  in  $S$  is an algebraic group containing  $\Gamma$  and the same argument using the minimality of  $P$  now implies  $\Gamma^Z = S$ , i.e.,  $\Gamma$  is Zariski dense in  $S$ . Setting  $I := \phi^{-1}(\bar{J})$  we have  $P/I = L/\bar{J} = S/\Gamma$  and the various constructions yield the diagram

$$\begin{array}{ccccccc}
 P & \xrightarrow{\phi} & P/H^\circ =: L & \xrightarrow{\psi} & L/R_L =: Q & \longrightarrow & Q/M^\circ =: S \\
 U & & U & & U & & U \\
 I & \longrightarrow & \bar{J} & \longrightarrow & M & \longrightarrow & \Gamma \\
 U & & U & & & & \\
 H & \longrightarrow & \Lambda & & & & 
 \end{array}$$

Since the groups  $L, \bar{J}$ , and  $\Lambda$  are complex Lie groups, the fibration  $P/H \rightarrow P/I$  is a locally trivial holomorphic bundle. The converse statement is an easy consequence of the fibration lemma. ††

REMARK. The proof does not show that  $P$  is unique. If  $G$  is complex (resp. real), then we suspect  $P = N$  (resp.  $P = U$ ), analogous to Tits' result [14] in the compact case.

**Corollary** *Suppose  $G$  is a connected Lie group and  $H$  is a closed subgroup such that  $G/H$  has an invariant complex structure and the fundamental group of  $G/H$  is solvable. Then  $e(G/H) \leq 2$ .*

PROOF: Assume that  $G/H$  has more than two ends. Then one has a double fibration  $G/H \rightarrow G/I \rightarrow G/P$  with  $P/I = S/\Gamma$ . From the long exact homotopy sequence of the fibration  $G/H \xrightarrow{I/H} G/I$  and the fact that  $I/H$  is connected one sees that  $\pi_1(G/H)$  solvable implies  $\pi_1(G/I)$  is also solvable. Since  $G/P$  is simply connected and  $\pi_2(G/P)$  is an abelian group, it follows from the long exact homotopy sequence of the fibration  $G/I \xrightarrow{P/I} G/P$  that  $\pi_1(P/I)$  is solvable too. But  $P/I = S/\Gamma$  and this would imply that  $\Gamma$  is solvable. Since  $S = \Gamma^Z$ , then  $S$  would also be solvable, a contradiction. Thus  $e(G/H) \leq 2$ . ††

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## Closures on finite algebras

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*We introduce a new type of closures on the set of all finitary operations on a finite universe. We study such closures, which admit only finitely many closed sets.*

### 1 Introduction

Let  $k$  be a positive integer. Put  $E_k = \{0, \dots, k - 1\}$ , denote  $P_k^{(n)}$  the set of all  $n$ -ary functions of the  $k$ -valued logic (i.e. maps  $f : E_k^n \rightarrow E_k$ ) and put  $P_k = \cup_{n=1}^{\infty} P_k^{(n)}$ . Following [1] define the following binary operation  $*$  and the unary operations  $\zeta, \tau, \Delta$  and  $\nabla$  on  $P_k$ . Let  $f \in P_k^{(n)}$  and  $g \in P_k^{(m)}$ . Then  $\zeta f \in P_k^{(n)}, \tau f \in P_k^{(n)}, \Delta f \in P_k^{(\max(n-1, 1))}, \nabla f \in P_k^{(n+1)}$ , and  $f * g \in P_k^{(n+m-1)}$  are determined by:  
for  $n = 1$

$$\tau f = \zeta f = \Delta f = f$$

while for  $n > 1$

$$\begin{aligned} (\zeta f)(x_1, \dots, x_n) &= f(x_2, \dots, x_n, x_1), \\ (\tau f)(x_1, \dots, x_n) &= f(x_2, x_1, x_3, \dots, x_n), \\ (\Delta f)(x_1, \dots, x_{n-1}) &= f(x_1, x_1, x_2, \dots, x_{n-1}), \\ (\nabla f)(x_1, \dots, x_{n+1}) &= f(x_2, x_3, \dots, x_{n+1}), \\ (g * f)(x_1, \dots, x_{n+m-1}) &= f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{n+m-1}) \end{aligned}$$

for all  $x_1, \dots, x_{n+m-1} \in E_k$ .

The algebra  $\underline{P}_k = \langle P_k; \zeta, \tau, \Delta, \nabla, * \rangle$  is called the *full iterative algebra*. An *iterative set* is a subalgebra of  $\underline{P}_k$  i.e. a subset  $F$  of  $P_k$  closed under  $\zeta, \tau, \Delta, \nabla$  and  $*$ .

For  $f \in P_k^{(n)}$  and  $g_i \in P_k^{(m_i)} (i = 1, \dots, n)$ , put  $r = m_1 + \dots + m_n$  and define  $h = f^0(g_1, \dots, g_n) \in P_k^{(r)}$  by setting  $h(a_1, \dots, a_r) = f(g_1(a_1, \dots, a_{m_1}), \dots, g_n(a_{r-m_n+1}, \dots, a_r))$  for all  $a_1, \dots, a_r \in E_k$ . For subsets  $M$  and  $\Phi$  of  $P_k$  denote  $[M]^\Phi$  the least iterative set containing  $M$  such that  $f^0(g_1, \dots, g_n) \in [M]^\Phi$  whenever

$n > 0, f \in \Phi \cap P_k^{(n)}$  and  $g_1, \dots, g_n \in [M]^\Phi$  (i.e.  $[M]^\Phi$  is the subalgebra of  $\langle P_k; \{\zeta, \tau, \Delta, \nabla, *\} \cup \{f^0 : f \in \Phi\} \rangle$  generated by  $M$ ). For fixed  $\Phi \subseteq P_k$  the map  $M \rightarrow [M]^\Phi$  is a closure on  $P_k$ .

For  $M \subseteq P_k$  denote by  $[M]$  the least iterative set containing  $M$ . Clearly  $[M]^\Phi = [M]$ .

**Example** Let  $k = 3$ . Denote by  $T_0$  the set of all functions preserving 0 i.e.  $T_0 = \{f \in P_3 | f(0, \dots, 0) = 0\}$ . Let  $s_1(x), s_2(x) \in P_3^{(1)}$  be defined by  $s_1(0) = s_2(0) = s_1(1) = s_2(1) = 0, s_1(2) = 1, s_2(2) = 0$ . Let the set  $M$  consist of  $s_1(x), s_2(x)$  and all functions differing from  $s_1(x)$  or  $s_2(x)$  only in fictitious variables. Then we have  $[M] = M, [T_0 \cup M] = T_0$  and  $[M]^{T_0}$  is the set of functions  $f(x_1, \dots, x_n)$  from  $T_0$  ( $n \geq 1$ ) such that for the equivalence  $\theta = \{(0,0), (0,1), (1,0), (1,1), (2,2)\}$  on  $E_3$  the equality  $f(b_1, \dots, b_n) = f(c_1, \dots, c_n)$  holds whenever  $(b_1, c_1), \dots, (b_n, c_n) \in \theta$ .

For  $1 \leq i \leq n$  the  $i$ -th  $n$ -ary projection  $e_i^n$  is defined by setting  $e_i^n(a_1, \dots, a_n) = a_i$  for all  $a_1, \dots, a_n \in E_k$ . Denote by  $J$  the set of all projections. An iterative set  $C$  is a clone if  $C \supseteq J$ . We have:

**Fact:** Let  $M, \Phi \subseteq P_k$  and  $M' = [M], \Phi' = [\Phi \cup J]$ . Then  $[M]^\Phi = [M']^{\Phi'}$  (i.e. without loss of generality we may assume that  $\Phi$  is a clone and  $M$  an iterative set) and if  $M$  is a clone then  $[M]^\Phi = [M \cup \Phi]$ .

