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AFFINE REGULARITY

by H.S.M. Coxeter, F.R.S.C.

in fond memory of FRIEDRICH BACHMANN

This sequel to *Affinely Regular Polygons* [2] was suggested by the question: Can an affinely regular polygon be defined in words which Euclid himself could have appreciated? The answer is Yes: such a polygon is the natural generalization of the parallelogram (which is the affinely regular quadrangle or "4-gon"). In fact, an n -gon $A_0A_1 \dots A_{n-1}$ is *affinely regular* if all the lines $A_\mu A_\nu$ with $\mu + \nu \equiv c \pmod{n}$ are parallel, for each value of c . The case $n = 5$ is illustrated in Figure 1; see also [1, §13.4 (Figur 13.4e & Übung 10)].

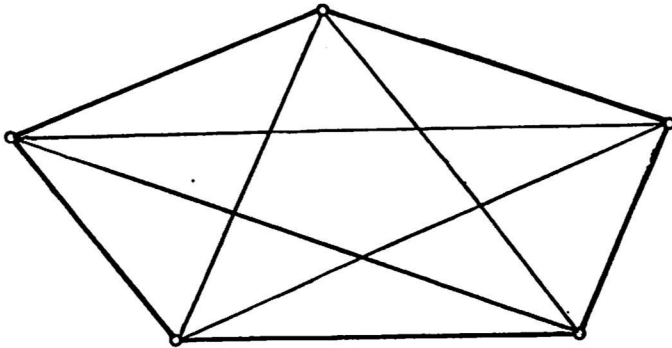


Figure 1: An affinely regular pentagon

1. The general affinely regular polygon

We recall [2, p.39] that an *affinity* maps the whole affine plane into itself while preserving incidence and therefore preserving parallelism: parallel lines are mapped into parallel lines. Any two triangles determine a unique affinity $ABC \rightarrow A'B'C'$, and we call this an *equiaffinity* if the triangles have the same area. For instance, triangles $A_0A_1A_2$ and $A_1A_2A_3$ determine an affinity

$$(1.1) \quad A_0A_1A_2 \rightarrow A_1A_2A_3$$

which is an equiaffinity if A_0A_3 is parallel to A_1A_2 . A polygon $A_0A_1A_2\dots$ is said to be *affinely regular* if it is mapped into itself by an equiaffinity which maps each vertex into the next.

Since the inverse equiaffinity $A_1A_2A_3 \rightarrow A_0A_1A_2$ maps each vertex into its predecessor, the polygon is more properly expressed as

$$(1.2) \quad \dots A_{-2}A_{-1}A_0A_1A_2 \dots$$

It is mapped into itself [2, p.43] by the affine *reflections*

$$(1.3) \quad \begin{cases} (A_0A_1) (A_{-1}A_2) (A_{-2}A_3) \dots, \\ (A_0A_2) (A_{-1}A_3) (A_{-2}A_4) \dots, \end{cases}$$

as well as by the basic equiaffinity

$$(1.4) \quad (\dots A_{-2}A_{-1}A_0A_1A_2 \dots),$$

which is their product [3, p.4]. Therefore the lines

$$A_0A_1, A_{-1}A_2, A_{-2}A_3, \dots$$

are all parallel, as also are the lines

$$A_0A_2, A_{-1}A_3, A_{-2}A_4, \dots$$

Applying the affinity (1.4) to this statement, we deduce that all the lines $A_\mu A_\nu$ with $\mu + \nu = c$ are parallel, for each value of c . Conversely, we have

Theorem 1. *If points $\dots, A_{-2}, A_{-1}, A_0, A_1, A_2, \dots$ are so arranged that, for every integer μ , the line $A_\mu A_{\mu+3}$ is parallel to $A_{\mu+1} A_{\mu+2}$, and $A_\mu A_{\mu+4}$ to $A_{\mu+1} A_{\mu+3}$, then the polygon $\dots A_0 A_1 A_2 \dots$ is affinely regular.*

Proof. Since $A_0 A_3$ is parallel to $A_1 A_2$, four consecutive vertices A_0, A_1, A_2, A_3 of such a polygon must form a trapezium (American 'trapezoid'). When this trapezium $A_0 A_1 A_2 A_3$ is given, the subsequent vertices are successively determined [2, Figure 3 on p. 43] as follows. For each $\mu \geq 0$, $A_{\mu+4}$ is the point where the line through A_μ parallel to $A_{\mu+1} A_{\mu+3}$ meets the line through $A_{\mu+1}$ parallel to $A_{\mu+2} A_{\mu+3}$. This procedure is clearly equivalent to repeated application of the equiaffinity (1.4) which maps the polygon (1.2) into itself.

Having proved this theorem, we see that its enunciation provides an elementary definition for an affinely regular polygon, equivalent to the following prettier (but redundant) definition:

A polygon $\dots A_{-2} A_{-1} A_0 A_1 A_2 \dots$ is affinely regular if all the lines $A_\mu A_\nu$ with $\mu + \nu = c$ are parallel, for each value of c .

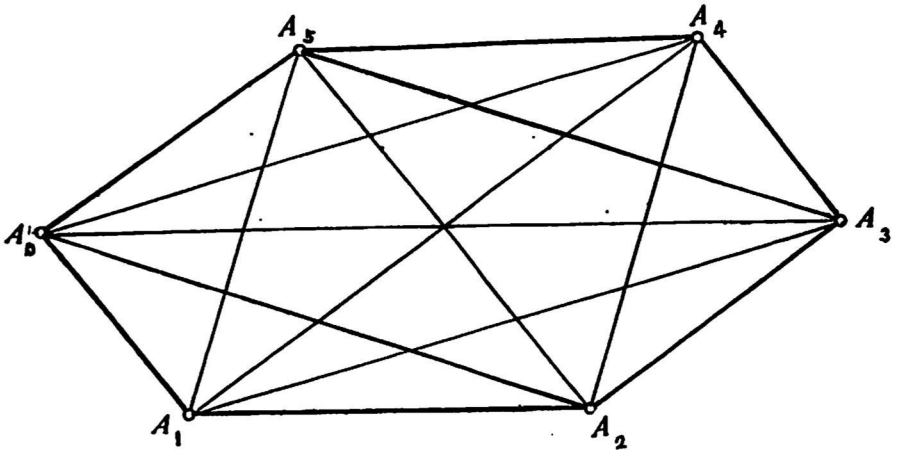


Figure 2: An affinely regular hexagon

2. The affinely regular n -gon

We know [2, pp. 42-48] that there are various types of affinely regular polygon, depending on the value of the real number

$$T = A_0 A_3 / A_1 A_2.$$

In particular, if $T = 1 + 2 \cos 2d\pi/n$, where $n/d > 2$ is rational, then the polygon closes with $A_n = A_0$. Accordingly, Theorem 1 implies

Theorem 2. *If a cycle of points A_0, A_1, \dots, A_{n-1} , with subscripts reduced modulo n , has the property that, for each residue μ the line $A_\mu A_{\mu+3}$ is parallel to $A_{\mu+1} A_{\mu+2}$, and $A_\mu A_{\mu+4}$ to $A_{\mu+1} A_{\mu+3}$, then the n -gon $A_0 A_1 \dots A_{n-1}$ is affinely regular.*

Of course, the properties mentioned are meaningless when $n = 3$: every triangle is affinely regular. Also the property concerning $A_\mu A_{\mu+4}$ can be ignored when $n = 4$ and is superfluous again when $n = 5$. The case $n = 6$ is illustrated in Figure 2.

Every n -gon has a centre: the common point of the axes of the two reflections (1.3). In fact, there are n such axes, all passing through the centre. This invariant point O of the equiaffinity (1.4) may also be described as the centroid of the n vertices and as the common centre of the circumscribed and inscribed ellipses [2, pp. 44,48]. Whereas a Euclidean regular n -gon has equal edges $A_0 A_1, A_1 A_2, \dots, A_{n-1} A_0$, an affinely regular n -gon has equal areas for the triangles $OA_0 A_1, OA_1 A_2, \dots, OA_{n-1} A_0$.

Any affinity (strain or shear), which maps the circumscribed ellipse into a circle, transforms the affine reflections (1.3) into ordinary (orthogonal) reflections, and the affinely regular n -gon into an ordinary regular n -gon. Hence [6, p.412]

Theorem 3. *Any affinely regular n -gon can be derived from a Euclidean regular n -gon by applying a suitable affinity; for instance, by parallel projections onto another plane.*

It follows that an affinely regular star polygon of density d (derived from $\{n/d\}$, where $n = 5$ or $n > 6$ and $d < \frac{1}{2}n$) has the same n vertices as a convex n -gon (derived from $\{n\}$).

Even without appealing to Theorem 3, we can see that, if n is even, any two opposite sides of an affinely regular n -gon belong to a parallelogram. It follows that the diameters joining pairs of opposite vertices are concurrent, all being bisected by the centre O . In other words, the polygon is centrally symmetric.

3. Gerber's generalization of Napoleon's Theorem

Leon Gerber [5] discovered a remarkable theorem which combines affine regularity with Euclidean regularity:

Construct regular n -gons outwardly (or inwardly) on the sides of any affinely regular n -gon. Their centres form the vertices of a regular n -gon.

The case $n = 3$, commonly attributed to Napoleon, was first proved by W. Fischer [4]. The case $n = 4$ was discovered by Thébault [7]. The cases $n = 5$ and $n = 6$ (see Figure 3) were rediscovered by K.R. Shankaranarayan Sastry in Addis Ababa, Ethiopia.

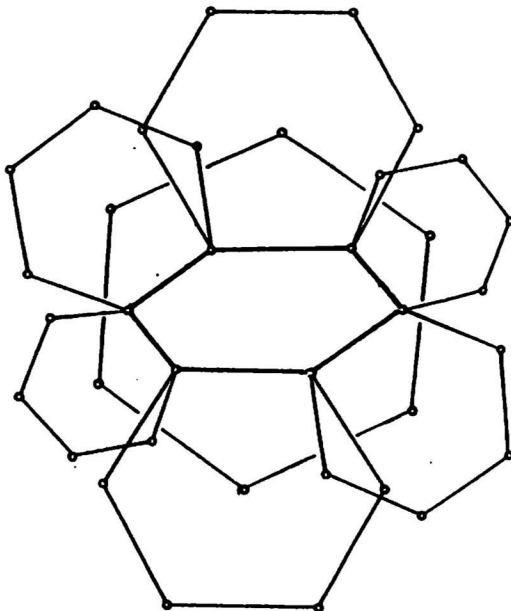


Figure 3: A hexagonal cluster of hexagons

References

- [1] H.S.M. Coxeter, *Unvergängliche Geometrie* (2. Aufl.), Birkhäuser, Basel 1981; translation of *Introduction to Geometry* (2nd ed.), Wiley, New York 1969.
- [2] H.S.M. Coxeter, Affinely regular polygons, *Abh. Math. Sem. Univ. Hamburg* **34** (1969), 38-58.
- [3] H.S.M. Coxeter, *Regular Complex Polytopes* (2nd ed.), Cambridge University Press 1991.
- [4] W. Fischer, Ein geometrischer Satz, *Arch. Math. Phys.* **40** (1863), 460-462.
- [5] L. Gerber, Napoleon's theorem and the parallelogram inequality for affine-regular polygons, *Amer. Math. Monthly* **87** (1980), 644-648.
- [6] B. Grünbaum, *Convex Polytopes*, Wiley, New York 1968.
- [7] V. Thébault, Solution to Problem 169, *National Math. Mag.* **12** (1937-38), 192-194.

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**ON CONVOLUTION OPERATORS WHICH ARE FAR FROM BEING
CONVOLUTION BY A BOUNDED MEASURE. EXPOSITORY MEMOIR**

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ABSTRACT: This paper is concerned, mostly but not exclusively, with the existence (and multitude) of convolution operators on $L^p(G)$ with support (see sequel) included in the closed set $F \subset G$, which are not given as convolution by bounded measures or norm limits of such. Here G stands for a Hausdorff locally compact group with identity e and left Haar measure dx , and $1 < p < \infty$.

