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CONTENTS

The master coloring

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Presented by G.F.D. Duff, F.R.S.C.

P.Erdős [1] observed that it is possible to color the plane with countably many colors with no monochromatic points in rational distance. He deduced this from an earlier graph-theoretical result of himself and A. Hajnal. On the line, the corresponding result is trivial, as "being in rational distance" is an equivalence relation there, with countable classes. Using more complicated (not graph-theoretical) arguments the result was later extended to $\mathbf{R}^3$ [2], and finally for $\mathbf{R}^n$ for every n [3].

Answering a question of P. Erdős, K. Kunen proved [4] that if the continuum hypothesis (CH, in short) holds, then there is a coloring of $\mathbf{R}^2$ with countably many colors, omitting monochromatic triangles with rational, $\neq 0$, area.

One can notice that the distance of points (x_1, y_1) and (x_2, y_2) is $r \geq 0$ iff $(x_1 - x_2)^2 + (y_1 - y_2)^2 - r^2 = 0$ and if r is rational, this is a polynomial with rational coefficients. There is a similar polynomial for the above mentioned result of K. Kunen. Namely, from Heron's formula one can get that the area of a triangle with sides a, b, c is t iff $p(a^2, b^2, c^2) = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16t^2 = 0$ and as a^2, b^2, c^2 are polynomials of the coordinates of the nodes, so is $p(a^2, b^2, c^2)$. In this note we show that, assuming CH, there is a coloring of $\mathbf{R}^n$ with countably many colors that works for every such polynomial simultaneously, assuming the polynomial satisfies a natural condition corresponding to the observation that we cannot and do not exclude pairs with zero distance, or triangles with zero area, in the above examples.

Notation. As follows, $\bar{a}$, $\bar{x}$, etc will denote elements of $\mathbf{R}^n$. $p(\bar{x}_1, \ldots, \bar{x}_t)$ is, therefore, a polynomial in tn variables. A polynomial $p(\bar{x}_1, \ldots, \bar{x}_t)$ is in class $\mathcal{P}$ if $p(\bar{a}, \ldots, \bar{a}) \neq 0$ holds for every $\bar{a} \in \mathbf{R}^n$.

Theorem. *(CH) There is a coloring* $f\colon \mathbf{R}^n \to \omega$ *such that if* $p(\bar{x}_1, \ldots, x_t) \in \mathcal{P}$ $(t \geq 2)$ *is a polynomial with rational coefficients , then there are no different, monochromatic points* $\bar{a}_1, \ldots, \bar{a}_t \in \mathbf{R}^n$ *with* $p(\bar{a}_1, \ldots, \bar{a}_t) = 0$.

Proof. If $p(\bar{x})$ is a polynomial, we let $X(p) = \{\bar{x} \in \mathbf{R}^n : p(\bar{x}) = 0\}$, and call these sets, as in algebraic geometry, *closed* sets.

By well-known closure principles it is possible to decompose $\mathbf{R}^n$ into the increasing, continuous union of countable sets A_α $(\alpha < \omega_1)$, and the set of all closed sets into the increasing continuous union of countable sets $\mathcal{X}_\alpha$ $(\alpha < \omega_1)$ such that there exist sets $F_\alpha \subseteq \mathbf{R}$ and the following conditions hold:

(1) F_α is a field ;
(2) $A_\alpha = F_\alpha^n$;
(3) if $p(\bar{x}) \in F_\alpha[\bar{x}]$, then $X(p) \in \mathcal{X}_\alpha$;
(4) the intersection of arbitrary many members of $\mathcal{X}_\alpha$ is in $\mathcal{X}_\alpha$;
(5) if $X \in \mathcal{X}_{\alpha+1} - \mathcal{X}_\alpha$ is infinite, then $|X \cap (A_{\alpha+1} - A_\alpha)| = \omega$.

These conditions can be met by a Skolem-type construction, at least, if (4) replaced with the weaker requirement where only the intersection of finitely many members is required to be in $\mathcal{X}_\alpha$. This, however, suffices, as by Hilbert's finite basis theorem, the intersection of an arbitrary number of closed sets is the intersection of some finitely many of them. (5) follows from the fact that an infinite closed set is uncountable, but in the argument below can be replaced by the weaker

(5') if $X \in \mathcal{X}_{\alpha+1} - \mathcal{X}_\alpha$ then either $X \subseteq A_{\alpha+1}$ or $|X \cap (A_{\alpha+1} - A_\alpha)| = \omega$.

We are going to color A_α by induction on α, satisfying

(6) f is one-to-one on $A_{\alpha+1} - A_\alpha$.

Let f be one-to-one on A_0. If f on A_α is already determined, enumerate $A_{\alpha+1} - A_\alpha$ as $\{\bar{a}_j : j < \omega\}$. We color $\bar{a}_j$ by induction on j. Assume that $\bar{a}_0, \dots, \bar{a}_{j-1}$ have already been colored. Let $Y(\bar{a}_j) \in \mathcal{X}_\alpha$ be the minimal closed set in $\mathcal{X}_\alpha$ such that $\bar{a}_j \in Y(\bar{a}_j)$ (exists by (4)). Assume that $Y(\bar{a}_j) \in \mathcal{X}_{\beta+1} - \mathcal{X}_\beta$, $\beta < \alpha$. Select an element $\bar{b} \in Y(\bar{a}_j) \cap (A_{\beta+1} - A_\beta)$ such that $f(\bar{b})$ is different from $f(\bar{a}_0), \dots, f(\bar{a}_{j-1})$ and let $f(\bar{a}_j) = f(\bar{b})$.

Assume now that $p(\bar{a}_1, \dots, \bar{a}_t) = 0$ for a polynomial $p \in \mathcal{P}$, and for the different points $\bar{a}_1, \dots, \bar{a}_t$ with $f(\bar{a}_1) = \dots = f(\bar{a}_t) = i$. We can assume, by (6), that $\bar{a}_1, \dots, \bar{a}_{t-1} \in A_\alpha$, $\bar{a}_t \in A_{\alpha+1} - A_\alpha$ for some $\alpha < \omega_1$. We can as well assume that α is minimal such that there is a counter-example for some polynomial $p \in \mathcal{P}$. As $\bar{a}_t$ solves $p(\bar{a}_1, \dots, \bar{a}_{t-1}, \bar{x}) = 0$, necessarily $\bar{a}_t \in X(p) \in \mathcal{X}_\alpha$. Then, $X(p) \supseteq Y(\bar{a}_t)$, by the minimality of this latter set, so, if $\bar{b}$ was the element in $Y(\bar{a}_t) \cap A_\alpha$ with $f(\bar{a}_t) = f(\bar{b})$, and by the minimality of α, $\bar{b}$ must be one of $\bar{a}_1, \dots, \bar{a}_{t-1}$, say, $\bar{b} = \bar{a}_{t-1}$. But then, we get a monochromatic solution of $q(\bar{a}_1, \dots, \bar{a}_{t-1}) = 0$, where $q(\bar{x}_1, \dots, \bar{x}_{t-1}) = p(\bar{x}_1, \dots, \bar{x}_{t-1}, \bar{x}_{t-1})$ is also in $\mathcal{P}$, a contradiction again to the minimality of α.

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SUR DES CRITERES DE COMMUTATIVITE DANS LES ALGEBRES DE BANACH

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Résumé : Etant donné une algèbre de Banach A, nous donnons ici des conditions suffisantes pour que A^2 soit contenu dans le centre de A et par conséquent le nil radical de A coincide avec l'ensemble des éléments nilpotents de A. Nous étudions ensuite d'autres conditions entraînant la commutativité de A.

I. Introduction. Dans la première partie de ce papier, nous reprenons l'étude des algèbres de Banach (complexes) vérifiant la condition suivante :

$$\|xy\| \leqslant c\|yx\| \qquad (C)$$

pour tous x,y éléments de l'élgèbre et où C est une constante positive. Cette condition a été considérée dans [4], [5] et [2]. Rappelons qu'un exemple donné dans [2] montre que la condition (C) n'implique pas la commutativité en général. Cependant nous montrons, , comme dans le cas commutatif, que si A est une algèbre de Banach vérifiant (C), alors le nil radical de A coincide avec l'ensemble des éléments nilpotents de A.

Maintenant puisque la condition (C) n'implique pas la commutativité en général, il est tout à fait naturel de chercher une condition de type (C) et qui implique la commutativité de toute algèbre de Banach la vérifiant. Dans ce sens, il est montré dans [3], qu'une algèbre de Banach A vérifiant :

$$\|xy+y\| \leqslant C'\|yx+y\| \qquad (C')$$

pour tous $x,y \in A$ et $c>0$, est nécessairement commutative. La deuxième partie de ce papier fait l'objet de l'étude de conditions similaires à (C').

II. Définitions. Une partie B d'une algèbre A est dite nilpotente s'il existe $n \in N$ tel que $B^n = \{0\}$.

Le nil radical d'une algèbre A noté N est défini comme étant l'intersection des idéaux premiers de A. Rappelons que N contient tout idéal à gauche ou à droite qui est nilpotent.

Dans tout ce qui suit C(A) désigne le centre de l'algèbre A.

III. Algèbres de Banach vérifiant (C). Le résultat suivant est une amélioration du théorème II.1. de [5].

Proposition III.1. Soit $(A,\|.\|)$ une algèbre de Banach et soit $(E,|.|)$ un espace vectoriel semi-normé. Supposons qu'il existe $T:A \to E$ une application et $c>0$ tels que :

$$\|xy\| \leq c|T(xy)|$$

pour tous $x,y \in A$; alors A^2 est contenu dans C(A). Si de plus A admet une unité approchée à gauche ou à droite alors A est commutative.

Preuve. Soit $\tilde{A} = A \oplus \mathbb{C}$ si A n'est pas unitaire, sinon on pose $\tilde{A}=A$. Pour tous $x,y,z \in A$, considérons $G : \mathbb{C} \to A$ l'application définie par $G(\lambda)=e^{\lambda z}xye^{-\lambda z}$. C'est une fonction entière bornée puisque $\|G(\lambda)\| \leq c|T(yx)|$ pour tout λ; donc G est constante, d'où $zxy = xyz$. Si A admet une unité approchée à gauche $(e_i)_i$ alors pour tous $x,y \in A$, nous avons $e_i xy = ye_i x$ pour tout i, d'où $xy = yx$.

Corollaire III.2. Soit $(A,\|.\|)$ une algèbre de Banach. Supposons qu'il existe une semi-norme $|.|$ sur A et $c>o$ tels que $\|xy\| \leq c|yx|$ pour tous $x,y\in A$, alors $A^2 \subseteq C(A)$.

Nous donnons maintenant le résultat concernant le nil radical.

Proposition III.3. Soit A une algèbre de Banach telle que $A^2 \subseteq C(A)$ et soit N le nil radical de A. Alors on a :

(i) N est l'ensemble des éléments nilpotents de A.

(ii) Pour tous $x,y\in A$, on a $xy-yx \in N$; en particulier si A est semi-première, alors elle est commutative.

Preuve. (i) On sait que tout élément de N est nilpotent. Inversement soit $a\in A$ tel qu'il existe $n\in \mathbb{N}$ tel que $a^n=o$; montrons que $(a)_1$ l'idéal à gauche engendré par a est nilpotent. En effet, soient x_1a+p_1a, x_2a+p_2a, ... , x_na+p_na n éléments de $(a)_1$. On a $(x_1a+p_1a)(x_2a+p_2a) = x_2'a^2+p_2'a^2$ où $x_2' = x_1x_2+p_2x_1+p_1x_2$ et $p_2'=p_1p_2$ (on utilise $A^2 \subseteq C(A)$). Nous obtenons à la fin :

$(x_1a+p_1a) \ldots (x_na+p_na) = x_n'a^n + p_n'a^n = o$, donc $(a)_1$ est nilpotent et par suite $a\in N$.

(ii) On a $(xy-yx)^2=o$ et d'après (i) $xy-yx\in N$.

Corollaire III.4. Soit A une algèbre de Banach vérifiant (C). Alors N est l'ensemble des éléments nilpotents et pour tous $x,y\in A$, $xy-yx\in N$.

Comme la condition (C) implique que $A^2 \subseteq C(A)$, on peut se demander ce qu'il en est si cette condition est vérifiée seulement sur une partie de A. Nous avons alors le résultat suivant:

Proposition III.5. Soit A une algèbre de Banach, I un idéal bilatère de A tel qu'il existe $c>0$ vérifiant $\|xy\|\leqslant c\|yx\|$ pour tous $x,y\in I$. Alors $I^2\subseteq C(A)$.

Preuve. Si A n'est pas unitaire, on fait les calculs dans $\tilde{A}=A\oplus\mathbb{C}$ et il est clair que I reste encore un idéal bilatère de $\tilde{A}$. Soient $z\in A$, x et y deux éléments de I et $G:\mathbb{C}\longrightarrow I$ l'application définie par $G(\lambda)=e^{-\lambda z}xye^{\lambda z}$; on a alors $\|G(\lambda)\|\leqslant c\|yx\|$, d'où $zxy=xyz$.

Voici maintenant un petit résultat motivé par le 1er problème de Hirschfeld-Zelazko ([1]).

Proposition III.6. Soient A une algèbre de Banach et I un idéal bilatère de A, supposons qu'il existe $c>0$ tel que

$$\|x\|\leqslant cr(x)$$

pour tout $x\in I$, r étant le rayon spectral. Alors $I\subseteq C(A)$.

Preuve. On suppose A unitaire, quitte à prendre $\tilde{A}=A\oplus\mathbb{C}$. Soient $x\in I$ et $z\in A$. Considérons $G:\mathbb{C}\longrightarrow I$ tel que $G(\lambda)=e^{\lambda z}xe^{-\lambda z}$. On a $\|G(\lambda)\|\leqslant cr(x)$; d'où $zx=xz$.

IV. Algèbres de Banach vérifiant (C'). Nous donnons ici un résultat similaire au théorème de [3].

Proposition IV.1. Soit $(A,\|.\|)$ une algèbre de Banach et soit $(E,|.|)$ un espace vectoriel semi-normé. Supposons qu'il existe une application $T: A\longrightarrow E$ et une constante $c>0$ tels que :

$$\|xy+y\|\leqslant c\|T(yx+y\|$$

pour tous $x,y\in A$. Alors A est commutative.

Preuve. Si A n'est pas unitaire, on fait les calculs dans $\tilde{A}=A \oplus \mathbb{C}$. Soient $x,y\varepsilon A$ et $G : \mathbb{C}\longrightarrow A$ définie par $G(\lambda) = e^{-\lambda x}ye^{\lambda x}$. on a $G(\lambda) = (e^{-\lambda x}-1)ye^{\lambda x} + ye^{\lambda x}$; et comme $(e^{-\lambda x} - 1)\varepsilon A$ et $ye^{\lambda x}\varepsilon A$, on a $\|G(\lambda)\| \leqslant c|T(y)|$. Donc G est constante, d'où $xy=yx$.

Corollaire IV.2. Soit $(A,\|.\|)$ une algèbre de Banach. Supposons qu'il existe une semi-norme $|.|$ sur A et une constante $c>o$ tels que :

$$\|xy + y\| \leqslant c\,|yx+y|$$

pour tous x, y A. Alors A est commutative.

Preuve. Il suffit de prendre E=A muni de la semi-norme $|.|$ et prendre T = identité.

