
 CONTENTS

H. BEN-EL-MECHAIECKH	
General fixed point and coincidence theorems for set-valued maps	237
S. SHOKRANIAN	
The twisted L^2 -Lefschetz number of Hecke operators	243
D.F. ANDERSON	
The kernel of $\text{Pic}(R_0) \rightarrow \text{Pic}(R)$ for R a graded domain	248
O.H. KEGEL and S. WUJIE	
On the orders of elements of a locally finite simple group	253
M. KANEMITSU, T. SUGATANI and K. YOSHIDA	
Note on the dual of an ideal in a Noetherian domain	255
R. SHARMA	
Second order parallel tensors on contact manifolds II	259
M.S. NIKULIN	
A chi-square goodness of fit test for natural exponential families with power variance function	265
C.L. STEWART	
On the number of solutions of polynomial congruences	271
G. MAKSA, K. NIKODEM and Z. PALES	
Results on t -Wright convexity	274
W. JAWORSKI	
Poisson and Furstenberg boundaries of random walks	279
Z. MOSZNER	
Sur une forme de la solution de l'équation de translation	285
Mailing Addresses	291
Index - Volume XIII	292

GENERAL FIXED POINT AND COINCIDENCE THEOREMS FOR SET-VALUED MAPS

by

H. Ben-El-Mechaiekh

Presented by P.A. Fillmore, F.R.S.C.

ABSTRACT. This note follows [1], [2], and [3]; its purpose is to put on record some general fixed point and coincidence theorems for an abstract class of set-valued maps defined on subsets of topological vector spaces, namely the class of approachable usc's. Our results include numerous known theorems for both the convex and non-convex cases. We essentially present generalizations of the Fan-Glicksberg-Kakutani fixed point theorem to non-necessarily convex maps, and we derive some very general coincidence properties for such maps. The results presented here find numerous applications to the theory of differential inclusions, as well as to non-convex minimax theory. Details will be published elsewhere.

1. PREREQUISITES. In this note, a space is a Hausdorff topological real vector space. Given a space E , $\mathcal{N}_E(0)$ will denote a base of open circled neighborhoods of the origin in E . Set-valued transformations are simply called maps and are denoted by capital letters ($A, B, \dots$), while small letters ($f, s, \dots$) are used for ordinary single-valued functions. Given a class $\mathcal{B}$ of maps and two sets X, Y , we write: $\mathcal{B}(X, Y) = \{A: X \rightarrow Y \mid A \in \mathcal{B}\}$; $\mathcal{B}(X) = \mathcal{B}(X, X)$, $\mathcal{B}^{-1}(X, Y) = \{A: X \rightarrow Y \mid A^{-1} \in \mathcal{B}\}$. Given two topological spaces X and Y , an usco from X into Y is an upper semicontinuous map $A: X \rightarrow Y$ with non-empty compact values.

An important tool in fixed point theory for set-valued maps and related topics is the approachability property of these transformations by single-valued continuous functions. The use of "approximate selections" goes back to the work of J. von Neumann on minimax theory in $\mathbb{R}^n$. This approach was extended to the infinite dimensional setting in numerous situations by several authors, such as A. Cellina, A. Cellina-A. Lasota, A. Mas-Colell, A. Granas-L. Górniewicz-W. Kryszewski, A. McLennan, L. Górniewicz-M. Lassonde, H. Ben-El-Mechaiekh-P. Deguire, etc... (see [1], [2], [3] and references therein).

This research was supported in part by a grant from the National Research Council of Canada

DEFINITION (1.1). Given $U \in \mathcal{K}_E(0)$ and $V \in \mathcal{K}_F(0)$, a function $s: X \rightarrow Y$ is said to be a (U, V) -selection of A if for each $x \in X$, $s(x) \in (A(x + U) \cap X) + V \cap Y$. We put: $Q(A; U, V) = \{s: X \rightarrow Y \mid s \text{ is a continuous } (U, V)\text{-selection of } A\}$, and we define the class of approachable maps by putting:

$$A \in \mathcal{A}(X, Y) \Leftrightarrow \begin{cases} \text{(i) } A \text{ is an usco from } X \text{ into } Y; \\ \text{(ii) } Q(A; U, V) \neq \emptyset \text{ for any } U \in \mathcal{K}_E(0) \text{ and any } V \in \mathcal{K}_F(0). \end{cases}$$

The class $\mathcal{A}$ satisfies nice stability properties for basic operations on maps such as restriction, composition product, cartesian product, domination of domain, etc... We present without proofs some of these properties crucial to the sequel.

PROPOSITION (1.2). (i) (Restriction). Let X, Y be two subsets of spaces E, F respectively, C a compact subset of X , and let $A: X \rightarrow Y$ be a usco map. Then for any $U \in \mathcal{K}_E(0)$, $V \in \mathcal{K}_F(0)$, there exist a finite subset $N = \{x_1, x_2, \dots, x_n\}$ of C and a set $U' \in \mathcal{K}_E(0)$, $U' \subset U$, such that for any subset D of X containing N , $s|_{C \cap D} \in Q(A|_{C \cap D}; U', V)$ provided $s \in Q(A; U, \frac{1}{2}V)$.

(ii) (Cartesian Product). Let I be an arbitrary index set and let $A_i \in \mathcal{A}(X, Y_i)$, $i \in I$, (Y_i being a subset of a space F_i). Then the product map $A: X \rightarrow Y = \prod_{i \in I} Y_i$ defined by $A(x) = \prod_{i \in I} A_i(x)$ is in $\mathcal{A}(X, Y)$.

(iii) (Composition Product). Let X, Y, Z be three subsets in three spaces E, F, G respectively, and let $A \in \mathcal{A}(X, Y)$, $B \in \mathcal{A}(Z, Y)$. If Z is compact, then $AB \in \mathcal{A}(Z, Y)$.

The class of approachable maps is quite large and contains several subclasses of maps useful in optimization and in the theory of differential equations and differential inclusions. We list now without proofs some interesting examples of approachable maps.

PROPOSITION (1.3). (I.3). (i) (Convex Case). If X is paracompact and Y is convex, then

$$\mathcal{X}(X, Y) = \{A: X \rightarrow Y \mid A \text{ is an usco with convex values}\} \subset \mathcal{A}(X, Y);$$

(ii) (Contractible Case). If X is a compact ANR and Y is convex, then

$$\mathcal{C}(X, Y) = \{A: X \rightarrow Y \mid A \text{ is an usco with contractible values}\} \subset \mathcal{A}(X, Y);$$

(iii) (Non-contractible Case). If X is a compact ANR and Y is arbitrary, then

$$\mathcal{D}(X, Y) = \{A: X \rightarrow Y \mid A \text{ is an usco with } PC_Y^\infty \text{ values}^{(1)}\} \subset \mathcal{A}(X, Y).$$

REMARKS. (1) It is clear that the inclusions in (B) are topological in nature and that no linear structure is needed (the convexity of Y is therefore not necessary and can be replaced by a weaker condition). In the finite dimensional case ($X \subset E = \mathbb{R}^m$, $Y \subset F = \mathbb{R}^n$) (ii) was obtained by Mas-Colell. In the metric spaces case, (iii) is due to Granas-Görmiewicz-Kryszewski. It was proven in [3] that both (ii) and (iii) hold if the domain X belongs to the larger class of spaces dominated by finite polyhedra. (2) If Y is an ANR, then $\mathcal{D}(X, Y)$ contains $\mathcal{C}(X, Y)$ (thus (ii) follows from (iii)) as well as the class of usc's with R_δ -values (decreasing sequence of compact contractible spaces) important in the theory of differential equations. (3) Also, if Y is an ANR, then $\mathcal{D}(X, Y)$ contains all usc's having trivial shape values. (4) The class $\mathcal{D}(X, Y)$ also contains the class of usc maps taking non-empty decomposable values in decomposable subsets of $L^1(T, E)$ defined by Bressan-Colombo (see reference in [3]).

2. FIXED POINT AND COINCIDENCE FOR APPROACHABLE MAPS. We present in this section some general fixed point and coincidence theorems for approachable maps. For related results, refer to [2] and [3].

Two maps $A, B: X \rightarrow Y$ are said to have a coincidence if $A(x_0) \cap B(x_0) \neq \emptyset$ for some $x_0 \in X$, or equivalently if $\text{Graph}(A) \cap \text{Graph}(B) \neq \emptyset$. If $A: X \rightarrow X$ is a self map, then $x_0 \in X$ is a fixed point for A if $x_0 \in A(x_0)$; the set of all fixed point of A is denoted by $\text{Fix}(A)$. Given a map $A: X \rightarrow X$ and a family $\mathcal{U}$ of subsets of X , a point $x_0 \in X$ is a $\mathcal{U}$ -fixed point for A if there exists a set $U \in \mathcal{U}$ such that (i) $x_0 \in U$ and (ii) $A(x_0) \cap U \neq \emptyset$. To relate this concept to that of fixed point, we have:

LEMMA (2.1). Let X be a regular topological space and $A: X \rightarrow X$ be a usc map with closed values. Assume that there exists a cofinal family of open covers $(\mathcal{U})$ of $A(X)$ in X such that A has a $\mathcal{U}$ -fixed point for each $\mathcal{U} \in \{\mathcal{U}\}$. Then $\text{Fix}(A) \neq \emptyset$.

Since each open cover of a compact subset of a uniform space is a uniform cover, it follows that every such space admits a cofinal family of open covers. Therefore, for compact maps, it is sufficient to establish the existence of "almost fixed points" in order to ensure the existence of a fixed point. The following fixed point property follows at once from the definition of the class $\mathcal{A}$ and the lemma above:

PROPOSITION (2.2). Let X be a compact subset of a space E , and let $A \in \mathcal{A}(X)$. Then $\text{Fix}(A) \neq \emptyset$ provided X has the fixed point property for single-valued continuous functions.

Recall that a subset X of a space E is almost convex if for any $U \in \mathcal{N}_E(0)$ and for any finite subset $\{z_1, z_2, \dots, z_n\}$ of X there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of X such that $x_i - z_i \in U$ for all i , and $\text{Conv}\{x_1, x_2, \dots, x_n\} \subset X$. If X is almost convex, then it is a dense subset of its convex hull $\text{Conv}(X)$. The next result generalizes a theorem of Himmelberg which itself extends the fixed point theorem of Fan-Glicksberg-Kakutani (see reference in [3]).

THEOREM (2.3). Let E be a locally convex space, and let C be a compact subset of E containing an almost convex dense subset X . If $A: C \rightarrow C$ is usc and $A|_X \in \mathcal{A}(X, C)$, then $\text{Fix}(A) \neq \emptyset$.

THEOREM (2.4). Let X be a convex subset of a locally convex space E , and let $A \in \mathcal{A}(X)$ be a compact map, then $\text{Fix}(A) \neq \emptyset$.

PROOF. Without loss of generality, suppose that the space E is complete, and let K be a compact subset of X containing $A(X)$. By the previous theorem and since the convex set $C = \overline{\text{Conv}(K)}$ is a compact subset of X , it is sufficient to show that the map $A_C: C \rightarrow C$ defined by $A_C(x) = A(x) \cap C$ is in $\mathcal{A}(C)$.

Let $U, V \in \mathcal{N}_E(0)$ be two arbitrary open neighborhoods of the origin in E . By the proposition (1.2)

(i), there exists a finite subset $N = \{x_1, x_2, \dots, x_n\}$ of C and a set $U' \in \mathcal{N}_E(0)$, $U' \subset U$, verifying:

$$\{x_i + \frac{1}{4}V\}_1^n \text{ is an open cover of } K, \text{ and } s|_C \in Q(A|_C; U, \frac{1}{4}V) \text{ provided } s \in Q(A; U', \frac{1}{8}V).$$

Let $s \in Q(A; U', \frac{1}{8}V)$, and let $\pi_V: \bigcup_1^n (x_i + \frac{1}{2}V) \rightarrow C$ be the Schauder projection associated with $\frac{1}{2}V$ (2). Clearly, the function $\pi_V s|_C: C \rightarrow C$ is well-defined and continuous. Moreover, it verifies the inclusions $\pi_V s(x) \in (s(x) + \frac{1}{2}V) \cap C \subset (A((x + U) \cap C) + \frac{3}{4}V) \cap C \subset (A((x + U) \cap C) + V) \cap C$ for all $x \in C$. $\square$

REMARK. As a particular case, the closed unit ball B^n , $n \geq 1$, has the fixed point property for the class $\mathcal{A}$, and one can deduce a very general fixed point theorem for approachable maps defined on some non-necessarily locally convex non-metrizable spaces (see [2] theorem (3.7)).

THEOREM (2.5). Let X be a subset of a space E , Y be a convex subset of a locally convex space F , and let $A, B: X \rightarrow Y$ be two uscos. If $A \in \mathcal{A}(X, Y)$ is compact and $B^{-1}|_{\text{Conv}(N)} \in \mathcal{A}(\text{Conv}(N), X)$ for any finite subset N of Y , then A and B have a coincidence.

PROOF. Given $V \in \mathcal{X}_F^c(0)$, let $N = \{y_i \in A(X), 1 \leq i \leq n\}$ be such that $\{y_i + \frac{1}{3}V\}_1^n$ is an open cover of the compact subset $K = \overline{A(X)}$ of Y . Let $\pi_V: \bigcup_1^n (y_i + \frac{2}{3}V) \rightarrow C_V = \text{Conv}(N)$ be the Schauder projection associated with $\frac{2}{3}V$. We claim that the map $A_V: X \rightarrow C_V$ given by $A_V(x) = (A(x) + \bar{V}) \cap C_V$ is in $\mathcal{A}(X, C_V)$. It is indeed routine to verify that A_V is an usco. In order to see that A_V is approachable, let s be any function in $Q(A; U, V)$ where $V \in \mathcal{X}_F^c(0)$ satisfies $V \subset \frac{1}{3}V$ and let $U \in \mathcal{X}_E^c(0)$ be arbitrary. Clearly, $s(X) \subset \bigcup_1^n (y_i + \frac{2}{3}V)$ and consequently, the function $\pi_V s: X \rightarrow C$ is well-defined and continuous. It verifies the inclusions $\pi_V s(x) \in (s(x) + \frac{2}{3}V) \cap C_V \subset (A((x+U) \cap X) + V) \cap C_V \subset A_V((x+U) \cap X)$ for all $x \in X$ that is $\pi_V s \in Q(A_V; U, W)$ for any $W \in \mathcal{X}_F^c(0)$. The set U being arbitrary, it follows that A_V is approachable.

By hypothesis, $B^{-1}|_{C_V} \in \mathcal{A}(C_V, X)$ and by the proposition (1.2) (iii) the product $A_V B^{-1}|_{C_V} \in \mathcal{A}(C_V)$. The existence of a fixed point for this map is provided by the proposition (2.2) and the Brouwer fixed point theorem. Such a fixed point is an "almost fixed point" for the compact usco AB^{-1} and the usual compactness argument based on the lemma (2.1) ends the proof. $\square$

We formulate now a coincidence theorem for an infinite collections of approachable maps.

PROPOSITION (2.6). Let $(X_i)_{i \in I}$ be a collection of compact spaces each of which being in a space E_i and such that for each finite subset I' of I , the product $X_{I'} = \prod_{i \in I'} X_i$ has the fixed point property for continuous functions. For each $j \in I$, let $X^j = \prod_{i \in I, i \neq j} X_i$ and let $A_j \in \mathcal{A}(X^j, X_j)$. Then $\bigcap_{j \in I} \text{Graph}(A_j) \neq \emptyset$.

