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A NOTE ON (P,ℓ)-CORRELATIONS

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ABSTRACT. Assuming a modest degree of transitivity, we show that a (P,ℓ)-correlation of a projective plane has order four unless the underlying ternary ring has characteristic two, in which case the correlation has order two.

In [1] Baer introduced the notions of (P,ℓ)-transitivity and (P,ℓ)-homogeneity. Suppose $\mathcal{P}$ is a projective plane. (See [2] for the necessary definitions.) We will sometimes write $P\mathbf{I}\ell$ when the point P is incident with the line ℓ. A *collineation* of $\mathcal{P}$ is a bijection mapping points to points and lines to lines which preserves incidence, while a *correlation* is a bijection interchanging lines with points which preserves incidence. The square of every correlation is clearly a collineation; a correlation whose square is the identity map is called a polarity. A collineation is (P,ℓ)-*central* with respect to a point-line pair (P,ℓ) if it fixes every point on ℓ and every line through P. $\mathcal{P}$ is (P,ℓ)-*transitive* for a point-line pair (P,ℓ) if, given any points Q and R not on ℓ which are collinear with but different from P, there is a (P,ℓ)-central collineation which maps Q to R. Finally, $\mathcal{P}$ is (P,ℓ)-*homogeneous* for P and ℓ which are not incident if it is (P,ℓ)-transitive and there exists a (P,ℓ)-correlation, *i.e.* one mapping every line m through P to its intersection with ℓ and every point Q on ℓ to the line PQ. Baer showed that [1, Theorem 5.2] if a projective plane $\mathcal{P}$ has distinct lines ℓ and m such that it is (P,ℓ)-homogeneous for all points P on m which are not on ℓ, then $\mathcal{P}$ is Pappian, *i.e.* coordinatized by a field.

(P,ℓ)-homogeneity was generalized as follows by Jónsson [3]: a projective plane $\mathcal{P}$ is (P,ℓ,μ)-*homogeneous*, where P may be incident with ℓ, if it is (P,ℓ)-transitive and there exists a correlation whose square is a (P,ℓ)-central collineation and which induces the

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function μ as the function from the lines through P to the points of ℓ. If P and ℓ are not incident, then (P,ℓ)-homogeneity is just (P,ℓ,μ)-homogeneity, where $m^\mu = m \cap \ell$ for every line m through P. Jónsson [4] showed that if $\mathcal{P}$ is (P,ℓ,μ)-homogeneous and $P\,\mathrm{I}\,\ell$, then $\mathcal{P}$ admits a (P,ℓ,μ)-polarity. The following theorem answers the corresponding question for (P,ℓ)-correlations with $P\,\not{\mathrm{I}}\,\ell$ in most projective planes $\mathcal{P}$.

Theorem. *Suppose OUV is a triangle in the projective plane $\mathcal{P}$, $\mathcal{P}$ is (U,OV)- and (V,OU)-transitive, and c is a (U,OV)-correlation. Then $c^4 = id_{\mathcal{P}}$ and we can choose coordinates so that, if (R,T) is the corresponding ternary ring, the following hold:*

(1) *(R,T) is linear with associative and commutative multiplication, and both distributive laws hold;*

(2) $c^2 = id_{\mathcal{P}} \iff \forall x \in R:\ x + x = 0.$

Proof. Choose $A\,\mathrm{I}\,UV$ different from U and V, and let $E = A^c \cap OA$. Coordinatize $\mathcal{P}$ with respect to the quadrangle $OUVE$ using Hall's method [2, p. 125] (so $O = (0,0)$, $E = (1,1)$, $U = (0)$, and $V = (\infty)$) and let (R,T) be the resulting ternary ring. Since $\mathcal{P}$ is $(U,OV) = ((0),[0])$-transitive, (R,T) is linear and has associative multiplication. As we also have $(V,OU) = ((\infty),[0,0])$-transitivity, we have that $a(b+c) = ab+ac$ for $a,b,c \in R$.

Define functions $\varphi,\psi : R \to R$ by $(u)^c = [u^\varphi]$ and $[m]^c = (m^\psi)$. Then φ and ψ are bijections and it follows from our choice of coordinates that $0^\varphi = 0^\psi = 0$, $1^\varphi = 1$, $(x,y)^c = [x^\psi, y]$, and $[m,b]^c = (m^\varphi, b)$. Thus, for any $x,y,m,b \in R$,

$$\begin{aligned} y = xm + b &\iff (x,y)\,\mathrm{I}\,[m,b] \\ &\iff (x,y)^c\,\mathrm{I}\,[m,b]^c \\ &\iff [x^\psi, y]\,\mathrm{I}\,(m^\varphi, b) \\ &\iff b = m^\varphi x^\psi + y. \end{aligned}$$

Substituting for b in $y = xm + b$ gives

$$(*) \qquad y = xm + (m^\varphi x^\psi + y).$$

Let $-a$ denote the right additive inverse of $a \in R$, *i.e.* $a + (-a) = 0$, and let $\sim a$ denote the left additive inverse of $a \in R$, *i.e.* $(\sim a) + a = 0$. Note that $- \sim a = a$ and $\sim -a = a$.

If $m = 1$ and $y = 0$ in $(*)$, $x + x^\psi = 0$, *i.e.* $x^\psi = -x$ since $1^\varphi = 1$. Setting $m = 1$, $y = -z$, and $x = \sim z$ in $(*)$ gives $-z = \sim z + (- \sim z + (-z)) = \sim z + (z + (-z)) = \sim z + 0 = \sim z$. Thus right and left additive inverses are equal in (R,T) and so $-(-a) = a$ for all $a \in R$. Since $a(b+c) = ab + ac$ for $a,b,c \in R$, $ab + a(-b) = a(b+(-b)) = a0 = 0$, so $a(-b) = -(ab)$ whenever $a,b \in R$.

Setting $y = 0$ and $x = 1$ in (*) now implies that $m + m^{\varphi}(-1) = m + -(m^{\varphi}) = 0$, so $m^{\varphi} = m$. Hence $\varphi = id_R$.

Setting $y = O$ in (*) now implies that $xm + -(mx) = 0$, so $xm = mx$. Hence multiplication is commutative in R. Also $(x,y)^{c^4} = ((x^{\psi})^{\psi}, y) = (-(-x), y) = (x,y)$, so $c^4 = id_{\mathcal{P}}$. Finally, since $(x,y)^{c^2} = (x^{\psi}, y) = (-x, y)$, we have that

$$\begin{aligned} c^2 = id_{\mathcal{P}} &\iff \forall x \in R : x = -x \\ &\iff \forall x \in R : x + x = 0. \end{aligned}$$

This completes the proof. $\square$

Note that the proof tells us how to construct (P,ℓ)-correlations, for P not incident with ℓ, in planes with enough transitivity: Fix a coordinate system so that $P = (0)$ and $\ell = [0]$, and define the correlation c by setting $(u,v)^c = [-u,v]$, $(u)^c = [u]$, $(\infty)^c = [\infty]$, $[m,b]^c = (m,b)$, $[u]^c = (-u)$, and $[\infty]^c = (\infty)$.

Since Desarguesian planes are (P,ℓ)-transitive for all point-line pairs (P,ℓ), the existence of a (P,ℓ)-correlation in a Desarguesian plane implies that it is (P,ℓ)-homogeneous. The following variant of Baer's result is thus an immediate consequence of the theorem.

Corollary. *Suppose F is a skewfield and $\mathcal{P}(F)$ is the Desarguesian plane coordinatized by F. Then $\mathcal{P}$ admits a (P,ℓ)-correlation c for some non-incident point-line pair (P,ℓ) if and only if F is commutative, i.e. F is a field and $\mathcal{P}(F)$ is Pappian. Moreover, $c^4 = id_{\mathcal{P}(F)}$ and $c^2 = id_{\mathcal{P}(F)} \iff F$ has characteristic 2.*

It remains open whether the hypotheses of the theorem can be weakened to require only (P,ℓ)-homogeneity.

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FREE HEYTING ALGEBRAS AS BI-HEYTING ALGEBRAS

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Abstract.[1] By generalizing the construction of [5] (and simultaneously giving simple alternative proofs), we describe the Heyting algebra freely generated by a finite distributive lattice by means of the inductive colimit of an iterated procedure which at each step freely adds new implications without destroying those previously introduced. We show, as a corollary, that such free algebras are bi-Heyting algebras, in the sense that the difference operation (dual to implication) is defined in them. Connections with the normal forms [3], [4] of the modal system S4 arise.

The category of finite distributive lattices[2] is dual to the category of finite posets, hence each finite distributive lattice can be represented in the form $\downarrow P$, where $\downarrow P$ is the set of downward closed subsets of a poset P ordered by inclusion (we usually omit the explicit mention of the partial ordering relation $\leq$ of a poset). If $a \in P$, $\downarrow a$ and a^c stand respectively for $\{b \mid b \leq a\}$ and $\{b \mid a \not\leq b\}$. We say that an order-preserving map $f : P \longrightarrow Q$ is *open* iff the inverse image morphism $f^{-1} : \downarrow Q \longrightarrow \downarrow P$ preserves relative relative pseudocomplementation, which is equivalent to the following condition entirely expressed in terms of elements:

$$\forall a \in P, \forall b \in Q\ (b \leq f(a) \Rightarrow \exists a' \in P (a' \leq a \ \&\ f(a') = b)).$$

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[2]Presence of 0 and 1 elements is considered part of the definition of a lattice.

The same condition can be also expressed as the commutativity of the following square of relations:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\geq"'] & Q \arrow[d, "\geq"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

Given another order preserving map $g : Q \longrightarrow R$, we say that f is *open relatively to g* (briefly *g-open*) iff f^{-1} preserves relative pseudocomplements of the kind $g^{-1}(C_1) \rightarrow g^{-1}(C_2)$ for $C_1, C_2 \in \downarrow R$. Again there is an easy characterization by elements, that is

$$\forall a \in P, \forall b \in Q \quad (b \leq f(a) \Rightarrow \exists a' \in P(a' \leq a \ \& \ g(f(a')) = g(b))).$$

In terms of relations it means the commutativity of the following square:

$$\begin{tikzcd} P \arrow[r, "f \geq"] \arrow[d, "\geq f"'] & Q \arrow[d, "g"] \\ Q \arrow[r, "g"'] & R \end{tikzcd}$$

If the inclusion $S \subseteq Q$ is g-open, then we say that S itself is *g-open* (here S is regarded as a poset with the restricted partial ordering relation). This means that S is g-open iff the following condition holds:

$$\forall s \in S, \forall b \in Q \quad (b \leq s \Rightarrow \exists s' \in S(s' \leq s \ \& \ g(s') = g(b))).$$

If we represent Q as partitioned into the g-fibers, the condition says that, whenever $\downarrow s$ meets an element of any fiber, then $(\downarrow s) \cap S$ must actually contain an element from that fiber. A fact useful in the computations is the following: is S is g-open and $s \in S$, then $(\downarrow s) \cap S$ is g-open too.

Here we describe the main construction: let $g : Q \longrightarrow R$ be an order-preserving map. Q^g is the poset of *g-open rooted subsets of* Q, ordered by inclusion (a subset of a poset is said to be rooted iff it has a greatest element). The order-preserving map $r^g : Q^g \longrightarrow Q$ that sends a rooted g-open subset onto its root is easily seen to be g-open. Now given a finite poset P we can form the sequence:

$$\cdots P_{i+1} \xrightarrow{r_{i+1}} P_i \xrightarrow{r_i} \cdots P_1 \xrightarrow{r_1} P_0 \xrightarrow{r_0} 1$$

by iterating the above construction. That is, we put: $P_0 = P$, $r_0 =$ the unique map into the one-point poset $\mathbf{1}$ and, inductively, $P_{i+1} = P_i^{r_i}$, $r_{i+1} = r^{r_i}$.

1. Theorem *The inductive colimit of the chain of distributive lattices*

$$\downarrow \mathbf{1} \xrightarrow{r_0^{-1}} \downarrow P_0 \cdots \xrightarrow{r_i^{-1}} \downarrow P_i \xrightarrow{r_{i+1}^{-1}} \downarrow P_{i+1} \cdots$$

is a Heyting algebra, in fact the Heyting algebra freely generated by $\downarrow P$. $\square$

The proof of the theorem is entierely contained in the following lemma:

2. Lemma *Let $g : Q \longrightarrow R$ be an order-preserving map between finite posets. Then the pair $\langle \downarrow Q^g, (r^g)^{-1} \rangle$ has the following universal property. Suppose we are given any other pair*

$$\langle D, \mu : \downarrow Q \longrightarrow D \rangle$$

such that

(i) *D is a distributive lattice containing relative pseudocomplements of the kind $\mu(C_1) \rightarrow \mu(C_2)$, for $C_1, C_2 \in \downarrow Q$;*

(ii) *$\mu(g^{-1}(D_1) \rightarrow g^{-1}(D_2)) = \mu(g^{-1}(D_1)) \rightarrow \mu(g^{-1}(D_2))$ for all $D_1, D_2 \in$ $\in \downarrow R$.*

Then there exists a unique lattice morphism $\mu' : \downarrow Q^g \longrightarrow D$ such that the triangle

$$\begin{tikzcd} & \downarrow Q \arrow[dl, "(r^g)^{-1}"'] \arrow[dr, "\mu"] & \\ \downarrow Q^g \arrow[rr, "\mu'"'] & & D \end{tikzcd}$$

commutes and such that $\mu'((r^g)^{-1}(C_1) \rightarrow (r^g)^{-1}(C_2)) = \mu(C_1) \rightarrow \mu(C_2) =$ $\mu'((r^g)^{-1}(C_1)) \rightarrow \mu'((r^g)^{-1}(C_2))$ for all $C_1, C_2 \in \downarrow Q$.

Proof. As Q is finite and as finitely generated distributive lattices are finite, we can limit ourselves to the case in which D is finite. But then we may equivalently prove the dual statement for finite posets, which is the following: given a finite poset T and a g-open order-preserving map $h : T \longrightarrow Q$, there exists a unique r^g-open order-preserving map h' such that the following triangle

$$\begin{tikzcd} T \arrow[rr, "h'"] \arrow[dr, "h"'] & & Q^g \arrow[dl, "r^g"] \\ & Q & \end{tikzcd}$$

commutes. The definition of h' is indeed forced by the commutativity of the triangle and by the fact that h' must be order-preserving and r^g-open:

$$h'(a) = \{h(b) \mid b \leq a\}.$$

The required verifications (that $h'(a)$ is a g-open rooted subset of Q, that the triangle commutes and that h' is order-preserving and r^g-open) are easy. □

The join-irreducible elements[3] of the Heyting algebra freely generated by a finite distributive lattice are exactly the equivalence classes (in the colimit) represented by the elements of the kind $\downarrow a$ which arise at the various steps of the construction of the above theorem. This is due to the fact for every order-preserving map $g : Q \longrightarrow R$, the map $r^g : Q^g \longrightarrow Q$ has a right adjoint which is also a section of it. Here is another consequence of the existence of such adjoint:

3. Corollary *The Heyting algebra freely generated by a finite distributive lattice is a bi-Heyting algebra (i.e. the dual operation of implication is defined in it).*

Proof. This follows from the fact that the r^g maps are co-open, i.e. that inverse image along them preserves the dual of implication (which is always defined in finite distributive lattices). □

Notice that, given $g : Q \longrightarrow R$, the join-irreducible elements $\downarrow S$ of Q^g admit the following representation in terms of the elements of $\downarrow R$:

$$\downarrow S = \bigcap_{a \notin S} ((r^g)^{-1}(\downarrow a) \rightarrow (r^g)^{-1}(a^c))$$

[3]We recall that join-irreducible elements are the non-zero elements x such that, for every y, z, if $x = y \vee z$, then either $x = y$ or $x = z$. When an element satisfies the analogous condition with respect to all existing (possibly infinitary) joins it is said to be completely join-irreducible. Such elements in free Heyting algebras are much less than the join-irreducible ones, as shown in [1]. In fact the classical effective representation of finitely generated free Heyting algebras, contained in [1], [3] and based on an analysis of finite Kripke models, consists in embedding such algebras into the complete lattice of the downward closed subsets of the poset of their completely join-irreducible elements (which are explicitly described).

