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ON THE FUNCTIONAL EQUATION $a[f(x+y)+f(x-y)-2f(x)-2f(y)] + b[f(xy)-f(x)f(y)] = 0$

Claus Hammer

Presented by J. Aczel, F.R.S.C.

Abstract. Let E be a unitary ring, divisible by two and three, F an integral domain and a, b two elements of F, different from zero. It is shown, that every nonconstant solution f: E -> F of the equation in the title is both multiplicative and quadratic. Multiplicativity

(1) $f(xy) = f(x)f(y)$

and the quadratic equation

(2) $f(x+y)+f(x-y) = 2f(x)+2f(y)$

are among the most important functional equations. Of course, every function satisfying (1) and (2), is a solution of the equation in the title, which is clearly a combination of the two.

Theorem. Every nonconstant solution f: E -> F of the functional equation

(3) $a[f(x+y)+f(x-y)-2f(x)-2f(y)] + b[f(xy)-f(x)f(y)] = 0$

($x, y \in E$) is multiplicative, hence it is also quadratic.

Proof. Let f be a nonconstant solution of (3). Setting in (3) $x = y = 0$, we obtain $f(0)[2a-b+bf(0)] = 0$. Supposing $f(0) \neq 0$, we would have $bf(0) = b-2a \neq 0$, because F has no divisors of zero and $b \neq 0$. With $y = 0$ in (3) we would get

$(b-2a)[f(x)-f(0)] = 0$. Hence f is a constant solution of (3). But this is a contradiction to our hypothesis. This proves $f(0) = 0$. Setting in (3) $y = 1$, we obtain

(4) $a[f(x+1)+f(x-1)] = (2a-b+bc)f(x)+2ac,$

where $c = f(1)$. With $x = 1$ in (4) and $d = 4a-b+bc$, we find

(5) $af(2) = cd.$

Now, multiplying (3) by a and using (5), $y = 2$ in (3) yields

(6) $a^2[f(x+2)+f(x+2)] = (2a^2+bcd)f(x)+2acd-abf(2x).$

Next we compute the left-hand side of (6), repeatedly making use of (4):

$$a^2[f(x+2)+f(x-2)] =$$

$$a^2[f((x+1)+1)+f((x+1)-1)+f((x-1)+1)+f((x-1)-1)-2f(x)] =$$

$$a(2a-b+bc)[f(x+1)+f(x-1)]+4a^2c-2a^2f(x) =$$

$$(2a^2+bcd-bd)f(x)+2acd.$$

Thus, comparing our result with (6), we obtain

(7) $af(2x) = df(x).$

By iteration, equation (7) yields

(8) $a^2f(4x) = adf(2x) = d^2f(x).$

Now we multiply (3) by a^2 and replace x, y by $2x$, $2y$, respectivly. Using (7) and (8), we derive

(9) $a^2d[f(x+y)+f(x-y)-2f(x)-2f(y)]+bd^2[f(xy)-f(x)f(y)] = 0.$

By (3), multiplied by ad, and (9) we conclude

(10) $d(d-a)[f(xy)-f(x)f(y)] = 0.$

Hence, if $d(d-a)$ is different from zero, equation (10) implies $f(xy) = f(x)f(y)$, that is, f is multiplicative. But supposing $d(d-a) = 0$, by (7) we would have $a(d-a)f(2x) = 0$. Now E is divisible by two, f is nonconstant and a is different from zero. So we obtain $a = d$. Thus, (7) changes to $f(2x) = f(x)$. Setting in (3) $x = y$, $x = 2y$, we get $-3af(y)+b[f(y^2)-f(y)^2] = 0$ and $a[f(3y)-3f(y)]+b[f(y^2)-f(y)^2] = 0$. Hence we obtain $f(3y) = 0$. This leads to a contradiction, because E is divisible by three and f is not a constant. This proves the theorem.

In the theorem we dealt only with nonconstant solutions of (3). However, it is easy to see, that the constant solutions of (3) are functions $f = c$ with an element $c \in F$, satisfying $c(2a-b+bc) = 0$. Thus as multiplicative constant solutions of (3) (where necessarily $c = 0$ or $c = 1$) we get $f = 0$, if char $F \neq 2$, $f = 0$ and $f = 1$, if char $F = 2$.

<u>Remark.</u> The theorem does not hold in its general form, if E is not divisible by two or is not divisible by three. We examine the case, where char $F \neq 2$ and $b = 2a$. Here equation (3) reduces to

(11) $f(x+y)+f(x-y)-2f(x)-2f(y)+2f(xy)-2f(x)f(y) = 0.$

Consider the following counterexamples:

1) $E = \mathbb{Z}$, the set of integers, which is neither divisible by two nor by three. Put $g: E \to F$, $g(x) = 0$, if x is even, and $g(x) = -1$, if x is odd.

2) $E = \{0,1\}$, the field with two elements, divisible by three, but not by two. Define $h: E \to F$ by $h(0) = 0$, $h(1) = -1$.

3) $E = \{0,1,2\}$, the field with three elements, divisible

by two, but not by three. Let char $F \neq 3$ and $2 \in F$ have an inverse in F. Define $k: E \to F$ by $k(0) = 0$, $k(1) = k(2) = -2^{-1}$.

It is easy to check, that g, h, k satisfy (11), but not (1).

In the case, where $E = F = R$ (the set of real numbers) the system (1), (2) of functional equations has been solved completely in [1]. Using the results of [1], we have the following

<u>Corollary.</u> The solutions $f: R \to R$ of (3) are given by

$$f(x) = 1-2ab^{-1} \text{ and } f(x) = [\mathrm{Re}\, w(x)]^2 + [\mathrm{Im}\, w(x)]^2,$$

where $w: C \to C$ is an arbitrary endomorphism of C (the set of complex numbers), that is, an arbitrary solution of

$$w(u+v) = w(u)+w(v) \text{ and } w(uv) = w(u)w(v) \quad (u,v \in C).$$

Notice that there exist noncontinous automorphisms $w: C \to C$ (cf. [2]).

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Tightness of Gaussian capacities on subspaces

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ABSTRACT: We consider the capacity associated with a classical Dirichlet form defined over a Hilbert space H equipped with a Gaussian measure μ. We give a sufficient condition for the tightness of this capacity on a subspace E with full μ measure.

1. Introduction.

Suppose, as in [6], that we have a classical Dirichlet form $\mathcal{E}$ over a real, separable Banach space E. This means that $\mathcal{E}$ is given by

$$\mathcal{E}(u,v) = \int_E \langle \nabla u, \nabla v \rangle_H d\mu \tag{1.1}$$

where μ is a finite measure on E, and H is a Hilbert space continuously embedded into E so that

$$E^* \hookrightarrow H^* \approx H \hookrightarrow E. \tag{1.2}$$

If u is of the form $u(z) = \phi(\ell_1(z), \ldots, \ell_k(z))$ with $\phi \in C_b^\infty(\mathbb{R}^k)$ and $\ell_1, \ldots, \ell_k \in E^* \backslash \{0\}$, then ∇u is defined to be the E^*-valued function $(\nabla u)(z) = \sum_{i=1}^k (\partial_i \phi)(\ell_1(z), \ldots, \ell_k(z))\ell_i$. (For more details and precise definitions we refer the reader to [4] or [6]).

In studying Dirichlet forms of the type (1.1) we find that while the space H is an intrinsic part of $\mathcal{E}$, there is no canonical choice for the space E which merely acts as a support for the measure μ. How then do we decide on a reasonable space E on which to model the form (1.1)? One important criterion is that $\mathcal{E}$-capacity should be tight on E, i.e. there should exist an increasing sequence K_n of compact subsets of E so that

$$\text{Cap}\,(E \backslash K_n) \longrightarrow 0. \tag{1.3}$$

In particular, this tightness is a necessary condition for the existence of an E-valued diffusion $\{X_t\}$ associated with $\mathcal{E}$ (see [1; Prop. 5.4]). In [4] we showed that if $\mathcal{E}$ is of the type (1.1) and if (1.2) holds, then Cap is tight on E. Thus, the capacity is tight on any separable Banach superspace E with full measure. One notable feature of this result is that it doesn't require anything of μ except that it is a finite measure.

What happens when (1.2) fails? Suppose that we have a Dirichlet form as in (1.1) where $\mu(H) = 1$. In contrast to superspaces, if E is a subspace of H such that $\mu(E) = 1$, it no longer follows automatically that Cap is tight on E (for a counterexample see [5; Prop. 2.3]).

In this situation the tightness question is more difficult and requires a more delicate analysis of the spaces E and H, and of the measure μ. In part 2 we give a tightness result for the case when μ is Gaussian and E is a Hilbert subspace of H. The unlabelled symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will always refer to the norm and inner product of H.

2. Tightness of capacity on subspaces.

We begin with a Dirichlet form $\mathcal{E}$ as in (1.1) and in addition assume the following.

$$\mu \text{ is a mean zero Gaussian measure on } H \tag{2.1}$$

$$E \text{ is a Hilbert subspace of } H \text{ such that } E \hookrightarrow H \text{ continuously and } \mu(E) = 1. \tag{2.2}$$

From (2.2) we conclude that there is an invertible, self-adjoint operator T on H so that

$$E = \mathcal{D}(T) \quad \text{and} \quad \langle z, w\rangle_E = \langle Tz, Tw\rangle_H. \tag{2.3}$$

Since $\mu(E) = 1$ and μ is Gaussian we have

$$\int \|Tz\|^2 \mu(dz) < \infty. \tag{2.4}$$

Proposition 1: If

$$\int \|\varphi(T)(z)\|^2 \mu(dz) < \infty, \tag{2.5}$$

where $\varphi(x) = x(\ln(x) \vee 1)^{1/2}$ for all $x \in \mathbb{R}$, then Cap is tight on E.

Proof: Let $\{E_\lambda : 0 \leq \lambda < \infty\}$ be the spectral decomposition of the self-adjoint operator T. For $j \geq 1$, define $P_j = E_j - E_{j-1}$, and for each j let $\{e_i^j\}_{i=1}^{n_j}$ be an orthonormal basis for $P_j H$. Here $1 \leq n_j = \dim(P_j H) \leq \infty$.

For $j \geq 1$ we set $\gamma_j^2 = 136(\ln(j^2) \vee 1) \int \|TP_j z\|^2 \mu(dz)$, and note that (2.5) implies $\sum \gamma_j^2 < \infty$.

For $n \geq 2$ define a continuous function u_n on H by

$$u_n(z) = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n \wedge n_j} (Tz, e_i^j)^2 \Big) \vee \gamma_j^2. \tag{2.6}$$

As $n \to \infty$, the sequence u_n increases pointwise and converges in $L^2(H; \mu)$ to the lower semicontinuous function

$$u(z) = \begin{cases} \sum_{j=1}^{\infty} \|TP_j z\|^2 \vee \gamma_j^2 & z \in E \\ \infty & z \in H \backslash E. \end{cases} \tag{2.7}$$

Now, for each n the function $u_n \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned} \mathcal{E}(u_n) &= \int \| \sum_{ij} 2(Tz, e_i^j) Te_i^j 1_{\{\sum_i (Tz, e_i^j)^2 \geq \gamma_j^2\}} \|^2 \, \mu(dz) \\ &\leq c \int \sum_j \| \sum_i (Tz, e_i^j) Te_i^j \|^2 1_{\{\|TP_j z\|^2 \geq \gamma_j^2\}} \, \mu(dz) \\ &= c \int \sum_j \|T^2 P_j z\|^2 1_{\{\|TP_j z\|^2 > \gamma_j^2\}} \, \mu(dz) \\ &\leq c \sum_j j^2 \int \|TP_j z\|^2 1_{\{\|TP_j z\|^2 > \gamma_j^2\}} \, \mu(dz) \\ &\leq c \sum_j j^2 \Big(\int \|TP_j z\|^4 \, \mu(dz)\Big)^{1/2} \big(\mu\{\|TP_j z\|^2 > \gamma_j^2\}\big)^{1/2} \\ &\leq c \sum_j j^2 \Big(\int \|TP_j z\|^2 \, \mu(dz)\Big) \, j^{-2} \\ &= c \int \|Tz\|^2 \, \mu(dz) < \infty. \end{aligned} \tag{2.8}$$

To obtain the final inequality above we used Fernique's results [2] on Gaussian seminorms (see appendix). This bound on $\mathcal{E}(u_n)$, along with the L^2 convergence, shows that $\{u_n\}$ is bounded in the $\mathcal{E}_1$ norm. By passing to a subsequence if necessary we may assume that the averages $N^{-1}\sum_{j=1}^{N} u_j$ converges strongly in $\mathcal{E}_1$ norm. These averages also converge pointwise to u and so we conclude that u is quasicontinuous and $u \in \mathcal{D}(\mathcal{E})$, with (2.8) giving a bound on $\mathcal{E}(u)$.

In particular [3; Lemma 3.1.5.] for $\lambda > 0$ we have

$$\text{Cap}\,(\|z\|_E^2 > \lambda) \leq \text{Cap}\,(u(z) > \lambda) \leq \mathcal{E}_1(u)/\lambda^2. \tag{2.9}$$

Now a similar argument shows that the function u^N defined by

$$u^N(z) = \sum_{j=1}^{N}\left[\left(\sum_{i=N+1}^{n_j}\langle Tz, e_i^j\rangle^2\right)\vee\gamma_j^2\right] + \sum_{j=N+1}^{\infty}\left[\left(\sum_{i=1}^{n_j}\langle Tz, e_i^j\rangle^2\right)\vee\gamma_j^2\right] \tag{2.10}$$

is quasicontinuous and belongs to $\mathcal{D}(\mathcal{E})$. By bounding $\mathcal{E}(u^N)$ as in (2.8), we can also show that $u^N \to 0$ in $\mathcal{E}_1$-norm. Thus by [3; Th. 3.1.4.] we can find a set F with $\text{Cap}\,(H\backslash F) < \epsilon$ and such that $u^N(z) \to 0$ uniformly on F. By (2.9) we can assume that F is a bounded set in E.

Since F is bounded, it is weakly metrizable with the metric $d_w(x,y) = \sum_{i,j} 2^{-(i+j)}|\langle T(x-y), e_i^j\rangle|$. Notice that $\|z\|_E^2 = \sum_{i,j}\langle Tz, e_i^j\rangle^2$ and so on F we have

$$\begin{aligned}\|x-y\|_E &\leq \Big(\sum_{i,j\leq N}\langle T(x-y), e_i^j\rangle^2\Big)^{1/2} + u^N(x-y)^{1/2}\\ &\leq C_N d_w(x,y) + u^N(x)^{1/2} + u^N(y)^{1/2}.\end{aligned} \tag{2.11}$$

For $\delta > 0$, choose N so large that $(u^N(x))^{1/2} \leq \delta/2$ for all $x \in F$, so

$$\|x-y\|_E \leq C_N d_w(x,y) + \delta. \tag{2.12}$$

Now F is bounded and therefore weakly totally bounded. Choose a finite set $\{x_1,\ldots,x_n\} \subseteq F$ so that

$$\left|\sup_{y\in F}\inf_{1\leq j\leq n} d_w(x_j,y)\right| \leq \delta/C_N. \tag{2.13}$$

Then $\sup_{y\in F}\inf_{1\le j\le n}\|x_j-y\|_E \le 2\delta$, so F is totally bounded and hence relatively compact in the strong topology. Setting $K=\overline{F}$ we conclude that K is a compact subset of E with Cap $(H\backslash K)\le\epsilon$. This establishes the required tightness.

3. Appendix: Inequalities for Gaussian Seminorms.

Lemma: Let μ be a mean zero Gaussian measure on H and suppose N is a finite seminorm on H. Then $E(N^2)<\infty$, and

$$t^2\ge 2E(N^2)\Rightarrow \mu(N>t)\le \exp(-t^2/68E(N^2)). \tag{3.1}$$

Also there is a universal constant c so that $E(N^4)\le c(E(N^2))^2$.

Proof: At first, suppose N has a continuous distribution function and let s be the median. This implies $E(N^2)\ge E(N^2;N>s)\ge s^2\mu(N>s)=s^2/2$. Now [2; Theorem 0.3.3.] says

$$t\ge s\Rightarrow \mu(N>t)\le\exp(-t^2/34s^2). \tag{3.2}$$

This shows that $E(N^2)<\infty$ and combines with the previous inequality to give (3.1). Theorem 0.3.3 of [2] also says that $E(\exp(N^2/84s^2))\le 2$ and since $x^2/2\le e^x$ for all $x\ge 0$ we get

$$E(N^4/2\cdot 84^2s^4)\le E(\exp(N^2/84s^2))\le 2. \tag{3.3}$$

This shows that $E(N^4)\le 4\cdot 84^2s^4\le 112{,}896\,(E(N^2))^2$. If N doesn't have a continuous distribution, consider the seminorm $N_\epsilon(t,x)=\epsilon|t|+N(x)$ on the space $\mathbb{R}\times H$. We equip $\mathbb{R}\times H$ with the product measure $\mu_z\times\mu$, where μ_z is the standard normal measure. Then N_ϵ has a continuous distribution and the conclusion of the lemma holds for N_ϵ. Letting $\epsilon\to 0$ will give us these results for N also.

