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SURVEY OF A NEW GALOIS CORRESPONDENCE: ATTACHED SUBGROUPS AND ENDOMORPHIC SUBSEMIGROUPS

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Abstract: Given a group G and its semigroup of endomorphisms $\mathcal{E}$, we attach to each subgroup H of G, the subsemigroup of endomorphisms $\mathrm{S}(H)$ which fix the cosets of H. Moreover, to each subsemigroup S of $\mathcal{E}$ we attach the smallest subroup $\mathbf{H}(S)$ whose cosets are fixed by each element $\sigma \in S$. Then the attached subgroups of G, $\mathfrak{A}(G)$, and the attached subsemigroups of $\mathcal{E}$, $\mathfrak{A}(\mathcal{E})$, are complete sublattices of the lattice of normal subgroups $\mathfrak{N}(G)$ of G, and the lattice of "normal" subsemigroups, $\mathfrak{N}(\mathcal{E})$ of $\mathcal{E}$. The mappings S and $\mathbf{H}$ are order-preserving inverse isomorphisms between $\mathfrak{A}(G)$ and $\mathfrak{A}(\mathcal{E})$. Examples are given showing that the attached sublattice $\mathfrak{A}(G)$ gives quite specific information about the structure of G. A generalized commutator series is defined, and connected with previous work of other authors who restricted themselves to the stability group of automorphisms. The concept of nil endomorphisms is used to obtain information, for relatively free groups G, as to when automorphisms of factor groups G/N can be lifted to automorphisms of G.

1. INTRODUCTION

Let G be any group, $\mathcal{E} = \mathrm{End}(G)$ its semigroup of endomorphisms, $\mathrm{Aut}(G)$ its group of automorphisms, $\mathrm{Inn}(G)$ its group of inner automorphisms, and $\mathrm{Inn}_G(H)$ the subgroup of inner automorphisms of G generated by the elements of the subgroup $H \leq G$. As usual we call the endomorphism $\iota : g \to g$ the identity endomorphism, the endomorphism $\theta : g \to 1$ the trivial endomorphism, and we use i_g to denote conjugation of G by the element g. We write the action of the endomorphism σ as a right exponent, and abbreviate $(x^\sigma)^{-1} = (x^{-1})^\sigma$ by $x^{-\sigma}$. The group of automorphisms $\mathrm{Aut}(G)$, as a subsemigroup of $\mathcal{E}$, can act on $\mathcal{E}$ by multiplication, i.e., composition. However, it also can act on $\mathcal{E}$ by exponentiation, i.e., conjugation. Hence, if $\alpha \in\mathrm{Aut}(G), \sigma \in \mathcal{E}$ and $g \in G$ then $g^{\alpha\sigma} = (g^\alpha)^\sigma$, while $\sigma^\alpha = \alpha^{-1}\sigma\alpha$. In analogy with subgroups, we call a subsemigroup of endomorphisms $S \leq \mathcal{E}$ *invariant* under $\alpha \in\mathrm{Aut}(G)$ if $S^\alpha \leq S$. It is usual to define an exponential action of an element $b \in G$ on G by $g^b = g^{i_b} = b^{-1}gb$; in an analogous fashion we define an

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action of an element $b \in G$ on $\mathcal{E}$ by $\sigma^b = \sigma^{i_b} = i_b{}^{-1}\sigma\, i_b$. Moreover, in analogy with subgroups, we call a subsemigroup of endomorphisms $S \leq \mathcal{E}$ *normal* if it is invariant under G, and *characteristic* if it is invariant under $\mathrm{Aut}(G)$.

It is customary to use $[a,b]$ to denote $a^{-1}b^{-1}ab = a^{-1}a^{i_b}$, which is the commutator of a and b. In an analogous manner, we will use $[g,\sigma]$ to denote $g^{-1}g^{\sigma}$, the *commutator* of g and σ, where $g \in G, \sigma \in \mathcal{E}$. We also use $[\sigma, g]$ for $[g,\sigma]^{-1} = g^{-\sigma}g$.

If $S \leq \mathcal{E}$ then we define the commutator of G and S by $[G,S] = gp([g,\sigma]; \forall g \in G, \sigma \in S)$. We denote the lattice of subgroups in G under inclusion, by $\mathfrak{L}(G)$ and the lattice of subsemigroups in $\mathcal{E}$ under inclusion, by $\mathfrak{L}(\mathcal{E})$. The lattices $\mathfrak{L}(G)$ and $\mathfrak{L}(\mathcal{E})$ are complete, i.e., every subset of elements has a glb and a lub.

If $H \leq G$ then $\mathrm{Stb}(H)$ is the subsemigroup of $\mathcal{E}$ consisting of those $\sigma \in \mathcal{E}$ which map H back into itself, i.e., $H^{\sigma} \leq H$. If $\sigma \in \mathrm{Stb}(H)$, then σ induces a mapping of $G : H$, the set of left cosets of H in G, into itself given by $gH \rightarrow g^{\sigma}H$. If for all $g \in G$ we have $g^{\sigma}H = gH$ (or, equivalently, $g^{-1}g^{\sigma} \in H$) we say that σ fixes the left cosets $G : H$. By making use of the mapping $g \rightarrow g^{-1}$, one can show that σ fixes the left cosets $G : H$ iff σ fixes the right cosets of H in G.

Clearly if H_1, H_2 are subgroups of G and $H_1 \leq H_2$ then σ fixes the cosets of H_1 implies that σ fixes the cosets of H_2; similarly if S_1, S_2 are subsemigroups of $\mathcal{E}$ and $S_1 \leq S_2$ then the cosets of H are fixed by the endomorphisms in S_2 implies that the cosets of H are also fixed by the endomorphisms in S_1.

The dual notions of subsemigroups S of endomorphisms σ in $\mathcal{E}$ fixing the cosets of a given subgroup H of G, and those subgroups H of G whose cosets are fixed by given endomorphisms σ in $\mathcal{E}$, define a "Galois correspondence" between $\mathfrak{L}(\mathcal{E})$ and $\mathfrak{L}(G)$, but one which preserves inclusion (because it is not the elements of H which are fixed, but rather its cosets).

2. Attached subgroups and semigroups

Definition 1. If S is a subsemigroup in $\mathcal{E}$ then $\mathbf{H}(S)$ is the smallest subgroup H in G such that each $\sigma \in S$ fixes the cosets of $G : H$. Computationally,

$$\mathbf{H}(S) = gp(g^{-1}g^{\sigma}; \forall g \in G, \sigma \in S) = [G,S].$$

We say that $\mathbf{H}(S)$ is *attached* to S. The partially ordered set of attached subgroups of G under inclusion, is denoted by $\mathfrak{A}(G)$.∎

Definition 2. If H is a subgroup of G then $\mathbf{S}(H)$ is the largest subsemigroup S in $\mathcal{E}$ such that each $\sigma \in S$ fixes the cosets of $G : H$. Computationally,

$$\mathbf{S}(H) = \{\sigma \in \mathcal{E}; \forall g \in G, [g,\sigma] \in H\}.$$

We say that $\mathbf{S}(H)$ is *attached* to H. The partially ordered set of attached subsemigroups of $\mathcal{E}$ under inclusion, is denoted by $\mathfrak{A}(\mathcal{E})$.∎

We note that if $H \lhd G$ then each element σ of $\mathrm{Stb}(H)$ induces a natural endomorphism of G/H, $(\sigma)\nu : gH \rightarrow g^{\sigma}H$; $\mathbf{S}(H)$ is just the kernel of the homomorphism ν from $\mathrm{Stb}(H)$ into $\mathrm{End}(G/H)$.

Corollary 1. *If $S_1 \leq S_2$, then $\mathbf{H}(S_1) \leq \mathbf{H}(S_2)$; if $H_1 \leq H_2$ then $\mathbf{S}(H_1) \leq \mathbf{S}(H_2)$. Moreover, if $H \leq G$ then $\mathbf{H}(\mathbf{S}(H)) \leq H$; if $S \leq \mathcal{E}$ then $S \leq \mathbf{S}(\mathbf{H}(S))$.*

Proof. Immediate.∎

Example 1. Let G be any group. Then G and the trivial subgroup $\{1\}$ are attached to $\mathcal{E}$ and $\{\iota\}$, respectively. Moreover, the terms of the lower central series of G are attached; more generally, if H is any normal subgroup of G then the commutator group $[G, H]$ is attached to the semigroup $\mathrm{Inn}_G(H)$.

Dually, $\mathcal{E}$ and $\{\iota\}$ are attached to G and $\{1\}$ respectively. Clearly, $\{\iota\}$ can be adjoined to any subsemigroup S without changing the attached subgroup; moreover, if any subsemigroup contains $\{\theta\}$ it cannot be attached unless it is $\mathcal{E}$. Hence, every attached semigroup contains $\{\iota\}$ and excludes $\{\theta\}$ unless it is $\mathcal{E}$.■

Example 2. Let $G = \langle x \,|\, x^q \rangle$, where $q = p^r$, and p is a prime. Then $\mathcal{E} = \{\sigma_t : x \to x^t\,; 0 \le t < q\}$. If t is a multiple of p then $\sigma_t^q : x \to 1$ is trivial. Hence, all subsemigroups S of $\mathcal{E}$ not containing the trivial endomorphism, are subgroups of $\mathrm{Aut}(G)$, which is known to be cyclic, and have generators σ_a where a is coprime to p. Then $H = \mathrm{H}(S)$ is the (normal) subgroup generated by x^{a-1}. But then $a - 1 = kp^n$, where k is coprime to p; we may assume $k = 1$ and obtain the same subgroup. Thus the attached subsemigroups in $\mathcal{E}$ are the subgroups of $\mathrm{Aut}(G)$, $S_a = \mathrm{smgp}(\sigma_a;\, a = 1 + p^n,\ 0 < n < r)$, and of course, $\mathcal{E}$.■

Example 3. Let G be a vector space over F, where F is Z_p, the integers mod p, or Q, the rationals. The subgroups of G are just the linear subspaces of G. Choose a basis for G over F. Then any endomorphism of G is a linear map, and can be represented by a matrix over F, and conversely. The automorphisms of G are the non-singular matrices, and a change of basis maps any matrix of an endomorphism into its conjugate by an automorphism. Given a subgroup H of G, the endomorphisms in its attached subsemigroup in $\mathcal{E}$ is obtained as follows: choose a basis for G which includes a basis for H, map the basis elements of H arbitrarily back into H, and map each complementary basis element of G into itself added to any linear combination of the basis elements of H. Thus up to conjugation by a fixed automorphism depending just on H, the matrices in $\mathrm{S}(H)$ have the form of a 2×2 block of submatrices A, B, C, and D, where A is the matrix of the mapping of H into itself, $B = 0$, C is the projection of the images of the complementary basis into H, and $D = I$, the identity matrix (care has to be taken to use rows for images, instead of columns, since our mappings act on the right). Every subgroup of G is attached.■

Since an endomorphism of G is determined by its effect on the generators of G, it is not surprising that the attached subgroup of $S \le \mathcal{E}$ can be expressed in terms of generators for G. Specifically, we have the following:

Lemma 1. *Let $X = \{x_i\}$ be a set of generators for G, and let S be a subsemigroup of $\mathcal{E}$. Then*

$$\mathrm{H}(S) = ngp(x^{-1}x^{\sigma};\, x \in X,\, \sigma \in S) = ngp([x,\sigma];\, x \in X,\, \sigma \in S) = [G, S].$$

Proof. We shall show first that $\mathrm{H}(S)$ is in the normal subgroup generated by $\{x^{-1}x^{\sigma}\}$; in fact, we show that for fixed $\sigma \in S$, every element of the form $g^{-1}g^{\sigma}$ is a product of conjugates in G of elements $x^{-1}x^{\sigma}$, and their inverses, $x^{-\sigma}x$, where $x \in X$. For this purpose, we use induction on the length n of $g \in G$ as a word in x_i and x_i^{-1}. If $n = 1$ then $g = x^{\epsilon}$ where $x \in X$ and $\epsilon = \pm 1$. If $\epsilon = 1$ the result is immediate, while if $\epsilon = -1$ we observe that $(x^{-1})^{-1}(x^{-1})^{\sigma} = xx^{-\sigma} = x(x^{-\sigma}x)x^{-1}$.

If g_1 has length $n+1$ then $g_1 = x^{\epsilon}g$ where g has length n, $x \in X$, and $\epsilon = \pm 1$. Hence,

$$g_1^{-1}g_1^{\sigma} = g^{-1}(x^{-\epsilon}x^{\epsilon\sigma})g \cdot g^{-1}g^{\sigma}, \tag{1}$$

and $\mathbf{H}(S)$ is in the given normal subgroup.

That the normal subgroup is in $\mathbf{H(S)}$, follows from rewriting equation (1) as

$$g^{-1}(x^{-\epsilon}x^{\epsilon\sigma})g = g_1^{-1}g_1^{\sigma} \cdot (g^{-1}g^{\sigma})^{-1}. \blacksquare$$

The duality between the attached subgroups of G and attached subsemigroups of $\mathcal{E}$ might lead one to suspect that the generators $[x_i, \sigma]$ whose conjugates generate the normal subgroup $[G, S]$ in Lemma 1 could be replaced by generators $[x_i, \tau_j]$ where $S = smgp(\{\tau_j\})$. However, this is not true in general, as the following example shows.

Example 4. *Let G be the vector space of 2-tuples over Z_3. Then $G = gp(x_1, x_2)$ where $x_1 = (1,0)$ and $x_2 = (0,1)$. Let $S = smgp(\sigma_1, \sigma_2)$, where*

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \sigma_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

An easy calculation shows that $[x_1, \sigma_1] = (0,0)$, $[x_2, \sigma_1] = (0,1)$, $[x_1, \sigma_2] = (0,1)$, $[x_2, \sigma_2] = (0,0)$. Hence, $N = ngp([x_i, \sigma_j]; 1 \le i, j \le 2) = gp((0,1))$. On the other hand, S contains $\sigma = \sigma_1\sigma_2$ where

$$\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

and $[x_2, \sigma] = (1,0)$ which is not in N. $\blacksquare$

We can find normal generators for $[G, S]$ that involve semigroup generators of S by using "repeated commutators" of the form $[g, \tau_1, \tau_2, \dots, \tau_n]$ where $g \in G$, and $\tau_i \in \mathcal{E}$. But first we must define them precisely, and establish some commutator identities.

Definition 3. *Let G be a group, $\mathcal{E}$=End(G); let X be a subset of G, T a subset of $\mathcal{E}$, x denote an element of X, and τ denote an element of T. Then we define a (left-normed)* repeated commutator of weight n in X and T *inductively as follows: $[x, \tau]$ is a commutator in X and T of weight two; if c is a repeated commutator of weight n in X and T, then $[c, \tau]$ is a repeated commutator in X and T of weight $n+1$.* $\blacksquare$

Lemma 2. *For any $x, y \in G$, and $\sigma, \tau \in S$*

$$[x^{-1}, \sigma] = x[x,\sigma]^{-1}x^{-1} = [x,\sigma]^{-x^{-1}} \tag{2}$$

$$[x, \sigma\tau] = [x,\sigma][x^{\sigma},\tau] = [x,\tau][x,\sigma][x,\sigma,\tau] \tag{3}$$

$$[xy, \sigma] = [x,\sigma]^{y}[y,\sigma] = [y,\sigma][x,\sigma]^{y^{\sigma}} \tag{4}$$

$$[x,\sigma]^{\alpha} = [x^{\alpha}, \sigma^{\alpha}], \qquad \textit{for any } \alpha \in Aut(G) \tag{5}$$

PROOF. The identities follow immediately by "free cancellation", after replacing each commutator involved, by its definition in terms of elements of G and of $\mathcal{E}$.■

Corollary 2. *Let G be a group, and let $\mathcal{E} = End(G)$. Let X be a set of group generators of G and $T = \{\tau_i\}$ be a set of semigroup generators for a subsemigroup $S \leq \mathcal{E}$.*

(i): *If $\sigma = \tau_1\tau_2\cdots\tau_r$ and $g \in G$, then $[g,\sigma]$ is a product of repeated commutators $[g,\tau_{i_1},\cdots,\tau_{i_m}]$, where $1 \leq i_1 < i_2 < \ldots < i_m \leq r$.*

(ii): *$[G,S]$ has as normal generators, repeated commutators in X and T, and their inverses.*

PROOF. Let $\sigma = \tau_1\tau_2\cdots\tau_r$, where $\tau_i \in T$. We use induction on r. If $r = 1$ then $[g,\tau_1]$ has the required form. Let us consider $[g,\tau\sigma]$, where $\tau \in T$. From equation (3) of Lemma 2, we have

$$[g,\tau\sigma] = [g,\sigma][g,\tau][g,\tau,\sigma]. \tag{6}$$

From the inductive assumption, it follows for all $g \in G$ that

$$[g,\sigma] = c_1(g)\cdots c_s(g),$$

where each $c_i(g) = [g,\tau_{i_1},\cdots,\tau_{i_m}]$ with $1 \leq i_j < i_{j+1} \leq r$. Hence, from equation (6) we have

$$[g,\tau\sigma] = c_1(g)\cdots c_s(g)[g,\tau]c_1([g,\tau])\cdots c_s([g,\tau]),$$

which has the required form. This establishes (i).

We now find normal generators for $[G,S] = gp([g,\sigma];\forall g \in G, \sigma \in S)$. Using Lemma 1 it follows that $[G,S]$ is normally generated by elements of the form $[x_i,\sigma]$ and their inverses, where $x_i \in X$ and $\sigma \in S$. Hence, using the result in (i), $[G,S]$ is normally generated by repeated commutators in X and T, and their inverses.■

We now apply our results on commutators to attached subgroups and subsemigroups.

Lemma 3. *Let $H \leq G$, and let $S \leq \mathcal{E}$.*

(i): *Every attached subgroup $\mathbf{H}(S)$ is normal in G.*

(ii): *If a subgroup H of G is invariant under $\alpha \in Aut(G)$, then so is $\mathbf{S}(H)$.*

(iii): *Dually, if $S \leq \mathcal{E}$ is invariant under $\alpha \in Aut(G)$, then so is $\mathbf{H}(S)$.*

PROOF. Statement (i) follows immediately from Lemma 1.

To show (ii), let σ be fixed; we use $[G,\sigma]$ to denote the set $\{[g,\sigma];\forall g \in G\}$. Then by Definition 2, $\sigma \in \mathbf{S}(H)$ iff $[G,\sigma] \leq H$. If now $H^\alpha \leq H$, then from equation (5) it follows that $[G,\sigma^\alpha] = [G^\alpha,\sigma^\alpha] = [G,\sigma]^\alpha \leq H^\alpha \leq H$. This implies that $\sigma^\alpha \leq S = \mathbf{S}(H)$ and $S^\alpha \leq S$. Hence, (ii) follows.

Finally, if $S^\alpha \leq S$ then $[g,\sigma]^\alpha = [g^\alpha,\sigma^\alpha] \in [G,S] = H = \mathbf{H}(S)$, which implies $H^\alpha \leq H$. Thus, (iii) follows.■

Although every attached subgroup is normal (and as we shall show in Corollary 3, every attached subsemigroup is normal), normality is necessary but not sufficient to be attached, as the following two examples show.