From a logical point of view, this formula, once applied to the sequence of the above theorem produced by a finitely generated free distributive lattice, gives *normal forms* for propositional intuitionistic logic. The analogy with the normal forms of the modal system $S4$ is very strict, because the elements of the posets $\{P_i\}_i$ of theorem 1 can be represented as commutative idempotent trees with labels in P_0 [4] and it turns out that they are exactly the trees satisfying the $S4$-conditions and, in addition, the condition that labels are decreasing along paths from the root to the leaves.

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[4] We associate with an element a of P_n a labelled tree $\psi(a)$ of height n recursively as follows (recall that as our trees are commutative and idempotent, immediate subtrees are a set, not a list): $\psi_0(a) = a$, $\psi_{n+1}(a) = \langle r_1 r_2 \cdots r_{n+1}(a), \{\psi_n(b)\}_{b \in a}\rangle$.

UNIVERSAL COVERING LOCALIC TOPOSES

MARTA BUNGE

Presented by J. Lambek, F.R.S.C.

Abstract. A connected locally connected topological space may have a trivial paths fundamental group but a non-trivial Chevalley group of automorphisms of a universal covering space. It is shown here that this "anomaly" dissapears if the topological space is replaced by its topos of sheaves. More generally, for a connected locally connected localic topos with a suitably defined universal covering topos, its paths fundamental group in the sense of Moerdijk and Wraith[21] is equivalent to the classifying topos of its Chevalley (or coverings) fundamental group as constructed by Bunge[5].

1. Paths versus coverings fundamental groups of a topos.

Let $\mathcal{S}$ be a fixed elementary topos. All toposes considered are assumed to be bounded over $\mathcal{S}$ and all geometric morphisms are over $\mathcal{S}$ (cf.[[11]). Also, all locales and continuous maps of locales (cf. [12]) are assumed to be in $\mathcal{S}$.

Consider the cosimplicial locale $\Delta^0 \rightrightarrows \Delta^1 \rightrightarrows \Delta^2 \ldots$, with Δ^n the standard n-simplex locale, constructed from the unit interval locale I (cf. [7]) in the usual manner (cf. [20]). It follows that Δ^n is compact regular hence exponentiable both as a locale (cf. [10]) and as the topos $\mathcal{S}[\Delta^n]$ of sheaves (cf.[13]). If $\mathcal{E}$ is a topos, the paths fundamental group of $\mathcal{E}$ is defined in [21] as the colimit (cf. [18]) of the simplicial (or descent) topos obtained from $\mathcal{E}$ by exponentiating to the simplicial topos obtained from the above cosimplicial locale by applying sheaves. Using more familiar notations for Λ^n in the cases n = 0, 1 and 2 and denoting the topos $\mathcal{S}[\Delta^n]$ of sheaves simply by Δ^n, this colimit will be denoted by

$$\ldots \mathcal{E}^{\Delta} \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} \mathcal{E}^{I} \rightrightarrows \mathcal{E} \to \Pi_1(\mathcal{E}) \qquad (1.1).$$

The main result about $\Pi_1(\mathcal{E})$ in [21] is that if $\mathcal{E}$ is connected locally connected, then $\Pi_1(\mathcal{E})$ is connected atomic.

For the coverings approach to the fundamental group of a topos there are several versions beginning with that of Grothendieck (cf. [9], [1]). The systematic use of locales in the study of toposes initiated by Joyal and Tierney[15] was exploited by Moerdijk[19] for the purpose of defining the coverings fundamental group of a pointed connected locally connected topos, and extended by Bunge[5] to the unpointed case. For a connected locally connected topos $\mathcal{E}$ and a connected cover U in $\mathcal{E}$, consider (cf. [5]) the pushout diagram of toposes given below:

$$\begin{array}{ccc} \mathcal{E}/_U & \to & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{S} & \to & \mathcal{G}_U . \end{array} \qquad (1.2)$$

By the Joyal-Tierney theorem (cf. [15]), $\mathcal{G}_U$ is the classifying topos (in the sense of [18], [4]]) of a (etale complete) discrete localic group G_U which is the group of automorphisms of the point $\mathcal{S} \to \mathcal{G}_U$ in the above pushout. With any generating filtered poset $\mathcal{C}$ of covers in $\mathcal{E}$ there is associated an inversely filtered system of toposes $\mathcal{G}_U$ and open surjections, whose limit is a topos $\mathcal{G}$ that is the classifying topos of a totally disconnected (cf. [14], [5]) localic groupoid $\pi_1(\mathcal{E})$ which represents cohomology of $\mathcal{E}$ with coefficients in discrete groups, as shown in [5]. The localic groupoid $\pi_1(\mathcal{E})$ (or, equivalently, its classifying topos $\mathcal{G} = \mathrm{Lim}\ \mathcal{G}_U$, is said to be the covering fundamental group of $\mathcal{E}$

2. Covering toposes.

The following notion of covering topos is adopted here for the purpose of comparing the paths and Chevalley group of a connected locally connected localic topos.

(2.1) Definition. Let $\mathcal{E}$ be a topos. A covering topos of $\mathcal{E}$ is a pair $<\mathcal{F}, p>$, where $\mathcal{F}$ is a topos and $p : \mathcal{F} \to \mathcal{E}$ is a geometric morphism with the (simplicial lifting) property : for any $n \geq 0$, the induced geometric morphism $p^{\Delta^n} : \mathcal{F}^{\Delta^n} \to \mathcal{E}^{\Delta^n}$ is an open surjection.

It is not true that for any cover U in a non localic topos $\mathcal{E}$, the slice topos $\mathcal{E}/_U$ is a covering topos of $\mathcal{E}$, not even if U is a locally constant cover (cf. [3]). For example, if $\mathcal{S}^G$ is the topos of G-sets for a connected group G, then $\mathcal{S}^G/_G \simeq \mathcal{S}$, and the canonical geometric morphism from the slice topos is identified with the unique point of $\mathcal{S}^G$. If this point were a covering topos of $\mathcal{S}^G$, the latter would be totally disconnected (in the sense of [5]), which is a contradiction, as $\mathcal{S}^G$ is connected locally connected and non-trivial. However, the following notion of covering locale (which is given in the spirit of the classical notion, cf. [8], [2], [17] and [16]), provides examples of localic covering toposes by taking sheaves. Covering toposes are stable under composition and pullbacks. Furthermore, the notion of a covering topos is stable under base change since all the constructions involved are stable.

(2.2) Definition. Let X be a locale. A covering locale of X is a pair $<Y, p>$, where Y is a locale and $p : Y \to X$ is an open surjection relative to which the frame $\mathcal{O}(X)$ has a basis of admissible elements, where an element $b \in \mathcal{O}(X)$ is admissible (relative to p) if $p^*(b) = \bigvee\{c^\alpha\}$ with $c^\alpha \in \mathcal{O}(Y)$ and $\exists_p(c^\alpha) = b$ for all α.

(2.3). Proposition. Let $<Y, p>$ be a covering locale of X. Then, the induced geometric morphism $p : \mathcal{S}[Y] \to \mathcal{S}[X]$ is a covering topos of $\mathcal{S}[X]$.

Proof. More generally, we show that for any locally compact locale K, the morphism $p^K : Y^K \to X^K$ of locales (hence the induced morphism of sheaves) is an open surjection.. The map $q = p^K$ is the transpose of the composite of p with the evaluation map $Y^K \times K \to Y$ and, as such, it has an explicit description given by Hyland[10], as follows. Let $q^* : \mathcal{O}(X^K) \to \mathcal{O}(Y^K)$ be the morphism of frames corresponding to q. On a subbasic open of $\mathcal{O}(X^K)$ (written formally in [10] as) $[a << f^*(b)]$, with $a \in \mathcal{O}(K)$ and $b \in \mathcal{O}(X)$, $q^*([a<<f^*(b)]) = \bigvee_{I_b}\{\bigwedge_J[a^\alpha<<f^*(c^\alpha)]\}$, where $p^*(b) = \bigvee_{I_b}\{c^\alpha\}$, and where $J \subseteq I_b$ is finite and $\bigvee_J\{a^\alpha\} = a$.. Define $\exists_q([a<<f^*(c)]) = [a<<f^*(\exists_p(c))]$, for $a \in \mathcal{O}(K)$ and $c \in \mathcal{O}(Y)$. We claim that $\exists_q$ is left adjoint to q^* with an identity counit, and that the adjoint pair satisfies Frobenius reciprocity (cf.[15]).

The adjointness (and surjectivity of q) is shown as follows using the axioms (cf.[10]) governing the expressions for the subbasic opens of the exponential locales considered.. First, $\exists_q\, q^*([a<<f^*(b)] = \bigvee_{I_b}\{\bigwedge_J[a^\alpha<<f^*(\exists_p(c^\alpha))]\} = \bigvee_{I_b}\{\bigwedge_J[a^\alpha<<f^*(b)]\} = [a<<f^*(b)]$. Secondly, $q^*\exists_q([a<<f^*(c)]) = q^*([a<<f^*(\exists_p(c))] = \bigvee_i\{q^*([a<<f^*(b_i)])\} = \bigvee_i\bigvee_{I_{b_i}}\{\bigwedge_J[a_i^{\alpha_i}<<f^*(c^{\alpha_i})]\} \geq [a<<f^*(c)]$, where $\exists_p(c) = \bigvee_i\{b_i\}$. This follows since $\bigvee_i\bigvee_{I_{b_i}}\{c^{\alpha_i}\} = \bigvee_i p^*(b^i) = p^*(\bigvee_i\{b^i\}) = p^*\exists_p(c) = c$, and since $\bigvee_{I_{b_i}}\{a_i^{\alpha_i}\} = a$, for each $i \in I_{b_i}$.

It remains to show Frobenius reciprocity of the adjoint pair $\exists_q \dashv q^*$ using the same property for the adjoint pair $\exists_p \dashv p^*$. We have:
$\exists_q([a_0<<f^*(c)] \wedge q^*([a_1<<f^*(b)])) = \exists_q([a_0<<f^*(c)] \wedge \bigvee_{I_b}\{\bigwedge_J[a^\alpha<<f^*(c^\alpha)]\}) =$
$\bigvee_{I_b}\{\bigwedge_J[(a_0 \wedge a_1{}^\alpha)<<f^*(\exists_p(c \wedge c^\alpha))]\} \geq [(a_0 \wedge a_1)<<f^*(\exists_p(c \wedge p^*(b))] \geq [(a_0 \wedge a_1)<<f^*(\exists_p c \wedge b)]$
$= [a_0<<f^*(\exists_p(c))] \wedge [a_1<<f^*(b)] = \exists_q([a_0<<f^*(c)] \wedge [a_1<<f^*(b)]$. This completes the proof. □

3. The Chevalley group of a connected locally connected localic topos

In both [3] and in [21], the example of the long circle (cf. [22]) is used to show the discrepancy between the Chevalley fundamental group of the space (cf. [8]), or the paths fundamental group of the topos of sheaves, either of which is identified with the group of integers, and the paths fundamental group of the space, which is trivial.. These comments led us to conjecture that, for the topos of sheaves on a connected locally connected topological space with a universal covering (suitably defined), the Chevalley and the paths fundamental groups agree. This conjecture is established here in the more general case of localic toposes, but the validity of the analogous statement in the general case remains an open question.

(3.1) Definition. Let $\mathcal{E}$ be a connected locally connected topos. $\mathcal{E}$ has a universal covering topos if for some $U \in \mathcal{C}$ in a generating filtered poset $\mathcal{C}$ of connected covers in $\mathcal{E}$, $\varphi_U : \mathcal{E}/_U \to \mathcal{E}$ is a covering topos, the projection $\mathit{Lim}\ \mathcal{G}_U \to \mathcal{G}_U$ is an equivalence of toposes, and the topos $\mathcal{E}/_U$ is paths simply connected in the sense that $\Pi_1(\mathcal{E}/_U) \simeq \mathcal{S}$. Under these assumptions, the coverings fundamental group $\pi_1(\mathcal{E})$ of $\mathcal{E}$ is said to be the Chevalley fundamental group of $\mathcal{E}$.

(3.2) Theorem. Let $\mathcal{E}$ be a connected locally connected localic topos with a universal covering topos $< \mathcal{E}/_U , \varphi_U >$. Then, the paths fundamental group $\Pi_1(\mathcal{E})$ of $\mathcal{E}$ is equivalent to the classifying topos of the Chevalley fundamental group $\pi_1(\mathcal{E})$ of $\mathcal{E}$.

Proof. Consider the diagram

$$
\begin{array}{ccc}
\vdots & & \vdots \\
(\mathcal{E}/_U)^{\Delta} & \to & \mathcal{E}^{\Delta} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
(\mathcal{E}/_U)^{I} & \to & \mathcal{E}^{I} \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{E}/_U & \to & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{S} & \to & \mathcal{G}_U
\end{array}
$$

where the lower square is the pushout of toposes defining $\mathcal{G}_U$ and where the vertical diagrams above

$\mathcal{E}/_U$ and $\mathcal{E}$ are the corresponding simplicial toposes. The horizontal arrows above the pushout diagram are of the form $(\varphi_{U})^{\Delta^*}$ and, by assumption, are all (open) surjections. The left vertical diagram above $\mathcal{S}$ is a colimit diagram since, by assumption, $\Pi_1(\mathcal{E}/_U) \simeq \mathcal{S}$. We claim the the right vertical diagram above $\mathcal{G}_U$ is also a colimit diagram, from which the desired equivalence $\Pi_1(\mathcal{E}) \simeq \mathcal{G}_U$ would follow.

As shown in [18], the colimit topos may be equivalently be described as a descent topos. With the notations $\varepsilon_0, \varepsilon_1 : \mathcal{E}^1 \to \mathcal{E}$ and $\delta_0, \delta_1, \delta_2 : \mathcal{E}^\Delta \to \mathcal{E}^1$ for the morphisms in the truncated simplicial complex on the right hand side of the diagram (and similarly for the left), the descent conditions for a geometric morphism $g: \mathcal{E} \to \mathcal{H}$ are given (cf. [18], [4]) by an iso 2-cell $\theta : g\varepsilon_0 \Rightarrow g\varepsilon_1$ satisfying the cocycle condition $\theta\delta_0 . \theta\delta_1 : = : \theta\delta_2$, where canonical isomorphisms are supposed to be inserted for the expression to be meaningful.. The descent topos for a descent diagram is a topos $\mathcal{H}$ with descent data (g, θ) for the descent diagram, that is universal among all toposes with such descent data.