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Limit Theorems for Extreme Order Statistics: Large Deviations and Asymptotic Expansions with Non-Uniform Estimates of Remainders

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Abstract : We construct asymptotic expansions with non-uniform remainder estimates taking into account large deviations in the limit theorem for the maximum of a sample of i.i.d. r.v.'s whose common d.f. has a right-hand tail of power type. Analogous expansions for the distribution of the minimum of the sample of i.i.d.r.v.'s belonging to the domain of attraction of the Weibull distribution are also obtained. The leading error term and asymptotics of large deviation probabilities for the maximum of the normal sample are derived.

Let $\{X_n, n \geq 1\}$ be i.i.d. random variables with common distribution function F; denote the corresponding order statistics by $X_n(n) \leq \ldots \leq X_1(n)$, and $X_1 + \ldots + X_n$ by S_n.

Gnedenko (1943) (cf. Theorem 4 threin) proved that if

$$(1) \qquad 1 - F(x) = c_{\alpha_1} \cdot x^{-\alpha_1} + o(x^{-\alpha_1})$$

as $x \to \infty$, where $c_{\alpha_1} > 0$ and $\alpha_1 > 0$ then $F_n(x) := P\{ X_1(n) \leq (c_{\alpha_1} \cdot n)^{1/\alpha_1} \cdot x \} \overset{d}{\to}$ $\Phi_{\alpha_1}(x) := \exp(-x^{-\alpha_1})$ as $n \to \infty$ uniformly in $x \in R^1_+$. It is also well known (cf. for instance Heyde (1968)) that if (1) and certain supplementary conditions are satisfied, then the following results on the asymptotic behavior of probabilities of large deviations are valid:

$$(2) \qquad \mathbb{P}(S_n > y) \sim n \cdot c_{\alpha_1} \cdot y^{-\alpha_1} \sim \mathbb{P}(X_1(n) > y)$$

as $n \to \infty$ for $y \geq L(n)$, where $L(n)$ is a positive monotone sequence that depends on certain parameters and increases to infinity as $n \to \infty$. Note that the second relationship in (2) is valid even as $n \to \infty$, $y/n^{1/\alpha_1} \to \infty$, if (1) is fulfiled, without any supplementary restriction (cf. for instance Section 2.3 of Resnick (1987)).

It seems reasonable to assume that if more precise information on the tail behavior of the function F is available (to compare with (1)) then more precise representations for $\mathbb{P}(S_n>y)$ and $\mathbb{P}(X_1(n)>y)$ as $n \to \infty$, that is, better than simple equivalence, can be derived. Various expansions for $\mathbb{P}(S_n>y)$ refining the first relationship in (2) have been constructed in a number of works by the author (cf. for instance Vinogradov (1992)) in the case in which the right-hand tail of F admits an expansion in negative powers of x:

$$(3) \qquad 1 - F(x) = \sum_{i=1}^{\ell} c_{\alpha_i} \cdot x^{-\alpha_i} + o(x^{-r})$$

as $x \to \infty$, where $c_{\alpha_1} > 0$, and $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \leq r$.

Here we construct asymptotic expansions for the distribution of $X_1(n)$ refining both the above mentioned result on weak convergence of F_n towards Φ_{α_1} obtained by Gnedenko (1943) and the second relationship in (2) in the case in which Condition (3) is valid. The corresponding error terms can be considered as unremovable errors generated by the lack of perfect information on the tail behavior of F. Hereafter we denote *different* functions from $\mathbb{R}^1_+ \otimes \mathbb{N}$ into $\mathbb{R}^1_+$ tending to zero as $n \to \infty$ by $\delta(x,n)$ and *different* positive constants by C.

<u>**Theorem 1.**</u> Let Condition (3) be fulfiled. Then

$$F_n(x) = \mathbb{P}(\, X_1(n) \leq (c_{\alpha_1} \cdot n)^{1/\alpha_1} \cdot x \,) = \left(1 - n^{-1} \cdot x^{-\alpha_1} \right)^n$$

$$\cdot \left(1 + \sum_{m=1}^{[r/\alpha_2] \vee [(r-\alpha_1)/(\alpha_2-\alpha_1)]} \frac{(-1)^m}{m!} \cdot \left(n \cdot \sum_{s=1}^{[r/\alpha_2]} \frac{1}{s} \cdot \left(\sum_{i=2}^{\ell} \frac{c_{\alpha_i}}{c_{\alpha_1}^{\alpha_i/\alpha_1}} \cdot n^{-\alpha_i/\alpha_1} \cdot x^{-\alpha_i} \right)^s \right.\right.$$

$$\cdot \sum_{k=0}^{[(r/m-s\cdot\alpha_2)/\alpha_1]\vee 0} (-1)^k\cdot\binom{-s}{k}\cdot n^{-k}\cdot x^{-k\cdot\alpha_1}\Bigg)^m\Bigg)\cdot\left(1+\delta(x,n)\cdot n^{-(r-\alpha_1)/\alpha_1}\cdot x^{-r/\alpha_1}\right),$$

where $\delta(x,n)\to 0$ as $n\to\infty$ uniformly on the rays $[C,+\infty)$; $a^{\vee}b$ stands for $\max(a,b)$; $\binom{-s}{0}:=1$, and $\binom{-s}{k}:=(-1)^k\cdot s\cdot(s+1)\cdot\ldots\cdot(s+k-1)/k!$ for integer $k\geq 1$.

<u>**Proof of Theorem 1.**</u> Obviously,

$$F_n(x)=F((c_{\alpha_1}\cdot n)^{1/\alpha_1}\cdot x)^n=\left(1-\sum_{i=1}^{\ell} c_{\alpha_i}\cdot c_{\alpha_1}^{-\alpha_i/\alpha_1}\cdot n^{-\alpha_i/\alpha_1}\cdot x^{-\alpha_i}+o(n^{-r/\alpha_1}\cdot x^{-r})\right)^n$$

(4)

$$=\exp\left\{n\cdot\log\left(1-\sum_{i=1}^{\ell} c_{\alpha_i}\cdot c_{\alpha_1}^{-\alpha_i/\alpha_1}\cdot n^{-\alpha_i/\alpha_1}\cdot x^{-\alpha_i}+o(n^{-r/\alpha_1}\cdot x^{-r})\right)\right\}$$

as $n\to\infty$, $x\cdot n^{1/\alpha_1}\to\infty$. Making transformations with the last expression and taking Taylor expansion of $\log(1-x)$ near zero we obtain that

$$F_n(x)=\exp\left\{n\cdot\log\left(1-n^{-1}\cdot x^{-\alpha_1}-\sum_{i=2}^{\ell} c_{\alpha_i}\cdot c_{\alpha_1}^{-\alpha_i/\alpha_1}\cdot n^{-\alpha_i/\alpha_1}\cdot x^{-\alpha_i}+o(n^{-r/\alpha_1}\cdot x^{-r})\right)\right\}$$

(5)

$$=\left(1-n^{-1}x^{-\alpha_1}\right)^n\cdot\exp\left\{n\cdot\log\left(1-\left(\sum_{i=2}^{\ell} c_{\alpha_i}\cdot c_{\alpha_1}^{-\alpha_i/\alpha_1}\cdot n^{-\alpha_i/\alpha_1}\cdot x^{-\alpha_i}+o(n^{-r/\alpha_1}\cdot x^{-r})\right)\right.\right.$$

$$\left.\left.\cdot\left(1-n^{-1}\cdot x^{-\alpha_1}\right)^{-1}\right)\right\}=\left(1-n^{-1}\cdot x^{-\alpha_1}\right)^n\cdot\exp\left\{-n\cdot\sum_{s=1}^{[r/\alpha_2]}\frac{1}{s}\cdot\left(\sum_{i=2}^{\ell}\frac{c_{\alpha_i}}{c_{\alpha_1}^{\alpha_i/\alpha_1}}\cdot n^{-\alpha_i/\alpha_1}\cdot x^{-\alpha_i}\right)^s\right.$$

$$\left.\cdot\left(1-n^{-1}\cdot x^{-\alpha_1}\right)^{-s}\right\}\cdot\left(1+\delta(x,n)\cdot n^{-(r-\alpha_1)/\alpha_1}\cdot x^{-r/\alpha_1}\right),$$

where $\delta(x,n)\to 0$ as $n\to\infty$ uniformly in $x\in R^1_+$ for which $x\cdot n^{1/\alpha_1}\to\infty$. Now, it follows from the Taylor expansion of the function e^x near zero that the second factor of the expression on the right-hand side of (5) can be represented as

$$\left\{1+\sum_{m=1}^{[r/\alpha_2]\vee[(r-\alpha_1)/(\alpha_2-\alpha_1)]}\frac{(-1)^m}{m!}\cdot\left(n\cdot\sum_{s=1}^{[r/\alpha_2]}\frac{1}{s}\cdot\left(\sum_{i=2}^{\ell}\frac{c_{\alpha_i}}{c_{\alpha_1}^{\alpha_i/\alpha_1}}\cdot n^{-\alpha_i/\alpha_1}\cdot x^{-\alpha_i}\right)^s\right.\right.$$

(6)

$$\cdot\left(1-n^{-1}\cdot x^{-\alpha_1}\right)^{-s}\Bigg]^{m}\Bigg\}\cdot\left(1+\delta(x,n)\cdot n^{-(r-\alpha_1)/\alpha_1}\cdot x^{-r/\alpha_1}\right)$$

where $\delta(x,n)\to 0$ as $n\to\infty$ uniformly on the rays $[C,+\infty)$. It remains to note that

$$\left(1-n^{-1}\cdot x^{-\alpha_1}\right)^{-s}=\sum_{k=0}^{[(r/m-s\cdot\alpha_2)/\alpha_1]\vee 0}(-1)^k\cdot\binom{-s}{k}\cdot n^{-k}\cdot x^{-k\cdot\alpha_1}$$

$$+R(x,n)\cdot\left(n^{-1}\cdot x^{-\alpha_1}\right)^{1+0\vee[(r/m-s\alpha_2)/\alpha_1]},$$

where $|R(x,n)|$ is uniformly bounded on the rays $[C,+\infty)$ for sufficiently large n. Together with (4)-(6) it yields the assertion of Theorem 1.□

Example (compare with Example 1 in Sec.6 of Smith (1982) and Exercise 2.4.2 of Resnick (1987)). Let (3) be fulfiled with $\ell = 2$, $\alpha_2 < 2\cdot\alpha_1$, and $r = \alpha_2$. Then

$$F_n(x)=\left(1-n^{-1}x^{-\alpha_1}\right)^n\cdot\left(1-\frac{c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2/\alpha_1}}\cdot n^{-(\alpha_2-\alpha_1)/\alpha_1}\cdot x^{-\alpha_2}\right)\cdot\left(1+\delta(x,n)\cdot n^{-(\alpha_2-\alpha_1)/\alpha_1}\cdot x^{-\alpha_2}\right),$$

where $\delta(x,n)\to 0$ as $n\to\infty$ uniformly on the rays $[C,+\infty)$ $(C>0)$.

Remark 1. Note that for any common distribution function F of i.i.d. random variables $\{X_n, n\ge 1\}$ the following representation for the probabilities of large deviations of $X_1(n)$ is valid, without the additional assumption that the distribution function of properly centered and normalized maximum $X_1(n)$ belongs to the domain of attraction of one of three limiting distributions:

(7) $$P(\,X_1(n) > y\,) = n\cdot(1-F(y)) + O((n\cdot(1-F(y))^2)$$

as $n\to\infty$, $y\to\infty$ such that $n\cdot(1-F(y))\to 0$.

Remark 2. For the case when Condition (3) is valid it follows from (7) that

(8) $$P\left\{X_1(n) > (c_{\alpha_1}\cdot n)^{1/\alpha_1}\cdot x\right\}=\sum_{\substack{1\le i\le \ell:\\ \alpha_i<2\alpha_1}}\frac{c_{\alpha_i}}{c_{\alpha_1}^{\alpha_i/\alpha_1}}\cdot n^{-(\alpha_i-\alpha_1)/\alpha_1}\cdot x^{-\alpha_i}+O\left(x^{-2\cdot\alpha_1}\right)$$

as $n\to\infty$, $x\to\infty$. It is easily seen that the expansion of our Theorem 1 is more precise (to compare with (8)) at least in the range of deviations in which x grows to infinity slower than any power of n.

Theorem 2. Let $\{X_n, n \geq 1\}$ have common standard normal distribution. Set $\Lambda(x) := \exp(-e^{-x})$; $a_n := (2\cdot\log n)^{1/2} - (\log\log n + \log(4\pi))/(8\log n)^{1/2} - [(\log\log n + \log(4\pi))^2 - 4(\log\log n + \log(4\pi)]/(8\cdot(2\log n)^{3/2})$; $b_n := a_n^{-1}$. Then

i) $P(X_n \leq a_n + b_n x) - \Lambda(x) = \Lambda(x)\cdot e^{-x}\cdot \dfrac{x^2/2+x+1}{2\cdot\log n} + o(1/\log n)$ as $n\to\infty$ uniformly in $x\in\mathbb{R}^1$;

ii) $P\left\{X_1(n) > (2\cdot\log n)^{1/2} - \frac{1}{2}\cdot(\log\log n + \log(4\pi))\cdot(2\cdot\log n)^{-1/2} + x\cdot(2\cdot\log n)^{-1/2}\right\}$

$\sim (1-\Lambda(x))\cdot Q(x,n)$ as $n\to\infty$, $x\to\infty$, where $Q(x,n) = 1$ for $x = o((\log n)^{1/2})$,

$Q(x,n) = \exp\left\{-\dfrac{(x-(\log\log n + \log(4\pi))/2)^2}{4\cdot\log n}\right\}$ for $x \geq C\cdot(\log n)^{1/2}$, $x = o(\log n)$,

and $Q(x,n) = \exp\left\{-\dfrac{(x-(\log\log n + \log(4\pi))/2)^2}{4\cdot\log n}\right\}\cdot\left(1+\dfrac{x-(\log\log n + \log(4\pi))/2}{2\cdot\log n}\right)^{-1}$

for $x \geq C\cdot\log n$.

Proof of Theorem 2. (i) is obtained by a slight refinement of the arguments by Hall (1979). (ii) Note that for $x = o((\log n)^{1/2})$ it follows from Prop. (2.10) of Resnick (1987). Set $y := (2\log n)^{1/2} + (x - \frac{1}{2}\cdot(\log\log n + \log(4\pi))/(2\log n)^{1/2}$. Then $n\cdot(1-\Phi(y)) \sim n\cdot\exp(-y^2/2)/((2\cdot\pi)^{1/2}\cdot y)$ as $y\to\infty$; the last expression is equal to $\exp\left\{-x-\dfrac{(x-(\log\log n + \log(4\pi))/2)^2}{4\cdot\log n}\right\}\cdot\left(1+\dfrac{x-(\log\log n + \log(4\pi))/2}{2\cdot\log n}\right)^{-1}$.

All that remains is to note that $\exp(-x) \sim 1-\Lambda(x)$ as $x\to\infty$. □

It appears that the technique applied in the proof of Theorem 1 can also be used in order to construct asymptotic expansions for the distribution of the minimum $X_n(n)$ under the fulfilment of the following condition:

(9) $$F(x) = \sum_{l=1}^{\ell} c_{\alpha_l}\cdot x^{\alpha_l} + o(x^r)$$

as $x\to 0$, where $c_{\alpha_1} > 0$, and $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \leq r$. Recall that in this case $P\left\{(c_{\alpha_1}\cdot n)^{1/\alpha_1}\cdot X_n(n) \leq x\right\}$ converges weakly towards the Weibull distribution with index α, $\Psi_\alpha(x) := 1-\exp(-x^\alpha)$ as $n\to\infty$ (cf. for instance Gnedenko (1943)).

Theorem 3. Let Condition (9) be valid. Then

$$P\left\{(c_{\alpha_1}\cdot n)^{1/\alpha_1}\cdot X_n(n) \leq x\right\} = 1 - \left(1 - x^{\alpha_1}/n\right)^n$$

$$\cdot\left(1+\sum_{m=1}^{[r/\alpha_2]\vee[(r-\alpha_1)/(\alpha_2-\alpha_1)]}\frac{(-1)^m}{m!}\cdot\left(n\cdot\sum_{s=1}^{[r/\alpha_2]}\frac{1}{s}\cdot\left(\sum_{i=2}^{\ell}\frac{c_{\alpha_i}}{c_{\alpha_1}^{\alpha_i/\alpha_1}}\cdot x^{\alpha_i}\cdot n^{-\alpha_i/\alpha_1}\right)^s\right.\right.$$

$$\left.\left.\cdot\sum_{k=0}^{[(r/m-s\cdot\alpha_2)/\alpha_1]\vee 0}(-1)^k\cdot\binom{-s}{k}\cdot x^{k\cdot\alpha_1}\cdot n^{-k}\right)^m\right)\cdot\left(1+\delta(x,n)\cdot x^{r/\alpha_1}\cdot n^{-(r-\alpha_1)/\alpha_1}\right),$$

where $\delta(x,n)\to 0$ as $n\to\infty$ uniformly on the intervals $[0,C]$.

Proof of Theorem 3 is the same as that of Theorem 1 and therefore is omitted.□

Remark 3. The representation for the distribution of the maximum $X_1(n)$ analogous to those of Theorems 1 and 3 when $1-F(x)=\sum_{i=1}^{\ell}c_{\alpha_i}\cdot(a-x)^{\alpha_i}+o((a-x)^r)$ as $x\to a$, where $a<\infty$, $c_{\alpha_1}>0$, and $0<\alpha_1<\alpha_2<\ldots<\alpha_\ell\le r$ is also valid.