Example 5. *Let $G = Q$, the rationals under addition. Then an endomorphism σ of Q is determined by the image of the integer 1; specifically, if $\sigma : 1 \to p/q$ then $\sigma : r/s \to rp/sq$. Hence, if $H = Z$, the subgroup of integers, then $[r/s, \sigma] = (-r/s) + rp/qs = (r/s)(1 - p/q)$ is an integer for all r/s iff $p/q = 1$, i.e., the only endomorphism which fixes the cosets $Q{:}Z$ is the identity, and its attached subgroup consists of 0 alone and hence Z is not attached.*∎

Example 6. *Let $G = S_3 = \langle a, b \mid a^3, b^2, (ab)^2\rangle$ and let $H = gp(a)$. $End(G) = \mathcal{E}$ consists of 12 endomorphisms, six of which are automorphisms mapping $a \to \{a, a^{-1}\}, b \to \{b, ba, ba^{-1}\}$, say $Aut(G) = \{\alpha_1, \ldots, \alpha_6\}$; five are not automorphisms and map $a \to 1$ and b into an element of order 2, say $S = \{\epsilon_1, \epsilon_2, \epsilon_3\}$; or map $b \to 1$ and a into an element of order 3, say $T = \{\epsilon_4, \epsilon_5\}$; the last is the trivial endomorphism θ. Now G has only 3 normal subgroups, G, $gp(a)$, and $\{1\}$. The corresponding attached subsemigroups of $\mathcal{E}$ are $\mathcal{E}$, $V = Aut(G) \cup S$, and $\{\iota\}$, each of which is normal in $\mathcal{E}$. However, neither $Aut(G)$ nor S occur as attached subsemigroups, although they are normal in $\mathcal{E}$.*∎

We denote the sublattice of $\mathfrak{L}(G)$ given by normal subgroups of G by $\mathfrak{N}(G)$, and the sublattice of $\mathfrak{L}(\mathcal{E})$ given by normal subsemigroups of $\mathcal{E}$ by $\mathfrak{N}(\mathcal{E})$, respectively.

We shall presently show that the the partially ordered sets $\mathfrak{A}(G)$ of attached subgroups of G and $\mathfrak{A}(\mathcal{E})$ of attached subsemigroups of $\mathcal{E}$ are isomorphic sublattices of $\mathfrak{N}(G)$ and $\mathfrak{N}(\mathcal{E})$, respectively. For that purpose we shall use:

Lemma 4. *If $\{H_i\}$ and $\{S_j\}$ are subgroups of G and subsemigroups of $\mathcal{E}$, respectively, then*

$$S(\cap H_i) = \cap S(H_i). \text{ Moreover, } \mathbf{H}(smgp_j(S_j)) = gp_j(\mathbf{H}(S_j)).$$

PROOF. First of all $\cap \mathrm{S}(H_i) = \{\sigma;\ g^{-1}g^\sigma \in H_i, \forall i\} = \{\sigma;\ g^{-1}g^\sigma \in \cap H_i\} = \mathrm{S}(\cap H_i)$.

To show the second equality, we use identity (3) of Lemma 2, namely, $[x, \tau\sigma] = [x, \tau][x^\tau, \sigma]$.∎

Theorem 1. *The partially ordered sets $\mathfrak{A}(\mathcal{E})$ and $\mathfrak{A}(G)$ are complete sublattices of $\mathfrak{N}(\mathcal{E})$ and $\mathfrak{N}(G)$, respectively, and are isomorphic under the inverse isomorphisms $\mathbf{H}$ and S.*

PROOF. To show that **H** and **S** are isomorphisms we need only check that **HoS** is the identity map on $\mathfrak{A}(G)$ while **SoH** is the identity map on $\mathfrak{A}(\mathcal{E})$. Let $H_1 \in \mathfrak{A}(G)$. Then by Corollary 1, $\mathbf{H}(\mathbf{S}(H_1)) \le H_1$. Since $H_1 \in \mathfrak{A}(G)$, there exists $S_1 \le \mathcal{E}$ such that $H_1 = \mathbf{H}(S_1)$. By using Corollary 1 again, we obtain $S_1 \le \mathbf{S}(\mathbf{H}(S_1))$, and so $H_1 = \mathbf{H}(S_1) \le \mathbf{H}(\mathbf{S}(\mathbf{H}(S_1)))$. Hence, we have $H_1 = \mathbf{H}(\mathbf{S}(\mathbf{H}(S_1)))$, and so **HoS** is the identity on $\mathfrak{A}(G)$. In a similar manner, one shows that **SoH** is the identity on $\mathfrak{A}(\mathcal{E})$. It follows from (i) and (ii) of Lemma 3 that $\mathfrak{A}(G)$ consists of normal subgroups, i.e., is in $\mathfrak{N}(G)$, and that the image of $\mathfrak{A}(G)$ under **S** consists of normal subsemigroups of $\mathcal{E}$, i.e., is in $\mathfrak{N}(\mathcal{E})$.

It is obvious that a partially ordered system in which every subset of elements has a glb (or dually a lub) is complete, since the glb of the upper bounds of a set of elements is its lub. Moreover, by Lemma 4, the glb in $\mathfrak{N}(\mathcal{E})$ of a set of elements in $\mathfrak{A}(\mathcal{E})$ is itself in $\mathfrak{A}(\mathcal{E})$, and so $\mathfrak{A}(\mathcal{E})$ is a complete sublattice of $\mathfrak{N}(\mathcal{E})$. Again by Lemma 4, dually, the lub in $\mathfrak{N}(G)$ of a set of elements in $\mathfrak{A}(G)$ is itself in $\mathfrak{A}(G)$, and so $\mathfrak{A}(G)$ is a complete sublattice of $\mathfrak{N}(G)$.∎

Corollary 3. *Every attached subsemigroup of $\mathcal{E}$ is normal.*

Proof. The result is immediate from Theorem 1.∎

Since $\mathfrak{A}(G)$ is a sublattice of $\mathfrak{N}(G)$ one may ask when $\mathfrak{A}(G) = \mathfrak{N}(G)$. The dual question, as we shall see requires some restrictions on the subsemigroups considered.

Definition 4. *A group G is called* full *iff every normal subgroup H of G is attached. Dually, the endomorphism semigroup End(G) is called* full *iff every normal subsemigroup excluding the trivial endomorphism but including the identity automorphism is attached.*∎

Example 6 shows that G being full does not imply that End(G) is full. On the other hand, Example 2 is one in which both G and End(G) are full, since there are precisely $r - 1$ proper subgroups, each of which is attached to a corresponding proper subgroup of Aut(G), and G itself which attaches to End(G). We shall show that relatively free groups are full, but that the converse is false. We shall also show that if we consider only subgroups of G attached to subsemigroups of Aut(G), then we may get a proper subset of the attached subgroups.

Lemma 5. *In a relatively free group G, any normal subgroup is attached, i.e., $\mathfrak{A}(G) = \mathfrak{N}(G)$.*

Proof. Any mapping of relatively free generators $\{x_i\}$ of G induces an endomorphism of G, and so the endomorphisms $\sigma : x_i \to x_i h_i$, where each h_i is in the normal subgroup H of G form a subsemigroup $S \leq \mathcal{E}$ whose attached subgroup is H.∎

Example 7. *Let G be the infinite dihedral group $\langle a, b \mid a^2, b^2 \rangle$. G is not relatively free, since when abelianized, it has exponent 2, and this would be true for G itself.*

We first show that G is full, i.e., any normal subgroup H of G is attached. Indeed, the normal subgroups of G are of the following type: $H_1 = gp((ab)^k)$, where k is a fixed positive integer, $H_2 = gp(a, (ab)^2), H_3 = gp(b, (ab)^2), H_4 = 1$, or $H_5 = G$. Then $H_i = \mathbf{H}(S_i)$ where $S_i = smgp(\sigma_i)$ and $\sigma_1 : a \to a(ab)^k, b \to b$; $\sigma_2 : a \to 1, b \to aba$; $\sigma_3 : a \to bab, b \to 1$; $\sigma_4 : a \to a, b \to b$; $\sigma_5 : a \to 1, b \to 1$. Thus G is full, which shows that the converse of Lemma 5 is false.

We shall now show that there are subgroups of G that are attached only to subsemigroups S which contain proper endomorphisms. Specifically, the subgroups H_2, H_3, and H_5 contain either a or b. Suppose S has only automorphisms and is attached to one of these subgroups. Under any automorphism α, both a and b must go into elements of order 2; such elements are of odd length as words in alternating symbols a and b, and so have the form $a(ba)^k$, or $b(ab)^k$. But since a and b generate G, from Lemma 1, $\mathbf{H}(S)$ is normally generated by $a^{-1}a^\alpha$ and $b^{-1}b^\alpha$ and these are in $ngp(ab)$. Hence, $\mathbf{H}(S) \leq ngp(ab)$ which does not contain a or b.∎

The structure of the lattice of attached subgroups is an isomorphic invariant of a group, and may, in certain cases, give quite specific information about the group.

Theorem 2. *If a finitely generated group G has no proper attached subgroups then G is either simple, or is a perfect group which is the central extension of a non-abelian simple group.*

PROOF. If G is simple then G has no proper normal subgroup, and since attached subgroups are normal, G has no proper attached subgroup.

Suppose then that G is not simple. We first show that G cannot be abelian. For this purpose, we note that if G has a retraction ρ, onto the subgroup H with kernel K, then the element hk with $h \in H$ and $k \in K$ has its commutator $[hk, \rho] = (hk)^{-1}(hk)^{\rho} = k^{-1}h^{-1}h = k^{-1}$. Thus the proper subgroup K is attached to the subsemigroup of $\mathcal{E}$ generated by ρ. But a proper direct product can be retracted onto any direct factor. Now, a finitely generated abelian group is a proper direct product, unless it is infinite cyclic, or finite cyclic of order p^s, where p is prime and $s > 1$. We show that such cyclic groups contain proper attached subgroups. In the case of an infinite cyclic group G, the subsemigroup of endomorphisms generated by $\sigma : g \to g^3$ has the proper subgroup $gp(g^2)$ as its attached subgroup. On the other hand, if G is cyclic of order p^s, then $\tau : g \to g^{p+1}$ generates a subsemigroup of automorphisms whose attached subgroup is $gp(x^p)$, which is proper. Thus G cannot be abelian.

We may assume then that G is not simple and not abelian. We show then that G has a non-trivial centre C which contains all proper normal subgroups of G. Indeed, let N be any proper normal subgroup of G. Now the attached subgroup of $\mathrm{Inn}_G(N)$ is $[G, N]$, which is contained in N, and so is proper, unless it is 1. Therefore, $[G, N] = 1$ and $N \leq C$. Thus, C is the maximum proper normal subgroup of G and G/C is simple. We now show that G is perfect. Since G is not abelian, $[G, G] \neq 1$. If $[G, G] \neq G$, it is a proper normal subgroup and so in C. But then G/C is both abelian and simple, and hence a prime cyclic group. A central extension by a cyclic group is abelian. Thus $G = [G, G]$ and G is perfect.∎

3. STABILITY SEMIGROUPS

Several authors have considered normal series in which the factor groups were fixed by a given subgroup of automorphisms (see, e.g., [4, 3, 7]). We shall consider such series for every subsemigroup of endomorphisms, but first we need some more properties of commutators.

Lemma 6. *Let $U, H \leq G$, $S \leq \mathcal{E}$. Then*

- **(a):** *$[U, S]$ is $Inn_G(U)$-invariant and S-invariant;*
- **(b):** *$[U, S] \leq U$ iff U is S-invariant;*
- **(c):** *$[U, S]$ is the smallest subgroup of U such that S fixes the U-cosets of $[U, S]$;*
  in particular, $[G, S(H)] \leq H$;
- **(d):** *$H \lhd G$ iff $[G, H] \leq [G, S(H)]$ iff for all U, $[U, H] \leq [U, S(H)]$;*
- **(e):** *$[G, S] = H(S)$ for all $S \leq \mathcal{E}$*
- **(f):** *$[G, S(H)] = H$ iff H is an attached subgroup in G;*
- **(g):** *For any $\alpha \in Aut(G)$ we have that $[U, S]^{\alpha} = [U^{\alpha}, S^{\alpha}]$;*
- **(h):** *If U^G and S^G denote the normal closure of a subgroup U and a subsemigroup S, respectively, then*

$$[U, S]^G \leq [U^G, S^G] = [U, S^G]^G = [U^G, S]^G; \tag{7}$$

(i): *If A=Aut(G) and U^A and S^A denote the characteristic closure of a subgroup U and a subsemigroup S, respectively, then*

$$(8) \qquad [U,S]^A \le [U^A,S^A] = [U,S^A]^A = [U^A,S]^A.$$

PROOF. To prove (a) we use the commutator identities (4) and (3) from Lemma 2 to obtain

$$[u_1,\sigma]^u = [u_1u,\sigma][u,\sigma]^{-1} \qquad \text{and} \qquad [u,\sigma]^\tau = [u,\tau]^{-1}[u,\sigma\tau].$$

For (b), the proof follows from the fact that $[u,\sigma] = u^{-1}u^\sigma \in U$ iff $u^\sigma \in U$.

To prove (c) we note that, from (a) we have $S \le \mathrm{Stb}([U,S])$, so that each $\sigma \in S$ defines a mapping on the cosets of $[U,S]$; since $u^{-1}u^\sigma \in [U,S]$ the U-cosets of $[U,S]$ are fixed by S. Moreover, if the U-cosets of a subgroup J of G are fixed by each $\sigma \in S$ then $u^{-1}u^\sigma \in J$, and so $[U,S] \le J$. In particular, $[G,\mathrm{S}(H)] \le H$.

For (d), if $H \lhd G$ then $\mathrm{Inn}_G(H) \le \mathrm{S}(H)$, and therefore, for all U, $[U,H] \le [U,\mathrm{S}(H)]$. On the other hand, if $[G,H] \le [G,\mathrm{S}(H)] \le H$ (this last containment follows from (c)), then we have $g^{-1}hg \in H$.

The result (e) follows immediately from Definition 1.

For (f), we note that from (e), $[G,S] = \mathbf{H}(S)$, and so $[G,\mathrm{S}(H)] = \mathbf{H}(\mathrm{S}(H))$. If this is H then H is obviously attached; if H is attached then by Theorem 1 this is H.

For (g), we use the identity (4) of Lemma 2 so that for every $\alpha \in \mathrm{Aut}(G)$, $[x,\sigma]^\alpha = [x^\alpha,\sigma^\alpha]$, and thus, $[U,S]^\alpha = [U^\alpha,S^\alpha]$.

To show (h) we note first that $[U,S]$ is monotonic increasing in both its variables U,S. Hence, $[U,S]^G \le [U^G,S^G]$. Next from the first part of (g),

$$[U^\alpha,S^\beta] = [U,S^{\beta\alpha^{-1}}]^\alpha = [U^{\alpha\beta^{-1}},S]^\beta.$$

If $\alpha,\beta \in \mathrm{Inn}(G)$, we get the remaining equalities.

To show (i), we repeat the argument for (h) but allow α,β to range independently over Aut(G), instead of over Inn(G).∎

As the next example shows, the first inequalities in (h) and (i) of Lemma 6 cannot, in general, be replaced by equalities.

Example 8. *Let $G = \langle a,b \mid a^2,b^2\rangle$ as in Example 7. Let $U = gp(aba) = \{1,aba\}$, and $\sigma : a \to b$ and $b \to a$, so that $S = smgp(\sigma) = \{\iota,\sigma\}$. Then $[U,S] = gp((ab)^3)$ which is normal and characteristic in G, while $[aba,\sigma^a] = ab \in [U,S^G]$ but not in $[U,S] = [U,S]^G$. Thus $[U,S^G] \ne [U,S]^G$. Similarly $[U,S^A] \ne [U,S]^A$, where A =Aut(G).*∎

Specifically, the authors, in [4, 3, 7], have considered series of subgroups

$$G = J_0 \ge J_1 \ge J_2 \ge \dots \ge J_k \ge J_{k+1} \ge \dots$$

and a subgroup of automorphisms A which fixes all the cosets $J_k : J_{k+1}$. A was called a *stability group* (although, *co-stability* group might be more appropriate) for this series. We shall consider a subsemigroup of endomorphisms which fixes all the cosets $J_k : J_{k+1}$ and shall speak of a *stability semigroup* for this series. We shall also call the series $\{J_i\}$ *stable* under S. It will be shown that every subsemigroup $S \leq \mathcal{E}$ has a *minimal stable series*. For this purpose we introduce a generalized *commutator series*.

Definition 5. *If S is a subsemigroup of endomorphisms of G then the series*

$$G = H_0 \geq H_1 \geq H_2 \geq \dots \geq H_k \geq H_{k+1} \geq \cdots$$

given by $H_{k+1} = [H_k, S] = [G, {}_{k+1}S]$, is called the commutator series *of S in G. If for some smallest integer n, $H_n = H_{n+1}$ then we say that the series has* length *n.*■

Theorem 3. *The commutator series $\{H_i\}$ of S in G is a descending series stable under S, each term of which is normal in the preceding. Moreover, if*

$$G = J_0 \geq J_1 \geq J_2 \geq \dots \geq J_k \geq J_{k+1} \geq \cdots$$

is any series in G stable under S, then $J_k \geq H_k$.

Proof. To show by induction on k that H_{k+1} is S-invariant and normal in H_k, we use (a) and (c) of Lemma 6 . By (a), if $k = 0$ then $H_1 = [G, S] = \mathbf{H}(S)$ is S-invariant and normal in G. Moreover by (c), $H_1 = \mathbf{H}(S) \leq J_1$. Again by (a) we have that $H_{k+1} = [H_k, S]$ is S-invariant and is normal in H_k.

By inductive hypothesis, $J_k \geq H_k$. Since S fixes the J_k-cosets of J_{k+1}, it fixes the H_k-cosets of J_{k+1}, and therefore by (c), $J_{k+1} \geq [H_k, S] = H_{k+1}$.■

Corollary 4. *If S is a normal (characteristic) subsemigroup of $\mathcal{E}$ then each term of the commutator series of S in G is normal (characteristic) in G.*

Proof. We first show the normality, using (h) of Lemma 6; the characteristic case is dealt with using (i). We use induction on k. Since $[G, S] = [G^G, S^G] = [G, S]^G$, we have our result for $k = 0$. Suppose by inductive hypothesis that $[G, {}_kS] = U$ is normal in G. From (h) we have $[U, S]^G = [U^G, S^G] = [U, S]$ so that $[G, {}_{k+1}S]$ is normal in G.■

For any subsemigroup $S \leq \mathcal{E}$ (perhaps neither normal, nor characteristic) we also have a *minimal stable series of normal (characteristic) subgroups* in G.

Lemma 7. *Let G be a group, and $\mathcal{E} = End(G)$. For $S \leq \mathcal{E}$, the minimal stable series of subgroups normal (characteristic) in G, is the commutator series of S^G, the normal closure of S in G (S^A, the characteristic closure of S in G).*

Proof. We first consider the normal case. By Corollary 4 the commutator series of S^G

$$G = N_0 \geq N_1 \geq N_2 \geq \dots \geq N_k \geq N_{k+1} \geq \cdots$$

consists of normal subgroups of G. Let

$$G = J_0 \geq J_1 \geq J_2 \geq \dots \geq J_k \geq J_{k+1} \geq \cdots$$

be any series of normal subgroups, stable under S. We use induction on k. For $k = 0$, $N_0 = J_0 = G$. By inductive hypothesis, $N_k \leq J_k$. Then

$$N_{k+1} = [N_k, S^G] \leq [J_k, S^G].$$

Since J_k is normal in G, we have from (h) (the characteristic case uses (i)) of Lemma 6

$$[J_k, S^G] = [J_k{}^G, S^G] = [J_k{}^G, S]^G = [J_k, S]^G.$$

Since S leaves the J_k-cosets of J_{k+1} fixed, we have by (c) of Lemma 6 that $[J_k, S] \leq J_{k+1}$. But J_{k+1} is normal in G, and so

$$N_{k+1} \leq [J_k, S]^G \leq J_{k+1}.$$

The normal case is now done; the characteristic case follows in a similar manner.∎

The commutator series of S in G may have infinite length, or finite length; it may end in the identity subgroup, or otherwise.