We show first that, for the geometric morphism $\sigma_U : \mathcal{E} \to \mathcal{G}_U$ defined by the pushout diagram (1.2), .there is an iso 2-cell $\theta : \sigma_U\varepsilon_0 \Rightarrow \sigma_U\varepsilon_1$, satisfying the cocycle condition.. Since $\mathcal{E}$ is localic over $\mathcal{S}$ and since any point of a topos is localic, all geometric morphisms in the above diagram are localic over $\mathcal{G}_U$, hence the diagram may be viewed as one of locales and continuous maps in the topos $\mathcal{G}_U$. By assumption, all geometric morphisms $(\varphi_U)^{\Delta^*}$ are open surjections between localic toposes over $\mathcal{G}_U$, hence also the continuous maps of locales in $\mathcal{G}_U$ which they determine (cf. [15], [20]) are open surjections and, as such, are regular epis. From the commutativity of the diagram and the identity $\gamma_U\varepsilon_0 = \gamma_U\varepsilon_1$,follows the identity $\sigma_U\varepsilon_0 = \sigma_U\varepsilon_1$ in the context of continuous maps of locales, using that the continuous map induced by $(\varphi_U)^1$ is a regular epi. The above identity between continuous maps gives rise, in turn, to an iso 2-cell $\theta : \sigma_U\varepsilon_0 \Rightarrow \sigma_U\varepsilon_1$ of the corresponding localic geometric morphisms. The cocycle condition $\theta\,\delta_0 . \theta\,\delta_1 : = : \theta\,\delta_2$ is now a consequence of the cocycle condition for the identity 2-cell $\gamma_U\varepsilon_0 \Rightarrow \gamma_U\varepsilon_1$, using the commutativity of the diagram and the surjectivity of $(\varphi_U)^\Delta$ as a geometric morphism..

It remains to prove that the descent data for $\sigma_U : \mathcal{E} \to \mathcal{G}_U$ given by $\theta : \sigma_U\varepsilon_0 \Rightarrow \sigma_U\varepsilon_1$, is universal. But this is an immediate consequence of the combined universal properties of the left vertical simplicial diagram whose colimit topos is $\mathcal{S}$, and of the pushout diagram defining $\mathcal{G}_U$ This completes the proof. □

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FONCTEUR PHOM : QUELQUES THEOREMES D'EXACTITUDE

BABACAR SARR

Presented by P. Ribenboim, F.R.S.C.

SOMMAIRE :

Cet article fait suite à celui publié dans le volume IX, N° 2 de comptes rendus mathématiques de l'Académie des Sciences (Société Royale du Canada) et intitulé : Homomorphismes p-petits de groupes abéliens p-torsion, p-réduits.

Le foncteur PHom(-,-) étudié ici est un sous-foncteur du foncteur Hom. Nous rappelons que si G et A sont deux groupes p-torsion, p-réduits, tels que $U_G = U_A$,

PHom $(G,A) \overset{def}{=} \{\phi \in \text{Hom}\,(G,A) \;/\; \phi$ est p-petit$\}$ est un sous-groupe de Hom (G,A).

Nous rappelons les définitions suivantes:

Soit $u = (u_o, u_1, \ldots, u_k, \ldots)$ une suite croissante d'entiers non négatifs.

$G(u) = \{x \in G \;/\; H_p^G(x) \geqslant u\}$. où $H_p^G(x)$ désigne la suite d'ULm de x dans G.

$U_G = \{u = (u_o, u_1, \ldots, u_k, \ldots) \;/\; G(u)$ est p-large dans $G\}$.

Soit G un groupe abélien, un sous-groupe L de G est dit p-large dans G si L est totalement invariant et si $G = B + L$ pour tout sous-groupe de p-base B de G.

$\phi : G \longrightarrow A$ est dit p-petit si $\ker \phi$ contient un sous-groupe p-large de G.

1 - FONCTEUR PHOM

Proposition 1.1 : Soit $\phi : X \longrightarrow C$ et $f : C \longrightarrow D$ où $\phi \in$ PHom (X,C) et f un homomorphisme quelconque, alors $f \circ \phi \in$ PHom (X,D).

<u>Proposition 1.2</u> : Soit $f : G \longrightarrow H$ un homomorphisme, $\phi : H \to X$ un homomorphisme p-petit. Si $U_H \subsetneq U_G$, alors $\phi \circ f \in PHom(G,X)$.

Dans la suite, les groupes p-torsion, p-réduits G considérés ont la même famille de suites u telles que $G(u)$ est p-large dans G. Un exemple de groupe p-torsion, p-réduit est la complétion $\hat{B}$ pour la topologie p-adique de $B = \bigoplus_{n=1}^{\infty} B_n$, avec $B_n = \bigoplus \mathbb{Z}(p^n)$ ou encore $\hat{B} = \varprojlim_n (B/p^nB)$.

Les sous-groupes p-larges de $\hat{B}$ sont de la forme $\hat{B}(u)$ où u est une suite strictement croissante d'entiers non négatifs présentant un nombre fini d'écarts.

Soit $\mathcal{C}$ la sous-catégorie de la catégorie des groupes abéliens dont les objets sont les groupes G p-torsion, p-réduits, ayant la même famille de suites u telles que $G(u)$ est p-large dans G. On peut en vertu des propositions 1.1 et 1.2 définir le sous-foncteur $PHom(-,-)$ du foncteur $Hom(-,-)$ de la façon suivante :

$$PHom : \mathcal{C} \times \mathcal{C} \longrightarrow Ab$$
$$(G,H) \longrightarrow PHom(G,H)$$

ou en fixant l'une ou l'autre des variables, on a :

$$PHom(X,-) : \mathcal{C} \longrightarrow Ab$$

$$\begin{array}{ccc} C & \longrightarrow & PHom(X,C) \\ f\downarrow & & \downarrow f_* \\ H & \longrightarrow & PHom(X,H) \end{array}$$

où $f_* : PHom(X,C) \longrightarrow PHom(X,H)$

$$\phi \longrightarrow f \circ \phi$$

et $PHom(-,X) : \mathcal{C} \longrightarrow Ab$

$$\begin{array}{ccc} G & \longrightarrow & PHom(G,X) \\ f\downarrow & & \uparrow f^* \\ H & \longrightarrow & PHom(H,X) \end{array}$$

où $f^* : PHom(H,X) \longrightarrow PHom(G,X)$

$$\phi \longrightarrow \phi \circ f.$$

<u>Proposition 1.3</u> : PHom $(X, C \oplus D)$ est isomorphe à PHom $(X,C) \oplus$ PHom (X,D).

<u>Proposition 1.4</u> : PHom $(G \oplus H,X)$ est isomorphe à PHom $(G,X) \oplus$ PHom (H,X).

2 - QUELQUES THEOREMES D'EXACTITUDE

Nous étudions dans ce qui suit une certaine classe de suites exactes qui sont envoyées sur des suites exactes par les foncteurs PHom $(-,X)$ et PHom $(X,-)$.

<u>Théorème 2.1</u> : Soit la suite exacte
$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$. Pour tout X, la suite
$0 \longrightarrow \text{PHom}(X,C) \xrightarrow{f_*} \text{PHom}(X,D) \xrightarrow{g_*} \text{PHom}(X,E)$ est exacte.

<u>Théorème 2.2</u> : Soit $0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$ une suite p-pure exacte. Pour tout X, la suite
$0 \longrightarrow \text{PHom}(X,C) \xrightarrow{f_*} \text{PHom}(X,D) \xrightarrow{g_*} \text{PHom}(X,E) \longrightarrow 0$ est exacte

<u>Remarque 2.3</u> : Si $0 \longrightarrow F \xrightarrow{f} G \xrightarrow{g} H \longrightarrow 0$ est une suite exacte, pour tout X, la suite
$0 \longrightarrow \text{PHom}(H,X) \xrightarrow{g^*} \text{PHom}(G,X) \xrightarrow{f^*} \text{PHom}(F,X)$ n'est pas toujours exacte.
En effet, g^* est injectif car $g^*(\phi) = \phi \circ g = 0$ implique $\phi(g(G)) = \phi(H) = 0$ et donc $\phi = 0$. On a par ailleurs $(f^* \circ g^*)(\phi) = f^*(\phi \circ g) = \phi \circ g \circ f = (g \circ f)^*(\phi) = 0$ pour tout ϕ, $f^* \circ g^* = 0$ et $\text{Im } g^* \subseteq \ker f^*$. Cependant on peut avoir $\text{Im } g^* \subsetneq \ker f^*$ comme le prouve l'exemple suivant.

Supposons que g est p-petit et que PHom (H,X) est contenu strictement dans Hom (H,X). Cette dernière condition est satisfait si H_p est non borné et si X contient un sous-groupe isomorphe à H.

Soit alors $\phi \in \mathrm{Hom}\,(H,X) \setminus \mathrm{PHom}\,(H,X)$ et $\psi = \phi \circ g$.
ψ n'appartient pas à $\mathrm{Im}\, g^*$. En effet, s'il appartenait à $\mathrm{Im}\, g^*$, il existerait ϕ_1 élément de $\mathrm{PHom}\,(H,X)$ tel que $g^*(\phi_1) = \psi$ par conséquent $g^*(\phi_1) = g^*(\phi)$ et $\phi_1 = \phi$, ce qui est faux car ϕ n'est pas p-petit.

<u>Théorème 2.4</u> : Soit la suite exacte

$$0 \longrightarrow F \xrightarrow{f} G \xrightarrow{g} H \longrightarrow 0$$

On suppose la condition (*) suivante satisfaite :
(*) il existe des entiers non négatifs m et r tels que pour tout $y \in H$ avec $e(y) \geq m$, il existe $x \in G$ tel que $g(x) = y$ et $e(x) \leq e(y) + r$. Alors la suite

$$0 \longrightarrow \mathrm{PHom}\,(H,X) \xrightarrow{g^*} \mathrm{PHom}\,(G,X) \xrightarrow{f^*} \mathrm{PHom}\,(F,X) \text{ est exacte.}$$

<u>Preuve</u>: D'après la remarque 2.3, il suffit de montrer que $\ker f^* \subseteq \mathrm{Im}\, g^*$. Soit alors ϕ un élément de $\mathrm{PHom}\,(G,X)$ tel que $f^*(\phi) = \phi \circ f = 0$. On a $\ker g = \mathrm{Im}\, f \subseteq \ker \phi$. Comme $\phi \circ f = 0$, il existe ψ tel que le diagramme suivant commute :

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "f"] & G \arrow[r, "g"] \arrow[d, "\phi"'] & H \arrow[r] \arrow[dl, dashed, "\psi"] & 0 \\ & & X & & \end{tikzcd}$$

Montrons que $\psi \in \mathrm{PHom}\,(H,X)$.
Etant donné que $\ker \phi \supseteq G(u)$ qui est p-large dans G (c'est-à-dire que $\phi \in \mathrm{PHom}\,(G,X)$), d'après le Théorème 1.5 (volume IX,N) 2, Avril 1987), il existe $v = (v_0, v_1, \ldots, v_k, \ldots)$ une suite strictement croissante avec $v_k = u_{k-1} + 1$ telle que pour $x \in G$, $e(x) \geq v_{k+r}$ implique $e(\phi(x)) < e(x) - (k+r)$. Et de plus $G = ((\ker \phi) \cap G(u)) + G_p$.

Soit $y \in H$, tel que $e(y) > \max\{m, v_{k+r}\}$. Par $(*)$, il existe $x \in G$ tel que $y = g(x)$ et $e(x) \leq e(y) + r$.
Par conséquent, $e(x) \geq e(g(x)) = e(y) > v_{k+r} \geq v_k$ et il s'ensuit que $e(\psi(y)) = e(\psi(x)) \leq e(x) - (k+r) \leq e(y) - k$.
Il reste à montrer que $H = ((\ker\psi) \cap H(u)) + H_p$. On sait que $G = ((\ker\phi) \cap G(u)) + G_p$ et que g étant surjectif,

$$\begin{aligned} H = g(G) &= g((G(u) \cap \ker\phi) + g(G_p) \\ &\subseteq (g(G(u)) \cap g(\ker\phi)) + H_p \\ &\subseteq (H(u) \cap g(\ker\phi)) + H_p. \end{aligned}$$

On a $g(\ker\phi) = \ker\psi$. En effet, soit $x \in \ker\psi$, $\psi(x) = 0$ et comme g est surjectif, il existe $x_1 \in G$ tel que $g(x_1) = x$.
Il s'ensuit que $\psi(x) = (\psi \circ g)(x_1) = \phi(x_1) = 0$.
Donc $x_1 \in \ker\phi$ et par suite $x = g(x_1)$ est un élément de $g(\ker\phi)$.
Soit maintenant $x \in g(\ker\phi)$. Il existe alors $x_1 \in \ker\phi$ tel que $x = g(x_1)$ et $\psi(x) = \psi \circ g(x_1) = \phi(x_1) = 0$, donc $x \in \ker\psi$.
Par conséquent $(H(u) \cap g(\ker\phi)) = H(u) \cap \ker\psi$ et $H = (H(u) \cap \ker\psi) + H_p$ donc ψ est p-petit.

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FAMILLE DE MÉTRIQUES CONFORMÉMENT PLATES ET RÉGULARITÉ

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Abstract

In this article we are interested in the solutions of the Yamabe equation for the conformal product on $S^1(T) \times S^{n-1}$, a circle of length T crossed with the standard sphere.
We show that only for the dimension values $n = 3, 4$ and 6 the solutions with two singularities are meromorphic and following R. Schoen analysis in [1] his number depends on T values.

§1. - Etant donnée une variété rémannienne compacte de dimension (M, g_0) de dimension $n > 2$, le problème de Yamabe consiste à mettre en évidence dans chaque classe conforme $C(g_0)$, une métrique qui soit à courbure scalaire constante. Une métrique $g \in C(g_0)$ si et seulement si $g = u^{\frac{4}{n-2}} g_0$ où u est une fonction C^∞ positive sur M de l'équation de Yamabe :

$$(Y) \qquad 4\frac{n-1}{n-2}\Delta_{g_0} u + R_0 u = R u^{\frac{n+2}{n-2}}.$$

où R est la courbure scalaire associée à g.

Δ_{g_0} étant l'opérateur de Laplace-Beltrami sur (M, g_0). Surtout, depuis la résolution complète de ce problème en 1984 par R. Schoen, on s'est intéressé aux solutions multiples lorsque la courbure scalaire R est strictement positive.

Les métriques conformément plates retiennent particulièrement l'attention, puisque la variété produit $S^1 \times S^{n-1}$ (du cercle par la sphère de dimension $n-1$) est la seule variété connue, sur laquelle une analyse de l'ensemble des solutions à ce problème semble possible; [1], §2.

En effet, la variété $S^1 \times S^{n-1}$ est revêtue sur $\mathbf{R} \times S^{n-1}$; l'application stéréographique composée avec celle qui à $x \in \mathbf{R}^n - \{0\}$ associe $\left(\operatorname{Log}|x|, \frac{x}{|x|}\right) \in \mathbf{R} \times S^{n-1}$ réalisent un difféomorphisme conforme entre $S^n - (0, \infty)$ et $\mathbf{R} \times S^{n-1}$. Il existe alors des solutions périodiques de l'équation de Yamabe qui sont toutes à symétrie radiale. Des arguments de réflexion d'Alexandroff entraîne que toutes les solutions singulières en seulement 2 points ont nécessairement cette symétrie. Une solution $u(t)$ de l'équation (Y) est singulière si elle

tend vers 0 ou ∞ pour certaines valeurs de t. Ainsi, le problème se réduit à la résolution d'une équation différentielle ordinaire, et à ses seules solutions.
Par ailleurs, Cafarelli - Gidas - Spruck [2] ont montré que toute solution de l'équation de Yamabe sur $S^n - (0, \infty)$ singulière en 0 ou ∞, est nécessairement singulière en 0 et ∞, de plus, c'est une fonction radiale.