INTRODUCTION: Let $CONV_p(G)$ denote the (bounded linear) operators on $L^p(G)$ which commute with translation from the right, equipped with operator norm (we follow Herz [Hz1] or see sequel). If $F \subset G$ is closed let $CONV_p(F)$ denote those $\phi \in CONV_p(G)$ with support, $\text{supp } \phi \subset F$ (see sequel). Denote by $M(F)$ the bounded Borel measures on F and if $\mu \in M(G)$ let $\lambda_p \mu \in CONV_p(G)$ be defined by $(\lambda_p \mu)f = \mu * f$ for $f \in L^p(G)$. Denote by $M_p(F) = \text{norm cl } \lambda_p(M(F)) \subset CONV_p(F)$, where cl means closure.

Our results in simplified version, are concerned with the question of when is $CONV_p(F) \sim M_p(F) \neq \phi$, i.e. when is the quotient space $CONV_p(F)/M_p(F) \neq \{0\}$ and furthermore, when is $CONV_p(F)/M_p(F)$ big, and in fact as big as it can be. Since for second countable G , $CONV_2(F)$ is a subspace of ℓ^∞ , a way to express the above is:

(*) "for which closed $F \subset G$ does the (quotient) Banach space $CONV_p(F)/M_p(F)$ have ℓ^∞ as a continuous linear image (or equivalently have ℓ^∞ as a quotient)"

If $F \subset G$ is "thin" one cannot expect this to happen. In fact let G be abelian and $p = 2$. If $F \subset G$ is compact and scattered then Loomis' lemma shows that $CONV_2(F) = M_2(F)$.

Furthermore if $G = T$ (the torus) and $F \subset T$ is any perfect Helson S -set (and there exist such by a result of J.P. Kahane [Ka2]) then even $CONV_2(F) = \lambda_2(M(F)) = M_2(F)$. In fact this is the case if G is any abelian nondiscrete group and $F \subset G$ is any perfect

Helson S -set. Such sets F are thin in the following sense: for any nondiscrete closed subgroup $H \subset G$ and any $a \in G$, the interior of F in aH is empty (i.e. $\text{int}_{aH}(F) = \emptyset$).

A weak version of the main result of this paper will express the fact that if G is any second countable group and $F \subset G$ is closed and if for some nondiscrete closed subgroup $H \subset G$ and $a, b \in G$, $\text{int}_{aHb}(F) \neq \emptyset$ then $\text{CONV}_p(F)/M_p(F)$ has ℓ^∞ as a quotient, for $1 < p < \infty$.

In fact moreover, $M_p(F)$ will be replaced by a much bigger space, and furthermore the result will not depend on whether F is a set of synthesis. In addition $\text{CONV}_p(F)$ can be replaced by a much smaller space $P_c \subset \text{CONV}_p(F)$ if $\sigma(P) = F$ (see sequel). As seen above, this is the best one can do in terms of the thinness of F and of the bigness of $P/M_p(F)$, since $P \subset \ell^\infty$ will hold if G is second countable.

There are many results in the literature which express the fact that for some closed $F \subset G$, $\text{CONV}_p(F) \sim \lambda_p(M(F)) \neq \emptyset$ (\sim is set theoretical difference).

(a) Global results i.e. $F = G$: As pointed out in G. Gaudry's Yale 1962 Harmonic Analysis lecture notes, an old result of M. Riesz states that the function $1_{Z^+} \in \ell^\infty(Z)$ where $Z^+ = \{0, 1, 2, \dots\}$, is a Fourier coefficient multiplier of $L^p(T)$ if $1 < p < \infty$ (i.e. $1_{Z^+}(\widehat{LP(T)}) \subset \widehat{LP(T)}$) yet for no $\mu \in M(T)$, is $1_{Z^+} = \hat{\mu}$ (since any such μ belongs to $L^1(T)$ by a theorem of F. and M. Riesz and then $\hat{\mu}(n) \rightarrow 0$ if $|n| \rightarrow \infty$). Thus $\text{CONV}_p(T) \sim \lambda_p(M(T)) \neq \emptyset$.

Results of Brainerd and R.E. Edwards [BE] as improved by Figa-Talamanca and Gaudry [FG] and J.F. Price [Pr1] imply that if G is compact and infinite or any locally compact abelian group then $\text{CONV}_p(G) \sim \lambda_p(M(G)) \neq \emptyset$. A result of N. Lohoue [Loh] implies that if G is a noncompact semisimple Lie group with finite centre then $\text{CONV}_p(G) \sim \lambda_p(M(G)) \neq \emptyset$.

All the above have been definitively improved by M. Cowling and J. Fournier in [CF] namely: For any infinite locally compact group G if $|q^{-1} - 2^{-1}| < |p^{-1} - 2^{-1}|$ there is some $\phi \in \text{CONV}_q(G)$ such that $\phi \notin \text{CONV}_p(G)$, consequently $\phi \notin \lambda_q(M(G))$ and $\text{CONV}_q(G) \sim \lambda_q(M(G)) \neq \emptyset$ for all $1 < q < \infty$.

If $p = 2$ and G is nondiscrete second countable, it has been proved by Ching Chou in [Ch2] that if $P = CONV_2(G)$ then $P/W_P(e)$ (a fortiori $P/M_2(G)$) has ℓ^∞ as a quotient and $P_c/W_{P_c}(e)$ is not norm separable, where e is the identity of G (see [Ch2] thm. 3.3 and Cor 3.6 respectively). Chou uses C^* algebraic methods which are not available if $p \neq 2$.

(b) Local results: If G is any nondiscrete locally compact group and $U \subset G$ is open with $F = \text{cl}U$ compact, then R.E. Edwards and J.F. Price's results in [EP] imply that there are norm separably many elements ϕ in $CONV_p(F) \sim \lambda_p(M(F))$ (for every $1 < p < \infty$) provided U admits a Rudin-Schapiro sequence (URS). If G is abelian and nondiscrete (URS) sequences exist for all open $U \subset G$. (URS) sequences are also constructed in [EP] for particular cases of G and certain open $U \subset G$. If G is arbitrary and $U \subset G$ is open, it is not known if (URS) sequences exist at all.

If $G = T$ the torus then Cowling and Fournier show in [CF] that there are $\phi \in CONV_q(G) \sim CONV_p(G)$ whose support is included in any symmetric Cantor set, and if G is not unimodular then $\text{supp } \phi$ can be chosen in the kernel of the modular function (where $|q^{-1} - 2^{-1}| < |p^{-1} - 2^{-1}|$).

Our theorems will improve substantially, in a sense, the above results. They are new even if $G = R^n$ or T^n and $p = 2$.

I. THE GENERAL CASE. We need some more definitions (consistent with Herz [Hz1]) to proceed to our main result.

Denote by $PM_p(G)$ the ultraweak (u.w.) closure of $\lambda_p(M(G))$ in $CONV_p(G)$ (where u.w. is the topology on $CONV_p(G)$ generated by the semi-norms $\phi \rightarrow |\sum_1^\infty (\phi f_n, g_n)|$ with $f_n \in L^p(G)$, $g_n \in L^{p'}(G)$ and $\sum_1^\infty \|f_n\|_p \|g_n\|_{p'} < \infty$, $1/p + 1/p' = 1$, $1 < p < \infty$). As is well known, see Herz [Hz1,2], if G is amenable or $p = 2$, and in many other cases, $PM_p(G) = CONV_p(G)$.

$PM_p(G)$ is the dual of the Banach function algebra $A_p(G)$ (see [Hz1]) and $(PM_p(G), w^*) = (PM_p(G), \text{u.w.})$. Furthermore $CONV_p(G)$ is an A_p module where $(u \cdot \phi, v) = (\phi, uv)$ if $u, v \in A_p(G)$ and $\phi \in PM_p(G)$. If G is abelian and $p = 2$, $A_2(G) = \widehat{L^1(\widehat{G})}$.

If $\phi \in CONV_p(G)$ we define the support of ϕ , $\text{supp } \phi$ by: If $u \in L^p(G)$ then $x \notin \text{supp}$

with compact support) such that $\text{supp } v \subset V$. If $\phi \in \text{CONV}_p(G)$ then $x \notin \text{supp } \phi$ iff there is a neighborhood U of c such that $x \notin \text{supp } \phi(u)$ for all $u \in C_c(G) \subset L^p(G)$ with $\text{supp } u \subset U$. If $\mu \in M(G)$ then $\text{supp } \lambda_p \mu = \text{supp } \mu$ as a measure as shown in Herz [Hz1], p. 120.

Denote for closed $F \subset G$, $\text{CONV}_p(F) = \{\phi \in \text{CONV}_p(G); \text{supp } \phi \subset F\}$ $PM_p(F) = PM_p(G) \cap \text{CONV}_p(F)$ and $PM_{p^*}(F) = w^* \text{cl lin}\{\lambda_p \delta_x, x \in F\}$ where δ_x is the point mass at x and cl [lin] denotes closure [linear span] respectively. Clearly, F is of spectral synthesis for $A_p(G)$ iff $PM_{p^*}(F) = PM_p(F)$, see [Hz1].

Let $M_p(F) = \text{norm cl } \lambda_p(M(F))$ and if $P \subset \text{CONV}_p(G)$ denote $P_c = \text{norm cl } \{\phi \in P; \text{supp } \phi \text{ is compact}\}$, $\sigma(P) = \{x \in G; \lambda_p \delta_x \in P\}$ and if $x \in G$

$$E_P(x) = \text{norm cl}\{\phi \in P; x \notin \text{supp } \phi\} \quad (E \text{ from ergodic}).$$

It so happens that if $P \subset \text{CONV}_p(G)$ is an u.w. closed $A_p(G)$ module with $\sigma(P) = F$ then $M_p(F) \subset \mathcal{C}(\lambda_p \delta_x) \oplus E_{P_c}(x)$ where \mathcal{C} is the complex field, for all $x \in F$. We hence denote $W_P(x) = \mathcal{C}(\lambda_p \delta_x) + E_P(x)$ if $P \subset \text{CONV}_p(G)$ (the sum is direct for all P we consider). The reader should note the following properties of the Banach space ℓ^∞ :

(a) If $X \subset Y \subset Z$ are Banach spaces and X has ℓ^∞ as a quotient so does Y . Since any operator $t: X \rightarrow \ell^\infty$ has an extension operator to all of Y by the injectivity of ℓ^∞ . And if Z/Y has ℓ^∞ as a quotient so does Z/X .

(b) ℓ^∞ is the biggest among all duals of separable Banach spaces since it contains any such (see [LT] p. 108) and $\ell^\infty = (\ell^1)^*$. Thus $PM_p(G) \subset \ell^\infty$ if G is second countable.

If F, H are subsets of G then the interior of F in H , $\text{int}_H(F)$, is defined by: $x \in \text{int}_H(F)$ iff there is some open $U \subset G$ such that $x \in H \cap U \subset F$.

We are now ready to state our main theorem:

Theorem M: Let G be a second countable locally compact group H a closed nondiscrete subgroup, $a, b \in G$. Let $P[Q]$ be a w^* [norm] closed A_p submodule of $PM_p(G)$ such that $P_c \subset Q \subset P$ and let $\sigma(P) = F$.

Then for any $x \in \text{int}_{aHb}(F)$, $Q/W_Q(x)$ (a fortiori $Q/M_p(F)$ if such a, b, H, x exist) has ℓ^∞ as a quotient and $TIM_Q(x)$ contains F .