PROOF. For any $x \in X = \prod_{i \in I} X_i$, let $x^j = p^j(x)$ be the j th-projection of x onto X^j . It is therefore clear that $\bigcap_{j \in I} \Gamma A_j \neq \emptyset$ if and only if for each $j \in I$, $\bar{x}_j \in A_j(\bar{x}^j)$ for some point $\bar{x} \in X$. By the proposition (1.2) (iii), and since X is compact, the map $\hat{A}_j = A_j p^j \in \mathcal{A}(X, X_j)$. Consequently, and by the

proposition (1.2) (ii), the map $A: X \rightarrow X$ defined by $A(x) = \prod_{i \in I} \hat{A}_i(x)$ is in $\mathcal{A}(X)$. By the proposition

(2.2), and since X has the fixed point property for continuous functions, $\text{Fix}(A) \neq \emptyset$; that is, for each $j \in I$, $\hat{x}_j \in \hat{A}_j(\hat{x}) = A_j(\hat{x}^j)$ for some point $\hat{x} \in X$ and the proof is complete. $\square$

When all the spaces X_i are convex and compact and the maps are of type K (convex case), we recover a theorem of Granas-Liu (see reference in [3]). As an illustration, we formulate now an intersection theorem for non-convex maps. By the proposition (1.3) (iii), the class $\mathcal{D}(X, Y)$ is contained in $\mathcal{A}(X, Y)$ whenever X is a compact ANR, and since any compact AR has the fixed point property for continuous function, it follows:

COROLLARY (2.7). Let $\{X_i\}_{i \in I}$ be a collection of compact AR's. For each $j \in I$, let $A_j \in \mathcal{D}(X^j, X_j)$. Then $\bigcap_{j \in I} \Gamma A_j \neq \emptyset$.

NOTES

(1) A subspace Z of a topological space Y is called $k\text{-PC}_X$ (k -proximally connected in Y) for $k \geq 0$, if for each neighborhood U of Z in Y there exists a neighborhood $V \subset U$ of Z in Y , such that the morphism induced on homotopy groups $i_*: \pi_k(V) \rightarrow \pi_k(U)$ is trivial. Z is called PC_Y^n if it is $k\text{-PC}_Y$ for all $0 \leq k \leq n$. Z is PC_Y^∞ if it is PC_Y^n for all n .

(2) Given a finite subset $N = \{c_1, c_2, \dots, c_n\}$ of a locally convex E , the Schauder projection associated with an open set $U \in \mathcal{N}_E(0)$ is the function $\pi_U: \bigcup_{i=1}^n (c_i + U) \rightarrow E$ given by $\pi_U(x) = \frac{\sum_{i=1}^n \mu_i(x) c_i}{\sum_{i=1}^n \mu_i(x)}$ for all x . Here, $\mu_i(x) = \max\{0, 1 - p_U(x - c_i)\}$, p_U being the Minkowski functional of U . It is readily seen that $\pi_U(x) - x \in U$ and that $\pi_U(x) \in C = \text{conv}(N)$ for all x .

REFERENCES

- [1] H. Ben-El-Mechaiekh and P. Deguire, *Approximation of non-convex set-valued maps*, C. R. Acad. Sci. Paris, t. 312., Série I, (1991), 379-384.
- [2] _____, *General fixed point theorems for non-convex set-valued maps*, C. R. Acad. Sci. Paris, t. 312., Série I, (1991), 433-438.
- [3] _____, *Approximation and fixed points for non-convex set-valued maps*, to appear.

H. Ben-El-Mechaiekh, Department of Mathematics, Brock University, St. Catharines, Ontario, L2S 3A1, CANADA.

Received July 26, 1991

THE TWISTED L^2 -LEFSCHETZ NUMBER OF HECKE OPERATORS

*Salahoddin Shokranian**

Presented by J. Arthur, F.R.S.C.

Abstract

In this paper we give a formula for the L^2 -Lefschetz number of Hecke operators on a twisted group, using the twisted version of Arthur's trace formula. Our main results are theorem 5, and theorem 6.

1. The present paper may be considered as a continuation of our previous paper [S]. We thus freely use the notations and the terminologies of [S]. When a notation is not explained we indicate a reference. Our references here will be the continuation of those in [S].

2. For $f \in C_c^\infty(G(\mathbb{R}))$ define the twisted operator

$$\pi_R(f)\pi_R(\epsilon) = \int_{G(\mathbb{R})} f(x)\pi_R(\epsilon^{-1}(x))dx.$$

Then, make the following assumption.

Assume that there is a function $f_\mu \in C_c^\infty(G(\mathbb{R}))$ (called twisted pseudo-coefficient) such that for a ϵ -stable π_R

$$\text{tr}(\pi_R(f_\mu)\pi_R(\epsilon)) = \begin{cases} (-1)^{r(G)} & \text{if } \pi_R \in \Pi_{\text{st}}(G(\mathbb{R})), \\ 0 & \text{otherwise.} \end{cases}$$

Note that, we are writing $r(G) = \dim(G(\mathbb{R})/K'(\mathbb{R}))$.

This assumption in particular permit us to relate $\chi_\mu^t(\pi_R)$ and the twisted trace $\text{tr}(\pi_R(f_\mu)\pi_R(\epsilon))$ by analytic continuation of tempered characters, i.e.,

$$\chi_\mu^t(\pi_R) = \text{tr}(\pi_R(f_\mu)\pi_R(\epsilon)).$$

Then, by the general argument as in [A], if we set

$$x = x_\infty x_0 \in G(\mathbb{R})G(A_0)$$

*Supported in part by NSERC-Canada, and I.P.M.-Iran.

and

$$(f_\mu \otimes h)(x) = f_\mu(x_\infty)h(x_0),$$

where $h \in \mathcal{H}$, we get:

$$\begin{aligned} \text{tr}(\pi(f_\mu \otimes h)\pi(\epsilon)) &= \left(\int_{G(\mathbf{A})} (f_\mu \otimes h)(x) \pi(\epsilon^{-1}(x)) dx \right) \\ &= \text{tr} \pi_R(f_\mu) \cdot \text{tr} \pi_0(h, \epsilon). \end{aligned}$$

We have thus proved the following lemma.

Lemma 1. With the notations as above we have

$$L(\epsilon, \mu, h) = \sum_{\pi \in \prod(G(\mathbf{A}))} m_{\text{disc}}(\pi) \text{tr}(\pi(f_\mu \otimes h)\pi(\epsilon)).$$

In particular this result enables us to calculate the number $L(\epsilon, \mu, h)$ by means of the twisted trace formula [CLL].

3. As it is known, the trace formula is an identity between two distributions [A1]. This is also the case for the twisted trace formula. To be more precise, let G^* be a non-connected reductive group over $\mathcal{Q}$. Let $\tilde{G}$ be a connected component of G^* such that $\tilde{G}(\mathcal{Q}) \neq \emptyset$. Let us write G^+ for the subgroup of G^* generated by $\tilde{G}$, and G^0 for the connected component of the identity in G^+ , where we assume that G^+ is reductive. The group G^+ is then called the twisted group over $\mathcal{Q}$. Thus ϵ -twisted group [S] is an example of our twisted group here. To see this let ϵ and G be as in ([S], §2). Then $G^\epsilon = G \rtimes \langle \epsilon \rangle$ is a non-connected algebraic group whose number of connected components is equal to ℓ , the order of $\langle \epsilon \rangle$. Denote the elements of G^* by $g \times t$ where $g \in G$, $t \in \langle \epsilon \rangle$. Then consider the connected component $\tilde{G}(t) = G \times t$, ($t \in \langle \epsilon \rangle$). It is clear that $G^+ = G^*$, and $G^0 = G \times e \cong G$, where e is the identity element of $\langle \epsilon \rangle$. In G^* the multiplication is defined by

$$(g_1 \times t_1)(g_2 \times t_2) = (g_1(t_1(g_2))) \times t_1 t_2,$$

where $t_1(g_2)$ is the action of $t_1 \in \langle t \rangle$ on $g_2 \in G$. Here we adopt the notions of [A3]. Let P be a parabolic subgroup of $G \cong G \times e$, P^+ is a parabolic subgroup of G^ϵ , and $\tilde{P} = P^+ \cap \tilde{G}$ is the parabolic subset of $\tilde{G} = G \times \epsilon$. Let $\tilde{M} = M^+ \cap \tilde{P}$, be the Levi component of $\tilde{P}$, where M^+ is the normalizer in G^+ of some Levi component M of P , defined over $\mathcal{Q}$. In fact both $\tilde{P}$ and $\tilde{M}$ are subvarieties of G defined over $\mathcal{Q}$. Now, let S be a finite set of valuations of

Q . Let $Q_S = \prod_{p \in S} Q_p$. With the notations of [A3], define the twisted weighted orbital integral

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_{1,\gamma}(Q_S) \backslash G(Q_S)} f(x^{-1} \gamma \epsilon(x)) v_M(x) dx,$$

where $v_M(x)$ is the twisted weight function defined by means of a twisted $(\bar{G}, \bar{M})$ -family.

4 The twisted trace formula is an identity which consists of a geometric expansion, and a spectral expansion. We recall that these expansions are the following:

$$I'(f) = \sum_{i \geq 0} \sum_{M \in L} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M, i)} a^M(\pi) I_M(\pi, f) d\pi$$

$$I'(f) = \sum_{M \in L} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (\bar{M}(Q))_{M, S}} a^M(S, \gamma) I_M(\gamma, f),$$

for the unexplained notations we refer the reader to [A4], [A5].

To exploit the twisted trace formula and to apply it for the determination of the twisted L^2 -Lefschetz number, we must have an invariant version of these identities. Such identities can be obtained once we have proved a twisted version of the Paley-Wiener theorem for real groups ([A5], App.). We thus assume that the twisted trace formula is in invariant form.

Now, let $\Pi(M^+(Q_S))$ denote the set of equivalence classes of irreducible (admissible) representations of $M^+(Q_S)$ consisting of those π whose restriction π^0 to $M^0(Q_S)$ remains irreducible. Then we write $\Pi_t(M(Q_S))$ for the subset of representations π in $\Pi(M^+(Q_S))$ such that π^0 is tempered. For such a π one defines the representation

$$\pi_\lambda(x) = \pi(x) e^{\lambda(H_G(x))},$$

where $x \in G^0(Q_S)$, and $\lambda \in a_{G, G}^*$ (cf. [A4]). Based on these notations we define the ϵ -twisted Fourier transform

$$f_G^t(\pi, X) = \int_{ia_{G, S}^*} \text{tr}(\pi_\lambda(f) \pi(\epsilon)) e^{-\lambda(X)} d\lambda,$$

where $X \in a_{G, S}$.

Let $\mathcal{H}(G(A))$ be the Hecke space of G (cf. [A4]). Suppose that $f \in \mathcal{H}(G(A))$, then f is a finite combination of the product $\prod_p f_p$, with $f_p \in$

$\mathcal{H}(G(Q_p))$. Say that f is t -cuspidal, if there are t prime numbers $p_i \in S$ ($i = 1, \dots, t$) such that

$$f_{p_i, M} = 0,$$

where $M \in L_0$ (for notations see [A4]).

Theorem 2. If f is 1-cuspidal, then

$$I'(f) = \sum_{t \geq 0} \sum_{\pi \in \Pi_{\text{disc}}(G, t)} a_{\text{disc}}^G(\pi) I_G(\pi, f).$$

Proof. See [A5], Th. 7.1 (a).

Lemma 3. Suppose that there is a prime $p \in Q$, such that $\text{tr}(\pi(f_\mu)\pi(\epsilon)) = 0$, for any representation $\pi \in \Pi_{\text{unit}}(G(Q_p))$ which is a constituent of a (properly) induced representation σ^G , where $\sigma \in \Pi_{\text{unit}}(M(Q_p))$. Then

$$I'(f) = \sum_{t \geq 0} \text{tr}(R_{\text{disc}, t}(f)R(\epsilon)),$$

where $R_{\text{disc}, t}$ is the regular representation of $G(A)$ on $L_{\text{disc}, t}^2(G(Q)A_{G, \infty} \backslash G(A))$.

For the unexplained notations and a proof see [A5], §4.

Lemma 4. Let f_μ be the twisted pseudo-coefficient, and $h \in \mathcal{H}$. Then $f_\mu \otimes h$ is 1-cuspidal.

Proof. This follows from the definition of t -cuspidality and our assumption of §2.

Theorem 5. (Splitting property). With the notations as above, we have

$$I_{\tilde{M}}(\gamma, f_\mu \otimes h) = I_{\tilde{M}}^G(\gamma, f_\mu) \hat{I}_{\tilde{M}}^{\tilde{M}}(\gamma, h_{\tilde{M}}).$$

Proof. The theorem follows by induction on the number of elements in S , and lemma 4.

Theorem 6. The twisted L^2 -Lefschetz number $L(\epsilon, \mu, h)$ is equal to

$$\sum_{\tilde{M} \in L} |W_0^{\tilde{M}}| |W_0^G|^{-1} \sum_{\gamma \in (\tilde{M}(Q))_{M, S}} a^{\tilde{M}}(S, \gamma) I_{\tilde{M}}^G(\gamma, f_\mu) \hat{I}_{\tilde{M}}^{\tilde{M}}(\gamma, h_{\tilde{M}}).$$

Proof. Let $(f_\mu \otimes h)^1$ be the restriction of $(f_\mu \otimes h)$ to the subgroup $G(A)^1$ (cf. [S]). Then, it follows from the work of Müller [Mü] that $f_\mu \otimes h$ defines

an operator of trace class in the discrete spectrum of $L^2(G(Q)\backslash G(A)^1)$. On the other hand, this operator is isomorphic to the operator defined by $f_\mu \otimes h$ on $L(\mu)$, (cf. [A]), whose trace is given by either one of the identities which define the trace formula. Hence, by Lemma 1

$$L(\epsilon, \mu, h) = I'(f_\mu \otimes h).$$

Then, by

$$I'(f_\mu \otimes h) = \sum_{M \in L} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(Q))_{M,S}} a^M(S, \gamma) I_M(\gamma, f)$$

together with the splitting property the theorem follows.

Here, I would like to thank J. Arthur for his help and encouragement. I have also to thank J.-P. Labesse for interesting discussions on the content of an earlier version of this paper. I thank the financial supports of the Institute for Theoretical Physics and Mathematics (I.P.M.), Tehran-Iran, while parts of this work were reviewed.

References

- [A3] J. Arthur, *Local behaviour of weighted orbital integrals*, Duke Math. J. vol. 56, n° 2 (1988) 223-293.
- [A4] _____, *The invariant trace formula I. Local theory*, J. Amer. Math. Soc. vol. 1 n° 2 (1988) 323-383.
- [A5] _____, *The invariant trace formula II. Global theory*, J. Amer. Math. Soc. vol. 2 n° 3 (1988) 501-544.
- [CLL] L. Clozel, J.-P. Labesse, and R.P. Langlands, *Morning Seminar on the Trace Formula*, Notes, IAS-Princeton (1983-84).
- [S] S. Shokranian, *L^2 -Cohomology and Hecke operators*, submitted to the Comptes Rendus of the Royal Society of Canada.

Departamento de Matemática
Universidade de Brasília
70910 Brasília - DF
BRASIL

Received July 19, 1991

THE KERNEL OF $\text{Pic}(R_0) \longrightarrow \text{Pic}(R)$ FOR R A GRADED DOMAIN

David F. Anderson*

Presented by J. Lambek, F.R.S.C.