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EXACT SOLUTIONS FOR UNSTEADY CONVECTION-DIFFUSION PROBLEMS

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Presented by K. Ranger, F.R.S.C.

ABSTRACT. New exact solutions for an unsteady diffusion equation with convection are obtained in the cases of a continuous line source and a point source of heat, respectively. The heat flux from the sources is a given function of time. Results of numerical computation are presented.

RÉSUMÉ. On présente de nouvelles solutions du problème non stationnaire de diffusion-convection pour les sources calorifiques linéaire et ponctuelle avec écoulement calorique fonction du temps. On présente aussi des résultats numériques.

Subject-classification: AMS(MOS): 35K05, 65R10.
Keywords: heat source, Laplace transform

1. Introduction. Many exact solutions to the heat equation without convection are known for different regions [1, 2]. However, the presence of a convective term like $\mathbf{v}\,\mathrm{grad}\,T$, where $\mathbf{v}$ is the flow velocity and T is the temperature of the medium or the concentration distribution from a source of soluble mass [3, 4], makes it much more difficult to obtain an analytical solution to a convection-diffusion equation. Solutions to convection-diffusion problems are important in ecological studies because they can be used to predict contaminant dispersion in tidal waters or to investigate the spread of a cloud of contaminant [5]. If $\mathbf{v}$ is a function of space and time, then a nonlinear system of partial differential equations is to be solved for the unknowns $\mathbf{v}$ and T; but, if $\mathbf{v} = \mathrm{const}$, the heat equation is linear.

Two exact solutions to an unsteady heat equation in an uniform stream were recently obtained [6] for the cases of a sudden constant source and a periodic source of heat, respectively.

In the present paper, the results of [6] are generalized to the following two heat sources: (a) a continuous line source and (b) an unsteady point source. Moreover, the temperature

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field in an uniform stream for a sudden heat source is expressed in a new form that is more convenient for computation than the one given in [6].

2. The continuous line source. The basic equation which describes the temperature, T, of a medium in an uniform stream is

$$\frac{\partial T}{\partial t'} + U\frac{\partial T}{\partial x'} = \kappa(T_{x'x'} + T_{y'y'} + T_{z'z'}), \tag{1}$$

where U is the constant stream velocity and κ is diffusivity.

By introducing the dimensionless variables

$$x = \frac{U}{2\kappa}x', \quad y = \frac{U}{2\kappa}y', \quad z = \frac{U}{2\kappa}z', \quad t = \frac{t'}{t_0}, \tag{2}$$

where t_0 is the characteristic time scale, (1) becomes

$$\left(\frac{2\kappa}{U^2 t_0}\right)\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{1}{2}\left(T_{xx} + T_{yy} + T_{zz}\right). \tag{3}$$

If we substitute

$$T(x,y,z,t) = Ce^{x}\varphi(x,y,z,t) \tag{4}$$

into (3), where the constant C and the function φ have to be determined, and if we let $t_0 = 2\kappa/U^2$, we have

$$\varphi_t + \varphi = \Delta\varphi. \tag{5}$$

In the case of a continuous line source of heat (see [1], p. 261) the heat flux $q(t)$ from the source is given by

$$q(t) = \lim_{r'\to 0}\left(-2\pi r'\lambda\frac{\partial T}{\partial r'}\right), \tag{6}$$

where (r', θ, z) are cylindrical polar coordinates and λ is conductivity. By using (2) and (4), (6) may be written as follows:

$$\lim_{r\to 0}\left(r\frac{\partial\varphi}{\partial r}\right) = -q(t), \tag{7}$$

where $r = Ur'/2\kappa$ and the constant C in (4) is chosen so that

$$\frac{U}{4\pi\kappa\lambda C} = 1.$$

Because the solution to (5) has to be bounded at infinity:

$$\lim_{r\to\infty} e^{r\cos\theta}\varphi = 0, \tag{8}$$

it follows from the boundary and initial conditions that φ is a function of r and t only (we assume that $\varphi|_{t=0} = 0$). Hence,

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}.$$

Applying the Laplace transform to (5) we have

$$(p+1)\bar{\varphi} = \frac{d^2\bar{\varphi}}{dr^2} + \frac{1}{r}\frac{d\bar{\varphi}}{dr}, \tag{9}$$

which is the modified Bessel equation of order zero and whose solution satisfying (8) has the form

$$\bar{\varphi}(r,p) = A(p)K_0\left(\sqrt{p+1}\,r\right), \tag{10}$$

where $A(p)$ is an arbitrary constant.

Now taking the Laplace transform of (7) and using (10) we obtain

$$\bar{\varphi}(r,p) = \bar{q}(p)K_0\left(\sqrt{p+1}\,r\right), \tag{11}$$

where $\bar{q}(p)$ is the Laplace transform of $q(t)$. The inverse Laplace transform, $\mathcal{L}^{-1}\{\bar{\varphi}(r,p)\}$, of (11) can be found by means of the convolution theorem and the following formula (see [7], p. 345, formula (2.10.125)):

$$\mathcal{L}^{-1}\left\{K_0\left(\sqrt{p+1}\,r\right)\right\} = \frac{e^{-t-r^2/4t}}{2t}.$$

Hence

$$\varphi(r,t) = \mathcal{L}^{-1}\{\bar{\varphi}(r,p)\} = \int_0^t q(t-\tau)\frac{e^{-\tau-r^2/4\tau}}{2\tau}\,d\tau. \tag{12}$$

In general, integral (12) cannot be evaluated in a closed form even in the simple case where $q(t) = q_0 = \text{const}$. In this simple case, if we introduce the new variable u by $\tau = r\exp(u)/2$ and use the following integral representation of the modified Bessel function of order zero,

$$K_0(x) = \int_0^\infty e^{-x\cosh u}\,du,$$

then the right-hand side of (12) becomes

$$\varphi(r,t) = \frac{q_0}{2}\left[K_0(r) + \int_0^{\ln\frac{2t}{r}} e^{-r\cosh u}\,du\right]. \tag{13}$$

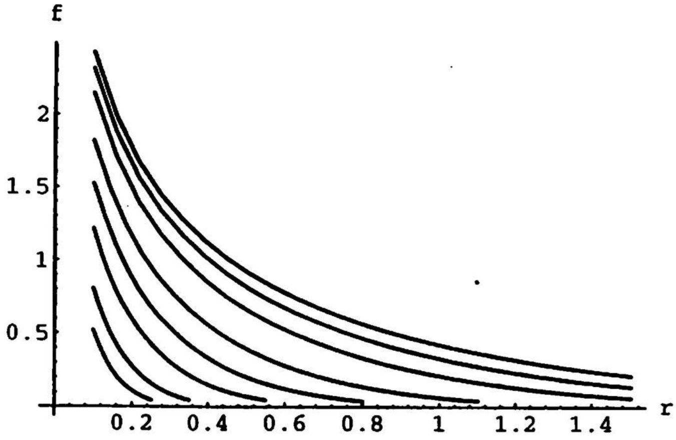

FIGURE 1. The dependence of $f(r,t)$ upon the radial distance, r, from the source for different values of time (from top to bottom: $t = \infty$; 1; 0.5; 0.2; 0.1; 0.05; 0.02; 0.01).

Computational results obtained by using (13) are shown in Fig. 1 for the function

$$f(r,t) = \frac{\varphi(r,t)}{q_0}.$$

It is to be noted that solution (13) tends to the finite limit $\varphi(r,\infty) = q_0 K_0(r)$ as $t \to \infty$, in contrast with the case $\mathbf{v}\,\mathrm{grad}\,T = 0$ where $T \to \infty$ as $t \to \infty$. It is also to be noted that the continuous line source model is not only a pure mathematical idealization; it is used, for example, in the transient hot-wire method for determining the thermal conductivity of fluids and gases [8].

3. A point source of heat. As mentioned in the introduction, two exact solutions are found in [6] for the cases of a periodic source and a sudden constant source of heat, respectively. In the case of a point heat source, equation (5) does not change, but $\Delta\varphi$ is written in spherical coordinates (ρ,θ,ψ). It is shown in [6] that φ is a function of ρ and t only. Condition (7) is replaced with

$$\lim_{\rho\to 0}\left(\rho^2\frac{\partial\varphi}{\partial\rho}\right) = -q(t) \tag{14}$$

(in this case $C = U/8\pi\kappa\lambda$). Condition (8) does not change (provided r is replaced with ρ). The Laplace transform of the solution is

$$\bar{\varphi}(\rho,p) = \frac{\bar{q}(p)}{\rho}e^{-\sqrt{p+1}\,\rho}, \tag{15}$$

and its inverse Laplace transform, which may be found by means of the convolution theorem, is

$$\varphi(\rho,t)=\frac{1}{2\sqrt{\pi}}\int_0^t e^{-\tau-\rho^2/4\tau}\frac{q(t-\tau)}{\tau^{3/2}}\,d\tau. \tag{16}$$

In particular, if $q(t)=q_0=\text{const}$, (16) becomes

$$\varphi(\rho,t)=\frac{q_0}{2\sqrt{\pi}}\int_0^t \frac{e^{-\tau-\rho^2/4\tau}}{\tau^{3/2}}\,d\tau. \tag{17}$$

This integral, which was found in [6], cannot be evaluated in a closed form but it can be transformed into a form which is more useful for numerical computations by means of the following integral (see [9]):

$$\int_0^t \frac{e^{-\alpha^2\tau-\beta^2/\tau}}{\tau^{3/2}}\,d\tau=\frac{\sqrt{\pi}}{2\beta}\left[e^{2\alpha\beta}\operatorname{erfc}\left(\alpha\sqrt{t}+\frac{\beta}{\sqrt{t}}\right)+e^{-2\alpha\beta}\operatorname{erfc}\left(\frac{\beta}{\sqrt{t}}-\alpha\sqrt{t}\right)\right], \tag{18}$$

where

$$\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}\,du$$

is the complementary error function. For completeness, to derive (18) it suffices to rewrite the right-hand side in the form

$$\begin{aligned}\int_0^t \frac{e^{-\alpha^2\tau-\frac{\beta^2}{\tau}}}{\tau^{3/2}}\,d\tau&=\frac{1}{\beta}\int_0^t e^{-(\alpha\sqrt{\tau}-\beta/\sqrt{\tau})^2-2\alpha\beta}\,d\left(\alpha\sqrt{\tau}-\frac{\beta}{\sqrt{\tau}}\right)\\&\quad-\frac{1}{\beta}\int_0^t e^{-(\alpha\sqrt{\tau}+\beta/\sqrt{\tau})^2+2\alpha\beta}\,d\left(\alpha\sqrt{\tau}+\frac{\beta}{\sqrt{\tau}}\right),\end{aligned} \tag{19}$$

and to let $z=\beta/\sqrt{\tau}-\alpha\sqrt{\tau}$ and $z=\beta/\sqrt{\tau}+\alpha\sqrt{\tau}$ in the second last and last integrals of (19), respectively. Similarly, it can be shown that

$$\int_0^t \frac{e^{-\alpha^2\tau-\beta^2/\tau}}{\tau^{1/2}}\,d\tau=\frac{\sqrt{\pi}}{2\alpha}\left[e^{-2\alpha\beta}\operatorname{erfc}\left(\frac{\beta}{\sqrt{t}}-\alpha\sqrt{t}\right)-e^{2\alpha\beta}\operatorname{erfc}\left(\frac{\beta}{\sqrt{t}}+\alpha\sqrt{t}\right)\right]. \tag{20}$$

By using (18), solution (17) can be represented in the following useful form:

$$\varphi(\rho,t)=\frac{q_0}{2\rho}\left[e^{\rho}\operatorname{erfc}\left(\sqrt{t}+\frac{\rho}{2\sqrt{t}}\right)+e^{-\rho}\operatorname{erfc}\left(\frac{\rho}{2\sqrt{t}}-\sqrt{t}\right)\right]. \tag{21}$$

Finally, using (18) and (20), we can express solution (16) in terms of the complementary error function in some cases where $q(t)$ is a function of time. We consider two examples.

(a) $q(t) = q_0 t$. In this case integral (16) is a linear combination of (18) and (20) so that

$$\varphi(\rho, t) = \frac{q_0}{2}\left[\left(\frac{t}{\rho} + \frac{1}{2}\right) e^{\rho} \operatorname{erfc}\left(\frac{\rho}{2\sqrt{t}} + \sqrt{t}\right) + \left(\frac{t}{\rho} - \frac{1}{2}\right) e^{-\rho} \operatorname{erfc}\left(\frac{\rho}{2\sqrt{t}} - \sqrt{t}\right)\right]. \tag{22}$$

(b) $q(t) = q_0 e^{-\alpha t}$, $0 < \alpha < 1$. In this case, it follows from (16) and (18) that

$$\varphi(\rho, t) = \frac{q_0}{2\rho} e^{-\alpha t}\left\{e^{\sqrt{1-\alpha}\rho} \operatorname{erfc}\left[\sqrt{(1-\alpha)t} + \frac{\rho}{2\sqrt{t}}\right] + e^{-\sqrt{1-\alpha}\rho} \operatorname{erfc}\left[\frac{\rho}{2\sqrt{t}} - \sqrt{(1-\alpha)t}\right]\right\}. \tag{23}$$

Formulae (22) and (23) give two simple solutions for the case of an unsteady point source.

4. Conclusion. In the present paper the exact solution to an unsteady heat equation with a convective term is found in the case of the continuous line source of heat $q(t)$, which is an arbitrary function of time. The results of [6] are generalized to the case of nonperiodic $q = q(t)$. The solution is presented in a computationally useful form in terms of the complementary error function.

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Homomorphisms compatible with covering maps of the two-torus

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Abstract. In this note we show that any two homomorphisms between matrix algebras over the continuous functions on the two dimensional torus which are compatible with the same covering map $\mathbb{T}^2 \to \mathbb{T}^2$ are inner equivalent. Some immediate consequences for C*-algebra inductive limits are given.

1. Introduction. The homomorphisms compatible* with a covering map have been introduced in [4], where it was shown that certain C*-algebra inductive limits $\varinjlim (C(\mathbb{T}^2, M_{n_k}), \Phi_k)$, with Φ_k compatible with a certain n_{k+1}/n_k-fold covering map $\phi_k : \mathbb{T}^2 \to \mathbb{T}^2$, are isomorphic with the tensor product of two Bunce-Deddens algebras [1]. The main step in that proof was the fact that if $\Phi_k, \Phi'_k : C(\mathbb{T}^2, M_{n_k}) \to C(\mathbb{T}^2, M_{n_{k+1}})$ are compatible with ϕ_k, then they are inner equivalent ([4, Lemma 3.2]), i.e. $\Phi_k(\cdot) = u\Phi'_k(\cdot)u^*$ for some unitary $u \in C(\mathbb{T}^2, M_{n_{k+1}})$. The above mentioned C*-algebra inductive limits were then completely classified modulo isomorphisms or stable isomorphisms in [5] in terms of K-theoretical invariants, using again the above unicity result, [4, Lemma 3.2].

Let $\Phi, \Psi : C(\mathbb{T}^2, M_n) \to C(\mathbb{T}^2, M_{npqr})$ be homomorphisms compatible with $\mathbb{T}^2 \ni (u,v) \overset{\phi}{\mapsto} (u^p, v^q) \in \mathbb{T}^2$, i.e. $\Phi(f \circ \phi \otimes 1_n) = \Psi(f \circ \phi \otimes 1_n) = f \otimes 1_{npqr}$, $f \in C(\mathbb{T}^2)$ (see [4]). The main result of this paper asserts that Φ and Ψ are inner equivalent. The proof given here uses winding numbers (and extends the corresponding result in [4]). The theorem proved in this note is one of the tools used by G. Gong and G. A. Elliott in [3] to show that certain inductive limits $\varinjlim C(\mathbb{T}^2, F_n)$, with the F_n 's finite dimensional C*-algebras, can be written as $\varinjlim C(\mathbb{T}, G_n)$, with the G_n 's finite dimensional C*-algebras.