The set of functions  $M \subseteq P_k$  is called a *closed* ( $\Phi$ -closed) set if  $[M] = M$  (if  $[M]^\Phi = M$ ).

The purpose of this paper is an "amplification" of the standard closure operator in  $P_k$ . It is known [4] that the traditional closure operator  $M \rightarrow [M]$  over  $P_k, k > 2$ , admits a continuum of closed sets. From this point of view, this closure operator is "weak" and it is interesting to consider "stronger" operators with at most countably many of closed sets. "Amplification" is performed by adding new operations which are defined by functions. Every function  $f$  from  $P_k$  induces an operation on  $P_k$  by 'substituting functions in the variables of  $f$ '.

A clone  $M$  is *maximal* if there is no closed set  $G$  such that  $M \subset G \subset P_k$ . Let  $h$  be a positive integer. A subset  $\rho$  of  $E_k^h$  (i.e. a set of  $h$ -tuples over  $E_k$ ) is an  $h$ -ary relation on  $E_k$ . A function  $f \in P_k^{(n)}$  preserves  $\rho$  if for every  $h \times n$  matrix  $X = [a_{ij}]$  over  $E_k$  whose columns are all  $h$ -tuples from  $\rho$  we have

$$(f(a_{11}, \dots, a_{1n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \rho$$

Denote by  $Pol\rho$  the set of all functions preserving  $\rho$ . It is known [3] that every maximal clone is of the form  $Pol\rho$ , where  $\rho$  belongs to one of the six families  $P, E, L, B, C$  and  $O$  of relations (where the relations are defined by permutations, equivalence relations, elementary Abelian  $p$ -groups, homomorphic inverse images of elementary  $h$ -ary relations, central relations, and by bounded orders (see [3] for more detailed description)).

## 2 Finite type clones

The clone  $\Phi$  is said to be a *finite type clone* if the number of clones containing  $\Phi$  is finite. If  $\Phi$  is not a finite type clone then it is said to be an *infinite type clone*.

**Lemma.** For any  $k, k \geq 2$ , if  $\Phi \subset P_k$  is an infinite type clone, then the set of all  $\Phi$ -closed sets is infinite.

**Proof.** If  $\Phi$  is an infinite type then the number of clones  $\Phi_i$  such that  $\Phi \subset \Phi_i$  is not finite. Since  $[\Phi_i]^* = \Phi_i$ , the number of  $\Phi$ -closed sets is infinite.  $\square$

**Theorem 1** Let  $\Phi \subseteq P_2$  be a clone. Then the set of all  $\Phi$ -closed sets is finite if and only if  $\Phi$  is a finite type clone.

**Proof.** ( $\Rightarrow$ ) Lemma. ( $\Leftarrow$ ) Let  $\Phi$  be a finite type clone in  $P_2$ . If  $M$  is a clone then  $[M]^* = [M \cup \Phi]$  (by the above fact) is one of the finitely many clones containing  $\Phi$ . Thus let  $M$  be a preiterative set that is not a clone. From Post's classification [2] we know that  $M$  consists of constant functions. Clearly then  $[M]^*$  consists of constant functions as well.  $\square$

For any  $\Phi \subseteq P_2$  we can describe all  $\Phi$ -closed sets. For  $k > 2$  the statement of Theorem 1 is not true. The following statement can be proved [7]:

**Theorem 2** For every  $k > 2$  there exists a finite type clone  $\Psi \subset P_k$  with  $2^{k_0}$   $\Psi$ -closed sets.

A set  $\Psi$  in Theorem 2 is the set of all functions whose restriction to  $E_2$  is either projection or a constant with value in  $E_2$ .

Let us consider now  $P_k$ -closed sets. For a equivalence  $\theta$  on  $E_k$  denote by  $L_\theta$  the set of all functions with the following property:  $f \in L_\theta \cap P_k^{(n)} \Leftrightarrow f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$  whenever  $(a_1, b_1), \dots, (a_n, b_n) \in \theta$ .

**Theorem 3** A subset  $M$  of  $P_k$  is  $P_k$ -closed if and only if there exists an equivalence  $\theta$  on  $E_k$  such that  $M = L_\theta$ .

**Proof.** ( $\Leftarrow$ ) It is evident that for any  $\theta$  the equality  $[L_\theta]^{P_k} = L_\theta$  holds.

( $\Rightarrow$ ) Let  $[M]^{P_k} = M$ . Clearly  $M$  contains all constant functions. Set  $\theta = \bigcap \{ \ker s : s \in M \cap P_k^{(1)} \}$ . Suppose that  $f \in M \cap P_k^{(n)}$  and  $(a_1, b_1), \dots, (a_n, b_n) \in \theta$ . Consider the sequence of  $n$ -tuples  $\alpha_i = (a_1, \dots, a_i, b_{i+1}, \dots, b_n)$  ( $i = 0, \dots, n$ ). For any  $i < n$  the equality  $f(\alpha_i) = f(\alpha_{i+1})$  holds because  $f(a_1, \dots, a_i, x, b_{i+2}, \dots, b_n) \in M \cap P_k^{(1)}$ . Therefore  $f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$ . Thus  $M \subseteq L_\theta$ . To complete the proof we show that every function  $f \in L_\theta \cap P_k^{(n)}$  also belongs to  $M$ . Suppose  $M \cap P_k^{(1)} = \{s_1, \dots, s_r\}$ . Put

$$\bar{s}(x_1, \dots, x_n) = (s_1(x_1), \dots, s_r(x_1), s_1(x_2), \dots, s_r(x_2), \dots, s_1(x_n), \dots, s_r(x_n)).$$

It is evident that  $\bar{s}(a_1, \dots, a_n) = \bar{s}(b_1, \dots, b_n)$  if and only if  $(a_1, b_1), \dots, (a_n, b_n) \in \theta$ . Thus there exists  $g \in P_k^{(nr)}$  such that  $f(x_1, \dots, x_n) = g(\bar{s}(x_1, \dots, x_n))$  and so  $f \in [M]^{P_k} = M$ .  $\square$

### 3 Closures generated by maximal clones

After obtaining results (Theorem 2 and 3), it is interesting to consider the  $\Phi$ -closed sets where  $\Phi$  is a maximal clone.

Denote  $P_k^{(n)}$  the set of all  $n$ -ary partial functions of the  $k$ -valued logic and put  $P_k = \bigcup_{n=1}^{\infty} P_k^{(n)}$ . Let  $\rho$  be an  $h$ -ary relation on  $E_k$ .  $f \in P_k^{(n)}$  preserves  $\rho$  if for every  $h \times n$  matrix  $X = [a_{ij}]$  over  $E_k$  whose rows are all in the domain of  $f$  and columns are all  $h$ -tuples from  $\rho$ , we have  $(f(a_{11}, \dots, a_{1n}), \dots, f(a_{h1}, \dots, a_{hn})) \in \rho$ .

If every partial function preserving a relation  $\rho$  can be extended to a full function preserving  $\rho$  we say  $\rho$  possesses the *extension property*.

A partial  $n$ -ary function  $f$  with domain  $D$  and exactly  $m$  values has the  $\alpha$ -property if for every  $m$ -element subset  $I = \{r_1, \dots, r_m\}$  of  $D$  with  $|f(I)| = m$ , the restriction of  $f$  to  $I$  is a partial projection (i.e. there exists  $1 \leq p \leq n$  such that  $f(r_i) = r_{ip}$  for all  $i = 1, \dots, m$ ).

If any partial function preserving a relation  $\rho$  and possessing the  $\alpha$ -property can be extended to a full function preserving  $\rho$  we say  $\rho$  possesses the  $\alpha$ -extension property.

Let  $\rho$  be an  $h$ -ary relation on  $E_k$ . The following proposition can be proved [7].