§2. - On considère sur $\mathbf{R} \times S^{n-1}$ la métrique produit $g_0 = dt^2 + dy^2$ où $(t, y) \in \mathbf{R} \times S^{n-1}$ et dy^2 étant la métrique de la sphère standard unité S^{n-1}, ainsi la tourbure scalaire $R(g_0) = (n-1)(n-2)$. Pour une solution $u(t)$ (qui est en fait une solution globale) où t désigne la distance à un point fixe, l'équation de Yamabe s'écrit :

$$\frac{d^2u}{dt^2} - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0 \tag{1}$$

On s'intéresse aux solutions positives sur $\mathbf{R}$ de l'équation (1). D'après le principe du maximum une telle solution ne peut s'annuler sans être identiquement nulle.
Cette équation admet 2 solutions évidentes, à savoir $u_0 = \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}$ dont la métrique associée $g_0\, u_0^{\frac{4}{n-2}}$ a pour courbure scalaire $n(n-1)$. La 2ème solution explicite est $u_1(t) = (\cos h\, t)^{-\frac{n-2}{2}}$, mais la métrique associée $g_1 = u_1^{\frac{4}{n-2}} g_0$ n'est pas complète sur $\mathbf{R} \times S^{n-1}$.
L'équation (1) peut s'écrire comme un système autonome :

$$\frac{d}{dt}(u, v) = X(u, v) = \left[v, \frac{(n-2)^2}{4}u - \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}}\right]$$

X a deux points critiques : $(0,0)$ et $(u_0, 0)$.
L'orbite correspondant à la solution $u_1(t)$ contient le point $(1,0)$ du plan (u, v) et approche $(0,0)$ quand t devient infini, elle borde une région Ω de ce plan. Toute orbite $\gamma_\alpha(t)$ pour $u > 0$ doit être incluse dans $\overline{\Omega}$ et vérifier $\gamma_\alpha(0) = (\alpha, 0)$ où $\alpha \in [u_0, 1]$; $\gamma_\alpha(t)$ est périodique et de période $T_{(\alpha)}$.
Conséquence de cette discussion pour les solutions de Yamabe sur la variété $S^1 \times S^{n-1}$, le rayon de S^{n-1} étant égal à 1 et long $S^1 = \int_0^T dt = T$, g_0 étant la métrique produit.
L'hypothèse fondamentale que Schoen a considérée est : la période $T(\alpha)$ est une fonction analytique croissante de $\alpha \in [u_0, 1]$. Dans ce cas, il existe $T_0 = \frac{2\pi}{\sqrt{n-2}}$ telle que pour tout $T \leq T_0$, la variété $S^1(T) \times S^{n-1}$ a une solution unique pour Yamabe, à savoir, constante $\times g_0$.
Pour $T \in [T_0, 2T_0]$, il existe deux solutions non équivalentes, la solution constante et celle de période T, cette dernière $u_T(x)$ n'étant pas invariante par rotation sur S^1, les

fonctions $u_T(x+\alpha)$ sont aussi solutions, elles s'obtiennent par un changement d'origine sur S^1 et, sont donc équivalentes (en ce sens que les métriques associées sont dans la même classe conforme).

D'une manière plus générale, pour $T \in ((k-1)T_0, k.T_0)$, il existe k solutions non équivalentes : la constante et les solutions de période $\frac{T}{i}$ où $i = 1, 2, \ldots, k-1$. Chacune de ces $(k-1)$ solutions appartient à une famille de solutions équivalentes paramétrées par les rotations de S^1.

Remarques : 1) Lorsque $T \in ((k-1)T_0, kT_0)$, seules les solutions période fondamentale T sont stables, elles sont donc minimisantes pour le problème de Yamabe, les autres solutions de période $\frac{T}{i}$ avec $i = 1, 2, \ldots, k-1$ étant instables de ce point de vue.

C'est précisément la solution de période T qui tend vers $u_1(t) = (\cos h\, t)^{-\frac{n-2}{2}}$ quand T tend vers ∞. Ainsi, le minimum de la fonctionnelle de Yamabe $\mu(S^1(T) \times S^{n-1})$ qui est un invariant conforme pour les métriques de volume 1 tend vers $\mu(S^n) = n(n-1)(\operatorname{vol} S^n)^{1/n}$ lorsque T tend vers ∞. Ce fait a déjà été établi par O. Koboyashi [4].

2) Si on s'intéresse aux solutions de l'équation (1) autres que celles ayant deux singularités, notons qu'il n'existe pas de solution ayant un seul point singulier. Lorsque l'ensemble des points singuliers Λ est fini et de cardinal $\#\Lambda > 2$, dans ce cas R. Schoen a établi que les solutions périodiques sont fortement asymptotiques à celles mise en évidence pour $\#\Lambda = 2$.

3) Si on fixe la longueur ℓ du cercle S^1 une solution périodique de période T de l'équation (1) peut être solution au problème de Yamabe; si $T \neq \ell$, celle-ci est une fonction sur S^1 mais non périodique. Ainsi, lorsque T est une fonction croissante du paramètre α, on peut mettre en évidence des familles de solutions sur S^1, $\{u_T/T \geq \ell\}$; $u_{T=\ell}$ est la solution constante à condition que $\ell = 2\pi\sqrt{\frac{n-1}{R}}$; R étant la courbure scalaire de (S^{n-1}, dy^2). Soit $C(g_\ell)$ la classe conforme de la métrique produit sur $S^1 \times S^{n-1}$, alors toutes les métriques $g = (u_T)^{\frac{4}{n-2}} g_\ell \in C(g_\ell)$ dès que $\operatorname{vol}(S^1 \times S^{n-1}, g) = \operatorname{vol}(S^1 \times S^{n-1}, g_\ell)$.

§3. - Dans cette partie, on se propose d'appliquer l'analyse de l'équation (1) aux dimensions $n = 3, 4$ et 6.

PROPOSITION. — *Soit (S^1, dt^2) le cercle de longueur T, (S^{n-1}, dy^2) la $(n-1)$-sphère standard de rayon 1 et, $T_0 = \frac{2\pi}{\sqrt{n-2}}$. Lorsque la dimension n est égale à 3, 4 ou 6* *et la longueur $T \in ((k-1)T_0, kT_0)$, alors il existe $(k-1)$ familles à un paramètre de solutions T-périodiques non constantes de l'équation de Yamabe sur la variété produit $(S^1 \times S^{n-1}, g_0)$ ayant deux singularités. De plus, ces solutions qui s'expriment à l'aide des fonctions elliptiques n'admettent que des pôles. Pour les autres valeurs de la dimension n, les solutions périodiques non constantes de (Y) ayant deux singularités ne sont pas méromorphes.*

Pour $n = 4$, l'équation (1) devient :

$$\frac{d^2u}{dt^2} - u + 2u^3 = 0 \tag{2}$$

Grâce aux fonctions de Jacobi, on peut écrire la famille de ses solutions périodiques, à savoir :

$$u_\alpha(x) = \frac{1}{\sqrt{2-\alpha^2}}\, dn\Big(\frac{x+\beta}{\sqrt{2-\alpha^2}}\Big) \quad \text{où} \quad \alpha \in [0,1]$$

dont la période T s'écrit :

$$T_\alpha = 2\sqrt{2-\alpha^2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\alpha^2\sin^2\theta}}$$

T_α est une fonction analytique réelle en $\alpha \in]0,1[$, ayant 2 points singuliers en $\alpha = 0$ et $\alpha = 1$. Un calcul de la dérivée donne :

$$\frac{dT}{d\alpha} = \frac{-2\alpha}{\sqrt{2-\alpha^2}}\int_0^{\pi/2} \frac{\cos 2\theta\, d\theta}{(1-\alpha^2\sin^2\theta)^{3/2}}.$$

Puis en effectuant des majorations sur les intervalles $\left(0, \frac{\pi}{4}\right)$ et $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, on trouve que : $\frac{dT}{d\alpha} \geq 0$.

Dans ces conditions, on peut adapter l'analyse développée au paragraphe précédent pour le cas $n = 4$, qui permet de déterminer exactement le nombre de solutions.

On obtient des résultats analogues lorsque la dimension $n = 6$, car les solutions de l'équation correspondante :

$$\frac{d^2u}{dt^2} - 4u + 6u^2 = 0 \tag{3}$$

peuvent s'exprimer au moyen de la fonction elliptique de Weierstrass $p(z)$ relative à $g_2 = \frac{4}{3}$, à savoir $u(t) = \frac{1}{3} - p(t+\alpha)$. Lorsque le paramètre g_3 vérifie l'inégalité : $\frac{64}{27} - 27g_3^2 > 0$, alors les solutions $u(t)$ sont périodiques réelles, et admettent un pôle double pour singularité.

Quant au cas $n = 3$, pour intégrer l'équation correspondante, on remarque que les orbites du système autonome ayant pour équation :

$$\left(\frac{du}{dt}\right)^2 = \frac{1}{4}(u^2 - u^6) + K$$

sont en fait reliées aux orbites du genre 1 d'équation

$$\left(\frac{dv}{dt}\right)^2 = v^2 - 1 + 4K v^3$$

par le changement $u^2 = \dfrac{1}{v}$.

Ainsi, $v = v(t)$ est une fonction méromorphe périodique réelle si $-\dfrac{1}{6\sqrt{3}} < K < 0$. Plus exactement, on montre que $v(t) = \dfrac{1}{K}(p(t + \alpha) - \dfrac{1}{12})$, où $p(t)$ est la fonction elliptique relative à $g_2 = \dfrac{1}{12}$ et $g_3 = \dfrac{216K^2 - 1}{216}$ auquel cas, les solutions de l'équation (1) correspondante à $n = 3$ admettent exactement deux pôles d'ordre $\dfrac{1}{2}$ comme singularités (cela résulte du fait que la fonction de Weierstrass admet deux zéros simples dans un parallélogramme des périodes).

On pourra vérifier là aussi, que la période T_α est une fonction croissante de paramètre; autrement dit, la détermination du nombre de solutions de l'équation de Yamabe et, l'analyse du paragraphe 2 s'appliquent au cas $n = 3$.

Remarques : 1) Lorsque la dimension n est égale à 3, 4 ou 6, la période d'une solution de l'équation correspondante dépend d'un seul paramètre. On verra que tel n'est pas le cas pour les autres valeurs de n, où les orbites sont en général des courbes de genre $g \geq 2$, dont on sait que le module est au moins égal à 3. C'est précisément le nombre de paramètres dont peut dépendre la période T d'une solution.

2) Par ailleurs, on sait d'après un théorème de Picard que pour tout réseau de périodes L de $\mathbf{C}^g$: $g \geq 2$, le corps des fonctions méromorphes sur $\mathbf{C}^g/L$ est réduit aux constantes. Autrement dit, une solution périodique de l'équation (1) admet au moins une singularité essentielle pour les dimensions n autres que 3, 4 et 6.

§4. - Les orbites du système autonome associé à l'équation (1) :

$$u'^2 = \left(\frac{n-2}{2}\right)^2 \left[u^2 - u^{\frac{2n}{n-2}}\right] + K. \tag{4}$$

Celles-ci sont périodiques seulement lorsque la constante K satisfait l'inégalité :

$$-\frac{2}{n}\left(\frac{n-2}{2}\right)^2 \left(\frac{n-2}{n}\right)^{\frac{n-2}{2}} < K < 0. \tag{5}$$

Pour étudier la nature algébrique de ces orbites, il y a lieu de distinguer deux cas selon la parité de n.

a) n impair :

On opère le changement $u = v^{-\frac{n-2}{2}}$ et, l'équation (4) devient :

$$v'^2 = (v^2 - 1) + \left(\frac{2}{n-2}\right)^2 K v^n.$$

On voit que seulement pour $n = 3$, la solution v est une fonction elliptique, donc méromorphe. Dans les autres cas, l'orbite non dégénérée est une courbe du genre $g = \dfrac{n-1}{2}$.

b) n pair :

On fait le changement $u = v^{-\frac{n-2}{4}}$ et, l'équation (4) devient :

$$v'^2 = 4(v^2 - v) + \left(\frac{4}{n-2}\right)^2 K v^{\frac{n+2}{2}}.$$

Pour les valeurs $n = 4$ et $n = 6$, la solution v est une fonction elliptique. Pour les autres valeurs de la dimension, l'orbite non dégénérée est une courbe de genre $g = \dfrac{n}{4}$ ou $g = \dfrac{n-2}{4}$ en général.

Notons que l'orbite dégénère lorsque le polynôme en v, $P(v)$ défini par

$$v'^2 = P(v)$$

admet une racine double. C'est le cas, notamment lorsque la constante K est réelle et prend les valeurs extrèmes de l'inégalité (5). Sinon, le genre est égal à g, auquel cas il existe $2g$ périodes indépendantes; l'une d'elles (seulement) est réelle lorsque l'inégalité (5) est satisfaite.

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Homomorphisms compatible with some covering maps

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Abstract. Let X be a compact, connected topological space and let S be a finite group acting freely on X. We show that any two $*$-homomorphisms $C(X,M_n) \to C(X/S,M_m)$ compatible with the canonical covering $X \to X/S$ are inner equivalent, provided that X,S and X/S satisfy some additional conditions.

1. Introduction and preliminaries. Let X be a compact, connected topological space and let S be a finite group acting freely on X. Let $\varphi: X \to X/S$ be the canonical quotient map, which is a k-fold covering, where k is the order of S. A $*$-homomorphism $\Phi: C(X,M_n) \to C(X/S,M_{nkr})$ $(r \in \mathbb{N})$ is called compatible with φ if $\Phi(f \circ \varphi \otimes 1_n) = f \otimes 1_{nkr}$, for any $f \in C(X/S)$. The class of $*$-homomorphisms compatible with a covering was introduced in [5] and studied in [1], [3], [5], [6] and [7]. In this note we give a description of the $*$-homomorphisms $\Phi: C(X,M_n) \to C(X/S,M_{nkr})$ compatible with φ, provided that X,S and X/S satisfy some additional conditions (see Theorem 1), and we deduce from this that any two such $*$-homomorphisms are inner equivalent (see Corollaries 1 and 2). Our results extend those in [3], where the case when r=1 was considered. They also extend the main result in [7], which asserts that any two $*$-homomorphisms $C(\mathbb{T}^2,M_n) \to C(\mathbb{T}^2,M_{nkr})$ compatible with the same k-fold covering $\mathbb{T}^2 \to \mathbb{T}^2$ are inner equivalent (the same thing but for r=1 was obtained in [5]). This result (i.e. [7, Theorem]) in the case $r \geq 2$, was one of the main tools used by G. A. Elliott and G. Gong in [4] to show that certain C^*- inductive limits $\varinjlim C(\mathbb{T}^2,F_n)$, with the F_n's finite dimensional C^*-algebras, can be written as $\varinjlim C(\mathbb{T},G_n)$ with the G_n's finite dimensional C^*-algebras.

In this paper we shall denote by M_n the C^*-algebra of $n \times n$ complex matrices, by 1_n its unit and by $P_r(M_n)$ the space of all orthogonal projections in M_n of rank r ($r \in \mathbb{N}$) endowed with the induced topology from M_n. If A and B are unital C^*-algebras, we shall denote by Hom(A,B) the space of all unital $*$-homomorphisms $A \to B$, endowed with the topology of pointwise-norm convergence. Let $\mathrm{Vect}_m(X)$ denote the set of isomorphism classes of complex vector bundles of rank m on the topological space X. We shall denote by $T_n\mathrm{Vect}_m(X)$ the subset of $\mathrm{Vect}_m(X)$ given by all vector bundles E such that $E \oplus E \oplus \ldots \oplus E$ (n-times) is isomorphic to the trivial bundle of rank nm.