2. Results. We shall prove the following:

Theorem. *If $\Phi, \Psi : C(\mathrm{T}^2, M_n) \to C(\mathrm{T}^2, M_{npqr})$ are homomorphisms compatible with the covering map $\mathrm{T}^2 \ni (u,v) \mapsto (u^p, v^q) \in \mathrm{T}^2$, where $p, q \in \mathrm{N}$, then they are inner equivalent.*

Proof: The proof is mainly inspired from [4, the proof of Lemma 3.2]. For any $1 \le k \le p$ and $1 \le \ell \le q$, we define the map

$$[0,2\pi]^2 \ni (t_1,t_2) \overset{z_{k,\ell}}{\mapsto} \left(\exp\left(i \cdot \frac{t_1 + (k-1)2\pi}{p}\right), \exp\left(i \cdot \frac{t_2 + (\ell-1)2\pi}{q}\right)\right) \in \mathrm{T}^2 .$$

It will be convenient to assume that the set of indices (k,ℓ) is endowed with the lexicographical order. Since Φ is compatible with $\mathrm{T}^2 \ni (u,v) \mapsto (u^p, v^q) \in \mathrm{T}^2$, it follows that we can define a continuous map $\theta : [0,2\pi]^2 \to \mathrm{Hom}\left(\bigoplus_1^{pqr} M_n, M_{npqr}\right)$ by:

$$\theta(t_1,t_2)\Big(\bigoplus_{(k,\ell)} f\big(z_{k,\ell}(t_1,t_2)\big)\Big) := \Phi(f)\big(\exp(it_1), \exp(it_2)\big)$$

for $t_1, t_2 \in [0,2\pi]$, $f \in C(\mathrm{T}^2, M_n)$. But θ defines in a canonical way a homomorphism $\bigoplus_1^{pqr} M_n \to C([0,2\pi]^2, M_{npqr})$, also denoted by θ. Since $[0,2\pi]^2$ is contractible, working e.g. as in [6] or as in [2], it follows that there are integers $s(k,\ell)$ $(1 \le k \le p,\ 1 \le \ell \le q)$ and a unitary u_Φ in $C([0,2\pi]^2, M_{npqr})$ such that:

$$\text{(1)} \quad \Phi(f)\big(\exp(it_1), \exp(it_2)\big) = u_\Phi(t_1,t_2)\Big(\bigoplus_{(k,\ell)} \Big(f\big(z_{k,\ell}(t_1,t_2)\big) \otimes 1_{s(k,\ell)}\Big)\Big) \cdot u_\Phi(t_1,t_2)^*$$

for $f \in C(\mathrm{T}^2, M_n)$, $t_1, t_2 \in [0,2\pi]$. Since for any $f \in C(\mathrm{T}^2, M_n)$ the map $\mathrm{T}^2 \ni (u,v) \mapsto \mathrm{tr}\,(\Phi(f)(u,v)) \in \mathrm{C}$ is continuous (here tr denotes the usual trace on M_{npqr}), it follows using also (1) that $s(k,\ell) = r$ for any k and ℓ. In a similar way we obtain that there is a unitary $u_\Psi \in C([0,2\pi]^2, M_{npqr})$ such that:

$$\Psi(f)\big(\exp(it_1), \exp(it_2)\big) = u_\Psi(t_1,t_2)\Big(\bigoplus_{(k,\ell)} \Big(f\big(z_{k,\ell}(t_1,t_2)\big) \otimes 1_r\Big)\Big) u_\Psi(t_1,t_2)^*$$

for $f \in C(\mathrm{T}^2, M_n)$, $t_1, t_2 \in [0,2\pi]$.

For every scalar valued path g avoiding 0, denote by $A(g)$ the angle swept by g with respect to the point 0. If in addition g is closed, we have $A(g) = 2\pi W(g)$ (where $W(g)$ is the winding number of g).

It will be enough to find maps $X_k, Y_k \in C([0,2\pi], U(r))$, $1 \leq k \leq pq$, such that:

(2) $X_k(0) = Y_k(0), \qquad 1 \leq k \leq pq,$

(3) $X_k(2\pi) = b'_{\lambda_1(k)}(0)\, Y_{\lambda_1(k)}(0)\big(b''_{\lambda_1(k)}(0)\big)^*, \qquad 1 \leq k \leq pq,$

(4) $Y_k(2\pi) = c'_{\lambda_2(k)}(0)\, X_{\lambda_2(k)}(0)\, \big(c''_{\lambda_2(k)}(0)\big)^*, \qquad 1 \leq k \leq pq,$

(5) $A\Big(\det\big(\frac{X_k b_{\lambda_1(k)} Y_{\lambda_1(k)}}{c_{\lambda_2(k)} X_{\lambda_2(k)} Y_k}\big)\Big) = 0, \qquad 1 \leq k \leq pq.$

Here the maps $b'_k, b''_k, c'_k, c''_k \in C([0,2\pi], U(r))$ are those given by the necessary relations:

$$u_\Phi(2\pi,\cdot)^* u_\Phi(0,\cdot) = \big(1_{C([0,2\pi],M_{nr})} \otimes S_1\big)\Big(\bigoplus_{k=1}^{pq}(1_n \otimes b'_k)\Big)$$

$$u_\Phi(\cdot,2\pi)^* u_\Phi(\cdot,0) = \big(1_{C([0,2\pi],M_{nr})} \otimes S_2\big)\Big(\bigoplus_{k=1}^{pq}\Big)(1_n \otimes c'_k))$$

$$u_\Psi(2\pi,\cdot)^* u_\Psi(0,\cdot) = \big(1_{C([0,2\pi],M_{nr})} \otimes S_1\big)\Big(\bigoplus_{k=1}^{pq}(1_n \otimes b''_k)\Big)$$

$$u_\Psi(\cdot,2\pi)^* u_\Psi(\cdot,0) = \big(1_{C([0,2\pi],M_{nr})} \otimes S_2\big)\Big(\bigoplus_{k=1}^{pq}\Big)(1_n \otimes c''_k))$$

where λ_1, λ_2 are the permutations and $S_1, S_2 \in U(pq)$ are the corresponding unitaries defined in [4, the proof of Lemma 3.2] and $b_k := (b''_k)^* b'_k$, $c_k := (c''_k)^* c'_k$, $1 \leq k \leq pq$. (We use the obvious identifications.)

Suppose that we have found maps X_k, Y_k which satisfy (2), (3), (4) and (5). Then, it is not difficult to see that the maps Z_k $(1 \leq k \leq pq)$ defined on the boundary δ of $[0,2\pi]^2 \subset \mathbb{R}^2$ by

$$\begin{aligned}
Z_k(t_1,0) &= X_k(t_1), & 0 \leq t_1 \leq 2\pi,\\
Z_k(2\pi,t_2) &= b'_{\lambda_1(k)}(t_2)\cdot Y_{\lambda_1(k)}(t_2)\big(b''_{\lambda_1(k)}(t_2)\big)^*, & 0 \leq t_2 \leq 2\pi,\\
Z_k(t_1,2\pi) &= c'_{\lambda_2(k)}(t_1)\cdot X_{\lambda_2(k)}(t_1)\big(c''_{\lambda_2(k)}(t_1)\big)^*, & 0 \leq t_1 \leq 2\pi,\\
Z_k(0,t_2) &= Y_k(t_2), & 0 \leq t_2 \leq 2\pi,
\end{aligned}$$

are continuous maps $\delta \to U(r)$ and satisfy the relations $W(\det Z_k) = 0$, $1 \leq k \leq pq$. Then, it follows that each Z_k has a continuous extension to the whole of $[0,2\pi]^2$, also denoted by Z_k, $Z_k : [0,2\pi]^2 \to U(r)$. Define $U : [0,2\pi]^2 \to U(npqr)$ by

$$U(t_1,t_2) = u_\Phi(t_1,t_2)\Big(\bigoplus_{k=1}^{pq}(1_n \otimes Z_k(t_1,t_2))\Big)u_\Psi(t_1,t_2)^*,$$

$t_1,t_2 \in [0,2\pi]$. If we put for any $t_1,t_2 \in [0,2\pi]$, $U(\exp(it_1), \exp(it_2)) := U(t_1,t_2)$, then we have that $U \in C(\mathbb{T}^2, U(npqr))$ and $\Phi(\cdot) = U\Psi(\cdot)U^*$.

Now, the existence of some maps X_k, Y_k satisfying (2), (3), (4) and (5) is obtained using similar arguments with those given in [4, the proof of Lemma 3.2].

Corollary 1. *Let $(p_k),(q_k),(r_k)$ be three increasing sequences of positive integers, with p_k (resp. q_k, resp. r_k) dividing p_{k+1} (resp. q_{k+1}, resp. r_{k+1}) for all $k \geq 1$. Set $L := \varinjlim\big(C(\mathbb{T}^2, M_{p_kq_kr_k}),\Phi_k\big)$, where each $\Phi_k : C(\mathbb{T}^2, M_{p_kq_kr_k}) \to C(\mathbb{T}^2, M_{p_{k+1}q_{k+1}r_{k+1}})$ is compatible with the covering $\mathbb{T}^2 \ni (u,v) \mapsto (u^{p_{k+1}/p_k}, v^{q_{k+1}/q_k}) \in \mathbb{T}^2$. Then L does not depend on the particular choice of the homomorphisms Φ_k and we have that $L \cong A \otimes B \otimes C$, with $A = \varinjlim\big(C(\mathbb{T}, M_{p_k}),\Psi_k\big)$ where each Ψ_k is an arbitrary homomorphism compatible with the covering $\mathbb{T} \ni u \mapsto u^{p_{k+1}/p_k} \in \mathbb{T}$, $B = \varinjlim\big(C(\mathbb{T}, M_{q_k}),\theta_k\big)$ where each θ_k is an arbitrary homomorphism compatible with the covering $\mathbb{T} \ni u \mapsto u^{q_{k+1}/q_k} \in \mathbb{T}$, and $C = \varinjlim(M_{r_k},\Lambda_k)$ where each Λ_k is unital.*

Notation. Let (n_k) be a strictly increasing sequence of positive integers, with n_k dividing n_{k+1} for all $k \geq 1$. We shall denote by $A((n_k))$ the corresponding Bunce-Deddens algebra, i.e. the Bunce-Deddens algebra of type (n_k), and by $U((n_k))$ the UHF-algebra of type (n_k).

Remark. Observe that if in the above corollary $p_k \to \infty$, then, by [4, Remark 3.3] and assuming that (p_k) is strictly increasing, we have $A \cong A((p_k))$, and that if $p_k \to p \in \mathbb{N}$, then obviously we have $A \cong C(\mathbb{T}, M_p)$.

Corollary 2. *Let* (p_k), $q_k)$, (r_k) *be three sequence as in the above corollary but which moreover are strictly increasing. Let* $a, b, c \in \mathbf{Q}_+$ *be such that* $abc = 1$ *and such that each number* ap_k, bq_k, cr_k $(k \geq 1)$ *is a strictly positive integer. Then we have:*

$$A((ap_k)) \otimes A((bq_k)) \otimes U((cr_k)) \cong A((p_k)) \otimes A((q_k)) \otimes U((r_k)) .$$

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Involutions in Integral Group Rings of Certain Dihedral Groups

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Abstract

The number of conjugacy classes of involutions in the integral group ring of dihedral 2-groups is calculated. As consequence the outer automorphism group of the group ring is given in a very explicit way. Furthermore the case of dihedral groups of prime index is investigated.

The Problem

The following is an abstract of parts of the author's forthcoming Ph. D. thesis.

Though the isomorphism problem and the Zassenhaus conjecture for integral group rings $\mathbb{Z}G$ of finite p-groups G for some prime number p have been positively answered by K. W. Roggenkamp and L. L. Scott [6] and the subgroup rigidity for the group ring over the p-adics $\hat{\mathbb{Z}}_pG$ is proved by A. Weiss [8], a lot of questions remain open. It is not clear up to what extent a finite subgroup of the group of units of augmentation 1, $V(\mathbb{Z}G)$, of the integral group ring is conjugate in $V(\mathbb{Z}G)$ to a subgroup of a group basis. That this is always so is a conjecture of H. Zassenhaus. It is also not clear how many $V(\mathbb{Z}G)$-conjugacy classes of group basis "contain" a given finite subgroup U of G.

In [5] K. W. Roggenkamp in collaboration with L. L. Scott developed a theory similar to A. Fröhlich's theory of Picard groups to answer these questions in terms of certain subgroups of class groups. For the dihedral groups $G = D_{2^n}$ of order 2^{n+1} these calculations are carried out for U being not central of order 2. Combining the obtained results with the theorems of S. Endo, T. Miyata and K. Sekiguchi [2] on the outer central automorphism group $Outcent(\mathbb{Z}D_{2^n})$ and those of A. Fröhlich, M. E. Keating and S.M.J. Wilson [3] on class groups of dihedral 2-groups we are able to give generators for $Outcent(\mathbb{Z}D_{2^n})$ in a very explicit way.

The Answer

We have to calculate $|Cl(C_{\mathbb{Z}D_{2^n}}(b))|$, the class number of the centralizer of a non central involution b of D_{2^n} in $\mathbb{Z}D_{2^n}$. This is done by writing the above centralizer $C_{\mathbb{Z}D_{2^n}}(b) =: C(n)$ iteratively as a pullback of subrings of group rings of groups of exponent 2 over the ring of algebraic integers $R_k = \mathbb{Z}[\zeta_k + \zeta_k^{-1}]$ with $\zeta_k^{2^{k-1}} + 1 = 0$. The second source of our calculation is the observation that $C_{\mathbb{Z}D_{2^n}}(b)$ is a tensorproduct of the integral group ring of the cyclic group of order 2 and a certain subring of the integral group ring of the cyclic group of order 2^n. We then use Milnor's Mayer-Vietoris sequences and several observations on the units in the occuring factors to determine $Cl(C_{\mathbb{Z}D_{2^n}}(b))$ up to isomorphism.

H. Cohn has conjectured [1] that the class number of R_k is equal to one and our results give a neccessary and sufficient criterion for the validity of this conjecture in terms of involutions in integral group rings. According to [5] we also have to calculate the kernel of the induction homomorphism from the class group of the centralizer $C(n)$ to the class group of $\mathbb{Z}D_{2^n}$. This kernel parametrizes the number of conjugacy classes of involutions which are rationally conjugate to b. We also determine the kernel of the homomorphism from the class group of the center of $\mathbb{Z}D_{2^n}$ to $Cl(C(n))$.

Let $h_k^+ = |Cl(\mathbb{Z}[\zeta_k + \zeta_k^{-1}])|$ be the class number of the maximal real subfield of the 2^k-th roots of unity. Let for any natural number $k \geq 3$ $D_k =< a, b | a^k = 1, b^2 = 1, baba = 1 >$ be the dihedral group of order $2k$.
We summarize our results as

Theorem. *The $V(QD_{2^n})$-conjugacy class of b splits into $2^{n-1} \cdot \prod_{k \leq n} h_k^+$ different $V(\mathbb{Z}D_{2^n})$-conjugacy classes. The set of conjugacy classes of group bases with a representative containing b has cardinality 2^{n-2}. Therefore the number of conjugacy classes of involutions rationally conjugate to b which are not part of a group basis is $2^{n-1} \cdot ((\prod_{k \leq n} h_k^+) - 1)$. The group $Outcent(\mathbb{Z}D_{2^n})$ is generated by conjugation with $1 + a - b$, which has order 2^{n-1}, and by conjugation with $1 + b \cdot (a + a^{-1})$, which has order 2^{n-2}.*

Therefore Cohn's conjecture is true up to a number n, if and only if every involution in $\mathbb{Z}D_{2^n}$ is part of a suitably chosen group basis. Representatives of all conjugacy classes of group bases can be written down by letting act $Outcent(\mathbb{Z}D_{2^n})$ on the standard group basis D_{2^n}. Cohn's conjecture is verified by F. J. van der Linden [7] for $n \leq 7$ without any further assumption and for $n \leq 8$ if one assumes the generalized Riemann hypothesis.

In the same manner we are able to treat the case $\mathbb{Z}D_p$, the integral group ring of a dihedral group of order $2p$, p being an odd prime. We denote by ζ_p a primitive p-th root of unity and by C_2 the cyclic group of order 2, and for any abelian group A we denote by $A_{[2]}$ the subgroup of A of elements of order 2 or 1. We obtain the

Theorem. *In $V(\mathbb{Z}D_p)$ there are exactly*

$$\frac{|Cl(\mathbb{Z}[\zeta_p + \zeta_p^{-1}]C_2)|}{|Cl(\mathbb{Z}[\zeta_p + \zeta_p^{-1}])|}$$

conjugacy classes of involutions that are in the group ring over the q-adic integers for each prime q conjugate to b. Among those conjugacy classes only $|(Cl(\mathbb{Z}[\zeta_p + \zeta_p^{-1}])_{[2]})|$ conjugacy classes consist of involutions contained in group bases.

Computer calculations using the results of F. J. van der Linden [7] show that for $p = 37$ there are exactly 2 conjugacy classes of involutions locally but not integrally conjugate to elements of group bases. We mention that the involution given in the paper of I. Hughes and K.E. Pearson [4] is 2–adically not conjugate to a group element of S_3.

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Gihman statistic and goodness-of-fit tests for grouped data

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Presented by P. Ribenboim, F.R.S.C.

Summary. We consider the properties of the of the statistic of Gihman, based on the empirical distribution function for grouped data, and its applications in nonparametric statistics.

1. Gihman statistic. We let $X_1, X_2, \ldots, X_n$ denote independent uniform [0,1] random variables with

$$P\{X_i \le x\} = x, \quad x \in [0,1], \tag{1}$$

and we denote by $X^{(\cdot)} = (X_{(n,0)}, X_{(n,1)}, \ldots, X_{(n,n)})^T$ the vector of order statistics obtained from the sample $X_1, \ldots, X_n$:

$$0 \equiv X_{(n,0)} \le X_{(n,1)} \le \ldots \le X_{(n,n)} \le X_{(n,n+1)} \equiv 1. \tag{2}$$

We denote by $F_n(x)$ the standard empirical distribution function of the sample $X_1, X_2, \ldots, X_n$:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{[X_i \le x]}, \quad x \in [0,1]. \tag{3}$$

Note (see, for example, Shorack and Wellner (1986)) that for any fixed x, $x \in [0,1]$, the statistic $nF_n(x)$ has a binomial distribution with parameters n and x so that

1) $EF_n(x) = x$ and $n\mathrm{Cov}(F_n(x), F_n(y)) = x \wedge y - xy, \quad 0 \le x, y \le 1;$

2) $F_n(x) \longrightarrow x$ with probability 1 for each x as $n \longrightarrow \infty$.