Lemma 8. *The k-th term of the commutator series of $S(G')$ in G is the $k+1$-st term of the lower central series of G, i.e., $[G, {}_kS(G')] = \gamma_{k+1}(G)$.*

PROOF. Let $S = S(G')$, which consists of those endomorphisms of G that induce the identity on the commutator quotient group G/G'. Then we shall show that $[G, {}_kS] = \gamma_{k+1}(G)$, by using induction on k. For $k = 0$ the result is immediate. By inductive hypothesis,

$$[G, {}_kS] = [[G, {}_{k-1}S], S] = [\gamma_k(G), S].$$

Since $\gamma_k(G)$ and S are normal, it follows from (h) of Lemma 6 that this last subgroup is a normal subgroup of G. Now $\gamma_k(G)$ is generated by repeated commutators $[g_1, \ldots, g_k]$ where $g_i \in G$ (see, e.g.,[6]). From Lemma 1, it follows that $[\gamma_k(G), S]$ is generated by conjugates of elements w of the form

$$[g_1, \ldots, g_k]^{-1}[g_1, \ldots, g_k]^\sigma = [g_1, \ldots, g_k]^{-1}[g_1c_1, \ldots, g_kc_k]$$

where $c_i \in G'$. It follows from the commutator calculus that w is in $\gamma_{k+1}(G)$ (see, e.g.,[6]). Hence, $[G, {}_kS] \leq \gamma_{k+1}(G)$.

The reverse inclusion follows from the fact that the inner automorphisms of G are in S, so that

$$\gamma_{k+1}(G) = [\gamma_k(G), \mathrm{Inn}(G)] \leq [\gamma_k(G), S] = [G, {}_kS].∎$$

Corollary 5. *A group G is nilpotent of class k iff the commutator series of $S(G')$ has length k and ends with 1. Moreover, G is residually nilpotent iff the intersection of the commutator series of $S(G')$ in G is 1.*

PROOF. This follows immediately from Lemma 8.∎

Lemma 9. *The Hall-Witt commutator identity*

$$[a, b, c^a][c, a, b^c][b, c, a^b] = 1$$

for elements a, b, c in G, holds also if a, c are elements of G but $b = \beta$ is an endomorphism of G.

PROOF. We first write the expression for the three commutators involved:

$$[a,\beta,c^{a}] = a^{-\beta}a\cdot a^{-1}c^{-1}a\cdot a^{-1}a^{\beta}\cdot a^{-1}ca;$$

$$[c,a,\beta^{c}] = a^{-1}c^{-1}ac\cdot(c^{-1}a^{-1}ca)^{\beta^{c}};$$

$$[\beta,c,a^{\beta}] = [c,\beta]\cdot a^{-\beta}\cdot[c,\beta]^{-1}\cdot a^{\beta} = c^{-1}c^{\beta}a^{-\beta}c^{-\beta}ca^{\beta}.$$

An element of G acts on an endomorphism by conjugating by the corresponding inner automorphism; hence

$$\beta^{c} = i_c^{-1}\beta i_c, \quad \text{so that} \quad x^{\beta^{c}} = c^{-1}c^{\beta}x^{\beta}c^{-\beta}c.$$

A straight forward substitution and free cancelation produces the identity element.∎

Corollary 6. *Let S be a normal subsemigroup of $\mathcal{E}$, and U and C be normal subgroups of G. Then*

$$[U,S,C] \le [C,S,U][C,U,S].$$

PROOF. The result follows immediately from Lemma 9.∎

Lemma 10. *If $S \le \mathcal{E}$ then*

$$\gamma_k([G,S^G]) \le [G,{}_kS^G].$$

PROOF. For $k=1$ this is immediate. By the inductive hypothesis,

$$\begin{aligned}\gamma_{k+1}([G,S^G]) &= [\gamma_k([G,S^G]),[G,S^G]]\\ &\le [[G,{}_kS^G],[G,S^G]] = [[G,S^G],[G,{}_kS^G]].\end{aligned}$$

We may then use Corollary 6, since S^G and $U=G$ are both normal, and by (h) of Lemma 6, $C=[G,{}_kS^G]$ is normal in G. Hence

$$[[G,S^G],[G,{}_kS^G]]$$

is contained in

$$[[G,{}_kS^G],S^G,G][[G,{}_kS^G],G,S^G].$$

But the terms of the commutator series of S^G are normal in G; and if $N \lhd G$, then $[N,G] \le N$. This allows us to remove G from the commutators $[[G,{}_kS^G],S^G,G]$ and $[[G,{}_kS^G],G,S^G]$ in which it appears. Thus,

$$[[G,{}_kS^G],S^G,G][[G,{}_kS^G],G,S^G]$$

is contained in

$$[G,{}_{k+1}S^G].$$∎

Theorem 4. *Suppose the series of normal subgroups*

$$G = J_0 \ge J_1 \ge J_2 \ge \ldots \ge J_k \ge J_k = 1$$

is stable under a subsemigroup S of $\mathcal{E}$. Then $[G,S]$ is nilpotent of class $\le k-1$.

PROOF. By (h) of Lemma 6, since $G^G = G$, we have that $[G,S] = [G,S]^G = [G,S^G]$. Now it follows from Lemma 10 that

$$\gamma_k([G,S]) = \gamma_k([G,S^G]) \leq [G,{}_kS^G] = N_k \leq J_k = 1,$$

by Lemma 7.■

In [4, Satz 4] the nilpotency of $[G,S]$ was proved for $S \leq \mathrm{Aut}(G)$.

4. Nil endomorphisms

Definition 6. *An endomorphism $\sigma \in \mathcal{E}$ is called a* nil endomorphism *if $\forall g \in G$, there exists a positive integer $k(g)$ such that*

$$[g,{}_{k(g)}\sigma] = 1.$$

The least such integer $k(g)$ is called the nility *of σ on g. Moreover, if σ is nil, then the integer*

$$k = \max_{g\in G}(k(g))$$

is called the nility *of σ on G.*■

We shall presently show that even though σ is nil, the nility of σ on G may be infinite; moreover, the product of two nil endomorphisms need not be nil.

It is well-known that the endomorphisms of an abelian group G, form a ring under the operations of addition and multiplication defined by

$$g^{\sigma_1+\sigma_2} = g^{\sigma_1}g^{\sigma_2}, \text{ and } g^{\sigma_1\sigma_2} = (g^{\sigma_1})^{\sigma_2}. \tag{9}$$

Example 9. *Let G be the group of a vector space over F, where F is the rationals or the integers mod p. Then the endomorphisms of G are just the linear maps of G over F. Given a fixed basis of G over F, we may represent the endomorphisms of G as matrices (of possibly infinite size). For any endomorphism σ, $[g,\sigma] = g^{\sigma-1}$, and $[g,{}_{k(g)}\sigma] = g^{(\sigma-1)^{k(g)}}$. Suppose that σ is nil and G has finite dimension with basis $\{g_1,\dots,g_n\}$. Let $k = \max(k(g_1),\dots,k(g_n))$ where $k(g_i)$ is the nility of σ on g_i. Since $(\sigma-1)^k$ is an endomorphism, it is clear that k is the nility of σ on G, and that σ is nil iff each eigenvalue of σ is 1. On the other hand, if G has a countably infinite dimension, then the endomorphism σ given by the infinite matrix, along whose main diagonal is a sequence of square $n \times n$ matrices with 1 on the main and super diagonals and 0 elsewhere, is nil on every element of G, but has infinite nility on G. In the case G has dimension 2, the two matrices*

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

are each nil, with nility 2 on G, whereas their product is nil iff a or b is 0.■

In the above example, since the eigenvalues of a nil endomorphism are $1 \neq 0$, a nil endomorphism is certainly a monomorphism. In fact, this result is true in general.

Lemma 11. *Let G be a group, $\mathcal{E} = End(G)$, $\sigma \in \mathcal{E}$, and let g be in the kernel of σ. Then $[g,{}_k\sigma] = g^{(-1)^k}$. If σ is nil endomorphism, then σ is a monomorphism.*

Proof. If g is in the kernel of σ then by induction on k, $[g, {}_k\sigma] = g^{(-1)^k}$. But since σ is nil, we have $g = 1$.∎

From Lemma 11, it follows that if $S = smgp(\sigma)$ then the intersection of the terms of the commutator series must contain the kernel of σ. However, that intersection need not be just the kernel as the following example shows.

Example 10. *As in the end of Example 9, let G be a two-dimensional vector space over F, where F is the rationals or the integers mod p. If*

$$\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$a = [1, 0]$, and $b = [0, 1]$, then $[a, \sigma] = b$ and $[b, \sigma] = a$. Since, a and b generate G it follows that for $S = smgp(\sigma)$, we have $[G, S] = G$, and $[G, {}_kS] = G \neq$ kernel of σ.∎

Our next theorem shows that a nil endomorphism is in fact an automorphism. To prove this we need the following remarks.

When a group G is not abelian, its endomorphisms no longer form a ring under the addition and multiplication given by equation (9). Indeed, $\sigma_1 + \sigma_2$ need not be an endomorphism, and need not be equal to $\sigma_2 + \sigma_1$.
However, the set $\mathbf{M}(S)$ of linear combinations with integer coefficients of the endomorphisms in a subsemigroup $S \leq \mathcal{E}$, is a subset of mappings of G under the action

$$g^{a_1\sigma_1 + a_2\sigma_2 + \ldots + a_n\sigma_n} = (g^{a_1})^{\sigma_1}(g^{a_2})^{\sigma_2} \cdots (g^{a_n})^{\sigma_n},$$

where a_i are integers, and $\sigma_i \in \mathcal{E}$. $\mathbf{M}(S)$ is closed under addition and multiplication, for which only the left distributive law holds. If ι is the identity map, then $n\iota$ is in the centre of the near-ring $\mathbf{M}(S)$. If S is a cyclic subsemigroup generated by σ we shall also denote the corresponding near-ring more simply by $\mathbf{M}(\sigma)$. Elements $n\sigma$ are in the centre of $\mathbf{M}(\sigma)$.

Theorem 5. *Every nil endomorphism is an automorphism.*

Proof. We have shown in Lemma 11 that every nil endomorphism is a monomorphism. Now let us fix $\sigma \in \mathcal{E}$. We shall show that for all $k \geq 1$, there exists $\mu(k), \nu(k)$ in $\mathbf{M}(\sigma)$ such that for all g in G

$$[g, {}_k\sigma] = g^{\sigma\mu(k)} g^{(-1)^k} g^{\sigma\nu(k)}.$$

Indeed, for $k = 1$, $\mu(1) = 0\iota$, and $\nu(1) = \iota$. By use of the inductive hypothesis, we have that

$$\begin{aligned}
[g, {}_{k+1}\sigma] &= (g^{\sigma\mu(k)} g^{(-1)^k} g^{\sigma\nu(k)})^{-1} (g^{\sigma\mu(k)} g^{(-1)^k} g^{\sigma\nu(k)})^{\sigma} \\
&= g^{\sigma(-\nu(k))} g^{(-1)^{k+1}} g^{\sigma(-\mu(k) + \mu(k)\sigma + (-1)^k\sigma + \nu(k)\sigma)}
\end{aligned}$$

as required.

Hence, $\forall k \geq 1$, $\exists \mu(k), \nu(k) \in \mathbf{M}(\sigma)$ such that $g^{-\sigma\mu(k)} g g^{-\sigma\nu(k)} = [g, {}_k\sigma]^{(-1)^k}$. Let now σ be a nil endomorphism. For each $g \in G$, $\exists c = k(g)$, such that σ satisfies $[g, {}_c\sigma] = 1$. Therefore,

$$g = g^{\sigma\mu(c)} g^{\sigma\nu(c)} = g^{\sigma(\mu(c) + \nu(c))} = g^{(\mu(c) + \nu(c))\sigma}, \tag{10}$$

where $\mu(c)$ and $\nu(c)$ depend on c (and thus on g). We now define the map δ by

$$g^{\delta} = g^{\mu(c)+\nu(c)}, \tag{11}$$

and therefore from equations (11) and (10) we have that for each $g \in G$,

$$g^{\delta\sigma} = g^{(\mu(c)+\nu(c))\sigma} = g.$$

This shows that σ is onto. Hence, $\sigma \in \mathrm{Aut}(G)$.■

Corollary 7. *If a subsemigroup $S \leq \mathcal{E}$ is generated by nil endomorphisms, then $S \leq Aut(G)$.*

PROOF. The result is immediate from Theorem 5.■

We now give some applications of Theorem 5 to the question, when can an automorphism of a factor group G/N be lifted to an automorphism of G. Specifically, several authors have considered this problem for characteristic N (see e.g.,[1, 2]). In that case every automorphism α of G induces an automorphism $(\alpha)\rho$ of G/N, and the problem asked under what conditions the map $\rho : \mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$ is onto, i.e., can every automorphism of G/N be lifted to an automorphism of G. This general problem is still open.

For a relatively free group G, if $N \lhd G$, we can lift every endomorphism of G/N to an endomorphism of G, and moreover, each such endomorphism is in $\mathrm{Stb}(N) \leq \mathcal{E}$. Therefore, we have a natural map $\nu : \mathrm{Stb}(N) \to \mathrm{End}(G/N)$, which is clearly onto. The question then becomes when is ρ, the restriction of ν to $\mathrm{Stb}(N) \cap \mathrm{Aut}(G)$, onto $\mathrm{Aut}(G/N)$.

The kernel of the map ν is precisely $\mathrm{S}(N)$, while the kernel of ρ is precisely $\mathrm{S}(N) \cap \mathrm{Aut}(G)$. Indeed we can use this to give sufficient conditions for a positive answer.

Theorem 6. *Let G be relatively free and let $N \lhd G$. If $S(N) \leq Aut(G)$, then the map*

$$\rho : Stb(N) \cap Aut(G) \to Aut(G/N)$$

is onto.

PROOF. Since G is relatively free, the map ν, of $\mathrm{Stb}(N)$ into $\mathrm{End}(G/N)$, is onto. Every map in $\mathrm{Aut}(G/N)$ can be written as $(\gamma)\nu$ for some $\gamma \in \mathrm{Stb}(N)$. Let $(\alpha)\nu$ be in $\mathrm{Aut}(G/N)$, and let $(\beta)\nu$ be its inverse, also in $\mathrm{Aut}(G/N)$, where $\alpha, \beta \in \mathcal{E}$. Since $(\alpha)\nu(\beta)\nu = (\alpha\beta)\nu = (\iota)\nu$, it follows that $\alpha\beta \in \mathrm{S}(N) \leq \mathrm{Aut}(G)$. But $\mathrm{Aut}(G)$ are just those elements of $\mathcal{E}$ which are bijections, i.e., are 1-1 and onto. Similarly, $\beta\alpha$ is a bijection. Now, both products $\alpha\beta$ and $\beta\alpha$ are bijections iff both factors α, β are bijections. Thus $\alpha, \beta \in \mathrm{Stb}(N) \cap \mathrm{Aut}(G)$, and ρ, which is ν restricted to $\mathrm{Stb}(N) \cap \mathrm{Aut}(G)$ is onto $\mathrm{Aut}(G/N)$.■

As the following example shows, the converse to the above theorem is not necessarily true.

Example 11. *Let G be a free group of rank two, and let $N = G'$, the commutator subgroup of G. Then the map $\rho : Aut(G) \to Aut(G/G')$ is onto (see, e.g.,[6]). However, $S(G')$ is not contained in $Aut(G)$. One way of showing this is to use the fact that the map $Aut(G) \to Aut(G/\gamma_k(G))$ is not onto for $k \geq 4$ (see [1, 2]).*

It follows then from Theorem 6, that $S(\gamma_k)$ is not in $Aut(G)$. But $S(G')$ contains $S(\gamma_k)$, and hence, is also not in $Aut(G)$.■

Corollary 8. *If G is a nilpotent relatively free group, and $N \lhd G'$ then the map*

$$\rho : Stb(N) \cap Aut(G) \to Aut(G/N)$$

is onto.

Proof. Since a normal subgroup of a characteristic subgroup is normal, $N \lhd G$. Thus our result follows from Theorem 6, once we establish that $\mathrm{S}(N)$ is contained in $\mathrm{Aut}(G)$. Now clearly $[G, {}_k\mathrm{S}(N)] \leq [G, {}_k\mathrm{S}(G')]$. But Corollary 5 implies that if G is nilpotent of class k then $[G, {}_k\mathrm{S}(G')] = 1$. Hence, $\mathrm{S}(N)$ consists of nil endomorphisms, which by Theorem 5 are contained in $\mathrm{Aut}(G)$.■

Corollary 9. *If G is a relatively free group with centre C, then the map ρ : $Aut(G) \to Aut(G/(G' \cap C))$ is onto.*

Proof. Let $N = G' \cap C \lhd G$ and let G be generated by $\{g_i\}$. By Theorem 6 and Theorem 5, it suffices to show that $\mathrm{S}(N)$ consists of nil endomorphisms of nility no more than two.

Indeed, $[G, \mathrm{S}(N)] \leq N$ by (c) of Lemma 6. Therefore, $[G, {}_2\mathrm{S}(N)] \leq [N, \mathrm{S}(N)]$ is generated by words $n^{-1}n^{\sigma}$ where $n \in N$ and $\sigma \in \mathrm{S}(N)$. But since $N \leq G'$, n is a product of commutators $k_j(g_i)$, and since $N \leq C$ and $\sigma \in \mathrm{S}(N)$, we have $(g_i)^{\sigma} = g_i\, c_i$ where $c_i \in C$. Hence,

$$\begin{aligned} n^{\sigma} &= \prod_j k_j(g_i)^{\sigma} = \prod_j k_j(g_i{}^{\sigma}) \\ &= \prod_j k_j(g_i\, c_i) = \prod_j k_j(g_i) = n. \end{aligned}$$

Thus $[n, \sigma] = n^{-1}n^{\sigma} = 1$ and so $[g, {}_2\sigma] = 1$.■

Other examples can be found in [5].

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VIRASORO-TOROIDAL ALGEBRAS AND VERTEX REPRESENTATIONS

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Presented by R.V. Moody, F.R.S.C.

Abstract: The theory of toroidal Lie algebras including their vertex operator representations was introduced by Moody, Eswara Rao and Yokonuma in [MEY] where they produced completely decomposable modules. In the present note the author extends the toroidal algebras by the Virasoro algebra (*Vir*) by constructing a semi-direct product (Virasoro-toroidal) Lie algebra. With the use of certain oscillator representations of *Vir* one finds that the Fock space is essentially completely reducible. A class of Lie algebras called Generalized Heisenberg algebras is introduced by means of which one can identify certain important subalgebras of the toroidal algebra.

§1 Toroidal algebras

Let $\dot{\mathfrak{g}}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. For $n \geq 1$ a toroidal Lie algebra, denoted $\tau_{[n]}$, can be realized as the universal covering algebra of the iterated loop algebra $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. It is known, [Ka], that the centre of $\tau_{[n]}$ is given by the module of differentials Ω_A/dA where $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. If $x, y \in \dot{\mathfrak{g}}$ and $a, b \in A$ then multiplication in $\tau_{[n]}$ is given by $[x \otimes a, y \otimes b] = [x,y] \otimes ab + (x|y)\overline{(da)b}$ where $(\cdot|\cdot)$ denotes the Killing form on $\dot{\mathfrak{g}}$ and $d : A \longrightarrow \Omega_A$ is the differential map. Next we fix a Cartan subalgebra $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{g}}$ and consider the subalgebra $\mathcal{A}_{[n]}$ of $\tau_{[n]}$ generated by the subspace $\dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. This leads to the following class of Lie algebras.