Abstract. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then $\ker(\text{Pic}(R_0) \longrightarrow \text{Pic}(R)) \cong \Gamma_1/\Gamma_2$ for certain subgroups Γ_1 and Γ_2 of $U(\Gamma) = \Gamma \cap -\Gamma$.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a (commutative) integral domain graded by an arbitrary torsionless grading monoid Γ . That is, Γ is a commutative cancellative monoid, written additively, and the quotient group $\langle \Gamma \rangle$ generated by Γ is a torsion-free abelian group. A commutative cancellative monoid Γ is torsionless if and only if Γ can be given a total order compatible with the monoid operation. We also make the harmless assumption that each R_α is nonzero. A general reference on torsionless grading monoids and Γ -graded rings is [4] (cf. [1], [2]). In general, the natural homomorphism $\varphi : \text{Pic}(R_0) \longrightarrow \text{Pic}(R)$ of Picard groups induced by inclusion is neither injective nor surjective. Here we determine $\ker \varphi$; this generalizes a result of Oda [5] (Corollary 2) for $\Gamma = \mathbb{Z}$. Let $U(\Gamma) = \Gamma \cap -\Gamma$ be the group of invertible elements of Γ . Then $\Gamma_1 = \{ \alpha \in U(\Gamma) \mid R_\alpha R_{-\alpha} = R_0 \}$ is a subgroup of $U(\Gamma)$. Moreover, $R_\alpha R_\beta = R_{\alpha+\beta}$ and

*Supported by a University of Tennessee Faculty Development Grant

$R_\alpha^{-1} (= R_0 : R_\alpha) = R_{-\alpha}$ for all $\alpha, \beta \in \Gamma_1$. Let $\Gamma_2 = \{ \alpha \in \Gamma \mid R_\alpha \cap U(R) \neq \emptyset \}$. Then Γ_2 is a subgroup of Γ_1 and $R_\alpha = u_\alpha R_0$ for any $u_\alpha \in R_\alpha \cap U(R)$.

Theorem. Let $\varphi : \text{Pic}(R_0) \longrightarrow \text{Pic}(R)$ be the natural homomorphism.

Then $\ker \varphi \approx \Gamma_1 / \Gamma_2$.

Proof. For $\alpha \in \Gamma_1$, choose $0 \neq t_\alpha \in R_\alpha$. Then $t_\alpha R_{-\alpha} \subset R_0$ and $(t_\alpha R_{-\alpha})(t_\alpha^{-1} R_\alpha) = R_0$, so $t_\alpha R_{-\alpha} \in \text{Inv}(R_0)$. Let $\psi : \Gamma_1 \longrightarrow \text{Pic}(R_0)$ be defined by $\psi(\alpha) = [t_\alpha R_{-\alpha}]$. Clearly $\psi(\alpha)$ is independent of the choice of t_α and ψ is a homomorphism.

$\text{im } \psi = \ker \varphi$: $R = R_0 R = R_\alpha R_{-\alpha} R \subset R_{-\alpha} R \subset R$ implies $R = R_{-\alpha} R$ for each $\alpha \in \Gamma_1$. Thus $(t_\alpha R_{-\alpha})R = t_\alpha R \in \text{Prin}(R)$. Hence $\text{im } \psi \subset \ker \varphi$. Conversely, let $[I] \in \ker \varphi$ with $I \subset R_0$. Then $IR = bR$ for some $0 \neq b \in R_p$ since IR is a nonzero homogeneous principal ideal. Hence $I = bR_{-p}$ and $I^{-1} = b^{-1}R_p$. Thus $\beta \in \Gamma_1$ and $\psi(\beta) = [bR_{-p}] = [I]$.

$\ker \psi = \Gamma_2$: Let $\alpha \in \Gamma_2$. Then $R_{-\alpha} = u_\alpha^{-1} R_0$ for any $u_\alpha \in R_\alpha \cap U(R)$. Thus $\psi(\alpha) = [u_\alpha R_{-\alpha}] = [R_0] = 0$. Conversely, suppose that $\psi(\alpha) = [t_\alpha R_{-\alpha}] = 0$. Then $t_\alpha R_{-\alpha} = xR_0$ for some $0 \neq x \in R_0$. Hence $R_{-\alpha} = z_\alpha R_0$ with $z_\alpha = xt_\alpha^{-1} \in R_{-\alpha}$. Thus $R_\alpha = R_{-\alpha}^{-1} = z_\alpha^{-1} R_0$ and $z_\alpha^{-1} \in R_\alpha \cap U(R)$, so $\alpha \in \Gamma_2$. $\square$

Corollary 1. The natural homomorphism $\varphi : \text{Pic}(R_0) \longrightarrow \text{Pic}(R)$ is injective if and only if $\Gamma_1 = \Gamma_2$.

Corollary 2. (Oda[5, Theorem 1.3]). Let $\Gamma = \mathbb{Z}$. Then $\Gamma_1 = d\mathbb{Z}$ and $\Gamma_2 = m\mathbb{Z}$ for some integers $d, m \geq 0$. Let $e = m/d$, and $e = 1$ if $d = 0$. Then $\ker \varphi = \mathbb{Z}/e\mathbb{Z}$.

An extension of rings $A \subset B$ is inert if whenever $xy \in A$ for nonzero $x, y \in B$, then $x = ru$ and $y = su^{-1}$ for some $r, s \in A$ and $u \in U(B)$. The extension $R_0 \subset R$ is inert if and only if $U(\Gamma) = \Gamma_2$ [1, p. 551], [2, Prop. 3.3]. Our next corollary, a special case of Corollary 1 above, recovers [1, Prop. 5.1(1)]. (However, φ may be injective without $R_0 \subset R$ being an inert extension [1, p. 562].)

Corollary 3. If $R_0 \subset R$ is an inert extension, then the natural homomorphism $\varphi : \text{Pic}(R_0) \longrightarrow \text{Pic}(R)$ is injective.

Remarks. (a) It may be shown that ψ is a natural isomorphism in the following sense. Let $A = \bigoplus_{\alpha \in S} A_\alpha$ and $B = \bigoplus_{\beta \in T} B_\beta$ be two graded integral domains and $f : A \longrightarrow B$ a ring homomorphism such that for each $\alpha \in S$, $f(A_\alpha) \subset B_\beta$ for some $\beta \in T$. Then f induces a monoid homomorphism $f' : S \longrightarrow T$ with $f'(U(S)) \subset U(T)$, $f'(S_1) \subset T_1$, and $f'(S_2) \subset T_2$. Then f' in turn induces a group homomorphism $\bar{f} : S_1/S_2 \longrightarrow T_1/T_2$ such that $\psi_B \circ \bar{f} = f'' \circ \psi_A$, where f'' is the

natural homomorphism from $\ker \varphi_A$ to $\ker \varphi_B$ induced by f .

(b) (Cf. [2, Section 3]) We may regrade $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ by $\Gamma' = \Gamma/U(\Gamma)$. Then $R = \bigoplus_{\gamma \in \Gamma'} R'_\gamma$ with $U(\Gamma') = 0$, $R'_0 = \bigoplus_{\gamma \in U(\Gamma)} R_\gamma$, and $R'_0 \subset R$ is an inert extension. Since $\text{Pic}(R'_0) = \text{HPic}(R)$ [1, Thm. 5.4], the study of the map $\text{Pic}(R_0) \longrightarrow \text{Pic}(R)$ can often be reduced to the case in which Γ is a group.

We end this note with an example which shows that for any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of abelian groups there is a graded integral domain R such that $B = \text{Pic}(R_0)$, $C = \text{Pic}(R)$, and $A = \ker \varphi$.

Example (cf. [3, p. 225-226]). Let R be a Krull domain with quotient field K . Then $G = \text{Div}(R)$ is a (free) abelian group under the usual v -product. For Γ any submonoid of G , define a subring $R(\Gamma)$ of the group ring $K[X; G]$ by $R(\Gamma) = \{f \in K[X; G] \mid f(l) \in l \text{ for all } l \in \Gamma\}$. Then $R(\Gamma)$ in the obvious manner is a Γ -graded integral domain with $R(\Gamma)_0 = R$, $U(\Gamma) = H = \Gamma \cap -\Gamma = \{l \in \Gamma \mid l^{-1} \in \Gamma\}$, $\Gamma_1 = H \cap \text{Inv}(R)$, and $\Gamma_2 = H \cap \text{Prin}(R)$. Thus $\ker \varphi = \ker(\text{Pic}(R) \longrightarrow \text{Pic}(R(\Gamma))) \cong \Gamma_1/\Gamma_2 = (H \cap \text{Inv}(R))/(H \cap \text{Prin}(R)) \cong ((H \cap \text{Inv}(R)) + \text{Prin}(R))/\text{Prin}(R)$.

Let B be any abelian group. By Claborn's Theorem, there is a Dedekind domain R with $\text{Pic}(R) = \text{Inv}(R)/\text{Prin}(R) \cong B$. For any subgroup A of B , $A = A''/\text{Prin}(R)$ for some subgroup A'' of $\text{Inv}(R)$ containing $\text{Prin}(R)$. For $\Gamma = A''$, the above construction

yields a Γ -graded integral domain $R(\Gamma)$ with $R(\Gamma)_0 = R$, $U(\Gamma) = \Gamma = \Gamma_1 = \Lambda^\times$, and $\Gamma_2 = \text{Prin}(R)$. Thus $\ker \varphi \approx \Gamma_1/\Gamma_2 = \Lambda$ and $\text{Pic}(R(\Gamma)_0) = \text{Pic}(R) \approx B$. As in [3, p. 225-226], it may be shown that $\varphi : \text{Pic}(R) \longrightarrow \text{Pic}(R(\Gamma))$ is surjective.

References

- [1] D. D. Anderson and D. F. Anderson, Divisorial ideals and invertible ideals in a graded integral domain, *J. Algebra* 76(1982), 549 - 569.
- [2] D. D. Anderson and D. F. Anderson, Grading integral domains, *Comm. Algebra* 11(1983), 1 - 19.
- [3] D. F. Anderson, Seminormal graded rings, II, *J. Pure Appl. Algebra* 23(1982), 221 - 226.
- [4] D.G. Northcott, Lessons on Rings, Modules and Multiplicities, (Cambridge University Press, 1968).
- [5] S. Oda, On Picard groups of $\mathbb{Z}$ -graded domains, preprint.

Department of Mathematics

The University of Tennessee

Knoxville, Tennessee 37996-1300 U.S.A.

Received June 22, 1991

ON THE ORDERS OF ELEMENTS OF A LOCALLY FINITE SIMPLE GROUP

OTTO H. KEGEL AND SHI WUJIE

Presented by P. Ribenboim, F.R.S.C.

Theorem: The locally finite simple group G is linear if and only if there is a natural number n that does not appear as order of an element of G . (This result depends on the classification of the finite simple groups.)

The set $\Pi_e(G)$ of orders of the elements of the finite group G contains substantial structural information about the group G ; many of the finite simple groups are indeed completely determined by this set; [5], [1], [4], [6].

Here we want to identify those locally finite simple groups G with $\prod_i(G) \subsetneq N$: These are precisely those admitting a faithful (finite-dimensional) linear representation.

That a locally finite simple linear group G satisfies $\Pi_e(G) \subsetneq N$ follows immediately from

Lemma: If the subgroup $G \subset GL(n, K)$ is locally finite, then either — if the characteristic of the field K is zero — G has an abelian normal subgroup of finite index, or — if the characteristic of K is a prime p — the unipotent elements of G are of bounded order.

The first part of this lemma follows from Jordan's theorem ([2], 36.13). If K is of characteristic $p \neq 0$, then every element of $GL(n, K)$ of order a power of p is unipotent and so obviously of order dividing p^n .

So for the proof of the above theorem we must show that if the locally finite simple group G is not linear, then it satisfies $\Pi_e(G) = N$.

For this, observe that G has a countable simple subgroup S that is not linear ([3], 4.4). By ([3], 4.5) there is an ascending sequence of finite subgroups $\{F_i\}$ of S with $F_i \subset F_{i+1}$, $\cup F_i = S$, F_{i+1} has a maximal normal subgroup M_{i+1} with $M_{i+1} \cap F_i = \{1\}$.

The group S then is a sort of limit of the sequence $\{S_i\}$ of finite simple groups $S_i = F_i/M_i$.

As there are only finitely many (two parameter) families of finite simple groups, one may assume — by the pigeon hole principle — that all groups S_i are in the same family (of Lie type). If the rank parameters n_i of the groups S_i were bounded, then S would be linear ([3], 4.7) contrary to assumption. If the rank parameters n_i of the groups S_i are unbounded, the Weyl groups W_i of S_i are unbounded; they involve alternating groups of unbounded degrees and thus contain — for n_i sufficiently large — elements of given order n . Thus the S_i — and so the F_i — contain elements of order n if i is large. We have proved that $N \subseteq \Pi_n(S) \subseteq \Pi_n(G) \subseteq N$.

Literature:

- [1] R. Brandl and Shi Wujie, The characterisation of $PSL(2, q)$ by its element orders. Submitted.
- [2] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras, Interscience Publishers, J. Wiley and Sons, New York and London, 1962.
- [3] O. H. Kegel and B. A. F. Wehrfritz, Locally finite groups, North Holland, Amsterdam, 1973.
- [4] C. E. Praeger and Shi Wujie, A characterization of some alternating and symmetric groups. Submitted.
- [5] Shi Wujie, A new characterization of the sporadic simple groups, Group Theory, Proc. Singapore Group Theory Conference 1987, 531–540, W. de Gruyter, Berlin, 1989.
- [6] Shi Wujie, The characterization of the sporadic simple groups by their element orders. Submitted.

Shi Wujie, Department of Mathematics, Southwest-China Normal University, Beibei, Chongqing, 630715, People's Republic of China

O. H. Kegel, Mathematics Department, University of Freiburg, Albertstrasse 23 b, D-7800 Freiburg, Germany

Received October 2, 1991

Note on the dual of an ideal in a Noetherian domain

Mitsuo KANEMITSU, Takasi SUGATANI and Ken-ichi YOSHIDA

Presented by K. Murasugi, F.R.S.C.

Introduction. Throughout this note R will be an integral domain with identity and quotient field K . Let I be a non-zero ideal of R . As usual, we denote the dual $R : I = \{x \in K : xI \subseteq R\}$ of I by I^{-1} , and $(I^{-1})^{-1}$ by I_v .

Huckaba and Papick [2] investigated the instances when I^{-1} is a subring of K , and asked if it is always the case that $I^{-1} = I : I$ whenever I^{-1} is a ring. A negative answer to this question was given by Anderson [1]. But in the affirmative direction, Anderson [1, Proposition 3.3] proved that if I is either radical or divisorial, then $I^{-1} = I : I$ whenever I^{-1} is a ring. We will continue in this direction for Noetherian domains to obtain some further results when I^{-1} is a ring. For convenience' sake, we will use the symbol $\mathcal{A}(R)$ to denote the collection of the ideals I of R whose dual I^{-1} is a ring. For undefined terminology see [3] and [5].

First of all, we recall a well-known fact due to Huckaba and Papick.

Proposition 1 ([2, Proposition 2.2]). *Let I be a non-zero ideal of R . Then $I \in \mathcal{A}(R)$ if and only if $I^{-1} = I_v : I_v$.*

Before proceeding to our first proposition, we note that Proposition 1 implies that if I is a non-zero ideal of a Noetherian domain R and $I \in \mathcal{A}(R)$, then I^{-1} is a finite R -module, and so is contained in the integral closure $\bar{R}$ of R .

Proposition 2. *Let R be a Noetherian domain. If $I \in \mathcal{A}(R)$, then $\sqrt{I} \in \mathcal{A}(R)$.*

Proof. Since I^{-1} is contained in the integral closure $\bar{R}$ of R , it follows that $\sqrt{I}^{-1}$ is also contained in $\bar{R}$. Let $p_1, \dots, p_n$ be prime ideals of $\bar{R}$ such that $\sqrt{I} = p_1 \cap \dots \cap p_n$, and let $P_1, \dots, P_n$ be prime ideals of $\bar{R}$ with $P_i \cap R = p_i$ $i = 1, \dots, n$, respectively. Now $\sqrt{I} \cdot \sqrt{I}^{-1} \subseteq P_i \cap R = p_i$, and so $\sqrt{I}^{-1} = \sqrt{I} : \sqrt{I}$, completing the proof.

Huckaba and Papick [2, p.14] showed that if I is an ideal of a Prüfer domain R and $I \in \mathcal{A}(R)$, then $\sqrt{I}^{-1} = I^{-1}$. However, in the Noetherian case, we may have $I^{-1} \neq \sqrt{I}^{-1}$. Let k be a field and $R = k[[X^3, X^5, X^7]]$ be the subring of the formal power series ring $k[[X]]$ over k . If $I = (X^5, X^6, X^7)$, then $I^{-1} = k[[X]]$ but $\sqrt{I}^{-1} = k[[X^2, X^3]]$. (Cf. [1, p.328]).