Le résultat suivant est une amélioration du corollaire 1 de [3]. En effet la technique utilisée dans [3] exige que l'unité approchée soit bilatère. Notre approche permet d'obtenir le même résultat en supposant seulement qu'elle soit à gauche où à droite.

Proposition IV.3. Soit $(A,\|.\|)$ une algèbre de Banach admettant une unité approchée à gauche (où à droite) et soit E un espace vectoriel normé. Soit $T : A\longrightarrow E$ une application linéaire continue. Les conditions suivantes sont équivalentes.

(i) Il existe $k>o$ tq $\|T(xy)\|\leqslant k\|yx\|$ pour tous $x,y\varepsilon A$.

(ii) $T(xy) = T(yx)$ pour tous $x,y\varepsilon A$.

Preuve. Il est évident que (ii) implique (i). Montrons la réciproque. Les calculs sont fait dans $\tilde{A}=A \oplus \mathbb{C}$. Soient $x,y,z\varepsilon A$ et $G:\mathbb{C}\longrightarrow E$ l'application définie par $G(\lambda)=T(e^{\lambda z}xye^{-\lambda z})$

$\| G(\lambda) \| \leq k \| yx \|$, donc G est constante ; d'où $T(zxy) = T(xyz)$. Soit $(e_i)_i$ l'unité approchée à gauche de A ; nous avons $T(ze_iy) = T(e_iyz)$ et puisque T est continue, nous obtenons $T(zy) = T(yz)$.

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Indecomposable R.A.Loops and their Loop Algebras*

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1 Introduction

Let R be a commutative (and associative) ring with unity and let L be a loop. The *loop algebra* of L over R was introduced in 1944 by R.H.Bruck as a means to obtain an important family of examples of non-associative algebras. They are defined in precisely the same way group rings are; namely, RL is the free R- module with basis L in which multiplication is introduced by extending that of L via the distributive laws.

Conditions for a loop algebra to be alternative were first studied by E. G. Goodaire [4] in the case where R has no 2-torsion. He showed that the fact that RL is alternative depends on the structure of L but not on R. Thus, a loop L is called a *ring alternative loop*, or simply, an *R.A. loop* if its loop ring over any ring R with the above condition is alternative.

In a subsequent paper, O. Chein and E. G. Goodaire [3] showed that R.A. loops are almost groups (they have a normal subgroup of index 2)

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and almost commutative (the commutator subloop is of order 2). Actually, they can be described as a particular instance of a well-known general construction of Moufang Loops (O. Chein [1], [2])

In this paper we state some recently obtained results. First we classify all indecomposable R.A. Loops, up to isomorphism. Next, we describe the structure of their rational loop algebras and finish by describing a subloop of finite index in the unit loop of the integral loop ring. The proof of these results can be found in [8] and [9] and will be published elsewhere.

2 Classifying Indecomposable R. A. Loops

We start by recalling that R.A. loops can be constructed from subgroups by means of a well-known general construction of Moufang Loops (O. Chein [1], [2]); details can be found in [5] and [6].

Theorem 2.1 *Let L be an R.A. loop. Then, there exists a group $G \subset L$ and an element $u \in L$ such that $L = G \cup Gu$, $G' = L' = \{1, s\} \subseteq \mathcal{Z}(G) = \mathcal{Z}(L)$ and $L/\mathcal{Z}(L) = C_2 \times C_2 \times C_2$ where C_2 denotes a cyclic group of order 2 (and consequently, $G/\mathcal{Z}(G) = C_2 \times C_2$).*

Furthermore, the map $^*: L \to L$ given by

$$g^* = \begin{cases} g & \text{if } g \in \mathcal{Z}(G) \\ sg & \text{if } g \notin \mathcal{Z}(G) \end{cases}$$

is an involution of L which extends linearly to RL. Setting $u^2 = g_0$, we have that $g_0 \in \mathcal{Z}(G)$ and multiplication in L is given by:

$$\begin{aligned} g(hu) &= (hg)u \\ (gu)h &= (gh^*)u \\ (gu)(hu) &= g_0 h^* g \end{aligned}$$

A loop constructed in such a way is denoted as $L(G, *, g_0)$. Conversely, given a group G and a map $^*: G \to G$ as above, the loop $L = L(G, *, g_0)$ is an R.A. loop.

Groups G such that $G/\mathcal{Z}(G) \cong C_p \times C_p$, p a rational prime, were studied in [10]. We quote two results, specializing for $p = 2$.

Lemma 2.2 *([10], Lemma 1.1) Let G be a group such that $G/\mathcal{Z}(G) \cong C_2 \times C_2$. Then $G' = \{1, s\} \subseteq \mathcal{Z}(G)$ is cyclic of order 2.* □

Theorem 2.3 *([10], Theorem 1.2) Let G be a group. Then $G/\mathcal{Z}(G) \cong C_2 \times C_2$ if and only if G can be written in the form $G = D \times A$ where A is abelian and D is an indecomposable 2-group such that $D = < x, y, \mathcal{Z}(D) >$ where $\mathcal{Z}(D) = C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$ with $C_{2^{m_i}}$ cyclic of order 2^{m_i}, $i = 1, 2, 3$; $m_1 \geq 1$; $m_2, m_3 \geq 0$ and $s = [x, y] \in C_{2^{m_1}}$, $x^2 \in C_{2^{m_1}} \times C_{2^{m_2}}$, $y^2 \in C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$.*

In what follows, we shall denote by t_i a generator of the cyclic group C_i, $1 \leq i \leq 3$.

Now, we turn our attention to indecomposable R.A. loops.

Theorem 2.4 *([8], Theorem 2.2) Let $L = L(G, *, g_0)$ be an indecomposable R.A. loop. Then $G = D \times C$ where D is an indecomposable 2-group and C is a cyclic group of order 2^n, $n \geq 0$. Also if $n > 0$ then $g_0 = dc$ with $d \in \mathcal{Z}(D), c \in C, c \neq 1$.*

We shall always use w to denote a generator of the cyclic group C.

With the notation above, all possible types of indecomposable R.A. loops are given by the following table (see [8]) :

Indecomposable R.A. Loops

	$\mathcal{Z}(D)$	x^2	y^2	G	$u^2 = g_0$
L_1	$< t_1 >$	1	1	D_1	1
L_2	$< t_1 >$	t_1	t_1	D_2	t_1
L_3	$< t_1 > \times < t_2 >$	1	t_2	D_3	1
L_4	$< t_1 > \times < t_2 >$	t_1	t_2	D_4	t_1
L_5	$< t_1 > \times < t_2 > \times < t_3 >$	t_2	t_3	D_5	1
L_6	$< t_1 > \times < t_2 > \times < t_3 >$	t_2	t_3	D_5	t_1
L_7	$< t_1 > \times < t_2 > \times < t_3 >$	t_2	t_3	$D_5 \times < w >$	w

3 Description of Rational Loop Algebras

Our main result regarding loop algebras of indecomposable R. A. Loops is the following.

Theorem 3.1 *Let L be an indecomposable R.A. loop. Then*

$$\mathbb{Q}L = \mathbb{Q}L(\frac{1+s}{2}) \oplus \mathbb{Q}L(\frac{1-s}{2})$$

and

1. $\mathbf{Q}L(\frac{1+s}{2}) \cong \mathbf{Q}(L/L') \cong \bigoplus_{2d||L|} a_d\mathbf{Q}(\xi_d)$, *with* a_d *equal to the number of cyclic factors of* L/L' *of order* d, *and* ξ_d *a primitive* d^{th}*-root of unity.*
2. $\mathbf{Q}L(\frac{1-s}{2}) = \Delta(L:L')$ *is a sum of* a_L *simple alternative algebras with* a_L *equal to the number of subgroups* H *in* $\mathcal{Z}(L)$ *such that* $\mathcal{Z}(L)/H$ *is cyclic and* $s \notin H$.
3. $\mathcal{Z}(\Delta(L:L')) \cong \bigoplus \mathbf{Q}(\xi_H)$, *where the direct sum runs over all subgroups* H *as in (2) and* ξ_H *is a primitive* $|\mathcal{Z}(L)/H|^{th}$*-root of unity.*

Furthermore:

if $L = L_i$, $i = 1, 3$ *or* 5, *then all simple components of* $\Delta(L:L')$ *are split Cayley-Dickson algebras.*

if $L = L_i$, $i = 2, 4, 6$ *or* 7, *then all simple components, but one, are split Cayley-Dickson algebras. The non-split component, in each case, is determined by a primitive central idempotent of the form* $e = \widehat{H}(\frac{1-s}{2})$. *We list below the corresponding subgroup in each case.*

Loop	H
L_2	$\{1\}$
L_4	$< t_1t_2 >$
L_6	$< t_1t_2 > \times < t_1t_3 >$
L_7	$< t_1t_2 > \times < t_1t_3 > \times < t_1w >$

We recall that the Cayley-Dickson matrix algebra over a field F is defined as

$$C(F) = \begin{bmatrix} F & F^3 \\ F^3 & F \end{bmatrix}$$

where F^3 denotes the set of 3-dimensional vectors over F, addition is defined componentswise and multiplication in $C(F)$ is given by:

$$\begin{bmatrix} a & V \\ W & b \end{bmatrix} \begin{bmatrix} a' & V' \\ W' & b' \end{bmatrix} = \begin{bmatrix} aa' + V \cdot W' & aV' + b'V - W \times W' \\ a'W + bW' + V \times V' & bb' + W \cdot V' \end{bmatrix}$$

(see [12, Theorem 2.4.7]).

In [8] concrete isomorphisms are given between the simple components of the rational loop algebras and the Cayley-Dickson algebra.

4 Units of Integral Loop Rings

For simplicity, we have restricted ourselves, in this section, to finite indecomposable R. A. loops with cyclic center.

First, we introduce some notation. In analogy with the associative case, we define the *General Linear Loop* over a commutative ring R to be the following:

$$GLL_2 = \{A \in C(R) \mid det(A) \text{ invertible in } R\}.$$

Let ξ be a primitive root of unity; for a loop $S \subseteq GLL_2(\mathbb{Z}[\xi])$, S containing the identity matrix I and also $-I$, we denote

$$S_{\overline{n(det)=1}} = \{A \in S \mid n(det(A)) = 1\}/\{I, -I\}$$

and

$$S_{det=1} = \{A \in S \mid det(A) = 1\},$$

where $n(\alpha)$ denotes the norm of $\alpha \in \mathbb{Z}[\xi]$.

Theorem 4.1 *The unit loop $U(\mathbb{Z}L)$ has a torsion-free normal subloop V such that the subloop generated by the central units and V is of finite index. Moreover, if L/L' has exponent 4, then V is a normal complement of $\pm L$ in $U(\mathbb{Z}L)$.*

Assume $L = \mathbf{L}_1$ with $o(t_1) = 2^{m_1}$ and let ξ be $a = 2^{m_1}$ root of unity. Then, V is isomorphic to:

$$\left\{\begin{bmatrix} 1+2a & 2V \\ 2W & 1+2b \end{bmatrix} \in \begin{bmatrix} 1+2\mathbb{Z}[\xi] & 2(\mathbb{Z}[\xi])^3 \\ 2(\mathbb{Z}[\xi])^3 & 1+2\mathbb{Z}[\xi] \end{bmatrix}_{\overline{n(det)=1}} \right| $$

$$\left. a+b \in 2\mathbb{Z}[\xi],\ V+W \in 2(\mathbb{Z}[\xi])^3 \right\},$$

if $m_1 > 1$ and isomorphic to

$$\left\{\begin{bmatrix} 1+2a & 2V \\ 2W & 1+2b \end{bmatrix} \in \begin{bmatrix} 1+2\mathbb{Z} & 2\mathbb{Z}^3 \\ 2\mathbb{Z}^3 & 1+2\mathbb{Z} \end{bmatrix}_{det=1} \right| $$

$$\left. a+b \in 2\mathbb{Z},\ V+W \in 2\mathbb{Z}^3 \right\},$$

if $m_1 = 1$.

In case $L = \mathbf{L}_2$ with $o(t_1) = 2^{m_1}$ then V is isomorphic to:

$$\left\{ \begin{bmatrix} 1+2a & 2V \\ 2W & 1+2b \end{bmatrix} \in \begin{bmatrix} 1+2\mathbf{Z}[\xi] & 2(\mathbf{Z}[\xi])^3 \\ 2(\mathbf{Z}[\xi])^3 & 1+2\mathbf{Z}[\xi] \end{bmatrix}_{\overline{n(det)=1}} \mid a+b \in 2\mathbf{Z}[\xi],\ v_2+w_2,\ v_1\xi+w_1, v_3\xi+w_3 \in 2\mathbf{Z}[\xi] \right\}.$$

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SOLUTION OF DIRECT ASYMMETRIC PROBLEMS IN EDDY CURRENT TESTING BY A PERTURBATION METHOD

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Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. A perturbation method is used to solve eddy current testing problems for a flaw of arbitrary shape situated in a conducting half-space when the electric conductivities of the flaw and the surrounding medium are nearly the same. A general formula for the change in impedance is obtained. Numerical results for a circular cylindrical flaw of finite length whose axis is shifted sideways with respect to the axis of the testing coil are given graphically.

RÉSUMÉ. On dépiste une faille de forme arbitraire située dans un demi-plan conducteur au moyen des courants de Foucault par la méthode des perturbations dans le cas où la conductivité électrique de la faille diffère peu de celle du milieu environnant et on donne une formule générale pour exprimer le changement de l'impédance. On présente sur graphique un exemple de failles cylindriques circulaires de longueur finie.

Subject-classification: AMS(MOS): 35K20, 65R10.
Keywords: eddy current nondestructive evaluation, method of small perturbations, Hankel transform

1. Introduction. Eddy current methods are widely used for quality testing of materials. If an electrically conducting medium is situated in the electromagnetic field of an excitation coil, eddy currents are induced in the conducting medium. The interaction of these currents with the coil's current changes the amplitude and phase of the latter current. The presence of flaws (cracks or regions with different electrical conductivities) in the medium leads to a change in the reading of the coil's current. Conversely, a change in the reading may indicate the presence of a flaw and, in some cases, give an estimate of the flaw's parameters. But, as in any inverse problem, it is generally difficult to determine the shape, or even the presence, of a flaw from an impedance change in the probe because it is not known how this change depends upon: (a) the parameters of the flaw, (b) the properties of the conducting medium, and (c) the relative position of the probe with respect to the flaw. In fact, inverse problems are often ill posed because their solutions, if they exist, may be many and may not depend continuously upon the input data. Therefore, in such cases, one often studies the direct problem in an attempt to tabulate solution patterns.

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For example, one minimizes (in some norm) the difference between experimental and theoretical (or numerical) data, and the unknown parameters of the flaw may be found by an iterative procedure applied to the minimizing function. Much attention has been given in recent years to numerical modeling of three-dimensional problems in eddy current testing [1–3]. But considerations are often restricted to axisymmetric problems because of large computational difficulties in three-dimensional asymmetric cases. In practice, however, the determination of a flaw is an asymmetric problem because the search is carried out by the movement of an eddy current probe along the exterior surface of the material to be inspected.

In this note, we use a perturbation method to obtain the solution of direct asymmetric eddy current testing problems.

2. The mathematical analysis. The equation for the vector potential $\hat{\mathbf{A}}$ in a conducting nonferromagnetic medium is

$$\Delta \hat{\mathbf{A}} = \mu_0 \sigma \frac{\partial \hat{\mathbf{A}}}{\partial t} - \mu_0 \hat{\mathbf{i}}^e, \tag{1}$$

where Δ is the three-dimensional Laplacian, σ is the electrical conductivity of the medium, μ_0 is the magnetic constant and $\hat{\mathbf{i}}^e$ is the density of the external current.