Practical use of the empirical distribution requires a large number, n, of observations $X_1, X_2, \ldots, X_n$. For this reason it is often very natural to group the observed data. The behavior of the corresponding empirical distribution function $G_n(x)$ of the grouped data is then of interest.

Let $p=(p_1,p_2,\dots,p_r,p_{r+1})^T$ be a vector of positive probabilities, $p_i>0$,

$$p_1+p_2+\dots+p_r+p_{r+1}=1, \tag{5}$$

where $r=r(n)\geq 1$. We set $x_0=0$, $x_{r+1}=1$,

$$x_j=p_1+p_2+\dots+p_j\ ,\quad j=1,\dots,r.$$

Thus we obtain a partition of $[0,1]$ into r+1 intervals

$$[0,x_1],\ (x_1,x_2],\dots,\ (x_{r-1},x_r],\ (x_r,x_{r+1}],\quad x_{r+1}=1. \tag{6}$$

Let $\nu=(\nu_1,\dots,\nu_r,\nu_{r+1})^T$ be a vector of frequencies obtained as result of grouping the random variables $X_1,X_2,\dots,X_n$ by the intervals in (6). We define the empirical distribution function $G_n(x)$ of the grouped data ν as:

$$G_n(x)=\begin{cases}0 & ,\quad x=x_0=0,\\ \dfrac{\nu_1+\nu_2+\dots+\nu_i}{n} & ,\quad x_{i-1}<x\leq x_i\ ,\quad i=1,2,3,\dots,r+1.\end{cases} \tag{7}$$

Let us consider now the statistic $Z_n=(Z_{n1},\dots,Z_{nr})^T$ of Gihman, where

$$Z_{ni}=\sqrt{n}[G_n(x_i)-x_i]=\sqrt{n}\left[\frac{\nu_1+\dots+\nu_i}{n}-(p_1+\dots+p_i)\right]. \tag{8}$$

It is evident that

$$EZ_n=(0,\dots,0)^T=0_r \quad\text{and}\quad EZ_nZ_n^T=\Sigma, \tag{9}$$

where

$$\Sigma=\left\|\begin{matrix}x_1 & x_1 & x_1 & \dots & x_1\\ x_1 & x_2 & x_2 & \dots & x_2\\ x_1 & x_2 & x_3 & \dots & x_3\\ \vdots & \vdots & \vdots & & \vdots\\ x_1 & x_2 & x_3 & \dots & x_r\end{matrix}\right\| - \left\|\begin{matrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_r\end{matrix}\right\|\cdot\|x_1,x_2,\dots,x_r\|. \tag{10}$$

2. Asymptotic properties of the statistic Z_n. We shall consider here the properties of Z_n, when $n\Rightarrow\infty$.

a) Let us suppose at first that $r=r(n)\longrightarrow\infty$ as $n\longrightarrow\infty$ so that the lengths of the grouping intervals (6) tend to zero uniformly and sufficiently fast, i.e., so that

$$\max_{1 \le i \le r+1} np_i \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{1}$$

We denote

$$D_n^* = \max_{1\le i\le r} |Z_{n,i}| \quad \text{and} \quad D_n = \sup_{0\le x\le 1} \sqrt{n}|F_n(x) - x|. \tag{2}$$

Theorem (Gihman, 1961). *If $r \Rightarrow \infty$ so that (1) is fulfilled as $n \Rightarrow \infty$, then statistics D_n and D_n^* are asymptotically equivalent:*

$$\lim_{n\to\infty} P\{ D_n^* \le z \} = \lim_{n\to\infty} P\{ D_n \le z \} = K(z), \tag{3}$$

where $K(z)$ is the Kolmogorov distribution function,

$$K(z) = \sum_{j=-\infty}^{+\infty} (-1)^j e^{-2j^2 z^2}, \; 0 < z < \infty.$$

Roughly speaking, under the conditions (1) one can use the statistic Z_n for large values of n to construct the well-known statistics ω^2 of Smirnov, W_n^2 of Anderson and Darling etc., considered, for example, by Darling (1952), Sherman (1950) and Birnbaum (1953).

b) Now let us suppose that the intervals (6) are fixed, with $r+1 \ge 2$. Then from (1.3), (1.4), (1.8) and the multivariate central limit theorem it follows that, in the limit as $n \Rightarrow \infty$, $\{Z_n\}$ has the r-dimensional normal distribution $N(0_r,\Sigma)$ with parameters given by (1.9). The covariance matrix Σ is nonsingular and its inverse $\Sigma^{-1} = \|\sigma^{ij}\|$ has elements σ^{ij} expressed by following formulae:

$$\begin{cases} \sigma^{ij} = 0, \quad |i - j| \ge 2, \\ \sigma^{i,i+1} = -\dfrac{1}{x_{i+1} - x_i} = -\dfrac{1}{p_{i+1}}, \quad i=1,\ldots,r-1, \\ \sigma^{i,i-1} = -\dfrac{1}{x_i - x_{i-1}} = -\dfrac{1}{p_i}, \quad i=2,\ldots,r, \\ \sigma^{ij} = -(\sigma^{i,i-1} + \sigma^{i,i+1}) = \dfrac{1}{x_{i+1} - x_i} + \dfrac{1}{x_i - x_{i-1}}, \; i=j. \end{cases} \tag{4}$$

Now we define the statistic Y_n^2 by setting

$$Y_n^2 = Z_n^T \Sigma^{-1} Z_n .$$

From the asymptotic normality of the statistic Z_n we obtain that

$$\lim_{n \Rightarrow \infty} P\{Y_n^2 \le x\} = P\{\chi_r^2 \le x\}. \tag{5}$$

It is easily to verify that Y_n^2 is the standard Pearson statistic:

$$Y_n^2 = \sum_{i=1}^{r+1} \frac{(\nu_i - np_i)^2}{np_i}. \tag{6}$$

We remark here that the same results can obtained if one uses random intervals as was considered, for example, in the papers of Jammalamadaka and Tiwari (1987), Nikulin (1973, 1974, 1991), Sherman (1950).

c) At last we consider the case when $r=r(n) \longrightarrow \infty$ as $n \longrightarrow \infty$, so that

$$\max_{1 \le i \le r+1} p_i \rightarrow 0 \quad \text{and} \quad \min_{1 \le i \le r+1} np_i \longrightarrow \infty. \tag{7}$$

Theorem (Tumanian, 1956). *If $r \Rightarrow \infty$ so that the condition (7) is satisfied when $n \Rightarrow \infty$, then*

$$\sup_{|x|<\infty} \left| P\{Y_n^2 \ge x\} - 1 + \Phi\left(\frac{x-r}{\sqrt{2r}}\right) \right| \longrightarrow 0, \quad n \longrightarrow \infty, \tag{8}$$

where $\Phi(x)$ is the distribution function of the standard normal law.

Remark. An extension of nonparametric tests to cover discrete case can be obtained easily. Let $X_1, X_2, \ldots, X_n$ be discrete or continuous random variables with distribution functions $F_1, F_2, \ldots, F_n$ respectively. Further let
$Y_1, Y_2, \ldots, Y_n$
be independent and identically uniformly distributed on [0,1] random variables and Y_i's are all independent on X_j's. Then the random variables

$$Z_i = F_i(X_i - 0) + Y_i[F_i(X_i) - F(X_i - 0)], \quad i = 1,2,\ldots,n; \tag{9}$$

are independent and uniformly distributed on [0,1]. The proof one can find, for example, in Shorack and Wellner (1986), Huber and Nikulin (1991).

Example. Let $X_1, X_2, \ldots, X_n$ be independent discrete random variables and we want to test the hypothesis

$$H_0: F_j(x) = P\{X_j \le x\} = \sum_{i=1}^{x} \binom{m_j+i-1}{m_j-1} p_j^{m_j}(1-p_j)^i, \; x = 0,1,2,\ldots,$$

where $p_1, p_2, \ldots, p_n$ and $m_1, m_2, \ldots, m_n$ are given, $0 < p_j < 1$. In other words under H_0 the random variable X_j follows negative binomial distribution with parameters m_j and p_j. If H_0 is true then we can write

$$F_j(x) = I_{p_j}(m_j, x+1) = \frac{1}{B(m_j, x+1)} \int_0^{p_j} u^{m_j-1}(1-u)^x du$$

and in this case using (9) we can construct the random variables

$$Z_j = F_j(m_j, X_j) + Y_j\,[F_j(m_j, X_j+1) - F_j(m_j, X_j)], \; j = 1,2,\ldots,n;$$

which one can uses to test H_0.

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REMARQUES SUR LA TRANSCENDANCE EN CARACTERISTIQUE FINIE

par

DENIS Laurent

Presented by P. Ribenboim, F.R.S.C.

Abstract : We give an elementary method to obtain transcendence results over the rationnal function field of characteristic $p > 0$. A special attention is care to the analogue of e and π constructed by Carlitz in [1]. Using this method and some known result of Wade [3], we prove among other things, that $e + \pi$ is transcendental over $F_q(T)$.

1 Position du problème

On désigne par $F_q[T]$ l'anneau des polynômes en une variable à coefficients dans le corps fini F_q , par $k = F_q(T)$ son corps des fractions, par $k_\infty = F_q((1/T))$ le complété de k pour la valuation (1/T)-adique v , que l'on prolonge à une clôture algébrique $\overline{k}$ (resp. $\overline{k_\infty}$) de k (resp. k_∞). On notera $|\alpha| = q^{-v(\alpha)}$, la valeur absolue d'un élément de $\overline{k_\infty}$.

Définition On dira qu'un élément α de $\overline{k_\infty}$ est transcendant s'il n'est pas dans $\overline{k}$.

On s'intéressera surtout aux éléments transcendants de k_∞ . On va voir comment l'action des automorphismes continus de k_∞ permet d'aboutir facilement à des résultats de transcendance.

§2 Préliminaires

On donne d'abord quelques lemmes d'indépendance linéaire.

LEMME 1 Soit $\sigma \in Gal(k_\infty/k)$, de corps des invariants k_σ . Cet automorphisme définit un endomorphisme du k_σ espace vectoriel k_∞ et des vecteurs propres associés à des valeurs propres distinctes sont linéairement indépendants.

Preuve: C'est clair.

LEMME 2 Sous les hypothèses précédentes, si on suppose que k_σ contient k , alors pour tout α dans k_∞ , $\sigma(\alpha)$ est transcendant équivaut à α est transcendant.

Preuve: Il suffit de conjuguer une éventuelle relation algébrique sur k .

Dans la suite on suppose toujours que k_σ contient k . On peut alors énoncé un premier lemme de transcendance :

LEMME 3 Soient α et β deux élements de k_∞ appartenant à des espaces propres associés sous l'action de σ à des valeurs propres algébriques distinctes .

1) Si α est transcendant alors $a\alpha + b\beta = c$ avec a,b dans $k_\sigma \cap \bar{k}$ et c dans $\bar{k}$ entraîne $a=0$.

2) Si de plus $\sigma(\beta)=\beta$ alors tout élement de $\alpha + k_\sigma(\beta)$ est transcendant.

Preuve :1) Supposons une relation $a\alpha + b\beta = c$ avec a,b,c dans $k_\sigma \cap \bar{k}$. On applique σ à cette égalité pour éliminer b comme $\sigma(c)$ est algébrique on obtient la conclusion.

2) On procède de même.

Les remarques qui vont conduire aux exemples de transcendance vont utiliser les cas particuliers suivant d'automorphismes.

LEMME 4 Soit ξ un élément de $(F_q)^*$, l'application :

$\sigma_\xi : T \to \xi T$ et qui est l'identité sur F_q se prolonge de manière unique en un morphisme de corps sur k_∞ qui est une isométrie. De plus le corps des invariants de σ_ξ est $F_q((1/T^a))$ où a est l'ordre de ξ dans F_q.

Preuve: La dernière assertion vient de l'unicité du développement en 1/T .

Regardons maintenant le comportement de certains polynômes qui interviennent souvent dans les fonctions transcendantes de la théorie.

Définitions : Pour tout entier $i \geq 1$, on pose :

$$[i]= T^{q^i}-T\ ,\quad L_k=\prod_{i=1}^{k}[i] \text{ et par récurence } D_k=[k](D_{k-1})^q \text{ et } D_0=1.$$

Des expressions précédentes, on déduit:

LEMME 5 Avec la notation du lemme 4 .

$$\sigma_\xi([i])= \xi[i]\ ,\ \sigma_\xi(L_k)=(\xi)^k(L_k)\ ,\ \sigma_\xi(D_k)=(\xi)^k(D_k)\ .$$

Les nombres dont on regarde la transcendance sont e, π et les valeurs de la fonction ζ de Carlitz.

On a les expressions suivantes :

$$\pi = \prod_{i=1}^{\infty}\left(1 - \frac{[i]}{[i+1]}\right) \; ; \; e = \sum_{h=0}^{\infty} \frac{1}{D_h} ; \; \zeta(s) = \sum_{h=0}^{\infty} \frac{(-1)^{hs}}{L_h^s} \quad (1 \leq s \leq q-1) .$$

On déduit alors du lemme 5 et de la continuité des automorphismes que :

LEMME 6 $\sigma_\xi(\pi) = \pi$; $\sigma_\xi(e) = \sum_{h=0}^{\infty} \frac{(\xi)^h}{D_h}$; $\sigma_\xi(\zeta(s)) = \sum_{h=0}^{\infty} \frac{(-1)^{hs}(\xi)^{-hs}}{L_h^s}$.

§ 3 Quelques nombres transcendants

Illustrons par quelques exemples les résultats de transcendance auquels on peut aboutir.

On suppose maintenant $\xi = -1$, σ_ξ est une involution. Donc pour tout a dans $\overline{k}_\infty$, $(\sigma_\xi - id)(a)$ est vecteur propre pour la valeur propre -1. On déduit alors du lemme 3 et de la transcendance des nombres e, π et $\zeta(s)$:

Corollaire 1 En caractéristique différente de 2 :

a) $(\sigma_\xi - id)(e) + \pi$ est transcendant.

b) $(\sigma_\xi - id)(e) + \zeta(s)$ est transcendant, pour s pair et $1 \leq s \leq q-1$.

c) $(\sigma_\xi - id)(\zeta(s)) + \pi$ est transcendant, pour s entier ≥ 1.

b) $(\sigma_\xi - id)(\pi) + \zeta(s)$ est transcendant, pour s pair et $1 \leq s \leq q-1$.

Des résultats un peu plus spectaculaires peuvent être déduit de la même méthode :

THEOREME 8 En caractéristique différente de 2 :

a) $e + \pi$ est transcendant.

b) tout élément de $e + F_q(T^2, \pi)$ est transcendant.

c) $e + \zeta(s)$ <u>est transcendant pour</u> s <u>pair</u> et $1 \leq s \leq q-1$.

<u>preuve</u> de a) , le reste est identique :

Si $e + \pi$ est algébrique alors $(\sigma_\xi)(e + \pi)$ aussi. On fait la différence de ces ceux éléments et on aboutit à $e-(\sigma_\xi)(e)$ qui est transcendant d'après les travaux de Wade (cf [4] et remarque §2.10 dans [3]).

Citons aussi quelque cas particuliers des résultats de J.YU (cf[5]) qu'on retrouve ici :

THEOREME 9 <u>En caractéristique différente de</u> 2 :

<u>Pour tout entier</u> $h \neq 0$, $\pi^h + \zeta(s)$ <u>est transcendant pour</u> s <u>impair et</u> $1 \leq s \leq q-1$.

<u>Preuve</u> : Comme au théorème 8 , on utilise ici la transcendance de $\zeta(s)-(\sigma_\xi)(\zeta(s))$ qui découle d'un analogue du théorème de [1] (modulo une modification car un terme sur deux est nul).

Signalons enfin que ces méthodes marchent beaucoup moins bien pour des produits, il est néanmoins possible de montrer :

<u>Corollaire 2</u> $e\,\pi$ est irrationnel.

<u>Preuve</u> : Sinon $e/(\sigma_\xi)(e)$ est rationnel et on conclut encore grâce à (§2.10 , [3]).

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On the degree of the Jones polynomial

Kunio Murasugi F.R.S.C.

Let L be an oriented link in S^3 and $V_L(t)$ the Jones polynomial of L [J]. Denote by $d_{\max} V_L(t)$ and $d_{\min} V_L(t)$ the highest and the lowest degree of terms $V_L(t)$. These degrees are estimated in [M1] in terms of the number of crossings and the signature $\sigma(L)$ of L. To be more precise, let $\tilde{L}$ be an (oriented) diagram of L on $\mathbb{R}^2$. Let $c_+(\tilde{L})$ and $c_-(L)$ denote, respectively, the number of positive and negative crossings in $\tilde{L}$. Then for any diagram $\tilde{L}$ of L, the following inequalities hold [M1]

(1) (a) $$d_{\max} V_L(t) \le c_+(\tilde{L}) - \frac{\sigma(L)}{2}$$

(b) $$d_{\min} V_L(t) \ge c_-(\tilde{L}) - \frac{\sigma(L)}{2} .$$

Equalities hold in (1) simultaneously if L is an alternating link and $\tilde{L}$ its reduced alternating diagram. However, if L is not a connected sum of alternating links, these inequalities are no longer the best possible. See Remark 2. V. Jones gives another estimate for $d_{\max} V_L(t)$ and $d_{\min} V_L(t)$. In fact, he proves the following proposition.