**Proposition 1** *If  $\rho$  possesses the  $\alpha$ -extension property then the set of all  $\text{Pol}\rho$ -closed sets is finite.*

Clearly, if  $\rho$  possesses the extension property then  $\rho$  possesses the  $\alpha$ -extension property. Therefore as a consequence from Proposition 1 we have:

**Proposition 2** *If  $\rho$  possesses the extension property then the set of all  $\text{Pol}\rho$ -closed sets is finite.*

It follows from Lemma and Proposition 2 that if  $\rho$  possesses the extension property then the clone  $\text{Pol}\rho$  is of finite type. It is easy to see that  $\rho \in \text{PUEUC}$  possesses the extension property. Therefore following Proposition 2 we obtain:

**Theorem 4** *If  $\rho \in \text{PUEUC}$  then the set of all  $\text{Pol}\rho$ -closed sets is finite.*

Note that there exist relations  $\rho \in \text{BUOL}$  which possess neither the extension nor the  $\alpha$ -extension property.

**Theorem 5** *If  $\rho \in \text{BUL}$  then the set of all  $\text{Pol}\rho$ -closed sets is finite.*

Theorem 5 can be proved with help of the following [5]:

**Proposition 3** *If  $\rho \in \text{BUL}$ ,  $M$  and  $M'$  are  $\text{Pol}\rho$ -closed sets and  $M \cap P_k^{(1)} = M' \cap P_k^{(1)}$  then  $M = M'$ .*

Let us now consider the  $\text{Pol}\rho$ -closed sets where  $\rho \in \mathcal{O}$ . The following proposition can be proved [5].

**Proposition 4** *If  $k \leq 7$  and  $\rho \in \mathcal{O}$  then  $\rho$  possesses the  $\alpha$ -extension property.*

As a consequence from Proposition 1 and Proposition 4 we have:

**Theorem 6** *If  $\rho \in O$ ,  $k \leq 7$ , then the set of all  $Pol\rho$ -closed sets is finite.*

For  $k > 7$  there exist relations from  $O$  such that the statement of Proposition 6 is not true. In the case of the relation  $\rho$ , defined by the order in Fig., we can find the sequence of functions ( $n = 3, 4, \dots$ )  $f_n(x_1, \dots, x_n)$  such that  $[f_n]^{Pol\rho} \neq [f_i]^{Pol\rho}$  where  $3 \leq i < n$  [5]. Therefore we have:

**Theorem 7** *There exists  $\rho \in O$ ,  $k > 7$ , such that the set of all  $Pol\rho$ -closed sets is infinite.*

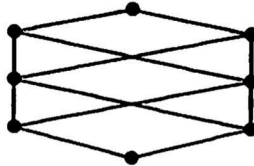


Fig.

**Theorem 8** *For  $k > 2$  there exists an infinite chain  $\Phi_1 \supset \Phi_2 \dots \supset \Phi_i \dots$  of clones in  $P_k$  such that the set of all  $\Phi_i$ -closed sets is finite for all  $i = 1, 2, \dots$*

*Proof.* For all  $\mu \geq 2$  set  $\rho_\mu = E_2^\mu \setminus \{(1, \dots, 1)\}$  and  $\Phi_\mu = Pol\rho_\mu$ . It can be shown that  $\Phi_2 \supset \Phi_3 \dots$ . We show that  $\rho_\mu$  has the  $\alpha$ -extension property. Let  $f$  be an  $n$ -ary partial function with domain  $D$  possessing the  $\alpha$ -property and preserving  $\rho_\mu$ . Extend  $f$  to  $f$  by setting  $f(x) = 0$  for all  $x \in E_2^n \setminus D$ . We show that  $f$  preserves  $\rho_\mu$ . Let  $X = [a_{ij}]$  be an  $\mu \times n$  matrix with rows  $r_1, \dots, r_\mu$  and all columns in  $\rho_\mu$ . We show that  $f(r_1), \dots, f(r_\mu) \in E_2$ . Suppose to the contrary that some  $q = f(r_i) \notin E_2$ . Then  $r_i \in D$  and we can choose  $I \subseteq D$  so that  $r_i \in I$  and  $|f(I)| = |im f|$ . By the  $\alpha$ -property,  $f$  restricted to  $I$  is a partial projection and so  $q = f(r_i) = a_{ip}$  for some  $1 \leq p \leq n$ . However,  $a_{ip} \in E_2$  because the  $p$ -th column of  $X$  belongs to  $\rho_\mu \subseteq E_2^\mu$ . This contradiction proves the claim. If at least one  $r_i \notin D$  then  $f(r_i) = 0$ , hence  $(f(r_1), \dots, f(r_\mu)) \in \rho_\mu$  and we are done. Thus let  $r_1, \dots, r_\mu \in D$ . Then

$$(f(r_1), \dots, f(r_\mu)) = (f(r_1), \dots, f(r_\mu)) \in \rho_\mu$$

because  $f$  preserves  $\rho_\mu$  (note that  $\rho_\mu$  does not possess the extension property).□

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## ON HUYGENS' PRINCIPLE FOR RELATIVISTIC WAVE EQUATIONS

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**ABSTRACT:** The history of attempts to resolve Hadamard's problem of determining all the second order linear partial differential equations of normal hyperbolic type which satisfy Huygens' minor premise is reviewed. The current status of the problem is described for conformally invariant relativistic wave equations on curved space-time including a description of a programme to prove a modified form of Hadamard's conjecture. A new necessary condition is given for the non-self-adjoint scalar equation to satisfy Huygens' principle. Some consequences of this condition are stated.

In 1923 Hadamard [1] reformulated Huygens' principle in the form of a syllogism and posed the general problem, as yet unsolved, of determining up to equivalence all the second-order linear hyperbolic partial differential equations in  $n$  independent variables that satisfy Huygens' minor premise (B). It is in this sense that Huygens' principle shall be understood in this article. We recall that such an equation may be written in coordinate invariant form as

$$\square u + A^a u_{,a} + C u = 0, \quad (1)$$

where  $\square$  denotes the Laplace Beltrami operator corresponding to the metric tensor  $g_{ab}$  of a  $n$ -dimensional Lorentzian space  $V_n$  of signature  $2 - n$ ,  $u$  the unknown scalar function and " $,$ " denotes the partial derivative with respect to the natural coordinator  $x^a$ . The coefficients  $g_{ab}$ ,  $A^a$ , and  $C$  and  $V_n$  are assumed to be of class  $C^\infty$ .

Cauchy's problem for equation (1) is the problem of determining a solution which assumes given values of  $u$  and its normal derivative on a given space-like  $(n-1)$ -dimensional submanifold  $S$ . These given values are called Cauchy data. The local existence and uniqueness of the solution of Cauchy's problem for (1) has been proved by Hadamard. A modern treatment using distributions has been given by Friedlander [2]. The considerations of this paper will be purely local.

Of particular importance in Cauchy's problem is the domain of dependence of the solution. In this regard, Hadamard has shown that for any  $x_0$ ,  $u(x_0)$  depends only on the data in the interior of the intersection of the past null conoid  $C^-(x_0)$  with  $S$  and on the intersection itself. Equation (1) is said to satisfy *Huygens' principle* iff for every Cauchy problem and every point  $x_0 \in V_n$  the solution depends only on the data in an arbitrarily small neighbourhood of  $S \cap C^-(x_0)$ . This definition is equivalent to the validity of Huygens' minor premise. An equation satisfying Huygens' principle is called a *Huygens' equation*. The most familiar of these equations are the ordinary wave equations that may be obtained from (1) by setting  $g_{ab} = \text{diag}(1, -1, \dots, -1)$ ,  $A^a = C = 0$ , and  $n = 2m$ ,  $m = 2, 3, \dots$ . Hadamard showed that in order that (1) be a Huygens' equation it is necessary that

$n$  be even and  $\geq 4$ . He also obtained a complicated necessary and sufficient condition for (1) to be a Huygens' equation and considered the problem of determining all the Huygens' equations. In this regard he wondered if every such equation was equivalent to the ordinary wave equation with the appropriate number of independent variables. This is often called "Hadamard's conjecture" in the literature. We recall that two equations of the form (1) are equivalent iff one may be transformed to the other by any of the following trivial transformations that preserve the Huygens' character of the equation:

(a) multiplication of both sides of (1) by non-vanishing factor  $e^{-2\psi(x)}$ , which transformation induces a conformal transformation of the metric:

$$\tilde{g}_{ab} = e^{2\psi} g_{ab}; \quad (2)$$

(b) replacement of the unknown function by  $\lambda u$ , where  $\lambda(x)$  is a non-vanishing function.