An excitation coil of radius R, with alternating current $i = I e^{j\omega t}$, $j = \sqrt{-1}$, is moved horizontally in free upper half-space, R_0, at height h above conducting lower half-space, $R_1 \cup R_2$, where R_2 is a convex flaw. It is assumed that the constant electrical conductivities, σ_1 and σ_2, of R_1 and R_2, respectively, are close so that $\epsilon = 1 - \sigma_2/\sigma_1$ is a small parameter naturally connected with the problem.

Introducing cylindrical polar coordinates (r, φ, z) with coil's center at $(0, \varphi, h)$, and separating the variable t,

$$\hat{\mathbf{A}}(r, \varphi, z, t) = \mathbf{A}(r, \varphi, z) e^{j\omega t}, \tag{2}$$

we have, from (1), the following vector equation:

$$\Delta \mathbf{A} + k^2 \mathbf{A} = -\mu_0 \mathbf{i}^e, \tag{3}$$

where $k^2 = -j\omega\sigma\mu_0$ and, with the Dirac measure $\delta(x)$, the vector $\mathbf{i}^e$ is

$$\mathbf{i}^e = I \delta(r - R) \delta(z - h) \mathbf{e}_\varphi. \tag{4}$$

In general, equation (3) is to be solved in each of the regions R_0, R_1 and R_2, with vector potential and normal derivatives continuous at the interface between the media. It is shown in [4] that a three-dimensional electromagnetic field can be described by a two-dimensional vector potential since the third component may be chosen arbitrarily; here it will be taken equal to zero. Since there are no external currents in the z-direction the vector potential may have the form

$$\mathbf{A}_i = \{A_{r,i}(r, \varphi, z), A_{\varphi,i}(r, \varphi, z), 0\}, \quad i = 1, 2, 3, \tag{5}$$

in the regions R_{i-1}, respectively, where the functions $A_{r,i}$ and $A_{\varphi,i}$ and their normal derivatives are continuous on the boundary between the media (these conditions imply the continuity of the tangential components of the magnetic field strength and the normal components of the magnetic displacement vector at the interface between the media). We denote by $z = z_2(r, \varphi)$ and $z = z_1(r, \varphi)$, $z_2 \geq z_1$, the equations describing the upper and lower parts of ∂R_2, respectively. The projection of R_2 on the plane $z = 0$ is a convex region D with smooth boundary ∂D. The nearer and farther parts, ACB and AEB, of ∂D are described in polar coordinates by the equations $r = r_1(\varphi)$ and $r = r_2(\varphi)$, respectively, and the rays OA and OB subtending D at the origin form angles φ_2 and φ_1, respectively, with the positive x-direction as shown in Fig.1a.

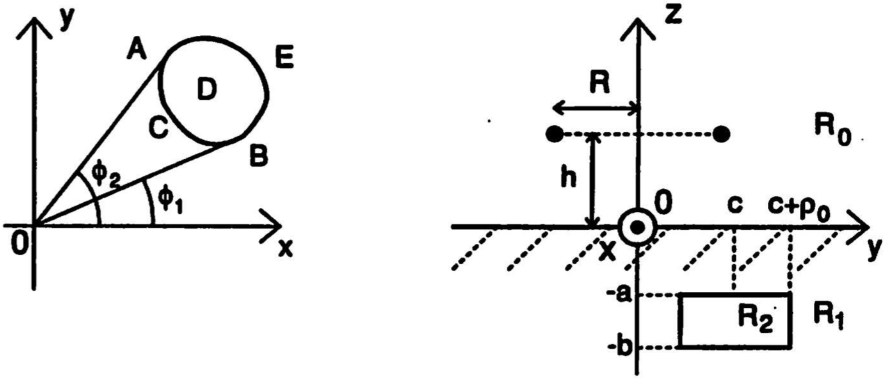

FIGURE 1A (LEFT) AND 1B (RIGHT). (a) D is projection of region R_2 on plane $z = 0$. (b) Cross-section of circular coil of radius R in free space R_0 at distance h above uniform conducting region R_1 containing shifted circular cylindrical flaw R_2 of radius ρ_0 and finite length $b - a$.

Setting $\epsilon = 1 - \sigma_2/\sigma_1$ and $k_i^2 = -j\omega\sigma_i\mu_0$, $i = 1, 2$, we have

$$k_2^2 = k_1^2(1 - \epsilon). \tag{6}$$

We seek the solution to (3), for $m = 0, 1, 2$, in powers of ϵ,

$$\begin{aligned} A_r^{(m)} &= A_{r_0}^{(m)} + \epsilon A_{r_1}^{(m)} + \cdots, \\ A_\varphi^{(m)} &= A_{\varphi_0}^{(m)} + \epsilon A_{\varphi_1}^{(m)} + \cdots. \end{aligned} \tag{7}$$

By substituting (6) and (7) into (3) and comparing the coefficients of each power of ϵ we obtain a system of equations for each such power.

In the absence of a flaw ($\epsilon = 0$), we have the following problem:

$$\Delta A_{\varphi_0}^{(0)} - \frac{A_{\varphi_0}^{(0)}}{r^2} = -\mu_0 I \delta(r - R)\,\delta(z - h), \tag{8}$$

$$\Delta A_{\varphi_0}^{(1)} - \frac{A_{\varphi_0}^{(1)}}{r^2} + k_1^2 A_{\varphi_0}^{(1)} = 0, \tag{9}$$

$$A_{\varphi_0}^{(0)}\Big|_{z=0} = A_{\varphi_0}^{(1)}\Big|_{z=0}, \qquad \frac{\partial A_{\varphi_0}^{(0)}}{\partial z}\Big|_{z=0} = \frac{\partial A_{\varphi_0}^{(1)}}{\partial z}\Big|_{z=0}. \tag{10}$$

The solution to (8)–(10) is given in [5], and, in general, it is the sum of two terms, the first corresponding to the vector potential of a solitary coil in unbounded free space, and the second representing the vector potential due to a uniform (flawless) conducting half-space. In the sequel, we consider only the second term of this solution. In particular, with $q = \sqrt{u^2 - k_1^2}$, we have

$$A_{\varphi_0}^{(1)}(r,z) = \mu_0 I R \int_0^\infty J_1(uR)\, J_1(ur)\, \frac{u}{u+q}\, e^{qz-uh}\, du. \tag{11}$$

It can be shown that the induced change in impedance,

$$Z_{\text{ind}} = \frac{j\omega}{I} \int_C \mathbf{A}\, d\mathbf{l}, \tag{12}$$

where C is the contour of the coil, is determined only by the A_φ-component, which is independent of φ, of the vector potential.

In the presence of a flaw ($\epsilon \neq 0$), if we substitute (6) and (7) into (3) and compare the coefficients of ϵ to the first power, we obtain a system of first-order approximation. We seek the solution to this system in the form of Fourier series:

$$A_{r_1}^{(m)}(r,z) = a_0^{(m)}(r,z) + \sum_{n=1}^{\infty} \left[a_{n1}^{(m)}(r,z)\cos n\varphi + a_{n2}^{(m)}(r,z)\sin n\varphi\right], \tag{13}$$

$$A_{\varphi_1}^{(m)}(r,z) = b_0^{(m)}(r,z) + \sum_{n=1}^{\infty} \left[b_{n1}^{(m)}(r,z)\cos n\varphi + b_{n2}^{(m)}(r,z)\sin n\varphi\right]. \tag{14}$$

Since we are interested only in the impedance change in the coil, it follows from the above considerations that we need only know the function $b_0^{(0)}(r,z)$ in order to compute this change. The determination of $b_0^{(m)}(r,z)$ reduces to the following boundary value problem

$$Lb_0^{(0)} - \frac{b_0^{(0)}}{r^2} = 0, \tag{15}$$

$$Lb_0^{(1)} - \frac{b_0^{(1)}}{r^2} + k_1^2 b_0^{(1)} = F(r,z), \tag{16}$$

$$b_0^{(0)}\Big|_{z=0} = b_0^{(1)}\Big|_{z=0}, \qquad \frac{\partial b_0^{(0)}}{\partial z}\Big|_{z=0} = \frac{\partial b_0^{(1)}}{\partial z}\Big|_{z=0}, \tag{17}$$

where

$$F(r,z) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(r,\varphi,z)\, d\varphi, \tag{18}$$

$$\Phi(r,\varphi,z)=\begin{cases} k_1^2 A_{\varphi 0}^{(1)}, & \text{if } M(r,\varphi,z)\in V, \\ 0, & \text{if } M(r,\varphi,z)\notin V, \end{cases}$$

and

$$L:=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{19}$$

Problem (15)–(19) is solved by the method of Hankel integral transform.

Computations were done for a shifted circular cylindrical flaw of radius ρ_0 and finite length, $b-a$, with centre shifted c units along the x-axis, as shown in Fig.1b, i.e.

$$\varphi_1=-\arctan\frac{\rho_0}{c}, \qquad \varphi_2=\arctan\frac{\rho_0}{c},$$

$$r_1=c\cos\varphi-\sqrt{\rho_0^2-c^2\sin^2\varphi}, \quad r_2=c\cos\varphi+\sqrt{\rho_0^2-c^2\sin^2\varphi},$$

$$z_1=-b, \qquad z_2=-a, \qquad 0>-a>-b.$$

Using (12) we obtain the following formula for the change in impedance due to the flaw:

$$\begin{aligned}\widetilde{Z}_{\text{ind}}=\beta^2\int_{\varphi_1}^{\varphi_2}d\varphi\int_0^\infty\frac{x\,J_1(\beta x)\,e^{-\alpha\beta x}}{x+\sqrt{x^2+j}}\,dx\int_0^\infty\frac{y\,J_1(\beta y)\,e^{-\alpha\beta y}}{y+\sqrt{y^2+j}}\times \\ \times\frac{e^{-\nu\beta\left(\sqrt{x^2+j}+\sqrt{y^2+j}\right)}-e^{-\gamma\beta\left(\sqrt{x^2+j}+\sqrt{y^2+j}\right)}}{\left(\sqrt{x^2+j}+\sqrt{y^2+j}\right)(x^2-y^2)}\times \\ \times\left\{\left[y\,\rho\,J_1(\beta x\rho)\,J_0(\beta y\rho)-x\,\rho\,J_0(\beta x\rho)\,J_1(\beta y\rho)\right]\Big|_{\rho=r_1(\varphi)}^{\rho=r_2(\varphi)}\right\}dy,\end{aligned} \tag{20}$$

where $\widetilde{Z}_{\text{ind}}=Z_{\text{ind}}/(\omega\mu_0 R)$ and

$$\alpha=\frac{h}{R}, \quad \beta=R\sqrt{\omega\sigma_1\mu_0}, \quad \nu=\frac{a}{R}, \quad \gamma=\frac{b}{R}, \quad r_0=\frac{\rho_0}{R}, \quad x_0=\frac{c}{R}.$$

In Fig. 2, the real and imaginary parts of $\widetilde{Z}_{\text{ind}}=R_{\text{ind}}+jX_{\text{ind}}$ are shown in the case $r_0=0.5$. It is seen that the abscissa, x_0, of the maximum of the absolute value of any curve remains in the interval $[0.9,1.0]$ as β varies in the interval $[0.5,2.0]$. This means that the maximum change in impedance occurs when the projection of the coil on the plane $z=-a$ intersects the axis of the flaw. It is also seen that the best identification of the flaw occurs when $\beta=2$ because the maximum is sharper in this case. Computation done with $\beta=3$ has shown that $|X_{\text{ind}}|$ increases and R_{ind} decreases as compared with the case $\beta=2$. Computation has also shown that a flaw is better identified if the coil's radius, R, is greater than the flaw's radius, ρ_0.

3. Conclusion. The method of small perturbations is used to compute the change in impedance due to a flaw whose conductivity is close to that of the surrounding medium. Computation shows that a flaw is better identified by an eddy current probe if the coil is vertically above the flaw and the radius of the coil is greater than that of the flaw.

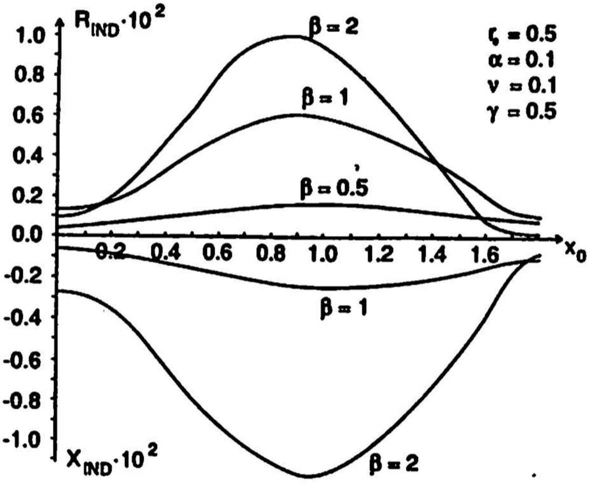

FIGURE 2. Dependence of $\Re \widetilde{Z}_{ind}$ and $\Im \widetilde{Z}_{ind}$ upon x_0 and β, at $r_0 = 0.5$.

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Similarity Reductions of the Two-Dimensional Toda Lattice

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Summary and introduction. Similarity solutions of integrable evolution type equations were studied in works [1-4] where the connections with Painlevé equations were emphasised. In [5], equations for similarity solutions of some 2+1 dimensional evolution equations were derived. Certain similarity solutions of the Korteweg -de Vries (KdV) equation were shown in [6] to have an operator representation $L + [L,A] = 0$, which was derived from the well known Lax representation for the KdV equation. Analagous operator representations can be obtained for similarity reductions of other integrable evolution equations as well. Similarity reductions of special interest are those for which the operator representation is equivalent to the Lax equation, a situation which confirms the integrability of the reduced system. In the present work we show that this property is fulfilled for the 2-D Toda lattice in the nonperiodic case and for analogous systems in the framework of semi-simple Lie algebras. For the similarity reduced periodic case (and its Lie-algebraic analogues) we found a simple qualitative description of the dynamics. Finally, we prove integrability of the similarity reduced (2+1)-D equation $q_{tt} + \alpha q_{xx} = (\exp(q))_{yy}$, which is a continuous limit of 2-D Toda lattice.

I. A generalized two-dimensional Toda lattice, associated with an arbitrary simple Lie algebra $\mathfrak{G}$, admits the Lax representations [7,8]

$$[L + \alpha\partial_x \, , \, \partial_t + A] = 0 \, . \tag{1}$$

Here L and A are vectors of Lie algebra $\mathfrak{G}$, determined by the formulae

$$L = p + \sum_{i=1}^{N} a_i(e_{\omega_i}+e_{-\omega_i}) \, , \quad A = s + \sum_{i=1}^{N} a_i(e_{\omega_i}-e_{-\omega_i}) \, . \tag{2}$$

The vectors $p(t,x)$ and $s(t,x)$ belong to the Cartan subalgebra H, $\omega_1,\dots,\omega_N \in H$ form an admissible set of roots Δ. The e_{ω_i} are root vectors. A set of roots Δ is called admissible, if for every pair of roots ω_i, $\omega_j \in \Delta$ the vector $\omega_i - \omega_j$ is not a root. The number N satisfies the condition $N \le n + 1$, where $n = \dim H$ is the rank of Lie algebra $\mathfrak{G}$.