§2 Generalized Heisenberg algebras

Let L be a geometric lattice, $\ell := \mathbb{C} \otimes_{\mathbb{Z}} L$ and fix $n \geq 1$. For each $r \in \mathbb{Z}^n$ let $\ell(r)$ (resp. $\mathfrak{z}'(r)$) be an isomorphic copy of ℓ (resp. $\mathbb{C}^n$). Thus, as γ

(resp. s) runs over a basis of ℓ (resp. $\mathbb{C}^n$), $\gamma(r)$ (resp. $z_s(r)$) runs over a basis of $\ell(r)$ (resp. $\mathfrak{z}'(r)$). Define $\mathfrak{z}' := \coprod_{r\in\mathbb{Z}^n} \mathfrak{z}'(r)$; $R := \coprod_{r\in\mathbb{Z}^n} \mathbb{C}z_r(r)$ and $\mathfrak{z} := \mathfrak{z}'/R$. Finally, form the $\mathbb{C}$-space $\mathcal{H}(L,n) := \left(\coprod_{r\in\mathbb{Z}^n} \ell(r)\right) \oplus \mathfrak{z}$ and introduce a multiplication on $\mathcal{H}(L,n)$ in such a way that for $\gamma,\eta\in\ell$ and $r,s\in\mathbb{Z}^n$, $[\gamma(r),\eta(s)] := (\gamma|\eta)z_r(r+s)$ and each $z_s(r)$ is central. We call $\mathcal{H}(L,n)$ the <u>Generalized Heisenberg algebra</u> associated to L and n.

Observe that $\mathrm{centre}\big(\mathcal{H}(L,n)\big) = \ell(0) \oplus \mathfrak{z} \oplus \left(\coprod_{\substack{r\in\mathbb{Z}^n \\ \gamma \in \mathrm{rad}(\cdot|\cdot)}} \mathbb{C}\gamma(r)\right)$ and $\dim_{\mathbb{C}}\mathfrak{z} = \begin{cases} 1, & n=1 \\ \infty, & n\geq 2. \end{cases}$

When $n=1$ one recovers the well-known Heisenberg algebra with generators $\{a(n): a\in\ell, n\in\mathbb{Z}\}$ and relations $[a(n),b(m)] = n(a|b)\delta_{n+m,0}c$ where c is a central symbol. We will denote $\mathcal{H}(L,1)$ by $\mathfrak{a}(L)$. Moreover, when $n=2$ and L is arbitrarily fixed of type A_m, B_m or E_m we will denote the algebra $\mathcal{H}(L,2)$ simply by $\mathcal{H}$.

Proposition: Let $\mathring{Q}$ be the root lattice of a simple finite dimensional Lie algebra $\mathring{\mathfrak{g}}$ over $\mathbb{C}$. Then, as Lie algebras, we have $\mathcal{H}(\mathring{Q},n) \cong \mathcal{A}_{[n]}$.

§3 The Lattices

Throughout this work $\mathring{Q}$ will denote a lattice of type A_m, D_m or E_m ; $Q := \mathring{Q}\oplus\mathbb{Z}\delta$ where $(Q|\delta) = 0$; $\Gamma := Q\oplus\mathbb{Z}\mu$ where $(\mathring{Q}|\mu) = 0 = (\mu|\mu)$ and $(\delta|\mu) = 1$ and finally, $\Lambda := \mathbb{Z}\mu\oplus\mathbb{Z}\delta$. Note that $\mathring{Q}$ is (positive) definite, Q is degenerate, Γ is nondegenerate and Λ is indefinite and all these lattices are even. We denote the complexifications of $\mathring{Q}$, Q, Γ and Λ by $\mathring{\mathfrak{h}}$, $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively.

§4 The Canonical Representation

Let $S(\mathfrak{a}(L)_-)$ denote the symmetric algebra of $\mathfrak{a}(L)_- := \coprod_{n>0} \ell(-n)$. For $\lambda\in\ell$ introduce the <u>Fock space</u> $V_L(\lambda) := \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(L)_-)$. $V_L(\lambda)$ becomes an $\mathfrak{a}(L)$-module in such a way that for $a,b\in\ell$, $n,m>0$ and $f\in S(\mathfrak{a}(L)_-)$ we have

$a(-n)\cdot(e^{\lambda}\otimes f) = e^{\lambda}\otimes a(-n)f$; $a(n)$ is the unique derivation of $V_L(\lambda)$ satisfying $a(n)\cdot(e^{\lambda}\otimes b(-m)) := n\delta_{n,m}(a|b)(e^{\lambda}\otimes 1)$ and finally, $a(0)$ acts as the scalar $(a|\lambda)$ and c acts as the identity. One can easily show that $V_L(\lambda)$ is irreducible if and only if L is nondegenerate.

§5 Vertex Operators

Let $\varepsilon : Q \times Q \longrightarrow \{\pm 1\}$ be a bimultiplicative map satisfying $\varepsilon(\alpha,\alpha) = (-1)^{(\alpha|\alpha)/2}$; $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{\varepsilon(\alpha,\beta)}$ and $\varepsilon(\alpha,\delta) = 1$, $\alpha,\beta\in Q$. Let $\mathbb{C}[Q]$ be the twisted group algebra [B] associated to ε: $e^{\alpha}\cdot e^{\beta} = \varepsilon(\alpha,\beta)\, e^{\alpha+\beta}$. Extend ε to a bimultiplicative map $\varepsilon : Q \times \Gamma \longrightarrow \{\pm 1\}$ and let $\mathbb{C}[\Gamma]$ be the $\mathbb{C}$-space with basis e^{γ}, $\gamma\in\Gamma$. We make $\mathbb{C}[\Gamma]$ into a $\mathbb{C}[Q]$-module in the obvious way. Form the <u>Full Fock space</u> $V(\Gamma) := \mathbb{C}[\Gamma]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-) = \coprod_{\lambda\in\Gamma} V_{\Gamma}(\lambda)$. Define, as in [MEY], the <u>vertex operators</u> $X(\alpha,z)$, $\alpha\in Q$, $z\in\mathbb{C}^{\times}$ with the so-called <u>moments</u> $X_n(\alpha) : V(\Gamma) \longrightarrow V(\Gamma)$ determined by the formal expansion $X(\alpha,z) = \sum_{n\in\mathbb{Z}} X_n(\alpha)z^{-n}$. For $\alpha,\beta \in Q$ and $n,m \in \mathbb{Z}$ one can establish [GO] the following commutation relations between the moments:

CR0 $[\alpha(k),X_m(\beta)] = (\alpha|\beta)X_{n+k}(\beta)$,

CR1 $[X_m(\alpha),X_n(\beta)] = 0$ if $(\alpha|\beta) \geq 0$,

CR2 $[X_m(\alpha),X_n(\beta)] = \varepsilon(\alpha,\beta)\, X_{n+m}(\alpha+\beta)$ if $(\alpha|\beta)=-1$,

CR3 $[X_m(\alpha),X_n(\beta)] = \varepsilon(\alpha,\beta)\left\{X_{n+m}(\alpha+\beta) + \sum_{k\in\mathbb{Z}} :\alpha(k)X_{m+n-k}(\alpha+\beta):\right\}$

whenever $(\alpha|\alpha) = (\beta|\beta) = -(\alpha|\beta) = 2$.

By proposition 4.3 of [MEY], we know that $V(\Gamma)$ is a $\tau_{[2]}$-module under the correspondences $e_1\otimes s^m t^n \longleftrightarrow X_m(\alpha_1+n\delta)$ and $-f_1\otimes s^m t^n \longleftrightarrow X_m(-\alpha_1+n\delta)$. Now for $\gamma\in\mathfrak{h}$ and $m,n\in\mathbb{Z}$ define the operators $T^{\gamma}_m(n\delta) := \sum_{k\in\mathbb{Z}} :\gamma(k)X_{-k+m}(n\delta):$ on $V(\Gamma)$.

Proposition: The assignment $\gamma(m,n) \longmapsto T^{\gamma}_m(n\delta)$, $n,m\in\mathbb{Z}$ determines a representation of $\mathcal{H}$ on $V(\Gamma)$.

§6 Oscillator Representations of *Vir*

Let L be a nondegenerate lattice and $\{u_i\}_{i=1}^{r}$ an orthonormal basis for ℓ. One knows [FLM] that the Virasoro operators $T_k := \sum_{j\in\mathbb{Z}} \sum_{i=1}^{r} :u_i(-j)u_i(j+k):$ for $k\in\mathbb{Z}$ define a representation of *Vir* on $V_\Gamma(\lambda)$, $\lambda\in\ell$ with central charge r. Applying this construction to an orthonormal basis for $\mathfrak{p}$ (resp. $\mathfrak{h}$), we obtain an infinite set of operators which we denote by $\{H_k\}$ (resp. $\{L_k\}$) acting on the Fock space $V_\Lambda(\alpha)$, $\alpha\in\mathfrak{p}$ (resp. $V_\Gamma(\lambda)$, $\lambda\in\mathfrak{h}$) and giving oscillator representations of *Vir* with central charge 2 (resp. m+2). In the sequel, when the Fock space $V_\Lambda(\alpha)$ is viewed as a module over *Vir* in the way just described, we will denote it simply by $W(\alpha)$. The key to the construction of the Virasoro-toroidal algebras are the commutation relations

(CR4) $[L_k, X_n(\alpha)] = \left\{ \frac{k}{2}(\alpha|\alpha)-(n+k)\right\}X_{n+k}(\alpha)$. For a proof see [GO] or [FK].

§7 Vertex Representations of Virasoro-toroidal algebras

Let $\tilde{\tau}_{[2]}$ (resp. $\tilde{\mathcal{H}}$) denote the Lie algebra of operators on $V(\Gamma)$ generated by $X_n(\alpha)$, L_k, $n,k \in \mathbb{Z}$, $\alpha \in \Delta^{re}$ (resp. $T_m^{\gamma}(n\delta)$; L_k, $n,m,k \in \mathbb{Z}$, $\gamma \in \mathring{\mathfrak{h}}$). Then $\tilde{\tau}_{[2]}$ (resp. $\tilde{\mathcal{H}}$) is a semi-direct product of *Vir* with $\tau_{[2]}$ (resp. $\mathcal{H}$). We call $\tilde{\tau}_{[2]}$ (resp. $\tilde{\mathcal{H}}$) a Virasoro-toroidal (resp. Virasoro-Heisenberg) algebra. For $\lambda\in\Gamma$ and $m\in\mathbb{Z}$ introduce the spaces $H(\lambda) := \mathbb{C}[\lambda+\mathbb{Z}\delta]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ and $K(m) := \mathbb{C}[m\mu+Q]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$. Observe that $V(\Gamma) = \coprod_{m\in\mathbb{Z}} K(m)$.

Proposition: If $\lambda \in \Gamma\setminus Q$ then $H(\lambda)$ is an irreducible $\tilde{\mathcal{H}}$-module and if $m\neq0$ then $K(m)$ is an irreducible $\tilde{\tau}_{[2]}$-module.

§8 The Structure of $H(\lambda)$, $\lambda\in Q$ and $K(0)$

In this section we will see that $H(\lambda)$, $\lambda\in Q$ and $K(0)$ are not irreducible modules over $\tilde{\mathcal{H}}$ and $\tilde{\tau}_{[2]}$ respectively by constructing a filtration of submodules of each that arises from a corresponding filtration of Vir-submodules of $W(n\delta)$, $n\in\mathbb{Z}$.

Let $M := S\left(\coprod_{n>0} \mathbb{C}\mu(-n)\right)$ and $D := \left(\coprod_{m>0} \mathbb{C}\delta(-m)\right)$ so that $\Theta = S\left(\coprod_{n>0} p(-n)\right) = MD$ where MD denotes the smallest subalgebra containing M and D. Next we introduce a collection of subspaces of M and D. For $j<0$ define $\Omega_j := (0) =: \Delta_j$; $\Omega_0 := \mathbb{C} =: \Delta_0$ and for $j>0$ let Ω_j (resp. Δ_j) denote the subspace of M (resp. D) spanned by all monomials $\mu(-n_1)\cdots\mu(-n_j)$, $n_k>0$ (resp. $\delta(-m_1)\cdots\delta(-m_j)$, $m_k>0$) of length j. Set $\Omega_{\le j} := \coprod_{k\le j} \Omega_k$ and for $r\in\mathbb{Z}$ put $\Theta^r := \coprod_{j\ge 0}\left(\Omega_{\le(j+r)}\Delta_j\right)$. Finally, for $r,n \in \mathbb{Z}$ introduce $W^r(n\delta) := \mathbb{C}e^{n\delta}\otimes_{\mathbb{C}}\Theta^r$ and $W_r(n\delta) := W^r(n\delta)/W^{r-1}(n\delta)$. Then one can prove

Proposition: For every $n\in\mathbb{Z}$, $\{W^r(n\delta):r\in\mathbb{Z}\}$ is a filtration of Vir-submodules of $W(n\delta)$ and each subquotient $W_r(n\delta)$ is unitary and completely reducible.

If (using the above proposition) we write $W_r(n\delta) = \coprod_{\gamma\in I_r} W_r^{\gamma}(n\delta)$ then it is not hard to see that $\forall\gamma\in I_r$, $W_r^{\gamma}(n\delta) \cong L(2,h_r)$, for some $h_r\ge 0$ where $L(c,h)$ denotes the unique irreducible highest weight module over Vir with highest weight (c,h). Next let $e^{n\delta}\otimes\xi_r^{\gamma}$ denote the highest weight vector that generates $W_r^{\gamma}(n\delta)$ and for $\lambda\in Q$ and $r\in\mathbb{Z}$ define $\mathbb{C}$-spaces $H^r(\lambda) := \mathbb{C}[\lambda+\mathbb{Z}\delta]\otimes_{\mathbb{C}}S(a(\mathring{Q})_-)\Theta^r$; $H_r(\lambda) := H^r(\lambda)/H^{r-1}(\lambda)$; $K^r(0) := \mathbb{C}[Q]\otimes_{\mathbb{C}}S(a(\mathring{Q})_-)\Theta^r$ and $K_r(0) := K^r(0)/K^{r-1}(0)$.

Proposition: (i) If $\lambda\in Q$ then $\{H^r(\lambda) : r\in\mathbb{Z}\}$ is a filtration of $\tilde{\mathcal{H}}$-submodules of $H(\lambda)$ and for each $r\in\mathbb{Z}$, $H_r(\lambda) = \coprod_{\gamma\in I_r} H_r^{\gamma}(\lambda)$ where $H_r^{\gamma}(\lambda) := \mathfrak{U}(\tilde{\mathcal{H}})\cdot(e^{\lambda}\otimes\xi_r^{\gamma})$.

(ii) $\{K^r(0) : r\in\mathbb{Z}\}$ is a filtration of $\tilde{\tau}_{[2]}$-submodules of $K(0)$ and for each $r\in\mathbb{Z}$, $K_r(0) = \coprod_{\gamma\in I_r} K_r^{\gamma}(0)$ where $K_r^{\gamma}(0) := \mathfrak{U}(\tilde{\tau}_{[2]})\cdot(e^0\otimes\xi_r^{\gamma})$.

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ON ASYMPTOTIC EXPANSIONS IN LIMIT THEOREMS ON LARGE DEVIATIONS FOR SUMS OF INDEPENDENT RANDOM VARIABLES IN THE CASE OF POWER TAILS

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Presented by Donald A. Dawson, F.R.S.C.

Abstract: Let $\{X_n, n \ge 1\}$ be i.i.d.r.v-s with common d.f. $F(x)$ having the right-hand tail of power type: $1 - F(x) \sim c_{\alpha_1} \cdot x^{-\alpha_1}$ as $x \to \infty$. Asymptotic expansions of $P\{X_1+\cdots+X_n>y\}$ for any $n \in \mathbb{N}$ and in various ranges of deviations under some additional constraints on F are presented when α_1 is a positive non-integer.

Let $\{X_n, n \ge 1\}$ be i.i.d. random variables with common distribution function $F(x)$ having asymptotics of the power type on the tails:

(1) $$1 - F(x) = c_{\alpha_1} \cdot x^{-\alpha_1} + o(x^{-\alpha_1});$$

(1′) $$F(-x) = d_{\alpha_1} \cdot x^{-\alpha_1} + o(x^{-\alpha_1})$$

as $x \to \infty$. Here α_1 is a positive non-integer, $c_{\alpha_1} \ge 0$, $d_{\alpha_1} \ge 0$ (i.e. cases of normal and non-normal stable laws are considered simultaneously). Set $S_n = X_1 + \ldots X_n$ and and assume without loss of generality that $EX_1 = 0$ if $\alpha_1 > 1$ and $DX_1 = 1$ if $\alpha_1 > 2$.

It is well known (cf. Heyde (1968), A.Nagaev (1969), Tkachuk (1975)) that in the case of fulfilment of (1) - (1′) large deviations of S_n from 0 occur mainly due to one large summand X_1 comparable with the whole sum S_n:

(2) $$P\{S_n > y\} \sim n \cdot P\{X_1 > y\} \sim n \cdot c_{\alpha_1} \cdot x^{-\alpha_1}$$

as $n \to \infty$ with $y \ge \Lambda(n)$. Here $\Lambda(n)$ is any positive sequence such that

$\Lambda(n)/n^{1/\alpha_1} \to \infty$ as $n \to \infty$ if $\alpha_1 < 2$, and $\Lambda(n) > \mathrm{Const}\cdot(n\cdot\log n)^{1/2}$ for any integer $n \ge 1$ if $\alpha_1 > 2$. Note that in the latter case we do not consider the full range of large deviations of S_n. Relationship (2) remains true for $n \in \mathbb{N}$ being fixed and $y \to \infty$ (cf. Feller (1971) Chapter 8, (8.14)).

In this work we obtain estimates of the error appearing from the change of $P\{S_n > y\}$ to $n\cdot c_{\alpha_1}\cdot x^{-\alpha_1}$ and construct asymptotic expansions of probabilities of large deviations of S_n. All our results are related to the case when $n \ge 1$ and the range of deviations y is greater than a certain function $\Lambda(n)$, that depends also on α_1, c_{α_1}, and d_{α_1}.

The following theorem is valid for any sequence of independent identically distributed random variables, without any constraint on their common distribution function.

Theorem 1. Let $\alpha_1 > 0$, $c_{\alpha_1} > 0$, $0 < \kappa < 2/3$ be any fixed real, $\phi(\cdot,\cdot)$ be some function from $\mathbb{N} \times R^1_+$ into R^1_+ such that for any integer $n \ge 1$ and for any real $y > 0$ $\quad 0 \le \phi(n,y) \le \kappa\cdot y$. Then for any integer $n \ge 1$ and for any real $y > 0$

$$
\begin{aligned}
| P\{S_n > y\} - n\cdot c_{\alpha_1}\cdot y^{-\alpha_1} | \quad &\le \quad P\left\{ \max_{1\le k\le n} S_k > y/3 \right\}^2 \\
&+ \binom{n}{2}\cdot P\{X_1 > y/3\}^2 + n\cdot c_{\alpha_1}\cdot x^{-\alpha_1}\cdot\Big(P\{S_{n-1} \le -2y/3\} \\
&+ (3^{\alpha_1}-1)\cdot(P\{S_{n-1} > 2y/3\} + P\{\phi(n,y) \le |S_{n-1}| \le 2y/3\}) \\
&+ \alpha_1\cdot(1-\kappa)^{-\alpha_1-1}\cdot\phi(n,y)/y \Big) + n\cdot \sup_{x\ge y/3} |1 - F(x) - c_{\alpha_1}\cdot x^{-\alpha_1}|.
\end{aligned}
\tag{3}
$$

Estimate (3) can have its own value though we will need only its corollaries refining some of the results by Feller (1971), Heyde (1968), A.Nagaev (1969), and Tkachuk (1975) cited above. Here we formulate only the simplest one related to the case $\alpha_1 \in (0,1) \cup (1,2)$.