Here $TIM_{\mathcal{Q}}(x) = \{\psi \in \mathcal{Q}^*; 1 = \|\psi\| = \psi(\lambda_p \delta_x), \psi(E_{\mathcal{Q}}(x)) = 0\}$ (TIM from topological invariant mean and \mathcal{Q}^* is the dual of the Banach space \mathcal{Q}) and $F \subset \ell^{\infty}$ is defined by: $F = \{\psi \in \ell^{\infty}; 1 = \|\psi\| = \psi(1), \psi(c_0) = 0\}$ where 1 also stands for the constant one sequence in ℓ^{∞} and $c_0 = \{x = (x_n) \in \ell^{\infty}; \lim_n x_n = 0\}$. Note that F is a w^* compact perfect set of the positive face of the unit sphere of ℓ^{∞} which is as big as it can be, namely $\text{card } F = \text{card } \ell^{\infty} = 2^c$ where c is the cardinality of the continuum.

Remarks: 1. The fact that $\mathcal{Q}/W_{\mathcal{Q}}(x)$ has ℓ^{∞} as a quotient is a much stronger result than the fact that $\mathcal{Q}/M_p(F)$ has same. Since it has been proved by Ching Chou in [Ch1] that if G is abelian, nondiscrete, $p = 2$ and $\mathcal{Q} = PM_2(G)$ then even $W_{\mathcal{Q}}(e)/M_2(F)$ has ℓ^{∞} as a quotient (and even contains ℓ^{∞}).

2. Our result expresses the fact that $\mathcal{Q}/W_{\mathcal{Q}}(x)$ has ℓ^{∞} as a continuous linear image. In this connection R. Haydon has constructed in [Ha] a Banach space $C(K)$, K compact Hausdorff, which has ℓ^{∞} as a quotient, does not contain an isomorph of ℓ^{∞} and yet has the Grothendieck property (i.e. w^* convergent sequences in $C(K)^*$ converge weakly). See also Talagrand [Ta].

3. In many cases, even the fact that $\mathcal{Q} \sim M_p(F) \neq \phi$ i.e. $\mathcal{Q}/M_p(F) \neq \{0\}$ is new and thus the fact that $\mathcal{Q}/M_p(F)$ is big, is an additional bonus.

4. If $p \neq 2$ then $PM_p(G)$ is a strikingly different space from $PM_2(G)$. For example any $f \in L^{\infty}(\hat{G})$ such that $|f(x)| = 1$ a.e. will generate an isometry $Sf : L^2(G) \rightarrow L^2(G)$ with $Sf \in PM_2(G)$, via Plancherel, by $f(g)(x) = f(x)g(x)$, for $g \in L^2(\hat{G})$. In contrast, if $p \neq 2$ then the only isometric operators $S : L^p(G) \rightarrow L^p(G)$ with $S \in PM_p(G)$ (for arbitrary G) are the operators $\alpha(\lambda_p \delta_x)$ with $x \in G$, $\alpha \in \mathcal{C}$ and $|\alpha| = 1$. This result of S.K. Parrot and R.S. Strichartz has been given a beautiful short proof by P. Eymard [Ey] p. 367-10.

For the rest of the paper we will discuss corollaries of the main theorem. It will yield results which are new even for $G = R$ or T and $p = 2$.

We need some more notations. Let $PF_p(G) = \text{norm cl } \lambda_p(L^1(G))$, $\mathcal{C}_p(G) = \{\phi \in \text{CONV}_p(G); \phi\phi' \in PF_p(G) \text{ for all } \phi' \in PF_p(G)\}$. By J. Delaporte [De] thm. 2.1 if we let $PM_{pc}(G) = (PM_p(G))_c$ then $PM_{pc}(G) \subset \mathcal{C}_p(G) \subset PM_p(G)$ and $\mathcal{C}_p(G)$ is a Banach subalgebra and $A_p(G)$ submodule of $PM_p(G)$.

If G is abelian and $\mathcal{F} : L^1(\hat{G}) \rightarrow A_2(G) = A(G)$ is Fourier transform then $\mathcal{F}^*(PM_2(G)) = L^\infty(\hat{G})$, $\mathcal{F}^*(\mathcal{C}_2(G)) = C(\hat{G})$ and $\mathcal{F}^*(PM_{2c}(G)) = UC(\hat{G})$ where $[UC(\hat{G})]C(\hat{G})$ are the bounded [uniformly] continuous functions on \hat{G} , respectively. Clearly, if we take $Q = \mathcal{C}_2(G)$, $P = PM_2(G)$ in theorem M then $P_c \neq Q \neq P$ if \hat{G} is not compact or discrete. This is the reason for the A_p submodule Q appearing in theorem M.

If $F \subset G$ is closed denote by $\mathcal{C}_p(F) = \mathcal{C}_p(G) \cap PM_p(F)$, $\mathcal{C}_{p^*}(F) = \mathcal{C}_p(G) \cap PM_{p^*}(F)$, $PM_{pc}(F) = PM_p(F) \cap PM_{pc}(G)$, $PM_{p^*c}(F) = PM_{p^*}(F) \cap PM_{pc}(G)$. It so happens that $(PM_p(F))_c \subset PM_{pc}(F)$ and $(PM_{p^*}(F))_c \subset PM_{p^*c}(F)$ with equality if G is amenable. Furthermore for all closed $F \subset G$, $(PM_p(F))_c \subset PM_{pc}(F) \subset \mathcal{C}_p(F) \subset PM_p(F)$ and $(PM_{p^*}(F))_c \subset PM_{p^*c}(F) \subset \mathcal{C}_{p^*}(F) \subset PM_{p^*}(F)$ and all the above spaces are norm closed A_p submodules of $PM_p(G)$.

Corollary 1: Let G be any second countable group, $H \subset G$ a closed nondiscrete subgroup, $a, b \in G$ and $F \subset G$ closed. let Q be any of the eight spaces $(PM_{p^*}(F))_c \subset PM_{p^*c}(F) \subset \mathcal{C}_{p^*}(F) \subset PM_{p^*}(F)$, $(PM_p(F))_c \subset PM_{pc}(F) \subset \mathcal{C}_p(F) \subset PM_p(F)$.

Then for any $x \in \text{int}_{aHb}(F)$, $Q/W_Q(x)$ (a fortiori $Q/M_p(F)$ if such a, b, H, x exist) have ℓ^∞ as a quotient and $TIM_Q(x)$ contains F .

The reader should note that $Q = (PM_{p^*}(F))_c$ is included in all other of the above Q . Thus $(PM_{p^*}(F))_c/M_p(F)$ having ℓ^∞ as a quotient directly implies the same for $L/M_p(F)$, for any normed space $L \supset (PM_{p^*}(F))_c$, a fortiori for all Q . This is not the case with $Q/W_Q(x)$. $Q/W_Q(x)$ having ℓ^∞ as a quotient for one Q does not imply the same for all the other Q .

What can one say about sets F such that $\text{int}_{aHb}(F) = \emptyset$ for all $a, b \in G$ and closed nondiscrete subgroups $H \subset G$? If $G = T$ and $F \subset T$ is any compact countable subset then by Loomis' lemma $CONV_2(F) = M_2(F)$ (a norm separable space). Furthermore, if $F \subset T$ is any perfect Helson S -set then $CONV_2(F) = \lambda_2(M(F)) = M_2(F)$. Clearly $\text{int}_T(F) = \emptyset$ for such F (note that T is the only closed nondiscrete subgroup of T).

There are however perfect sets $F \subset T$ with $\text{int}(F) = \emptyset$ such that $CONV_2(F) \neq M_2(F)$. The Cantor middle third set imbedded in T (or any perfect symmetric subset $F \subset T$, [Gmc] p. 88) satisfies this, by a result of G.S. Woodward [Wol]. Does, for such F ,

$CONV_2(F)/M_2(F)$ have ℓ^∞ as a quotient? We do not know. However, there is a chance for this to occur, due to the fact that for such F and second countable amenable G , $CONV_p(F)$ has ℓ^∞ as a quotient. This fact and substantially more is proved in

Theorem 2: Let G be second countable and amenable. Let P be a w^* closed $A_p(G)$ submodule of $PM_p(G)$. If $\sigma(P)$ contains some compact perfect set then P has ℓ^∞ as a quotient and does not have the *WRNP*.

As known, the dual X^* of the Banach space X is said to have the *WRNP* iff X does not contain an isomorph of ℓ^1 .

Theorem 2 is an improvement, for second countable G , of a result of F. Piquard-Lust, who has shown that if $1 \leq p \leq 2$ and G is abelian and $F \subset G$ is compact and not scattered then $CONV_p(F)$ does not have the *RNP* (see [P2]). Theorem 2 improves this result even for $p = 2$ and G abelian and second countable. This is the case since by a result of R.C. James and J. Lindenstrauss there exist separable Banach spaces X such that X^* has the *WRNP* yet does not have the *RNP* (see [Ro] p. 806). We also note that theorem 2 cannot be much improved since $PM_2(T)$ is isometric to ℓ^∞ .

The next result concerns β closed $A_p(G)$ submodules $Q \subset \mathcal{C}_p(G)$. These have been defined and studied by Jacques Delaporte in his beautiful thesis [De].

Let β be the topology on $PM_p(G)$ defined by: $\phi_\alpha \rightarrow \phi$ in β iff $\|(\phi_\alpha - \phi)\eta\| \rightarrow 0$ for all $\eta \in PF_p(G)$. If $p = 2$ and G is abelian then β on $PM_2(G)$ corresponds via \mathcal{F}^* , where $\mathcal{F} : L^1(\hat{G}) \rightarrow A_2(G) = A(G)$ is Fourier transform, to the strict Buck topology on $L^\infty(\hat{G})$, namely, $f_\alpha \rightarrow 0$ iff $\|f_\alpha g\|_\infty \rightarrow 0$ for all $g \in C_0(\hat{G}) = \{f \in C(\hat{G}), f(x) \rightarrow 0 \text{ at } \infty\}$. Our main theorem together with results of Delaporte [De] yield

Theorem 3: Let G be second countable and amenable, $H \subset G$ a nondiscrete closed subgroup, $a, b \in G$. Let $Q \subset \mathcal{C}_p(G)$ be a β closed $A_p(G)$ submodule and denote $\sigma(Q) = F$. Then for any $x \in \text{int}_{aHb}(F)$, $Q/W_Q(x)$ (a fortiori $Q_c/M_p(F)$ if such a, b, x, H exist) has ℓ^∞ as a quotient and $TIM_Q(x)$ contains F .

If G is amenable, it can be shown that, for $\phi \in \mathcal{C}_p(G)$, $Q_\beta(\phi) = \beta \text{ cl}(A_p(G) \cdot \phi)$ is a β closed A_p module to which this theorem applies. In this case $\sigma(Q_\beta(\phi)) = \text{supp } \phi$. If $P(\phi) = w^* \text{ cl}(A_p(G) \cdot \phi)$ it follows from results of Delaporte in [De] that $Q_\beta(\phi) = P(\phi) \cap \mathcal{C}_p(G)$

thus $(P(\phi))_c \subset Q_\beta(\phi) \subset P(\phi)$ and the main theorem can be applied. If $\phi = \lambda_p \mu$ where $\mu \in M(G)$ with $F = \text{supp } \mu$ (supp μ as a measure), it can be seen that $P(\lambda_p \mu) = PM_{p_0}(F)$ and $Q_\beta(\lambda_p \mu) = \mathcal{E}_{p_0}(F)$ and the result follows from Corollary 1.

We will need to consider in the sequel $W_p(G)$ submodules of $PM_p(G)$ where $W_p(G) = PF_p(G)^*$. $W_p(G)$ is a function algebra on G and $A_p(G) \subset W_p(G)$ is an ideal. It is hard to identify functions in $W_p(G)$. $W_2(G) = PF_2(G)^*$ (the dual of a C^* algebra) is better known. In fact $W_2(G) = \text{lin } W_2^+(G)$ where W_2^+ are the continuous positive definite functions on G associated with the left regular representation. We will embed $W_2(G)$ in $W_p(G)$ if G is amenable (or $p = 2$ and G is arbitrary) and then $V_2^+ = V_2^+(G) = \text{ext}\{w \in W_2^+(G); w(e) = 1\}$ will play an important role (where ext denotes extreme points). We will consider a subspace $D_{\mathcal{F}}(a)$ of $E_{\mathcal{F}}(a)$ which involves V_2^+ and is more tractable than $E_{\mathcal{F}}(a)$.