Equations (1), (2) after the substitution $\rho = q_t$, $s = \alpha q_x$,

$a = \exp((q,\omega_i))$, where the vector $q(t,x)$ belongs to the Cartan subalgebra H , are transformed into the generalized two-dimensional Toda lattice [1,2]:

$$q_{tt} - \alpha^2 q_{xx} = -2 \sum_{i=1}^{N} e^{2(q,\omega_i)} \omega_i \,. \tag{3}$$

The explicit form of the right-hand side in (3), depending on the structure of the simple Lie algebra $\mathfrak{g}$, coincides with that for the generalized 1-dimensional Toda lattice, constructed in [9]. For the Lie algebra of type A_n system (3) has the form

$$q_{k_{tt}} - \alpha^2 q_{k_{xx}} = -2\left(e^{2(q_{k+1}-q_k)} - e^{2(q_k-q_{k-1})}\right) . \tag{4}$$

II. We consider special solutions of Eq. (3), determined by the formulae

$$q(t,x) = q(r) \,, \quad r = t^2 - x^2/\alpha^2 \,. \tag{5}$$

Eq. (3) for these solutions is reduced to the equation

$$r q_{rr} + q_r = -\frac{1}{2} \sum_{i=1}^{N} e^{2(q,\omega_i)} \omega_i \,. \tag{6}$$

In new coordinates $\rho_+ = t + x/\alpha$, $\rho_- = t - x/\alpha$ the variable r appears as $r = \rho_+\rho_-$, so solutions (5) in some sense might be called similarity solutions of Eq. (3).

Solutions of Eq. (6) may be continued across the point $r = 0$ at not all values of $q(0)$, $q_r(0)$, but only through a subspace satisfying the condition

$$q_r(0) = -\frac{1}{2} \sum_{i=1}^{N} e^{2(q(0),\omega_i)} \omega_i. \tag{7}$$

The presence of such a subspace is typical of all similarity solutions [10,11].

Eq. (6) for $r = 0$ is transformed after the substitution $\tau = \sqrt{|r|}$ into the equation

$$q_{\tau\tau} + \frac{1}{\tau} q_\tau = -2\sigma \sum_{i-1}^{N} e^{2(q,\omega_i)} \omega_i \,, \tag{8}$$

where $\sigma = \operatorname{sign} r$. Eq. (8) is equivalent to the following system

$$p_\tau = -\frac{1}{\tau} p - \partial H/\partial q \,, \quad q_\tau = \partial H/\partial p \,, \tag{9}$$

where the function $H(p,q)$ has the form

$$H = \frac{1}{2}(p,p) + \sigma \sum_{i=1}^{N} e^{2(q,\omega_i)} \,. \tag{10}$$

System (9) is a Hamiltonian system with time-dependent friction

$-p/\tau$. As a consequence of system (9) we get the equation

$$dH/d\tau = -(p,p)/\tau \le 0 \, . \tag{11}$$

Therefore the function $H(p,q)$ is the Liapunov function for the system (9). So all the trajectories of system (9) as $\tau \to \infty$ tend either to infinity or to singular points of this system, determined by the conditions

$$p = 0 \, , \; \partial H/\partial q = 2\sigma \sum_{i=1}^{N} e^{2(q_0,\omega_i)} \omega_i = 0 \, . \tag{12}$$

If $N \le n$, then vectors $\omega_1,\ldots,\omega_n$ are linearly independent, so equations (12) have no solutions and all trajectories of Eq. (9) tend to infinity in the space of vectors $q(\tau)$ as $\tau \to \infty$.

If $N = n + 1$, then the following relation holds

$$k_1\omega_1 + k_2\omega_2 + \ldots + k_{n+1}\omega_{n+1} = 0 \, , \tag{13}$$

where all k_i are positive integers [12]. From Eq. (12) it follows

$$2(q_0,\omega_i) = \ln c + \ln k_i \, . \tag{14}$$

Using equality (13) we get from (14)

$$\ln c = -\left(\sum_{i=1}^{N} k_i \ln k_i\right)\left(\sum_{i=1}^{n+1} k_i\right)^{-1} . \tag{15}$$

Then vector q_0 is uniquely determined by the equation (14).

The function H (10) is positive definite if $\sigma = +1$; in this case in view of Eq. (11) all trajectories of the dynamical system (9) for $N = n + 1$ tend to the attractive singular point $p = 0$, $q = q_0$ as $\tau \to \infty$.

Moreover Eq. (6) after the substitution $t = \ln|r|$ takes the form (σ=sign r)

$$q_{tt} = -\frac{\delta}{2} e^t \sum_{i=1}^{N} e^{2(q,\omega_i)} \omega_i \, . \tag{16}$$

This non-autonomous equation has the Hamiltonian form

$$p_t = -\partial H/\partial q \, , \; q_t = \partial H/\partial p \tag{17}$$

with the Hamiltonian function

$$H = \frac{1}{2}(p,p) + \frac{\delta}{4} e^t \sum_{i=1}^{N} e^{2(q,\omega_i)} \, . \tag{18}$$

If $N \le n$, then there exists a vector $q_1 \in H$, satisfying the relations $(q_1,\omega_i) = 1$, $i = 1,\ldots,N$. Then Eq. (16) after the substitution

$$q = \bar{q} - \frac{1}{2} t q_1 \tag{19}$$

assumes the form of a generalized Toda lattice [3]

$$\bar{q}_{tt} = -\frac{1}{2}\sum_{i=1}^{N} e^{2(\bar{q},\omega_i)}\omega_i . \qquad (20)$$

Thus we have proven the following

Proposition 1. The equations of a two-dimensional Toda lattice (3) are reducible for the special solutions (5) to the generalized Toda lattice equations with friction (9), to the non-autonomous Toda lattice (16), and in the special case $N \leq n$ are equivalent to the generalized Toda lattice (20), and hence are integrable.

III. Eq. (4) for $N \leq n$ has similarity solutions of the following form

$$q(t,x) = \bar{q}(\zeta) - q_1 \ln t , \quad \zeta = x/t , \qquad (21)$$

where the vector q_1 satisfies the conditions $(q_1,\omega_i) = 1$, $i = 1,\dots,n$. Eq. (4), after the substitution of the formulae (21), is reduced to the equation

$$(\zeta^2-\alpha^2)\bar{q}'' + 2\zeta\bar{q}' + q_1 = -2\sum_{i=1}^{N} e^{2(\bar{q},\omega_i)}\omega_i . \qquad (22)$$

This equation has two singular surfaces $\zeta = \pm\alpha$. Trajectories of Eq. (22) may be continued through these surfaces only under the following conditions $(\sigma=\pm 1)$

$$2\sigma\alpha\bar{q}'(\sigma\alpha) + q_1 = -2\sum_{i=1}^{N} e^{2(\bar{q}(\sigma\alpha),\omega_i)}\omega_i . \qquad (23)$$

IV. Let us now consider the equation

$$q_{tt} + \alpha q_{xx} = (e^{q})yy , \qquad (24)$$

where $q(t,x,y)$ is some smooth function. This equation was studied in connection with some problems of general relativity [13,14] and as the continuous limit [15] of the two-dimensional Toda lattice (4). Equation (24) for partial similarity solutions of the form

$$q(t,x,y) = q(r,y) , \quad r = t^2 + x^2/\alpha \qquad (25)$$

is reduced to the equation

$$rq_{rr} + q_r = \frac{1}{4}(e^{q})yy . \qquad (26)$$

After substitution $\tau = \ln|r|$ we get from (26)

$$q_{\tau\tau} = \frac{\sigma}{4} e^{\tau}(e^{q})yy , \quad \sigma = \sin r . \qquad (27)$$

This equation after substituting $q = \bar{q} - \tau$ takes the form

$$\bar{q}_{\tau\tau} = \frac{\sigma}{4}(e^{\bar{q}})yy . \qquad (28)$$

The integrability of Eq. (28) was proved in [15]. In conclusion we find that the special solutions (25) of Eq. (24) are described by the integrable equation (28).

V. Eq. (24) has also partially similarity solutions of the form

$$q(t,x,y) = q(\zeta,y) - 2\ln t \,, \quad \zeta = x/t \,, \tag{29}$$

or

$$q(t,x,y) = q\,(\zeta,\sigma) \,, \quad \xi = x/t \,, \quad \sigma = y/t \tag{30}$$

Eq. (24) for solutions (29) takes the form

$$(\zeta^2+\alpha)q_{\zeta\zeta} + 2\zeta q_\zeta = (e^q)_{yy} \,. \tag{31}$$

While Eq. (24) after substitution (30) is reduced to the equation

$$(\zeta^2+\alpha)q_{\zeta\zeta} + 2\sigma\zeta q_{\sigma\zeta} + \sigma^2 q_{\sigma\sigma} + 2\zeta q_\zeta + 2\sigma q_\sigma = (e^q)_{\sigma\sigma} \,. \tag{32}$$

Eq. (32) for similarity solutions of the form

$$q(\zeta,\sigma) = q(r) - 2\ln c \,, \quad r = \zeta + c\sigma \,, \quad c = \text{const} \tag{33}$$

is reduced to the equation

$$(r^2+\alpha)q_{rr} + 2rq_r = (e^q)_{rr} \,, \tag{34}$$

which is the derivative of the expression

$$(r^2+\alpha)q_r = (e^q)_r + \ell \,, \quad \ell = \text{const} \tag{35}$$

and so has simplest solution $q = \ln(r^2+\alpha)$, $b = 0$.

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A remark on a mean value property

Maciej Sablik

Presented by J. Aczel, F.R.S.C.

Abstract. J. Aczél in [1] and S. Haruki in [5] showed that functional equations (3) and (4) characterize quadratic polynomials among real functions. In the present note we extend these results to the case where variables and values of the unknown functions are in some abelian groups.

It is a well known fact that quadratic polynomials $f(x) = ax^2+bx+c, x \in \mathbb{R}$, satisfy the following two equations

$$\frac{f(x)-f(y)}{x-y} = f'\left(\frac{x+y}{2}\right) \tag{1}$$

and

$$\frac{f(x)-f(y)}{x-y} = \frac{f'(x)+f'(y)}{2} \tag{2}$$

for all $x, y \in \mathbb{R}, x \neq y$. Several authors dealt with functional equations of which (1) and (2) are special cases, viz.

$$f(x)-g(y) = h(x+y)(x-y) \tag{3}$$

and

$$f(x)-g(y) = (\Phi(x)+\Gamma(y))(x-y) \tag{4}$$

for all $x, y \in \mathbb{R}, x \neq y$. J. Aczél proved in [1] that without any regularity assumption f, g, h satisfy (3) if and only if $f(x) = g(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$, and $h(x) = ax + b$. A similar result was obtained by S. Haruki in [5]. He proved also that (3) is equivalent to the equation

$$f(x)-g(y) = (\Phi(x)+\Phi(y))(x-y).$$

Further investigations of problems motivated by the above mentioned results can be found e.g. in J. Aczél - M. Kuczma [2] and M. Kuczma [6]

(x, y are in a real interval and $h(x+y)$ is replaced by $k(\eta(x,y))$ where η is the harmonic or geometric mean, see also M. Kuczma [7]). G. E. Cross and Pl. Kannappan consider in [3] a generalization of (4):

$$f(x) = \sum_{k=0}^{n-2} g_k(y)(x-y)^k + (\phi(y) + \gamma(x))(x-y)^{n-1}$$

for $x, y \in \mathbb{R}, x \neq y$ and arbitrary $n \geq 2$. Again without any regularity assumed, f, g, h have to be polynomials of suitable degree. A similar result concerning the equation

$$f(x) = \sum_{k=0}^{n-2} g_k(y)(x-y)^k + h(x + (n-1)y)(x-y)^{n-1}$$

for $x, y \in \mathbb{R}, x \neq y$ and arbitrary $n \geq 2$, which is a generalization of (3) has been obtained recently by R. Ger (cf. [4]).

In the present note we are going to solve (3) and (4) in the case where x, y are in an abelian group. Of course, we have to make precise what the multiplication on the right hand sides means. This question turns out to be obvious if we notice that in the real case $h(u)$ and $\Phi(u) + \Gamma(v)$ may be identified with linear mappings "$t \to h(u)t$" and "$t \to (\Phi(u) + \Gamma(v))t$" (by the way, the linearity actually is a regularity assumption when compared to additivity for instance).

Proofs of theorems are elementary and in the case of Theorem 1 the argument is almost the same as in [1].

THEOREM 1. *Let* $(G, +)$ *and* $(H, +)$ *be abelian, uniquely 2-divisible groups. Then the functions* $f, g : G \to H$ *and* $h : G \to \mathrm{Hom}(G, H)$ *satisfy (3) for all* $x, y \in G, x \neq y$ *if and only if* $f(x) = g(x) = a_o + a_1(x) + a_2(x, x)$ *and* $h(x) = a_1 + a_2(x, \cdot)$ *for every* $x \in G$, *where* $a_o \in H$ *is a constant,* $a_1 \in \mathrm{Hom}(G, H)$ *and* $a_2 : G \times G \to H$ *is symmetric and biadditive.*

Proof. It is a straightforward matter to check that f, g, h which have the forms given in the assertion actually satisfy (3). To show the converse let f, g, h satisfy (3). We can follow the argument used by J. Aczél in [1] to get $f(x) = g(x), f(x) = f(0) + h(x)(x)$ and

$$[h(x+y) - h(0)](x-y) = [h(x-y) - h(0)](x+y) \tag{5}$$

for every $x, y \in G$. Putting $u = x + y$ and $v = x - y$ we see that (5) becomes

$$[h(u) - h(0)](v) = [h(v) - h(0)](u) \tag{6}$$

for every $u, v \in G$. Let $a_2 : G \times G \to H$ be defined by

$$a_2(u,v) = \frac{1}{2}(h(u) - h(0))(v).$$

In view of (6) a_2 is symmetric and biadditive. Put $a_o = f(0) \in H$ and $a_1 = h(0) \in \mathrm{Hom}(G, H)$. It is easy to see that f, g and h can be expressed as in the assertion. This concludes the proof.

Passing to the equation (4) we prove first a theorem on the case where (4) holds for all $x, y \in G$.