Now recall that an element x of a Noetherian domain R is called integral over an ideal I of R if x satisfies a monic equation $x^n + a_1 x^{n-1} + \dots + a_n = 0$ with $a_i \in I^i$, $i = 1, \dots, n$. Then the set $\bar{I}$ of the elements integral over I is called the integral closure of I . It is well known that $\bar{I} = t^{-1} \overline{R(I)} \cap R$, where $\overline{R(I)}$ means the integral closure of the Rees ring $R(I) = R[tI, t^{-1}]$ with respect to the ideal I . It should be noted that if I is an ideal of R , then $I \subseteq \bar{I} \subseteq \sqrt{I}$. If $I = \bar{I}$, then I is called integrally closed.

Let k be a field, and consider the polynomial ring $k[X, Y]$ in two indeterminates X, Y over k . Let $I = (X^2, XY + Y^2)$. It is readily seen that $\bar{I}$ is neither a radical nor a divisorial ideal, but satisfies that $\bar{I}^{-1} = \bar{I} : \bar{I} (= R)$. In fact the following holds.

Proposition 3. *Let R be a Noetherian domain and I be an ideal of R . If I is integrally closed and $I \in \mathcal{A}(R)$, then $I^{-1} = I : I$.*

Proof. Since I is integrally closed, we have that $I = t^{-1} \overline{R(I)} \cap R$. Let $a \in I^{-1}$. It then follows that $a \in \bar{R}$ and so $aI \subseteq t^{-1} \overline{R(I)} \cap I^{-1}$. Now $aI \subseteq R$, and hence $aI \subseteq I$. Thus $I^{-1} = I : I$.

Before stating our next theorem, we need some terminology. For any $I \in \mathcal{A}(R)$, we put $\mathcal{A}_I = \{J \in \mathcal{A}(R) : J = xI \text{ for some } x \in K\}$. Further, for a ring S we denote by $U(S)$ the unit group of S .

Theorem 4. *Let R be a Noetherian domain and $I \in \mathcal{A}(R)$. There exists a one-to-one correspondence between $\mathcal{A}_I$ and $U(I^{-1})/U(I : I)$.*

Proof. If $I = (0)$ or $I = R$, then the assertion is trivial. Therefore we may assume that I is a non-zero proper ideal of R . Let $x \in U(I^{-1})$, and

put $J = xI$. It then follows that $J^{-1} = x^{-1}I^{-1}$, and so J^{-1} is just the ring I^{-1} , since $x \in U(I^{-1})$. Hence $J \in \mathcal{A}_I$. Now let $y \in K$ such that $J = yI \in \mathcal{A}(R)$. Then Proposition 1 yields that $J^{-1} = J_v : J_v = I_v : I_v = I^{-1}$. On the other hand, $J^{-1} = y^{-1}I^{-1}$. Thus $y \in U(I^{-1})$. Next, let $z \in U(I^{-1})$ such that $zI = I$. Clearly, $z^{-1}I = I$, and so $z \in U(I : I)$. Finally, let $z \in U(I : I)$. Then $zI \subseteq I$ and $z^{-1}I \subseteq I$, and so $zI = I$. Thus we have the required one-to-one correspondence between $\mathcal{A}_I$ and $U(I^{-1})/U(I : I)$.

Next, we show that if R is a local Noetherian domain containing a field of cardinal ≥ 3 , then for any I of $\mathcal{A}(R)$, $\mathcal{A}_I$ is the singleton $\{I\}$ if and only if $I^{-1} = I : I$.

A lemma is needed.

Lemma 5. *Let T be a semilocal ring and S be a subring of T . Assume that S contains a field of cardinal ≥ 3 . If $U(T) = U(S)$, then $T = S$.*

Proof. Let $m_1, \dots, m_r$ be the maximal ideals of T . Let $f_i : T \rightarrow T/m_i$ be the canonical mapping for each $i = 1, \dots, r$. Let $a \in T$. Now each T/m_i has more than two elements, we can find non-zero $t_i \in T/m_i$ with $t_i \neq f_i(a)$ for any $i = 1, \dots, r$. Then by Chinese remainder theorem we have an element $b \in U(T)$ satisfying that $f_i(b) = t_i$. Since $f_i(a - b) \neq 0$, it follows that $a - b \in U(T) = U(S)$. Now $b \in U(T) = U(S) \subset S$, and hence $a \in S$. This proves that $T = S$.

Corollary 6. *If R is a local Noetherian domain containing a field of cardinal ≥ 3 , then for any I of $\mathcal{A}(R)$, $\mathcal{A}_I$ is the singleton $\{I\}$ if and only if $I^{-1} = I : I$.*

Proof. Since I^{-1} is integral over R , it follows from Mori-Nagata theorem [5, Theorem 33.10] that I^{-1} is a semilocal ring. Then our assertion is an immediate consequence of Theorem 4 and Lemma 5.

We close this note with the following Remarks: Let I be an ideal of a Noetherian domain R . (i) If $\text{grade } I \geq 2$, then $I^{-1} = I : I (= R)$. (See [3, Exercises 1 and 2, p.102]). (ii) If I satisfies that $(I^{-1})_p = (I : I)_p$ for any prime divisor p of I , then $I^{-1} = I : I$.

To see this, assume that $I^{-1} \neq I : I$. Let $p_1, \dots, p_n$ be the prime divisors of I , and let p be a prime divisor of the R -module $I^{-1}/(I : I)$. Then we can find some p_i which contains p . In fact, if this were not the

case, then we would have an element $x \in p$ not contained in any p_i . Now p is a prime divisor of $I^{-1}/(I : I)$. We have an element $t \in I^{-1}$ such that $p = \{a \in R : at \in I : I\}$. But this in turn shows that $xt \in I : I$. Now $tI \subseteq R$ and x is not contained in any p_i . Hence $tI \subseteq I$, a contradiction. Thus p is contained in some p_i . Therefore at this p_i , $(I^{-1})_p \neq (I : I)_p$. This completes Remark (ii).

We are grateful to H. Yanagihara for teaching us the improvement of the proof of Lemma 5.

References

- [1] D. F. Anderson, When the dual of an ideal is a ring, *Houston J. Math.*, 9(1983), 325-332.
- [2] J. A. Huckaba and I. J. Papick, When the dual of an ideal is a ring, *manuscripta math.*, 37(1982), 67-85.
- [3] I. Kaplansky, *Commutative Rings*, Boston, 1970.
- [4] Y. Koyama, T. Sugatani and K. Yoshida, Some remarks on divisorial and seminormal overrings, *Comm. in Algebra*, 13(1985), 795-810.
- [5] M. Nagata, *Local Rings*, Interscience, 1962.

M. Kanemitsu
Department of Mathematics,
Aichi University of Education,
Igaya-cho, Kariya 448, JAPAN

T. Sugatani
Department of Mathematics,
Toyama University,
Gofuku, Toyama 930, JAPAN

K. Yoshida
Department of Applied Mathematics,
Okayama University of Science,
Ridai-cho, Okayama 700, JAPAN

Received September 19, 1991

SECOND ORDER PARALLEL TENSORS ON CONTACT MANIFOLDS II*

Ramesh Sharma

Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. In an earlier paper the author proved (i) a second order symmetric parallel tensor on a K-contact manifold is a constant multiple of the associated metric tensor and (ii) there is no non-zero parallel 2-form on a Sasakian manifold. In this paper we generalize these results on a contact metric manifold whose ξ -sectional curvature $K(\xi, X)$ vanishes nowhere and is independent of the choice of X .

1. INTRODUCTION

In 1926 Levy [5] proved that a second order symmetric parallel non-singular tensor on a real space form is a constant multiple of the metric tensor. The author [6] generalized Levy's result by proving that a second order parallel (not necessarily symmetric and non-singular) tensor on an n -dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. It was also proved in [6] that a second order parallel tensor on a complex space form is a linear combination (with constant coefficients) of the Kaehlerian metric and the fundamental 2-form. Recently the author [7] proved:

THEOREM 1. On a K-contact manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.

THEOREM 2. On a Sasakian manifold there is no non-zero parallel 2-form.

*Research supported in part by a Faculty Fellowship of the University of New Haven.
AMS Subject Classification Codes: 53C, 58A.

In this paper we generalize Theorems 1 and 2 as

THEOREM. Let M be a contact metric manifold whose ξ -sectional curvature $K(\xi, X)$ is nowhere vanishing and is independent of the choice of X . Then a second order parallel tensor on M is a constant multiple of the associated metric tensor.

Examples of contact metric 3-manifolds with constant non-zero ξ -sectional curvature < 1 have been recently provided by D. E. Blair and H. Chen [3].

Note that this Theorem holds in particular when M is K -contact (i.e. $K(\xi, X) = 1$) and also that it generalizes Theorem 2 and in turn the following result for $p = 2$ on a contact metric manifold as hypothesized in the Theorem.

THEOREM (Blair and Goldberg [4]). There are no covariant constant p -forms on a compact Sasakian manifold M^{2n+1} for $1 \leq p \leq 2n$.

As an immediate corollary of our Theorem we have the following result.

THEOREM (Tanno [8]). If the Ricci tensor field is parallel in a K -contact Riemannian manifold, then it is an Einstein space.

2. CONTACT MANIFOLDS

By a contact manifold we mean an odd-dimensional C^∞ -manifold M^{2n+1} together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. For a contact form η we know that there exists a unique vector field ξ (characteristic vector field) on M^{2n+1} such that $\eta(\xi) = 1$ and $(d\eta)(\xi, X) = 0$. A Riemannian metric g is said to be an associated metric if there exists a tensor field φ of type (1,1) such that $(d\eta)(X, Y) = g(X, \varphi Y)$, $\eta(X) = g(X, \xi)$ and $\varphi^2 = -I + \eta \otimes \xi$. The manifold M^{2n+1} with structure (φ, ξ, η, g)

is called a contact metric manifold. Following [1] we define a tensor field h by $h = \frac{1}{2} \mathcal{L}_\xi \varphi$. It is known that h is self-adjoint and anti-commutes with φ . Also $h\xi = 0$. A contact metric manifold for which ξ is Killing, is said to be K-contact. Further, ξ is Killing with respect to g if and only if $h = 0$. A contact metric manifold is K-contact if and only if the sectional curvature $K(\xi, X)$ of a plane section containing ξ is equal to 1. If ∇ and R denote respectively the Riemannian connection of g and its curvature tensor, then for a general contact metric manifold (see [1] and [2])

$$(2.1) \quad \nabla_X \xi = -\varphi X - \varphi hX$$

$$(2.2) \quad R(\xi, X)\xi - \varphi R(\xi, \varphi X)\xi = 2(h^2 + \varphi^2)X$$

3. PROOF OF THE THEOREM

Let α denote a parallel (0,2)-tensor field on a contact metric manifold M with $K(\xi, X)$ as a nonvanishing function independent of X . As $\nabla \alpha = 0$, we have

$$(3.1) \quad \alpha(R(W, X)Y, Z) + \alpha(Y, R(W, X)Z) = 0$$

for arbitrary vector fields X, Y, Z and W on M . Let A denote the (1,1)-tensor defined by $g(AX, Y) = \alpha(X, Y)$. By hypothesis we have that

$$(3.2) \quad R(\xi, X)\xi = k\{\eta(X)\xi - X\}$$

where k is a smooth function on M . Using (3.2) in (2.2) shows

$$(3.3) \quad h^2 = (k-1)\varphi^2.$$

First we consider the case when α is symmetric. Substituting $W = Y = Z = \xi$ in (3.1) and using (3.2) we find $k\{\eta(X)\alpha(\xi, \xi) - \alpha(\xi, X)\} = 0$, so that, since $k \neq 0$ on M ,

$$(3.4) \quad A\xi = \alpha(\xi, \xi)\xi.$$

That $\alpha(\xi, \xi)$ is constant, can be checked as follows: $X\{\alpha(\xi, \xi)\} = 2\alpha(\nabla_X \xi, \xi) = 2\alpha(-\varphi X - \varphi hX, \xi) = -2g(-\varphi X - \varphi hX, A\xi) = 0$, where we used (2.1) and (3.4). Next,

differentiating (3.4) covariantly along X , using (2.1) and replacing X by φX we obtain

$$(3.5) \quad A(X - hX) = \alpha(\xi, \xi)(X - hX).$$

Replacing X by hX in (3.5) and adding the new equation to (3.5) gives $A(X - h^2X) = \alpha(\xi, \xi)(X - h^2X)$. Now, the use of (3.3) in the last equation and the fact $k \neq 0$ on M shows $AX = \alpha(\xi, \xi)X$, i.e. $\alpha = \alpha(\xi, \xi)g$. It remains to consider the case when α is anti-symmetric. We note $\nabla_{\varphi X}(A\xi) = A(\nabla_{\varphi X}\xi) = A(X - hX) - \eta(X)A\xi$, by using (2.1) and the fact that $h\varphi = -\varphi h$. Hence

$$(3.6) \quad A(X - hX) = \eta(X)A\xi + \nabla_{\varphi X}(A\xi).$$

Replacing X by hX in (3.6) and adding the resulting equation to (3.6) we have (since $\eta(hX) = 0$)

$$A(X - h^2X) = \eta(X)A\xi + \nabla_{\varphi X}(A\xi) + \nabla_{\varphi hX}(A\xi).$$

Using (3.3) in the last equation shows

$$(3.7) \quad kAX = k\eta(X)A\xi + \nabla_{\varphi X}(A\xi) + \nabla_{\varphi hX}(A\xi).$$

At this point we would like to prove $|A\xi| = \text{constant on } M$. Turning back to (3.1), substituting $W = Y = \xi$ and using (3.2), we find

$$kg(AX, Z) = k\eta(X)g(A\xi, Z) + g(A\xi, R(\xi, X)Z).$$

The use of (3.7) in the above equation provides

$$g(\nabla_{\varphi X}A\xi, Z) + g(\nabla_{\varphi hX}A\xi, Z) = g(A\xi, R(\xi, X)Z).$$

The substitution $Z = A\xi$ results in $g(\nabla_{\varphi X + \varphi hX}A\xi, A\xi) = 0$. Replacing X by φX and noting the fact $\nabla_{\xi}\xi = 0$ (can be seen from (2.1)) and finally using (3.3) one obtains $g(\nabla_{X-hX}A\xi, A\xi) = 0$. Replacing X by hX , adding the resulting equation to the original equation and then using (3.3) yields $g(\nabla_X A\xi, A\xi) = 0$, whence $X(|A\xi|) = 0$. Thus $|A\xi|$ is constant on M . So, as $g(A\nabla_X \xi, A\xi) = 0$ we have by replacing X by φX that

$$(3.8) \quad g(X - hX, A^2\xi) = \eta(X)g(\xi, A^2\xi).$$

Replacing X by hX in (3.8) and adding the resulting equation to (3.8) and then using (3.3) we obtain $g(\varphi^2X, A^2\xi) = 0$ and hence $\varphi(A^2\xi) = 0$. Therefore

$$(3.9) \quad A^2\xi = -|A\xi|^2\xi$$

noting $|A\xi| = \text{constant}$, as proved earlier. Differentiating (3.9) now covariantly along X and then replacing X by φX we get

$$(3.10) \quad A^2(X - hX) = -|A\xi|^2(X - hX).$$

Finally, replacing X by hX , adding the resulting equation to (3.10) and then using (3.3) we obtain $A^2 = -(|A\xi|^2)I$. Recall $|A\xi| = \text{constant}$ on M . If $|A\xi| \neq 0$ then $\left[\frac{1}{|A\xi|}\right]A$ defines a Kaehlerian structure on M . Hence M is even-dimensional. But M is odd-dimensional and we arrive at a dimensional contradiction. Thus $|A\xi| = 0$ and in turn $A\xi = 0$. Using this last consequence in (3.7) we conclude $A = 0$, i.e. $\alpha = 0$. Now if α is any second order tensor then its symmetric and anti-symmetric parts are given by $\alpha_s(X, Y) = \frac{1}{2}(\alpha(X, Y) + \alpha(Y, X))$ and $\alpha_a(X, Y) = \frac{1}{2}(\alpha(X, Y) - \alpha(Y, X))$. Thus if α is parallel, so are α_s and α_a . Hence using the consequences of the two cases considered we conclude that α is a constant multiple of g . This completes the proof.