If G is amenable, a powerful result of C. Herz [Hz3] allows one to imbed $A_2(G)$ in $A_p(G)$ by function identification, with decrease in norm, and then $PM_p(G)$ can be considered as a linear subspace of $PM_2(G)$ with $\|\phi\|_{PM_2} \leq \|\phi\|_{PM_p}$ if $\phi \in PM_p$. It follows then that $W_2(G) \subset W_p(G)$ (and clearly $\ell_a W_2 \subset W_2$ if $a \in G$, where $(\ell_a f)(x) = f(ax)$, for all $x \in G$ and $f \in C(G)$).

Definition: If $\mathcal{P} \subset PM_p(G)$ is a linear subspace, let $D_{\mathcal{P}}(a) = \text{norm cl lin } \{\phi - v \cdot \phi; \phi \in \mathcal{P}, v \in \ell_{a^{-1}} V_2^+\}$. If $a \in \sigma(\mathcal{P})$ and $D_{\mathcal{P}}(a) \subset \mathcal{P}$ let $IM_{\mathcal{P}}(a) = \{\psi \in \mathcal{P}^*; 1 = \|\psi\| = \psi(\lambda_p \delta_a), \psi(D_{\mathcal{P}}(a)) = 0\}$, the invariant means on \mathcal{P} at a .

The reader will find it instructive to see after the next lemma the identification of $D_{\mathcal{F}}(a)$ for abelian G and $p = 2$. With the above definition in mind one can show

Lemma 4: Let $\mathcal{P} \subset PM_p(G)$ be a norm closed $W_p(G)$ submodule and $a \in \sigma(\mathcal{P})$. If G is amenable or $p = 2$ then $D_{\mathcal{F}}(a) \subset E_{\mathcal{F}}(a)$ and $TIM_{\mathcal{P}}(a) \subset IM_{\mathcal{P}}(a)$. If in addition $\mathcal{P} \subset PM_{pc}(G)$ then $D_{\mathcal{F}}(a) = E_{\mathcal{F}}(a)$ and $IM_{\mathcal{P}}(a) = TIM_{\mathcal{P}}(a)$.

We also note that any norm closed $A_p(G)$ module $\mathcal{P} \subset PM_p(G)$ is a $W_p(G)$ module if $\mathcal{P} \subset PM_{pc}(G)$ or if \mathcal{P} is w^* closed. There exist norm closed $W_p(G)$ submodules of $\mathcal{E}_p(G)$ for which $D_{\mathcal{F}}(a) \neq E_{\mathcal{F}}(a)$ as will be presently shown for abelian G and $p = 2$.

Let G be abelian, $p = 2$, and $\mathcal{F} : L^1(\hat{G}) \rightarrow A_2(G) = A(G)$ denote Fourier transform. Then $\mathcal{F}^*(PM_2(G)) = L^\infty(\hat{G})$, $\mathcal{F}^*(\mathcal{C}_2(G)) = C(G)$ and $\mathcal{F}^*(PM_{2c}(G)) = UC(\hat{G})$. Furthermore, for any norm closed $A_p(G)$ submodule $P \subset PM_2(G)$ and $a \in G$ $\mathcal{F}^*(E_P(a)) = \text{norm cl lin } \{h - (\bar{a}f) * h; 0 \leq f \in L^1(\hat{G}), \int f(\chi) d\chi = 1, h \in \mathcal{F}^*P\}$ and $\mathcal{F}^*(D_P(a)) = \text{norm cl lin } \{h - a(\chi)\ell_\chi; \chi \in \hat{G}, h \in \mathcal{F}^*P\}$ if P is in addition a $W_2(G)$ module. Thus $\mathcal{F}^*(D_P(a)) \subset \mathcal{F}^*(E_P(a))$ with equality if $P \subset PM_{2c}(g)$. (If $a = e$ this last fact is known to "amenabilists" under the label: "Any invariant mean on $UC(G)$ is topologically invariant," see [Pa].) If we take $P = \mathcal{C}_2(G)$ then by a result of Liu and Van Rooij [LR] there exist invariant means on $C(\hat{G})$ which are not topologically invariant i.e. $\mathcal{F}^*(E_P(e)) \neq \mathcal{F}^*(D_P(a))$, if \hat{G} is not discrete.

If $Q \subset PM_p(G)$ denote by $V_Q(a) = \mathcal{C}(\lambda_p \delta_a) + D_Q(a)$. If $p = 2$ or G is amenable and Q is a norm closed W_p module then the sum is direct and $V_Q(a)$ is a closed subspace of $W_Q(a)$.

Corollary 5: Let G be second countable, $H \subset G$ a nondiscrete closed subgroup $a, b \in G$, $F \subset G$ closed and $x \in \text{int}_{aHb}(F)$.

Let P be any one of $(PM_{p^*}(F))_c$, $PM_{p^*c}(F)$, $PM_{p^*}(F)$ or $(PM_p(F))_c$, $PM_{pc}(F)$, $PM_p(F)$ and Q any one of $\mathcal{C}_{p^*}(F)$ or $\mathcal{C}_p(F)$. Then

(a) $P/W_P(x)$ and $Q/W_Q(x)$ have ℓ^∞ as a quotient and $TIM_P(x)$ and $TIM_Q(x)$ contain F .

(b) If $p = 2$ or G is amenable then $P/V_P(x)$ also has ℓ^∞ as a quotient and $IM_P(x)$ also contains F . If in addition Q is a W_p module (if G is abelian and $p = 2$ Q is such) $Q/V_Q(x)$ has ℓ^∞ as a quotient and $IM_Q(x)$ contains F .

We note that Corollary 5(a) holds for any w^* [norm] closed $A_p(G)$ module $P[Q]$ with $\sigma(P) = F$ and $P_c \subset Q \subset P$. And Corollary 5(b) holds if G is amenable and Q is in addition a $W_p(G)$ module, which is the case if $Q = Q_\beta(\phi)$ for $\phi \in PM_p(G)$ with compact support (or if F is compact).

Are there any (many) elements which belong simultaneously for all $1 < p < \infty$, to $PM_p(F) \sim \lambda_p(M(F))$? Theorem 5.7 of Edwards and Price [EP] implies that if $F = \text{cl } U$ is compact and U is open in G and admits a URS sequence then there are norm separably many such elements.

We are able to improve this result in that F may be much thinner and no URS sequences are needed. The price we pay is that G needs to be second countable and amenable and the range of p is bounded.

If G is amenable then Herz's theorem C [Hz3] allows one to conclude that $PM_p(G) \subset PM_q(G)$ if $1 < p \leq q \leq 2$ or $2 \leq q \leq p < \infty$, with contraction of norms. Furthermore, if $\phi \in PM_p$ then by the definition of support, $\text{supp } \phi$ in $PM_p(G)$ is the same as $\text{supp } \phi$ in $PM_q(G)$. Thus $PM_p(F) \subset PM_q(F)$ if p, q are as above. Consider in the sequel $PM_p(G)$ for all $1 < p < \infty$ to be a subset of $PM_2(G)$. The above observations will yield now

Corollary 6: Let G be amenable and second countable, $a, b \in G$, $H \subset G$ a closed nondiscrete subgroup. Let $p \leq 2$ [$p \geq 2$] and $P \subset PM_p(G)$ be a w^* closed A_p submodule with $\sigma(P) = F$. If $\text{int}_{aHb}(F) \neq \phi$ then $P_c \sim M_p(F) \subset \{\cap(PM_q(F) \sim \lambda_q(M(F))); p \leq q \leq 2$ [$2 \geq q \geq p$]) and all above sets have the cardinality c .

Proof: Our main theorem implies that $\text{card}(P_c \sim M_p(F)) = c$ since $\text{card } PM_p(G) = c = \text{card } P/M_p(F)$ (since $P/M_p(F)$ has ℓ^∞ as a quotient). But for any p, q $\lambda_p(M(F)) = \lambda_q(M(F))$. Hence $P_c \sim M_p(F) \subset PM_p(F) \sim \lambda_p(M(F)) \subset \{\cap(PM_q(F) \sim \lambda_q(M(F))); p \leq q \leq 2, [2 \geq q \geq p]\}$.

Note that even $P_c \sim (\mathcal{C}(\lambda_p \delta_x) + E_{P_c}(x)) \subset P_c \sim M_p(F)$ if $x \in \text{int}_{aHb}(F)$. ■

The proof of the main theorem is based on the following result of this author (see [Gr] p. 158):

Theorem: Let X be a Banach space, $s_n : X \rightarrow X$ a sequence of operators $K \subset X$ be bounded convex and consider $X \subset X^{**}$. Let $A_0 = \{y \in w^* \text{cl } K; s_n^{**} y = 0, (y, x_n^*) = \alpha_n \text{ for all } n \geq 1\}$ where $x_n^* \in X^*$, $\alpha_n \in \mathcal{C}$.

If $A_0 \subset \{w^* \text{cl } K\} \sim \{w^* \text{seq cl } K\}$ then there exists an onto operator $t : X^* \rightarrow \ell^\infty$ such that $t\{\cup_{n=1}^\infty s_n^* X^* \cup \{x_n^*\}_1^\infty \cup K^0\} \subset c$ (where $K^0 = \{x^* \in X^*; (x^*, K) = 0\}$) and $c = \{a = (a_n) \in \ell^\infty; \lim a_n \text{ exists}\}$ and such that $t^* : \ell^{\infty*} \rightarrow X^{**}$ is a w^* - w^* continuous norm isomorphism into, such that $t^*(F) \subset A_0$.

II. THE CASE THAT G IS ABELIAN. All through this section G will be abelian. Herz's theorem [Hz3] or in fact the Riesz-Thorin theorem implies that $A_2(G) \subset A_p(G)$ and

if $u \in A_2(G)$ $\|u\|_{A_p} \leq \|u\|_{A_2}$ for $u \in A_2(G)$ and by duality $PM_p(G) \subset PM_2(G)$ for $1 < p < \infty$, with $\|\phi\|_{PM_2} \leq \|\phi\|_{PM_p}$ for $\phi \in PM_p(G)$.

Let $\mathcal{F} : L^1(\hat{G}) \rightarrow A_2(G)$ denote Fourier transform. Then $\mathcal{F}^* : PM_2(G) \rightarrow L^\infty(\hat{G})$ a w^* - w^* bicontinuous isometry onto. Denote $\mathcal{L}_p^\infty(\hat{G}) = \mathcal{F}^*(PM_p(G))$, a pointwise subalgebra of $L^\infty(\hat{G})$. Equip $\mathcal{L}_p^\infty(\hat{G})$ with the norm induced from $PM_p(G)$. Thus, if $\mathcal{F}^*(\phi) = f$ with $\phi \in PM_p(\hat{G})$ then $\|f\| = \|\phi\|_{PM_p}$.

$\mathcal{L}_p^\infty(\hat{G})$ with pointwise multiplication (a subalgebra of $L^\infty(\hat{G})$) equipped with the PM_p induced norm, is a commutative Banach algebra and $M(\hat{G})$ module thus $M(\hat{G}) * \mathcal{L}_p^\infty(\hat{G}) \subset \mathcal{L}_p^\infty(\hat{G})$. (Since if $\mathcal{F}_s : M(\hat{G}) \rightarrow W_2(G)$ denotes Fourier Stieljes transform (see [Ru1]) and $\nu \in M(\hat{G})$ and $\phi \in PM_p(G)$ then $\mathcal{F}^*[(\mathcal{F}_s \nu) \cdot \phi] = \nu^\vee * \mathcal{F}^* \phi$. Clearly $\mathcal{L}_2^\infty(\hat{G}) = L^\infty(\hat{G})$.)