THEOREM 2. *Let $(G,+)$ and $(H,+)$ be abelian groups and assume that H is uniquely 2-divisible. The functions $f, g : G \to H$ and $\Phi, \Gamma : G \to \mathrm{Hom}(G,H)$ satisfy (4) for all $x, y \in G$ if and only if $f(x) = g(x) = a_o + a_1(x) + a_2(x,x)$, $\Phi(x) = b_1 + a_2(x,\cdot) + a_3(x,\cdot)$ and $\Gamma(x) = a_1 - b_1 + a_2(x,\cdot) - a_3(x,\cdot)$ where $a_o \in H$ is a constant, $a_1, b_1 \in \mathrm{Hom}(G,H)$, $a_2 : G \times G \to H$ is symmetric and biadditive, and $a_3 : G \times G \to H$ is skew-symmetric and biadditive.*

Proof. It is easy to check that f, g, Φ and Γ which have forms as in the assertion satisfy (4). Let us prove the converse. Setting $x = y$ in (4) yields $f(x) = g(x), x \in G$. Thus f satisfies

$$f(x) - f(y) = (\Phi(x) + \Gamma(y))(x - y) \tag{7}$$

for all $x, y \in G$. Interchanging x and y in (7) and adding the obtained equality to (7) we get

$$2(f(x) - f(y)) = [\Phi(x) + \Gamma(x) + \Phi(y) + \Gamma(y)](x - y). \tag{8}$$

Define $\Pi : G \to H$ by $\Pi(x) = \Phi(x) + \Gamma(x)$. Putting $y = 0$ into (8) we get

$$2f(x) = 2f(0) + [\Pi(x) + \Pi(0)](x). \tag{9}$$

Substituting (9) into (8) we infer easily that

$$[\Pi(x) - \Pi(0)](y) = [\Pi(y) - \Pi(0)](x) \tag{10}$$

for all $x, y \in G$. Define $a_2 : G \times G \to H$ by

$$a_2(x, y) = \frac{1}{2}[\Pi(x) - \Pi(0)](y).$$

In view of (10), a_2 is symmetric and biadditive. Further, define $a_1 : G \to H$ by $a_1(x) = \Pi(0)(x)$. Then a_1 is additive. If we put $a_o = f(0)$, we get from (9), after division by 2, $f(x) = a_o + a_1(x) + a_2(x, x)$ for every $x \in G$. If we take into account the definitions of Π, a_1, a_2, and (10), we get

$$\Gamma(x) = \Pi(x) - \Phi(x) = a_1 + 2a_2(x, \cdot) - \Phi(x)$$

for every $x \in G$. Now, substituting formulas for f and Γ into (7), we obtain

$$(\Phi(x) - \Phi(y))(x - y) = a_2(x - y, x - y). \tag{11}$$

An easy calculation shows that (11) implies

$$[\Phi(x) - \Phi(0) - a_2(x, \cdot)](y) = -[\Phi(y) - \Phi(0) - a_2(\cdot, y)](x) \tag{12}$$

for all $x, y \in G$. Define $a_3 : G \times G \to H$ by

$$a_3(x, y) = [\Phi(x) - \Phi(0) - a_2(x, \cdot)](y).$$

It follows from (12) and the definition of a_2 that a_3 is biadditive and skew-symmetric. Putting $b_1 = \Phi(0) \in \mathrm{Hom}(G, H)$ we can write $\Phi(x) = b_1 + a_2(x, \cdot) + a_3(x, \cdot)$ for every $x \in G$. Inserting the form of Φ into the formula for Γ we have $\Gamma(x) = a_1 - b_1 + a_2(x, \cdot) - a_3(x, \cdot)$ for every $x \in G$. This concludes the proof.

The assumption that (4) holds for all $x, y \in G$ is essential as is shown by the following examples.

EXAMPLES. Let us consider two cases.
(i) If G is of order 2 and $H \neq \{0\}$ is uniquely divisible by 2 then $\mathrm{Hom}(G, H) = \{0\}$. Write $G = \{0, c\}$ and define $f, g : G \to H$ by $f(0) = g(c) \neq f(c) = g(0)$, and let $\Phi = \Gamma = 0 \in \mathrm{Hom}(G, H)$. Then (4) is satisfied for all $x, y \in G, x \neq y$, but $f \neq g$.
(ii) Let $G = \{0, a, 2a\}$ be a group of order 3 and suppose that a uniquely 2-divisible group H contains a subgroup $F = \{0, b, 2b\}$ of order 3. Define $f : G \to H$ by $f(x) = 2b, x \in G$, and $g : G \to H$ by $g(y) = b, y \in G$. Further, define $\Phi, \Gamma \in \mathrm{Hom}(G, H)$ putting $\Phi(0) = \Gamma(0) = \alpha, \Phi(a) = \Gamma(a) = \beta$ and

$\Phi(2a) = \Gamma(2a) = \beta$, where $\alpha(x) = 0, x \in G$, and $\beta(0) = 0, \beta(a) = b, \beta(2a) = 2b$. Then $f \neq g$ but f, g, Φ, Γ satisfy (4) for all $x, y \in G, x \neq y$.

We will show that situations presented in the examples are exceptional and in other cases (4) considered for $x \neq y$ is equivalent to (4) considered for all $x, y \in G$.

THEOREM 3. *Let $(G, +)$ be an abelian group which is not cyclic of order 2 and let $(H, +)$ be an abelian uniquely 2-divisible group. If G is not cyclic of order 3 or H contains no cyclic subgroup of order 3, the functions $f, g : G \to H$ and $\Phi, \Gamma : G \to \mathrm{Hom}(G, H)$ satisfy (4) for all $x, y \in G, x \neq y$ if and only if they are as in the assertion of Theorem 2.*

Proof. It is obvious that functions f, g, Φ and Γ which are of the forms given in the assertion satisfy (4) for all $x, y \in G$. To prove the converse assume that $f, g : G \to H$ and $\Phi, \Gamma : G \to \mathrm{Hom}(G, H)$ satisfy (4) for $x, y \in G, x \neq y$.

Suppose first that G is cyclic of order 3. Then H contains no subgroup of order 3 and thus $\mathrm{Hom}(G, H) = \{0\}$. Consequently, we have $\Phi = \Gamma = 0$ and $f(x) = g(y)$ for all $x, y \in G, x \neq y$. Hence it follows that $f = g = \mathrm{const} =: a_o$. Thus the assertion holds with $a_1 = 0 \in \mathrm{Hom}(G, H)$ and $a_2 = a_3 = 0 \in H^{G \times G}$.

From now on assume that G is neither of order 2 nor 3. Putting $y = 0$ and then $x = 0$ into (4) we get $f(x) = g(0) + \Gamma(0)(x) + \Phi(x)(x)$ for every $x \neq 0$, and $g(y) = f(0) + \Phi(0)(y) + \Gamma(y)(y)$ for every $y \neq 0$. These formulas together with (4) and additivity of $\Phi(x)$ yield

$$f(0) - g(0) = [\Gamma(y + z) - \Gamma(y) - \Gamma(z) + \Gamma(0)](x) =: A_{y,z}(x)$$

for all $x, y, z \in G$ such that $0 \notin \{y, z, y + z\}$ and $x \notin \{0, y, z, y + z\}$. Choosing $y \in G$ so that $0 \notin \{y, 2y, 3y\}$ and using additivity od $A_{y,y}$ we can prove that $f(0) = g(0)$. Thus $A_{y,z}(x) = 0$ for every $x \notin \{y, z, y + z\}$. A detailed examination of possible cases shows that $A_{y,z} = 0$ for all $y, z \in G$. Hence the function $a : G \times G \to H$ defined by $a(x, y) = (\Gamma(y) - \Gamma(0))(x)$ is biadditive. Denote by a_2 the symmetric part of a and by a_3 its skew-symmetric part. If, moreover, $a_o = f(0) = g(0), a_1 = \Gamma(0) + \Phi(0)$ and $b_1 = \Phi(0)$ then one can easily show that f, g, Γ and Φ are as in the assertion. This concludes the proof.

REMARK. Let us observe that, in the case where G and H are linear spaces [topological groups] and Hom(G, H) in the above theorems is replaced by L(G, H), the space of linear mappings from G to H [$A(G, H)$, the space of additive continuous mappings] then the words "*additive*" and "*biadditive*" in the statements can be replaced by "*linear*" and "*bilinear*" ["*additive continuous*" and "*biadditive continuous*"], respectively. This is an obvious conclusion from the proofs of our theorems.

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Hölder continuous solutions of functional equations

by Antal Járai

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Abstract. In this work it is proved that the solutions f of the functional equation

$$f(t) = h(t, y, f(g_1(t,y)), \dots, f(g_n(t,y))),$$

that are locally Hölder continuous with exponent $0 < \alpha < 1$, are locally Hölder continuous with exponent $2\alpha/(1+\alpha)$, too.

As it is treated in Aczél's classical book [1961], regularity is very important in the theory and practice of functional equations. The regularity problem of functional equations with two variables can be formulated as follows (see Aczél [1984] and Járai [1986]):

Problem. Let T and Z be open subsets of $\mathbf{R}^s$ and $\mathbf{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, 2, \dots, n)$ and $h : D \times Z^{n+1} \to Z$ be functions. Suppose that

(1)
$$f(t) = h(t, y, f(y), f(g_1(t,y)), \dots, f(g_n(t,y))) \text{ whenever } (t,y) \in D;$$

(2) h is analytic;

(3) g_i is analytic and for each $t \in T$ there exists a y for which $(t,y) \in D$ and $\frac{\partial g_i}{\partial y}(t,y)$ has rank s $(i = 1, 2, \dots, n)$.

Is it true that every f, which is measurable or has the Baire property is analytic?

The following steps may be used:

(I) Measurability implies continuity.

(II) Almost open solutions are continuous.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p times continuously differentiable solutions are $p+1$ times continuously differentiable.

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(VI) Infinitely many times differentiable solutions are analytic.

The complete answer to this problem is unknown. The problems corresponding to (I), (II), (IV) and (V) are solved in Járai [1986]. In the same paper, partial results in connection with (III) are treated. A partial result in connection with (VI) is treated in Járai [1988] (in Hungarian).

In this paper we deal with locally Hölder continuous solutions. The result is a new step in (III). In the paper Járai [1992] the same problem is treated for real solutions, using the fundamental lemma of the theory of Campanato spaces, which is a generalisation of the famous classical Morrey lemma from the regularity theory of partial differential equations. More general results are proved in this paper in an elementary way.

For simplicity, all norms on normed spaces will be denoted by $|\ |$. A function f mapping an open subset of $\mathbf{R}^s$ into $\mathbf{R}^k$ is called locally Hölder continuous with exponent $0 < \alpha \leq 1$, if each point of its domain has a neighbourhood V such that $\sup_{x,y \in V} |f(x) - f(y)|/|x-y|^\alpha < \infty$. Any constant not less then this supremum is called a (local) Hölder-constant for f. In the case $\alpha = 1$ Hölder continuous functions and Hölder constants are also called Lipschits functions and Lipschits constants, respectively. It is well-known, that continuously differentiable functions are locally Lipschits.

Lemma. Let V, W and U open subsets of $\mathbf{R}^s$, $\mathbf{R}^k$ and $\mathbf{R}^r$, respectively, $f : U \to \mathbf{R}$ a continuous function, $g : V \times W \to U$ a twice continuously differentiable function, $t_0 \in V$, $y_0 \in W$, and suppose, that $\frac{\partial g}{\partial y}(t_0, y_0)$ has rank r. Let S be a simplex with nonvoid interior contained in the unit ball of $\mathbf{R}^k$. Then there exists a convex neighbourhood $V_0 \subset V$ of t_0, a real number $R_0 > 0$ and a constant C, such that for each R for which $0 < R < R_0$ the function

$$t \mapsto \int_{RS+y_0} f(g(t,y))\,dy$$

is continuously differentiable on V_0 with gradient bounded by CR^{k-1}.

Proof. The proof depends on a theorem concerning differentiation of parametric integrals. The more general form of this theorem can be found in the paper Járai [1991]. For a somewhat weaker version (which is still enought for our present proof) see the paper Járai [1986], theorem 5.1.

Let g_t denote the partial function $y \mapsto g(t,y)$, and let $g_t(y) = x = (x_1, x_2, \dots, x_r)$. Without loss of generality we may suppose that the Jacobian

$$\frac{\partial(x_1, x_2, \dots, x_r)}{\partial(y_1, y_2, \dots, y_r)}$$

is positive at the point (t_0, y_0). Let us choose open balls V_0 and W_0 with centers t_0 and y_0, respectively, such that this Jacobian is positive whenever $t \in V_0$ and $y \in W_0$. Let R_0 be the radius of W_0. Let $x^* = (x_1^*, \dots, x_r^*, x_{r+1}^*, \dots, x_k^*)$ be defined in the following way: $x_i^* = x_i$ if $1 \leq i \leq r$ and $x_i^* = y_i$ if $r < i \leq k$. We shall denote the mapping $y \mapsto x^*$ by g_t^*. Let $p(x^*) = x$. It is clear that the Jacobian

$$\frac{\partial(x_1^*, \dots, x_k^*)}{\partial(y_1, \dots, y_k)}$$

is equal to the Jacobian above, and hence nonzero if $t \in V_0$ and $y \in W_0$. Similarly as in the proof of theorem 5.2 in the paper Járai [1986], not hard to prove that decreasing the radius of V_0 and W_0 if necessary, we may suppose that g_t^* is a homeomorphism of W_0 onto an open subset of $\mathbf{R}^k$ whenever $t \in V_0$. Using the substitution $x^* = g_t^*(y)$ we have

$$F(t) = \int_{RS+y_0} f(g(t,y))\,dy = \int_{g_t^*(RS+y_0)} f(p(x^*))J((g_t^*)^{-1}(x^*))\,dx^*.$$

Here J denotes the Jacobian. Let $h^*(t,x) = f(p(x^*))J((g_t^*)^{-1}(x^*))$ and let us use the theorem in the paper Járai [1991] concerning differentiation with respect to the parameter of the parametric integral

$$F(t) = \int_{g_t^*(RS+y_0)} h(t,x^*)\,dx^*.$$

We get that F is continuously differentiable. Using the proof of this theorem we can get an explicit formula for the partial derivatives of F, and write it as the sum of two integrals. The first integral is a volume integral over $RS + y_0$ and the integrand is a continuous function of the partial derivative $\frac{\partial h}{\partial t_i}$ and the derivatives of g. The second integral is a boundary integral over $\partial(RS + y_0)$ and the integrand is a continuous function of h and the partial derivatives of g. If V_0 and W_0 are small enough, these continuous functions are bounded and we have the expected estimate.

Theorem. Let $0 < \alpha < 1$, and let Z, Z_i be open subsets of Euclidean spaces ($i = 1, 2, \dots, n$). Let T, Y and X_i be open subsets of $\mathbf{R}^s$, $\mathbf{R}^k$ and $\mathbf{R}^{r_i}$, respectively. Let D be an open subset of $T \times Y$. Consider the functions $f : T \to Z$, $f_i : X_i \to Z_i$ ($i = 1, \dots, n$), $g_i : D \to X_i$ ($i = 1, \dots, n$), $h : D \times Z_1 \times Z_2 \times \cdots \times Z_n \to Z$. Suppose, that

(1) for each $(t,y) \in D$,

$$f(t) = h(t, y, f_1(g_1(t,y)), \dots, f_n(g_n(t,y)));$$

(2) h is twice continuously differentiable;

(3) g_i is twice continuously differentiable on D and for each $t \in T$ there exists a y such that $(t,y) \in D$ and $\frac{\partial g_i}{\partial y}(t,y)$ has rank r_i for $i = 1, \dots, n$;

(4) the functions f_i, $i = 1, \dots, n$ are locally Hölder continuous with exponent α.

Then f is locally Hölder continuous with exponent $2\alpha/(\alpha + 1)$.