<u>**Corollary 1.**</u> Let $\alpha_1 \in (0,1) \cup (1,2)$, Conditions (1) - (1′) be fulfiled, and $EX_1 = 0$ if $\alpha_1 > 1$. Then there exist positive constants K_1 and K_2 such that for any integer $n \ge 1$ with $y \ge K_1 \cdot n^{1/\alpha_1}$

$$| P\{S_n > y\} - n\cdot c_{\alpha_1} \cdot y^{-\alpha_1} | \le K_2 \cdot (n \cdot y^{-\alpha_1})^{1+1/(1+\alpha_1)}$$

(4)

$$+ n \sup_{x \ge y/3} |1 - F(x) - c_{\alpha_1} \cdot x^{-\alpha_1}|.$$

Estimate (4) easily follows from (3) if we set $\phi(n,y) = y\cdot(y/n^{\alpha_1})^{-\alpha_1/(1+\alpha_1)}$ and then apply Theorem 3 of Petrov (1975b) and Theorems 1.1-1.2 of S.Nagaev (1979).

Let us note that the last terms on the right-hand sides of (3) and (4) are 'unremovable errors' generated by the lack of perfect information on the tail behavior of $F(\cdot)$. It seems reasonable to assume that if more precise information on the tail behavior of function $F(\cdot)$ is available (as compared with (1)-(1′)) other terms on the right-hand sides of (3) and (4) can be written down more precisely, i.e. the further refinements for $P\{S_n > y\}$ can be deduced from them. To this end, we introduce the following assumption on the asymptotics of the right-hand tail of $F(\cdot)$:

$$1 - F(x) = \sum_{i=1}^{l} c_{\alpha_i} \cdot x^{-\alpha_i} + o(x^{-r}) \tag{5}$$

as $x \to \infty$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_l \le r$.

<u>**Theorem 2.**</u> Let Conditions (5) and (1′) be fulfiled with $\alpha_1 \in (0,1) \cup (1,2)$, and $EX_1 = 0$ if $\alpha_1 > 1$. Then

$$P\{S_n > y\} = n \cdot \sum_{i=1}^{l} c_{\alpha_i} \cdot y^{-\alpha_i} - \binom{n}{2} \cdot \frac{\Gamma(1-\alpha_1)}{\Gamma(1-2\alpha_1)} \cdot c_{\alpha_1}^2 \cdot y^{-2\alpha_1}$$

$$- 2\cdot\binom{n}{2}\cdot \frac{\Gamma(1-\alpha_1)\cdot\Gamma(2\alpha_1)}{\Gamma(\alpha_1)} \cdot c_{\alpha_1} \cdot d_{\alpha_1} \cdot y^{-2\alpha_1} + r_1(n,y) + r_2(n,y).$$

Here we use the fact that the analytic continuation of gamma function $\Gamma(\cdot)$ onto $\mathbb{C} \setminus \{0; -1; -2; \dots\}$ can be defined by $\Gamma(z) = \Gamma(z+1)/z$ for *Re* z being negative non-integer, $\Gamma(z) = \infty$ for *Re* z being non-positive integer; the remainders $r_1(\cdot,\cdot)$ and $r_2(\cdot,\cdot)$ are such that

i) there exist positive constants K_1 and K_2 such that for any integer $n \ge 1$ with $y \ge K_1 \cdot n^{1/\alpha_1}$ $|r_1(n,y)| \le K_2 \cdot n \cdot \sup\limits_{x \ge y/5} |1 - F(x) - \sum\limits_{i=1}^{l} c_{\alpha_i} \cdot x^{-\alpha_i}|$;

ii) there exists function $K(\cdot)$ from $\mathbb{R}^1_+$ into $\mathbb{R}^1_+$ such that for any real $\varepsilon > 0$ and for any integer $n \ge 1$ with $y \ge K(\varepsilon) \cdot n^{1/\alpha_1}$ $|r_2(n,y)| \le \varepsilon \cdot (n \cdot y^{-\alpha_1})^2$.

The analogous result is also valid for non-integer $\alpha_1 > 2$.

<u>Theorem 3.</u> Let Conditions (5) and (1′) be fulfiled with non-integer $\alpha_1 \in (2,\infty)$, $EX_1 = 0$, $DX_1 = 1$. In the case of non-integer $\alpha_1 > 3$ we assume also the fulfilment of Condition (C): $\limsup\limits_{|t| \to \infty} |E\exp\{itX_1\}| < 1$. Then

$$P\{S_n > y\} = n \cdot \sum_{i=1}^{l} c_{\alpha_i} \cdot y^{-\alpha_i} \cdot \left(1 + \sum_{m=2}^{N_0(\alpha_1,\kappa)} (-1)^m \cdot \binom{-\alpha_i}{m} \cdot y^{-m} \cdot (n-1)^{m/2} \right.$$

$$\left. \cdot \int_{\infty}^{\infty} v^m \cdot d\left\{ \Phi(v) + \sum_{\nu=1}^{m \wedge [\alpha_1]-2} Q_\nu(v)/(n-1)^{\nu/2} \right\} \right) - \binom{n}{2} \cdot \frac{\Gamma(1-\alpha_1)}{\Gamma(1-2\alpha_1)} \cdot c^2_{\alpha_1} \cdot y^{-2\alpha_1}$$

$$- 2 \cdot \binom{n}{2} \cdot \frac{\Gamma(1-\alpha_1) \cdot \Gamma(2\alpha_1)}{\Gamma(\alpha_1)} \cdot c_{\alpha_1} \cdot d_{\alpha_1} \cdot y^{-2\alpha_1} + r_1(n,y) + r_2(n,y),$$

where $\Phi(\cdot)$ is the Laplace function, the formulas for computing the functions $Q_\nu(\cdot)$ can be found in Petrov (1975a) (Chapter 6, (1.13)), $\binom{-\alpha_i}{m}$ is the coefficient under t^m in the Taylor expansion of function $(1+t)^{-\alpha_i}$ near zero, $a \wedge b$ stands for the minimum from a and b, $N_0(\alpha_1,\kappa) = [\alpha_1 + (\alpha_1-2)/(2\kappa)]$ ($[x]$ denotes the integer part of x), the remainder $r_1(\cdot,\cdot)$ is the same as in Theorem 2, and the remainder $r_2(\cdot,\cdot)$ is such that there exists function $K(\cdot,\cdot)$ from $\mathbb{R}^1_+ \otimes \mathbb{R}^1_+$ into $\mathbb{R}^1_+$ such that for any real $\varepsilon > 0$, for any real $\kappa > 0$, and for any integer $n \geq 1$ with $y \geq K(\varepsilon,\kappa)\cdot n^{1/2+\kappa}$ $|r_2(n,y)| \leq \varepsilon\cdot(n\cdot y^{-\alpha_1})^2$.

In order to get more detailed information on the asymptotic behavior of the right-hand large deviations of S_n (to compare with Theorem 2) we need to set supplementary constraints on the asymptotics of $F(x)$ as $x \to -\infty$. It is natural to assume the fulfilment of the left-hand analog of (5):

(5′) $$F(-x) = \sum_{i=1}^{l} d_{\alpha_i}\cdot x^{-\alpha_i} + o(x^{-r})$$

as $x \to \infty$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_l \leq r$.

Set $r_* = \alpha_1\cdot([r/\alpha_1]+1)$ and introduce the following condition:

(6) $$\begin{cases} \text{all non-trivial linear combinations of } \{\alpha_1,\ldots,\alpha_l\} \\ \text{with non-negative integer coefficients which} \\ \text{are less than or equal to } r_*, \text{ are not integers.} \end{cases}$$

Let us denote by $\{\beta_j\}$ the set of positive numbers that do not exceed r_* and can be represented in the form $\beta_j = \sum_i m_{ji}\cdot\alpha_i + n_j$, where m_{ji} and n_j are non-negative integers, $\sum_i m_{ji} > 0$.

<u>Theorem 4.</u> Let Conditions (5), (5′), (6) be satisfied with $\alpha_1 \in (0,1)\cup(1,2)$, and $EX_1 = 0$ if $\alpha_1 > 1$. Then there exist sets of polynomials $\{c_{\beta_j}\}$ and $\{c_{\beta_j}\}$ such that

$$P\{S_n > y\} = \sum_j c_{\beta_j}(n)\cdot y^{-\beta_j} + r_1^+(n,y) + r_2^+(n,y);$$

$$P\{S_n < -y\} = \sum_j d_{\beta_j}(n)\cdot y^{-\beta_j} + r_1^-(n,y) + r_2^-(n,y),$$

where the remainders $r_1^+(n,y)$ and $r_1^-(n,y)$ are such that there exist positive constants K_1 and K_2 such that for any integer $n \geq 1$ with $y \geq K_1\cdot n^{1/\alpha_1}$

$$|r_1^+(n,y)| \leq K_2\cdot n\cdot \sup_{x\geq\theta y} |1 - F(x) - \sum_{i=1}^{l} c_{\alpha_i}\cdot x^{-\alpha_i}|;$$

$$|r_1^-(n,y)| \le K_2 \cdot n \cdot \sup_{x \le -\theta y} \left| F(-x) - \sum_{i=1}^{1} d_{\alpha_i} \cdot |x|^{-\alpha_i} \right|,$$

where $\theta = 1/(2\cdot[r/\alpha_1]+3)$, and the second remainders are such that there exists function $V(\cdot)$ from R_+^1 into R_+^1 such that for any positive $\varepsilon > 0$ and for any integer $n \ge 1$ with $y \ge V(\varepsilon)\cdot n^{1/\alpha_1}$ $|r_2^\pm(n,y)| \le \varepsilon\cdot(n\cdot y^{-\alpha_1})^{[r/\alpha_1]+1}$.

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SUR LA CARACTERISATION EMPIRIQUE DES EXTREMES.

GANE SAMB LO.

Presented by P. Ribenboim, F.R.S.C.

Résumé. Soit X_1, X_2, ... une suite d'observations indépendantes d'une variable aléatoire X avec $\mathbb{P}(X\le x)=F(x)$, $x\in\mathbb{R}$. Nous proposons une classe de statistiques qui caractérisent la loi limite des extrêmes $X_{n,n}=\max(X_1,\ldots,X_n)$. Puis nous déterminons la normalité asymptotique de ces statistiques, permettant ainsi l'applicabilité des tests dérivés de ces resultats.

I- INTRODUCTION ET RESULTATS.

Les applications statistiques de la théorie des valeurs extrêmes reposent principalement sur la connaissance de la loi limite du maximun des observations indépendantes $X_{n,n}=\max(X_1,\ldots,X_n)$ d'une quantité aléatoire X avec $\mathbb{P}(X\le x)=F(x)$, $x\in\mathbb{R}$. Par exemple $X_{n,n}$ peut être le maximun des moyennes annuelles des températures relevées dans une région ou le maximun des moyennes saisonnières des crues observées d'un fleuve donné, etc....

Les lois limites non-dégénérées de $X_{n,n}$ sont maintenant fort connues(Gnedenko, 1943). Avant de les préciser, rappelons que $X_{n,n}$ converge en type vers une variable aléatoire (v.a.) Z si et seulement si

$$(1) \qquad \exists\ (a_n>0,\ b_n)_{n\ge1},\quad (X_{n,n}-b_n)/a_n \xrightarrow{d} Z \text{ quand } n\to+\infty,$$

où $\xrightarrow{d}$ (resp. $\overset{d}{=}$) désigne la convergence (resp. l'égalité) en distribution. Z représente ici une classe appelée type limite de $X_{n,n}$ en ce sens que toute autre v.a. Z' vérifiant (1) est une fonction affine de Z. Soit $H(x)=\mathbb{P}(Z\le x)$, $x\in\mathbb{R}$. On dira que F appartient au domaine d'attraction de Z, noté $F \in D(H)$.

Il est bien établi maintenant (Voir Resnick(1987), p.9) que si (1) est vraie alors nécessairement $H(x)=\Lambda(x)=\exp(-e^{-x})$, $x\in\mathbb{R}$, (type de Gumbel) ou $H(x)=\phi_\gamma(x)=\exp(-x^{-\gamma})$, $x\in\mathbb{R}_+$, $\gamma>0$, (type de Fréchet de paramètre γ) ou $H(x)=\psi_\gamma(x)=\exp((-x)^\gamma)$, $x\in\mathbb{R}_-$, $\gamma>0$, (type de Weibull de paramètre γ).

De plus plusieurs versions de conditions nécessaires et suffisantes de la validité de (1) ainsi que des caractérisations des des suites a_n et b_n sont disponibles (voir De Haan(1970) et Resnick(1987)).

Notre but est de d'obtenir une caractérisation statistique de (1) à partir uniquement des observations, i.e., par des statistiques dont la normalité asymptotique multivariée est caractérisée. Dans toute la suite les limites ont lieu quand n tend vers $+\infty$ et les entiers k, ℓ, m et p sont soit fixés soit fonction de n, par exemple

(K) $1\le k=k(n)\le n$, $k(n)/n\to 0$; . (L) $1\le \ell=\ell(n)\le k(n)$, $\ell(n)^2/k(n)\to 0$.

Introduisons ces notations avant de donner les statistiques.

$X_{1,n}\le\ldots\le X_{n,n}$ sont les statistiques d'ordre associées à $X_1,\ldots,X_n$; $Y_{i,n}$, $i=1,\ldots,n$, sont les statistiques d'ordre de $Y_i=\log X_i$, $i=1, \ldots,n$, où nous avons supposé, sans perte de généralité, que $X\ge 1$ presque sûrement; enfin $G(y)=\mathbb{P}(Y\le y)$. Définissons maintenant

$$A_n(1,k,\ell)=k^{-1}\sum_{i=\ell+1}^{i=k}\sum_{j=\ell+1}^{j=i} j\rho_{ij}(Y_{n-i+1,n}-Y_{n-i,n})(Y_{n-j+1,n}-Y_{n-j,n}),$$

$$T_n(2,k,\ell)=\frac{1}{k}\sum_{i=\ell+1}^{i=k} i(Y_{n-i+1,n}-Y_{n-i,n}); \quad T_n(1,k,\ell)=T_n(2,k,\ell)/\sqrt{A_n(1,k,\ell)};$$

$$T_n(3,k,\ell)=(Y_{p-k(p),p}-Y_{m-k(m)})/T_m(2,k(m),\ell(m));$$

$$T_n(4)=Y_{n,n}; \; T_n(5)=T_n(2,\ell,1);$$

$$T_n(6)=\frac{T_n(2,k,\ell)}{(Y_{n-\ell,n}-Y_{n-k,n})}+\frac{A_n(1,k,\ell)}{(Y_{n-\ell,n}-Y_{n-k,n})^2}+\frac{1}{n^{\nu}(Y_{n-\ell,n}-Y_{n-k,n})};$$

$$T_n^*(6)=\frac{T_n(2,k,\ell)}{(Y_{n-\ell,n}-Y_{n-k,n})}+\frac{A_n(1,k,\ell)}{(Y_{n-\ell,n}-Y_{n-k,n})^2}+\frac{(y_0-Y_{n-\ell,n})}{(y_0-Y_{n-k,\ell})},$$

où $\rho_{ij}=1/2$ si j=i et 1 sinon, k, ℓ, p et m sont des entiers dépendants de n, ν est un réel positif quelconque et $T_n^*(6)$ n'est défini que si $y_0=\log(\sup\{x,\ F(x)<1\})<+\infty$. Enfin, posons

$U_n=(T_n(1,k,\ell),T_n(2,k,\ell),T_n(3,k,\ell),T_n(4),T_n(5),T_n(6))$;

$U_n^*=(T_n(1,k,\ell),T_n(2,k,\ell),T_n(3,k,\ell),T_n(4),T_n(5),T_n^*(6))$.

Voici la caractérisation statistique des extrêmes.

Théorème 1. Soit $k=[n^{\alpha}]$, $\ell=[n^{\beta}]$, $0<\delta<0.5<\beta<\alpha<1$, $1<\beta+\delta<\alpha+\delta<2$, $\tau=2-\alpha-\delta$, $\nu=\beta/2$, $p(n)=n+[n^{\tau}]$, $m(n)=n-[n^{\tau}]$. Nous avons

1) $F\in D(\Lambda)$ ssi $U_n \xrightarrow{P} (1,0,0,y_0,0,0)$, $0\le y_0\le+\infty$;

2) $F\in D(\phi_{\gamma})$, $\gamma>0$, ssi $U_n \xrightarrow{P} (1,1/\gamma,0,+\infty,1/\gamma,0)$;

3) $F\in D(\psi_{\gamma})$, $\gamma>0$, ssi $U_n \xrightarrow{P} (c,0,0,y_0,0,0)$, $y_0<\infty$; $\gamma=-2+c^2/(c^2-1)$.

Maintenant, avant d'exposer les lois limites de ces statistiques, rappelons d'une part les représentations de G^{-1}, fonction inverse généralisée de G, pour $F\in\Gamma=D(\Lambda)\cup D(\phi)\cup D(\psi)$, avec $D(\phi)=\bigcup_{\gamma>0} D(\phi_{\gamma})$ and $D(\psi)=\bigcup_{\gamma>0} D(\psi_{\gamma})$ (voir Lô(1991)).

$F\in D(\phi_{\gamma})$ ssi $G^{-1}(1-u)=\log c+\log(1+f(u))-(\log u)/\gamma+\int_u^1 b(t)t^{-1}\,dt, 0<u<1$;

$F\in D(\psi_{\gamma})$ ssi $y_0=\log x_0=:\log((\sup\{x,\ F(x)<1\})<\infty$ et

$$y_0-G^{-1}(1-u)=c\ (1+f(u))\ u^{1/\gamma}\exp(\int_u^1 b(t)t^{-1}\,dt),\ 0<u<1;$$

$F\in D(\Lambda)$ ssi $G^{-1}(1-u)=d-s(u)+\int_u^1 s(t)t^{-1}dt,\ 0<u<1,$

$$\text{avec } s(u)=c\,(1+f(u))\,\exp(\textstyle\int_u^1 b(t)t^{-1}\,dt),\ 0<u<1;$$

où, dans les trois cas, $d\in\mathbb{R}$, $c\in\mathbb{R}_+^*$, $(f(u),b(u))\to(0,0)$ quand $u\to 0$. Ainsi, chaque élément F de Γ est associé à un couple (f,b). On écrira $F\equiv(f,b)\in\Gamma$.

D'autre part, représentons les statistiques d'ordre $Y_{i,n}$ par celles d'une suite de variables aléatoires indépendantes uniformes sur $(0,1)$, U_1, U_2,.... de la manière ci-dessous

$$\{Y_{n-i+1,n},\ 1\le i\le n,\ n\ge 1\}=\{G^{-1}(1-U_{i,n}),\ 1\le i\le n,\ n\ge 1\}.$$

Enfin notons

$$R(x,z,G)=\int_x^z \frac{1-G(t)}{1-G(x)}\,dt,\ x<z\le y_o,\ \text{avec } R(x,y_o,G)\equiv R(x);$$

$$W(x,z,G)=\int_x^z\int_y^z \frac{1-G(t)}{1-G(x)}\,dt\,dy,\ x<z\le y_o,\ \text{avec}\ W(x,y_o,G)\equiv W(x);$$

$$\tilde{x}_n=G^{-1}(1-U_{k+1,n});\ x_n=G^{-1}(1-k/n),\ \tilde{z}_n=G^{-1}(1-U_{\ell+1,n});\ z_n=G^{-1}(1-\ell/n).$$

Nous exposons seulement les lois limites des principales statistiques $T_n(1)$, $A_n(1)$ et $T_n(2)$ avec la convention que toute fonction de γ est relative à $F\in D(\psi_\gamma)$ pour $0<\gamma<\infty$ et à $F\in D(\Lambda)\cup D(\phi)$ pour $\gamma=+\infty$.