The conjecture has been proved in the case  $n = 4$ ,  $g^{ab}$  constant by Mathisson [3], Hadamard [4] and Asgeirsson [5]. However, it has been disproved by Stellmacher [6] who gave counter examples for  $n = 6, 8, \dots$ , and by Günther [7] who provided a family of counter examples in the physically interesting case  $n = 4$ , based on the plane wave space-time with metric

$$ds^2 = 2dv[du + (Dx^2 + D\bar{x}^2 + cx\bar{x})dv] - 2dzd\bar{z}, \quad (3)$$

where  $D \neq 0$ , and  $c$  are functions only of  $v$ . McLenaghan [8] subsequently showed that any Huygens' equation (1) on a conformally empty background space-time is necessarily equivalent to the wave equation  $\square u = 0$ , or the plane wave space-time with metric (3). These results have been extended to Maxwell's equation

$$d\omega = 0, \quad \delta\omega = 0, \quad (4)$$

where  $d$  denotes the exterior derivative,  $\delta$  the co-derivative and  $\omega$  the Maxwell 2-form, and to Weyl's neutrino equation

$$\nabla^B{}_{\Lambda} \phi_B = 0, \quad (5)$$

where  $\nabla^B{}_{\Lambda}$  denotes the covariant derivative on 2-spinors and  $\phi_{\Lambda}$  a valence one two-spinor by the work of Günther and Wunsch [9][10][11][12], who have developed criteria for the validity of Huygens' principle for these equations analogous to that for the scalar equation (1).

Hadamard's problem for each of the above equations, is that of determining all the space-times for which Huygens' principle is true for the particular equation. In view of the conformal invariance of the principle, the determination can only be effected up to an arbitrary conformal transformation (2).

Huygens' principle is valid for the conformally invariant wave equation

$$\square u + \frac{R}{6}u = 0, \quad (6)$$

where  $R$  denotes the curvature scalar and for equations (4) and (5) on any conformally flat space-time and also on any conformally plane wave space-time [7][11][12]. These are the only known space-times for which the principle is valid for (4), (5) and (6). It has been proved [8][12][13] that the only conformally empty space-time for which the principle holds for any of the three equations in question, are those conformal to the plane wave space-time with metric (3). However, Hadamard's

problem remains open in the general case. A detailed review of the status of the problem up to 1985 is given in Günther [14].

The results described in the preceding paragraphs suggest that it may be true that every space-time on which any of the equations (6), (4) or (5) satisfies Huygens' principle, is conformally related to the plane wave space-time or is conformally flat and that any Huygens' equation (1) is equivalent to  $\square u = 0$ , on a plane wave or flat background space-time. Carminati and McLenaghan [15] have outlined a programme to prove this modified Hadamard's conjecture based on the conformally invariant Petrov classification of the Weyl conformal curvature tensor  $C_{abcd}$  of space-time [16][17].

The procedure for proof of the conjecture consists in considering separately space-times of the five possible Petrov types. To date the conjecture has been settled for (6) on type *N* space-times with partial results for the equations (4) and (5). It has also been settled for all three equations on type *D* space-times and on type *III* space-times under a certain mild assumption. At present only partial results are available for type *II* space-times.

**THEOREM 1.** [15][18][19][20][21] *The validity of Huygens' principle for the conformally invariant scalar wave equation (6), or Maxwell's equations (4), or Weyl's neutrino equation (5) on any algebraically special space-time implies that the repeated principle null congruences of the Weyl tensor (defined by the null vector  $l^a$ ) are geodesic, shear-free and hypersurface orthogonal.*

In the case of Petrov types *N*, *D* and *III* we have the further results:

**THEOREM 2.** [15][23] *Any non-self-adjoint scalar wave equation on any Petrov type *N* background space-time satisfies Huygens' principle if and only if it is equivalent to the wave equation*

$$\square u = 0 \quad (7)$$

on an exact plane wave space-time with metric (3).

The analogous result for equations (4) and (5) is that the space-time is conformally related to a special generalized plane wave space-time. The detailed result is given in [15].

**THEOREM 3.** [18][20][21] *There exist no Petrov type *D* space-times on which the conformally invariant scalar wave equation (6), Maxwell's equation (4), or Weyl's equation (5), satisfy the Huygens' principle.*

**THEOREM 4.** [15] *The validity of Huygens' principle for the conformally invariant scalar wave equation (6), or Maxwell's equations (4) or Weyl's neutrino equation (5) on any Petrov type *III* space-time implies that the space-time is conformally related to one in which every repeated principal null vector field  $l_a$  of the Weyl tensor is recurrent, that is*

$$l_{[a}l_{b]c} = 0. \quad (8)$$

The above result is extended to equation (1) by the following theorem which appears to be new.

**THEOREM 5.** *The validity of Huygens' principle for any non-self-adjoint equation (1) on any type *III* background space-time implies that the space-time is conformally related to one in which every repeated null vector field of the Weyl tensor is recurrent.*

These theorems are a consequence of necessary conditions for equations (1), (5), or (6) to satisfy Huygens' principle obtained by a number of authors [8][24][9][12][25] [26][27].

We shall give here only the conditions for equation (1).

$$I \quad C = \frac{1}{2}A^a{}_{;a} + \frac{1}{4}A_a A^a + \frac{1}{6}R, \quad (9)$$

$$II \quad H^h{}_{a;h} = 0, \quad (10)$$

$$III \quad S_{abb;h} - \frac{1}{2}C^h{}_{cd}{}^i L_{hi} = -5(H_{ab}H_b{}^h - \frac{1}{4}g_{ab}H_{hi}H^{hi}), \quad (11)$$

$$IV \quad TS[3S_{abb}H^h{}_c + C^h{}_{ab}{}^i H_{chi}] = 0, \quad (12)$$

$$V \quad TS[3C_{hcd;lm}C^h{}_{ef}{}^i{}^m + 8C^h{}_{cd}{}^i{}_{;l}S_{hij} + 40S_{cd}{}^h S_{efh} - 8C^h{}_{cd}{}^i S_{hij} - \\ 24C^h{}_{cd}{}^i S_{efh;ij} + 4C^h{}_{cd}{}^i C_l{}^{mab} L_{fm} + 12C^h{}_{cd}{}^i D^m{}_{efi} L_{hkm} + \\ 12H_{hcd} H^h{}_f - 16H_{hcd} H^h{}_{e;f} - 84H^h{}_c C_{cdi} H^i{}_f - 18H_{bc} H^h{}_d L_{ef}] = 0, \quad (13)$$

where

$$H_{ab} := A_{(a,b)}, \quad (14)$$

$$C_{abcd} := R_{abcd} - 2g_{(a(b}L_{c)j}), \quad (15)$$

$$L_{ab} := -R_{ab} + \frac{R}{6}g_{ab}, \quad (16)$$

$$S_{abc} := L_{a[b;c]}. \quad (17)$$

In the above  $R_{abcd}$  denotes the Riemann curvature tensor,  $R_{ab}$  the Ricci tensor and  $TS [ ]$  the operator which takes the trace free symmetric part of the enclosed tensor.