We stress that \mathcal{F}^* stands for \mathcal{F}^* restricted to $PM_p(G)$ for some fixed $1 < p < \infty$ and is thus an isometry from $PM_p(G)$ onto $\mathcal{L}_p^\infty(\hat{G})$.

If $p \neq 2$, functions $f \in \mathcal{L}_p^\infty(\hat{G})$ are very difficult to characterize. Necessary and sufficient conditions for some $\psi \in \ell^\infty(Z)$ to belong to $\mathcal{L}_p^\infty(Z)$ seem to be unknown. Perhaps the closest to such, is a powerful theorem of E. Stein ([St] or [La] p. 131).

Definition: (a) If $F \subset G$ is closed let $\mathcal{UC}_p(\hat{G}, F) = \mathcal{F}^*(PM_{pc}(F))$, $\mathcal{C}_p(\hat{G}, F) = \mathcal{F}^*(\mathcal{C}_p(F))$, $\mathcal{L}_p^\infty(\hat{G}, F) = \mathcal{F}^*(PM_p(F))$, $\mathcal{UC}_{p*}(\hat{G}, F) = \mathcal{F}^*(PM_{p*c}(F))$, $\mathcal{C}_{p*}(F) = \mathcal{F}^*(\mathcal{C}_{p*}(F))$, $\mathcal{L}_{p*}^\infty(\hat{G}, F) = \mathcal{F}^*(PM_{p*}(F))$, and $\mathcal{B}_p(\hat{G}, F) = \mathcal{F}^*(M_p(F))$. (Since G is abelian, $PM_{pc}(F) = (PM_p(F))_c$ and $PM_{p*c}(F) = (PM_{p*}(F))_c$.) If $F = G$ then omit F and write $\mathcal{UC}_p(\hat{G})$ instead of $\mathcal{UC}_p(\hat{G}, G)$ et al. for the other spaces. Denote by $\mathcal{C}_p^0(\hat{G}) = \mathcal{F}^*(PF_p(G))$ (thus $\mathcal{C}_2^0(\hat{G}) = \mathcal{C}_0(\hat{G})$).

(a) Define the (strict) β topology on $\mathcal{L}_p^\infty(\hat{G})$ by: $f_\alpha \rightarrow f$ iff $\|(f_\alpha - f)g\| \rightarrow \infty$ for all $g \in \mathcal{C}_p^0(\hat{G})$. Clearly, the isometry $\mathcal{F}^* : PM_p(G) \rightarrow \mathcal{L}_p^\infty(\hat{G})$ is also a β - β homeomorphism.

It so happens that \mathcal{P} is a norm closed $A_p(G)[W_p(G)]$ submodule of $PM_p(G)$ iff $\mathcal{P} = \mathcal{F}^*(\mathcal{P})$ is a $L^1(\hat{G})$ $[M(\hat{G})]$ norm closed submodule of $\mathcal{L}_p^\infty(\hat{G})$.

(c) If $\mathcal{P} \subset \mathcal{L}_p^\infty(\hat{G})$ and $a \in G$ and $(\bar{a}f)(\chi) = \overline{a(\chi)}f(\chi)$ for $f \in L^1(\hat{G})$, and $(\ell_\chi h)(\chi') = h(\chi\chi')$ for $\chi, \chi' \in \hat{G}$, $h \in L^\infty(\hat{G})$, let

$$\mathcal{E}_\mathcal{P}(a) = \text{norm cl lin}\{h - (\bar{a}f) * h; h \in \mathcal{P}, 0 \leq f \in L^1(\hat{G}), \int f(\chi) d\chi = 1\}$$

$$\mathcal{D}_\mathcal{P}(a) = \text{norm cl lin}\{h - a(\chi)\ell_\chi h; h \in \mathcal{P}, \chi \in \hat{G}\}$$

Proposition 1 (i) Let P be a norm closed $A_p(G)$ [$W_p(G)$] submodule of $PM_p(G)$ and $\mathcal{P} = \mathcal{F}^*(P)$. Then $\mathcal{F}^*(E_P(a)) = E_{\mathcal{P}}(a)$ [$\mathcal{F}^*(D_P(a)) = D_{\mathcal{P}}(a)$, thus $D_{\mathcal{P}}(a) \subset E_{\mathcal{P}}(a)$ in this case], for any $a \in G$.

(ii) If $P \subset PM_{pc}(G)$ is a norm closed $A_p(G)$ (hence $W_p(G)$) module and $\mathcal{P} = \mathcal{F}^*(P)$ then $E_{\mathcal{P}}(a) = D_{\mathcal{P}}(a)$.

Definition: Let $\mathcal{P} \subset \mathcal{L}_p^\infty(\hat{G})$ be a norm closed $L^1(\hat{G})$ [$M(\hat{G})$] module. If $a \in G$ is such that $\bar{a} \in \mathcal{P}$ let, respectively, $IM_{\mathcal{P}}(a)$ [$TIM_{\mathcal{P}}(a)$] = $\{\psi \in \mathcal{P}^*; \psi(\bar{a}) = 1 = \|\psi\|, \psi(D_{\mathcal{P}}(a)) = 0$ [$\psi(E_{\mathcal{P}}(a)) = 0$]].

Remarks: (1) Note that $\psi(D_{\mathcal{P}}(a)) = 0$ [$\psi(E_{\mathcal{P}}(a)) = 0$] is equivalent to $\psi(\ell_\chi h) = \overline{a(\chi)}\psi(h)$ [$\psi((\bar{a}f) * h) = \psi(h)$] for all $h \in \mathcal{P}$ and $\chi \in \hat{G}$ [for all $0 \leq f \in L^1(\hat{G})$, $\int f d\chi = 1$].

(2) If $\{P \subset PM_{pc}(G)\}$, $P \subset PM_p(G)$ is a norm closed [$A_p(G)$] $W_p(G)$ module and $\mathcal{P} = \mathcal{F}^*(P)$, then [$TIM_{\mathcal{P}}(a) = IM_{\mathcal{P}}(a)$], $TIM_{\mathcal{P}}(a) \subset IM_{\mathcal{P}}(a)$.

Definition: If $h \in \mathcal{L}_p^\infty(\hat{G})$ let $\Sigma(h) = G \cap \beta \text{cl}(L^1(\hat{G}) * h)$. It so happens that if $h = \mathcal{F}^* \phi$ then $\text{supp } \phi = \Sigma(\bar{h})$ and if $\phi \in PM_{pc}(G)$ then $\Sigma(\bar{h}) = G \cap \beta \text{cllin}\{\ell_\chi \bar{h}; \chi \in \hat{G}\}$. Also, $\mathcal{F}^*(\lambda_p \mu) = \mathcal{F}_s(\mu)$ for all $\mu \in M(G)$, hence, $\text{supp } \lambda_p \delta_x = \{x\} = \Sigma(\overline{\mathcal{F}_s \delta_x})$ is obvious. Furthermore, $B_p(\hat{G}, F) = \text{norm cl}\{\mathcal{F}_s \mu; \mu \in M(F)\} \subset \mathcal{L}_p^\infty(\hat{G})$, and $\mathcal{L}_p^\infty(\hat{G}, F) = \{h \in \mathcal{L}_p^\infty(\hat{G}); \Sigma(\bar{h}) \subset F\}$, if $F \subset G$ is closed.

If $Q \subset \mathcal{L}_p^\infty(\hat{G})$ let $W_Q(d) = \mathcal{C}\bar{d} + E_Q(d)$ and $V_Q(d) = \mathcal{C}\bar{d} + D_Q(d)$.

Theorem 3: Let G be abelian second countable, $H \subset G$ a closed nondiscrete subgroup $a \in G$. Let $P \subset PM_p(G)$ be a w^* closed A_p submodule, $\sigma(P) = F$ and $\mathcal{P} = \mathcal{F}^*(P)$, $\mathcal{P}_c = \mathcal{F}^*(P_c)$. Let Q be any norm closed $L^1(\hat{G})$ submodule of $\mathcal{L}_p^\infty(\hat{G})$ such that $\mathcal{P}_c \subset Q \subset \mathcal{P}$.

Then for any $d \in \text{int}_{aH}(F)$, $Q/W_Q(d)$ (a fortiori $Q/B_p(\hat{G}, F)$, if such a, H, d exist) has ℓ^∞ as a quotient and $TIM_Q(d)$ contains F .

If in additon Q is a $M(\hat{G})$ submodule ($Q = \mathcal{P}_c$ or \mathcal{P} are such) then $Q/V_Q(d)$ also has ℓ^∞ as a quotient and $IM_Q(d)$ contains F , if $d \in \text{int}_{aH}(F)$.

Corollary 4: Let G be abelian, second countable, $F \subset G$ closed, $a \in G$, $H \subset G$ a nondiscrete closed subgroup and Q be any one of $\mathcal{U}\mathcal{C}_p(\hat{G}, F)$, $\mathcal{L}_p^\infty(\hat{G}, F)$, $\mathcal{U}\mathcal{C}_{p^*}(\hat{G}, F)$, $\mathcal{L}_{p^*}^\infty(\hat{G}, F)$ or $\mathcal{C}_p(\hat{G}, F)$, $\mathcal{C}_{p^*}(\hat{G}, F)$. Let $\mathcal{P} \subset \mathcal{C}_p(\hat{G})$ be any β closed $L^1(\hat{G})$ submodule such that $F = G \cap \{\bar{h}; h \in \mathcal{P}\}$.

Then for any $d \in \text{int}_{aH}(F)$, $\mathcal{Q}/\mathcal{W}_{\mathcal{Q}}(d)$ and $\mathcal{P}/\mathcal{W}_{\mathcal{P}}(d)$ (a fortiori $\mathcal{Q}/\mathcal{B}_{\mathcal{P}}(\hat{G}, F)$ and $\mathcal{P}/\mathcal{B}_{\mathcal{P}}(\hat{G}, F)$ if such a, d, H exist) have ℓ^{∞} as a quotient and $\text{TIM}_{\mathcal{Q}}(d)$ and $\text{TIM}_{\mathcal{P}}(d)$ contain F .

If in addition $\mathcal{Q}[\mathcal{P}]$ is a $M(\hat{G})$ module (which is the case if $p = 2$ or if F is compact, or if \mathcal{Q} denotes only the first four spaces) then $\mathcal{Q}/\mathcal{V}_{\mathcal{Q}}(d)$ and $[\mathcal{P}/\mathcal{V}_{\mathcal{P}}(d)]$ also has ℓ^{∞} as a quotient and $[\text{IM}_{\mathcal{P}}(d)], \text{IM}_{\mathcal{Q}}(d)$ contain F .

Remarks: 1. If $f \in \mathcal{L}_p^{\infty}(\hat{G})$ let $\mathcal{Q}_{\beta}(f) = \beta \text{cl}(L^1(\hat{G}) * f)$. It so happens that if $h \in \mathcal{UC}_{\mathcal{P}}(\hat{G})$ then $\mathcal{Q}_{\beta}(\bar{h}) = \beta \text{cl lin}\{\ell_{\chi}\bar{h}; \chi \in \hat{G}\}$. If $F = \mathcal{Q}_{\beta}(\bar{h}) \cap G$ then $\mathcal{P} = \mathcal{Q}_{\beta}(h)$ satisfies the conditions of Corollary 4.

2. If $p = 2$, any β closed translation invariant linear subspace of $C(\hat{G}) = C_2(\hat{G})$ is a $M(\hat{G})$ submodule of $C(\hat{G})$.