Proposition 1 [J, Proposition 14.2]. *Let β be an n-braid and $\hat{\beta}$ the closure of β. Let e_+ and e_- denote, respectively, the sum of positive and negative exponents in β. Write $e = e_+ - e_-$. Then*

(2) (a) $$d_{\max} V_{\hat{\beta}}(t) \le e_+ + \frac{e+n-1}{2}$$

(b) $$d_{\min} V_{\hat{\beta}}(t) \ge -e_- + \frac{e-n+1}{2}$$

T. Fiedler [F] recently obtained a much better estimate for a closed braid $\hat{\beta}$ as follows

Proposition 2 [F, Theorem 1]. *Let $\delta^+(\beta)$ (and $\delta^+(\beta)$) denote the number of distinct (Artin) braid generators appearing in β with positive (and negative) exponent. Then*

(3) (a) $$d_{\max} V_{\hat{\beta}}(t) \le e_+ + \frac{e+n-1}{2} - \delta^+(\beta)$$

(b) $\quad d_{\min} V_{\hat{\beta}}(t) \geq -e_- + \dfrac{e-n+1}{2} + \delta^-(\beta)$

Since at least one of $\delta^+(\beta)$ or $\delta^-(\beta)$ is positive, (3) is a considerable improvement of (2). In this short note, we will show that (3) is a consequence of the estimate (4) below proven in [M1, (13.2)]

(4) (a) $\quad d_{\max} V_L(t) \leq \dfrac{1}{2}\left\{p(\Gamma) + r_P + s_P - \dfrac{p(\Gamma) - n(\Gamma) - 3\omega(\tilde{L})}{2}\right\}$

(b) $\quad d_{\min} V_L(t) \geq \dfrac{1}{2}\left\{-\left(n(\Gamma) + r_N + s_N\right) - \dfrac{p(\Gamma) - n(\Gamma) - 3\omega(\tilde{L})}{2}\right\}$

For the notation, see [M1]. Since (1) is also a consequence of (4) [M1, Theorem 13.3], we see that all known estimates of $d_{\max} V_L(t)$ and $d_{\min} V_L(t)$ are consequences of (4).

Now, let L be the closure of an n-braid β and $\tilde{L}$ the link diagram obtained naturally from a braid representation of L. Let Γ be a plane graph in $\mathbf{R}^2$ associated with $\tilde{L}$. We classify the domains of $\mathbf{R}^2 - \Gamma$ into shaded or unshaded domains. We assume that the unbounded domain is shaded. See [M1, §12]. We only prove (3)(a), since a proof of (3)(b) is analogous.

Now in order to prove (3)(a), it suffices to show the following inequality

(5) $\quad \dfrac{1}{2}\left\{p(\Gamma) + r_P + s_P - \dfrac{p(\Gamma) - n(\Gamma) - 3\omega(\tilde{L})}{2}\right\} \leq e_+ + \dfrac{1}{2}(e+n-1) - \delta^+(\beta)$.

First we express $p(\Gamma)$, $n(\Gamma)$ and $\omega(\tilde{L})$ in terms of e_+, e_- and e. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be Artin's generators of the n-braid group B_n. Define $p(\sigma_i)$ and $n(\sigma_i)$, respectively, as the sum of positive and negative exponents of σ_i appearing in β. Then $e_+ = \sum_{i=1}^{n-1} p(\sigma_i)$ and $e_- = \sum_{i=1}^{n-1} p(\sigma_i)$. It is then easy to see

$$(6) \qquad \begin{cases} p(\Gamma) = \sum\limits_{i=\text{odd}} n(\sigma_i) + \sum\limits_{j=\text{even}} p(\sigma_j) , \\ n(\Gamma) = \sum\limits_{i=\text{odd}} p(\sigma_i) + \sum\limits_{j=\text{even}} n(\sigma_j) \end{cases}$$

Furthermore, noting that $\omega(\tilde{L}) = e = e_+ - e_-$ and $p(\Gamma) + n(\Gamma) = e_+ + e_-$, (5) is now equivalent to

(7) $\quad r_P + s_P \leq e_+ + n - 1 - 2\delta^+(\beta)$.

Note that $r_P+1-s_P = |V|-p(\Gamma)$ (Euler characteristic) and $|V| = \sum_{i=\text{odd}} \{p(\sigma_i)+n(\sigma_i)\}+\varepsilon$, where $\varepsilon = 0$ or 1 depending on whether n is even or odd. See Figure 1. Therefore,

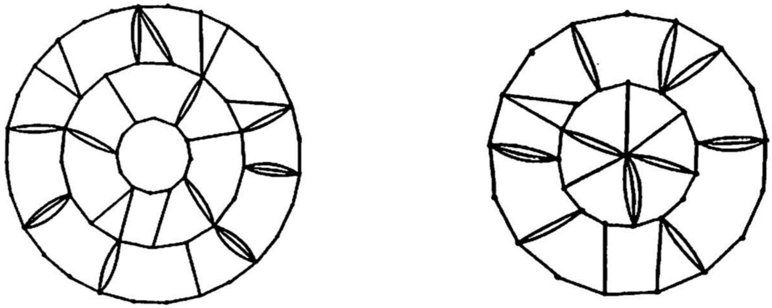

Figure 1

(7) is reduced to

(8) $$s_P + \delta^+(\beta) \leq \sum_{j=\text{even}} p(\sigma_j) + \left[\tfrac{n}{2}\right].$$

Now, by definition, $s_P + 1$ is the number of domains in which $\mathbb{R}^2$ is divided by the maximal positive subgraph P of Γ. Note that Γ consists of $\left[\frac{n+1}{2}\right]$ concentric circles C_i, $i = 1,2,\ldots,\left[\frac{n+1}{2}\right]$, and edges joining vertices on adjacent two circles C_i and C_{i+1}, $i = 1,2,\ldots,\left[\frac{n+1}{2}\right]-1$. Obviously, $\sum_{j=\text{even}} p(\sigma_j)$ gives the number of those positive edges which do not belong to any circle C_i.

Now we evaluate $s_P + 1$. First consider the special case where all the circles C_i consist of positive edges alone. In this case, no generator σ_{2i+1} appears in β with a positive exponent. Since $\delta^+(\beta)$ is, in our case, the number of $p(\sigma_{2_j})$ such that $p(\sigma_{2_j}) \geq 1$ and since each positive edge corresponding to σ_{2_i} in β divides the annulus bounded by C_i and C_{i+1}, we see easily that $s_P + \delta^+(\beta) = \sum_{j=\text{even}} p(\sigma_j) + \left[\frac{n}{2}\right]$. This proves (8) for the special case.

Next, we consider the general case. A graph Γ is obtained from the special graph by replacing some positive edges in C_i by negative edges. If we replace one positive edge of C_i by a negative edge, then δ^+ increases by at most one, but s_P decreases by at least one, and therefore

$$s_P + \delta^+(\beta) \leq \sum_{j=\text{even}} p(\sigma_j) + [\frac{n}{2}] \,.$$

This completes our proof.

Remark 1. Equality holds in (8) (or in (5)) if $p(\sigma_j) \leq 1$ for all odd j.

Remark 2. We compare two estimates (1) and (3) for a non-alternating torus knot of type (r,q), where $3 \leq q < r$ and g.c.d $(r,q) = 1$. If we represent L as the closure of a q-braid $\beta = (\sigma_1\sigma_2\ldots\sigma_{q-1})^r$, then $c_+(\tilde{L}) = r(q-1)$ and $c_-(\tilde{L}) = 0$. ($r(q-1)$ is the minimal number of crossings L can have, [M2].) Furthermore, $e_+ = r(q-1)$, $e_- = 0$, $e = e_+$, $\delta^+(\beta) = q-1$ and $\delta^-(\beta) = 0$. Since $-\sigma(L) \leq (r-1)(q-1)$, a simple computation shows that

$$\text{(9)} \quad \text{(a)} \quad c_+(\tilde{L}) - \frac{\sigma(L)}{2} \leq e_+ + \frac{e+n-1}{2} - \delta^+(\beta)$$

$$\text{(b)} \quad -c_-(\tilde{L}) - \frac{\sigma(L)}{2} \leq -e_- + \frac{e-n+1}{2} + \delta^-(\beta)$$

Therefore, (1)(a) is a better estimate than (3)(a) for $d_{\max} V_L(t)$, while (3)(b) is better than (1)(b) for $d_{\min} V_L(t)$.

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CONVOLUTEURS CONTINUS ET GROUPES QUOTIENTS

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Presented by Edmond E. Granirer, F.R.S.C.

0. Introduction.

Soient G un groupe localement compact, $1 < p < \infty$, $CV_p(G)$ l'ensemble des p-convoluteurs de G et $PF_p(G)$ le sous-espace des pseudofonctions (l'adhérence normique de $L^1(G)$ dans $CV_p(G)$). Le sous-espace $C_p(G) = \{T \in CV_p(G) \mid T\,PF_p(G) \subset PF_p(G)\}$ a été essentiellement introduit par Granirer [8]. Si G est abélien, par la transformée de Fourier qui identifie $CV_2(G)$ à $L^\infty(\widehat{G})$, $C_2(G)$ correspond à $C^b(\widehat{G})$, l'espace des fonctions continues bornées sur $\widehat{G}$. Dans [4], nous avons commencé à étudier $C_p(G)$ notamment à l'aide d'une topologie "stricte" β (en analogie avec la topologie stricte de Buck). Cette topologie localement convexe découle naturellement de notre définition de $C_p(G)$: β est donnée par la famille de semi-normes $T \longmapsto |||TS|||_p$, S parcourant $PF_p(G)$. β s'avère particulièrement bien adaptée à l'étude de $C_p(G)$ et est également intéressante en soi.

Dans ce travail, nous détaillons certains résultats annoncés dans [4] : $(C_p(G), \beta)$ est complet, son dual s'identifie à $W_p(G)$, le dual normique de $PF_p(G)$. On compare également les couplages de dualité entre $C_p(G)$ et $W_p(G)$, respectivement $A_p(G)$ et $PM_p(G)$ (§2). L'application de Anker $R : CV_p(G) \longrightarrow CV_p(G/H)$ (pour G moyennable et H sous-groupe normal fermé) est étudiée au paragraphe 3. R n'est pas intrinsèque en général, mais il est connu que, si G est abélien, $\widehat{R(T)} = \mathrm{Res}_{H^\perp}\widehat{T}$ pour tout $T \in CV_p(G)$ avec $\widehat{T}$ continu. On montre que $R : C_p(G) \longrightarrow C_p(G/H)$ est un homomorphisme intrinsèque β-continu d'algèbres. On obtient également le résultat suivant : $R : CV_p(G) \longrightarrow CV_p(G/H)$ est ultrafaiblement continue si et seulement si H est compact.

1. Notations et conventions.

Chaque auteur, ou presque, adoptant des conventions différentes concernant les convoluteurs et $A_p(G)$, nous ferons les choix suivants : Nous reprenons pour l'essentiel les notations de [4, §1] moyennant quelques adaptations afin de les rendre compatibles avec [5], [6] et [7] :

– L'algèbre de Figà-Talamanca Herz $A_p(G)$ est définie comme suit :

$$A_p(G) = \left\{ \sum_{n=1}^{\infty} \overline{k}_n * \check{\ell}_n \;\Big|\; k_n \in L^p(G),\ \ell_n \in L^{p'}(G) \text{ et } \sum_{n=1}^{\infty} \|k_n\|_p \|\ell_n\|_{p'} < \infty \right\}.$$

– Le couplage entre $A_p(G)$ et $PM_p(G)$ est donné par $<\overline{k} * \check{\ell}, T>_{A_p(G),PM_p(G)} = \overline{(T\tau_p k, \tau_{p'}\ell)}$ où $\tau_p k = \Delta_G^{-1/p}\check{k}$ et $(k, \ell) = \int_G k(x)\overline{\ell(x)}dx$ pour tout $k \in L^p(G)$ et $\ell \in L^{p'}(G)$. En particulier, $<u, \lambda_G^p(\mu)>_{A_p(G),PM_p(G)} = \mu^*(u)$ pour tout $u \in A_p(G)$ et $\mu \in M^1(G)$. Le couplage entre $PF_p(G)$ et $W_p(G)$ sera choisi de telle sorte que $<\lambda_G^p(f), v>_{PF_p(G),W_p(G)} = (f, \check{v})$ pour tout $f \in L^1(G)$ et $v \in W_p(G)$.

– La structure de $A_p(G)$-module sur $CV_p(G)$ est telle que pour tout $\alpha \in \mathbb{C}$, $u \in A_p(G)$ et $T \in CV_p(G)$, on a $(\alpha u)T = \overline{\alpha}(uT)$.

– Remarquons que $\mathrm{supp}\,\lambda_G^p(\mu) = (\mathrm{supp}\,\mu)^{-1}$ pour tout $\mu \in M^1(G)$.

Pour le surplus, on se fixe une fois pour toutes une unité approchée $(f_i)_{i\in I}$ de $L^1(G)$ vérifiant les propriétés suivantes : $f_i \geq 0$, $f_i \in C_{00}(G)$, $f_i^* = f_i$, $\|f_i\|_1 = 1$ et pour tout V voisinage de e il existe i_0 tel que $\mathrm{supp}\, f_i \subset V$ pour tout $i \geq i_0$. Si f est une fonction sur G, on note $\widetilde{f}$ la fonction définie par $\widetilde{f}(x) = \overline{f(x^{-1})}$ pour tout $x \in G$. On note $\lambda(X)$ la mesure de Haar de toute partie mesurable X de G. Notons finalement que les couplages de dualité seront généralement sesquilinéaires et que, si X et Y sont deux espaces de Banach, $\mathcal{L}(X,Y)$ sera l'ensemble des applications linéaires continues de X dans Y et $\mathcal{L}(X)=\mathcal{L}(X,X)$.

2. Résultats généraux sur $C_p(G)$.

Dans cette section, nous nous contentons de donner les preuves de quelques résultats sur $C_p(G)$ et la topologie stricte β énoncés dans [4, §2]; pour plus de détails sur la topologie stricte, on peut se reporter à [12].

Théorème 2.1. *$(CV_p(G),\beta)$ et $(C_p(G),\beta)$ sont complets.*

PREUVE. – Il suffit de montrer l'assertion pour $CV_p(G)$ vu que $C_p(G)$ y est β-fermé. Soit $(T_j)_{j\in J}$ une β-suite de Cauchy, pour tout $f \in L^1(G)$, il existe $T_f \in CV_p(G)$ avec $|||T_j\lambda_G^p(f)-T_f|||_p \longrightarrow 0$. Soit $m\in L^p(G)$, il existe $f\in L^1(G)$, $m'\in L^p(G)$ avec $m=\lambda_G^p(f)m'$; par suite, $T_jm=T_j\lambda_G^p(f)m' \longrightarrow T_fm'$ pour $\| \ \|_p$. Posons alors $Tm=T_fm'=\lim_j T_jm$. Il est alors clair que $T\lambda_G^p(f)=T_f$ pour tout $f\in L^1(G)$. On en déduit que T est continu; soit en effet $(m_n)_{n=1}^{\infty}\subset L^p(G)$ avec $\|m_n\|_p\longrightarrow 0$; par [10, 32.23], il existe $f\in L^1(G)$ et $(m'_n)_{n=1}^{\infty}\subset L^p(G)$ avec $\|m'_n\|_p\longrightarrow 0$ et $m_n=\lambda_G^p(f)m'_n$; ainsi $\|Tm_n\|_p\le |||T_f|||_p\|m'_n\|_p\longrightarrow 0$. Et, finalement, $T\in CV_p(G)$ et $T_j\longrightarrow T$ pour β. □

On peut en déduire le résultat suivant qui nous sera utile pour l'étude du dual de $(C_p(G),\beta)$. Soit $\mathrm{Hom}_{L^1(G)}(L^1(G),CV_p(G))=\{P\in\mathcal{L}(L^1(G),CV_p(G)) \mid P(f*g)=f\bullet P(g)$ pour tout f, $g\in L^1(G)\}$.

Proposition 2.2. *Pour tout $T\in CV_p(G)$ et $f\in L^1(G)$, posons $P_T(f)=f\bullet T$. L'application $T\longmapsto P_T$ est alors une isométrie de $CV_p(G)$ sur $\mathrm{Hom}_{L^1(G)}(L^1(G),CV_p(G))$; c'est aussi un homéomorphisme de $(CV_p(G),\beta)$ sur $\mathrm{Hom}_{L^1(G)}(L^1(G),CV_p(G))$ muni de la topologie forte d'opérateurs.* □

Passons à l'étude du dual de $(C_p(G),\beta)$.