2. Results. The technique used in this paper is mainly inspired from [3].

Let X be a compact, connected topological space and let S be a finite group acting freely on X. If k is the order of S, then the canonical quotient map $\varphi: X \to X/S$ is a k-fold covering.

Lemma 1. Let $\Phi: C(X) \to C(X/S, M_m)$ be a $*$-homomorphism compatible with $\varphi: X \to X/S$. Then $m=kr$ for some $r \in \mathbb{N}$ and there is a continuous map $p: X \to P_r(M_m)$ such that:

$$\Phi(f)(\varphi(x)) = \sum_{s \in S} f(s \cdot x)p(s \cdot x), f \in C(X), x \in X.$$

Proof: Let $S = \{s_1, s_2, \ldots, s_k\}$. Since Φ is compatible with φ, using [6], we can define in a correct way a map:

$$\theta: X \to \mathrm{Hom}(\mathbb{C}^k, M_m)$$

by:

$$\theta(x)\left(\bigoplus_{i=1}^{k} f(s_i x)\right) = \Phi(f)(\varphi(x)), x \in X, f \in C(X).$$

It is not difficult to prove (using again [6]) that θ is continuous. Using the canonical identification $C(X, \mathrm{Hom}(\mathbb{C}^k, M_m)) = \mathrm{Hom}(\mathbb{C}^k, C(X, M_m))$ and [6, Proposition 3.1] it follows that $(\exists)$ an open finite covering $(U_i)_{i \in I}$ of X, $(\exists) u_i \in C(U_i, U(m))$ and $(\exists) e_s = e_s^* = e_s^2 \in M_m \subset C(X, M_m)$ such that:

$$(1) \quad \Phi(f)(\varphi(x)) = u_i(x)\left(\sum_{s \in S} f(s \cdot x)e_s\right)u_i(x)^*$$

when $x \in U_i$ and $f \in C(X)$ (we used also the fact that X is connected). For any $s \in S$ define $p_s: X \to P(M_m)$:=the space of all orthogonal projections in M_m by:

(2) $p_s(x) = u_i(x)e_s u_i(x)^*$ if $x \in U_i$.

Observe that the definition of p_s is correct (indeed, if $x \in U_i \cap U_j$, we have $u_i(x)e_s u_i(x)^* = u_j(x)e_s u_j(x)^*$ since $\sum_{s\in S} f(sx)u_i(x)e_s u_i(x)^* = \sum_{s\in S} f(sx)u_j(x)e_s u_j(x)^* (= \Phi(f)(\varphi(x)))$ for any $f \in C(X)$) and hence p_s is continuous.

Fix $t \in S$. Since $\Phi(f)(\varphi(x)) = \Phi(f)(\varphi(t\cdot x))$ for any $f \in C(X)$ and any $x \in X$, using (1) and (2) it follows that:

$$p_t(x) = p(tx),\ x \in X$$

where $p := p_f$ and f is the neutral element of the group S. Since X is connected, the continuous map $X \ni x \mapsto \text{rank } p(x) = \text{tr } p(x) \in \mathbb{N}$ is constant. The equality $\sum_{s\in S} p_s(x) = 1_m$, $x \in X$ implies then that $m=kr$ where rank $p_s(x) = \text{rank } p(sx) = r$ for any $s \in S$ and any $x \in X$.

Lemma 2. Let X and S be as in Lemma 1, let $r \in \mathbb{N}$, and suppose in addition that for any $s \in S$ the map $X \ni x \mapsto s\cdot x \in X$ induces the identity on $Vect_r(X)$. Let $x_0 \in X$ and let $p: X \to P_r(M_{kr})$ be a continuous map such that $\sum_{s\in S} p(sx) = 1_{kr}$, $x \in X$. Assume that $T_k Vect_r(X)$ reduces to the trivial vector bundle. Then there is a unitary u in $C(X, M_{kr})$ such that:

$$p(sx) = u(x)^* p(sx_0)u(x),\ x \in X,\ s \in S.$$

Proof: The proof is similar with that of [3, Lemma 2.1]. Set $e_s(x) = p(s\cdot x)$, $s \in S$, $x \in X$ and let f be the neutral element of the group S. Since for any $s \in S$ the map $X \ni x \mapsto s\cdot x \in X$ induces the identity on $Vect_r(X)$, it follows that each projection $e_s (s \in S)$ is equivalent with e_f in $C(X, M_{kr})$. Then, $(\exists) e_{s,f} \in C(X, M_{kr})$ such that $e_{s,f}^* e_{s,f} = e_f$ and $e_{s,f} e_{s,f}^* = e_s$. Defining $e_{s,t} = e_{s,f} e_{t,f}^*$ we obtain a system $\{e_{s,t}\}_{s,t\in S}$ of matrix units in $C(X, M_{kr})$. Identifying $C^*\{e_{s,t}(x_0) | s,t \in S\} \subset M_{kr}$ with M_k, we may define two $*$-homomorphisms $\Phi, \Phi_0 : M_k \to C(X, M_{kr})$ given by $\Phi_0(e_{s,t}(x_0)) = 1_{C(X)} \otimes e_{s,t}(x_0)$, $\Phi(e_{s,t}(x_0)) = e_{s,t}$. Since $T_k Vect_r(X)$ reduces to the trivial vector bundle, it follows from [1] (see also [2, Proposition 1]) that $\Phi_0(\cdot) = u\Phi(\cdot)u^*$ for some unitary u in $C(X, M_{kr})$. Hence $p(s\cdot x) = u(x)^* p(s\cdot x_0)u(x)$, $x \in X$, $s \in S$.

Remark. We recall that the condition that $T_k Vect_r(Y)$ reduces to a single element holds provided that Y is homotopy equivalent to a finite CW-complex of dimension $\leq 2r$ and the

K-theory group $K^0(Y)$ does not have k-torsion (see [1] and also [2]). This fact follows using stability properties of vector bundles. ■

Let X and S be as in Lemma 2. We shall keep the notations used in the proof of Lemma 2. Set $e^0_{s,t} := e_{s,t}(x_0)$ and $e^0_s := e_{s,s}(x_0)$, $s,t \in S$. Let $\rho: S \to B(\ell^2(S))$ be the right regular representation of S. We shall identify $B(\ell^2(S))$ with $C^*\{e^0_{s,t}: s,t \in S\} \cong M_k$ so that $\rho(r)^* e^0_{s,t}\rho(r) = e^0_{sr,tr}$. Lemma 2 implies that, as in [3], each $w_s(x) := u(sx)u(x)^*\rho(s)^*$ commutes with all the projections e^0_t, $t \in S$, hence $(\exists) w_{t,s} \in C(X, e^0_t M_{kr} e^0_t)$ such that $w_s(x) = \sum_{t\in S} w_{t,s}(x)$, $x \in X$. An easy computation shows that: $w_s(t\cdot x) = w_{st}(x)\rho(s)w_t(x)^*\rho(s)^*$ or, equivalently:

(3) $w_{r,st}(x) = w_{r,s}(t\cdot x)\rho(s)w_{rs,t}(x)\rho(s)^*$, $r,s,t \in S$, $x \in X$.

We want to find a unitary v in $C(X, M_{kr})$ such that:

(4) $v(s\cdot x) = \rho(s)v(x)$, $x \in X$, $s \in S$

and:

(5) $p(x) = v(x)^* p(x_0) v(x)$, $x \in X$.

Observe that if we define $v(x) = \left(\sum_{s\in S} d_s(x)\right)u(x)$, $x \in X$, where:

$d_t(x) := \rho(t)^* w_{f,t}(x)\rho(t)$, $t \in S$, $x \in X$

and f is the neutral element of S, then (4) follows from (3) and (5) from Lemma 2. Now we can prove the following result:

*Theorem 1. Let $\Phi: C(X) \to C(X/S, M_{kr})$ $(r \in \mathbb{N})$ be a *-homomorphism compatible with the covering $\varphi: X \to X/S$, where X and S are as in Lemma 2. Then there is some continuous map $v: X \to U(kr)$ such that:*

$$v(s\cdot x) = \rho(s)v(x),\ x \in X,\ s \in S,$$
$$\Phi(f)(\varphi(x)) = v(x)^*\left(\sum_{s\in S} f(sx)e^0_s\right)v(x),\ f \in C(X),\ x \in X.$$

Proof: The proof follows from Lemma 1, Lemma 2 and the above discussion.

Corollary 1. *Let $\Phi, \Psi : C(X) \to C(X/S, M_{k^r})(r \in \mathbb{N})$ be two $*$-homomorphisms compatible with the covering $\varphi : X \to X/S$, where X and S are as in Lemma 2. Then Φ and Ψ are inner equivalent, i.e. $\Phi(\cdot) = u\Psi(\cdot)u^*$ for some unitary $u \in C(X/S, M_{k^r})$.*

Corollary 2. *Let $\Phi, \Psi : C(X, M_n) \to C(X/S, M_{nk^r})(r \in \mathbb{N})$ be two $*$-homomorphisms compatible with the covering $\varphi : X \to X/S$, where X and S are as in Lemma 2 and, in addition, $T_n Vect_{k^r}(X/S)$ reduces to the trivial vector bundle. Then Φ and Ψ are inner equivalent.*

Proof: The proof follows from Corollary 1 and [1] (see also [2, Theorem 2]). ■

Obviously, slight extensions of [3, Theorem 3.1 and Corollary 3.2] can be obtained.

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ON THE STABILITY OF THE HOMOGENEOUS MAPPING

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ABSTRACT. The problem of Ulam -Hyers stability of the homogeneous mapping is discussed.

S.M.Ulam ([4]) has posed the problem of the stability of the linear mapping, which has been solved by D.H.Hyers in [2] (for other generalizations see e.g. [1], [3]). In this note we investigate a similar problem for homogeneous functions.

The symbols: N, R, R_0, R_+ will stand for the set of all natural, real, real different from zero or real nonnegative numbers respectively. Denote

$$U_v := \{\alpha \in R : \alpha^v \text{ exists}\} \quad \text{for} \quad v \in R_0 .$$

LEMMA. *Let E be a linear space and F a normed space (over R). Let $f: E \to F$ and $h: R \times E \to R_+$ satisfy the inequality*

$$(1) \quad \| f(\alpha x) - \alpha^v f(x) \| \le h(\alpha, x)$$

for all $(\alpha, x) \in U_v \times E$, where $v \in R_0$ is fixed. Then

$$(2) \quad \| f(\alpha^n x) - \alpha^{nv} f(x) \| \le \sum_{s=0}^{n-1} |\alpha|^{sv} h(\alpha, \alpha^{n-s-1} x)$$

for all $n \in N$ and $(\alpha, x) \in U_v \times E$.

Proof. For $n=1$ (2) follows from (1). Now we have, by (2), for $n+1$:

$$\| f(\alpha^{n+1} x) - \alpha^{(n+1)v} f(x) \| \le \| f(\alpha^{n+1} x) - \alpha^v f(\alpha^n x) \| +$$
$$+ |\alpha|^v \| f(\alpha^n x) - \alpha^{nv} f(x) \| \le h(\alpha, \alpha^n x) + \ldots + |\alpha|^{nv} h(\alpha, x),$$

i.e., by the induction principle (2) is true for all $n \in N$.

THEOREM. Let the assumptions of Lemma be satisfied and let F be a Banach space. Suppose that for some $0 \neq \beta \in U_v$ the series

$$(3) \quad \sum_{n=1}^{\infty} |\beta|^{-nv} h(\beta, \beta^n x)$$

pointwise converges for every $x \in E$ and

$$(4) \quad \liminf_{n \to \infty} \left\{ |\beta|^{-nv} h(\alpha, \beta^n x) \right\} = 0$$

for all $(\alpha, x) \in U_v \times E$. Then there exists exactly one v-homogeneous mapping $g: E \to F$:

$$g(\alpha x) = \alpha^v g(x) \text{ for all } (\alpha, x) \in U_v \times E$$

such that

$$(5) \quad \| g(x) - f(x) \| \leq \sum_{n=1}^{\infty} |\beta|^{-nv} h(\beta, \beta^{n-1} x) \quad \text{for } x \in E .$$

Proof. Define for $n \in N$

$$(6) \quad g_n(x) := \beta^{-nv} f(\beta^n x), \quad x \in E.$$

From (2) we get for $n \in N$ and $x \in E$

$$(7) \quad \| g_n(x) - f(x) \| \leq \sum_{s=1}^{n} |\beta|^{-sv} h(\beta, \beta^{s-1} x) .$$

Now, from (2) for $n, m \in N$, $n > m$ we obtain

$$\| g_n(x) - g_m(x) \| \leq |\beta|^{-nv} \| f(\beta^n x) - \beta^{(n-m)v} f(\beta^m x) \|$$

$$\leq \sum_{s=m+1}^{\infty} |\beta|^{-sv} h(\beta, \beta^{s-1} x) ,$$

which means that $\{ g_n(x) \}$ is a Cauchy sequence for every $x \in E$.

Let $g(x) := \lim_{n \to \infty} g_n(x)$ for $x \in E$. For all $\alpha \in U_v$ and all $x \in E$, in view of (6), (1) and (4), we have

$$g(\alpha x) - \alpha^v g(x) = \lim_{n \to \infty} \left\{ \beta^{-nv} \left[f(\alpha \beta^n x) - \alpha^v f(\beta^n x) \right] \right\} = 0,$$

i.e. g is a v-homogeneous mapping. Moreover, from (7) we get (5).

Now assume that there exist two v-homogeneous mappings $g_i: E \to F$, $i=1,2$ such that (5) holds for $g = g_i$, $i=1,2$. Then we have for $m \in N$

$$\| g_1(x) - g_2(x) \| = |\beta|^{-mv} \| g_1(\beta^m x) - g_2(\beta^m x) \| \le$$

$$\le |\beta|^{-mv} \{ \| g_1(\beta^m x) - f(\beta^m x) \| + \| g_2(\beta^m x) - f(\beta^m x) \| \} \le$$

$$\le 2 \sum_{s=m+1}^{\infty} |\beta|^{-sv} h(\beta, \beta^{s-1} x) .$$

Consequently, the convergence of the series (3) implies that $g_1(x) = g_2(x)$ for $x \in E$, and completes the proof.

COROLLARY 1. <u>Let the assumptions of Lemma be satisfied with</u> $h(\alpha, x) = \delta + |\alpha|^v \varepsilon$ <u>for some</u> $\delta, \varepsilon \in R_+$ <u>and let</u> F <u>be a Banach space. Then there exists exactly one</u> v<u>-homogeneous function</u> $g: E \to F$ <u>such that</u>

$$\| g(x) - f(x) \| \le \varepsilon \quad \text{for} \quad x \in E . \tag{8}$$

Proof. Assume that $v > 0$. From Theorem, for every $2 \le \beta = m \in N$ there exists v-homogeneous function

$$g_m(x) := \lim_{n \to \infty} m^{-nv} f(m^n x) \quad \text{for} \quad x \in E ,$$

such that

$$\| g_m(x) - f(x) \| \le (\delta + m^v \varepsilon) (m^v - 1)^{-1} , \quad x \in E . \tag{9}$$

Now we shall show that, for every $2 \le m, r \in N$, $g_m = g_r$. We have by (9) for $n \in N$

$$\| g_m(x) - g_r(x) \| = 2^{-nv} \| g_m(2^n x) - g_r(2^n x) \| \le$$

$$\le 2^{-nv} [(\delta + m^v \varepsilon)(m^v - 1)^{-1} + (\delta + r^v \varepsilon)(r^v - 1)^{-1}] ,$$

whence, if $n \to \infty$, we get $g_m = g_r$.