Proof. We have to prove that for each point $t_0 \in T$ the function f is Hölder continuous on a neighbourhood of t_0 with exponent $2\alpha/(1+\alpha)$. Replacing f_i with the coordinates we may suppose without restricting generality that $Z_i \subset \mathbf{R}$ for $i = 1, \dots, n$. Let us choose y_0 by (3) for t_0. By our lemma there exist a convex open neighbourhood V_0 of t_0, a simplex with nonvoid interior contained in the unit ball, and constants $R_0 > 0$ and C such that for the closed ball W_0 with center y_0 and radius R_0 we have $V_0 \times W_0 \subset D$, and the mappings

$$t \mapsto \int_{RS+y_0} f_i(g_i(t,y))\,dy$$

are continuously differentiable on V_0 with gradient bounded by CR^{k-1}. Decreasing V_0 and R_0 if necessary we may suppose that $R_0 \le 1$ and g_i is a Lipschitz function with Lipschitz constant L on $V_0 \times W_0$. Similarly, we may suppose that f_i is Hölder continuous with exponent α and Hölder constant H and $|f_i|$ bounded by B on $g_i(V_0 \times W_0)$, moreover on a convex closed set containing

$$V_0 \times W_0 \times f_1(g_1(V_0 \times W_0)) \times \cdots \times f_n(g_n(V_0 \times W_0))$$

the functions $\frac{\partial h}{\partial z_i}$ are Lipschitz continuous with Lipschitz constant L'_i, and the functions $\left|\frac{\partial h}{\partial t}\right|$ and $\left|\frac{\partial h}{\partial z_i}\right|$ are bounded by B'_0 and B'_i, respectively ($i = 1, 2, \dots, n$). Let us fix R_0, V_0, W_0 and y_0. We shall prove that f is locally Hölder continuous on V_0 with exponent $2\alpha/(1+\alpha)$. Let t, t' denote arbitrary elements of V_0 and let $0 < R < R_0$. Let us integrate the two sides of the functional equation over the simplex $RS + y_0$. We have

$$cR^k f(t) = \int_{RS+y_0} h(t, y, f_1(g_1(t,y)), \dots, f_n(g_n(t,y)))\,dy,$$

where $c > 0$ is the measure of the simplex S. Hence

$$\begin{aligned}|f(t) - f(t')| = \frac{1}{cR^k}\bigg| \int_{RS+y_0} & h(t, y, f_1(g_1(t,y)), \dots, f_n(g_n(t,y))) \\ & - h(t', y, f_1(g_1(t',y)), \dots, f_n(g_n(t',y)))\,dy\bigg|.\end{aligned}$$

To get a good upper estimate for the left hand side we need an upper estimate for the difference

$$h(t, y, f_1(g_1(t,y)), \dots, f_n(g_n(t,y))) - h(t', y, f_1(g_1(t',y)), \dots, f_n(g_n(t',y))).$$

We may apply the Taylor theorem for the function h with points

$$z = (t, y, z_1, \dots, z_n) \quad \text{and} \quad z' = (t', y, z'_1, \dots, z'_n)$$

where $t', t \in V$, $y \in W$, $z_i = f_i(g_i(t,y))$ and $z_i' = f_i(g_i(t',y))$ for $i = 1,\ldots,n$. We have

$$h(z)-h(z') = \int_0^1 \frac{\partial h}{\partial t}(\tau z+(1-\tau)z')(t-t')\,d\tau+\sum_{i=1}^{n}\int_0^1 \frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')(z_i-z_i')\,d\tau.$$

Using this and omitting variables we have

$$\begin{aligned} cR^k|f(t') - f(t)| = \bigg|\int_{RS+y_0}\Big(&\int_0^1 \frac{\partial h}{\partial t}(\tau z + (1-\tau)z')(t-t')\,d\tau \\ &+ \sum_{i=1}^{n}\int_0^1 \frac{\partial h}{\partial z_i}(\tau z + (1-\tau)z')(z_i - z_i')\,d\tau\Big)\,dy\bigg|. \end{aligned}$$

Using the triangle inequality, we get $n+1$ terms on the right hand side. For the first term we get the trivial upper bound $cR^kB_0'|t'-t|$, where B_0' is an upper bound of $\left|\frac{\partial h}{\partial t}\right|$. Let $z_i^0 = f_i(g_i(t,y_0))$ $(i = 1,2,\ldots,n)$, and let $z^0 = (t, y_0, z_1^0, \ldots, z_n^0)$. If h_i' denotes the value of the partial derivative $\frac{\partial h}{\partial z_i}$ at the point z^0, than the other terms can be rewritten in the form

$$\int_{RS+y_0}\int_0^1\left(\frac{\partial h}{\partial z_i}(\tau z + (1-\tau)z') - h_i'\right)(z_i - z_i')\,d\tau\,dy + h_i'\int_{RS+y_0}|z_i - z_i'|dy.$$

First we give an upper estimate for the absolute value of the first term of this sum. An upper estimate of $|z_i - z_i'|$ is $H(L|t-t'|)^\alpha$, where H is a Hölder-constant for f_i and L is a Lipschitz-constant for g_i. Hence

$$\begin{aligned} &\left|\int_{RS+y_0}\int_0^1 \frac{\partial h}{\partial z_i}(\tau z + (1-\tau)z') - h_i')(z_i - z_i')\,d\tau\,dy\right| \\ &\le H(L|t-t'|)^\alpha \int_{RS+y_0}\int_0^1\left|\frac{\partial h}{\partial z_i}(\tau z + (1-\tau)z') - \overline{h}_i'\right|\,d\tau\,dy. \end{aligned}$$

Furthermore we need to estimate the difference $\left|\frac{\partial h}{\partial z_i}(\tau z + (1-\tau)z') - \frac{\partial h}{\partial z_i}(z^0)\right|$. This is not greater then L_i' multiplied by the norm of $\tau z + (1-\tau)z' - z^0$, that is, L_i' times the maximal distance between the vectors z' and $z^0 = (t, y_0, z_1^0, \ldots, z_n^0)$, where L_i' is a Lipschitz-constant for $\frac{\partial h}{\partial z_i}$. The maximal distance between z' and z^0 can be estimated by $|t-t'| + R + nH(L(|t-t'|+R))^\alpha$. Hence we get the upper bound

$$cR^kH(|t-t'|L)^\alpha L_i'(|t-t'| + R + nH(L(|t-t'|+R))^\alpha)$$

for the first term.

To get an upper bound for the second term, we need an upper bound for the absolute value of

$$\int_{RS+y_0}(z_i - z_i')dy = \int_{RS+y_0} f_i(g_i(t,y)) - f_i(g_i(t',y))\,dy,$$

because $|h'_i|$ is trivially bounded by the upper bound B'_i of $\left|\frac{\partial h}{\partial z_i}\right|$. From the lemma we get the upper bound $|t-t'|CR^{k-1}$ for this integral.

Summing up all this estimates, we get

$$|f(t)-f(t')| \le B'_0|t-t'| + H(L|t-t'|)^\alpha \sum_{i=1}^{n} L'_i(|t-t'| + R + nH(L(|t-t'|+R))^\alpha)$$
$$+\sum_{i=1}^{n} B'_i|t-t'|C/R.$$

If $|t-t'| \le R$ this can be rewritten as

$$|f(t)-f(t')| \le C_0|t-t'| + C_1|t-t'|^\alpha R^\alpha + C_2|t-t'|/R,$$

where C_0, C_1 and C_2 do not depend on t, t', and R. If we choose R such that it satisfy the condition $R = |t-t'|^{(1-\alpha)/(1+\alpha)}$, then we have

$$|f(t)-f(t')| \le (C_0+C_1+C_2)|t-t'|^{2\alpha/(1+\alpha)}$$

whenever $|t-t'| < R_0^{(1+\alpha)/(1-\alpha)}$ and $t, t' \in V_0$. This proves that f is locally Hölder continuous on V_0 which implies the theorem.

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SUR DES SOLUTIONS GLOBALES ET DES SOLUTIONS LOCALES DE L'ÉQUATION DE TRANSLATION

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Presented by J. Aczel, F.R.S.C.

Résumé: On compare la notion de la solution locale de l'équation de translation avec la notion de la solution globale de cette équation et on donne quelques conditions suffisantes pour que la solution locale soit prolongeable à la solution globale.

M.Lothar Berg en considérant l'équation de translation multidimensionnelle de la forme

(1) $$F(F(x,t),s)=F(x,t+s),$$

où $F: R^n \times R^m \longrightarrow R^n$,a défini [1] une solution locale de cette équation au point $x_0 \in R^n$ comme une fonction F telle qu'ils existent un entourage (une boule ouverte ou R^n) Δ de x_0 et un entourage (une boule ouverte ou R^m) Δ_2 de $0 \in R^m$ tels que

(2) $$\forall x \in \Delta_1 \ \forall t,s \in \Delta_2: \ F(F(x,t),s)=F(x,t+s).$$

On considère la solution locale aussi dans [3].

La fonction F est dite une solution globale si (2) a lieu pour $\Delta_1=R^n$ et $\Delta_2=R^m$.Cette note est consacrée à la comparaison de ces deux notions.

On voit que chaque solution globale est en même temps locale.Si F est une solution locale elle doit être définie au moins sur l'ensemble $E:=[F(\Delta_1 \times \Delta_2) \times \Delta_2] \cup [\Delta_1 \times (\Delta_2+\Delta_2)]$,puisque les membres droite et gauche dans (1) doivent être définis pour chaque $x \in \Delta_1$ et chaques $t,s \in \Delta_2$ et en dehors de cet ensemble F peut être arbitraire.On peut donc construire une solution locale qui n'est pas globale de la manière suivante. Nous considérons une solution F_1 globale pour laquelle ils existent des entourages Δ_1 et Δ_2 pour lesquels $E\neq R^n$ (voir pour ce but (2)) et posons $F=F_1$ sur E et F arbitraire sur $R^n\setminus E$, mais telle que F n'est pas une solution globale. La solution locale F, reçue de cette façon, considérée seulement sur E, peut être naturellement prolongée à la solution globale (p.ex. F_1). On se pose donc le problème de donner un exemple d'une solution locale F qui, considérée seulement sur E ne peut pas être prolongeable à la solution globale. Et voilà un exemple: Soit pour $n=m=1$

$$(3)\qquad F(x,t)=\begin{cases} x+t & \text{si } |x+t|<4, \\ 0 & \text{si } 4\le|x+t|, \end{cases}$$

$x_0\in R, \Delta_1=(-1,1), \Delta_2=(-3,3)$. Soit $x\in\Delta_1, t,s\in\Delta_2$, alors $F(x,t)=x+t$ et

$$F(F(x,t),s)=F(x+t,s)=\left\{\begin{matrix} x+t+s & \text{si } |x+t+s|<4 \\ 0 & \text{si } 4\le|x+t+s| \end{matrix}\right\}=F(x,t+s),$$

donc (2) a lieu. De plus $E=(-4,4)\times(-3,3)\cup(-1,1)\times(-6,6)$.

Prenons que $F_{|E}$ est prolongeable à une solution globale F_1. Nous avons $F_1(0,4)=F(0,4)=0=F(0,5)=F_1(0,5)$ et puisque F_1 remplit (1) on a:

$$0=F(0,0)=F_1(0,0)=F_1(F_1(0,4),-4)=F_1(F_1(0,5),-4)=F_1(0,1)=F(0,1)=1,$$

donc une contradiction.

Il se pose donc le problème sous quelles conditions une solution locale, considérée sur E, peut être prolongeable à une solution globale?

Nous allons démontrer à ce sujet le

Théorème 1. Si F est une solution locale de (1) et

$$(4)\qquad F(\Delta_1\times\Delta_2)\subset\Delta_1,$$

$$(5)\qquad \forall t_1,\dots,t_k,s_1,\dots,s_l\in\Delta_2+\Delta_2\ \forall x\in\Delta_1\ t_1+\dots+t_k=s_1+\dots+s_l \longrightarrow F(F(F\dots F(x,t_1),t_2),\dots,t_{k-1}),t_k)= =F(F(F\dots F(x,s_1),s_2),\dots,s_{l-1}),s_l),$$

alors F est prolongeable à une solution globale.

Démonstration.

La fonction F, étant définie sur E, est d'après (4) définie sur $\Delta_1\times(\Delta_2+\Delta_2)$. Posons pour $t\in(\Delta_2+\Delta_2)+(\Delta_2+\Delta_2)$, donc pour $t=t_1+t_2$, où $t_1,t_2\in\Delta_2+\Delta_2$ et pour x de Δ_1

$$F_1(x,t)=F(F(x,t_1),t_2).$$

Cette définition est correcte, puisque si $t=t_1+t_2=s_1+s_2$, où $t_1,s_1,t_2,s_2\in\Delta_2+\Delta_2$, alors d'aprés (5)

$$F(F(x,t_1),t_2)=F(F(x,s_1),s_2).$$

De plus pour $t\in\Delta_2+\Delta_2$, donc $t=t_1+t_2$, où $t_1,t_2\in\Delta_2$, et x de Δ_1 on a

$$F_1(x,t)=F_1(x,t_1+t_2)=F(F(x,t_1),t_2)=F(x,t_1+t_2)=F(x,t),$$

alors $F_{1|\Delta_1\times(\Delta_2+\Delta_2)}=F_{|\Delta_1\times(\Delta_2+\Delta_2)}$.

Nous allons démontrer que F_1 a la même propriété que (5), mais avec $\Delta:=(\Delta_2+\Delta_2)+(\Delta_2+\Delta_2)$ au lieu de $\Delta_2+\Delta_2$. En effet soit $t_1,\dots t_k,s_1,\dots,s_l\in\Delta$ et $t_1+\dots+t_k=s_1+\dots+s_l$, d'où $t_i=t_i^*+t_i^{**}$ $(i=1,\dots,k)$ et $s_j=s_j^*+s_j^{**}$ $(j=1,\dots,l)$, où $t_i^*,t_i^{**},s_j^*,s_j^{**}\in\Delta_2+\Delta_2$. Puisque

$$t_1^*+t_1^{**}+\dots+t_k^*+t_k^{**}=s_1^*+s_1^{**}+\dots+s_l^*+s_l^{**},$$

donc d'aprés (5):

$$F(F(F(\dots F(x,t_1^*),t_1^{**}),,\dots,t_k^*),t_k^{**})=$$
$$F(F(F(\dots F(x,s_1^*),s_1^{**}),\dots,s_l^*),s_l^{**}).$$

Il en résulte la thèse puisque nous avons d'aprés la définition de F_1: $F(F(z,u),v)=F_1(z,u+v)$ pour chaque z de Δ_1 et u,v de $\Delta_2+\Delta_2$.

Définissons Δ^ν comme il suit: $\Delta^1:=\Delta_2,\Delta^{\nu+1}:=\Delta^\nu+\Delta^\nu$ pour $\nu=1,2,\dots$. En prenant F_1 au lieu de $F,\Delta_2+\Delta_2$ au lieu de Δ_2 nous pouvons définir sur $\Delta_1\times\Delta^4$ de la manière analogue la fonction F_2 qui est égale à F_1 sur $\Delta_1\times\Delta^3$ et qui remplit (5) avec Δ^4 au lieu de $\Delta_2+\Delta_2$. On peut définir de même par induction la fonction F_ν sur $\Delta_1\times\Delta^{\nu+2}$ qui est égale à $F_{\nu-1}$ sur

$\Delta_1 x\Delta^{\nu+1}$ et qui remplit (5) avec $\Delta^{\nu+2}$ au lieu de $\Delta_2+\Delta_2$.

La fonction $\Phi=\bigcup_{\nu=1}^{\infty}F_\nu$ est évidemment définie sur Δ_1xR^m, puisque $\bigcup\Delta^\nu=R^m$, étant un prolongement de $F_{|E}$. Nous allons montrer qu'elle est une solution de (1) sur Δ_1xR^m.