3. Let $\mu \in M(G)$ and $\bar{h} = \mathcal{F}_s\mu = \mathcal{F}^*(\lambda_{\mathcal{P}}\mu)$. Then $\mathcal{Q}_n(h) = \text{norm cl lin}\{\ell_{\chi}h; \chi \in \hat{G}\} \subset \mathcal{B}_{\mathcal{P}}(\hat{G}, F)$ where $F = \text{supp } \lambda_{\mathcal{P}}\mu = \text{supp } \mu$ (supp μ as a measure). By Cor. 4, if $\text{int}_{aH}(F) \neq \emptyset$ for some $a \in G$ and nondiscrete closed subgroup $H \subset G$, $\mathcal{Q} = \mathcal{Q}_{\beta}(h) = \beta \text{cl lin}\{\ell_{\chi}h; \chi \in \hat{G}\}$ is so much bigger than $\mathcal{Q}_n(h)$ that $\mathcal{Q}_{\beta}(h)/\mathcal{B}_{\mathcal{P}}(\hat{G}, F)$ (and even $\mathcal{Q}_{\beta}(h)/\mathcal{W}_{\mathcal{Q}}(d)$ if $d \in \text{int}_{aH}(F)$) has ℓ^{∞} as a quotient. If however $p = 2$ and $F = \text{supp } \mu$ is a perfect compact Helson S set (see [Ka1], [Ru] (7.8.8) and p. 188) then $\mathcal{Q}_{\beta}(h) = \mathcal{L}_2^{\infty}(\hat{G}, F) = \mathcal{B}_2(\hat{G}, F) = \mathcal{F}_s(M(F)) = \mathcal{V}_{\mathcal{Q}}(d) = \mathcal{W}_{\mathcal{Q}}(d)$ for $d \in F$. Thus $\mathcal{Q}_{\beta}(h)$ is much smaller in this case.

The above corollary will yield results which are new even if $\hat{G} = Z$ or R .

(a) $G = T^n$ and the compact case. Reverse the roles of G and \hat{G} and let \hat{G} be compact. Denote then $\ell_p^{\infty}(G) = \mathcal{L}_p^{\infty}(G) = \mathcal{F}^*(PM_{\mathcal{P}}(\hat{G}))$ and if $F \subset \hat{G}$ is closed denote $\ell_p^{\infty}(G, F) = \mathcal{L}_p^{\infty}(G, F)$ and $\ell_{p^*}^{\infty}(G, F) = \mathcal{L}_{p^*}^{\infty}(G, F)$ where $\mathcal{F} : \ell^1(G) \rightarrow A_2(\hat{G}) = A(\hat{G})$ is Fourier transform. Denote $\ell^{\infty}(G, F) = \ell_2^{\infty}(G, F)$ and $\ell_*^{\infty}(G, F) = \ell_{2^*}^{\infty}(G, F)$. Since \hat{G} is compact, $PM_{\mathcal{P}c}(\hat{G}) = \mathcal{C}_{\mathcal{P}}(\hat{G}) = PM_{\mathcal{P}}(\hat{G})$. Thus for any $h \in \ell_p^{\infty}(G) \subset \ell^{\infty}(G)$ $\Sigma(h) = \hat{G} \cap \beta \text{cl lin}\{\ell_{\chi}h; \chi \in G\}$. And for any norm closed translation invariant (equivalently, $\ell^1(G)$ module) subspace $\mathcal{P} \subset \ell_p^{\infty}(G)$, $\mathcal{E}_{\mathcal{P}}(x) = \mathcal{D}_{\mathcal{P}}(x)$ and $\mathcal{V}_{\mathcal{P}}(x) = \mathcal{W}_{\mathcal{P}}(x)$.

Corollary 5. Let G be discrete, countable, $H \subset \hat{G}$ a nondiscrete closed subgroup, $\alpha \in \hat{G}$ and $F \subset \hat{G}$ closed. Let $\ell_{p\alpha}^\infty(G, F) \subset \mathcal{Q} \subset \ell_p^\infty(G, F)$ be any β closed translation invariant subspace.

Then for any $\chi \in \text{int}_{\alpha H}(F)$, $\mathcal{Q}/\mathcal{V}_\mathcal{Q}(\chi)$ (a fortiori $\mathcal{Q}/\mathcal{B}_p(G, F)$ if such α, χ, H exist) has ℓ^∞ as a quotient and $\text{TIM}_\mathcal{Q}(\chi) = \text{IM}_\mathcal{Q}(\chi)$ contains F .

Remarks: 1. In particular we can take $\mathcal{Q} = \ell_p^\infty(G, F)$ or $\ell_{p\alpha}^\infty(G, F)$ or $\mathcal{Q} = \mathcal{Q}_\beta(f) = \beta \text{cl lin}\{\ell_x f; x \in G\}$ where $f \in \ell_p^\infty(G)$ and $F = \Sigma(\bar{f})$. Or we can take $p = 2$ and $\mathcal{Q} = \ell^\infty(G, F)$ or $\ell_*^\infty(G, F) = w^* \text{cl lin}\{\chi; \bar{\chi} \in F\}$.

2. Note that $\mathcal{D}_\mathcal{Q}(\chi) = \text{norm cl lin}\{f - \chi(x)\ell_x f; f \in \mathcal{Q}, x \in G\} = \mathcal{E}_\mathcal{Q}(\chi)$, and $\mathcal{B}_p(G, F) = \text{norm cl } \mathcal{F}_s(M(F))$ (norm cl in $\ell_p^\infty(G)$ or in $\ell^\infty(G)$ if $p = 2$).

3. Corollary 6 is best possible in terms of the thinness of F , since any compact infinite group contains a perfect Helson S set F by Kahane and Salem [Ka1] and Rudin [Ru] (7.8.8). For such F , $\mathcal{Q} = \ell_2^\infty(G, F) = \ell_{2\alpha}^\infty(G, F) = \mathcal{F}_s(M(F)) = \mathcal{B}_2(G, F) = \mathcal{V}_\mathcal{Q}(\chi)$, for all $\chi \in F$. Hence $\text{TIM}_\mathcal{Q}(\chi) = \text{IM}_\mathcal{Q}(\chi)$ contains a unique element for all $\chi \in F$. Clearly $\text{int}_{\alpha H}(F) = \emptyset$ for all $\alpha \in \hat{G}$ and all non-discrete closed subgroups $H \subset \hat{G}$.

4. If $1 \in \text{int}_{\alpha H}(F)$ then $\mathcal{D}_\mathcal{Q}(1) = \text{norm cl}\{f - \ell_x f; f \in \mathcal{Q}; x \in G\}$ and $\text{IM}_\mathcal{Q}(1)$ is the set of honest-to-goodness invariant means on \mathcal{Q} . Thus $\text{IM}_\mathcal{Q}$ is big, namely it contains F . This result cannot be proved by the usual methods which use intersection properties of translates of subsets of G (see Paterson [Pa] Ch. 7) since \mathcal{Q} (which could be $\ell^\infty(G, F)$ or $\ell_*^\infty(G, F)$ if $p = 2$) need not be even an algebra of functions (take $G = \mathbb{Z}$ and $F \neq T$).

The following result for $G = \mathbb{Z}^n$ seems to be new even if $n = 1$ and $p = 2$:

Corollary 5': Let $G = \mathbb{Z}^n$, $H \subset T^n$ a closed nondiscrete subgroup, $F \subset T^n$ closed and $\alpha \in T^n$. Let $\ell_{p\alpha}^\infty(\mathbb{Z}^n, F) \subset \mathcal{Q} \subset \ell_p^\infty(\mathbb{Z}^n, F)$ be any β closed translation invariant linear subspace.

Then for any $\chi \in \text{int}_{\alpha H}(F)$, $\mathcal{Q}/\mathcal{V}_\mathcal{Q}(\chi)$ (a fortiori $\mathcal{Q}/\mathcal{B}_p(\mathbb{Z}^n, F)$ if such α, χ, H exist) has ℓ^∞ as a quotient and $\text{TIM}_\mathcal{Q}(\chi) = \text{IM}_\mathcal{Q}(\chi)$ contains F .

Remarks: 1. It has been proved by J.P. Kahane [Ka2] that if $n \geq 2$ there exists a continuous curve $F \subset T^n$ (and if $n \geq 3$ even a smooth curve, and if $n > 2k$ even a k dimensional manifold, see also O.C. McGehee [Mc]) which is a Helson set. Thus

$Q = \ell_{2\epsilon}^{\infty}(Z^n, F) = \mathcal{F}_s(M(F)) = B_2(Z^n, F) = \mathcal{V}_Q(\chi)$ for all $\chi \in F$, and $TIM_Q(\chi) = IM_Q(\chi)$ contains a unique element for all $\chi \in F$.

2. Let $n = 1$, $T = \{e^{i\theta}; -\pi < \theta \leq \pi\}$, $I_\epsilon = \{e^{i\theta}; -\epsilon \leq \theta \leq \epsilon\}$ for $0 < \epsilon < \pi$. Thus $\mathcal{V}_Q(e^{i\theta}) = \mathcal{C}e^{-i\theta} \oplus \mathcal{D}_Q(e^{i\theta})$ where $\mathcal{D}_Q(e^{i\theta}) = \text{norm cllin}\{f - e^{in\theta}\ell_n f; n \in Z, f \in Q\}$.

Recall that T has no proper closed subgroups. The following is new even for $p = 2$:

Corollary 5'' Let $F \subset T$ be closed, $\ell_p^{\infty}(Z, F) \subset Q \subset \ell_p^{\infty}(Z, F)$ a translation invariant β closed linear subspace.

Then for any $e^{i\theta} \in \text{int } F$, $Q/\mathcal{V}_Q(e^{i\theta})$ (a fortiori $Q/B_p(Z, F)$) if $\text{int } F \neq \emptyset$ has ℓ^{∞} as a quotient and $IM_Q(e^{i\theta})$ contains F .

Consequently, if $I_\epsilon \subset F$ (or $I_\epsilon = F$) for some $0 < \epsilon < \pi$ and $Q = \ell_p^{\infty}(Z, F)$ or $\ell_p^{\infty}(Z, F)$ then $Q/\mathcal{V}_Q(1)$ (a fortiori $Q/B_p(Z, F)$) has ℓ^{∞} as a quotient and $IM_Q(1)$ is big, namely contains F .

Remarks: 1. Note that $\ell_p^{\infty}(Z, I_\epsilon)$ is not an algebra if $0 < \epsilon < \pi$, and even the fact that for all $1 < p < \infty$, $\ell_p^{\infty}(Z, I_\epsilon)/B_p(Z, I_\epsilon)$ has ℓ^{∞} as a quotient seems to be new.

2. Let $F \subset T$ be closed with $\text{int } F = \emptyset$. Then $\ell^{\infty}(Z, F) = B_2(Z, F) = \mathcal{F}_s(M(F))$ if F is a Helson S-set. And yet, $\ell^{\infty}(Z, F) \neq B_2(Z, F)$ may also happen as is the case if $F \subset T$ is any perfect "symmetric" set. It has been shown by G.S. Woodward in [Wol] that any perfect symmetric set F (such as the Cantor middle third set imbedded in T , see [GMc] p. 88) satisfies that $Q = \ell^{\infty}(Z, F) \neq \mathcal{V}_Q(e^{i\theta}) \supset B_2(Z, F)$ for some $e^{i\theta} \in F$ (thus $\text{card } IM_Q(e^{i\theta}) > 1$). It is not clear to us if $\ell^{\infty}(Z, F)/\mathcal{V}_Q(e^{i\theta})$ has ℓ^{∞} as a quotient for this $e^{i\theta}$.

(b) **The case $G = R^n$:** Any straight line in R^n (R denotes the additive reals) is a coset of a nondiscrete closed subgroup. $I \subset R^n$ is an open interval if it is a nonvoid open subinterval of some straight line in R^n .

If $x, y \in R^n$ let $e_x(y) = e^{ix \cdot y}$, $\bar{e}_x = e_{-x}$. Thus $\widehat{R^n} = \{e_x; x \in R^n\}$. If $F \subset R^n$ let $e_F = \{e_x; x \in F\}$. If $Q \subset \mathcal{L}_p^{\infty}(R^n)$ let $\mathcal{V}_Q(e^x) = \mathcal{C}\bar{e}_x + \mathcal{D}_Q(e^x)$ and $\mathcal{W}_Q(e_x) = \mathcal{C}\bar{e}_x + \mathcal{E}_Q(e_x)$.