The proof of Theorem 5 required the derivation of a new necessary condition given below.

$$VI \quad TS[36C^h{}_{ab}{}^i C_{icdm;h} H^m{}_e - 6C^h{}_{ab}{}^i{}_{;c} C_{ide}{}^m H_{hm} - 138S_{ab}{}^h C_{hcd} H^i{}_e + \\ 6S_{abb} H^h{}_{c;de} + 6C^h{}_{ab}{}^i{}_{;c} H_{hd;je} - 24S_{abb;ic} H^h{}_{d;e} + \\ 12C^h{}_{ab}{}^i L_{hcd} H^i{}_{d;e} - 9C^h{}_{ab}{}^i{}_{;c} L_{hd} H_{ie} - 9S_{abb} L_{cd} H^h{}_e] = 0 \quad (18)$$

It is believed that conditions *I* through *VI* should imply that any Huygens' equation (1) on any type *III* background space-time is equivalent to the conformally invariant equation (6). Theorem 5 is a preliminary result in this direction.

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**On the von Staudt-Clausen theorem.**

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1. The von Staudt-Clausen theorem states that the denominator of the Bernoulli number  $B_{2k}$  is the product of those different primes  $p$  for which  $p-1$  divides  $2k$  and more precisely

$$B_{2k} + \sum_{p-1|2k} \frac{1}{p}$$

is an integer.

The  $n$ -th Bernoulli polynomial  $B_n(t)$  is defined by

$$\frac{x e^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n.$$

Then  $B_n(0) = B_n$  is the  $n$ -th Bernoulli number and we have  $B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$ .

In this note we will give a simple proof of the following generalization of the von Staudt-Clausen theorem to the Bernoulli polynomials :

**THEOREM .** *Let  $s > 0$  and  $t \geq 0$  be integers,  $p$  denote a prime number.*

*If  $n$  is an even positive integer or  $n$  is equal to 1, then*

$$s^n B_n\left(\frac{t}{s}\right) + \sum_{\substack{p-1|n \\ p \nmid s}} \frac{1}{p} \text{ is an integer.} \tag{1.1}$$

*If  $n \geq 3$  is odd then  $s^n B_n\left(\frac{t}{s}\right)$  is an integer.*

**Remarks.**

1. Putting  $s=t=1$  in (1.1) we get the von Staudt-Clausen theorem since

$$B_n(1) = (-1)^n B_n.$$

2. We have by Fermat's theorem for  $s$  positive integer

$$(s^n - 1) \sum_{\substack{p-1|n \\ p \nmid s}} \frac{1}{p} \text{ is an integer and obtain thus the Almkvist-Meurman}$$

theorem (see [1], theorem 2) that  $s^n (B_n(\frac{t}{s}) - B_n)$  is an integer. (1.3)

The Almkvist's and Meurman's proof of (1.3) is elementary but rather complicated. So, we give here a new and short proof applying the von Staudt-Clausen Theorem.

3 .Since the Hurwitz zeta-function for a positive integer  $n$

$$\zeta(1-n, a) = - \frac{B_n(a)}{n} \quad \text{we have the following}$$

**COROLLARY.** For  $t$  and  $s$  positive integers

$$s^n (n\zeta(1-n, \frac{t}{s}) + B_n) \quad \text{is an integer.} \tag{1.4}$$

(for odd  $n \geq 3$  see [1] theorem 9).

In the proof of Theorem we use the following known (see [2] p.271 and 273) property of sums of binomial coefficients

**LEMMA.** Let  $n$  be a natural number and let  $p$  be an arbitrary prime number. Denote

$$S_p(n) = \sum_{k=1}^n \binom{n}{k(p-1)} \quad (k = \lfloor \frac{n}{p-1} \rfloor).$$

Then

$$S_p(n) \equiv 0 \pmod{p} \quad \text{if } p-1 \nmid n \quad \text{and} \quad S_p(n) \equiv 1 \pmod{p} \quad \text{if } p-1 \mid n. \tag{1.5}$$

### 2. Proof of Theorem .

The idea of the proof is to do induction on  $t$  with  $s$  fixed via the addition formula

$$B_n(x+y) = \sum_{m=0}^n \binom{n}{m} B_m(x) y^{n-m} \quad \text{with } x = \frac{t}{s} \quad \text{and } y = \frac{1}{s}.$$

We have

$$s^n B_n(\frac{t+1}{s}) = \sum_{m=0}^n \binom{n}{m} B_m(\frac{t}{s}) s^m. \tag{2.1}$$

That our theorem holds for  $t=0$  is clear by the use of the von Staudt - Clausen Theorem and the Fermat Theorem since  $B_n(0) = B_n$ .

So assume that it holds for  $t$  and consider first  $s$  even. By the induction hypothesis and by (2.1) we see that we get an integer by adding the following number to  $s^n B_n(\frac{t+1}{s})$  :

$$\sum_{\substack{m=0 \\ m \text{ is even}}}^n \binom{n}{m} \sum_{\substack{p-1 \mid m \\ p \nmid s}} \frac{1}{p} = \sum_{\substack{p \nmid s \\ p-1 \mid n}} \frac{1}{p} \sum_{m=0}^n \binom{n}{m} = \sum_{\substack{p-1 \mid n \\ p \nmid s}} \frac{1}{p} + A_1$$

where using (1.5)  $A_1$  is a certain integer. So the result holds also for  $t+1$ .

Considering next the case  $s$  odd with  $p=2$  and  $p > 2$  separately we see that we get an integer by adding the following number to  $s^n B_n(\frac{t+1}{s})$  :

$$\sum_{m=1}^n \sum_{\substack{m=0 \\ \text{or } m \text{ even}}} \binom{n}{m} \sum_{\substack{p-1|m \\ p \nmid n}} \frac{1}{p} = \sum_{p \nmid n} \frac{1}{p} \sum_{\substack{m=0 \\ p-1|m \\ \text{or } m \text{ even}}}^n \binom{n}{m} = \sum_{\substack{p \nmid n \\ p > 2}} \frac{1}{p} \sum_{m=0}^n \binom{n}{m} + \frac{n}{2} + \\ + \frac{1}{2} \sum_{m=0}^n \binom{n}{m} = \sum_{p-1|n} \frac{1}{p} + \frac{n}{2} + 2^{n-2} + A_2$$

where  $A_2$  is a certain integer by lemma and the result follows for  $t+1$ . By the use of mathematical induction we get Theorem.

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## PROPRIETES D'EQUILIBRE FORT DANS LES JEUX DIFFERENTIELS

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Quand on utilise l'équilibre de Nash en qualité de solution dans les jeux différentiels sans coalition, tout joueur doit être convaincu que tous ses partenaires du jeu vont utiliser leurs stratégies de l'ensemble des stratégies d'équilibres de Nash. Ce défaut peut être éliminé si on suit le principe de l'équilibre fort donné par C.BERG [ 1 ].

Dans le présent article on étudie les propriétés de l'équilibre fort sur la base de la formalisation mathématique des jeux différentiels proposée par N.N.Krassovski [ 2 ].

### 1. Formalisation d'équilibre fort

On considère un jeu différentiel à deux joueurs

$$(1.1) \quad \langle (1.2), \Sigma, (U_j)_{j=1,2}, (J_j - F_j(x|0))_{j=1,2} \rangle, (J_j = \beta_j)_{j=1,2} + ct$$

où (1.2) désigne les numéros des joueurs avec des intérêts opposés.  $\Sigma$  est le système de contrôle défini par le système d'équations différentielles  $\dot{x} = f(t, x, u_1, u_2)$ ,  $t \in [0, \theta]$ ,  $x \in R^n$ ,  $u_j$  - fonction de contrôle de  $i$ -ème joueur ( $i=1,2$ );  $(U_j)$  est l'ensemble des stratégies de  $i$ -ème joueur;  $J_j$  - fonction gain de  $i$ -ème joueur.