Il résulte de (5) pour F_ν et $t_1=t, t_2=s, s_1=t+s$ que

$$F_\nu(F_\nu(x,t),s)=F_\nu(x,t+s) \tag{6}$$

si $x\in\Delta_1, t,s,t+s\in\Delta^{\nu+2}$. Soit $x\in\Delta_1, t,s\in R^m$. Il existe un ν tel que $t,s,t+s\in\Delta^{\nu+2}$, alors

$\Phi(x,t)=F_\nu(x,t), \Phi(\Phi(x,t),s)=F_\nu(F_\nu(x,t),s), \Phi(x,t+s)=F_\nu(x,t+s)$,

donc la fonction Φ remplit (1) d'après (6).

Si $\Delta_1=R^n$ la fonction Φ est une solution globale de (1), étant un prolongement de $F_{|E}$. Si $\Delta_1\neq R^n$, soit $x_0\in R^n\setminus\Delta_1$ et la fonction

$$\Phi^*(x,t)=\begin{cases}\Phi(x,t) & \text{pour } (x,t)\in\Delta_1xR^m,\\ x_0 & \text{pour } (x,t)\in R^nxR^m\setminus\Delta_1xR^m\end{cases}$$

est une solution exigée. La démonstration du théorème est donc achevée.

Remarquons que la condition (5) est évidemment nécessaire pour qu'il existe un prolongement de la solution locale, même que la supposition (4) n'est pas telle. En effet si $\Delta_1\neq R^n$ et $x_0\in R^n\setminus\Delta_1$ la fonction $F(x,t)=x_0$ est une solution locale pour Δ_2 arbitraire, ayant un prolongement et ne remplissante pas de la condition (4).

Il est vraie aussi le

Théorème 2. *Si une solution locale F de (1) remplit (4) et*

(7) $\forall x\in\Delta_1\ \forall t_1,\dots,t_k\in\Delta_2:\ t_1+\dots+t_k\in\Delta_2+\Delta_2$ et
$F(x,t_i)=F(x,0)$ pour $i=2,\dots,k$, $\longrightarrow F(x,t_1)=F(x,t_1+\dots+t_k)$,

(8) $\forall x\in\Delta_1\ \exists y\in F(x,\Delta_2)\ \exists 0\neq t\in\Delta_2:\ F(x,t)=y$,

alors F est prolongeable à la solution globale de (1).

Démonstration.

Remarquons au commencement que d'après (8)

$$\forall x\in\Delta_1\ \exists 0\neq t\in\Delta_2:\ F(x,t)=F(x,0).$$

En effet prenons pour x de Δ_1 un $y\in F(x,\Delta_2)$ d'après (8), d'où $y=F(x,u)$ pour un u de Δ_2. Soit t d'après (8) tel que $0\neq t\in\Delta_2$ et $F(y,t)=y$. Il en résulte que

$$F(x,0)=F(y,-u)=F(F(y,t),-u)=F(y,t-u)=F(F(y,-u),t)=F(x,t).$$

Désignons par $G(x)$, pour $x\in\Delta_1$, le sous-groupe additif de $(R^m,+)$ engendré par l'ensmble $E(x):=\{t\in\Delta_2:\ F(x,t)=F(x,0)\}$. Puisque $E(x)\neq\{0\}$ il existe pour chaque $t\in R^m$ un $t^*\in\Delta_2$ tel que $t-t^*\in G(x)$. Posons pour x de Δ_1

$$F_1(x,t)=F(x,t^*). \tag{9}$$

La fonction F_1 est bien définie sur $\Delta_1\times R^m$ puisque si $t-t^*\in G(x)$, $t-t^{**}\in G(x)$ et $t^*,t^{**}\in\Delta_2$, alors $t^*-t^{**}\in G(x)$, donc d'après (7): $F(x,t^*-t^{**})=F(x,0)$, d'où

$$F(x,t^{**})=F(F(x,0,)t^{**})=F(F(x,t^*-t^{**}),t^{**})=$$
$$F[F(F(x,t^*),-t^{**}),t^{**}]=F(F(x,t^*),0)=F(x,t^*).$$

Nous démontrerons que F_1 remplit (1) sur $\Delta_1\times R^m$. Au commencement nous allons démontrer que

$$G(x)=G(F(x,t^*)) \tag{10}$$

pour chaque x de Δ_1 et t^* de Δ_2. En effet $u\in G(x)$ est équivalente à la condition qu'ils existent $t_1,\dots,t_k$ de Δ_2 tels que $u = t_1+\dots+t_k$ et $F(x,t_i) = F(x,0)$ pour $i=1,\dots,k$ et $u\in G(F(x,t^*))$ désigne qu'ils existent $s_1,\dots,s_l$ de Δ_2 tels que $u=s_1+\dots+s_l$ et $F(F(x,t^*),s_j)=F(x,t^*)$ pour $j=1,\dots,l$. Si nous avons $t_1,\dots,t_k$ comme plus haut il suffit prendre $k=l$ et $s_i=t_i$ et invérsement pour avoir l'équivalence

$$u\in G(F(x,0)) \longleftrightarrow u\in G(F(x,t^*)).$$

Nous avons d'après la définition (9): $F_1(x,t)=F(x,t^*)$, où $t^*\in\Delta_2$ et $t-t^*\in G(x)$,

$$F_1(F_1(x,t),s)=F_1(F(x,t^*),s)=F(F(x,t^*),s^*)=F(x,t^*+s^*),$$

où $s^*\in\Delta_2$ et $s-s^*\in G(F(x,t^*))=G(x)$, et $F_1(x,t+s)=F(x,(t+s)^*)$, où $(t+s)^*\in\Delta_2$ et $(t+s)-(t+s)^*\in G(x)$. Puisque $(t^*+s^*)-(t+s)^*\in G(x)$, alors $t^*+s^*=(t+s)^*+t_2+\ldots+t_k$, où $t_i\in\Delta_2$ et $F(x,t_i)=F(x,0)$ pour $i=2,\ldots,k$, donc d'après (7): $F(x,(t+s)^*)=F(x,t^*+s^*)$, d'où F_1 remplit (1) sur $\Delta_1\times R^m$.

On a pour x de Δ_1 et $t\in\Delta_2$: $F_1(x,t)=F(x,t^*)$ pour t^* tel que $t^*\in\Delta_2$ et $t-t^*\in G(x)$. Puisque on peut prendre ici $t^*=t$ nous avons $F_1(x,t)=F(x,t)$ pour x de Δ_1 et t de Δ_2.

Nous avons pour x de Δ_1 et $t\in\Delta_2+\Delta_2$, d'où $t=t_1+t_2$ avec $t_1,t_2\in\Delta_2$, que

$$F_1(x,t)=F_1(x,t_1+t_2)=F_1(F_1(x,t_1),t_2)=$$
$$F(F(x,t_1),t_2)=F(x,t_1+t_2)=F(x,t),$$

donc F_1 est un prolongement de $F_{|E}$.

Si $\Delta_1=R^n$ la démonstration du théorème 2 est achevée, si $\Delta_1\neq R^n$ nous terminons le raisonnement comme en fin de la démonstration du théorème 1.

Remarquons que la supposition (7) est évidemment nécessaire pour qu'une solution locale F ait un prolongement globale. Au contraire la supposition (8) n'est pas telle, comme cela montre pour $n=m$ la fonction $F(x,t)=x+t$.

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Higher Schur- Multiplicators of Nilpotent Dihedral Groups

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Summary. Let G_c be the nilpotent dihedral group of class c given by a free presentation $1 \to R \to F \to G_c \to 1$. We calculate for all values of m and c, the Baer-invariants $B_m(G_c)$ of G_c as follows: for $m \leq c-1$, $B_m(G_c)$ is cyclic of order 2^m; for $m \geq c$, it is an abelian extension of a cyclic group of order 2^{c-1} by an elementary abelian 2-group of rank $r(m+1)$, the rank of the m+1-th lower central factor of F.

Introduction.

Let G be a group given by a free presentation : $1 \to R \to F \to G \to 1$. The *higher Schur -multiplicators* $B_m(G)$, $m \geq 1$, are defined to be the quotients $R \cap \gamma_{m+1}(F) / [R, {}_m F]$ which are precisely the Baer-invariants of G with respect to the variety of nilpotent groups, and are known to be independent of the choice of the presentation of G (Baer [1]). The case m = 1 is the well-known Hopf's formula for the second integral homology group H_2, and yields the Schur -multiplicator of G. Calculation of these Baer-invariants of a given group G is a recognized difficult problem (see, for instance, [3], [2]) and very little seems to be known in this direction. Recently, Rosset [5] has

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induced a renewed interest in the problem by exploring some results on the *higher lower central quotients* $\Gamma_m(G) = \gamma_{m+1}(F)/[R, {}_mF]$, and raising questions regarding the structure of $\Gamma_m(G)$. The Baer- invariants of the dihedral group of class two were calculated by Moghaddam [4]. In this paper, we calculate the Baer- invariants $B_m(G_c)$ of all nilpotent dihedral groups G_c of nilpotency class $c \geq 1$.We prove that $B_m(G_c)$ is cyclic of order 2^m for $m \leq c-1$, whereas, for $m \geq c$, it is an abelian extension of a cyclic group of order 2^{c-1} by an elementary abelian 2-group of rank $r(m+1)$, the rank of the lower central factor $\gamma_{m+1}(F)/\gamma_{m+2}(F)$ of F. Since the quotient $\Gamma_m(G_c) = \gamma_{m+1}(F)/[R, {}_mF]$ coincides with the Baer-invariant $B_m(G_c)$ for $m \geq c$, we obtain the structure of $\Gamma_m(G_c)$ as a corollary.

Preliminaries and Technical Lemmas

Let G_c, $c \geq 1$, denote the nilpotent dihedral group of class c. Then G_c is given by a free presentation: $1 \to R \to F \to G_c \to 1$, where $F = \langle x, y ; \emptyset \rangle$ is free and R is the normal closure in F given by $R = \langle x^2, y^2, \gamma_{c+1}(F) \rangle^F$.

It is routine to verify that R is also the normal closure $\langle x^2, y^2, [x, y]^{2^{c-1}} \rangle^F$, and that the order of G_c is 2^{c+1}. Put $S = \langle x^2, y^2 \rangle^F$, so $R = \gamma_{c+1}(F)S$.

For $m \geq 1$, define

$$\rho_m(S) = [S, {}_{m-1}F] = [S, F, \ldots, F] \quad (m\text{-}1 \text{ repeats of } F).$$

This yields the central series $S = \rho_1(S) > \rho_2(S) > \rho_3(S) > \ldots\ldots$

We shall need the following technical lemma.

Lemma 1. For $a_t, b_t, c_t \in \{x, y\}$ and $i, j, k \geq 0$ with $i + j + k = n - 4 \geq 0$, the following congruences hold modulo $\gamma_{n+2}(F)\rho_n(S)$:

(i) $[x, y, x, a_1, \ldots, a_{n-2}] \equiv [x, y, y, a_1, \ldots, a_{n-2}]$;

(ii) $[x, y, a_1, \ldots, a_i, x, x, b_1, \ldots, b_j, y, c_1, \ldots, c_k]$

$\equiv [x, y, a_1, \ldots, a_i, x, b_1, \ldots, b_j, y, y, c_1, \ldots, c_k]$;

(iii) $[x, y, a_1, \ldots, a_i, x, x, b_1, \ldots, b_j, x, c_1, \ldots, c_k]$

$\equiv [\, x , y , a_1, \dots, a_i , x, b_1, \dots, b_j, x, x, c_1, \dots, c_k \,]$;

(iv) $[\, x , y , a_1, \dots, a_i , y , y, b_1, \dots, b_j, y, c_1, \dots, c_k \,]$

$\equiv [\, x , y , a_1, \dots, a_i , y, b_1, \dots, b_j, y , y , c_1, \dots, c_k \,]$;

(v) $[\, [\, x , y , {}_{m-3}\, y] , [x, y] , a_1, \dots, a_{n-m} \,] \equiv 1$;

(vi) $[\, x , y , {}_{m-2}\, y , x, a_1, \dots, a_{n-m}] \equiv [\, x , y , {}_{m-1}\, y , a_1, \dots, a_{n-m}]$.

Proof. For the proof of (i) we expand the commutators

$[\, x^2 , y , a_1, \dots, a_{n-2} \,] \in \rho_n(S)$ and $[\, x , y^2 , a_1, \dots, a_{n-2} \,] \in \rho_n(S)$

modulo $\gamma_{n+2}(F)$ and compare the two expansions.

The proof of (ii) follows similarly by expanding and comparing

$[\, x , y , a_1, \dots, a_i , x^2, b_1, \dots, b_j, y, c_1, \dots, c_k \,]$

and $[\, x , y , a_1, \dots, a_i , x, b_1, \dots, b_j , y^2, c_1, \dots, c_k \,] \in \rho_n(S)$.

The proofs of (iii) and (iv) are identical to that of (ii) after appropriately choosing the two elements of $\rho_n(S)$ to compare.

To prove (v) we first consider the case when $m-3 = 2k$. By repeated application of (iv) we transfer k of the y-entries in $[\, x , y , {}_{m-3}\, y , [x, y] , a_1, \dots, a_{n-m}]$ to obtain the equivalent commutator $[\, [\, x , y , {}_k y \,] , [x, y , {}_k y \,] , a_1, \dots, a_{n-m}]$ which is trivial. When $m-3 = 2k+1$, we can likewise transform $[[\, x , y , {}_{m-3}\, y \,] , [x, y \,] , a_1, \dots, a_{n-m}]$ to the equivalent commutator $[\, [\, x , y , {}_{k+1} y \,] , [x, y , {}_k y \,] , a_1, \dots, a_{n-m}]$ which, in turn, is congruent to the square $[\, x , y , {}_k y \,], [x, y , {}_k y], a_1, \dots, a_{n-m}]^2$ ($= 1$) obtained from the expansion of $[\, [\, x , y , {}_{k-1} y, y^2 \,] , [x, y , {}_k y] , a_1, \dots, a_{n-m}] \in \rho_n(S)$.

Finally, for the proof of (vi), we write

$[\, x , y , {}_{m-2}\, y , x, a_1, \dots, a_{n-m}]$

$\equiv [\, x , y , {}_{m-3}\, y , x , y, a_1, \dots, a_{n-m}] \, [\, [\, x , y , {}_{m-3}\, y] , [\, x , y], a_1, \dots, a_{n-m}]$

$\equiv [\, x , y , {}_{m-3}\, y , x , y , a_1, \dots, a_{n-m}]$ (by (v)).

On the other hand, by repeated appllication of (iv) the resulting commutator

$[\, x , y , {}_{m-3}\, y , x , y, a_1, \dots, a_{n-m}]$

can be transformed to the commutator

$[\, x , y , x , {}_{m-2}\, y, a_1, \dots, a_{n-m}]$

which, by (i), is congruent to the desired commutator

$[x, y, y,_{m-2} y, a_1, \ldots, a_{n-m}]$.

This completes the proof of the lemma.

We can now prove our key lemma.

Lemma 2. Let u_{n+1} denote the commutator $[x, y,_{n-1} y]$. Then, for each $n \geq 3$,

(i) $\gamma_{n+1}(F) \equiv \langle u_{n+1} \rangle$ (modulo $\gamma_{n+2}(F)\, \rho_n(S)$)

(ii) $u_{n+1}{}^2 \equiv 1$ (modulo $\gamma_{n+2}(F)\, \rho_n(S)$).

Proof. The proof of (ii) follows instantly from the expansion of $[x, y,_{n-2} y, y^2]$ which is an element of $\rho_n(S)$. For the proof of (i), it clearly suffices to prove that every left-normed commuatator $[x, y, a_1, \ldots, a_{n-1}]$ with $a_i \in \{x, y\}$ is congruent, modulo $\gamma_{n+2}(F)\, \rho_n(S)$, to the given commutator $u_{n+1} = [x, y,_{n-1} y]$.