REMARK. Thus the question of second order parallel tensors on a general contact metric manifold is still open. However we note the following: Let α be a second order symmetric parallel tensor on a contact metric manifold and A be the $(1,1)$ -tensor such that $\alpha(X, Y) = g(AX, Y)$. As $\nabla A = 0$ we have (3.1). Setting $Y = Z = \xi$, we find $g(R(W, X)\xi, A\xi) = 0$, so that $R(\xi, A\xi)X = 0$ for arbitrary vector field X . Thus in particular $R(\xi, A\xi)\xi = 0$ which shows that the sectional curvature $K(\xi, A\xi) = 0$, if $A\xi$ is neither zero nor collinear with ξ .

ACKNOWLEDGMENT. The author is very much thankful to Professor David E Blair for valuable suggestions.

REFERENCES

- [1] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. 509, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
- [2] D. E. Blair, Two remarks on contact metric structures, Tôhoku Math. J. 29 (1977), 319-324.
- [3] D. E. Blair and H. Chen, A classification of 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$, II, Preprint.
- [4] D. E. Blair and S. I. Goldberg, Topology of almost contact manifolds, J. Diff. Geom. 1 (1967), 347-354.
- [5] H. Levy, Symmetric tensors of the second order whose covariant derivatives vanish, Annals of Maths. 27 (1926), 91-98.
- [6] R. Sharma, Second order parallel tensor in real and complex space forms, Internat. J. Math. & Math. Sci. 12 (1989), 787-790.
- [7] R. Sharma, Second order parallel tensors on contact manifolds, Algebras, Groups and Geometries 7 (1990), 145-152.
- [8] S. Tanno, Locally symmetric K-contact Riemannian manifolds, Proc. Japan Acad. 43 (1967), 581-583.

Department of Mathematics
University of New Haven,
West Haven, CT 06516, USA

Received October 8, 1991

A CHI-SQUARE GOODNESS-OF-FIT TEST FOR NATURAL EXPONENTIAL FAMILIES WITH POWER VARIANCE FUNCTION

M.S. NIKULIN

Presented by P. Ribenboim, F.R.S.C.

Summary. Here we construct, using the technique developed in papers of Nikulin, Voinov and Greenwood (1973, 1987, 1989, 1990, 1991), a chi-square goodness-of-fit test for the class of natural exponential families (NEF's) generated by stable distributions.

1. Introduction. Suppose that we have i.i.d.r.v.'s $X_1, X_2, \dots, X_n$ and the distribution function of X_1 belongs to the family $\{F(x; \theta), \theta \in \Theta \subseteq \mathbb{R}^1\}$, where

$$F(x; \theta) = P\{X_1 \leq x\} = \int_{-\infty}^x \exp(-u\theta + \psi(\theta)) dG(u), \quad x \in \mathbb{R}^1. \quad (1)$$

where $G(x)$ is taken to be a cumulative distribution function on not depending on the *natural parameter* θ . Morris (1982), Barndorff-Nielsen (1978), Bar-Lev & Enis (1986) have studied this class of distributions and called $\{F(x; \theta); \theta \in \Theta\}$ a *natural exponential family* (NEF) generated by G . The *cumulant generating function* $\psi(\theta)$ gives (1) unit probability and one can easily verify that

$$\mu = E_{\theta} X_1 = \frac{d\psi(\theta)}{d\theta}, \quad \sigma^2 = \text{Var}_{\theta} X_1 = E_{\theta} (X_1 - \mu)^2 = - \frac{d^2 \psi(\theta)}{d\theta^2}. \quad (2)$$

for any $\theta \in \Theta_0 = \text{Int} \Theta$. Let $V(\mu) = \text{Var} X_1$ denote the variance of X_1 expressed in terms of μ . The function $V(\mu)$ on its domain $\Omega_{\psi}(\Theta)$ is called the *power variance function* (PVF), if $V(\mu)$ has the form

$$V(\mu) = c\mu^{\beta}, \quad \mu \in \Omega, \quad (3)$$

for some constants $c > 0$ and $\beta \geq 0$, which are called the *scale and power parameters*, respectively. Following Barndorff-Nielsen (1978), Hougaard (1986), Morris (1982) and Bar-Lev & Enis (1986), a class (1) of NEF with $V(\mu)$ in the

form (3) is called the class of NEF-PVF's. For example, if $\beta=0$, we have the family of normal distributions. If $\beta=1$, we obtain the family of Poisson distributions, and for $\beta=2$ we obtain the family of gamma distribution. As was shown by Hougaard (1986), Bar-Lev and Enis (1986), for $\beta \in (0,1)$ no NEF-PVF's exists, but for any $\beta \in (1,2)$ we obtain a NEF-PVF's of compound Poisson distributions generated by a gamma distributions. When $\beta > 2$, we obtain the subclass of NEF-PVF distributions *generated by stable distribution on R_+^1* , which are unimodal and absolutely continuous with respect to Lebesgue measure on R_+^1 . We consider here namely this subclass of all NEF's on the positive numbers. As it is well known a stable distribution on R_+^1 has a stable index $\alpha \in (0,1)$ and as it was shown by Pollard (1946) its probability density $g_\alpha(x)$ is given by formula:

$$g_\alpha(x) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sin(\pi \alpha k) \frac{\Gamma(\alpha k + 1)}{x^{\alpha k}}, \quad x > 0. \quad (4)$$

We suppose that the generated distribution $G(x)$ in (1) is the stable one on R_+^1 with the density $g(x) = \frac{1}{\sigma} g_\alpha\left(\frac{x}{\sigma}\right)$, σ is known, $\sigma > 0$. In this case we obtain that the distribution $F(x; \theta)$, $\theta \in \Theta = [0, \infty)$, is unimodal and absolutely continuous with respect to Lebesgue measure on R_+^1 and the density $f(x; \theta) = F'(x; \theta)$ is

$$f(x, \theta) = \begin{cases} g(x) \exp\{-x\theta + \psi(\theta)\} & , x > 0, \\ 0, & x \leq 0 \end{cases} \quad (5)$$

where $\psi(\theta) = (\sigma\theta)^\alpha$, $\theta \in \Theta = [0, \infty)$. It is easy to verify that

$$V(\mu) = c\mu^\beta, \quad (6)$$

where $\mu = E_\theta X_1 = \alpha \sigma^\alpha \theta^{\alpha-1}$, $c = (1-\alpha)(\alpha \sigma^\alpha)^{1/(\alpha-1)}$, $\beta = \frac{2-\alpha}{1-\alpha} > 2$, $0 < \alpha < 1$.

The Laplace transform $\ell(s)$ of the random variable X_1 is

$$\ell(s) = E_\theta[\exp(-sX_1)] = \exp\left\{-\sigma^\alpha[(\theta + s)^\alpha - \theta^\alpha]\right\}. \quad (7)$$

2. Theorem of Washio-Morimoto-Ikeda. This theorem gives us the UMVUE's which we need to construct a chi-square test, as was done in the papers of

Nikulin, Voinov and Greenwood (1973,1987,1989,1990,1991). At first we remark that in view of the factorization theorem of Neyman, the family (1.5) possesses the complete sufficient statistic $S_n = X_1 + X_2 + \dots + X_n$ for the parameter θ . As was shown by Bar-Lev & Enis (1986) and Hougaard (1986) the density $f_n(x;\theta)$ of S_n is

$$f_n(x;\theta) = \begin{cases} n^{-1/\alpha} g(n^{-1/\alpha} x) \exp\{-x\theta + \psi(n^{1/\alpha}\theta)\}, & x > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

A theorem of Washio, Morimoto and Ikeda (1956) gives us a general expression for the UMVUE of a function $u(\theta)$ (see also Morris (1982), Zaks (1971), Voinov & Nikulin(1989)). Note that the characteristic function $\varphi_n(t,\theta)$ of S_n is

$$\varphi_n(t,\theta) = \exp\{n\psi(\theta) - n\psi(\theta - it)\}. \quad (2)$$

Theorem (Washio,Morimoto,Ikeda). Assume that the following conditions are satisfied:

- 1) $u(z)$, $z=\theta+it$, is analytic in a strip whose real part is $\Theta_0 = \text{int}\Theta$, $t \in \mathbb{R}$.
- 2) the characteristic function $\varphi_n(t,\theta)$ of the complete sufficient statistic S_n is such that

$$\lim_{|t| \rightarrow \infty} \varphi_n(-t,\theta)u(\theta+it) = 0$$

uniformly in any subinterval of Θ_0 and

$$\int_{-\infty}^{+\infty} |\varphi_n(-t,\theta)u(\theta+it)| dt < \infty, \quad \theta \in \Theta_0.$$

Then the essentially unique minimum quadratic risk unbiased estimator (UMVUE)

$\hat{u} = T(S_n)$ of $u(\theta)$ is

$$T(S_n) = \begin{cases} \frac{n^{\alpha} W(S_n)}{g(n^{-1/\alpha} S_n)}, & \text{if } g(n^{-1/\alpha} S_n) > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where

$$W(x) = E^{-1} \left\{ u(\theta) e^{-n\psi(\theta)} \right\} = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} u(z) \exp\{zx - n\psi(z)\} dz. \quad (4)$$

Using this theorem, one can find (see, also, Zacks (1971), Voinov & Nikulin (1989)) the UMVUE's $\hat{\varphi}^2$ for the variance $\text{Var}X_1$ and $\hat{f}(x)$ for the density $f(x; \theta)$. They are

$$\hat{\varphi}^2 = \frac{\beta\alpha}{g(S_n n^{-1/\alpha}) \Gamma(1-\alpha)} \int_0^{S_n} g(y n^{-1/\alpha}) (S_n - y)^{1-\alpha} dy. \quad (5)$$

and

$$\hat{f}(x) = \begin{cases} \left(\frac{n}{n-1} \right)^{1/\alpha} g(x) g[(S_n - x)(n-1)^{-1/\alpha}] \frac{1}{g(S_n n^{-1/\alpha})}, & 0 < x < S_n, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

3. A chi-square test for H_0 . Following the papers of Nikulin, Voinov and Greenwood (1973, 1987, 1989, 1990, 1991), we construct a chi-square test for testing H_0 , according to which the distribution of X_1 is (1.5). Taking into account the observed value s of the sufficient statistic S_n , we choose the vector $p = (p_1, \dots, p_r)^T$ such that $p_1 + \dots + p_r = 1$, $p_i > 0$, $r \geq 2$, and construct non-empty classes

$$J_1 = (0, x_1], J_2 = (x_1, x_2], \dots, J_r = (x_{r-1}, x_r], \quad x_r = s,$$

for grouping the data $X_1, \dots, X_n$, where $x_1, \dots, x_{r-1}$ are quantiles of the estimated distribution defined by

$$\int_0^{x_i} \hat{f}(x) dx = p_1 + \dots + p_i, \quad i=1, \dots, r-1.$$

Further, define the vectors $v = (v_1, \dots, v_r)^T$, $q = (q_1, \dots, q_r)^T$, where

$$q_i = \int_{J_i} x \hat{f}(x) dx, \quad i=1, \dots, r,$$

and v is a vector of frequencies arising as the result of grouping the data $X_1, \dots, X_n$ into the classes $J_1, \dots, J_r$. From the above-mentioned papers by

Nikulin, Voinov and Greenwood, we have the next

Theorem . If $n \Rightarrow \infty$ so that $S_n = O(n)$ then the conditional distribution of the vector $\frac{1}{\sqrt{n}}(v - np)$, given S_n , is $AN(0, B)$, where

$$B = P - pp^T - \frac{1}{\hat{\varphi}^2} \left[\frac{1}{n} S_n p - q \right] \left[\frac{1}{n} S_n p - q \right]^T.$$

P is a diagonal matrix with elements $p_1, \dots, p_r$ on the main diagonal, $\text{rank} B = r-1$ and $\hat{\varphi}^2$, given by (2.5), is the UMVUE of $\text{Var}X_1$.

We are now in a position to construct a chi-square type statistic for testing H_0 . As was done in other papers of Nikulin & Voinov (for example, (1973, 1989, 1990, 1991)), we construct the quadratic form

$$Y_n^2 = \frac{1}{n} (v - np)^T B^- (v - np),$$

where B^- is a generalized inverse matrix of B . By direct calculation we find

$$Y_n^2 = \sum_{i=1}^r \frac{(v_i - np_i)^2}{np_i} + \frac{1}{n\kappa} \left(\sum_{i=1}^r \frac{q_i}{p_i} v_i - S_n \right)^2, \quad (2)$$

where

$$\kappa = \hat{\varphi}^2 + \left(\frac{1}{n} S_n \right)^2 - \sum_{i=1}^r \frac{q_i^2}{p_i}. \quad (3)$$

From the above theorem we can deduce that under H_0 if $n \Rightarrow \infty$ the limiting distribution of Y_n^2 will be chi-square with $r-1$ degrees of freedom.

Bibliography

- Bar-Lev S.K., Enis P. (1986). "Reproducibility and natural exponential families with power variance functions," *Ann. Statist.*, 14, 1507-1522.
- Barndorff-Nielsen O. (1978). *Information and Exponential Families*. J.Wiley.
- Hougaard, P. (1986). "Survival models for heterogeneous populations derived from stable distributions," *Biometrika*, 73, 387-396.

- Morris C.N. (1982). "Natural exponential families with quadratic variance functions," *Ann. Statist.*, **10**, 65-80.
- Nikulin M.S. (1973). "On a chi-square test for continuous distributions". *Theory of Probability and its Applications*, **18**, #3, 638-639.
- Nikulin M.S., Voinov V.G. (1989). " χ^2 -tests for exponential distributions". In: *Springer-Verlag Lecture Notes in Mathematics*, v.1412, 239-258.
- Nikulin M.S. (1990). L'exhaustivité, l'estimation sans biais et le test d'ajustement du chi-deux. *Seminaire 89-90, Statistique des processus en milieu medical. Paris-V*, 176-198.
- Nikulin M.S., Greenwood P.E. (1990). A Guide to Chi-squared Testing. *Technical Report #94, The Department of Statistics, The University of British Columbia, Vancouver, Canada*, 199 pp.
- Nikulin M.S. (1991). Some recent results on chi-squared tests. *Queen's Papers in Pure and Applied Mathematics*, # 86, Queen's University, Kingston, Ontario, Canada, 74 pp.
- Pollard H. (1946). "The representation of $\exp(-x^\lambda)$ as a Laplace transform". *Bull. Amer. Math. Soc.*, **52**, 10, p.908-910.
- Voinov V.G., Nikulin M.S. (1989). Unbiased estimators and their applications. *Moscow, Nauka*, 440 pp., (in russian).
- Washio Y., Morimoto H. and Ikeda N. (1956). "Unbiased estimation based on sufficient statistics," *Bull. Math. Statist.*, **6**, p.69-94.
- Zacks, S. (1971). *The Theory of Statistical Inference*. N.Y., J.Wiley.
- Adresse de l'auteur: *The George L. Edgett Statistical Laboratory at Queen's University*, Kingston, Ontario, K7L 3N6, Canada.
- The Laboratory of Statistical Methods, S'.Petersburg Branch of The Steklov Mathematical Institute*, Fontanka 27, S'.Petersburg, Russia.

On the number of solutions of polynomial congruences

by C.L. Stewart *, F.R.S.C.