Théorème 2. Soit $F\in\Gamma$. Si

(K1) $1<k=k(n)<n$, $k\to+\infty$ and $K/n\to 0$ and

(K2) $1\le\ell<k$, $\exists\eta$, $0<\eta<1/2$, $\ell/k^{1/2-\eta}\to 0$,

Alors, pour $0<\gamma\le\infty$,

$$R(x_n)^{-1}k^{1/2}(T_n(2,k,\ell)-\mu(\bar{k}))\ \xrightarrow{d}\ N(0,\sigma_o^2(\gamma)),$$

$$\text{avec } \mu(\bar{k})=nR(\tilde{x}_n)(1-G(\tilde{x}_n))/k\ \text{ et } \sigma_o^2(\gamma)=2(\gamma+1)/(\gamma+2),\ 0<\gamma\le+\infty;$$

$$W(x_n)^{-1}\ k^{1/2}(A_n(1,k,\ell)-\tau(\tilde{k}))\ \xrightarrow{d}\ N(0,\sigma_2^2(\gamma)),$$

$$\text{avec } \tau(\tilde{k})=nW(\tilde{x}_n)(1-G(\tilde{x}_n))/k \text{ et } \sigma_2^2(\gamma)=\frac{6(\gamma+1)(\gamma+2)}{(\gamma+3)(\gamma+4)};$$

$$k^{1/2}(T_n(1,k,\ell)-\mu(\tilde{k})/\ \mu(\tilde{k})^{1/2})\ =\ \xrightarrow{d}\ N(0,\sigma_4^2(\gamma)),$$

$$\text{avec } \sigma_4^2(\gamma)=\frac{2\gamma^3+10\gamma^2+32\gamma+24}{4(\gamma+1)(\gamma+3)(\gamma+4)},\ 0<\gamma\le+\infty.$$

Le théorème suivant caractérise la possibilité de remplacer $\tilde{x}_n$ par x_n dans $\mu(\tilde{k})$ et $\tau(\tilde{k})$ pour obtenir $\mu(k)$ et $\tau(k)$. Pour cela posons $\gamma_n(k)=\sqrt{n}\ \{f(U_{k+1,n})-f(k/n)\}$ et pour $0<\gamma\le+\infty$,

$$\sigma_1^2(\gamma)=\frac{\gamma^3+\gamma^2+2}{\gamma^2(\gamma+2)};\quad \sigma_3^2(\gamma)=\frac{5\gamma^4+11\gamma^3+4\gamma^2+7\gamma+12}{\gamma^2(\gamma+3)(\gamma+4)};\quad \sigma_5^4(\gamma)=\frac{\gamma^3+\gamma^2+2\gamma}{4(\gamma+1)(\gamma+3)(\gamma+4)}.$$

<u>Théorème 3</u>. Supposons que les conditions du Théorème 2 soient satisfaites. Les propositions suivantes sont équivalentes.

1) $\gamma_n(k)\ \xrightarrow{P}\ 0$.

2) $R(x_n)^{-1}k^{1/2}\{T_n(2,k,\ell)-\mu(k)\}\ \xrightarrow{d}\ N(0,\sigma_1^2(\gamma))=N_1,\ 0<\gamma\le\infty.$

3) $W(x_n)^{-1}\ k^{1/2}(A_n(1,k,\ell)-\tau(k))\ \xrightarrow{d}\ N(0,\sigma_3^2(\gamma))=N_2,\ 0<\gamma\le\infty.$

De plus, chacune d'elle implique

3) $k^{1/2}(T_n(1,k,\ell)-\mu(k)/\mu(k)^{1/2})\ \xrightarrow{d}\ N(0,\sigma_5^2(\gamma))=N_3,\ 0<\gamma\le\infty.$

Enfin, le vecteur (N_1,N_2,N_3) est gaussien avec les covariances (s_{ij}), $s_{ij}=\mathbb{E}N_iN_j$, $1\le j,i\le 3$ citées dans l'ordre s_{11}, s_{12} et s_{23} :

$$\frac{2\gamma^3+4\gamma^2+18\gamma+18}{\gamma^2(\gamma+3)};\quad \left(\frac{\gamma+2}{\gamma+1}\right)^{1/2}\frac{2\gamma}{(\gamma+1)(\gamma+2)};\quad -\left(\frac{\gamma+1}{\gamma+2}\right)\ \frac{\gamma(\gamma-5)}{2(\gamma+3)(\gamma+4)},\ 0<\gamma\le\infty.$$

II- INDICATIONS SUR LES DEMONSTRATIONS.

Les démonstrations complètes sont données dans Lô(1991). Le lecteur intéressé y est indiqué. La démonstration du Théorème 1 par exemple repose sur les identités suivantes

(2) $T_n(2,k,\ell)=\frac{n}{k}\int_{\tilde{X}_n}^{\tilde{Z}_n} 1-G_n(t)\,dt$, $A_n(1,k,\ell)=nk^{-1}\int_{\tilde{X}_n}^{\tilde{Z}_n}\int_{y}^{\tilde{Z}_n} 1-G_n(t)\,dt\,dy$,

où G_n est la fonction de répartition empirique basée sur $Y_1,\dots Y_n$. De là, le principe consiste à chercher les conditions nécéssaires et suffisantes pour que $A_n(1,k,\ell)\sim W(x_n)$, $T_n(2,k,\ell)\sim R(x_n)$ et pour que $\lim_{n\to\infty} W(x_n)/R(x_n)^2=\lim_{x\to y_0} W(x)/R(x)^2$. Avec cette dernière limite, on pourra utiliser la caractérisation analytique de De Haan(1970) (voir son Théorème 2.6.1).

Les normalités asymptotiques sont obtenues à partir des identités (2) combinées avec l'approximation de Csörgö-Csörgö-Horvàth-Mason (1986) du processus empirique et du processus des quantiles uniformes par une même suite de ponts Browniens. En exprimant, la composante gaussienne de chaque statistique en fonction des mêmes ponts Browniens, la normalité asymptotique multivariées'obtient de façon naturelle.

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On Distributive Categories

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Presented by J. Lambek, F.R.S.C.

Recently some people have become interested in a class of categories called (pre)distributive categories. We feel that some approach is missing from the very begining, namely, using Lambek's techniques to develop his internal logic. We take the opportunity to do this and clarify some points. This note is divided in two parts. The first part introduces the notion of a predistributive category and shows that the functional completnees theorem (see [LS86] part I) remains true in this context. The second part introduces the notion of natural numbers object (denoted by N) and calculates some very basic facts about this notion. We show for instance that any finite coproduct of the terminal object (1) is a retract of N. Moreover, we calculate the class of numerical functions representable in this category. We shall see that the representable functions are exactly the primitive recursive functions. We prove this by showing that the free predistributive category with NNO is the Karoubi envelope of the free cartesian category with NNO. Finally, I would like to express my sincere thanks to María del Karmen for her support and help, and to Jim Lambek for his comments.

1 Distributive Categories

Since there are many definitions of the notion of a distributive category and in order that this paper is self contained, we introduce the notion of a predistributive category.

Definition 1.1

By a predistributive category we mean a cartesian category C with binary coproducts, such that the following is true: If A, B, C are C-objects then the canonical arrow

$$(A+B)\times C \leftarrow (A\times C)+(B\times C)$$

is an isomorphism.

Remark. Notice that we are not assuming we have an initial object, in fact this is crucial for the construction of the free predistributive category, see part II. Moreover, a distributive category is a predistributive category having an initial object.For different definitions and comments see for instance [Coc91] and [Wal91]. As we said in the introduction our main motivation is to use Lambek's calculus. Let us suppose we start with a (pre)distributive category C and suppose $x : 1 \to A$ is any indeterminate of type A (A is a C-object), we have then the following:

Proposition 1.2

If C is a (pre)distributive category and x is and indeterminate of type A then the polynomial category $C[x]$ is again a (pre)distributive category.

Proof (sketched). We know $C[x]$ is a cartesian category. Suppose C, B are $C[x]$-objects we shall see that $C+B$ is still a coproduct in $C[x]$.Suppose $f(x): C \to D$ and $g(x): B \to D$ are two arbitrary $C[x]$-arrows then by functional completeness there are two C-arrows $f: A \times C \to D$ and $g: A \times B \to D$ satisfying $f(x) \equiv f\langle xI_C, 1_C\rangle$, $g(x) \equiv g\langle xI_B, 1_B\rangle$. Now, since $(A \times C)+(A \times B)$ is a coproduct, there is a unique C-arrow $h:(A \times C)+(A \times B) \to D$, making the diagram

$$\begin{tikzcd} A \times C \arrow[r, "\kappa_{A\times C}"] \arrow[dr, "f"'] & (A \times C)+(A \times B) \arrow[d, dashed, "h"] & A \times B \arrow[l, "\kappa'_{A\times B}"'] \arrow[dl, "g"] \\ & D & \end{tikzcd}$$

commute. Now, since C is a predistributive category we get an arrow $h: A \times (C+B) \to D$ and then we define $h(x): C+B \to D$ as $h\langle xI_{C+B}, 1_{C+B}\rangle$ and we may calculate $h(x)\kappa_C = f(x)$ and $h(x)\kappa_B = g(x)$. Moreover, $h(x)$ is unique since h is unique.

This result can be viewed as a direct generalization of proposition 8.1 of [LS86], since in any distributive category the initial object is always strict (see [Coc91] or [Wal91]) and since we can deal with variables we can actually introduce a boolean calculus as in [LS86] section 8 p.66, for instance we can think of arrows $p: 1 \to 2$ as propositions or truth-values and introduce the classical propositional connectives $\neg: 2 \to 2$, $\wedge: 2 \times 2 \to 2$ etc, see [LS86] for more details and [Coc91] for a different approach.

We are now ready to introduce the natural numbers object.

2 The natural numbers object

We take the oportunity to made some comments about the natural numbers object (NNO) in categories which are not necesarily cartesian closed. For some strange reason, many people believe that the classical Lawvere definition of a NNO suffies when we are dealing with cartesian categories. This is not the case, the situation is even worse when we deal with a *weak* natural numbers object; we refer the reader to [Rom90] for a study and properties of such different notions. We will introduce as in [Rom89] the notion of a (weak) natural numbers object.

Definition 2.1

Let C be any (pre) distributive category, by a natural numbers object (NNO) in C we mean two arrows $0: 1 \to N$, $s: N \to N$ such that given any pair of arrows $f: A \to B$, $g: (A \times N) \times B \to B$, there is a unique arrow $h \equiv J_{A,B}(f,g)$ making the following diagrams commute:

$$
\begin{tikzcd}
 & A \times N \arrow[dd, "h"] \\
A \arrow[ur, "(1_A, 0_A)"] \arrow[dr, "f"'] & \\
 & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times N \arrow[r, "1_A \times s"] \arrow[d, "(1_{A\times N}, h)"'] & A \times N \arrow[d, "h"] \\
(A \times N) \times B \arrow[r, "g"'] & B
\end{tikzcd}
$$

Where, if A is any object of C, we denote by 0_A the following composition: $A \to 1 \to N$. We call the C-morphism s the successor function. If we have the existence but not necessarily the uniqueness of h, we shall speak of a weak NNO. As a direct consequence of proposition 1.2 we know that if $x : 1 \to C$ is an indeterminate arrow over C then $1 \to N \to N$ is also a NNO (weak NNO) in $C[x]$, see for instance [Rom89].

Suppose now we have a predistributive category with NNO; then, since finite coproducts of the terminal object exist, we know that *finite* sets are available. This result was proved in fact in a different kind of categories by the author, namely: if C is a cartesian category with NNO and *equalizers*, then finite sets are available (see proposition 1.3 of [Rom91]). Now, there is something important concerning coproducts in a cartesian category with natural numbers object, we state this as an easy lemma.

Lemma 2.2 Let C be an arbitrary cartesian category with NNO, then the following is true:

1. $1 + N$ exists and is isomorphic to N.
2. $N + (N \times N)$ exists and is isomorphic to $N \times N$.
3. $N \times N$ is isomorphic to N.
4. $N + N$ exists and is isomorphic to N.

Therefore the only thing we need in order to get a predistributive category are finite coproducts of the terminal object. We shall see that the *Karoubi* envelope (see [LS86] for the definition and properties of the Karoubi envelope) of the free cartesian category with NNO is in fact the free predistributive category with NNO. We begin first with the following:

Proposition 2.3 Let C be any cartesian category with NNO then the *Karoubi* envelope of C is again cartesian and has a NNO, namely the same as C.

Proof Denote by $K(C)$ the Karoubi envelope then clearly $K(C)$ is cartesian, the only problem is to show it has a NNO. If A is an arbitrary object of $K(C)$ then the diagram $A \to A \times N \to A \times N$ belongs to $K(C)$. Now, if $A \to B \to B$ is an arbitrary diagram in $K(C)$, then, since N is a NNO in C, there is a unique C-morphism $h : A \times N \to B$ such that $h(1_A, 0_A) = f$ and $h(1_A \times s) = gh$. The problem of course is to show that h belongs to $K(C)$, but this can be proved using the uniqueness of h, hence $K(C)$ has a NNO.Now, the second remark we made is that any finite coproduct of the terminal object is a retract of N. Denote by $\underline{n}$ the coproduct of 1 (n-times). We have then the following:

Lemma 2.4

Let C be any (pre)distributive category with NNO then every object $\underline{n}$ is a retract of N.

Proof Before we start the proof, we want to make the following remark: if we take an arbitrary finite coproduct of 1, we will use the following notation: $\underline{n+1}$ is the coproduct of $\kappa_1 : 1 \to \underline{n+1}$ and $\kappa'_n : \underline{n} \to \underline{n+1}$. Clearly there are many ways of embedding $\underline{n}$ into N. Let us take for instance the following construction from $\underline{2}$ into N: we take the arrow $\psi_2 : 2 \to N$ whose components are $0 : 1 \to N$ and $s0 : 1 \to N$. Suppose, we have defined $\psi_n : \underline{n} \to N$ then $\psi_{n+1} : \underline{n+1} \to N$ is the (unique) arrow whose components are $\kappa_1 : 1 \to \underline{n+1}$ and ψ_n.A left inverse for ψ_{n+1}, $\phi_{n+1} : N \to \underline{n+1}$ is the (unique) arrow whose components are $\kappa_1 : 1 \to \underline{n+1}$ and $\kappa'_n \circ \phi_n : N \to \underline{n+1}$. We make use the fact that N is the coproduct of $0 : 1 \to N$ and $s : N \to N$. For instance, $\phi_2 : N \to \underline{2}$ is nothing but $[\kappa_1, \kappa'_1 \circ !_N]$. The reader can check $\psi_n \circ \phi_n = 1_{\underline{n}}$. If we denote by e_n the composition $\phi_n \circ \psi_n$ then clearly this is an idempotent. Moreover, any arrow $f : \underline{n} \to \underline{m}$ induces a morphism in the Karoubi envelope of C, namely $\psi_m \circ f \circ \phi_n : e_n \to e_m$. If we apply the last construction to the *free cartesian category* with NNO (denoted by F(C)), then we have defined a functor $H : F(C) \to K(F(C))$. Now, by lemma 2.2 and proposition 2.3, $K(F(C))$ has finite products, a NNO and finite coproducts and the isomorphism stated in definition 1.1 is obvious, hence it is a predistributive category. By lemma 2.4, $K(F(C))$ is isomorphic to the *free predistributive category* with NNO. We have shown the following:

Proposition 2.5 The free predistributive category with NNO (denoted by $\underline{P}$) is isomorphic to the Karoubi envelope of the free cartesian category with NNO (denoted by $K(F(C))$).

In particular, since a numerical function $f : N^n \to N$ is *representable* in the free cartesian category with NNO iff it is primitive recursive (see [Rom89]) we know that f is representable in $\underline{P}$ iff f is primitive recursive.

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SOME UNIQUENESS THEOREMS FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

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1. INTRODUCTION AND STATEMENT OF RESULTS.

1.1. If an entire function f of one complex variable and exponential type less than π satisfies $f(n)=0$ for any $n\in\mathbb{N}$, Carlson's theorem ([2] p.171) states that $f\equiv 0$. If f satisfies $\Re e\, f(n)=\Re e\, f(n+i)=0$ for any $n\in\mathbb{Z}$, it follows from a result of Boas (see [3]) that f is a constant function: $f\equiv ib$ for some real number b. Under the additional hypothesis that $\sum_{n=-\infty}^{+\infty}|\Im m\, f(n)|<\infty$, Trembinska proved in [7] that it is enough for f to vanish in the whole plane that $\Re e\, f(n+i)$ vanishes only on a subset of $\mathbb{Z}$ of density greater than τ/π (with $\tau<\pi$ the exponential type of f). Moreover, this result still holds when $\Re e\, f(n+i)$ is replaced by $\Im m\, f(n+i)$ in the above condition (see theorem 2 of [7]). Boas and Trembinska obtained in [4] another uniqueness theorem for entire functions of one complex variable and exponential type$<\pi$: if such a function f satisfies $\sum_{n=-\infty}^{+\infty}|f(n)|<\infty$, $\Re e\, f(n+i)=0$ (resp. $\Im m\, f(n+i)=0$) for any integer $n<0$ (resp. $n\geq 0$) and $\Re e\, f(n)=0$ for any $n\in\mathbb{Z}$, then $f\equiv 0$.

In the case of entire functions of *several* complex variables, there exists a Carlson type theorem: an entire function f in $\mathbb{C}^N$ ($N\in\mathbb{N}, N\geq 1$) of exponential type less than π vanishes identically provided that $f(\nu)=0$ for any $\nu\in\mathbb{N}^N$. The uniqueness set $\mathbb{N}^N$ can even be replaced by a subset of the form $\{\nu=(\nu_1,\dots,\nu_N)\in\mathbb{N}^N:\nu_j\geq\mu_j\ (j=1,\dots,N)\}$ for some $\mu=(\mu_1,\dots,\mu_N)\in\mathbb{N}^N$ (see [1] p.364).

In [8], Trembinska proved the following uniqueness theorem which extends results of [7] ($N=1$) to the case $N=2$: *Let* $f(z_1,z_2)$ *be an entire function of exponential type less than* π, *and let* $f(x_1,x_2)$ *belong to* $L^2(\mathbb{R}^2)$. *Let* $f(m,n)=u(m,n)+iv(m,n)$, $u(m,n)=0$, *for all integers* m,n, *and* $\sum_{(m,n)\in\mathbb{Z}^2} |v(m,n)| <\infty$. *If* $u(m+i,n+i)=0$ *for all integers* m,n *then* $f(z_1,z_2)=0$.

Note that the same assertion holds when "$u(m+i,n+i)=0$" is replaced by "$v(m+i,n+i)=0$" in the above sentence (see [8] p.464).

In the first part of this note, we show that Trembinska's theorem remains valid without the condition "$f(x_1,x_2)$ belong to $L^2(\mathbb{R}^2)$". Moreover, the points (m,n) and $(m+i,n+i)$ where u and v have to vanish can be replaced by $(m+\alpha_1,n+\alpha_2)$ and $(m+\beta_1,n+\beta_2)$ with α_j and β_j complex numbers such that $\Im m\ \alpha_j\neq\Im m\ \beta_j$ $(j=1,2)$. Before stating our results, let's specify some notations.