Let $\ell(x)$ denote the number of x-occurences in the commutator $[x, y, a_1, \ldots, a_{n-1}]$. We prove by induction on $\ell(x) \geq 1$ that the commutator $[x, y, a_1, \ldots, a_{n-1}]$ is congruent to the commutator $[x, y,_{n-1} y]$. When $\ell(x) = 1$, there is nothing to prove. For the inductive step consider a commutator $[x, y, a_1, \ldots, a_{n-1}]$ with $2 \leq \ell(x) = k \leq n$. Then, since

$$[x, y, a_1, \ldots, a_{n-1}] = [x, y,_i y, x, a_{i+2}, \ldots, a_{n-1}]$$
$$\equiv [x, y,_{i+1} y, a_{i+2}, \ldots, a_{n-1}] \text{ (by Lemama 1(vi))},$$

and since the number of x-entries in $[x, y,_{i+1} y, a_{i+2}, \ldots, a_{n-1}]$ is k-1, the proof follows by the induction hypothesis.

Calculation of the Baer-invariants

We can now compute the Baer-invariants $B_m(G_c)$ of the nilpotent dihedral groups G_c given by a free presentation : $1 \to R \to F \to G_c \to 1$, where $F = \langle x, y ; \emptyset \rangle$ is free and $R = \gamma_{c+1}(F)S$, $S = \langle x^2, y^2 \rangle^F$.

From definition, we have

$$\begin{aligned} B_m(G_c) &\equiv R \cap \gamma_{m+1}(F) / [R, {}_m F] \\ &= \gamma_{c+1}(F)S \cap \gamma_{m+1}(F) / [\gamma_{c+1}(F)S, {}_m F] \\ &= \gamma_{c+1}(F)S \cap \gamma_{m+1}(F) / \gamma_{c+1+m}(F)\, \rho_{m+1}(S), \end{aligned}$$

so that

$$B_m(G_c) = \begin{cases} \gamma_{c+1}(F)\ \rho_{m+1}(S) / \gamma_{c+1+m}(F)\, \rho_{m+1}(S), & \text{when } m < c \\ \\ \gamma_{m+1}(F) / \gamma_{m+1+c}(F)\, \rho_{m+1}(S), & \text{when } c \leq m. \end{cases}$$

We state and prove our main result,

Theorem. For $m \leq c-1$, $B_m(G_c)$ is cyclic of order 2^m whereas, for $m \geq c$, $B_m(G_c)$ is an abelian extension of a cyclic group of order 2^{c-1} by an elementary abelian 2-group of rank $r(m+1)$, the rank of the lower central factor $\gamma_{m+1}(F)/\gamma_{m+2}(F)$ of F. In particular, for $m \geq c$, $B_m(G_c) \cong C_{2^c} \oplus C_2 \oplus \ldots \oplus C_2$ ($r(m+1) - 1$ copies of C_2).

Proof. Case I $(0 \leq m \leq c-1)$.

Since

$$\begin{aligned} B_m(G_c) &= \gamma_{c+1}(F)\ \rho_{m+1}(S) / \gamma_{c+1+m}(F)\, \rho_{m+1}(S) \\ &= \textstyle\prod_{1 \leq i \leq m} \gamma_{c+i}(F)\ \rho_{m+1}(S) / \gamma_{c+1+i}(F)\, \rho_{m+1}(S), \end{aligned}$$

and since, by Lemma 2, each quotient

$$\gamma_{c+i}(F)\ \rho_{m+1}(S) / \gamma_{c+1+i}(F)\, \rho_{m+1}(S)$$

is cyclic of order 2 generated by the coset $[x, y, {}_{c+i-2}\,y]\ \gamma_{c+1+i}(F)\, \rho_{m+1}(S)$, the proof follows.

Case II $(1 \leq c \leq m)$.

In this case,

$$\begin{aligned} B_m(G_c) &= \gamma_{m+1}(F) / \gamma_{m+1+c}(F)\, \rho_{m+1}(S) \\ &= \gamma_{m+1}(F) / \gamma_{m+2}(F)\, \rho_{m+1}(S) \\ &\qquad \times \textstyle\prod_{1 \leq i \leq c-1} \gamma_{m+i+1}(F)\ \rho_{m+1}(S) / \gamma_{m+i+2}(F)\, \rho_{m+1}(S). \end{aligned}$$

Since,

$$\gamma_{m+1}(F) / \gamma_{m+2}(F)\, \rho_{m+1}(S) \cong C_2 \oplus \ldots \oplus C_2 \ (r(m+1) \text{ copies})$$

and each of

$$\gamma_{m+i+1}(F)\ \rho_{m+1}(S) / \gamma_{m+i+2}(F)\, \rho_{m+1}(S) \cong C_2 \ (\text{Lemma } 2),$$

the proof follows.

In particular,

$$B_m(G_c) \cong C_{2^c} \oplus C_2 \oplus \ldots \oplus C_2 \ (r(m+1) - 1 \text{ copies of } C_2).$$

Finally, since

$$\Gamma_{m+1}(G_c) = \gamma_{m+1}(F)/[\gamma_{c+1}(F)S,{}_mF] = \gamma_{m+1}(F)/\gamma_{m+1+c}(F)\,\rho_{m+1}(S),$$

we restate the second part of our theorem as,

Corollary.(cf. Rosset [5]) $\Gamma_{m+1}(G_c) \cong C_{2^c} \oplus C_2 \oplus \ldots \oplus C_2$
($r(m+1)-1$ copies of C_2).

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The Diophantine equation $cx^4+dy^4=z^p$

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1. Introduction

Let p be an odd prime, c and d be two square-free positive integers such that $(c,d)=1$, and let $h(-cd)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-cd})$.

In [1], Powell proved that the Diophantine equation

$$x^4 \pm y^4 = z^p$$

has no integral solutions x,y,z, if $(x,y)=1$, $p \nmid xyz$ and $p \not\equiv \pm 1 \pmod 8$ for the case $x^4+y^4=z^p$.

In [2], Terai and Osada proved that the Diophantine equation

$$x^4+dy^4=z^p,\ (x,y)=1,\ p\nmid xy$$

has no integral solutions x,y,z, if (a) $d \not\equiv 3 \pmod 4$, $p \not\equiv \pm 1 \pmod 8$, $p \nmid h(-d)$ and $2 \mid y$; or (b) $d \equiv 2 \pmod 8$, $p \equiv 3 \pmod 8$ or $p \equiv 5 \pmod{16}$, $p \nmid h(-d)$.

In the present paper we shall prove the following

Theorem 1. If $cd \not\equiv 3 \pmod 4$, $p \not\equiv \pm 1 \pmod 8$ and $p \nmid h(-cd)$, then the Diophantine equation

$$cx^4+dy^4=z^p,\ (x,y)=1 \tag{1}$$

has no integral solutions x,y,z with $p \nmid xy$.

Clearly, the results of Powell [1] for the equation $x^4+y^4=z^p$ and Terai and Osada [2] are some special cases of Theorem 1.

We also have

Theorem 2. If $cd \not\equiv 3 \pmod 8$, $p \not\equiv \pm c^{(p-1)/2} \pmod 8$ and $p \nmid h(-cd)$, then the Diophantine equation

$$cx^2+dy^4=z^p,\ (x,y)=1 \tag{2}$$

has no non-zero integral solutions x,y,z with $p \mid x$, $2 \mid y$.

Since $3^4+2\cdot 5^4=11^3$, hence the conditions $p\nmid xy$, or $p \mid x$, $2 \mid y$ in Theorems 1-2 are essential for the equations (1) and (2) to be without any integral solutions.

2. Lemma

In this section we prove the following

Lemma. Suppose $cd \not\equiv 3 \pmod 4$, or $cd \not\equiv 3 \pmod 8$ and $2 \nmid z$. If $p \nmid h(-cd)$, then every integer solution x, y, z of the equation

$$cx^2 + dy^2 = z^p, \quad (x, y) = 1 \tag{3}$$

can be expressed as $z = ca^2 + db^2$,

$$x\sqrt{c} + y\sqrt{-d} = (a\sqrt{c} + b\sqrt{-d})^p, \tag{4}$$

where a and b are rational integers.

Proof. The principal ideals $[cx + y\sqrt{-cd}]$ and $[cx - y\sqrt{-cd}]$ have the greatest common ideal divisor $[c, \sqrt{-cd}]$, because $[c] = [c, \sqrt{-cd}]^2$, $(x, y) = 1$, c, d are two square-free positive integers and $(c, d) = 1$. From (3) it then follows

$$[cx + y\sqrt{-cd}] = [c, \sqrt{-cd}] \cdot \mathcal{A}^p,$$

where $\mathcal{A}$ is an ideal of the field $\mathbb{Q}(\sqrt{-cd})$. Further we get

$$[cx + y\sqrt{-cd}]^2 = [c] \cdot \mathcal{B}^p, \tag{5}$$

where $\mathcal{B} = \mathcal{A}^2$. Since $p \nmid h(-cd)$, from (5) we see that $\mathcal{B}$ is a principal ideal of $\mathbb{Q}(\sqrt{-cd})$. Hence we obtain the ideal equation

$$[cx + v\sqrt{-cd}]^2 = [c] \cdot [(u + v\sqrt{-cd})/2]^p, \tag{6}$$

where u and v are rational integers, $u \equiv v \pmod 2$. Since $cd \neq 3$, from (6) we have

$$(cx + y\sqrt{-cd})^2 = c((u_1 + v_1\sqrt{-cd})/2)^p, \quad u_1 \equiv v_1 \pmod 2. \tag{7}$$

By means of (7) we derive

$$(u_1 + v_1\sqrt{-cd})/2 = ((a_1\sqrt{c} + b_1\sqrt{-d})/2)^2, \quad a_1 \equiv b_1 \pmod 2.$$

Inserting this expression in (7) we get

$$x\sqrt{c} + y\sqrt{-d} = ((a_1\sqrt{c} + b_1\sqrt{-d})/2)^p, \quad a_1 \equiv b_1 \pmod 2, \tag{8}$$

where $z = (ca_1^2 + db_1^2)/4$. If $a_1 \equiv b_1 \equiv 1 \pmod 2$, then

$$4z \equiv c + d \pmod 8,$$

which is impossible when $cd \not\equiv 3 \pmod 4$, or $cd \not\equiv 3 \pmod 8$ and $2 \nmid z$. Hence $a_1 \equiv b_1 \equiv 0 \pmod 2$, and put $a_1 = 2a$, $b_1 = 2b$. This yields (4) by (8). The lemma is proved.

3. Proof of Theorems

Proof of Theorem 1. From the lemma it follows that

$$x^2\sqrt{c}+y^2\sqrt{-d}=(a\sqrt{c}+b\sqrt{-d})^p.$$

Hence we have

$$x^2=a\sum_{j=0}^{(p-1)/2}\binom{p}{2j}a^{p-(2j+1)}c^{(p-1)/2-j}b^{2j}(-d)^j=aA, \qquad (9)$$

$$y^2=b\sum_{j=0}^{(p-1)/2}\binom{p}{2j+1}a^{p-(2j+1)}c^{(p-1)/2-j}b^{2j}(-d)^j=bB. \qquad (10)$$

It is easily seen that $(a, A)=1$ and $(b, B)=1$, since $p\nmid xy$. Hence from (9).(10) we obtain

$$a=\varepsilon_1 a'^2,\ A=\varepsilon_1 A'^2,\ b=\varepsilon_2 b'^2,\ B=\varepsilon_2 B'^2, \qquad (11)$$

where $\varepsilon_1, \varepsilon_2\in\{-1.1\}$. a'. A'. b' and B' are some non-zero integers.

We show that $2\mid xy$ as follows. Suppose $2\nmid xy$, from (9). (10) we have $2\nmid abAB$. Since

$$A\equiv\sum_{j=0}^{(p-1)/2}\binom{p}{2j}=2^{p-1}(\bmod 2)$$

is even when $2\nmid cd$. thus $2\mid cd$. Without loss of generality, we may assume that $2\nmid c$. $2\mid d$. Then it follows from (9).(10) and (11) that

$$A=\varepsilon_1 A'^2\equiv a^{p-1}c^{(p-1)/2}+\binom{p}{2}a^{p-3}c^{(p-3)/2}b^2(-d)+\binom{p}{4}a^{p-5}c^{(p-5)/2}\cdot b^4(-d)^2\ (\bmod 8), \qquad (12)$$

$$B=\varepsilon_2 B'^2\equiv\binom{p}{1}a^{p-1}c^{(p-1)/2}+\binom{p}{3}a^{p-3}c^{(p-3)/2}b^2(-d)+\binom{p}{5}a^{p-5}c^{(p-5)/2}\cdot b^4(-d)^2\ (\bmod 8). \qquad (13)$$

Suppose $p\equiv 3(\bmod 8)$. By(12), we easily see that

$$\varepsilon_1\equiv c-3d\ (\bmod 8). \qquad (14)$$

and by (13),

$$\varepsilon_2\equiv 3c-d\ (\bmod 8). \qquad (15)$$

From (14) and (15) it follows immediately that

$$3\varepsilon_1\equiv\varepsilon_2(\bmod 8),$$

which is impossible.

Suppose $p\equiv 5 \pmod 8$. From (13). we have

$$\varepsilon_2\equiv 5+4+4\equiv 5 \pmod 8).$$

which is a contradiction since $\varepsilon_2\in\{-1,1\}$. Hence $2|xy$. and without loss of generality. we may assume that $2\nmid x$. $2|y$.

It is easily seen from (9). (10) that $2|b$. $2\nmid aAB$. Therefore it follows from (11) that

$$\varepsilon_1 A'^2=A\equiv a^{p-1}c^{(p-1)/2} \pmod 8), \quad \varepsilon_2 B'^2=B\equiv pa^{p-1}c^{(p-1)/2} \pmod 8),$$

so $c^{(p-1)/2}\equiv\varepsilon_1 \pmod 8)$ and $p\equiv\varepsilon_2 c^{(p-1)/2} \pmod 8)$. i.e. $p\equiv\varepsilon_1\varepsilon_2=\pm 1 \pmod 8)$. This contradicts our assumption $p\not\equiv\pm 1 \pmod 8)$.

This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose that our assumptions are all satisfied. Then $2\nmid z$ since $2|y$. $(x,y)=1$ and c is a square-free. From the lemma and proof of Theorem 1, we have

$$x=aA\ , \quad y^2=bB. \tag{16}$$

where A.B are defined in (9) and (10). Since $p|x$. $2|y$ and $(x,y)=1$. we see that $2\nmid x$ and $(b,B)=1$. Hence from (16) we obtain.

$$b=\varepsilon b'^2. \quad B=\varepsilon B'^2, \tag{17}$$

where $\varepsilon\in\{-1,1\}$. b' and B' are two non-zero integers. Clearly. from (16) we have $2|b$. $2\nmid aB$ since $2|y$ and $2\nmid x$. Therefore (17) gives

$$\varepsilon B'^2=B\equiv pa^{p-1}c^{(p-1)/2} \pmod 8).$$

i.e. $p\equiv\pm c^{(p-1)/2} \pmod 8)$. This contradicts our assumption $p\not\equiv\pm c^{(p-1)/2}$ (mod8). which completes the proof of Theorem 2.

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