Corollary 7: Let $G = R^n$, $F \subset R^n$ closed. Let Q be any of: $\mathcal{L}_p^{\infty}(R^n, e_F)$, $\mathcal{UC}_p(R^n, e_F)$, $\mathcal{L}_{p^*}^{\infty}(R^n, e_F)$, $\mathcal{UC}_{p^*}(R^n, e_F)$ or $\mathcal{C}_p(R^n, e_F)$, $\mathcal{C}_{p^*}(R^n, e_F)$ and let \mathcal{P} be any of the first four spaces. Let \mathcal{R} be any β closed $L^1(R^n)$ submodule of $\mathcal{C}_p(R^n)$ with $F = \{x; \bar{e}_x \in \mathcal{R}\}$.

If F contains some open interval I then for any $x \in I$, $\mathcal{Q}/\mathcal{W}_{\mathcal{Q}}(e_x)$, $\mathcal{R}/\mathcal{W}_{\mathcal{R}}(e_x)$, and $\mathcal{P}/\mathcal{V}_{\mathcal{P}}(e_x)$ (a fortiori $\mathcal{S}/\mathcal{B}_{\mathcal{P}}(R^n, e_F)$ if $\mathcal{S} = \mathcal{P}$, \mathcal{R} or \mathcal{Q}) has ℓ^∞ as a quotient and $IM_{\mathcal{P}}(e_x)$, $TIM_{\mathcal{Q}}(e_x)$, and $TIM_{\mathcal{R}}(e_x)$ all contain F .

If $T = \mathcal{R}$ or $C_p(R^n, e_F)$ or $C_{p^*}(R^*, e_F)$ are in addition $M(R^n)$ modules (If F is compact or $p = 2$, T is such.) then $T/\mathcal{V}_T(e_x)$ also has ℓ^∞ as a quotient and $IM_T(e_x)$ also contains F .

Remarks: 1. Compact perfect Helson S subsets $F \subset R^n$ do not contain any open subintervals. They satisfy $\mathcal{Q} = \mathcal{L}_2^\infty(R^n, e_F) = \mathcal{V}_{\mathcal{Q}}(e_x) = \mathcal{B}_2(R^n, e_F) = \mathcal{F}_s(M(e_F))$ for all $x \in F$.

2. If $f \in UC_p(R^n)$ (or if $p = 2$ and $f \in C_2(R^n) = C(R^n)$) and $\mathcal{R} = \mathcal{Q}_\beta(f) = \beta \text{cl lin}\{\ell_x f; x \in R^n\}$ is such that $F = \{x \in R^n; e_x \in \mathcal{Q}_\beta(\bar{f})\}$ then \mathcal{R} satisfies the conditions of Corollary 7.

3. Note that $\mathcal{D}_{\mathcal{Q}}(e_x) = \text{norm cl lin}\{h - e_x(y)\ell_y h; y \in R^n, h \in \mathcal{Q}\}$.

A closed set $F \subset G$ is p -ergodic if for $\mathcal{Q} = PM_p(F)$, $PM_p(F) = \mathcal{C}(\lambda_p \delta_x) \oplus E_{\mathcal{Q}}(x)$ for all $x \in F$. 2-ergodic sets for abelian G have been studied by G.S. Woodward in [Wo 1,2,3]. It is clear from [Wo2] that if F contains an open set $U \subset G$ then for any $x \in U$, $PM_2(F) \neq \mathcal{C}(\lambda_2 \delta_x) \oplus E_{\mathcal{Q}}(x)$. It does not follow from [Wo2] that $PM_2(F)/\mathcal{C}(\lambda_2 \delta_x) \oplus E_{\mathcal{Q}}(x)$ has ℓ^∞ as a quotient even in this case ($p = 2$).

Important examples of perfect 2-ergodic sets $F \subset R$ whose every closed subset obeys synthesis and yet every "portion" of which is not a Helson set are the "Sigtuna" sets of I. Katznelson see [GMc] p. 394. (The closed set E is a "portion" of F if for some open interval I of R , $E = I \cap F$.)

Open questions: (a) Characterize those closed $F \subset R^n$ (or $F \subset G$ where G is abelian) for which $\mathcal{L}_{p^*c}^\infty(R^n, F)/\mathcal{B}_p(R^n, F)$ has ℓ^∞ as a quotient.

(b) A brilliant result of T. Korner as improved by Saeki [S] shows that every nondiscrete abelian G contains a compact Helson set F which disobeys synthesis. If $P = PM_2(F)$ does $P/M_2(F)$ (or $P/\mathcal{C}(\lambda_2 \delta_a) \oplus E_P(a)$ for some $a \in F$) have ℓ^∞ as a quotient?

The proofs of the above results will be published elsewhere.

References

- [BE] B.Brainerd - R.E. Edwards: Linear operators which commute with translations. Part 1. J. Australian Math. Soc. (1966) 289-327.
- [Ch1] Ching Chou: Weakly almost periodic functions and Fourier Stiltjes algebras of locally compact groups. Trans. Amer. Math Soc. 274 (1982) 141-157.
- [Ch2] Ching Chou: Topological invariant means on the Von Neumann algebra $VN(G)$. Trans. Amer. Math. Soc. 273 (1982) 207-229.
- [CF] M.E. Cowling - J.J.F. Fournier: Inclusions and noninclusions of spaces of convolution operators. Trans. Amer. Math. Soc. 221 (1976) 56-95.
- [De] J. Delaporte: Convoluteurs continus et topologie stricte. Thèse. Université Lausanne 1989.
- [EP] R.E. Edwards - J.F. Price: A naively constructive approach to boundedness principles with applications to harmonic analysis. L'Enseignement mathématique 16 (1970) 255-296.
- [Ey] P. Eymard: Algèbres A_p et convoluteurs de L^p . Seminaire Bourbaki 22e année, 1969/70, no. 367.
- [FG] A. Figa-Talamanca - G.I. Gaudry: Multipliers and sets of uniqueness of L^p . Michigan Math. J. 17 (1970) 179-191.
- [G,Mc] C.C. Graham - O. Caruth McGehee: Essays in commutative harmonic analysis. Springer-Verlag N.Y. 1979.
- [Gr] E.E. Granirer: Geometric and topological properties of certain w^* compact convex subsets of double duals of Banach spaces which arise from the study of invariant means. Illinois J. Math. 30 (1986) 148-174.
- [Ha] R. Haydon: A nonreflexive Grothendieck space which does not contain ℓ^∞ . Israel J. Math. 40 (1981) 65-73.
- [Hz1] C. Herz: Harmonic synthesis for subgroups. Ann. Inst. Fourier Grenoble 37 (1973) 91-123.
- [Hz2] C. Herz: Une généralisation de la notion de transformée de Fourier-Stiltjes. Ann. Inst. Fourier Grenoble 24 (1974) 145-157.
- [Hz3] C. Herz: The theory of p -spaces with an applicaton to convolution operators. Trans. A.M.S. 154 (1971) 69-82.
- [HR] E. Hewitt - K.A. Ross: Abstract Harmonic Analysis Vols. I,II. Springer-Verlag 1970.
- [Ka1] J.P. Kahane - R. Salem: Sur les ensembles linéaires ne portant pas de pseudomeasures. C.R. Acad. Sci. Paris 243 (1956) 1185-1187.
- [Ka2] J.P. Kahane: Sur les rearrangement de fonctions de la classe A. Studia Math. 31 (1968) 287-293.

- [Ka3] J.P. Kahane: *Séries de Fourier Absolument Convergentes*, Springer-Verlag 1970.
- [Ko] T.W. Körner: A pseudofunction on a Helson set. *Astérisque Math. France* 5 (1973) 3-224.
- [La] Ronald Larsen: *An introduction to the theory of multipliers*. Springer-Verlag, 1971.
- [LT] J. Lindenstrauss - L. Tzafriri: *Classical Banach Spaces, Vol. I*. Springer-Verlag
- [LR] T.S. Liu - A van Rooij: Invariant means on locally compact groups. *Monatsh. Math.* 78 (1974) 356-359.
- [Loh] N. Lohoué: Estimations L^p des coefficients des représentation et opérateurs de convolution. *Advances in Math.* 38 (1980) 178-221.
- [Mc] O. Carruth McGehee: Helson sets in T^n . *Conference on Harmonic Analysis*. College Park, Maryland 1971. Springer Lecture Notes, No. 266, 229-237.
- [Pa] A.L.T. Paterson: *Amenability*. Amer. Math. Soc. *Mathematical Surveys and Monographs* No. 29, 1988.
- [P2] F. Piquard-Lust: Means on $CV_p(G)$ - Subspaces of $CV_p(G)$ with RNP and Schur property. Preprint Orsay (1989).
- [Pr1] J.F. Price: Some strict inclusions between spaces of L^p multipliers. *Trans. Amer. Math. Soc.* 152 (1970) 321-330.
- [Ro] H.P. Rosenthal: Some recent discoveries in the isomorphic theory of Banach spaces. *Bull. A.M.S.* 84 (1978), 803-831.
- [Ru1] W. Rudin: *Fourier Analysis on Groups*. John Wiley and Sons 1960.
- [S] Sadahiro Saeki: Helson sets which disobey spectral synthesis. *Proc. Amer. Math. Soc.* 47 (1975) 371-377.
- [St] E. Stein: On the limits of sequences of operators. *Ann. of Math.* 74 (1961) 140-170.
- [T] M. Talagrand: Un nouveau $C(K)$ qui possède la propriété de Grothendieck. *Israel J. Math.* 37 (1980) 181-191.
- [Wo1] G.S. Woodward: Une classe d'ensembles épars. *C.R. Acad. Sci. Paris* 274 (1972) 221-223.
- [Wo2] G.S. Woodward: Invariant means and ergodic sets in Fourier analysis. *Pacific J. Math.* 54 (1974) 281-299.
- [Wo3] G.S. Woodward: The generalized almost periodic part of an ergodic function. *Studia Mathematica* 50 (1974) 103-116.

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ANNEAUX HENSELIENS ET CLOTURE SEPARABLE

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R E S U M E (N°2)

Nous présentons quelques résultats qui expriment des liens entre la théorie de Galois des anneaux commutatifs et les anneaux henséliens. Nous montrons à quelle condition la clôture séparable d'un anneau local est hensélienne puis à quelle condition on peut généraliser la construction de Nagata de l'hensélisé d'un anneau local. On exprime enfin l'hensélisé strict d'un anneau local avec sa clôture séparable.

I-INTRODUCTION

Les anneaux en question sont unitaires à élément unité et les homomorphismes d'anneaux sont unitaires. Soit A un tel anneau et p un idéal premier de A on note A_p le localisé de A en p . A étant un anneau local de corps résiduel $k = A/m$, on dit que A est hensélien si toute A -algèbre finie est décomposée (c.a.d. produit d'anneaux locaux). On appelle hensélisé d'un anneau local A un anneau local hA hensélien, hA dominant A et tel que tout homomorphisme local de A dans un anneau local hensélien B se factorise par hA ($u: A \rightarrow B$ et $i: A \rightarrow hA$ il existe $v: hA \rightarrow B$ tel que $u = v \circ i$).

Un anneau local intégralement clos A est hensélien si et seulement si toute extension entière intègre de A est locale. (L2)

II-POLYNOMES DE DECOMPOSITION

On peut avoir une autre approche des anneaux henséliens,

Soit $P(X) = a_0 + a_1X + \dots + X^n$ un polynôme unitaire $\in A[X]$. On dit que $P(X)$ est f.i. s'il vérifie : $P(0) = a_0 \in m$ l'idéal maximal de A et $P'(0) = a_1 \in m$, (où P' est la dérivée de P). Et alors L'anneau local A est hensélien si et seulement si tout polynôme f.i. de $A[X]$ admet une racine dans l'idéal maximal de A . Cette racine est alors l'unique racine relevant la racine nulle du polynôme réduit modulo m . (L)

Soit $f(X)$ un polynôme f.i. $\in A[X]$. Désignant par x_f la classe de X modulo $(f(X))$ on remarque que l'idéal $(m, x_f)A[x_f]$ de $A[x_f] = A[X]/(f(X))$ est maximal.