1.2. Given a compact set K in $\mathbb{C}^N$, let $\mathrm{Exp}(\mathbb{C}^N,K)$ denote the space of all entire functions f of N complex variables satisfying the following estimate: for every $\varepsilon>0$, there exists $M_\varepsilon\geq 0$ such that

$$|f(z)| \leq M_\varepsilon \exp(H_K(z)+\varepsilon\|z\|)$$

for any $z=(z_1,\ldots,z_N)\in\mathbb{C}^N$, with $\|.\|$ a norm in $\mathbb{C}^N$ and H_K the support function of K, defined by $H_K(z)=\sup_{\zeta\in K} \Re e\,\langle z,\zeta\rangle$ (where $\langle z,\zeta\rangle=z_1\zeta_1+\ldots+z_N\zeta_N$).

There is a Carlson type theorem for functions belonging to $\mathrm{Exp}(\mathbb{C}^N,K)$ with K a compact convex set contained in $\{z\in\mathbb{C}^N: |\Im m\ z_j|<\pi\ (j=1,\ldots,N)\}$: such a function vanishes identically provided that it vanishes on $\mathbb{N}^N$ (see [1], théorème 3.1.1.). We will make use of another uniqueness theorem: let K be a compact convex set contained in $\{\rho e^{i\theta}:0\leq\rho<\frac{\pi-|\theta|}{|\sin\theta|},|\theta|\leq\pi\}^N$ and $f\in\mathrm{Exp}(\mathbb{C}^N,K)$ such that $(D^\nu f)(\nu)=0$ for any $\nu\in\mathbb{N}^N$, with $D^\nu=\left(\frac{\partial}{\partial z_1}\right)^{\nu_1}\cdots\left(\frac{\partial}{\partial z_N}\right)^{\nu_N}$, then $f\equiv 0$ (see theorem 1 of [6]).

For any $z=(z_1,\ldots,z_N)\in\mathbb{Z}^N$ and $z'=(z'_1,\ldots,z'_N)\in\mathbb{Z}^N$, $z+z'$ will stand for $(z_1+z'_1,\ldots,z_N+z'_N)$ and $\bar{z}$ for $(\overline{z_1},\ldots,\overline{z_N})$.

1.3. We shall prove in § 2.1. the following theorem which enlarges Trembinska's one:

THEOREM A. *Given* $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{C}^N$ *and* $\beta=(\beta_1,\ldots,\beta_N)\in\mathbb{C}^N$ *such that* $\Im m\ \alpha_j\neq\Im m\ \beta_j$ $(j=1,\ldots,N)$, *let* K *be a compact convex set contained in* $\{z\in\mathbb{C}^N: |\Im m\ z_j|<\pi\ (j=1,\ldots,N)\}$ *such that* $K_j\cap\mathbb{R}$ *is contained in* $]-\pi/\tau_j,\pi/\tau_j[$ *with* K_j *the* j^{th} *projection of* K *and* $\tau_j=|\Im m(\alpha_j-\beta_j)|$ *(resp.* $\tau_j=2|\Im m(\alpha_j-\beta_j)|$*)* $(j=1,\ldots,N)$. *If an entire function* $f\in\mathrm{Exp}(\mathbb{C}^N,K)$ *satisfies*

$\Re e\, f(\nu+\alpha)=\Re e\, f(\nu+\beta)=0$ *(resp.* $\Re e\, f(\nu+\alpha)=\Im m\, f(\nu+\beta)=0)$ *for any* $\nu\in\mathbb{N}^N$*, then* f *is a constant function:* $f\equiv c$ *for some real number* c *(resp.* f *vanishes identically in* $\mathbb{C}^N$*).*

The proof of theorem A relies on the Avanissian-Gay transform of analytic functionals (see [1]) and on a result concerning periodic entire functions of exponential type (théorème 3.4.1. of [1]).

COROLLARY A. *Let* K *be a compact convex set contained in* $\{z\in\mathbb{C}^N: |\Im m\, z_j|<\pi\ (j=1,\dots,N)\}$ *and* $f\in \mathrm{Exp}(\mathbb{C}^N,K)$ *such that* $\sum_{\nu\in\mathbb{Z}^N}|f(\nu)|<\infty$*. If* $\Re e\, f(\nu+\alpha)=\Re e\, f(\nu+\beta)=0$ *(resp.* $\Re e\, f(\nu+\alpha)=\Im m\, f(\nu+\beta)=0)$ *for any* $\nu\in\mathbb{N}^N$ *(*α *and* β *defined as in theorem A), then* f *vanishes identically in* $\mathbb{C}^N$.

The proof of corollary A (§ 2.2.) makes use of a result, relative to analytic functionals and their Fourier-Borel transform, which was obtained by Yoshino (proposition 2 of [9]) with the help of the Avanissian-Gay transform of analytic functionals.

1.4. Finally, we prove another result of the same type than theorem A:

THEOREM B. *With* α, β *and* τ_j $(j=1,\dots,N)$ *defined as in theorem A, let* $f\in \mathrm{Exp}(\mathbb{C}^N,K)$ *where* K *is a compact convex set contained in* $\{\rho e^{i\theta}: 0\le\rho<\frac{\pi-|\theta|}{|\sin\theta|}, |\theta|\le\pi\}^N$ *such that* $K_j\cap\mathbb{R}$ *is contained in* $]-\pi/\tau_j,\pi/\tau_j[$ *with* K_j *the* j^{th} *projection of* K $(j=1,\dots,N)$*. If* $\Re e((D^\nu f)(\nu+\alpha))=\Re e((D^\nu f)(\nu+\beta))=0$ *(resp.* $\Re e((D^\nu f)(\nu+\alpha))=\Im m((D^\nu f)(\nu+\beta))=0)$ *for any* $\nu\in\mathbb{N}^N$*, then* f *is a constant function:* $f\equiv c$ *for some real number* c *(resp.* f *vanishes identically in* $\mathbb{C}^N$*).*

COROLLARY B. *Let* K *be a compact convex set contained in* $\{\rho e^{i\theta}: 0\le\rho<\frac{\pi-|\theta|}{|\sin\theta|}, |\theta|\le\pi\}^N$ *and* $f\in \mathrm{Exp}(\mathbb{C}^N,K)$ *such that* $\sum_{\nu\in\mathbb{Z}^N}|f(\nu)|<\infty$*. If* $\Re e((D^\nu f)(\nu+\alpha))=\Re e((D^\nu f)(\nu+\beta))=0$ *(resp.* $\Re e((D^\nu f)(\nu+\alpha))=\Im m((D^\nu f)(\nu+\beta))=0)$ *for any* $\nu\in\mathbb{N}^N$ *(*α *and* β *defined as in theorem A), then* f *vanishes identically in* $\mathbb{C}^N$.

2. PROOF OF THE RESULTS.

2.1. PROOF OF THEOREM A. Let f_α and f_β denote the entire functions in $\mathbb{C}^N$ defined by :

$$f_\alpha(z)=\frac{1}{2}\left(f(z+\alpha)+\overline{f(\bar{z}+\alpha)}\right)$$

and

$$f_\beta(z)=\frac{1}{2}\left(f(z+\beta)+\overline{f(\bar{z}+\beta)}\right)\ \left(\text{resp. } f_\beta(z)=\frac{1}{2i}\left(f(z+\beta)-\overline{f(\bar{z}+\beta)}\right)\right).$$

for any $z\in\mathbb{C}^N$, with $z+\alpha$ and $\bar{z}$ defined as in 1.2. We have $f_\alpha(x)=\Re e\, f(x+\alpha)$ and $f_\beta(x)=\Re e\, f(x+\beta)$ (resp. $f_\beta(x)=\Im m\, f(x+\beta)$) for any $x\in\mathbb{R}^N$. As $H_K(z+z')\le H_K(z)+H_K(z')$, f_α and f_β belong to $\mathrm{Exp}(\mathbb{C}^N,K)$. As they vanish on $\mathbb{N}^N$, it follows that $f_\alpha\equiv f_\beta\equiv 0$ (see 1.2.). Hence $f(z+2i\Im m(\alpha-\beta))=f(z)$ (resp. $f(z+4i\Im m(\alpha-\beta))=f(z)$) for any $z\in\mathbb{C}^N$, where $z+2i\Im m(\alpha-\beta)$ stands for $(z_1+2i\Im m(\alpha_1-\beta_1),\dots,z_N+2i\Im m(\alpha_N-\beta_N))$.

Theorem 4.5.3. of [5] asserts that f is the Fourier-Borel transform $\hat{T}$ of an analytic functional T carried by K. Its Avanissian-Gay transform $G_K(T)$ is holomorphic in $\prod_{j=1}^N \mathbb{C}\setminus\exp(-K_j)$ (see [1], théorème 1.3.1.). It follows from theorem 3.4.1. of [1] that $G_K(T)$ is a linear combination of rational functions of the form $\prod_{j=1}^N\left(e^{\pi v_j/\tau_j}-z_j\right)^{-1}$ ($v\in\mathbb{Z}^N$, $|v_j|\le\tau_j b_j/\pi$ with b_j the ray of a disc in $\mathbb{C}$ centered at 0 and containing K_j ($j=1,\dots,N$)). Hence $G_K(T)(z)=c\prod_{j=1}^N(1-z_j)^{-1}$ ($c\in\mathbb{C}$). In other words, f is a constant function.

2.2. PROOF OF COROLLARY A. Let T be an analytic functional carried by K, such that $\hat{T}=f$. Since $\sum_{v\in\mathbb{Z}^N}|f(v)|<\infty$, we have $\limsup_{|v|\to+\infty}|f(v)|^{1/|v|}\le 1$ ($v\in\mathbb{Z}^N$, $|v|=|v_1|+\dots+|v_N|$). According to proposition 2 of [9], T is carried by $L=\prod_{j=1}^N\,]-\pi,\pi[\cap K_j$ with K_j the j^{th} projection ($j=1,\dots,N$) of K, in other words $f\in\mathrm{Exp}(\mathbb{C}^N,L)$ and we are allowed to apply theorem A.

2.3. PROOF OF THEOREM B. Let f_α and f_β be defined as in the proof of theorem A. They belong to $\mathrm{Exp}(\mathbb{C}^N,K)$ and satisfy $(D^v f_\alpha)(x)=\Re e((D^v f)(x+\alpha))$ and $(D^v f_\beta)(x)=\Re e((D^v f)(x+\beta))$ (resp. $(D^v f_\beta)(x)=\Im m((D^v f)(x+\beta))$) for any $x\in\mathbb{R}^N$ and $v\in\mathbb{N}^N$. Since $(D^v f_\alpha)(v)=(D^v f_\alpha)(v)=0$ for any $v\in\mathbb{N}^N$, it follows that $f_\alpha\equiv f_\beta\equiv 0$ (see 1.2.). We conclude as in the proof of theorem A.

2.4. PROOF OF COROLLARY B. It is patterned on the proof of corollary A.

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SQUARES AND DOUBLE-SQUARES IN LUCAS SEQUENCES

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Abstract. For all odd relatively prime parameters P and Q, we determine the terms in the Lucas sequences $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ such that $U_n, 2U_n, V_n$ and $2V_n$ are squares.

1. Introduction.

Let P and Q be non-zero relatively prime integers, α and β ($\alpha > \beta$) be the zeros of $x^2 - Px + Q$. The Lucas sequence and companion Lucas sequence are defined by

$$U_n = U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = V_n(P,Q) = \alpha^n + \beta^n, \ n \geq 0.$$

The sequences satisfy the recurrence relations

$$U_0 = 0, \ U_1 = 1, \ U_n = PU_{n-1} - QU_{n-2}$$

and

$$V_0 = 2, \ V_1 = P, \ V_n = PV_{n-1} - QV_{n-2}.$$

Our purpose in this paper is to report that we have obtained a complete answer, for all odd P and Q, to the question "What are the square terms in the sequences $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$?". We have also determined the square terms in $\{2U_n(P,Q)\}$ and $\{2V_n(P,Q)\}$.

It is known, based on the method of Baker on linear forms in logarithms that there exists an effectively computable constant $C = C(P,Q,k)$ such that if $U_n(P,Q) = k\square$ or $V_n(P,Q) = k\square$, then $n < C$ (see [9]); however the "computable" constant C is extremely large. Our results establish that, for P and Q odd, the bound is actually 12 when $k = 1$ or 2.

Progress in determining the square terms had been made in certain special cases. Cohn [3] showed that $U_{12} = 144$ is the only perfect square for $n > 1$ when $P = 1$, $Q = -1$ (i.e., among the Fibonacci numbers) (see, also Alfred [1], Burr [2] and Wyler [12]), and determined the square terms of both $\{U_n(P,\pm1)\}$ and $\{V_n(P,\pm1)\}$ for P odd [4], [5]. Cohn also determined the square terms for a restricted but infinite set of even values of P with $Q = \pm1$ [6]. The square terms are known, too, for $P = 2$ and $Q = -1$ (see [11]), and for $P = Q + 1$ (in which case $U_n = \frac{Q^n-1}{Q-1}$ and $V_n = Q^n + 1$) [7] and [8].

The determination of the square terms of these sequences is equivalent to solving certain Diophantine equations; this relationship is discussed in the articles cited.

2. The Theorems.

We assume that P and Q are odd relatively prime integers, and that for all
$u \geq 1$, $V_{2^u} > 0$. This latter condition holds if $D = P^2 - 4Q > 0$, and it may be noted that all sequences $\{U_n\}$ or $\{V_n\}$ who terms are positive have a positive discriminant D; however, the converse is not true (that is, our result is not restricted to sequences containing only positive terms). In the statement of the following theorems, we use $\square$ for "a square".

THEOREM 1. *If $n \geq 0$, $U_n(P,Q) = \square$ if and only if*

(i) $n = 0$ or 1,

(ii) $n = 2$ and $P = \square$,

(iii) $n = 3$ and $P^2 - Q = \square$,

(iv) $n = 6$, $P = 3\square$, $P^2 - Q = 2\square$ and $P^2 - 3Q = 6\square$; *this implies* $Q \equiv 1 \pmod{24}$, *or*

(v) $n = 12$, $P = \square$, $P^2 - Q = 2\square$, $P^2 - 2Q = 3\square$, $P^2 - 3Q = \square$ and $(P^2 - 2Q)^2 - 3Q^2 = 6\square$; *these conditions imply that* $Q \equiv -1 \pmod{120}$.

THEOREM 2. *If $n \geq 0$, $U_n = 2\square$ if and only if*

(i) $n = 0$,

(ii) $n = 3$ and $P^2 - Q = 2\square$, or

(iii) $n = 6$, $P = \square$, $P^2 - Q = 2\square$ and $P^2 - 3Q = \square$; *this implies* $Q \equiv -1 \pmod{24}$).

THEOREM 3. *If $n \geq 0$, then $V_n = \square$ if and only*

(i) $n = 1$ and $P = \square$,

(ii) $n = 3$, and either $P = \square$ and $P^2 - 3Q = \square$, or $P = 3\square$ and $P^2 - 3Q = 3\square$; *this implies that* $Q \equiv 3 \pmod 4$, *or*

(iii) $n = 5$, $P = 5\square$ and $P^4 - 5P^2Q + 5Q^2 = 5\square$; *this implies that* $P \equiv Q \equiv 5 \pmod 8$.

THEOREM 4. *If $n \geq 0$, then $V_n = 2\square$ if and only if*

(i) $n = 0$,

(ii) $n = 3$, and either $P = \square$ and $P^2 - 3Q = 2\square$, or $P = 3\square$ and $P^2 - 3Q = 6\square$; *the first condition is possible iff* $P \equiv 1 \pmod 8$ *and* $Q \equiv 3$ *or* $5 \pmod 8$ *and the second condition iff* $P \equiv 1 \pmod{24}$ *and* $Q \equiv 1$ *or* $3 \pmod 8$, *or*

(iii) $n = 6$, $P^2 - 2Q = 3\square$ and $(P^2 - 2Q)^2 - 3Q^2 = 6\square$; *this implies* $Q \equiv 3 \pmod 4$.

3. The Proofs.

The proofs use elementary methods only. We obtain certain congruences permitting us to examine the residue class of, for example, U_n with respect to an odd modulus M. If U_n is a square, and the Jacobi symbol $(U_n|M)$ is defined, then $(U_n|M) = +1$. We indicate how to select, for each $n > 1$, a modulus M such that by using certain identities relating U_n and V_n, the Jacobi symbol $(U_n|M)$ is equal to -1 for all values other than those listed in the theorem above (showing $U_n \neq \square$).

We also discuss the pairs (P,Q) with the property that $U_n(P,Q) = \square$ or $2\square$, or $V_n(P,Q) = \square$ or $2\square$ (for $n \leq 12$ listed in the theorems above), providing, in most cases a parametric representation of P and Q giving all such pairs.

Details of the proofs are contained in [10].

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On a topological invariant of complex Lie groups and solv-manifolds

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Presented by J.S. Halperin, F.R.S.C.

Abstract. For a connected complex Lie group G define d_G to be the real codimension of a maximal compact subgroup of G. We prove that if $d_G \leq 2$, then G is abelian and belongs to a certain list, see theorem 1. We also show that if X is a complex solv-manifold with d_X defined to be the dimension of the fiber of a certain vector bundle, see [2] or [13], then $d_X \leq 2$ and X holomorphically separable imply that X is $\mathbf{C}, \mathbf{C}^*$ or $\mathbf{C}^* \times \mathbf{C}^*$.

I. Introduction

Given a Lie group G one has the Iwasawa decomposition $G = K \times \mathbf{R}^{d_G}$, where K is a maximal compact subgroup of G. Since K is unique up to conjugation, the invariant d_G is independent of the choice of K. The main purpose of this note is to prove that if G is a connected complex Lie group with $d_G \leq 2$, then G is abelian and belongs to a certain list. For G compact ($d_G = 0$) this is classical.

We also discuss the case of complex solv-manifolds. For our purposes we note that Auslander-Tolimieri [2] and Mostow [13] proved that any solv-manifold fibers as a vector bundle over a compact solv-manifold. Thus for a solv-manifold X one can define the invariant d_X to be the dimension of the fiber of this bundle. We study the case where X is holomorphically separable and satisfies $d_X \leq 2$. Since a holomorphically separable complex solv-manifold is Stein, see [6], a simple argument using homology shows that one has $d_X \geq n$, where n is the complex dimension of X. Thus the possibilities for the spaces X which can occur follow from the classification of complex solv-manifolds in dimensions one and two [3], and these are $\mathbf{C}, \mathbf{C}^*$ and $\mathbf{C}^* \times \mathbf{C}^*$.

This paper was inspired by a fibration which exists for (real) homogeneous spaces G/H, where G is a connected Lie group and H is a closed subgroup with a finite number of connected components. Then the homogeneous space $X := G/H$ fibers as a vector bundle over a compact homogeneous space, see Mostow [12] and also Karpelevich [8], and one can define an invariant of the space X to be the real dimension d_X of the fiber of this bundle. Homogeneous spaces X of complex algebraic groups with $d_X = 2$ were classified in [1]. Holomorphically separable homogeneous spaces of complex Lie groups with $d_X = 2$ will be studied elsewhere using the results of this paper; in particular, corollary 1 and theorem 2.

II. Complex Lie groups

As we noted above the following is classical in the compact case. The case $d_G = 1$ was proved in [5] - here we present a different proof.

Theorem 1 *Suppose G is a connected complex Lie group with $d_G \leq 2$. Then G is abelian and belongs to the following list:*

0) *$d_G = 0$ and G is a compact complex torus.*

1) *$d_G = 1$ and G is either $\mathbf{C}^* \times T$ or a Cousin group which fibers as a $\mathbf{C}^*$-bundle over a torus*

2) *$d_G = 2$ and G is either $\mathbf{C} \times T$, $(\mathbf{C}^*)^2 \times T$, $\mathbf{C}^* \times$ (a Cousin group), the latter fibering as a $\mathbf{C}^*$ – bundle over a torus or a Cousin group which fibers as a $(\mathbf{C}^*)^2$-bundle over a torus.*

Remark: Given G with $d_G = 3$ it no longer follows that G must be abelian. For, now $SL(2, \mathbf{C})$ can occur as well as $\mathbf{C} \times \mathbf{C}^*$ with a solvable structure. Also it is not enough to have a real Lie group which has only a left invariant complex structure, as one sees even in the compact case!