Supposons que les conditions suivantes sont vérifiées:

**Conditions A.** La fonction  $f: [0, \theta] \times R^n \times P_1 \times P_2 \rightarrow R^n$  est continue. Les ensembles  $P_j$  sont des compacts sur  $R^{m_j}$ . Pour tout sous-ensemble borné  $G$  dans l'espace de la position  $(t, x) \in [0, \theta] \times R^n$  il existe  $\lambda(G) > 0$  telle que

$$\| f(t^{(1)}, x^{(1)}, u_1, u_2) - f(t^{(2)}, x^{(2)}, u_1, u_2) \| \leq \lambda(G) (\| x^{(1)} - x^{(2)} \| + |t^{(1)} - t^{(2)}|)$$

quelque soit la position  $(t^{(j)}, x^{(j)}) \in G$  ( $j = 1, 2$ ) et  $u_j \in P_j$ .

Pour  $\forall t \in [t_0, \theta]$ ,  $x \in R^n$ ,  $u_j \in P_j$  il y a inégalité

$$\| f(t, x, u_1, u_2) \| \leq \gamma (1 + \| x \|), \quad \gamma = ct > 0.$$

On appelle stratégie  $U_j$  de  $i$ -ème joueur chaque fonction multiforme  $u_j: t, x \in P_j$ . La correspondance entre les stratégies  $U_j$  et les fonctions  $u_j(t, x)$  nous allons noter par  $U_j + u_j(t, x)$ . Le couple de stratégies  $(U_1, U_2)$  s'appelle parfois situation du jeu (1.1).

Soit  $U_1 + u_1(t, x)$  une stratégie de premier joueur. Soit  $\Delta^{(r)}$  un partage d'intervalle  $[t_0, \theta]$  avec les points  $\tau_i^{(r)}$ , c'est-à-dire

$$t_0 = \tau_0^{(r)} < \tau_1^{(r)} < \dots < \tau_k^{(r)} = \theta. \text{ Le mouvement par pas}$$

$x_{\Delta(r)}(\cdot, t^{(r)}, x^{(r)}, U_1, u_2^{(r)}(\cdot)) = x(\cdot, \Delta^{(r)}) = x(t, \Delta^{(r)}, t^{(r)}, t \leq \theta$

represent d'après [ 3 ] une solution de l'équation

$$(1.2) (t, \Delta^{(r)}) - x(t_j^{(r)}, \Delta^{(r)}) + \int_{t_j^{(r)}}^t [ \tau, x(\tau, \Delta^{(r)}), u_1(t_j^{(r)}, x(t_j^{(r)}, \Delta^{(r)})) u_2^{(r)}(\tau) ] dt$$

ou  $t_j^{(r)} \leq t \leq t_{j+1}^{(r)}$  ( $j = 0, 1, \dots, k^{(r)} - 1$ ),  $x(t^{(r)}, \Delta^{(r)}) = x^{(r)}$ , ( $r = 0, 1, 2, \dots$ )  
 où  $u_2^{(r)}(t)$  est une fonction mesurable de Borel telle que  $u_2^{(r)}(t) \in P_2$ ,  $\forall t \in [t^{(r)}, \theta]$ .

Si les conditions A sont satisfaites, le système des équations (1.2) admet un et un seul mouvement continu  $x(t, \Delta^{(r)})$ , qu'on peut élargir sur l'intervalle  $[t_0, \theta]$  [ 3, p.53 ]

Le mouvement  $x(t, t_0, x_0, U_1)$ ,  $t_0 \leq t \leq \theta$  du système  $\Sigma$ , provoqué d'une stratégie  $U_1$  par la position initiale  $(t_0, x_0)$ , s'appelle chaque fonction continue  $x[\cdot] = (x(t), t_0 \leq t \leq \theta)$  pour laquelle il existe une suite de mouvements par pas  $(x_{\Delta(r)}(\cdot, t^{(r)}, x^{(r)}, U_1, u_2^{(r)}(\cdot)))$  uniformément convergente à cette fonction pour  $t^{(r)} \rightarrow t_0$ ,  
 $\|x^{(r)} - x_0\| \rightarrow 0$ ,  $\sup |t_{j+1}^{(r)} - t_j^{(r)}| \rightarrow 0$ , quand  $r \rightarrow \infty$

Dans ([ 2 ], [ 3 ]) on demontre que le faisceau  $\mathbb{M}[t_0, x_0, U_1]$  de mouvements est un compact continu dans  $C_n[t_0, \theta]$ , alors l'ensemble  $X[t_0, t_0, x_0, U_1] = \mathbb{M}[t_0, x_0, U_1] \cap (t = \theta)$  represent un compact dans  $R^n$ .

Analogiquement on définit le mouvement  $x[\cdot, t_0, x_0, U_2]$  du système  $\Sigma$  provoqué de la stratégie  $U_2 + u_2(t, x)$  de position initiale  $(t_0, x_0)$ . Dans la formalisation du mouvement  $x[\cdot, t_0, x_0, U_1, U_2]$  provoqué de la situation  $U = (U_1, U_2) \in \mathbb{M}$  de la position  $(t_0, x_0)$  il faut poser dans (1.2)  $u_2^{(r)}(t) = u_2^{(r)}(t_j^{(r)}, x(t_j^{(r)}, \Delta^{(r)})$

L'intersection  $X[t_0, t_0, x_0, U_1, U_2]$  du faisceau de mouvement  $[\cdot, t_0, x_0, U_1, U_2]$  est un compact dans  $R^n$ . Les fonctions gains  $F_i = F_i(x[\theta])$  ( $i = 1, 2$ ) ou  $x[\theta] \in X[t_0, t_0, x_0, U_1]$  ou  $x[\theta] \in X[t_0, t_0, x_0, U_1, U_2]$  sont définies sur les ensembles fermés et bornés  $X[t_0, t_0, x_0, U_i]$  ( $i = 1, 2$ )  $X[t_0, t_0, x_0, U_1, U_2]$

**Conditions B** Supposons que les fonctions  $F_i(x)$  sont continues dans  $R^n$ . Plus loin on suppose que les conditions A et B soit vérifiées. Ainci tous les composants du jeu différentiel (1.1) sont définis et le probleme de chaque joueur est le choix de stratégie pour laquelle sa fonction-gain est maximale.

Dans les jeux différentiels sans coalition avec la somme non nulle il n'existe pas un seul concept en qualité de solution du jeu.

Dans un grand nombre des articles scientifiques en qualité de solution du jeu différentiel on utilise l'équilibre de Nash . Cependant une telle notion utilisée en qualité de solution dans les jeux différentiels possède certain défauts [4] . Par exemple si un joueur utilise dans un jeu la stratégie d'équilibre de Nash , il doit être certain , que son partenaire de jeu utilise aussi sa stratégie de la même ensemble de stratégies d'équilibres de Nash. Dans un grand nombre des problèmes pratiques une telle certitude est rare, c'est pourquoi dans le présent travail on propose pour les jeux différentiels de position

une autre notion - solution du jeu appelée équilibre fort (équilibre de Berge) suivant la définition pour le point d'équilibre fort [1,p.96]

**Définition** Le complete de stratégies  $U^* = (U_1^*, U_2^*) \in \mathbb{M}$  s'appelle équilibre fort (équilibre de Berg ) de jeu différentiel (1.1) avec la position initiale  $(t_0, x_0) \in [0, \theta] \times \mathbb{R}^n$  , si

$$(1.3) \quad \max F_1(x|\theta, t_0, x_0, U_1^*, U_2) \leq \min F_1(x|\theta, t_0, x_0, U^*) \quad \forall U_2 \in \mathbb{M}_2$$

$$(1.4) \quad \max_{x|\cdot} F_2(x|\theta, t_0, x_0, U_1, U_2^*) \leq \min_{x|\cdot} F_2(x|\theta, t_0, x_0, U^*) \quad \forall U_1 \in \mathbb{M}_1$$

L'apparition de max et de min dans (1.3) et (1.4) est du au faite que, quelque soit la situation  $(U_1, U_2) \in \mathbb{M}$  les extrémités " droites" du mouvement  $X| \theta, t_0, x_0, U_1, U_2$  dans  $\mathbb{R}^n$  , où sont définis les fonctions-gain. D'autre part max et min s'obtient dans (1.3) et (1.4) grace à la continuité des fonctions  $F_i(x)$ .