Put $g(x) = g_2(x)$, $x \in E$. Then from (9), letting $n \to \infty$, we obtain the estimation (8), which was to be proved.

EXAMPLE. Take $f(x) = \sin x$, $x \in R$. Then

$$|\sin(\alpha x) - \alpha^v \sin x| \leq 1 + |\alpha|^v \quad \text{for } (\alpha, x) \in U_v \times R$$

but f is not a v-homogeneous function.

COROLLARY 2. Let the assumptions of Lemma be satisfied with $h(\alpha, x) = \delta + |\alpha|^v \varepsilon$ for some $\delta, \varepsilon \in R_+$ and let F be a Banach space. If $\delta = 0$ or $\varepsilon = 0$, then

$$f(\alpha x) = \alpha^v f(x) \quad \text{for all } (\alpha, x) \in (U_v \setminus \{0\}) \times E. \tag{10}$$

Proof. Suppose that $\delta = 0$. Then we have

$$\| f(\alpha x) - \alpha^v f(x) \| \leq |\alpha|^v \varepsilon \quad \text{for } (\alpha, x) \in U_v \times E.$$

Hence, for every $y \in E$ and $\alpha \in U_v \setminus \{0\}$, inserting $\frac{y}{\alpha}$ instead of x, we get

$$\| f(y) - \alpha^v f(\tfrac{y}{\alpha}) \| \leq |\alpha|^v \varepsilon \tag{11}$$

and consequently (for $v > 0$)

$$f(y) = \lim_{\alpha \to 0} \left[\alpha^v f(\tfrac{y}{\alpha})\right] \quad \text{for } y \in E. \tag{12}$$

Therefore, for $(\beta, x) \in (U_v \setminus \{0\}) \times E$, we obtain

$$f(\beta x) = \lim_{\alpha \to 0} \left[\alpha^v f(\tfrac{\beta x}{\alpha})\right] = \lim_{\alpha \to 0} \left[\beta^v \left(\tfrac{\alpha}{\beta}\right)^v f(\tfrac{\beta x}{\alpha})\right] =$$

$$= \beta^v f(x),$$

i.e. (10) holds true.

If $v < 0$, then from (11) we get

$$\lim_{|\alpha| \to \infty} \left[\alpha^v f(\tfrac{y}{\alpha})\right] = f(y) \quad \text{for } y \in E$$

and similarly one can verify that (10) is satisfied.

Now, if $\varepsilon = 0$, we have the inequality

$$\| \alpha^{-v} f(\alpha x) - f(x) \| \leq |\alpha|^{-v} \delta \quad \text{for } (\alpha, x) \in (U_v \setminus \{0\}) \times E .$$

Hence, for $v > 0$,

$$f(x) = \lim_{|\alpha| \to \infty} \left[\alpha^{-v} f(\alpha x) \right], \quad x \in E ,$$

and, for $v < 0$,

$$f(x) = \lim_{\alpha \to 0} \left[\alpha^{-v} f(\alpha x) \right], \quad x \in E .$$

One can easily check that f satisfies (10), which completes the proof.

REMARK. For $v = 0$, except for the case $E = F = R$, the problem remains open.

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WEIGHTED ASYMPTOTICS OF PARTIAL SUM PROCESSES IN $D[1,\infty)$

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Abstract: We summarize optimal weighted asymptotics for partial sum processes by a standard Wiener process $\{W(t),\ 0 \leq t < \infty\}$ in $D(0,1]$. Considering functions h on $[1,\infty)$ such that $\limsup_{t\to\infty} |W(t)|/h(t) < \infty$ a.s. enables us to obtain weighted approximations, and hence also weak convergence, of partial sum processes in $D[1,\infty)$ as well. The admissible class of weight functions is seen to be bigger than that for asymptotics near 0.

1. Introduction and summary of results near 0. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables (i.i.d.r.v.'s) with $EX_1 = 0$, $EX_1^2 = 1$, and partial sums $S(n) = X_1 + \ldots + X_n$. After Donsker's theorem, the question arises under what conditions does weak convergence continue to hold $n^{-1/2}S(nt)/q(t)$, $0 < t \leq 1$, where $q(t)$ is a nonnegative function on $(0,1]$ approaching zero as $t \to 0$.

The motivation for studying weighted partial sums comes from earlier studies of Chibisov (1964), O'Reilly (1974) and others concerning the asymptotics of weighted empirical and quantile processes. O'Reilly (1974) proved the weak convergence of weighted partial sum processes in $C(0,1]$ under the assumption of $E|X_1|^3 < \infty$. For an extension of the Komlós, Major and Tusnády [KMT] (1975, 1976) approximation of partial sums to weighted supremum norm approximations with $E|X_1|^r < \infty$ for some $r > 2$, which improve also the just mentioned result of O'Reilly (1974) in terms of the optimal class of weight functions as in Csörgő, Csörgő, Horváth and Mason [CsCsHM] (1986), we refer to Csörgő and Horváth (1988) and the references given there.

A new method of proof had to be developed for obtaining weighted approximations of $n^{-1/2}S(nt)$ under the assumption of two moments only. The proof of Theorem A is based on a strong approximation theorem of Major (1979). As a corollary we obtain the optimal weighted version of Donsker's theorem in supremum metrics. These results were announced in Szyszkowicz (1991) and summarized in detail in Szyszkowicz (1992a,d).

Let Q be the class of positive functions on $(0,1]$, i.e., $\inf_{\delta\leq t\leq 1} q(t) > 0$ for all $0 < \delta < 1$, which are nondecreasing in a neighbourhood of zero. Let also

$$I(q,c) = \int_0^1 t^{-1} \exp(-ct^{-1}q^2(t))dt,\ c > 0.$$

THEOREM A. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0 \quad \text{and} \quad EX_1^2 = 1.$$

(a) A standard Wiener process $W(t)$ can be so constructed that with $q \in Q$ we have

$$\sup_{0<t\leq 1} |n^{-1/2}(S(nt) - W(nt))|/q(t) = o_P(1)$$

if and only if $I(q,c) < \infty$ for all $c > 0$.

(b) *Let $q \in Q$ be continuous. Then*

$$\sup_{0<t\leq 1} |n^{-1/2}S(nt)|/q(t) \to \sup_{0<t\leq 1} |W(t)|/q(t)$$

if and only if $I(q,c) < \infty$ for some $c > 0$.

Obviously part (a) of Theorem A implies weak convergence of any continuous in sup-norm functional of $n^{-1/2}S(nt)/q(t)$ to the corresponding functional of $W(t)/q(t)$ with $q \in Q$ and such that $I(q,c) < \infty$ for all $c > 0$. However, for the sup- functional itself the class of possible weight functions is bigger. Such a phenomenon was first noticed and proved for weighted empirical and quantile processes by CsCsHM (1986), and then by Csörgő and Horváth (1988) for partial sums with $E|X_1|^r < \infty$ for some $r > 2$.

The optimal conditions for weighted L_p-convergence and approximation of the empirical and quantile processes were given by Csörgő, Horváth and Shao (1991). Considering weighted L_p-approximations of the partial sums $n^{-1/2}S(nt)$ when only two moments are assumed to be finite, Szyszkowicz (1992b,c) obtained the following result.

THEOREM B. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0 \quad \text{and} \quad EX_1^2 = 1.$$

We assume that $0 < p < \infty$ and q is positive on $(0,1]$.

(a) *A standard Wiener process $\{W(t),\ 0 \leq t < \infty\}$ can be so constructed that*

$$\int_0^1 |n^{-1/2}(S(nt) - W(nt))|^p/q(t)dt = o_P(1) \tag{1.1}$$

if and only if

$$\int_0^1 t^{p/2}/q(t)dt < \infty. \tag{1.2}$$

(b) *Let $\{W(t),\ 0 \leq t < \infty\}$ be a standard Wiener process. Then*

$$\int_0^1 |n^{-1/2}S(nt)|^p/q(t)dt \xrightarrow{\mathcal{D}} \int_0^1 |W(t)|^p/q(t)dt \tag{1.3}$$

if and only if (1.2) *holds.*

We note that the statements (1.1), (1.2) and (1.3) of Theorem B are equivalent.

2. Weighted asymptotics near ∞. Obviously, all the results from Section 1 for $t \in (0,1]$ can be stated on $(0,T]$ for any $0 < T < \infty$. Since $W(t) \to \infty$ as $t \to \infty$, there is no weak convergence of $n^{-1/2}S(nt)$ on $[1,\infty)$. However, introducing appropriate weight functions opens up the possibility of studying such phenomena near infinity. We study asymptotics of weighted partial sum processes $n^{-1/2}S(nt)/h(t)$, $1 \le t < \infty$, where $h(t)$ is a non-negative function on $[1,\infty)$ and $h(t) \to \infty$ as $t \to \infty$. While our weighted L_p-approximations near infinity will be complete analogs of those near zero, the optimal class of weight functions for weak convergence near infinity in the sup-norm will be seen to be bigger than the corresponding class for the local case ($t \downarrow 0$). This is due to the fact that near infinity there is no *a priori* need for a Chibisov-O'Reilly type theorem. Namely, we obtain approximations in probability, and hence also weak convergence of our weighted partial sum processes on $D[1,\infty)$ whenever

$$\limsup_{t\to\infty} |W(t)|/h(t) < \infty \quad \text{a.s.}$$

For details we refer to Szyszkowicz (1992d).

A function $h : [1,\infty) \to (0,\infty)$ will be called positive if $\inf_{1\le t\le K} h(t) > 0$ for all $1 < K < \infty$.

THEOREM 2.1. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0,\ EX_1^2 = 1,\ E|X_1|^r < \infty \textit{ for some } r > 2.$$

Then a standard Wiener process $\{W(t);\ 0 \le t < \infty\}$ can be so constructed that with a function $h(t)$ on $[1,\infty)$ which is positive and such that

$$\limsup_{t\to\infty} t^{1/2}/h(t) < \infty$$

we have

$$\sup_{1\le t<\infty} |n^{-1/2}(S(nt) - W(nt))|/h(t) = o(1) \quad \text{a.s.}$$

From Theorem 2.1 we conclude the weak convergence of weighted partial sum processes $n^{-1/2}S(nt)/h(t)$ to $W(t)/h(t)$ whenever the limiting process is finite. In order to make this statement more precise, we let $\mathcal{H}$ be the class of those positive functions h on $[1,\infty)$ for which $h(t)/t$ is non-increasing in a neighbourhood of infinity. We introduce the following integral:

$$I_\infty(h,c) = \int_1^\infty t^{-1}\exp(-ct^{-1}h^2(t))dt, \quad 0 < c < \infty.$$

For a global description of the behaviour of a Wiener process near infinity we refer to Csörgő, Shao and Szyszkowicz (1991). By Theorem 2.1 we obtain the following result.

COROLLARY 2.1. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0, \; EX_1^2 = 1, \; E|X_1|^r < \infty \text{ for some } r > 2$$

and let $\{W(t), \; 0 \le t < \infty\}$ be a standard Wiener process. Then with $h \in \mathcal{H}$ we have

$$n^{-1/2}S(nt)/h(t) \xrightarrow{\mathcal{D}} W(t)/h(t) \quad \text{on } D[1, \infty)$$

if and only if $I_\infty(h, c) < \infty$ for some $c > 0$.

We note that this result is not completely analogous to the case when t is approaching 0, where the corresponding class of possible weight functions was smaller.

The proof of Theorem 2.1 is based on the KMT (1975, 1976) approximation. It cannot be carried out if we assume the existence of two moments only. To handle this case, we use the Theorem of Major (1979) and obtain the following result.

THEOREM 2.2. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0, \quad EX_1^2 = 1.$$

Let $h \in \mathcal{H}$ and $h(t)/t^\alpha$ be non-decreasing near infinity for some $0 < \alpha \le 1/2$.

(a) A standard Wiener process $\{W(t), \; 0 \le t < \infty\}$ can be constructed in such a way that if $I_\infty(h, c) < \infty$ for some $c > 0$, then

$$\sup_{1 \le t < \infty} |n^{-1/2}(S(nt) - W(nt))|/h(t) = o_P(1).$$

(b) *With $\{W(t), \; 0 \le t < \infty\}$ being a standard Wiener process, we have*

$$n^{-1/2}S(nt)/h(t) \xrightarrow{\mathcal{D}} W(t)/h(t) \quad \text{in } D[1, \infty),$$

if and only if $I_\infty(h, c) < \infty$ for some $c > 0$.

Assuming montonicity of $h(t)/t^\alpha$ for some $0 < \alpha \le 1/2$ is less restrictive, of course, than the same assumption with $\alpha = 1/2$. Obviously $h(t)/t^\alpha$ being non-decreasing for some $0 < \alpha \le 1/2$ implies monotonicity of $h(t)$ itself. If we require only $h(t)$ to be non-decreasing near infinity, we obtain a Chibisov-O'Reilly type theorem.

THEOREM 2.3. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0, \quad EX_1^2 = 1.$$

Let $h \in \mathcal{H}$ and $h(t)$ be non-decreasing in a neighbourhood of infinity. If $I_\infty(h,c) < \infty$ for all $c > 0$, then a standard Wiener $\{W(t),\ 0 \le t < \infty\}$ can be so constructed that

$$\sup_{1 \le t < \infty} |n^{-1/2}(S(nt) - W(nt))|/h(t) = o_P(1).$$

Considering L_p functionals of partial sum processes we obtain the following result.

THEOREM 2.4. *Let $0 < p < \infty$ and $X_1, X_2, \ldots$ be i.i.d.r.v.'s such that*

$$EX_1 = 0, \quad EX_1^2 = 1.$$

A standard Wiener process $\{W(t),\ 0 \le t < \infty\}$ can be so constructed that with $h(t)$ on $[1,\infty)$ which is positive and such that

$$\int_1^\infty t^{p/2}/h(t)dt < \infty \tag{2.1}$$

we have

$$\int_1^\infty |n^{-1/2}(S(nt) - W(nt))|^p/h(t)dt = o_P(1).$$

(b) Let h be a positive function on $[1,\infty)$. Then, as $n \to \infty$, we have

$$\int_1^\infty |n^{-1/2}S(nt)|^p/h(t)dt \xrightarrow{\mathcal{D}} \int_1^\infty |W(t)|^p/h(t)dt$$

if and only if (2.1) holds.

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A NOTE ON ENTROPY

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ABSTRACT. We study the problem of minimizing the entropy with respect to a given probability measure over a convex set. We prove that under weak conditions, when a solution exists, it is unique. We also find the general form of the solution under additional conditions.

1. INTRODUCTION

The problem of minimizing the entropy with respect to a given probability measure over a convex set appears in many areas of mathematics, statistics and physics: theory of automata, bayesian analysis, statistical mechanics, large deviations, etc.

In this Note we prove in Theorem 1, under weak conditions on the convex set, that if at least one solution exists, then it is unique. We also find the general form of the solution in Theorem 2, under additional assumptions. These results are proved in Section 2. Finally, in Section 3, we give some applications of these two theorems.