To prove the theorem we use the holomorphic reduction of connected complex Lie groups. Given a connected complex Lie group G we define an equivalence relation $\sim$ on G: $g_1 \sim g_2 \iff f(g_1) = f(g_2)$ for every $f \in \mathcal{O}(G)$, where $g_1, g_2 \in G$. Alternatively one may introduce the subgroup

$$G_0 := \{g \in G | f(g) = f(e) \text{ for every } f \in \mathcal{O}(G)\}, \tag{1}$$

since it is clear that $G/\sim = G/G_0$. Morimoto showed that G_0 is a closed connected complex central subgroup of G and thus the quotient G/G_0 is again a complex Lie group which is holomorphically separable by the definition of G_0, see [11]. As well he proved that $\mathcal{O}(G_0) \cong \mathbf{C}$, i.e. G_0 is a Cousin group.

We use the characterization of Stein Lie groups given by Matsushima–Morimoto [10] in order to understand the base of the above fibration. Suppose G is a connected complex Lie group and K is a maximal compact subgroup of G. Let $\mathfrak{k}$ be the corresponding subalgebra of $\mathfrak{g}$. Then one has $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, i\mathfrak{k}] \subset i\mathfrak{k}$. Let $\mathfrak{m} := \mathfrak{k} \cap i\mathfrak{k}$. Clearly $\mathfrak{m}$ is an ideal in $\mathfrak{k}$ and thus a subalgebra of $\mathfrak{g}$, so the distribution defined by $\mathfrak{m}$ is integrable. If $\mathfrak{m}$ is positive dimensional, it follows by the maximum principle that the group G cannot be holomorphically separable. In [10] Matsushima and Morimoto showed more: Suppose G is a connected complex Lie group. Then the following conditions are equivalent:

a) G is holomorphically separable

b) K is totally real, i.e. $\mathfrak{m} = (0)$

c) G is Stein

We also use the following lemma. For a maximal compact subgroup K of G the notation $\tilde{K}$ is used to denote the connected complex subgroup of G which has Lie algebra $\mathbf{k} + i\mathbf{k}$. In passing we recall that Matsushima [9] showed that such a subgroup is closed.

Lemma 1 *Suppose G is a connected complex Lie group with $G = \tilde{K}$ and H is a closed connected complex normal solvable subgroup of G such that $G/H \cong (\mathbf{C}^*)^q$. Then G is abelian.*

Proof: As H and G/H are solvable, G is also solvable and so is K. Since K is also compact, it is abelian. But then $G = \tilde{K}$ is also abelian. ††

Proof of theorem 1: We suppose that G is a connected complex Lie group with $d_G \leq 2$. Consider the "Steinizator" subgroup G_0 of Morimoto defined above in (1). Then we have a fibration $G \to G/G_0$, where the base G/G_0 is a holomorphically separable Lie group and so is Stein by the result of Matsushima–Morimoto. Now we have

$$d_G = d_{G_0} + d_{G/G_0},$$

by basic properties of the Iwasawa decomposition, see [7]. Thus $d_{G/G_0} \leq 2$ and this implies that the codimension of any maximal compact subgroup of G/G_0 is at most two. But because G/G_0 is Stein, any maximal compact subgroup is totally real and thus its dimension is at most half the real dimension of the group G/G_0. So we have the following possibilities for the group G/G_0:

0) $d_{G/G_0} = 0 \Longleftrightarrow G/G_0 = \{e\}$

1) $d_{G/G_0} = 1 \Longleftrightarrow G/G_0 = \mathbf{C}^*$

2) $d_{G/G_0} = 2 \Longleftrightarrow G/G_0 = \mathbf{C}$ or $\mathbf{C}^* \times \mathbf{C}^*$

Assume $d_G = 2$ and the base $G/G_0 \cong \mathbf{C}$. Then the fiber is a compact complex torus T, which is central [11]. Now there are different ways to see that G is a direct product of T with $\mathbf{C}$. We use the following. One has the short exact sequence

$$0 \longrightarrow T \xrightarrow{\iota} G \xrightarrow{\tau} \mathbf{C} \longrightarrow 0,$$

where ι is the inclusion homomorphism and τ is the holomorphic reduction map. There exist one parameter subgroups $\sigma : \mathbf{C} \to G$ and we choose one that is a section of the bundle $G \to G/G_0$, i.e. such that $\tau \circ \sigma = \mathrm{id}$. Define

$$p : T \times \mathbf{C} \to G, \quad (t, z) \mapsto \iota(t) + \sigma(z).$$

We have to check that p is a group isomorphism. First note that for all $z_1, z_2 \in \mathbf{C}$, since σ is a homomorphism, one has

$$p(0, z_1 + z_2) = \sigma(z_1 + z_2) = \sigma(z_1) + \sigma(z_2) = p(0, z_1) + p(0, z_2). \tag{2}$$

Also for $t \in T$ and $z \in \mathbf{C}$ one has

$$p(t, z) = \iota(t) + \sigma(z) = t + \sigma(z) = t + p(0, z). \tag{3}$$

Hence for $t_i \in T$ and $z_i \in \mathbf{C}$ with $i = 1, 2$ it follows that

$$\begin{aligned}
p(t_1 + t_2, z_1 + z_2) &= t_1 + t_2 + p(0, z_1 + z_2) && \text{by (3)} \\
&= t_1 + t_2 + p(0, z_1) + p(0, z_2) && \text{by (2)} \\
&= t_1 + p(0, z_1) + t_2 + p(0, z_2) && \text{since } T \text{ is central} \\
&= p(t_1, z_1) + p(t_2, z_2) && \text{by (3) again}
\end{aligned}$$

and thus p is a group homomorphism. The fact that p is a bijection is obvious.

If $d_G = 2$ and $\mathcal{O}(G) \cong \mathbf{C}$, then $G = \mathbf{C}^n/\Gamma$, where $\Gamma = \langle v_1, \dots, v_{2n-2} \rangle_{\mathbf{Z}}$, is a *Cousin group*. Now any Cousin group fibers as a $(\mathbf{C}^*)^k$-bundle over a torus (here $k = 2$). To see this let $K := \langle v_1, \dots, v_{2n-k} \rangle_{\mathbf{R}}/\Gamma$ be the maximal compact subgroup of G and M the maximal connected complex subgroup of K. Let $\tilde{G}$ be the universal covering group of G and denote preimages in $\tilde{G}$ by tildes. Choose linearly independent vectors $w_1, \dots, w_k$ in $\tilde{G}$ such that, if $W_{\mathbf{R}} := \langle w_1, \dots, w_k \rangle_{\mathbf{R}}$ and $W_{\mathbf{C}} := \langle w_1, \dots, w_k \rangle_{\mathbf{C}}$, then $\tilde{G} = \tilde{M} \oplus W_{\mathbf{C}}$ and $\tilde{K} = \tilde{M} \oplus W_{\mathbf{R}}$. Let $H := W_{\mathbf{C}}/W_{\mathbf{C}} \cap \Gamma \cong (\mathbf{C}^*)^k$. Because $H \cap K = W_{\mathbf{R}}/W_{\mathbf{R}} \cap \Gamma = (S^1)^k$ is closed in K, the subgroup H is closed in G. Then $G \xrightarrow{H} G/H$ is the desired bundle.

There are two other cases with $d_G = 2$. If the base of the holomorphic reduction of G is $\mathbf{C}^*$, then its fiber is a Cousin group with two ends. And if this base is $\mathbf{C}^* \times \mathbf{C}^*$, then its fiber is a compact complex torus T. Matsushima [9] proved that any connected complex Lie group G is biholomorphic to $\tilde{K} \times \mathbf{C}^s$, where K is a maximal compact subgroup of G and s is a nonnegative integer. Now if s is positive, then $d_G = 2$ implies $s = 1$. Since T is a subgroup of $\tilde{K}$, we have the fibration $\mathbf{C}^* \times \mathbf{C}^* = G/T \to G/\tilde{K} = \mathbf{C}$. But then $\mathbf{C}^* \times \mathbf{C}^*$ contains a compact, one-dimensional, complex submanifold, which is an obvious contradiction. Therefore $s = 0$, $G = \tilde{K}$ and thus by lemma 1 it follows that G is abelian. The claimed structure is clear, since abelian complex Lie groups always split as direct products: $G = \mathbf{C}^p \times (\mathbf{C}^*)^q \times G_0$.

If $d_G = 0$ or 1, there are no $\mathbf{C}$'s. Thus $G = \tilde{K}$ and the result follows easily. ††

Corollary 1 *Suppose G is a connected Stein Lie group (i.e. a holomorphically separable complex Lie group) with $d_G \leq 2$. Then G is $\mathbf{C}, \mathbf{C}^*$ or $\mathbf{C}^* \times \mathbf{C}^*$.*

III. Complex solv-manifolds

First we need a result about Stein manifolds which are vector bundles. Note that this is independent of the existence of a group action.

Lemma 2 *Let X be a connected Stein manifold with $n := \dim_{\mathbb{C}} X$ which fibers as a vector bundle of rank d. Then*

$$n \leq d \leq 2n \tag{4}$$

Proof: Since X is Stein, one has $H_p(X) = 0$ for $p > n$ and $H_n(X)$ has no torsion. Let M denote the compact base of the vector bundle. It follows from the given fibration that $H_p(X) \cong H_p(M)$ for $p \geq 0$. Hence if M is orientable, then $H_{2n-d}(X)$ is not zero, and if M is not orientable, then $H_{2n-d}(X)$ is zero and $H_{2n-d-1}(X)$ has torsion, while $H_p(X) = 0$ for all larger values of p. These considerations imply that one has

$$2n - d \leq n,$$

from which one gets the first inequality of the lemma. The second is clear. ††

Now let $X = G/H$ be a solv-manifold, i.e. G is a connected (real) solvable Lie group and H is a closed subgroup. Then there exists a vector bundle

$$X = G/H \xrightarrow{\mathbb{R}^d} M, \tag{5}$$

where M is a compact solv-manifold, see Auslander–Tolimieri [2] and Mostow [13]. We now classify complex solv-manifolds X with $d_X \leq 2$ and $\mathcal{O}(X)$ having maximal rank. For $d_X = 1$ and the bundle in (5) orientable, see also [4].

Theorem 2 *Suppose $X = G/H$ is a complex solv-manifold, i.e. G is a connected complex solvable Lie group and H is a closed complex subgroup, with $d_X \leq 2$ and $\mathcal{O}(X)$ has maximal rank. Then X is biholomorphic to* $\mathbb{C}, \mathbb{C}^*$ *or* $\mathbb{C}^* \times \mathbb{C}^*$.

Proof: In this setting X is Stein [6] and from inequality (4) we have $n \leq d_X$. Thus $d_X \leq 2$ implies $n = 1$ or 2. The result is then a direct consequence of the list of homogeneous Riemann surfaces and the classification of complex solv-manifolds in dimension two, e.g. see [3, Theorem 2.1, p. 62]. ††

Remark: The case $d_X = 3$ with $\mathcal{O}(X)$ having maximal rank can also be classified, but is left to the reader.

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IMAGINARY VERMA MODULES FOR AFFINE LIE ALGEBRAS

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Presented by P. Ribenboim, F.R.S.C.

In the following note we discuss a family of modules over an Affine Lie algebra which possesses weight spaces of both finite and infinite dimension. Detailed proofs will appear in a subsequent paper.

Let $A = (a_{ij})_{i,j=0}^{n}$ be an Affine Cartan matrix, $\mathfrak{G} = \mathfrak{G}(A)$ be a corresponding Affine Lie algebra with Cartan subalgebra H, root system Δ and one-dimensional centre $C = \mathbb{C}c$.

Let $\delta = \sum_{i=0}^{n} \kappa_i \alpha_i$ be a minimal positive imaginary root in Δ where $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ is the set of simple roots of Δ indexed in such a way that $\kappa_0 = 1$ and $-\alpha_0 + \delta \in \Delta$ or $\frac{1}{2}(-\alpha_0 + \delta) \in \Delta$.

Set $\psi := \sum_{i=1}^{n} \alpha_i^* - \left(\sum_{i=1}^{n} \kappa_i\right)\alpha_0^*$ where $\alpha_i^* \in H$, $\alpha_i^*(\alpha_j) = \delta_{ij}$. Consider a set $N = \{\alpha \in \Delta \mid \psi(\alpha) > 0\} \cup \{\kappa\delta \mid \kappa \in \mathbb{Z}_+\} \subset \Delta$. One can see that $N \cup (-N) = \Delta$, $N \cap (-N) = \phi$.

It is well known that irreducible $\mathfrak{G}$-modules with highest weight are unique irreducible quotients of Verma modules associated with the set of positive roots for some choice of the base of Δ. The purpose of the present paper is to study Verma modules associated with N which we shall call Imaginary Verma Modules and their irreducible quotients. These modules are the particular case of the modules associated with partitions of Δ (i.e., decompositions $\Delta = P \cup (-P)$ where P is closed under addition of roots and $P \cap (-P) = \phi$), which were introduced by V. G. Kac and H. P. Jakobsen [1,2] and also by the present author [3,4].

Let $\mathfrak{G} = \sum_{\alpha \in \Delta} \mathfrak{G}_\alpha \oplus H$ be a root decomposition of $\mathfrak{G}$ and let $\mathfrak{G}_{\pm N} = \sum \mathfrak{G}_\alpha$ for $\alpha \in \pm N$. Then $\mathfrak{G} = \mathfrak{G}_N \oplus H \oplus \mathfrak{G}_{-N}$.

Denote by $U(\mathfrak{G})$ the universal enveloping algebra of $\mathfrak{G}$. Let $\lambda \in H^*$. Consider $\mathbb{C}$ as a one-dimensional $H \oplus \mathfrak{G}_{-N}$-module under the action $(h+x)\cdot 1 = \lambda(h)\cdot 1$ for any $h \in H$, $x \in \mathcal{G}_{-N}$.

Define a $\mathcal{G}$-module

$$\bar{M}(\lambda) = U(\mathcal{G}) \otimes_{U(\mathcal{H}\oplus\mathcal{G}_{-N})} \mathbb{C}$$

associated with N and λ. We shall call this an <u>Imaginary Verma module</u>.

Module $\bar{M}(\lambda)$ is a $U(\mathcal{G}_N)$-free 1-generated module. Moreover, $\dim \bar{M}(\lambda)_\lambda = 1$; $0 < \dim \bar{M}(\lambda)_{\lambda+\kappa\delta} < \infty$ for any integer $\kappa > 0$; if $\bar{M}(\lambda)_\mu \neq 0$, $\mu \neq \lambda+\kappa\delta$ for any $\kappa \geq 0$ then $\dim \bar{M}(\lambda)_\mu = \infty$. Like a Verma module, $\bar{M}(\lambda)$ has a unique maximal submodule. Denote the unique irreducible quotient of $\bar{M}(\lambda)$ by $\bar{L}(\lambda)$.

Choose $h_i \in H$ such that $\alpha_j(h_i) = a_{ij}$, $1 \leq i \leq n$, $0 \leq j \leq n$.

Let $T \subset \{1,2,\dots,n\}$, $P(T)$ be a closed subset of Δ which contains N and $P(T) \cap (-P(T)) = \left(\sum_{i\in T} \mathbb{Z}\alpha_i + \mathbb{Z}\delta\right) \cap \Delta$. Consider $\mathcal{G}_{-P(T)} = \sum_{\alpha\in -P(T)} \mathcal{G}_\alpha$, $H_T \subset H$ spanned by h_i, $i \in T$ and $\lambda \in H^*$ such that $H_T \oplus \mathbb{C} \subset \operatorname{Ker}\lambda$.

Let $\mathbb{C}$ be an $H \oplus \mathcal{G}_{-P(T)}$-module under the action $(h+X)\cdot 1 = \lambda(h)\cdot 1$ for all $X \in \mathcal{G}_{-P(T)}$, $h \in H$.

Define a $\mathcal{G}$-module

$$M(\lambda,T) = U(\mathcal{G}) \otimes_{U(H\oplus\mathcal{G}_{-P(T)})} \mathbb{C}$$

associated with N, T and λ.

One can check that if $M(\lambda,T)_\mu \neq 0$ then $\dim M(\lambda,T)_\mu = 1$ for $\mu = \lambda + \sum_{j\in T} n_j\alpha_j + \alpha_i + \kappa\delta$ where $\mu - \lambda \in \Delta$, $i \in \{1,2,\dots,n\}\setminus T$, κ, $n_j \in \mathbb{Z}$, $n_j \geq 0$ and $\dim M(\lambda,T)_\mu = \infty$ in other cases.

Module $M(\lambda,T)$ has a unique maximal submodule. Denote by $L(\lambda,T)$ the unique irreducible quotient $M(\lambda,T)$.

If $H_T \oplus \mathbb{C} \subset \operatorname{Ker}\lambda$ then $\bar{L}(\lambda) \cong L(\lambda,T)$. We can now formulate our main result:

<u>Theorem 1</u>: 1) $\bar{M}(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.

2) Let $T \subset \{1,2,\dots,n\}$, $H_T \oplus \mathbb{C} \subset \operatorname{Ker}\lambda$. The module $M(\lambda,T)$ is irreducible if and only if $H_{T'} \not\subset \operatorname{Ker}\lambda$ for any $T' \supsetneq T$.

Note, that if $T = \{1,2,\dots n\}$ then $\bar{L}(\lambda)$ is a trivial one-dimensional module.

Remark. The Theorem 1 is also true if $M(\lambda)$ and $M(\lambda,T)$ are viewed as a $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$-modules. In this case statement 2) of the Theorem 1 was also proved in [1].

Assume now that $C \subset \operatorname{Ker} \lambda$ but $H_T \not\subset \operatorname{Ker} \lambda$ for any $T \subset \{1,2,\dots,n\}$. In this case any submodule K of $\bar{M}(\lambda)$ is generated by $K \cap \sum_{m=0}^{\infty} \bar{M}(\lambda)_{\lambda+m\delta}$.

This implies

Theorem 2: Let $\lambda \in H^*$, $\lambda(c) = 0$, $\lambda(h_i) \neq 0$ for $1 \le i \le n$. Then

1) $\bar{M}(\lambda)$ has an infinite composition series.

2) The modules $M(\lambda + \kappa\delta,\phi)$ with multiplicities $m_\kappa = \dim \bar{M}(\lambda)_{\lambda+\kappa\delta}$ and non-negative integers κ exhaust all irreducible subquotients of $\bar{M}(\lambda)$.

3) $\operatorname{Hom}_{\mathfrak{g}}(\bar{M}(\mu)), \bar{M}(\lambda)) \neq 0$ if and only if $\mu = \lambda + \kappa\delta$ for some non-negative κ and $\dim \operatorname{Hom}_{\mathfrak{g}}(\bar{M}(\lambda + \kappa\delta), \bar{M}(\lambda)) = \dim \bar{M}(\lambda)_{\lambda+\kappa\delta}$.

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Corrigendum

E.E. GRANIRER

On convolution operators which are far from being convolution by a bounded measure. Expository memoir.

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On p. 188 insert after the last row:
"It so happens that all the above $\phi \notin \lambda_q(M(G))$ belong to $\mathbb{M}_q(G)$"

On p. 189 insert after the last row:
"u, iff for some neighborhood V of x, $\int uvdx = 0$ for all $v \in C_s(G)$ (continuous functions"

On p. 191 row 9 from top, replace:
$\mathbb{M}_2(F)$ by $\mathbb{M}_2(G)$.

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