Théorème 2.3. *Tout élément v de $W_p(G)$, dual de $(PF_p(G),|||\ |||_p)$, est strictement continu sur $PF_p(G)$ et définit donc, par prolongement, un élément du dual de $(C_p(G),\beta)$. $W_p(G)$ s'identifie alors au dual de $(C_p(G),\beta)$; la topologie forte de dualité de $(C_p(G),\beta)'$ étant la topologie normique $\|\ \|_{W_p(G)}$. Le couplage de dualité est donné par $<T,v>_{C_p(G),W_p(G)} = \lim_i <T\lambda_G^p(f_i),v>_{PF_p(G),W_p(G)}$ et satisfait $|<T,v>_{C_p(G),W_p(G)}|\le |||T|||_p\|v\|_{W_p(G)}$.*

PREUVE. – 1) Remarquons tout d'abord que tout élément de $(C_p(G),\beta)'$ donne, par restriction à $PF_p(G)$, un élément de $W_p(G)$.
2) Comme $PF_p(G)$ est un $L^1(G)$-module à gauche, $W_p(G)$ devient un $L^1(G)$-module à droite par la relation $<T,v\bullet f>_{PF_p(G),W_p(G)} = <f\bullet T,v>_{PF_p(G),W_p(G)}$ pour tout $T\in PF_p(G)$, $v\in W_p(G)$ et $f\in L^1(G)$; de plus un calcul explicite donne que $v\bullet f=f^{\circ}*v$. On déduit alors, grâce à Cohen-Hewitt et à la continuité de la translation dans $W_p(G)$ [3, theorem 3], que $W_p(G)\bullet L^1(G)=W_p(G)$. Soit alors $v\in W_p(G)$, il existe $f\in L^1(G)$ et $w\in W_p(G)$ avec $v=w\bullet f$. Pour tout $T\in PF_p(G)$, on a $|<T,v>_{PF_p(G),W_p(G)}|\le |||f\bullet T|||_p\|w\|_{W_p(G)}$ et $T\longmapsto <T,v>_{PF_p(G),W_p(G)}$ est strictement continu sur $PF_p(G)$ et se prolonge donc en un élément de $(C_p(G),\beta)'$.
3) On peut alors procéder à l'identification de $W_p(G)$ avec $(C_p(G),\beta)'$ (en tant qu'espaces vectoriels). Comme, pour tout $T\in C_p(G)$, $T\lambda_G^p(f_i)\longrightarrow T$ pour β, la dualité entre $C_p(G)$ et $W_p(G)$ sera donnée par: $<T,v>_{C_p(G),W_p(G)}=\lim_i <T\lambda_G^p(f_i),v>_{PF_p(G),W_p(G)}$, et, en particulier, $|<T,v>_{C_p(G),W_p(G)}|\le |||T|||_p\|v\|_{W_p(G)}$. La coïncidence des topologies découle alors de la proposition 2.2 et du principe de la borne uniforme. □

Lemme 2.4. *Soient $T \in CV_p(G)$ à support compact et $v, w \in W_p(G)$ avec $v = w$ sur un voisinage U de* $\operatorname{supp} T$, *alors* $<T, v>_{C_p(G), W_p(G)} = <T, w>_{C_p(G), W_p(G)}$.

PREUVE. – a) Supposons tout d'abord que $T \in PF_p(G)$. Il existe $u \in A_p(G) \cap C_{00}(G)$ avec $u = 1$ sur un voisinage de $\operatorname{supp} T$ et $\operatorname{supp} u \subset U$; par suite, $uT = T$. Soit $\epsilon > 0$, il existe $g \in C_{00}(G)$ avec

$$|||T - \lambda_G^p(g)|||_p < \frac{\epsilon}{(1 + \|u\|_{A_p(G)})(1 + \|v\|_{W_p(G)} + \|w\|_{W_p(G)})}.$$

On en déduit, vu que $u(v - w) = 0$:

$$|<T, v>_{C_p(G), W_p(G)} - <T, w>_{C_p(G), W_p(G)}| \le$$
$$|||T - \lambda_G^p(\check{u}g)|||_p \left(\|v\|_{W_p(G)} + \|w\|_{W_p(G)}\right) + |<\lambda_G^p(g), u(v - w)>_{C_p(G), W_p(G)}| < \epsilon .$$

b) Le cas général se déduit en approximant T par $T\lambda_G^p(f_i) \in PF_p(G)$. □

Théorème 2.5. *Pour tout $T \in PM_p(G)$, $u \in A_p(G)$ et $v \in W_p(G)$, on a* $<uT, v>_{C_p(G), W_p(G)} = \overline{<uv, T>}_{A_p(G), PM_p(G)}$.

PREUVE. – Supposons tout d'abord u à support compact. Soit $w \in A_p(G)$ avec $v = w$ sur un voisinage de $\operatorname{supp} u$. Par le lemme 2.4, on a: $<uT, v>_{C_p(G), W_p(G)} = \overline{<w, uT>}_{A_p(G), PM_p(G)} = \overline{<uv, T>}_{A_p(G), PM_p(G)}$. Le cas général se déduit alors par approximation. □

3. Relations entre $C_p(G)$ et $C_p(G/H)$. Application R de Anker.

Supposons dorénavant G moyennable et soit H un sous-groupe fermé et normal de G. L'application quotient de G dans G/H est notée ω: $\omega(x) = xH = \dot{x}$ pour tout $x \in G$. On choisira les mesures de Haar dh et $d\dot{x}$ sur H et G/H de telle sorte que $dx = dh\, d\dot{x}$.
Nous renvoyons à [6, §3] et [7, §3] pour les notations ($f_{r,s}$, $F_{r,s}$, $A_{K,\epsilon}$, $\mathcal{R}$), la construction et les propriétés de l'application R de Anker de $CV_p(G)$ dans $CV_p(G/H)$ (voir aussi [1, p. 72-79] et [2, p. 635-636]). Notons encore, pour tout K compact de G et $\epsilon > 0$, $\mathcal{U}_{K,\epsilon}$ l'ensemble des U, compacts de G, tels que $\lambda(U) > 0$ et $\lambda(xU \Delta U) < \epsilon\lambda(U)$ pour tout $x \in K$; $\overline{\mathcal{A}}_{K,\epsilon}$ désignera l'adhérence de $\mathcal{A}_{K,\epsilon}$ dans l'espace des formes trilinéaires continues sur $CV_p(G) \times L^p(G/H) \times L^{p'}(G/H)$ muni de la topologie faible de dualité avec $CV_p(G) \times L^p(G/H) \times L^{p'}(G/H)$.
L'étude de R sur $C_p(G)$ repose sur le lemme suivant:

Lemme 3.1. *Soient $T \in C_p(G)$ et $f \in C_{00}(G)$, alors* $R(T\lambda_G^p(f)) = R(T)\lambda_{G/H}^p(T_H f)$.

PREUVE. – Soient m, $n \in C_{00}(G/H)$ et $\delta > 0$. Posons $m' = \lambda_{G/H}^p(T_H f)m = R(\lambda_G^p(f))m$. Il existe V voisinage de e dans G avec

$$\|m' - m'_{\dot{z}}\, \Delta_{G/H}^{1/p'}(\dot{z})\|_p < \frac{\delta}{5(1 + |||T|||_p\, \|n\|_{p'})}$$

pour tout $z \in V$ et il existe $i \in I$ avec

$$\operatorname{supp} f_i \subset V \qquad \text{et} \qquad \|f_i * f - f\|_1 < \frac{\delta}{5(1 + |||T|||_p\, \|m\|_p\, \|n\|_{p'})}.$$

Il existe $w \in A_p(G)$ et $S \in cv_p(G)$ avec $T\lambda_G^p(f_i) = wS$. La preuve du théorème 6 de [9] montre l'existence de K compact de G et de $\epsilon > 0$ avec

$$\|(g^{1/p} * (g^{1/p'})^{\vee})w - w\|_{A_p(G)} < \frac{\delta}{5(1 + |||S|||_p\, \|m'\|_p\, \|n\|_{p'})},$$

ceci *pour tout* $g \in L^1(G)$, $g \ge 0$, satisfaisant $\|{}_z g - g\|_1 < \epsilon^p$ pour tout $z \in K$. On peut supposer, de plus, que

$$\epsilon < \frac{\delta}{5(1 + |||T|||_p\, \|m\|_p\, \|n\|_{p'})} \qquad \text{et} \qquad V \subset K .$$

De $\mathcal{R} \in \overline{\mathcal{A}}_{K,\epsilon^p}$, on tire l'existence de $U \in \mathcal{U}_{K,\epsilon^p}$ avec $|\mathcal{R}(T,m',n) - F_{r,s}(T,m',n)| < \frac{\delta}{5}$, où $\tau_p r = g^{1/p}$, $\tau_{p'} s = g^{1/p'}$, $g = \lambda(U)^{-1}1_U$. On a $g \geq 0$, $\|_z g - g\|_1 < \epsilon^p$ pour tout $z \in K$ et donc, également, $\|_z(\tau_p r) - \tau_p r\|_p < \epsilon$. On a donc pour $u = g^{1/p} * (g^{1/p'})^\vee = \tau_p r * (\tau_{p'} s)^\vee \in A_p(G)$:

$$\|uw - w\|_{A_p(G)} < \frac{\delta}{5(1 + |||S|||_p \|m'\|_p \|n\|_{p'})}.$$

On peut alors procéder à différentes estimations:

1) $|(R(T\lambda_G^p(f))m,n) - (R(T\lambda_G^p(f_i * f))m,n)| \leq |||R(T\lambda_G^p(f) - T\lambda_G^p(f_i * f))|||_p \|m\|_p \|n\|_{p'} < \frac{\delta}{5}$.
2) De $T\lambda_G^p(f_i)$, $\lambda_G^p(f) \in PF_p(G)$, on tire que $R(T\lambda_G^p(f_i * f))m = R(T\lambda_G^p(f_i))m'$. Par suite:
$|(R(T\lambda_G^p(f_i * f))m,n) - (R(u(T\lambda_G^p(f_i)))m',n)| \leq |||wS - uwS|||_p \|m'\|_p \|n\|_{p'} < \frac{\delta}{5}$.
3) La proposition 3 (iii) de [7] montre que $R(u(T\lambda_G^p(f_i))) = f_{r,s}(T\lambda_G^p(f_i))$. La partie (ii) de cette même proposition permet alors d'estimer:

$$|(R(u(T\lambda_G^p(f_i)))m',n) - (f_{r,s}(T)m',n)| = |(f_{r,s}(T\lambda_G^p(f_i))m',n) - (f_{r,s}(T)m',n)| \leq$$
$$|||T|||_p \|s\|_{p'} \|n\|_{p'} \left(\|r\|_p \sup_{z \in \operatorname{supp} f_i} \|m' - m'_z \Delta_{G/H}^{1/p}(\dot{z})\|_p + \|m\|_p \sup_{z \in \operatorname{supp} f_i} \|r_{z^{-1}} \Delta_G^{-1/p}(z) - r\|_p \right) < \frac{2\delta}{5}.$$

4) Rappelons que $|(f_{r,s}(T)m',n) - (R(T)\lambda_{G/H}^p(T_H f)m,n)| = |F_{r,s}(T,m',n) - \mathcal{R}(T,m',n)| < \frac{\delta}{5}$.
5) Finalement, de ce qui précède, on tire que $|(R(T\lambda_G^p(f))m,n) - (R(T)\lambda_{G/H}^p(T_H f)m,n)| < \delta$.
Ainsi, $(R(T\lambda_G^p(f))m,n) = (R(T)\lambda_{G/H}^p(T_H f)m,n)$ pour tout $m, n \in C_{00}(G)$, d'où la conclusion.
□

On peut alors énoncer le théorème principal sur R restreint à $C_p(G)$:

Théorème 3.2. *R est une application intrinsèque de $C_p(G)$ dans $C_p(G/H)$; $R : (C_p(G),\beta) \longrightarrow (C_p(G/H),\beta)$ est un homomorphisme continu d'algèbres, qui se dualise de la manière suivante: $R^* : W_p(G/H) \longrightarrow W_p(G)$, $v \longmapsto v \circ \omega$.*

PREUVE.– Soient $T \in C_p(G)$, $S \in PF_p(G)$ et $\epsilon > 0$. Du lemme 3.1, on tire que $R(TS) = R(S)R(T)$. Soient alors R et R_1, deux applications de Anker, $T \in C_p(G)$ et $f \in L^1(G)$; de $\operatorname{Res}_{PF_p(G)} R = \operatorname{Res}_{PF_p(G)} R_1$, on tire que $R(T)\lambda_{G/H}^p(T_H f) = R_1(T)\lambda_{G/H}^p(T_H f)$, d'où $R(T) = R_1(T)$; ainsi R est intrinsèque sur $C_p(G)$. Soient $T \in C_p(G)$ et $g \in L^1(G/H)$. Il existe $f \in L^1(G)$ avec $T_H f = g$. On a alors $R(T)\lambda_{G/H}^p(g) = R(T\lambda_G^p(f)) \in PF_p(G/H)$. Ainsi $R(T) \in C_p(G/H)$ et de $|||R(T)\lambda_{G/H}^p(g)|||_p \leq |||T\lambda_G^p(f)|||_p$, on tire que $R : C_p(G) \longrightarrow C_p(G/H)$ est β-continue. Soient S, $T \in C_p(G)$ et $f \in L^1(G)$; on a $R(ST)\lambda_{G/H}^p(T_H f) = R(ST\lambda_G^p(f)) = R(S)R(T\lambda_G^p(f)) = R(S)R(T)\lambda_{G/H}^p(T_H f)$ et ainsi $R(ST) = R(S)R(T)$. Soit $v \in W_p(G/H)$; pour tout $x \in G$, on a $(R^*(v))^\sim(x) = \langle R(\lambda_G^p(x)), v \rangle_{C_p(G/H),W_p(G/H)} = \tilde{v}(\dot{x})$, d'où $R^*(v) = v \circ \omega$.
□

Remarque. Supposons G abélien. Citons les résultats obtenus par Anker et Derighetti:
i) *Si $T \in CV_p(G)$ est tel que $\widehat{T} \in UC^b(\widehat{G})$, alors $\widehat{R(T)} = \operatorname{Res}_{H^\perp} \widehat{T}$* ([1, p. 77] et [6, p. 11]).
ii) *Si $T \in CV_p(G)$ est tel que $\widehat{T}$ est continue en chaque point de $H^\perp$, alors $\widehat{R(T)}$ coïncide localement presque partout avec $\operatorname{Res}_{H^\perp} \widehat{T}$* [Derighetti, non publié].
Ainsi, en particulier, R est intrinsèque sur de tels T. Le théorème 3.2 est donc assez proche de ces résultats.

Proposition 3.3. *i) $R(T) = \langle T, 1_G \rangle_{C_p(G),W_p(G)} \operatorname{id}_{L^p(G/H)}$ si $T \in C_p(G)$ et $\operatorname{supp} T \subset H$.*
ii) $R(uT) = \overline{\langle u.T \rangle}_{A_p(G),PM_p(G)} \operatorname{id}_{L^p(G/H)}$ pour tout $T \in PM_p(G) = CV_p(G)$ et $u \in A_p(G)$ tels que $\operatorname{supp} uT \subset H$.

PREUVE. – i) Considérons l'application linéaire et β-continue $\psi : C_p(G) \longrightarrow C_p(G/H)$, $T \longrightarrow <T, 1_G>_{C_p(G), W_p(G)} \mathrm{id}_{L^p(G/H)}$. Pour tout $h \in H$, on a $\psi(\lambda_G^p(h)) = \mathrm{id}_{L^p(G/H)} = R(\lambda_G^p(h))$; ainsi ψ et R coïncident sur l'espace vectoriel β-fermé engendré par $\{\lambda_G^p(h) \mid h \in H\}$. H étant de synthèse pour $A_p(G)$, par le théorème 8 de [4], on a que, pour tout $T \in C_p(G)$ avec $\mathrm{supp}\, T \subset H$, $R(T) = \psi(T)$, d'où la conclusion.
ii) Découle immédiatement de i) et du théorème 2.5. □

Derighetti a montré que $R(cv_p(G)) = cv_p(G/H)$ et, de plus, pour tout $T \in cv_p(G/H)$ et $\epsilon > 0$, il existe $S \in cv_p(G)$ avec $R(S) = T$ et $|||S|||_p \leq |||T|||_p + \epsilon$ [6, théorème 6]. Par contre, nous ignorons si $R(C_p(G)) = C_p(G/H)$; mais on peut remarquer que les techniques de [6] permettent d'obtenir ces résultats pour $PF_p(G)$. Comme corollaire, on peut alors citer les résultats suivants :

Proposition 3.4. *$R : PF_p(G) \longrightarrow PF_p(G/H)$ se dualise en $R^* : W_p(G/H) \longrightarrow W_p(G)$, $v \longrightarrow v \circ \omega$. R^* est une isométrie, l'image de R^* est exactement l'ensemble des éléments H-périodiques de $W_p(G)$.* □

La proposition 3.4 permet de donner une caractérisation de $\mathrm{Ker}\,(\mathrm{Res}_{PF_p(G)} R)$ en analogie avec celle de Reiter [11, 6.4, p.81] pour $\mathrm{Ker}\, T_H$:

Proposition 3.5. *$\mathrm{Ker}\,(\mathrm{Res}_{PF_p(G)} R)$ est l'espace vectoriel normiquement fermé engendré par chacune des parties suivantes de $PF_p(G)$:*
$\{\lambda_G^p({}_hf - f) \mid f \in C_{\infty}(G),\ h \in H\}$; $\{\lambda_G^p(f_h \Delta_G(h) - f) \mid f \in C_{\infty}(G),\ h \in H\}$. □

De manière similaire, on peut caractériser $\mathrm{Ker}\,(\mathrm{Res}_{C_p(G)} R)$ à l'aide de la topologie stricte :

Proposition 3.6. *$\mathrm{Ker}\,(\mathrm{Res}_{C_p(G)} R)$ est l'espace vectoriel strictement fermé engendré par chacune des parties suivantes de $C_p(G)$:*
$\{\lambda_G^p(xh) - \lambda_G^p(x) \mid x \in G,\ h \in H\}$; $\{\lambda_G^p(hx) - \lambda_G^p(x) \mid x \in G,\ h \in H\}$;
$\{\lambda_G^p({}_hf - f) \mid f \in C_{\infty}(G),\ h \in H\}$; $\{\lambda_G^p(f_h \Delta_G(h) - f) \mid f \in C_{\infty}(G),\ h \in H\}$. □

Terminons par l'étude du cas où H est compact.