Before stating our main results, we need to introduce some definitions and notations.

From now on, X is a Polish space, i.e. a complete and separable metric space, $\mathcal{B}$ is the associated Borel σ-algebra on X, $\mathcal{B}_b$ stands for the space of bounded and measurable real functions on X, and $M(X)$ denotes the space of probability measures on $(X, \mathcal{B})$ equipped with the weak star topology.

Following Donsker and Varadhan (1975), we define the entropy of Q with respect to P (hereafter denoted by $h(Q; P)$) by

$$h(Q;P) = \sup_{u \in \mathcal{B}_b} \int u dQ - \log(\int e^u dP) \tag{1}$$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Suppose C is a convex subset of $M(X)$ and let $I(C) = \inf_{Q \in C} h(Q;P)$. If $I(C) < \infty$, we define $S_C = \{Q \in M(X) \colon h(Q;P) = I(C)\}$. Any member of S_C will be called a solution.

We are now in a position to state our main results.

2. Main Results

We first state a proposition recalling some well-known properties of entropy.

Proposition 1. *Let X be a Polish space and suppose that $P \in M(X)$ is fixed. Then*

(1) *$h(Q;P) < \infty$ iff Q is absolutely continuous with respect to P (denoted by $\ll P$), and $\log \frac{dQ}{dP} \in L^1(Q)$. In the later case,*

$$h(Q;P) = \int \left(\log \frac{dQ}{dP}\right) dQ\,;$$

(2) *$h(Q;P) \geq 0$, and $h(Q;P) = 0$ iff $Q = P$;*

(3) *the mapping $Q \mapsto h(Q;P)$, $Q \in M(X)$, is lower semicontinuous and convex;*

(4) *for any $a > 0$, $K_a = \{Q \in M(X) \colon h(Q;P) \leq a\}$ is compact.*

Let $C_a = C \cap K_a$ and let $C_\infty = \bigcup_{a \geq 0} C_a$. Recall that $Q_1, Q_2 \in M(X)$ are equivalent ($Q_1 \sim Q_2$ for short) iff $Q_1 \ll Q_2$ and $Q_2 \ll Q_1$.

Theorem 1. *Suppose $I(C) < \infty$ and C_a is closed for all $a \geq 0$. Then*

(1) *If $Q_1 \in C_\infty$ and $Q_0 \in S_C$, then $h(Q_1;Q_0) < \infty$;*

(2) *S_C contains only one element, i.e. the solution is unique.*

Proof. It follows easily from Proposition 1 that C_a is convex and compact since C_a is closed by hypothesis. The proof of item (1) is similar to the proof of Lemmma 2.2 in Donsker and Varadhan (1975) but we include it for sake of completeness. So suppose that $Q_1 \in C_\infty$ and $Q_0 \in S_C$. Set $a_i = \frac{dQ_i}{dP}$, $i = 0, 1$. Let $Q_\epsilon = (1-\epsilon)Q_0 + \epsilon Q_1$. Then $Q_\epsilon \in C_\infty$ since

$$\phi(\epsilon) = h(Q_\epsilon;P) \leq (1-\epsilon)h(Q_0;P) + \epsilon h(Q_1;P) < \infty$$

using the convexity of $h(\cdot;P)$. Set $a_\epsilon = \frac{dQ_\epsilon}{dP}$. Therefore

$$\log a_\epsilon \in L^1(Q_\epsilon) \subset L^1(Q_0) \cap L^1(Q_1) \text{ if } 0 < \epsilon < 1$$

Now it is easy to see that

$$\log^+ a_\epsilon \leq \log 2 + \log^+ a_{\frac{1}{2}} \in L^1(Q_0) \cap L^1(Q_1) \qquad 0 \leq \epsilon \leq 1 \tag{2}$$

and

$$\log^- a_\epsilon \leq \log\left(\frac{1}{1-\epsilon}\right) + \log^- a_0 \qquad 0 \leq \epsilon < 1 \tag{3}$$

Next we see that $\phi(\cdot)$ is differentiable in $(0,1)$ and

$$\phi'(\epsilon) = \frac{d\phi(\epsilon)}{d\epsilon} = \int \log a_\epsilon dQ_1 - \int \log a_\epsilon dQ_0 \qquad 0 < \epsilon < 1$$

Since $\phi(\epsilon) \geq \phi(0)$ and $h(\cdot\,;P)$ is convex, it follows that $\phi'(\epsilon) \geq 0$. Hence

$$\int \log a_\epsilon dQ_1 \geq \int \log a_\epsilon dQ_0 \qquad 0 < \epsilon < 1 \tag{4}$$

Therefore

$$\int \log^- a_\epsilon dQ_1 \leq \int \log^- a_\epsilon dQ_0 + \int \log^+ a_\epsilon dQ_1 - \int \log^+ a_\epsilon dQ_0, \quad 0 < \epsilon < 1 \tag{5}$$

Now $\lim_{\epsilon \downarrow 0} \log^\pm a_\epsilon = \log^\pm a_0$ and it follows from Fatou's Lemma and from (2) and (3) that we can let $\epsilon \downarrow 0$ in (5) to obtain

$$\int \log^- a_0 dQ_1 \leq \int \log^- a_0 dQ_0 + \int \log^+ a_0 dQ_1 - \int \log^+ a_0 dQ_0 < \infty \tag{6}$$

where we have used the bounded convergence theorem to prove that the r.h.s. of (5) converges to the r.h.s. of (6). It follows from (2) - (6) that $Q_1(\{a_0 > 0\}) = 1$ and rearranging terms in (6) we obtain

$$\int \log a_0 dQ_1 \geq \int \log a_0 dQ_0 = h(Q_0; P) \tag{7}$$

Therefore $\frac{dQ_1}{dQ_0} = \frac{a_1}{a_0}$ and

$$\log \frac{dQ_1}{dQ_0} = \log a_1 - \log a_0 \in L^1(Q_1)$$

Hence $h(Q_1; Q_0)$ is finite and

$$h(Q_1; Q_0) = \int \log a_1 dQ_1 - \int \log a_0 dQ_1 \leq h(Q_1; P) - h(Q_0; P) \tag{8}$$

using (7). To prove uniqueness of the solution, suppose that $Q_0, Q_1 \in S_C$. Then using item (1) and inequality (8) we get

$$0 \leq h(Q_1; Q_0) \leq h(Q_1; P) - h(Q_0; P) = 0$$

It follows from Proposition 1 that $Q_0 = Q_1$, completing the proof of the theorem. □

Having settled the question of uniqueness, we are ready to study the general form of the solution.

Theorem 2. *Suppose $I(C) < \infty$ and C_a is closed for all $a \geq 0$. Also suppose that L is a closed linear subspace of $L^1(Q)$ ($S_C = \{Q\}$) such that $1 \in L$ and such that for all $k \in L^\infty(Q)$ satisfying $\int ku\,dQ = 0$, for all $u \in L$, we have $Q_\epsilon \in C$, where*

$$\frac{dQ_\epsilon}{dQ} = 1 + \epsilon k$$

whenever $|\epsilon| < \frac{1}{2\|k\|_\infty}$. Under these conditions, $\log \frac{dQ}{dP} \in L$.

Proof. Suppose that $k \in L^\infty(Q)$ is given and that $\int ku\,dQ = 0$, for all $u \in L$. Suppose also that $|\epsilon| < \frac{1}{2\|k\|_\infty}$. Then we see that

$$\frac{1}{2} \leq \frac{dQ_\epsilon}{dQ} \leq \frac{3}{2}$$

Therefore $Q_\epsilon \in C_\infty$ and $\psi(\epsilon) = h(Q_\epsilon; P)$ is differentiable at 0 and

$$0 = \psi'(0) = \int k \log \frac{dQ}{dP} dQ + \int k dQ = \int k \log \frac{dQ}{dP} dQ$$

since $1 \in L$ and $\psi(\epsilon) \geq \phi(0)$, $|\epsilon| < \frac{1}{2\|k\|_\infty}$. Thus for every $k \in L^\infty(Q)$ such that $\displaystyle\int ku\,dQ = 0$, for all $u \in L$, we have

$$\int k \log \frac{dQ}{dP} dQ = 0 \tag{9}$$

Since L is a closed subspace of $L^1(Q)$, it follows from (9) and the Hahn-Banach Theorem that $\log \frac{dQ}{dP} \in L$, completing the proof. □

3. Some Applications

Throughout this section, f is a continuous function from X to $\mathbb{R}^p$ satisfying one of the following assumptions:

(A1) $\int e^{<\lambda, f>} dP < \infty$, for all $\lambda \in \mathbb{R}^p$;

(A2) there exists $\delta > 0$ such that $\int e^{\theta|f|} dP < \infty$, for all $0 \leq \theta < \delta$.

We see that if f satisfies (A2) and $\theta \in (0, \delta)$, then using (1) we obtain

$$\sup_{Q \in K_a} \theta \int_{|f| \leq M} |f| dQ \leq a + \log \left(\int_{|f| \leq M} e^{\theta|f|} dP + P(|f| > M) \right)$$

Letting $M \uparrow \infty$ we get

$$\sup_{Q \in K_a} \int |f| dQ \leq \frac{a}{\theta} + \frac{1}{\theta} \log \left(\int e^{\theta|f|} dP \right) \qquad 0 < \theta < \delta \tag{10}$$

Moreover, if f satisfies (A1), then replacing $|f|$ by $|f| 1_{\{|f| > M\}}$ in (10), we obtain

$$\lim_{M \uparrow \infty} \sup_{Q \in K_a} \int_{|f| > M} |f| dQ = 0$$

proving the continuity of the mapping $Q \mapsto \int f dQ$ restricted to K_a.

The first category of convex sets we will consider are convex sets appearing in large deviations. These sets are defined as follows:

$$C = \left\{ Q \in M(X); f \in L^1(Q) \text{ and } \int f dQ = b \right\}$$

Corollary 1. *Suppose that f satisfies (A1) and $I(C) < \infty$. Then $S_C = \{Q\}$ and there exists $\lambda \in \mathbb{R}^p$ and $A \in B$ such that $Q(A) = 1$ and*

$$\frac{dQ}{dP} = \frac{e^{<\lambda, f>}}{Z_\lambda} \qquad \text{on } A \tag{11}$$

for some positive constant Z_λ. If f satisfies (A2) and $Q \in C$ is such that $\frac{dQ}{dP}$ satisfies (11) and $Q \sim P$, then $S_C = \{Q\}$.

Proof. Suppose first that f satisfies (A1) and $I(C) < \infty$. Then it is easy to see that C_a is closed for any $a \geq 0$. It follows from Theorem 1 that $S_C = \{Q\}$. Next set $L = \{a + < \lambda, f >; a \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}^p\}$. Then one can see easily that L satisfies all requirements on Theorem 2. Therefore $\log \frac{dQ}{dP} \in L$ proving the first part. Suppose now that $\frac{dQ}{dP} = \frac{e^{<\lambda, f>}}{Z_\lambda}$ and that $Q \sim P$. Then $h(Q; P) = -\log Z_\lambda$. Moreover if $R \in C_\infty$, then

$$0 \leq h(R; Q) = \int \log\left(\frac{\frac{dR}{dP}}{\frac{dQ}{dP}}\right) dR = h(R; P) - h(Q; P)$$

proving that $h(R; P) \geq h(Q; P)$ with equality iff $R = Q$. Hence $S_C = \{Q\}$. □

Remark. In Donsker and Varadhan (1976), it is proven that if f satisfies (A1), then

$$I(C) = \sup_\lambda < \lambda, b > - \log\left(\int e^{<\lambda, f>} dP\right)$$

For our second example, we take a product space of Polish spaces, i.e.

$$X = X_1 \times X_2 \times \cdots \times X_n$$

and $P = P_1 \otimes P_2 \otimes \cdots \otimes P_n$. Let π_i be the canonical projection of X onto X_i, $1 \leq i \leq n$.

We will consider convex sets of the form:

$$C = \left\{ Q \in M(X); f \in L^1(Q), \int f dQ = 0 \text{ and } Q \circ \pi_i^{-1} = P_i, \ 1 \leq i \leq n \right\}$$

Corollary 2. *Suppose that f satisfies (A1) and $I(C) < \infty$. Then $S_C = \{Q\}$ and if there exists $Q_1 \in C_\infty$ such that $Q_1 \sim P$, then there exist $g_i \in L^1(P_i)$, $i = 1, \dots, n$ and $\lambda \in \mathbb{R}^p$ such that*

$$\frac{dQ}{dP} = \left(\prod_{i=1}^{n} e^{g_i \circ \pi_i}\right) e^{<\lambda, f>}, \text{ } P \text{ almost surely} \tag{12}$$

Moreover if f satisfies (A2) and $Q \in C$ is such that $\frac{dQ}{dP}$ satisfies (12), then $S_C = \{Q\}$.

Proof. Repeating the arguments in Corollary 1, we see that C_a is closed for any $a \geq 0$, proving that $S_C = \{Q\}$ if $I(C) < \infty$. Now suppose that $Q_1 \in C_\infty$ and $Q_1 \sim P$. It follows from Theorem 1 that $h(Q_1; Q) < \infty$, so $Q \sim P$. Set $N = \{\sum_{i=1}^n g_i \circ \pi_i; g_i \in L^1(P_i),\ 1 \leq i \leq n\}$. Using the same arguments as in Lemma 2.3 in Donsker and Varadhan (1975), we obtain that N is a closed subspace of $L^1(Q)$ since $Q \sim P$. Also let $F = \{< \lambda, f >; \lambda \in \mathbb{R}^p\}$. Then F is a finite dimensional subspace of $L^1(Q)$, and it follows from Theorem 1.42 in Rudin (1973) that $L = N + F$ is a closed subspace. Moreover L satisfies the requirements of Theorem 2 so we conclude that $\log \frac{dQ}{dP} \in N + F$. Finally if f satisfies (A2) and $Q \in C$ is such that $\frac{dQ}{dP}$ satisfies (12), then for any $R \in C_\infty$,

$$0 \leq h(R; Q) = \int \log\left(\frac{\frac{dR}{dP}}{\frac{dQ}{dP}}\right) dR = h(R; P) - h(Q; P)$$

proving that $h(R; P) \geq h(Q; P)$ with equality iff $R = Q$. Hence $S_C = \{Q\}$. $\square$

Conjecture. $N = \left\{\sum_{i=1}^{n} g_i \circ \pi_i; g_i \in L^1(P_i),\ 1 \leq i \leq n\right\}$ *is closed in $L^1(Q)$ for any $Q \in C_\infty$.*

If this conjecture is true, then we can restate Corollary 2 as follows:

Corollary 3. *Suppose that f satisfies (A1) and $I(C) < \infty$. Then $S_C = \{Q\}$ and there exist $g_i \in L^1(P_i)$, $i = 1, \dots, n$ and $\lambda \in \mathbb{R}^p$ such that*

$$\frac{dQ}{dP} = \left(\prod_{i=1}^{n} e^{g_i \circ \pi_i}\right) e^{<\lambda, f>}, \text{ } Q \text{ almost surely}$$

Moreover if f satisfies (A2) and $Q \in C$ is such that $\frac{dQ}{dP}$ satisfies (12), then $S_C = \{Q\}$.

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A Non-Uniform Estimate Taking into Account Large Deviations in the Limit Theorem on Non-Normal Convergence to the Normal Law

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Abstract: We construct asymptotic expansions taking into account large deviations in the limit theorem on non-normal convergence to the normal law. Some refining terms of our expansions are of logarithmic type.