Let f be a polynomial with integer coefficients, degree $r(\geq 2)$ and non-zero discriminant D . Let p be a prime number and let p^l denote the largest power of p which divides D . Assume that p does not divide the content of f . For each positive integer k we denote by $N(k)$ the number of solutions of the congruence

$$(1) \quad f(x) \equiv 0 \pmod{p^k}$$

in congruence classes modulo p^k .

In 1921 Nagell [2] and Ore [3] proved that for all positive integers k ,

$$N(k) \leq rp^{2l}.$$

This was improved by Sándor [4] in 1952 to

$$(2) \quad N(k) \leq rp^{l/2}$$

for $k > l$. In 1981 Huxley [1] obtained (2) for all positive integers k . For any real number x let $[x]$ denote the greatest integer less than or equal to x . We have recently shown [5] that

$$(3) \quad N(k) \leq 2p^{[l/2]} + r - 2$$

* Research supported in part by a Killam Research Fellowship and by Grant A3528 from the Natural Sciences and Engineering Research Council of Canada.

for all positive integers k . Estimate (3) is, in general, best possible as the following example shows. Let r be an integer with $r \geq 2$, let p be a prime number with $p > r$ and let m be a positive integer. Put

$$f(x) = (x + p^m)(x + 2p^m)(x + 3) \cdots (x + r).$$

Then l , the p -adic order of the discriminant of f , is $2m$. Let k be an integer with $k > l$. The complete solution of the congruence (1) is given by $x \equiv -p^m \pmod{p^{k-m}}$ or $x \equiv -2p^m \pmod{p^{k-m}}$ or $x \equiv -i \pmod{p^k}$ for $i = 3, \dots, r$ hence, for this example,

$$N(k) = 2p^m + r - 2 = 2p^{l/2} + r - 2,$$

whenever $k > l$.

Let r, k and l be integers with $r \geq 2, k \geq 1$ and $l \geq 0$. Define $T = T(r, k, l)$ by

$$T = \begin{cases} \left\lfloor \frac{l}{2} \right\rfloor & \text{if } k \geq l, \\ \left\lfloor \frac{l}{(j+1)(j+2)} \right\rfloor + \left\lfloor \frac{j}{j+2} \right\rfloor k & \text{if } \frac{l}{j} \geq k \geq \frac{l}{j+1} \text{ for } j = 1, \dots, r-2, \\ \left\lfloor \left(\frac{r-1}{r} \right) k \right\rfloor & \text{if } \frac{l}{r-1} \geq k \geq 1, \end{cases}$$

and note that

$$T = \min \left(\left\lfloor \left(\frac{r-1}{r} \right) k \right\rfloor, \min_{j=0, \dots, r-2} \left(\left\lfloor \frac{l}{(j+1)(j+2)} \right\rfloor + \left\lfloor \frac{j}{j+2} \right\rfloor k \right) \right).$$

Theorem Let f be a polynomial with integer coefficients, degree $r (\geq 2)$ and non-zero discriminant D . Let p be a prime number and let l be the p -adic order of D . Assume that p does not divide the content of f . Let k be a positive integer. There is an integer t with $0 \leq t \leq r$ and there are non-negative integers $b_1, \dots, b_t$ and $u_1, \dots, u_t$ such that the complete solution of the congruence (1) is given by the t congruences

$$x \equiv b_i \pmod{p^{k-u_i}},$$

for $i = 1, \dots, t$. Further

$$(4) \quad 0 \leq u_i \leq T,$$

for $i = 1, \dots, t$ and

$$(5) \quad u_1 + \dots + u_t \leq \min \left(2 \left\lfloor \frac{l}{2} \right\rfloor, r \left[\left(\frac{r-1}{r} \right) k \right] \right).$$

The above result is a special case of Theorem 2 of [5]. Since

$$N(k) = p^{u_1} + p^{u_2} + \dots + p^{u_t},$$

we may use (4) and (5) to deduce (3) and indeed to sharpen (3) when $k \leq l$. For details we refer to Corollary 2 of [5].

References

- [1] Huxley, M.N., A note on polynomial congruences. In : Recent progress in analytic number theory, Volume 1, Halberstam, H., Hooley, C. eds., pp. 193-196. London: Academic Press, 1981.
- [2] Nagell, T., Généralisation d'un théorème de Tchebicheff. *Journ. de Math.* 8 (1921) 343-356.
- [3] Ore, O., Anzahl der Wurzeln höherer Kongruenzen. *Norsk Matematisk Tidsskrift*, 3 Aagang, Kristiana (1921) 343-356.
- [4] Sándor, G., Über die Anzahl der Lösungen einer Kongruenz. *Acta Math.*, 87 (1952) 13-17.
- [5] Stewart, C.L., On the number of solutions of polynomial congruences and Thue equations, *J. Amer. Math. Soc.*, to appear.

C.L. Stewart

Department of Pure Mathematics
The University of Waterloo
Waterloo, Ontario
Canada
N2L 3G1

Received October 4, 1991

RESULTS ON t -WRIGHT CONVEXITY

Gy. Maksa, K. Nikodem and Zs. Páles

Presented by J. Aczél, F.R.S.C.

Abstract. In [5] J. Matkowski posed the following problem: Is every t -Wright-convex function, with a fixed t from $]0, 1[$, Jensen-convex? The aim of the present note is to show that the answer is positive for all rational $t \in]0, 1[$ and for certain algebraic numbers, however it is not true for all those values of t that are either transcendental or there is an algebraic (maybe complex) conjugate t^* of t such that $|t^* - 1/2| \geq 1/2$.

Let X be a real vector space, D be a convex non-empty subset of X and $t \in [0, 1]$ be a fixed number. A function $f : D \rightarrow \mathbb{R}$ is said to be t -convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad x, y \in D;$$

t -Wright-convex if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y), \quad x, y \in D;$$

Jensen-convex if

$$f\left(\frac{x+y}{2}\right) \leq f(x) + f(y), \quad x, y \in D.$$

A function f is called *convex* (*Wright-convex*) if it is t -convex (t -Wright-convex) for all $t \in [0, 1]$. It is clear that every convex function is Wright-convex and every Wright-convex function is Jensen-convex. We have also the following characterization of Wright-convex functions due to C. T. Ng [6]: A function f is Wright-convex if and only if it is of the form $f = g + a$, where g is convex and a is additive (cf. also [2], [7]). In [5] J. Matkowski asked if for every $t \in]0, 1[$ t -Wright-convex functions are Jensen-convex (cf. also [8, Problem 15]). Of course it is true for $t = 1/2$. It is also known that every t -convex function is Jensen-convex. This is a consequence of the result of N. Kuhn [4] (cf. also [1, Lemma 1] for a direct proof). The present note consists of two parts:

In the first section we give some properties of the set of all $t \in]0, 1[$ for which a given function is t -Wright-convex. As a consequence, we obtain a positive answer to the problem of Matkowski for all rational $t \in]0, 1[$ and for certain algebraic values of t . (We note that

algebraic means algebraic over $\mathbb{Q}$ everywhere in this paper.) Some partial results on this topic were remarked earlier at the Debrecen-Graz Meeting in April, 1991, and later at the 29th International Symposium on Functional Equations, June 3-10, 1991, Wolfville, Canada (cf. [8, Remark 24]).

In the second part we show that if t is either transcendental or the distance of some of the algebraic conjugates of t from $1/2$ is at least $1/2$, then there is always a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is t -Wright convex but Jensen-concave and not Jensen-convex.

Section 1. Given a function $f: D \rightarrow \mathbb{R}$ we consider the set

$$W_f := \{t \in]0, 1[: f \text{ is } t\text{-Wright-convex}\}$$

Clearly, the set W_f is equal to the whole interval $]0, 1[$ if f is a Wright-convex function; it can be also empty (for instance if f is strictly concave). We have the following

Theorem 1. *Let D be a convex subset of a real vector space. For every function $f: D \rightarrow \mathbb{R}$ the following implications hold:*

- (1) *If $t \in W_f$, then $1 - t \in W_f$.*
- (2) *If $s, t \in W_f$, then $st + (1 - s)(1 - t) \in W_f$.*
- (3) *If $t \in W_f$, then $nt/(k + (n - k)t) \in W_f$, for all $n, k \in \mathbb{N}$.*

Proof. Implication (1) is trivial. To prove (2), assume that $s, t \in W_f$ and fix $x, y \in D$. Using s -Wright-convexity and then t -Wright-convexity we get

$$\begin{aligned} f(x) + f(y) &\geq f(sx + (1 - s)y) + f((1 - s)x + sy) \\ &\geq f(t(sx + (1 - s)y) + (1 - t)((1 - s)x + sy)) + f((1 - t)(sx + (1 - s)y) + t((1 - s)x + sy)) \\ &= f((st + (1 - s)(1 - t))x + (s + t - 2st)y) + f((s + t - 2st)x + (st + (1 - s)(1 - t))y). \end{aligned}$$

This shows that $st + (1 - s)(1 - t) \in W_f$.

To prove (3), assume that $t \in W_f$, take $n, k \in \mathbb{N}$, and fix $x, y \in D$. Define

$$x_{i,j} := x + \frac{j + (i - j)t}{k + (n - k)t}(y - x)$$

for all $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, k\}$. Observe that

$$tx_{i,j} + (1 - t)x_{i+1,j+1} = x_{i,j+1} \quad \text{and} \quad (1 - t)x_{i,j} + tx_{i+1,j+1} = x_{i+1,j}$$

for $i \in \{0, \dots, n-1\}$ and $j \in \{0, \dots, k-1\}$. Hence, in view of t -Wright-convexity, we have

$$f(x_{i,j+1}) + f(x_{i+1,j}) \leq f(x_{i,j}) + f(x_{i+1,j+1})$$

for $i \in \{0, \dots, n-1\}$ and $j \in \{0, \dots, k-1\}$.

Adding up these inequalities for $i = 0, \dots, n-1$, we get

$$f(x_{0,j+1}) + f(x_{n,j}) \leq f(x_{0,j}) + f(x_{n,j+1}).$$

Adding up again with respect to $j = 0, \dots, k-1$, we obtain

$$f(x_{0,k}) + f(x_{n,0}) \leq f(x_{0,0}) + f(x_{n,k}).$$

This means that

$$f\left(\frac{nt}{k+(n-k)t}x + \frac{k-kt}{k+(n-k)t}y\right) + f\left(\frac{k-kt}{k+(n-k)t}x + \frac{nt}{k+(n-k)t}y\right) \leq f(x) + f(y).$$

Thus $nt/(k+(n-k)t) \in W_f$, which was to be proved.

As an immediate consequence of the above theorem we obtain the following

Corollary. *If a function $f : D \rightarrow \mathbb{R}$ is t -Wright-convex with rational $t \in]0, 1[$, then it is Jensen-convex.*

Proof. Assume that f is t -Wright-convex with a rational $t \in]0, 1[$. Since the number $t/(1-t)$ is rational and positive, there exist $n, k \in \mathbb{N}$ such that

$$nt = k(1-t). \quad (*)$$

By implication (3) of Theorem 1 we have $nt/(k+(n-k)t) \in W_f$. But, in view of (*), $nt/(k+(n-k)t) = nt/(nt+nt) = 1/2$. Thus $1/2 \in W_f$, which means that f is Jensen-convex.

Remarks. 1. The third implication in Theorem 1 can be formulated equivalently as follows: If $t \in W_f$, then $t/(r+(1-r)t) \in W_f$ for all positive rational numbers r . This shows that if the set W_f is non-empty, then it is dense in $]0, 1[$.

2. There exist irrational numbers $t \in]0, 1[$ for which t -Wright-convexity implies Jensen-convexity. For instance, if $t = (2 + \sqrt{2})/4 \in W_f$ then, by implication (2) of the Theorem 1, $t^2 + (1-t)^2 = 3/4 \in W_f$. Thus, in view of the Corollary, $1/2 \in W_f$. This holds true also for higher order algebraic numbers, but the general form of those numbers is not yet known.

Section 2. If $t \in \mathbb{R}$ is an algebraic number then the *algebraic conjugates* of t are the roots of the minimal polynomial of t (see [3]). For an arbitrary value $t \in \mathbb{R}$, denote by $L(t)$ the smallest subfield of $\mathbb{R}$ that contains t . We shall need the following

Lemma. [3, page 106] *Let $t \in \mathbb{R}$ be arbitrary and let $0 \neq t^* \in \mathbb{C}$ be either one of the algebraic conjugates of t (if t is algebraic) or a transcendental number (if t is transcendental). Then there exists a homomorphism $\varphi : L(t) \rightarrow \mathbb{C}$ with $\varphi(t) = t^*$.*

Using this result we arrive at

Theorem 2. *If either $t \in]0, 1[$ is transcendental or it is algebraic and one of its algebraic conjugates is outside the open disc $\{x \in \mathbb{C} : |x - 1/2| < 1/2\}$ then there exists a t -Wright-convex, Jensen-concave quadratic function $f : \mathbb{R} \rightarrow]-\infty, 0]$ which is not Jensen-convex.*

Proof. Applying the Lemma, we can find a homomorphism $\varphi : L(t) \rightarrow \mathbb{C}$ such that $\varphi(t) \neq 0$ and $|\varphi(t) - 1/2| \geq 1/2$, that is

$$\operatorname{Re} \varphi(t) \leq |\varphi(t)|^2. \quad (**)$$

Let $\mathcal{H}$ be a Hamel basis in $\mathbb{R}$ over $L(t)$ and let

$$a(x) = \sum_{i=1}^n \varphi(l_i), \quad \text{if} \quad x = \sum_{i=1}^n l_i h_i, \quad (l_i \in L(t), h_i \in \mathcal{H}, \quad i = 1, \dots, n).$$

Then the function $a : \mathbb{R} \rightarrow \mathbb{C}$ turns out to be a noncontinuous additive function satisfying

$$a(lx) = \varphi(l)a(x), \quad l \in L(t), x \in \mathbb{R}.$$

Define

$$f(x) = -|a(x)|^2, \quad x \in \mathbb{R}.$$

Then, an easy calculation yields

$$f(x) + f(y) - f(lx + (1-l)y) - f((1-l)x + ly) = 2(|\varphi(l)|^2 - \operatorname{Re} \varphi(l)) |a(x-y)|^2$$

for $x, y \in \mathbb{R}$, $l \in]0, 1[\cap L(t)$. Putting $l = t$, $(**)$ implies that f is t -Wright-convex. On the other hand $\varphi(1/2) = 1/2$, therefore putting $l = 1/2$ we get that f is Jensen-concave. Since a is not identically zero, hence f cannot be Jensen-convex.

Example. The algebraic conjugate of $t = (1 + \sqrt{2})/3$ is $(1 - \sqrt{2})/3 < 0$, therefore the t -Wright-convexity does not imply the Jensen convexity for this value of t .

References

1. Z. Daróczy, Zs. Páles, *Convexity with given infinite weight sequences*, *Stochastica*, 11(1987), 5-12.
2. Z. Kominek, *On additive and convex functionals*, *Radovi Mat.*, 3(1987), 267-279.
3. M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.
4. N. Kuhn, *A note on t -convex functions*, *General Inequalities 4* (Proc. of the 4th International Conference on General Inequalities, Oberwolfach, 1983), 269-276.
5. J. Matkowski, *On α -Wright convexity and the converse of the Minkowski's inequality*, *Aequationes Math.*, (to appear).
6. C. T. Ng, *Functions generating Schur-convex sums*, *General Inequalities 5* (Proc. of the 5th International Conference on General Inequalities, Oberwolfach, 1986), 433-438.
7. K. Nikodem, *On some class of midconvex functions*, *Ann. Polon. Math.*, 50(1989), 145-151.
8. *Report of Meeting* (The 29th International Symposium on Functional Equations, June 3-10, 1991, Wolfville, Canada), *Aequationes Math.*, (to appear).