Conformément à cette définition chaque joueur obtient le gain maximal quand son partenaire a la meilleure conduite. Si ce partenaire utilise une stratégie différente de celle d'équilibre fort. le gain de ce joueur dans le cas général diminue.

## 2. Propriétés de l'équilibre fort

**Propriété 2.1** Le gain des joueurs dans la situation d'équilibre fort est uniforme.

En effet si dans le second membre des égalités (1.3) et (1.4) on pose  $U_2 = U_2^*$  et  $U_1 = U_1^*$  on obtient

$$(2.1) \quad \max_{x|\cdot} F_i(x|\theta, t_0, x_0, U^*) = \min_{x|\cdot} F_i(x|\theta, t_0, x_0, U^*) = F_i(x|\theta, t_0, x_0, U^*) \quad (i=1,2)$$

**Propriété 2.2**  $\max F_j(x|\theta, t_0, x_0, U_j^*) = F_j(x|\theta, t_0, x_0, U^*) \quad (i=1,2)$

En effet si dans (1.3) on pose  $U_2 = U_2^*$  alors nous obtenons

$$(2.3) \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*, U_2 + P_2) - \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*) \leq \min_{x|\cdot} F_1(x|\theta, t_0, x_0, U^*)$$

D'autre part d'après [2]  $X(\theta, t_0, x_0, U_1^*) \supseteq X(\theta, t_0, x_0, U_1^*, U_2^*)$  alors

$$(2.4) \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*) \geq \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*, U_2^*) \geq \min_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*, U_2^*)$$

De (2.3), (2.4) et (2.1) il suit l'égalité (2.2). La raison du jeu dans (2.2) est la suivante: Quelque soit la conduite d'un partenaire dans un jeu différentiel, le gain maximal de chaque participant ne dépasse pas son gain dans l'équilibre fort. C'est -à-dire le gain de chaque joueur dans l'équilibre fort est stable par rapport aux écarts du partenaire.

### Propriété 2.3

$$(2.5) \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U^*) \geq \max_{U_2} \min_{x|\cdot} F_1(x|\theta, t_0, x_0, U_2)$$

$$(2.6) \max_{x|\cdot} F_2(x|\theta, t_0, x_0, U^*) \geq \max_{U_1} \min_{x|\cdot} F_2(x|\theta, t_0, x_0, U_1)$$

Pour établir l'inégalité (2.5) nous allons considérer la stratégie  $\min \max U_1$ :

$$\max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*) \geq \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*)$$

A l'aide de (1.3), (2.7) et (2.2) nous obtenons

$$F_1(x|\theta, t_0, x_0, U^*) = \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*) \geq \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1^*) \geq \min_{U_1} \max_{x|\cdot} F_1(x|\theta, t_0, x_0, U_1)$$

D'après la démonstration dans [3, p.240]

$$\min_{U_1} \max_{x|\cdot} F_1(\theta, t_0, x_0, U_1) \geq \max_{U_2} \min_{x|\cdot} F_1(\theta, t_0, x_0, U_2)$$

d'où on obtient la vérité d'inégalité (2.5) L'inégalité (2.6) est démontrée de la même façon.

La raison de jeu dans (2.5) est la suivante: Dans la situation de l'équilibre fort, un joueur obtient un gain en tout cas supérieur à ce qu'on peut lui être garantie par ses partenaires.

Propriété 2.4 La situation d'équilibre fort pour les jeux différentiels antagonistes coïncide avec le point-selle.

En effet dans les jeux compétitifs

$$(2.8) F_1(x) + F_2(x) = F(x) \quad \forall x \in R^n$$

par définition [4, p.25] la situation  $(U_1^*, U_2^*)$  est point-selle du jeu compétitif  $\Gamma_c = \langle (1, 2), \Sigma, (\mathbb{U}_i), (i = 1, 2), J = F(x|\theta) \rangle$  de la position initiale  $(t_0, x_0)$ , si l'inégalité

(2.9)  $F(x|\theta, t_0, x_0, U_1, U_2) \leq F(x|\theta, t_0, x_0, U_1, U_2) \leq F(x|\theta, t_0, x_0, U_1, U_2)$   
 est valable pour chaque mouvement  $x[\cdot, t_0, x_0, U_1, U_2]$ . De (2.8) et (2.9)  
 il suit  $F_1(x|\theta, t_0, x_0, U_1, U_2) = F(x|\theta, t_0, x_0, U_1, U_2) \geq$   
 $\geq \max_{x[\cdot]} F(x|\theta, t_0, x_0, U_1, U_2) = \max_{x[\cdot]} F_1(x|\theta, t_0, x_0, U_1, U_2)$  pour  $\forall U_2 \in \mathbb{U}_2$

D'ici et d'après [4, p.26] on obtient

$$(2.10) \quad \min_{x[\cdot]} F_1(x|\theta, t_0, x_0, c) \geq \max_{x[\cdot]} F_1(x|\theta, t_0, x_0, U_1, U_2) \quad \text{pour } \forall U_2 \in \mathbb{U}_2$$

A l'aide d'égalité  $\max_{x[\cdot]} [ - F(x) ] = - \min_{x[\cdot]} F(x)$  d'une manière  
 analogue nous pouvons établir

$$(2.11) \quad \min_{x[\cdot]} F_2(x|\theta, t_0, x_0, c) \geq \max_{x[\cdot]} F_2(x|\theta, t_0, x_0, U_1, U_2) \quad \text{pour } \forall U_1 \in \mathbb{U}_1$$

D'après la définition (1.1) les inégalités (2.10) et (2.11)  
 désignent que le point-selle  $(U_1^*, U_2^*)$  représente la situation d'équilibre  
 fort du jeu  $\Gamma_c$ . De la même manière on peut établir que la situation de  
 équilibre fort des jeux différentiels compétitifs devient point-selle.

Ainsi la notion de l'équilibre fort (équilibre de Berg) dans les  
 jeux différentiels de position apparaît finalement très large, parce qu'  
 elle englobe comme cas particulier la notion point-selle.

Les propriétés précédentes montrent que l'équilibre fort peut  
 être utilisé comme solution dans les jeux différentiels sans coalition.

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## The Schläfli Double-Six Configurations

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*Presented by H.S.M. Coxeter, F.R.S.C.*

In [2], Hilbert and Cohn-Vossen discuss the application of Schläfli's double-six [3] to the construction of a certain space point-and-line  $(30_2, 12_5)$  configuration. Figure 1 (taken from their book with permission) shows the thirty points and twelve lines of a model of this configuration, with five points on each line and two lines passing through each point. "Lines" here (and in the following discussion) are abbreviated to the segments in which we are interested, but for projective configuration figures they should be thought of as extending indefinitely beyond the end points of the segments. The cube outlined in Figure 1 is not a part of the configuration but is helpful for visualization purposes and also for the construction of the model.

Figure 2 is intended to illustrate as clearly as possible the double-six property, namely that the twelve lines are divided into two sets of six each where no line of either set has a point in common with another line of that set, and where one line of each set passes through each of the thirty points of the configuration.

Figure 3 illustrates a property of this configuration (not mentioned in [2]) that can also be helpful. The solid, dotted, and dashed lines clearly show that this  $(30_2, 12_5)$  is made up of three space quadrilaterals.

In Figures 2 and 3 the  $(30_2, 12_5)$  configuration has been projected onto a plane. In correspondence related to [1] Coxeter conjectured it could be reciprocated into 30 lines forming a configuration  $(12_5, 30_2)$ . This may appear an easy task. Although the two regular hexagons of Figure 4 with inscribed equilateral triangles, whose vertices are joined one to one, do form a  $(12_5, 30_2)$ , it does not have the double six property. Alterations made as in Figure 5 do lead to the double six property, but there are uncomfortably many

intersections at the central crossover point. Thus Figure 5 is not obtained by reciprocating Figures 1,2 or 3.