0. Introduction. There is the one particular result among those on weak convergence towards the normal law for the classical scheme of summation of independent random variables $\{X_n, n \ge 1\}$ with common distribution function F. It is related to the case when the tails of F are regularly varying functions of index $\alpha = -2$. Obviously, for this case in order to establish weak convergence to the normal law for properly centered and normalized sum $S_n := X_1 + \ldots + X_n$ a normalizing sequence that differs from $(n \cdot VX_1)^{1/2}$ should be chosen, since $VX_1 = \infty$. For example, in a special case of power tails of index $\alpha = -2$ such that

(0.1) $$1 - F(x) \sim c \cdot x^{-2};$$

(0.1′) $$F(-x) \sim d \cdot x^{-2}$$

as $x \to \infty$ it follows from Theorem 2.6.5 of Ibragimov and Linnik (1971) that for the random sequence $\zeta_n := (S_n - n \cdot EX_1)/((c+d) \cdot n \cdot \log n)^{1/2}$ the central limit theorem is valid (as $n \to \infty$). In other words, for this sequence non-normal convergence to the normal law holds.

The corresponding result on precise asymptotics (up to equivalence) for the probabilities of two-sided large deviations of S_n from zero under fulfilment of (0.1) - (0.1′) was obtained in Tkachuk (1975). In particular, Theorem 2 of Tkachuk (1975) implies that if $EX_1 = 0$ then for any $\varepsilon > 0$

(0.2) $$P(|S_n| > y) \sim P(\max_{1 \le i \le n} |X_i| > y) \sim n \cdot P(|X_1| > y)$$

as $n \to \infty$ with $y \ge ((c + d + \varepsilon) \cdot n \cdot \log n \cdot \log \log n)^{1/2}$. Note that the probabilistic interpretation of (0.2) consists in the fact that large deviations of $|S_n|$ occur due to one large summand whose absolute value is comparable with $|S_n|$.

In this work we construct an asymptotic expansion in the central limit theorem for the sequence ζ_n with non-uniform estimate of the remainder under fulfilment of certain supplementary restrictions on the asymptotics of the tails of function F (cf. Theorem 1.2 of Section 1). This result can have its own value due to the novelty of some refining terms of the asymptotic expansion as well as due to the fact that it yields a more precise representation (compare with Relationships (0.2) obtained in Tkachuk (1975)) for the probabilities of large deviations of S_n (cf. Corollary 1.4 in Section 1). Thereupon, we apply Theorem 1.2 for the derivation of the asymptotic expansions for the right-hand side of (0.2) (cf. Theorem 2.1 in Section 2).

Let us note that the present article is close conceptually to the author's work Vinogradov (1990) where the analogous problems related to the case of power tails with integer index $\alpha \le -3$ were considered. However, only the fact that both cases of integer $\alpha \le -3$ and $\alpha = -2$ can be treated by use of the same technique may not provide enough reason for a simultaneous consideration of these cases. Note that the case $\alpha = -2$ is much more complicated even when establishing the results on weak convergence. In order to avoid duplicating the proofs, we refer the reader to Vinogradov (1990), whenever possible. End of proofs is marked by □.

1. Formulation and Proof of Theorem 1.2. Throughout this section we set mild supplementary constraints on the tail behavior of the common distribution function F of X_n's (to compare with (0.1)-(0.1′)). Namely, let us assume that

$$1 - F(x) = c\cdot x^{-2} + O(x^{-2-\kappa}); \tag{1.1}$$

$$F(-x) = d\cdot x^{-2} + O(x^{-2-\kappa}) \tag{1.1'}$$

as $x \to \infty$, where $e := c + d > 0$, and $\kappa \in (0,1)$ is fixed. Let us also assume without loss of generality that $EX_1 = 0$.

The following rate of convergence in (0.2) is easily derived from Theorem 1 of Vinogradov (1992), Relationship (0.2), and Theorem 3 of Petrov (1975):

Proposition 1.1. Let Conditions (1.1) - (1.1′) be fulfilled, $e > 0$, and $EX_1=0$. Then there exist positive constants K_1 and K_2 such that for any integer $n \ge 2$ with $y \ge (K_1\cdot n\cdot\log n\cdot\log\log(e\cdot n))^{1/2}$ the following estimates hold:

$$|P(S_n > y) - n\cdot c\cdot y^{-2}| \le K_2\cdot n\cdot y^{-2}\cdot n^{1/3}\cdot y^{-2/3} + n\cdot \sup_{x\ge y/3} |1 - F(x) - c\cdot x^{-2}| \; ; \tag{1.2}$$

$$|P(S_n < -y) - n\cdot d\cdot y^{-2}| \le K_2\cdot n\cdot y^{-2}\cdot n^{1/3}\cdot y^{-2/3} + n\cdot \sup_{x\le -y/3} |F(x) - d\cdot |x|^{-2}| \, . \tag{1.2'}$$

Now, in order to formulate the main result of this section (Theorem 1.2) we introduce the following quantities:

$$E_+ = \int_0^1 (e^{iz} - \sum_{m=0}^{2} ((iz)^m/m!)\cdot d(-z^{-2}) + \int_0^\infty (e^{iz} - 1 - iz)\cdot d(-z^{-2}) ;$$

$$E_- = \int_0^1 (e^{-iz} - \sum_{m=0}^{2} ((-iz)^m/m!)\cdot d(-z^{-2}) + \int_0^\infty (e^{-iz} - 1 + iz)\cdot d(-z^{-2}) .$$

Set

$$H(x) := \int_{-\infty}^{x} dv\cdot\left(\int_{-\infty}^{\infty} \frac{1}{2\pi}\cdot e^{-t^2/2-itv}\cdot((it)^2\cdot\log\frac{1}{|t|} + \frac{1}{e}\cdot(c\cdot E_{\mathrm{sgn}\,t} + d\cdot E_{-\mathrm{sgn}\,t}))\cdot dt\right);$$

$$b := \int_{-\infty}^{-1} x^2\cdot d(F(x) - d\cdot|x|^{-2}) + \int_{-1}^{1} x^2\cdot d\,F(x) + \int_1^\infty x^2\cdot d(F(x) + c\cdot x^{-2}) .$$

Note that b is finite under the fulfilment of (1.1)-(1.1′), since all the integrals on the right-hand side of the last formula being convergent. Hereinafter we refer to b as the *second pseudomoment*, since it possesses some properties of the second moment (for example, it coincides with the coefficient under $(it)^2/2!$ in the Taylor expansion of the characteristic function f(t) of X_1 near zero (cf. (1.6) below)). Set $F_n(x) := P(\zeta_n \le x)$ (recall that $\zeta_n = S_n/(e\cdot n\cdot\log n)^{1/2}$). Then the following result is valid:

<u>Theorem 1.2.</u> Let Conditions (1.1)-(1.1′) be fulfilled, $e > 0$, $EX_1 = 0$, and r.v. X_1 is not a lattice variable. Then

a) for any integer $n \ge 2$ and for any real x

$$\left| F_n(x) - \Phi(x) - \frac{\log\log n + \log e + b/e}{2\cdot\log n}\cdot\Phi''(x) - \frac{H(x)}{\log n}\right| \le \frac{\delta(n)}{(1+|x|)^2\cdot\log n}, \tag{1.3}$$

where $\Phi(\cdot)$ is the Laplace function, and $\delta(n) \to 0$ as $n \to \infty$;

b) function $H(\cdot)$ has the following asymptotics on the tails as $x \to \infty$:

$$-H(x) = \frac{c}{e}\cdot x^{-2} + O(x^{-3}\cdot\log x) ; \tag{1.4}$$

$$H(-x) = \frac{d}{e}\cdot x^{-2} + O(x^{-3}\cdot\log x) \tag{1.4'}$$

<u>Remark 1.</u> It is interesting to compare (1.3) with uniform estimates that can be derived from Theorems 2 and 5 of Hall (1983) (see also Theorem 4.12 of Hall (1982)). In particular, Theorem 4.12 of Hall (1982) implies that under the conditions of our Theorem 1.2 the following estimate holds:

$$P(S_n \le C_n\cdot x + D_n) - \Phi(x) - \frac{\log\log n}{2\cdot\log n}\cdot\Phi''(x) - \frac{\psi(x,n)}{2\cdot\log n}\cdot\Phi''(x) = o\left(\frac{1}{\log n}\right)$$

as $n \to \infty$ uniformly in $x \in R^1$, where $C_n := \sup\{a:\ a^{-2}\cdot E(X_1^2\cdot\chi(|X_1| \le a)) \ge n^{-1}\} \sim ((c+d)\cdot n$

$\cdot\log n)^{1/2}$ as $n \to \infty$, $D_n := n\cdot E(X_1\cdot\chi(|X_1| \le C_n)) \sim n\cdot(c-d)\cdot\log n$ as $n \to \infty$, $\psi(\cdot,\cdot)$ is a certain uniformly bounded function from $R^1\times N$, and $\chi(A)$ stands for the indicator of set A. On the other hand, the asymptotic expansion of Theorem 5 of Hall (1983) contains a refining term $2n\cdot P(|X_n| > C_n)\cdot\tilde{\omega}_0\Phi(x)$, where the operator $\tilde{\omega}_0$ is defined by Formula (4) in Höglund (1970). Note that this term is analogous to the term $H(x)/\log n$ in (1.3). We emphasize that the just mentioned results by Hall provide uniform estimates of remainders, whereas our Estimate (1.3) is non-uniform.

<u>Proof of Theorem 1.2.</u> Proof of (b) is carried out by the stationary phase method. It is similar to that of Theorem 1.1.b of Vinogradov (1990) and therefore is omitted. □

Now, in order to establish the assertion of Point (a) we need

<u>Proposition 1.3.</u> Let all the conditions of Theorem 1.2 be fulfilled. Then there exists a positive constant K such that for any integer $n \ge 2$ and for any real x the left-hand side of (1.3) does not exceed

$$K/(n^{\kappa/2}\cdot(1 + \log n)^{1+\kappa/2}) \,. \tag{1.5}$$

<u>Proof of Proposition 1.3</u> is straightforward. It relies on the Smoothing Inequality and the following representation of $f(t)$ as $t \to 0$:

$$\begin{aligned} f(t) &= 1 + b\cdot(it)^2/2 + e\cdot(it)^2\cdot\log\frac{1}{|t|} \\ &+ |t|^2\cdot(c\cdot E_{\mathrm{sgn}\, t} + d\cdot E_{-\mathrm{sgn}\, t}) + O(|t|^{2+\kappa}) \,. \end{aligned} \tag{1.6}$$

Then Point (a) of Theorem 1.2 follows from the uniform Estimate (1.5), Relationships (1.4)-(1.4′) of Point (b), and Inequalities (1.2)-(1.2′) for the probabilities of right-hand and left-hand large deviations of S_n. In fact, (1.5) yields (1.3) in the range of deviations $|x| \le \delta_1(n)\cdot(n\cdot\log n)^{\kappa/4}$, where $\delta_1(n) \to 0$ as $n \to \infty$. On the other hand, changing y to $x\cdot(e\ \cdot n\cdot\ \log n)^{1/2}$ in Formulas (1.2) - (1.2′) we obtain the corresponding bounds for $|\ 1 - F_n(x) - \frac{c}{e}\cdot x^{-2}\cdot(\log n)^{-1}\ |$ and $|\ F_n(-x) - \frac{d}{e}\cdot x^{-2}\cdot(\log n)^{-1}\ |$ for $x \ge ((K/e)\cdot\log\ \log\ (e\ \cdot n))^{1/2}$. The subsequent application of Relationships (1.4) - (1.4′) yields the required estimates for $|1 - F_n(x) + H(x)/\log n|$ in the range of deviations $x \ge ((K/e)\cdot\log\ \log\ (e\cdot n))^{1/2}$ and for $|F_n(x) - H(x)/\log n|$ in the range of deviations $x \le -((K/e)\cdot\log\ \log\ (e\cdot n))^{1/2}$. It only remains to combine these estimates with the exponential decay of the tails of Φ and Φ''. □

<u>Corollary 1.4.</u> Let all the conditions of Theorem 1.2 be fulfilled . Then

$$\begin{aligned} P(S_n > y) &= 1 - \Phi(y/(e\cdot n\cdot\log\log n)^{1/2}) \\ &- \frac{\log\log n + \log e + b/e}{2\cdot\log n}\cdot\Phi''(y/(e\cdot n\cdot\log n)^{1/2}) + n\cdot c\cdot y^{-2} + o(n\cdot y^{-2}) \,. \end{aligned} \tag{1.7}$$

Remark 2. Note that under Conditions (1.1) -(1.1′) the representation for $P(S_n > y)$ given by (1.7) is more precise in comparison with that given by Theorem 2 of Tkachuk (1975) (cf. also (0.2)) in the range of deviations $y/(n\cdot\log n)^{1/2} \to \infty$, $y \le const\cdot(n\cdot\log n\cdot\log\log n)^{1/2}$. This is on account of the right-hand side of (1.7), which provides an *asymptotic expansion* for $P(S_n > y)$ in this range of deviations.

2. Formulation of Theorem 2.1. It is easily seen that in the range of deviations $y/(n\cdot\log n\cdot\log\log n)^{1/2} \to \infty$ as $n \to \infty$ Corollary 1.4 provides only the asymptotics of $P(S_n > y)$ up to equivalence (compare with (0.2)). On the other hand, we can not guarantee their accuracy better than up to $const\cdot n\cdot y^{-2-\kappa}$ (that is not much different from $o(n\cdot y^{-2})$) at least because that is the case even for $n = 1$ (cf. (1.1)). Therefore, let us require that

$$1 - F(x) = \sum_{i=1}^{\ell} c_{\alpha_i}\cdot x^{-\alpha_i} + o(x^{-r}), \tag{2.1}$$

as $x \to \infty$, where $2 = \alpha_1 < \alpha_2 < \dots < \alpha_\ell \le r$.

Theorem 2.1. Let Conditions (2.1) and (1.1′) be fulfilled with $\alpha_1 = 2$, $e_2 > 0$, $EX_1 = 0$, and r.v. X_1 is not a lattice variable. Then for any integer $n \ge 2$ and for any real $y > 0$

$$P(S_n > y) = n\cdot\sum_{i=1}^{\ell} c_{\alpha_i}\cdot y^{-\alpha_i} + 6\cdot n\cdot(n-1)\cdot\log(n-1)\cdot c_2\cdot e_2\cdot y^{-4}$$

$$+ 9\cdot n\cdot(n-1)\cdot\log\log(e\cdot(n-1))\cdot c_2\cdot e_2\cdot y^{-4}$$

$$+ 6\cdot n\cdot(n-1)\cdot\log\log\log(e^e\cdot(n-1))\cdot c_2\cdot e_2\cdot y^{-4} + r_1(n,y) + r_2(n,y),$$

where $r_1(\cdot,\cdot)$ is such that for any integer $n \ge 2$ and for any real $y > 0$ $|r_1(n,y)| \le n\cdot\sup_{x\ge y/3}\left|1 - F(x) - \sum_{i=1}^{\ell} c_{\alpha_i}\cdot x^{-\alpha_i}\right|$, and $r_2(\cdot,\cdot)$ is such that there exists a function $K(\cdot)$ from R^1_+ into R^1_+ such that for any real $\theta > 0$, for any integer $n \ge 2$, and for $y \ge K(\theta)\cdot n^{1/2+\theta}$ $|r_2(n,y)| \le K_1\cdot n^2\cdot y^{-4}$.

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