Kazimierz Nikodem
Department of Mathematics
Technical University
PL-43-309 Bielsko-Biala
Poland

Gyula Maksa and Zsolt Páles
Institute of Mathematics
Kossuth Lajos University
H-4010 Debrecen, Pf.12
Hungary

Received November 7, 1991

POISSON AND FURSTENBERG BOUNDARIES OF RANDOM WALKS

Wojciech JAWORSKI

Presented by G.A. Elliott, F.R.S.C.

Abstract: We study properties of the group action on the Poisson and Furstenberg boundaries of random walks.

1. Definitions and notations

Let G be a locally compact second countable group and $(X, \mathcal{A})$ a Borel G -space with a quasiinvariant probability measure α_0 ($\mathcal{A}$ denotes the Borel sets). A transition probability P on X is said to be G -invariant if $P(gx, gA) = P(x, A)$ for all $x \in X$, $g \in G$, and $A \in \mathcal{A}$. By a random walk on X we mean a Markov chain T with state space X and G -invariant transition probabilities $P_1, P_2, \dots$. We set $(X^\infty, \mathcal{A}^\infty) = \left(\prod_{n=0}^\infty X, \prod_{n=0}^\infty \mathcal{A} \right)$ and consider this product as a G -space with the G -action $g(x_n)_{n=0}^\infty = (gx_n)_{n=0}^\infty$. The Markov measure α with starting measure α_0 is quasiinvariant on $(X^\infty, \mathcal{A}^\infty)$. To simplify our exposition we shall additionally assume that the measures $\alpha_n = \alpha_0 P_1 \dots P_n$ ($\alpha_n(A) = \int \alpha_{n-1}(dx) P_n(x, A)$), $n = 1, 2, \dots$, are equivalent to α_0 (this is automatically true for random walks on G and its homogeneous spaces G/H). We shall write $L^\infty(X)$ for the G -space $L^\infty(X, \mathcal{A}, \alpha_0) = L^\infty(X, \mathcal{A}, \alpha_1) = \dots$.

The asymptotic σ -algebra is defined by $\mathcal{A}^{(a)} = \bigcap_{n=0}^\infty \sigma(X_n, X_{n+1}, \dots)$, where $X_n: X^\infty \rightarrow X$ stand for the canonical projections. The invariant σ -algebra reads $\mathcal{A}^{(i)} = \{ A \in \mathcal{A}^\infty \mid \vartheta^{-1}(A) = A \}$ where $\vartheta: X^\infty \rightarrow X^\infty$ is the shift $\vartheta((x_n)_{n=0}^\infty) = (x_{n+1})_{n=0}^\infty$. Clearly, $\mathcal{A}^{(i)} \subseteq \mathcal{A}^{(a)}$ and both $\mathcal{A}^{(i)}$ and $\mathcal{A}^{(a)}$ are invariant under the G -action on X^∞ .

DEFINITION 1.1. The Poisson boundary $L_a^\infty(T)$ of the random walk T is defined to be the G -space $L^\infty(X^\infty, \mathcal{A}^{(a)}, \alpha)$. The Furstenberg boundary $L_1^\infty(T)$ is defined to be the G -space $L^\infty(X^\infty, \mathcal{A}^{(1)}, \alpha)$.

A random walk is stationary when $P_1 = P_2 = \dots = P$. In this case and under the assumption that $\alpha_0 P \sim \alpha_0$, the measure $\vartheta\alpha$, $(\vartheta\alpha)(A) = \alpha(\vartheta^{-1}(A))$, is equivalent to α . Hence, ϑ induces a shift automorphism $\theta f = f \cdot \vartheta$ of $L_a^\infty(T)$. θ commutes with the G -action and $L_1^\infty(T)$ can be identified with the fixed point algebra of θ .

2. Properties of the group action on the Poisson boundary

The group action on the Poisson boundary inherits many of the properties of the group action on $L^\infty(X)$.

Let $L_1^1(\mathcal{Y})$ denote the set of probability measures in a given $L^1(\mathcal{Y})$.

DEFINITION 2.1. A G -space $L^\infty(\mathcal{Y})$ is called approximately transitive if for every pair $\nu_1, \nu_2 \in L_1^1(\mathcal{Y})$ and every $\epsilon > 0$ there exists $\nu \in L_1^1(\mathcal{Y})$ such that the L^1 -distances between ν_i , $i = 1, 2$, and the convex hull $\text{co}(G\nu)$ of the orbit $G\nu$ are both less than ϵ .

Approximate transitivity was discovered in [1] as a property characterizing the flow of weights of ITPFI factors. We note that if $L^\infty(X)$ is a transitive G -space, i.e., isomorphic to $L^\infty(G/H)$ for a closed subgroup $H \subseteq G$, then $L^\infty(X)$ is approximately transitive. Every approximately transitive G -space is ergodic.

DEFINITION 2.2. A G -space $L^\infty(\mathcal{Y})$ is called amenable if there exists a positive equivariant linear map $\mathfrak{A}: L^\infty(G) \otimes L^\infty(\mathcal{Y}) \rightarrow L^\infty(\mathcal{Y})$, such that $\mathfrak{A}(1 \otimes f) = f$ for all $f \in L^\infty(\mathcal{Y})$.

This definition of amenability follows [2]; another known definition [3] is equivalent to ours in the case of countable G , and always follows from ours. We note that $L^\infty(G)$ is always an amenable G -space and $L^\infty(G/H)$ is amenable iff H is an amenable subgroup.

THEOREM 2.3. *If $L^\infty(X)$ is an ergodic (respectively: approximately transitive, amenable) G -space then $L^\infty_\alpha(T)$ is also ergodic (respectively: approximately transitive, amenable).*

We remark that, in general, transitivity of $L^\infty(X)$ is not inherited by the Poisson boundary.

For a proof of Theorem 2.3 see [2,4]. (Note: the proof of amenability in [2] has a gap. A correct proof is given in [4].)

3. Special properties of the Furstenberg boundary

For the remaining part of our paper we shall be concerned exclusively with stationary random walks.

Theorem 2.3 remains valid for the Furstenberg boundary. In fact, for every stationary random walk T one can always find another stationary random walk T' such that $L^\infty_1(T)$ is G -isomorphic to $L^\infty_1(T')$. However, when $L^\infty(X)$ is a transitive G -space, the Furstenberg boundary in addition to being approximately transitive has often the following stronger property:

DEFINITION 3.1. *A G -space $L^\infty(Y)$ is said to have property AT_* if there exists $v \in L^1_1(Y)$ such that $\text{co}(Gv)$ is dense in $L^1_1(Y)$.*

THEOREM 3.2. *If $L^\infty(X)$ has property AT_* then so does $L^\infty_1(T)$.*

Obviously, when X is a countable transitive G -space, then AT_* is a property of $L^\infty(X)$ and, hence, also of $L^\infty_1(T)$. However, the AT_*

property holds under more general conditions:

Let $X = G$ and $\mu = P(e, \cdot)$, where e is the identity of G . The random walk T on G defined by P is called *spread out* if there is an $n = 1, 2, \dots$ such that the n -th convolution power μ^n is non-singular with respect to the Haar measure.

THEOREM 3.3. *If T is spread out then $L_1^\infty(T)$ has property AT_* .*

There is a straightforward generalization of this theorem to random walks on homogeneous spaces G/H .

The AT_* property is a rather strong one:

THEOREM 3.4. *Let $(Y, \mathcal{B})$ be a Borel G -space with a σ -finite G -invariant measure α . Assume that $L^\infty(Y)$ has property AT_* . Then α is purely atomic. If $(Y, \mathcal{B})$ is a countably separated Borel space, then G has a countable conull orbit in Y .*

Let $X = G$ and $\mu = P(e, \cdot)$. The random walk T defined by P is called *adapted* if no proper closed subgroup of G contains $\text{supp} \mu$. Theorems 3.3 and 3.4 have the following consequences.

COROLLARY 3.5. *Let T be an adapted spread out random walk on G . If there exists a σ -finite G -invariant measure on $(X^\infty, \mathcal{A}^{(1)})$, equivalent to α_0 , then $L_1^\infty(T)$ is trivial.*

COROLLARY 3.6. *The Furstenberg boundary of an adapted random walk on a countable group G is either trivial or properly ergodic.*

By Corollary 3.5 the known examples [5,6] of random walks on G with non trivial boundary are examples of G -actions with no σ -finite G -invariant measure. We remark that Corollary 3.6 fails

for uncountable groups, e.g., for adapted spread out random walks on semisimple Lie groups [5].

Proofs of Theorems 3.2-3.6 can be found in [4].

4. The relation between the Poisson and Furstenberg boundaries

The Furstenberg boundary of a stationary random walk can be defined as the fixed point algebra of the shift automorphism θ acting on the Poisson boundary. In the case of spread out random walks on G the action of θ admits the following factorization.

THEOREM 4.1. *Let T be a spread out random walk on G and let C be the cyclic group $(\theta^j)_{j \in \mathbb{Z}}$. Then the Poisson boundary $L_a^\infty(T)$ is isomorphic to $L_f^\infty(T) \otimes L^\infty(C)$ so that θ corresponds to $\text{id} \otimes \theta'$ where $\theta': L^\infty(C) \rightarrow L^\infty(C)$ is the rotation $\theta' f(c) = f(c\theta^{-1})$.*

This result has a generalization to random walks on homogeneous spaces G/H . Its proof relies on an application of one of Derrienic's 0-2 laws [6] to show that the action of θ is, essentially, a smooth action on any standard pointwise realization of $L_a^\infty(T)$ (the smoothness fails without the spread out assumption). We remark that the G action on $L_f^\infty(T) \otimes L^\infty(C)$ arising from the isomorphism with $L_a^\infty(T)$ can be described, in terms of pointwise realizations of $L_a^\infty(T)$ and $L_f^\infty(T)$, as a skew product action. Under certain additional conditions for the measure $\mu = P(e, \cdot)$ the situation is simpler:

Denote by S the open semigroup of elements $g \in G$ such that for some k μ^k ($\mu = P(e, \cdot)$) dominates a multiple of the Haar measure on a neighbourhood of g (when G is countable, S is the semigroup generated by $\text{supp} \mu$). We say that the random walk T defined by P satisfies condition S if $SS^{-1} = G$.

THEOREM 4.2. *Let T be a stationary spread out random walk on G satisfying condition S. Let N denote the smallest closed normal subgroup of G such that $P(e, \xi N) = 1$ for some $\xi \in G$. Then the Poisson boundary is G -isomorphic to the tensor product $L_1^\infty(T) \otimes L^\infty(G/N)$ equipped with the product G -action. Moreover, G/N is a cyclic group and θ corresponds to $\text{id} \otimes \theta'$ where θ' is the rotation $\theta': L^\infty(G/N) \rightarrow L^\infty(G/N)$, $(\theta' f)(x) = f(x(\xi N))$ defined by the coset ξN containing $\text{supp } f$.*

We remark that there exist examples of adapted random walks on countable G where Theorem 4.2 fails, i.e., the period of θ does not coincide with the index of N in G .

A proof of Theorem 4.2 is given in [4].

References

- [1] A. Connes and E.J. Woods, *Approximately transitive flows and ITPFI factors*. Ergodic Th. Dynam. Sys. 5 (1985), 203-236.
- [2] A. Connes and E.J. Woods, *Hyperfinite von Neumann algebras and Poisson boundaries of time dependent random walks*. Pacific J. Math. 137 (1989), 225-243.
- [3] R. Zimmer, *Ergodic theory and semisimple groups*. Birkhauser, Boston 1984
- [4] W. Jaworski, *Poisson and Furstenberg boundaries of random walks*. Thesis, Queen's University 1991.
- [5] V.A. Kalmanovich and A.M. Vershik, *Random walks on discrete groups: boundary and entropy*. Ann. Prob. 11 (1983), 457-490.
- [6] R. Azencott, *Espaces de Poisson des groupes localement compacts*. Lecture Notes in Math., vol. 148, Springer 1970.
- [7] Y. Derriennic, *Lois zero ou deux pour les processus de Markov. Applications aux marches aleatoires*. Ann. Inst. H. Poincare 12 (1976), 111-129.

SUR UNE FORME DE LA SOLUTION DE L'EQUATION DE TRANSLATION

Zenon Moszner

Presented by J. Aczel, F.R.S.C.

Résumé: En considérant les théorèmes 1 et 2 de J. Aczel dans [1] (p.367-370), on donne des conditions nécessaires et suffisantes pour que la solution $F: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ de l'équation de translation

$$F(x, U+V) = F(F(x, U), V)$$

soit de la forme

$$F(x, U) = k^{-1}[k(x) + CU],$$

où $k: \mathbb{R}^n \rightarrow \mathbb{R}^n$ est une bijection et C une matrice constante Δ dimensions $n \times m$, de rang égal $\Delta \min(n, m)$. On donne aussi une généralisation du problème considéré.

Introduction

J. Aczel dans [1] (p.367-370) a démontré les résultats suivants:

Théorème 1. Si $n > m$ et si pour une constante $A \in \mathbb{R}^m$ l'équation $F\left(\begin{smallmatrix} A \\ \eta \end{smallmatrix}\right), Y = x$, où $\eta \in \mathbb{R}^{n-m}$, $Y \in \mathbb{R}^m$ et $x \in \mathbb{R}^n$, peut être pour chaque x résolue uniquement par rapport $\Delta Y = \begin{pmatrix} Y \\ \eta \end{pmatrix}$, alors

$$(1) \quad F(x, U) = k^{-1}[k(x) + CU],$$

avec une fonction bijective arbitraire $k: \mathbb{R}^n \rightarrow \mathbb{R}^n$ et une matrice constante arbitraire $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ (C_1 est une matrice carrée non singulière Δ dimensions m), est la solution générale de

$$(2) \quad F(x, U+V) = F(F(x, U), V),$$

où $x \in \mathbb{R}^n$ et $U, V \in \mathbb{R}^m$.

Théorème 2. Aussi dans le cas où $n < m$, la solution générale de (2) est la fonction (1), $k: R^n \rightarrow R^n$ étant une bijection arbitraire et $C = (C_i C_j)$ une matrice constante arbitraire (C_i est une matrice carrée non singulière à dimensions n) , s'il existe un $a \in R^n$ tel que $P(a, \begin{pmatrix} U \\ \mu \end{pmatrix}) = x$, où $x \in R^n, U \in R^n$ et $\mu \in R^{n-m}$, possède une solution unique $U = g(\begin{pmatrix} x \\ \mu \end{pmatrix})$, g étant une fonction de x et μ continue ou bornée.

On sait [2] que chaque solution de l'équation de translation

$$(3) \quad P(x, U.V) = P(P(x, U), V),$$

où $P: \Gamma \times G \rightarrow \Gamma$, Γ est un ensemble arbitraire et $(G, ., e)$ forme un groupe, est de la forme

$$(4) \quad P(x, U) = g_k^{-1} [g_k(h(x)) . U] \text{ pour } h(x) \in \Gamma_k,$$

où $h: \Gamma \rightarrow \Gamma$ est telle que $h(h(x)) = h(x)$,

$$(5) \quad h(\Gamma) = \bigcup_{k \in K} \Gamma_k, \quad \Gamma_k \cap \Gamma_s = \emptyset \text{ pour } k \neq s$$

et pour chaque $k \in K$ il existe un sous-groupe G_k de G dont l'indice est égal à la puissance de Γ_k et g_k est une bijection de Γ_k à la famille des classes d'équivalence à droite du groupe G par rapport au sous-groupe G_k .

Nous avons d'après [3] et [4] les conclusions suivantes:

1. $P(., U)$ est univalente si et seulement si $P(x, e) = x$ pour chaque x de Γ (c'est la condition d'identité).

2. $P(x, .)$ est univalente si et seulement si $G_k = \{e\}$ pour chaque k de K .

3. La solution P de (3) est disjointe, c.à d.

$$\forall x \in \Gamma \quad \forall U_1, U_2 \in G \quad \{P(x, U_1) = P(x, U_2) \implies \forall y \in \Gamma : P(y, U_1) = P(y, U_2)\},$$

si et seulement si

$$G_k = N_k = \{U \in G : \forall x \in \Gamma : P(x, U) = x\} \text{ pour chaque } k \text{ de } K.$$

Résultats fondamentaux

Nous allons donner généralisations des théorèmes cités.