Many years ago one of the writer's students had constructed a model of Figure 1. Placing the model in a symmetrical position, gives Figure 6 (analogous to Figure 3) and Figure 7, the latter directly as the reciprocal of Figure 1. The end point designation of the sets of points by solid dots and by circles in Figure 7 is sufficient to show the double-six property that no two points of either set have a line in common and that each of the thirty segments has a point of one set as one end point and a point of the other set as the other end point. In other words, we "reciprocate" Schläfli's matrix of points

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{pmatrix},$$

a matrix of lines, such that two points  $P_i$  and  $Q_j$  are joined if  $i = j$ , or, more simply for the complete property, that *two of the twelve points are joined if and only if their symbols occur neither in the same row nor in the same column*. The actual positioning of these labels is aided by Figure 6, and variations in these locations will produce variations in Figure 7, i.e., different locations for the unconnected points  $P_i$  and  $Q_j$  when  $i = j$ .

Figure 7 can, of course, be thought of as a space configuration (as can Figures 4 and 5), but the writer has not tried to make models of any of these! He has, however, suggested to several sculptors who work with steel rods that two large-scale models of Figures 2 and 3 — using colored rods and with a descriptive plaque — might be a sculpture of general interest. So far there are apparently no takers of the suggestion.

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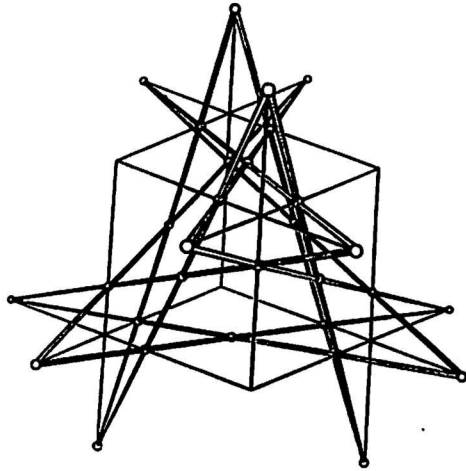


Figure 1

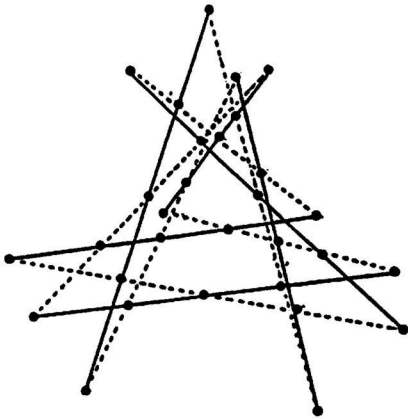


Figure 2

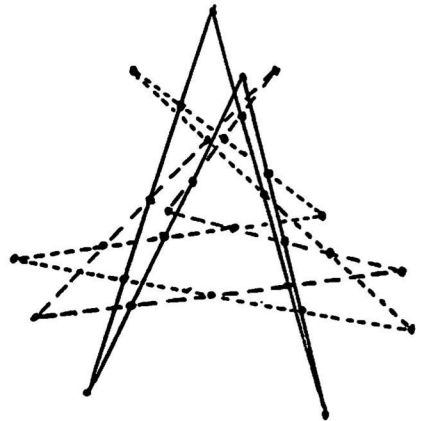


Figure 3

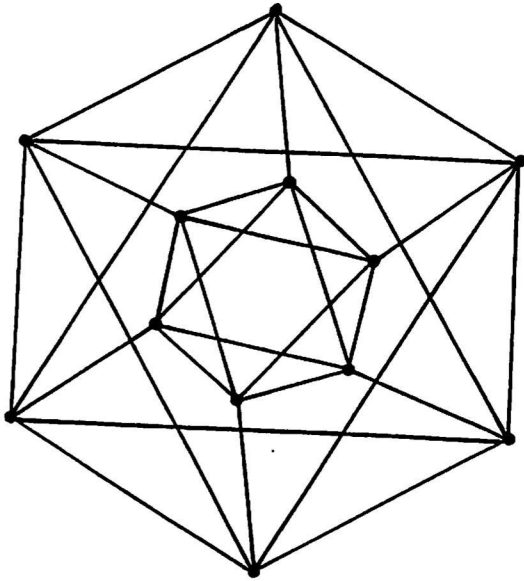


Figure 4

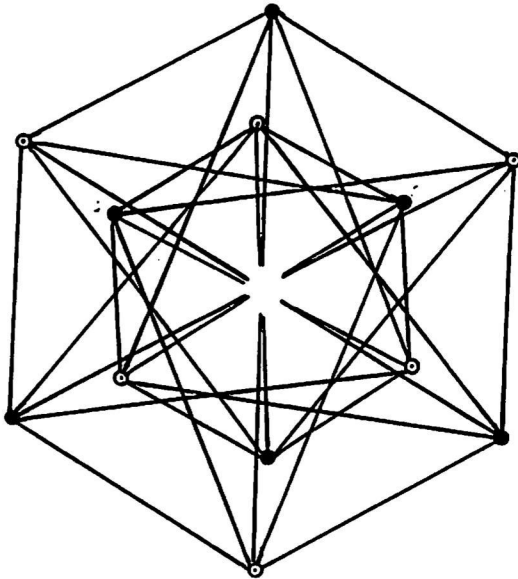


Figure 5

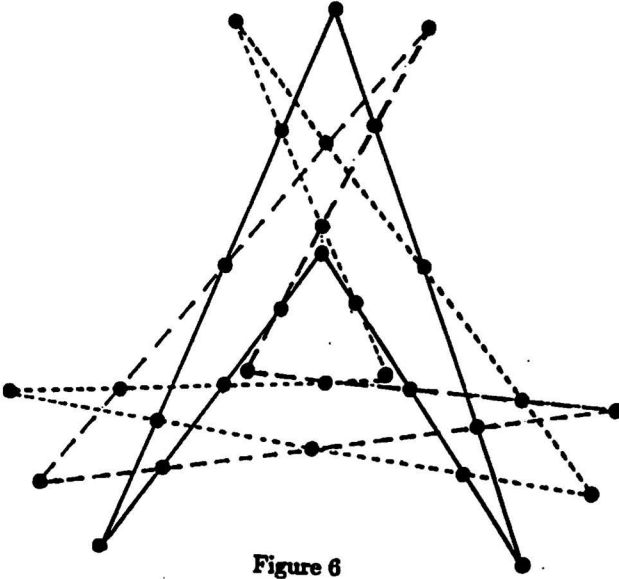


Figure 6

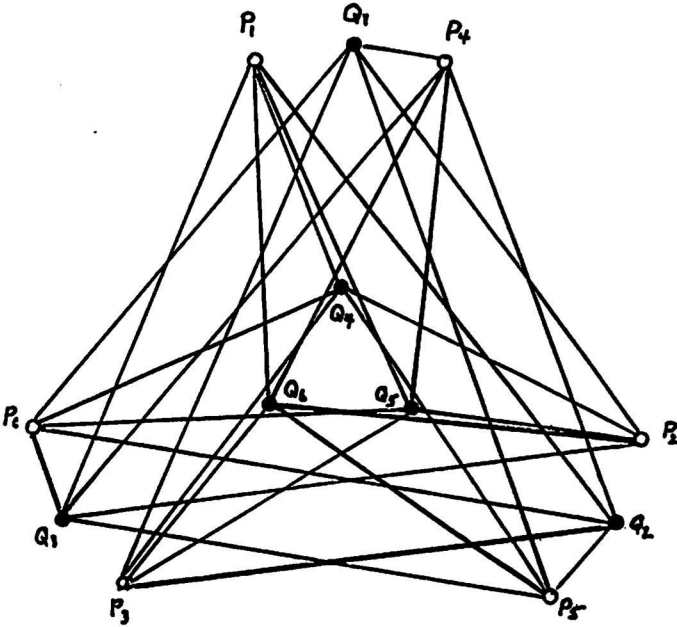


Figure 7

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