Proposition 3.7. *$R : PM_p(G) \longrightarrow PM_p(G/H)$ est ultrafaiblement continue si et seulement si H est compact.*

PREUVE. – 1) Supposons tout d'abord R ultrafaiblement continue. Il s'ensuit qu'il existe $\phi : A_p(G/H) \longrightarrow A_p(G)$ linéaire et faiblement continue telle que $<\phi(u), T>_{A_p(G), PM_p(G)} = <u, R(T)>_{A_p(G/H), PM_p(G/H)}$ pour tout $u \in A_p(G/H)$ et $T \in PM_p(G)$. Un calcul explicite donne que $\phi(u) = u \circ \omega$. Par suite, pour tout $u \in A_p(G/H)$, $u \circ \omega \in A_p(G) \subset C_o(G)$ et donc H est compact.

2) Supposons H compact, alors $\phi : A_p(G/H) \longrightarrow A_p(G)$, $u \longmapsto u \circ \omega$, est une isométrie de $A_p(G/H)$ sur l'ensemble des éléments H-périodiques de $A_p(G)$ [9, proposition 6]. On va montrer que $<\phi(u), T>_{A_p(G), PM_p(G)} = <u, R(T)>_{A_p(G/H), PM_p(G/H)}$ pour tout $u \in A_p(G/H)$ et $T \in PM_p(G)$. Il suffit de le vérifier pour $u = \overline{k} * \check{\ell}$, avec k et $\ell \in C_{\infty}(G/H)$. Comme dans la preuve du lemme 3.1, il existe K compact de G et $\epsilon > 0$ avec

$$\|(g^{1/p} * (g^{1/p'})^{\check{}}\,)(u \circ \omega) - u \circ \omega\|_{A_p(G)} < \frac{\delta}{2(1 + |||T|||_p)},$$

ceci *pour tout* $g \in L^1(G)$, $g \geq 0$, satisfaisant $\|{}_xg - g\|_1 < \epsilon^p$ pour tout $x \in K$. De $\mathcal{R} \in \overline{\mathcal{A}}_{K,\epsilon^p}$, on tire l'existence de $U \in \mathcal{U}_{K,\epsilon^p}$ avec $|\mathcal{R}(T, \tau_p k, \tau_{p'} \ell) - F_{r,s}(T, \tau_p k, \tau_{p'} \ell)| < \dfrac{\delta}{2}$ où $r = \tau_p(g^{1/p})$, $s = \tau_{p'}(g^{1/p'})$, $g = \lambda(U)^{-1} 1_U$, g satisfaisant $\|{}_xg - g\|_1 < \epsilon^p$ pour tout $x \in K$ (cf la preuve du lemme 3.1). On a ainsi

$$\|(\tau_p r * (\tau_{p'} s)^\smile)(u \circ \omega) - u \circ \omega\|_{A_p(G)} < \frac{\delta}{2(1 + |||T|||_p)},$$

ce qui entraîne que $|<u \circ \omega, (\tau_p r * (\tau_{p'} s)^\smile)T>_{A_p(G),PM_p(G)} - <u \circ \omega, T>_{A_p(G),PM_p(G)}| < \frac{\delta}{2}$. Comme $(\tau_p r * (\tau_{p'} s)^\smile)T \in cv_p(G)$, on déduit des théorèmes 2.5 et 3.2 et de la proposition 3 (iii) de [7] que $<u \circ \omega, (\tau_p r * (\tau_{p'} s)^\smile)T>_{A_p(G),PM_p(G)} = <u, f_{r,s}(T)>_{A_p(G/H),PM_p(G/H)}$. De plus, $F_{r,s}(T, \tau_p k, \tau_{p'} \ell) = \overline{<k * \check{\ell}, f_{r,s}(T)>}_{A_p(G/H),PM_p(G/H)} = \overline{<u \circ \omega, (\tau_p r * (\tau_{p'} s)^\smile)T>}_{A_p(G),PM_p(G)}$ et $\mathcal{R}(T, \tau_p k, \tau_{p'} \ell) = \overline{<u, R(T)>}_{A_p(G/H),PM_p(G/H)}$.
Par suite, $|<u, R(T)>_{A_p(G/H),PM_p(G/H)} - <u \circ \omega, T>_{A_p(G),PM_p(G)}| < \delta$ et ainsi $R = \phi^*$ est ultrafaiblement continue. □

Corollaire 3.8. *Supposons H compact. Alors $R = \phi^*$ où $\phi^* : PM_p(G) \longrightarrow PM_p(G/H)$ est l'application duale de $\phi : A_p(G/H) \longrightarrow A_p(G)$, $u \longmapsto u \circ \omega$. En particulier, R est intrinsèque sur $PM_p(G)$.* □

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IRREDUCIBLE FACTORS AND P-ADIC POLES OF HIGHER ORDER BERNOULLI POLYNOMIALS

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Abstract. By analyzing the p-adic singularity pattern of the coefficients of the higher order Bernoulli polynomials, we have determined all instances of p-Eisenstein behavior, thereby generalizing known irreducibility results.

1. Introduction.

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad ,$$

and the arbitrary order Bernoulli polynomials $B_n^{(\omega)}(x)$ by

$$\left(\frac{t}{e^t-1}\right)^{\omega} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\omega)}(x)\frac{t^n}{n!} \quad . \quad [9,10]$$

These are called higher order, or more properly first or higher order, if $\omega = \alpha \in \{1,\ldots,n\}$. It is well known that $B_n^{(\omega)}(x)$ is a polynomial with rational coefficients, of degree n in x and ω, which is monic in x. It is also well known that $B_n^{(\omega)}(x)$ splits if $\omega = 0$ or $\omega = n+1$. (All factoring in this paper refers to $\mathbf{Q}$ unless otherwise indicated.) Also $(2x-\omega)|B_n^{(\omega)}(x)$ if n is odd, and $x(2x-1)(x-1)|B_n(x)$ if n is odd and > 1.

It had been conjectured that $B_n(x)$ is irreducible if n is even, and that $B_n(x)/x(2x-1)(x-1)$ is irreducible if n is odd and > 3. However, it was shown in [3] that the latter conjecture is false for $n = 11$. The former conjecture is still open. We proved in [1] that $B_n^{(\omega)}(x)$ is absolutely irreducible if n is even and > 0, and that $B_n^{(\omega)}(x)/(2x-\omega)$ is absolutely irreducible if n is odd and > 1. In this paper we announce

p–Eisenstein criteria that enable us to prove certain instances of rational irreducibility consistent with these results. Details of the proofs are given in [2].

Several authors (cf. [3–8]) have considered irreducibility of particular first and higher order Bernoulli polynomials and of certain factors. Their proofs usually invoke p–Eisenstein criteria for sufficient conditions, rely on substantial p–congruence properties of the Bernoulli numbers, and make heavy use of the classical von Staudt–Clausen theorem. Our results on factorization are more general than those in the literature, and typically involve necessary and sufficient conditions. Furthermore, they don't depend on the congruence machinery, but depend instead on detailed analysis of the p–adic poles of the coefficients.

We have been able to show that, viewing the coefficients from the highest degree down, for any $\omega = \alpha \in \{1, \ldots, n\}$

(1) there are no gaps in the orders of the poles,

(2) the poles occur with increasing order, and

(3) the first occurrences of poles of increasingly higher order can be algorithmically determined from the base p expansion of n.

The p–Eisenstein behavior occurs if and only if the highest order pole is simple. The preceding analysis can be used to determine <u>all</u> instances of p–Eisenstein behavior, thereby giving best possible irreducibility results of this type.

2. Main Results.

All numbers in this paper will be assumed to be positive integers, unless otherwise indicated. If p is a prime, let ν_p denote the <u>standard p-adic valuation</u> of **Q**. We say that r has a <u>pole of order b</u> if $\nu_p(r) = -b$.

Let $S(n) = S(n,p)$ be the <u>p-adic digit sum</u> of n, i.e. if $n = \sum n_i p^i$, where the digits satisfy $0 \le n_i \le p-1$, then $S(n) = \sum n_i$. We say that l is a <u>segment of</u> n starting in place u if $l = \sum_{i=u}^{v} n_i p^i$, including $l = 0$ if $v < u$. If $u = 0$ we call l a <u>bottom segment</u>.

If $S(n) \ge p-1$, we consider the <u>Kimura function</u> $N(n) = N(n,p)$ which can be defined as follows: if r is the largest number such that $\sum_{i=0}^{r} n_i \le p-1$, then

$$N(n) = \sum_{i=0}^{r} n_i p^i + (p-1-\sum_{i=0}^{r} n_i)p^{r+1}.$$

It follows that $S(N(n)) = p-1$ for our use of $N(n)$, which is more restrictive than Kimura's [5]. It is not hard to show then that $N(n) =$ smallest $t > 0$ such that $p-1|t$ and $p \not| \binom{n}{t}$, which is essentially Kimura's definition.

Write $B_n^{(\omega)}(x) = \sum_{i=0}^{n} c_i(\omega)x^{n-i}$, and if $\omega = \alpha \in \{1,\dots,n\}$, let $c_i = c_i(\alpha)$.

Define the <u>pole sequence</u> $i_1, i_2, \dots, i_b$ for $\omega = \alpha$ to be those i that give first occurrences of successively higher order poles. Thus $i_1 =$ smallest i such that $\nu_p(c_i) < 0$, $i_2 =$ smallest i such that $\nu_p(c_i) < \nu_p(c_\lambda)$ for all $0 \le \lambda \le i_1$, etc., i.e. $i_{j+1} =$ smallest i such that $\nu_p(c_i) < \nu_p(c_\lambda)$ for all $0 \le \lambda \le i_j$, with $i_0 = 0$ for convenience.

Theorem 1. *The following results characterize the pole sequence.*

(i) $\nu_p(c_i) = -j$ if $i = i_j$,

(ii) $S(i_j - i_{j-1}) = p-1$ *and* $S(i_j) = j(p-1)$,

(iii) $i_j =$ *smallest* i *such that* $p-1|i$, $\frac{S(i)}{p-1} \ge j$, $p \not| \binom{n}{i}$, *and* $p \not| \binom{\alpha-n-1}{\frac{i}{p-1}}$, *and*

(iv) $i_{j+1} - i_j = N\left(n - i_j - \sum_{\mu=0}^{j} l_\mu\right)$ *for a (unique) sequence* $l_0, l_1, \dots, l_{b-1}$ *such that* l_j *is a segment of* $n - i_j$ *starting in the place of the highest digit of* i_j.

If $f(x) = \sum_{i=0}^{n} c_i x^{n-i} \in \mathbf{Q}[x]$, we say $f(x)$ is <u>p-Eisenstein down to degree d</u> if $\nu_p(c_0) = 1$, $\nu_p(c_i) > 0$ if $0 < i < n-d$, $\nu_p(c_{n-d}) = 0$, and $\nu_p(c_i) \ge 0$ for $i > n-d$. (In particular, $f(x)$ is <u>p-Eisenstein</u> if $d = 0$.) It is well known that if $f(x)$ is p-Eisenstein down to degree d, then it has an irreducible factor of degree $\ge n-d$ (so p-Eisenstein implies irreducible).

By abuse of language, we will say that $B_n^{(\alpha)}(x)$ is <u>p-Eisenstein down to degree d</u> if $pB_n^{(\alpha)}(x)$ is. This amounts to having $n-d$ as the entire pole sequence of $B_n^{(\alpha)}(x)$, i.e. the first (simple) pole is in degree d and there is no higher order (double) pole. The following theorem, giving all instances of p–Eisenstein behavior, follows easily.

Theorem 2. *If $B_n^{(\alpha)}(x)$ is p-Eisenstein down to degree d, then $d = n - N(n-l)$ for some bottom segment l of n such that $S(n-l) \geq p-1$. (In particular, there is no p-Eisenstein behavior if $S(n) < p-1$.) Conversely if $d = n - N(n-l)$, then $B_n^{(l+1)}(x)$ is p-Eisenstein down to degree d; furthermore, $B_n^{(\alpha)}(x)$ is p-Eisenstein down to degree d iff the following three conditions holds:*

(i) $p \mid \binom{\alpha - n - 1}{\frac{N(n-l')}{p-1}}$ *for all bottom segments l' of n such that $l' < l$,*

(ii) $p \nmid \binom{\alpha - n - 1}{\frac{N(n-l)}{p-1}}$, *and*

(iii) $p \mid \binom{\alpha - n - 1}{\frac{N(n-l-l_1-N(n-l))}{p-1}}$ *if l_1 is any bottom segment of $n - l - N(n-l)$.*

In the following three corollaries, the bottom segment $l = 0$.

Corollary 1. *(Kimura) $B_n(x)$ is p-Eisenstein down to degree $n - N(n)$.*

Corollary 2. *(McCarthy) $B_n(x)$ is p-Eisenstein iff $S(n) = p-1$.*

Corollary 3. *The following are equivalent:*

(i) $S(n) = p - 1$.

(ii) *There exists $\alpha \in \{1, \ldots, n\}$ such that $B_n^{(\alpha)}(x)$ is p-Eisenstein.*

(iii) *$B_n(x)$ is p-Eisenstein.*

(iv) *$B_n^{(n)}(x)$ is p-Eisenstein.*

Furthermore, if these conditions hold, then $B_n^{(\alpha)}(x)$ is p-Eisenstein iff $p \nmid \binom{\alpha - n - 1}{\frac{n}{p-1}}$.

In the next three corollaries, $l = n_0$ the lowest digit. In the second, $l = n_0 = 1$. In the third, $l = n_0 = 1$ and $p = 2$.

Corollary 4. *If $n = m + l$ where $l \neq 0$ and $p \nmid (n)_l$, the following are equivalent:*

(i) $S(m) = p - 1$ *and* $p \mid m$.

(ii) *There exists $\alpha \in \{1, \ldots, n\}$ such that $B_n^{(\alpha)}(x)$ is p-Eisenstein down to degree l.*

(iii) $B_n^{(l+1)}(x)$ *is p–Eisenstein down to degree l.*

Furthermore, if these conditions hold, then $B_n^{(\alpha)}(x)$ *is p–Eisenstein down to degree l iff* $p \not| \binom{\alpha - n - 1}{\frac{m}{p-1}}$ *and* $p | \binom{\alpha - n - 1}{\frac{N(n)}{p-1}}$.

Corollary 5. *Suppose that* $p|n-1$ and $S(n-1) = p-1$. *Then* $B_n^{(2)}(x)$ *is p–Eisenstein down to degree* 1. *Furthermore,* $B_n^{(\alpha)}(x)$ *is p–Eisenstein down to degree* 1 *iff* $p \not| \binom{\alpha - n - 1}{\frac{n-1}{p-1}}$ *and* $p | \binom{\alpha - n - 1}{\frac{n-i}{p-1}}$ *where i is the largest p-power* $< n$.

Corollary 6. *If* $n = 2^r + 1$, *with* $r > 0$, *then* $B_n^{(\alpha)}(x)$ *is p–Eisenstein down to degree* 1 *iff* α *is even.*

The preceding corollaries gives a sufficient condition to prove that $B_n^{(\alpha)}(x)/(2x - \alpha)$ is irreducible for certain odd n and certain α. For example, corollary 5 applies if $n = 13$ by taking $p = 3$ with $\alpha = 2, 3, 11, 12$, while corollary 6 applies if $n = 2^r + 1$ and α is even.

3. The Proofs.

Expand $B_n^{(\alpha)}(x+1)$ in terms of the falling factorials $(x)_{n-i}$, namely if

$$B_n^{(\alpha)}(x+1) = \sum_{i=0}^{n} b_i (x)_{n-i}$$

then (b_i) has the same pole sequence as (c_i), with the same successively higher order poles.

In [1] we established the explicit formula

$$|b_i| = \sum_{w(u)=i} (n)_i \binom{\alpha - n - 1}{d(u)} \binom{d(u)}{u_1 u_2 \dots} \frac{1}{2^{u_1} 3^{u_2} \dots} \quad ,$$

where the sum is over all non-negative integer sequences $(u) = (u_1, u_2, \dots)$, eventually zero, with <u>weight</u> $w(u) = \sum_j j u_j = i$ and arbitrary <u>degree</u> $d(u) = \sum_j u_j$.

The main idea in our proof of Theorem 1 is to show that the successively higher order poles occur only in the terms where $u_j = 0$ for all $j \neq p-1$. It follows immediately that $p - 1 | i$ for these terms. The following lemma also plays a role.

Lemma. *Let $n > 0$ and suppose that $p - 1 | n$. Then* $p \nmid \binom{-n}{\frac{n}{p-1}}$ *iff* $S(n) = p - 1$.

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