Théorème 3. La solution $F(x, U)$ de l'équation (2) pour $n > m$, est de la forme (1) si et seulement si

(6) $F(., U)$ est univalente pour tout U de R^m ,

(7) $F(x, .)$ est univalente pour tout x de R^n ,

(8) la puissance de la famille $\mathfrak{F}$ des fibres transitives de F , c.à d. des ensembles $\{F(x, U) : U \in R^m\}$, est égale à la puissance du continu.

Démonstration de "seulement si".

Il est évident que pour $U=0$ on a $F(x, 0)=x$, d'où $F(., U)$ est univalente. Il résulte de $k[F(x, U)]=k(x)+CU$ que $F(x, .)$ est univalente. D'après l'univalence de $F(., U)$ la réunion des fibres transitives de F est égale à R^n , puisque k est une bijection, $k[F(x, U)]=k(x)+CU$ et les fibres transitives sont toutes disjointes, donc $\mathfrak{F}$ a la puissance du continu.

Démonstration de "si".

D'après nos suppositions, F a la forme (4):

$$F(x, U) = g_l^{-1}[g_l(x) + U] \text{ pour } x \in \Gamma_l,$$

où g_l , pour l de L , est une bijection de Γ_l à R^m , $\{\Gamma_l\}_{l \in L}$ forme une décomposition de R^n aux ensembles non-vides et disjoints. Puisque les Γ_l forment des fibres transitives de F , donc la puissance de L est égale au continu. Soit h une bijection de L à R^{n-m} . La fonction

$$k_l(x) = \begin{pmatrix} g_l(x) \\ k(l) \end{pmatrix} \text{ pour } x \text{ de } \Gamma_l$$

est une bijection de Γ_l à un hyperplan de R^n à dimensions m . La fonction $k = \bigcup_{l \in L} k_l$ est une bijection de R^n . Supposons que $x \in \Gamma_l$, alors aussi

$$g_l^{-1}[g_l(x) + U] = F(x, U) \in \Gamma_l,$$

d'où

$$k[F(x, U)] = \begin{pmatrix} g_l(x) + U \\ k(l) \end{pmatrix} = \begin{pmatrix} g_l(x) \\ k(l) \end{pmatrix} + \begin{pmatrix} U \\ 0 \end{pmatrix} = k(x) + \begin{pmatrix} U \\ 0 \end{pmatrix}.$$

En posant que $\begin{pmatrix} U \\ 0 \end{pmatrix} = CU$, il en résulte que

$$P(x, U) = k^{-1}[k(x) + CU],$$

c.q.f.d.

Pour $n=m$, les fibres transitives seront confondues et avec la condition $P(x, e) = x$ nous aurons la puissance de $\mathbb{B}$ égale à 1.

Nous avons donc

Théorème 4. La solution $P(x, U)$ de l'équation (2), où $n=m$, est de la forme (1) si et seulement si (7) a lieu et

(9) P est transitive, c.à d. $\forall x, y \in R^n \exists U \in R^m: P(x, U) = y$.

Supposons maintenant le cas où $n < m$. Pour généraliser le théorème 2, nous allons appliquer le théorème 2 dans [2] (dans ce théorème la supposition que G est indénombrable a été omise !) comme $(R^n, +)$ et $(R^m, +)$ sont isomorphes.

Nous avons donc le

Théorème 5. La solution $P(x, U)$ de l'équation (2), où $n < m$, est de la forme

$$(10) \quad P(x, U) = k^{-1}[k(x) + l(U)],$$

où $k: R^n \rightarrow R^n$ est une bijection et l est un homomorphisme de R^m à R^n , si et seulement si (6) a lieu et de plus

(11) tous les sous-groupes de stabilité de P , c'est-à-dire les ensembles $\{U \in R^m: P(x, U) = x\}$, sont égaux et ils forment un espace vectoriel sur le corps des nombres rationnels Q .

(12) la famille des fibres transitives de P a la puissance égale à 1 ou transfinie.

Nous donnons un exemple qui montre que les suppositions (6), (11) et (12) sont plus faibles que la supposition du théorème 2. Soit $n = 2$ et $m = 3$ et posons que $G = \{(U_1, U_2, U_3) \in R^3: U_3 = 0\}$. Soit q une bijection de R^2 sur R^3/G et $P(x, U) = q^{-1}[q(x) + U]$. On peut facilement vérifier que les conditions (6), (11) et (12) sont

satisfaites. Il n'existe pas un x de R^2 tel que $F[x, (U_1, U_2, U_3)] = y \in R^2$ a la solution unique par rapport à (U_2, U_3) pour chaque U_1 de R et chaque y de R^2 . En effet pour x donné, nous avons

$$F[x, (U_1, U_2, 0)] = g^{-1}[g(x) + (U_1, U_2, 0)] = x$$

pour chaque U_1, U_2 de R .

On peut généraliser le théorème 5 de la façon suivante.

Théorème 6. Soit donné un groupe $(H, +)$. La solution F de l'équation de translation (3) sur le groupe $(G, +)$ est de la forme

$$(13) \quad F(x, U) = \phi^{-1}[\phi(x) + c(U)],$$

où $\phi: \Gamma \rightarrow H$ est une injection et $c: G \rightarrow H$ est un homomorphisme, si et seulement si (6) a lieu et

(14) il existe un monomorphisme i du groupe G/N dans H , où

$$N := \{U \in G : \forall x \in \Gamma : F(x, U) = x\},$$

(15) la puissance de la famille $\mathfrak{F}$ des fibres transitives de F est plus petite ou égale à l'indice de $i(G/H)$ dans H ,

(16) F est disjointe.

Démonstration de "si".

Soit $i: G/H \rightarrow H$ un monomorphisme, soit K tel que dans (5) et prenons $\psi: K \rightarrow H$, telle que $\psi(k_1)$ et $\psi(k_2)$ appartiennent à différentes classes d'équivalence à gauche de H par rapport à $i(G/N)$. Prenons F de la forme (4) et définissons

$$c(U) = i(UN) \text{ et } \phi(x) = \psi(k) + i g_k(x) \text{ pour } x \in \Gamma_k.$$

D'après (16) chaque g_k est définie sur G/N . Nous avons

$$\phi^{-1}[\psi(k) + i g_k(x) + i(UN)] = \phi^{-1}[\psi(k) + i(g_k(x) + UN)] = g_k^{-1}[g_k(x) + UN] = F(x, U),$$

$$\text{car } \phi[g_k^{-1}(g_k(x) + UN)] = \psi(k) + i(g_k(x) + UN).$$

Démonstration de "seulement si".

Le monomorphisme i est donné par la formule $i(UN) = c(U)$. La définition est correcte puisque N est égal au noyau de c .

Pour démontrer (15) remarquons que d'après (12)

$$\phi[F(x,U)] = \phi(x) + c(U),$$

d'où chaque fibre transitive de F (c.à d. chaque ensemble $\{F(x,U) : U \in G\}$) est transformée par ϕ en une classe d'équivalence à gauche du groupe H par rapport au sous-groupe $c(G) = i(G/N)$. Puisque ϕ est injective, les fibres transitives différentes (comme disjointes) sont transformées par ϕ en classes diverses, d'où (15).

La condition (6) est évidente. Puisque chaque G_k est égal au noyau de c , la condition (16) a aussi lieu.

Remarque. Si F est disjointe, il existe toujours un groupe H remplissant (14) et (15). En effet posons $H = (G/N) \times K$ (le produit cartésien), où dans K on donne une structure du groupe quelconque. Dans ce cas $i(a) = (a, k_0)$, si $a \in G/N$ et k_0 est l'élément neutre de K , est le monomorphisme exigé.

Remarquons encore que (13) est bien définie si et seulement si $\phi(\Gamma) + c(G) \subset \phi(\Gamma)$ et dans ce cas elle est une solution de (3).

Travaux cités

- [1] J. Aczél, Lectures on functional equations and their applications. Accademic Press, New York and London, 1966.
- [2] Z. Moszner, Structure de l'automate plein, réduit et inversible, Aequationes Math. 9 (1973), 46-59.
- [3] Z. Moszner, Une généralisation d'un résultat de J. Aczél et M. Hosszú sur l'équation de translation, Aequationes Math. 37 (1989), 267-278.
- [4] Z. Moszner, Sur les propriétés complémentaires des solutions de l'équation de translation, Ann. Polon. Math. 52 (1990), 27-36.

Mailing Addresses

1. D.F. Anderson Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37996-1300, U.S.A.
2. H. Ben-El-Mechaiekh Department of Mathematics
Brock University
St. Catharines, Ontario L2S 3A1, Canada
3. W. Jaworski Department of Mathematics
University of Ottawa
Ottawa, Ontario K1N 6N5, Canada
4. H. Kanemitsu Department of Mathematics
Aichi University of Education
Igaya-cho, Kariya 448, Japan
5. O.H. Kegel Mathematics Department
University of Freiburg
Albertstrasse 23b
Freiburg D-7800, Germany
6. G. Maksa Institute of Mathematics
Kossuth Lajos University
E-4010 Debrecen, Pf. 12, Hungary
7. Z. Moszner PL-37-084, Krakow
Kazimierza W87/4, Poland
8. K. Nikodem Department of Mathematics
Technical University
PL-43-309 Bielsko-Biala, Poland
9. H.S. Nikulin Laboratory of Statistical Methods
St. Petersburg Branch, Steklov Mathematical Institute
Fontanka 27, St. Petersburg, Russia
10. Z. Páles Institute of Mathematics
Kossuth Lajos University
E-4010 Debrecen, Pf. 12, Hungary
11. R. Sharma Department of Mathematics
University of New Haven
West Haven, CT 06516, U.S.A.
12. S. Shokranian Departamento de Matemática
Universidade de Brasília
70910 Brasília, DF, Brasil
13. C.L. Stewart Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
14. T. Sugatani Department of Mathematics
Toyoama University
Gofulku, Toyoama 930, Japan
15. S. Wujiu Department of Mathematics
Southwest China Normal University
Beibei, Chongqing 630715, P.R. China
16. K. Yoshida Department of Applied Mathematics
Okayama University of Science
Ridai-cho, Okayama 700, Japan

Articles Index - Volume XIII

G. Almkvist	
Values of Bernoulli polynomials and Hurwitz's zeta function at rational points	104
D.F. Anderson	
The kernel of $\text{Pic}(R_0) \rightarrow \text{Pic}(R)$ for R a graded domain	248
H. El Azhari	
Sur la continuité des homomorphismes d'algèbres	99
W. Beekmann	
Sums of FK-spaces	25
A. Benhissi	
Quelques propriétés des séries formelles généralisées	13
A. Benhissi	
La clôture séparable du corps des séries formelles généralisées	66
A. Benhissi	
Le Théorème de Rolle sur le corps des séries formelles généralisées	109
H. Ben-El-Mechaiekh	
General fixed point and coincidence theorems for set-valued maps	237
H. Breitenfellner	
A unified Möbius inversion formula	39
H. Chang	
Low Reynolds number viscous jets	49
S.-C. Chang	
Sums of FK-spaces	25
R. Chouikha	
Métriques conformément plates et équations de Yang-Mills	7
H.S.M. Coxeter	
Affine regularity	181
M. Csörgö	
Criteria for limit inferior of small increments of Banach space valued stochastic processes	173
G. Díaz	
Passage d'une mesure d'approximation à une mesure de transcendance	131
D.E. Dobbs	
The probability of invertibility for a matrix over a finite commutative ring	217

D. Drissi	
Sur l'inégalité d'Allan Sinclair pur les opérateurs Hermitiens	30
A.G. Earnest	
On the theta series of positive quaternary quadratic forms	33
C. Farsi	
Fixed point subalgebras of the rotation algebra	75
C. Farsi	
Fixed point subalgebras of the rotation algebra - corrigendum	234
I. Fleischer	
A lattice theoretic look at some ring theoretical radicals	93
P. Fuchs	
A note on the Granville-Heath-Brown theorem	61
E.E. Granirer	
On convolution operators which are far from being convolution by a bounded measure. Expository memoir.	187
A. Grytczuk	
On Fermat's equation in the first case	141
J. Gutiérrez	
A polynomial decomposition algorithm over factorial domains	81
K. Inkeri	
On Fermat's equation in the first case	141
W. Jaworski	
Poisson and Furstenberg boundaries of random walks	279
A. Joyal	
Strong stacks and classifying spaces	2
A. Joyal	
A categorical theory of cumulative hierarchies of sets	55
M. Kanemitsu	
Note on the dual of an ideal in a Noetherian domain	255
O.H. Kegel	
On the orders of elements of a locally finite simple group	253
M.J. Lancaster	
The probability of invertibility for a matrix over a finite commutative ring	217
K. Liu	
Quantum central extensions	135
G. Maksa	
Results on t-Wright convexity	274

D.R. Masson	
A generalization of Ramanujan's best theorem on continued fractions	167
A. Meurman	
Values of Bernoulli polynomials and Hurwitz's zeta function at rational points	104
M. Mignotte	
Passage d'une mesure d'approximation à une mesure de transcendance	131
M. Mignotte	
A note on the equation $x^2 - 1 = y^2(z^2 - 1)$	157
I. Moerdijk	
A categorical theory of cumulative hierarchies of sets	55
R.A. Mollin	
Class numbers bounded below by the divisor function - corrigenda	234
Z. Moszner	
Sur une forme de la solution de l'équation de translation	285
M. M'Zari	
Algorithme pour calculer les points de Heegner de $Y_0(N)$	43
K. Nikodem	
Results on t-Wright convexity	274
M.S. Nikulin	
A chi-square goodness-of-fit test for natural exponential families with power variance function	265
G. Nipp	
On the theta series of positive quaternary quadratic forms	33
K.-G. Nolte	
Universal appearance of periodic orbits in the Lorenz equations	125
D. Nour El Abidine	
Sur le groupe des classes d'un anneau intègre	69
Z. Páles	
Results on t-Wright convexity	274
M.S. Samaali	
Anneaux henséliens et clôture séparable	209
M.S. Samaali	
Quelques propriétés de la clôture séparable d'un anneau connexe	151
Q.-M. Shao	
Criteria for limit inferior of small increments of Banach space valued stochastic processes	173
R. Sharma	
Second order parallel tensors on contact manifolds II*	259

R.K. Sharma	
On solvable Lie ideals of a ring	115
S. Shokranian	
L^2 -cohomology and Hecke operators	229
S. Shokranian	
The twisted L^2 -Lefschetz number of Hecke operators	243
J.B. Srivastava	
On solvable Lie ideals of a ring	115
C.L. Stewart	
On the number of solutions of polynomial congruences	271
P. Streit	
On the characteristic functions of the Kotz type distributions	121
H. Su	
On the classification of C^* -algebras of real rank zero: inductive limits of matrix algebras over graphs	223
T. Sugatani	
Note on the dual of an ideal in a Noetherian domain	255
B. Szyszkowicz	
Empirical type processes and contiguity	161
B. Szyszkowicz	
Weighted stochastic processes under contiguous alternatives	211
H. Tierney	
Strong stacks and classifying spaces	2
J.-M. Tricot	
Un modèle d'accord entre observateurs sur un échelle nominale	146
N. Watling	
Fixed point subalgebras of the rotation algebra	75
N. Watling	
Fixed point subalgebras of the rotation algebra - corrigendum	234
A. Wiedeman	
Some order of Auslander type ≥ 2	19
S. Wujie	
On the orders of elements of a locally finite simple group	253
M. Yamada	
An approach to Wieferich's condition	87
K. Yoshida	
Note on the dual of an ideal in a Noetherian domain	255

Paging of Vol. XIII

(1)	1-60	(2)	61-92	(3)	93-120
(4)	121-180	(5)	181-236	(6)	237-296