On a une expression explicite de cet hensélisé comme suit :

$${}^hA = \bigcup_{f \in I} A[x_f]_{((m, x_f)A[x_f])}$$

On peut améliorer ce résultat très connu en se limitant à certains polynômes f.i. à savoir ceux qui sont irréductibles dans l'ensemble des polynômes unitaires de $A[X]$.

II-1-Définition

Un polynôme unitaire de $A[X]$ est dit de décomposition s'il est f.i. et irréductible dans l'ensemble des polynômes unitaires de $A[X]$.

On peut voir à titre d'exemple que si $P(X) = a_0 + a_1X + X^2$ est un polynôme de décomposition $\in A[X]$ i.e. $P(X)$ irréductible $a_0 \in m$ et $a_1 \notin m$ il est séparable car :

$P(X) = X(\bar{a}_1 + X)$ les racines de P sont simples dans toute clôture de k ($k = A/m$) ; (0 et $-\bar{a}_1$ dans k).

II-2 -Lemme

Soit A un anneau local intégralement clos de corps des fractions K et (K'/K) une extension galoisienne finie. Soit A' la fermeture intégrale de A dans K' , q_1, \dots, q_r les idéaux maximaux de A' , $A^z(q_i)$ l'anneau de décomposition de q_i , a et b des éléments de $A^z(q_i)$, $a \in q_i$ et $a \notin q_j, i=2, \dots, j=r$, $b \in q_j, i=2, \dots, j=r$ et $b \notin q_1$ et $1 = a+b$, $P = \text{Irr}(a, A)$ $F = \text{Irr}(b, A)$ alors

il y a équivalence entre :

- i) Le polynôme $F(X)$ de $A[X]$ est séparable.
- ii) Le polynôme $P(X)$ de $A[X]$ est séparable.
- iii) $[K^Z(q_1) : K] = r$ est ≤ 2 .

De plus s'il en est ainsi $A^Z(q_1) = A[a]$. Ref[Lemme .III.2.1.(T)].

III- EXTENSIONS GALOISIENNES D'UN ANNEAU LOCAL HENSELIEN

Construction de Nagata de l'hensélisé d'un anneau local: Soit A un anneau local intégralement clos de corps des fractions K $L_S = \Omega_f(K)$ la clôture algébrique séparable de K (notation dans (T)) et Ω_S la fermeture intégrale de A dans L_S . Soit q_S un idéal maximal de Ω_S et $H_S = G^Z_A(q_S)$ le groupe de décomposition de q_S par rapport à A ($H_S = \{ \sigma \in \text{Gal}(L_S/K) / \sigma(q_S) = q_S \}$). L'anneau de décomposition de q_S par rapport à A est noté $A^Z(q_S)$ et c'est l'ensemble des éléments de Ω_S invariants par ceux de H_S :

$A^Z(q_S) = \Omega_S^{H_S}$ Alors l'hensélisé hA de A est isomorphe à $\Omega_S^{H_S}_{(q_S \cap \Omega_S^{H_S})}$ le localisé de $\Omega_S^{H_S}$ en l'idéal $(q_S \cap \Omega_S^{H_S})$. Pour un anneau

local quelconque B on a $B \cong A/a$ avec A local et intégralement clos et $hB \cong hA/a.h$ C'est la construction de Nagata (N) de l'hensélisé de A . Dans l'intention de généraliser cette construction on démontre :

III.1. Proposition

Soit A un anneau local et S une extension de A , localement fortement séparable et connexe. Alors les assertions suivantes sont équivalentes :

- i) A est hensélien.
- ii) S est local et hensélien.
- iii) $\Omega_f(A)$ est local et hensélien Ref[Corollaire .III.3.4 (T)].

Pour démontrer ce résultat on l'établit d'abord pour un anneau local intégralement clos puis par passage au quotient on le démontre dans le cas général.

III.2. Proposition

Soit A un anneau local intégralement clos, $\Omega_j(A)$ la clôture séparable de A , q un idéal maximal de $\Omega_j(A)$ et $H = H(q)$ le groupe de décomposition de q sur A . Les assertions suivantes sont équivalentes :

i) L'anneau $\Omega_j(A)_q$ est hensélien.

ii) L'hensélisé hA de A est isomorphe à $\Omega_j(A)^H / (q \cap \Omega_j(A)^H)$

iii) La fermeture intégrale de A dans hA est localement fortement séparable sur A [Proposition III.4.7.(T)].

III-8-Définition

Soit A un anneau local intégralement clos.

On dit que A est préhensélien s'il vérifie les propriétés équivalentes de la Proposition III.4.7.]

Exemple d'anneau Préhensélien :

Soit A un anneau local intégralement clos tel que tout polynôme de décomposition sur A est de degré ≤ 2 .

Alors A est préhensélien.

IV.1 HENSELISE STRICT D'UN ANNEAU LOCAL

On dit qu'un anneau local A est strictement hensélien s'il est hensélien et si son corps résiduel est séparablement clos (\mathbb{R}).

Exemple Soit A un anneau local intégralement clos. On va donner la construction de l'hensélisé strict de A qu'on note : ${}^{hs}A$. Soit K le corps des fractions de A . $L_S = \Omega_j(K)$ et Ω_S la fermeture intégrale de A dans L_S . Soit q_S un idéal maximal de Ω_S et $T_S = G_A^T(q_S)$ le groupe d'inertie de q_S par rapport à A . On a donc i) ${}^{hs}A \cong \Omega_S^{T_S} / (q_S \cap \Omega_S^{T_S})$, ii) si \mathfrak{a} est un idéal

de A : ${}^{hs}(A/\mathfrak{a}) = {}^{hs}A/\mathfrak{a}.{}^{hs}A$. [Lemme IV.1.2. (T)].

D'abord on exprime ${}^{hs}A$ dans le cas A un anneau local intégralement clos et hensélien.

IV.1. Proposition

Soit A un anneau local int gralement clos et hens lien. Alors la cl ture s parable de A, $\Omega_j(A)$, est l'hens lis  strict de A. c'est   dire qu'on a : $h_{SA} = \Omega_j(A)$. Ref[Proposition IV.1.3.(T)].

IV.2. Th or me

Soit A un anneau local hens lien. L'hens lis  strict de A est sa cl ture s parable $\Omega_j(A)$.

c.a.d. : $h_{SA} = \Omega_j(A)$. Ref[Th or me IV. 1.4.(T)].

IV.3. Corollaire .

Soit A un anneau local, h_A l'hens lis  de A et h_{SA} l'hens lis  strict de A. Soit $k = A/m$ le corps r sidual de A. Alors h_{SA} est une extension galoisienne de h_A de groupe de Galois G v rifiant : $G = \text{Gal}(h_{SA}/h_A) = \text{Gal}(\Omega_j(k)/k)$. Et de plus on a $h_{SA} = \Omega_j(h_A)$. Ref[Corollaire . IV.1.5. (T)].

IV. 4. Proposition

Soit A un anneau local int gralement clos pr hens lien ; et soit q un id al maximal de $\Omega_j(A)$. Alors l'hens lis  strict de A not  h_{SA} est isomorphe   $\Omega_j(A)_q$ Ref[Proposition IV. 2.1(T)].

Pour un anneau local quelconque :

IV.6. Th or me .IV.3.2.

Soit A un anneau local int gralement clos et soit q un id al maximal de $\Omega_j(A)$. On a $h_{SA} = \Omega_j(h_A) = h_{(\Omega_j(A)_q)} = h_A[\Omega_j(A)]$ [Th or me .IV.3.2.(T)].

IV. 7. Corollaire IV. 3.3.

Soit A un anneau local int gralement clos et hens lien. Soient K le corps des fractions de A et K' une extension galoisienne finie de K. Soit A' la fermeture int grale de A dans K'. Soit q' l'id al maximal de A'

et T le groupe, d'inertie de q' .

Alors l'anneau d'inertie de q' , A'^T , est une extension galoisienne de A . Ref[Corollaire IV. 3.3.(T)].

BIBLIOGRAPHIE

- (A.B) M. Auslander and D.Buchsbaum, On ramification theory in Noetherian rings, Amer.J. Math 81.(1959). 749-765.
- (B)1 - N. Bourbaki .Algèbre Commutative . Chapitre 1, Chapitre 2 Hermann
- (B)2 - N. Bourbaki .Algèbre Commutative . Chapitre 5, Chapitre 6. Hermann
- (B)3 -N. Bourbaki .Algèbre . Chapitre IV, Chapitre V Hermann
- (C.H) S.U.Chase , D.K. Harrison and A.Rosenberg .Galois Theory and Galois cohomology of commutative rings .Memoirs of Amer Math.Soc. 52.1965
- (I.D.M.) F.Demeyer , E.Ingraham . Separable Algebras over Commutative Rings . Lecture Notes . 181 .Springer Verlag .
- (J) G.J.Janusz .Separable Algebras over commutative Rings Trans.Amer. Math.Soc. 122.1966.461-479 .
- (L)J.P.Lafon .Anneaux Henséliens .Bull. Soc. Math de France .1963 ,77- 107
- (L)J.P.Lafon . Anneaux Henséliens Séminaire P. Dubreil et C; Pisol .Algèbre et théorie des nombres. Année 1956-57 Exposé n° 19.
- (T) Cloture séparable et hensélisé d'un anneau local : Thèse soutenue le 14-12 1990 à l' Université Paul Sabatier (U.P.Sciences) de Toulouse (N° d'ordre 765) par Samaali mohamed Salah

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WEIGHTED STOCHASTIC PROCESSES UNDER CONTIGUOUS ALTERNATIVES

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Abstract: We present optimal results for the asymptotic weighted sup-norm behaviour of partial sum processes based on independent observations, ranks and sequential ranks under the null assumption of observations being identically distributed, as well as under a class of contiguous alternatives.

1. Introduction. Let X_1, X_2, \dots be independent random variables. We wish to test the null hypothesis $H_0 : X_i, 1 \leq i \leq n$, have the same distribution F , versus the alternative hypothesis $H_1 : X_i, 1 \leq i \leq n$, have the respective distribution functions F_{in} , where we assume that all F_{in} are absolutely continuous with respect to the distribution function F and

$$(1.1) \quad \left[\frac{dF_{in}}{dF}(F^{-1}(u)) \right]^{1/2} = 1 + \frac{1}{2n^{1/2}} g_n(t, u), \quad \frac{i-1}{n} < t \leq \frac{i}{n},$$

where $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 < u \leq 1$, $F^{-1}(0) = F^{-1}(0^+)$. We assume also that there exists a function $g \in L^2[0, 1]^2$ such that

$$(1.2) \quad \int_0^1 g(t, u) du = 0 \quad \text{for almost all } t \in [0, 1]$$

and

$$(1.3) \quad \int_0^1 \int_0^1 \left[g_n(t, u) - g\left(\frac{[nt]}{n}, u\right) \right]^2 dudt \rightarrow 0, \quad n \rightarrow \infty.$$

It is well known that the sequence of direct products $F_{1n} \times \dots \times F_{nn}$, $n = 1, 2, \dots$ is contiguous (for the notion of contiguity see e.g. Le Cam, 1960, 1986; Greenwood and Shirayev 1985; Hájek and Šidak, 1967; Roussas, 1972) to the sequence $F \times \dots \times F$ (cf. Oosterhoff and Van Zwet, 1975).

Let Q^* be the class of functions q defined on $(0, 1)$ which are nondecreasing near zero and nonincreasing near one, and

$$(1.4) \quad \inf_{\delta \leq t \leq 1-\delta} q(t) > 0 \quad \text{for all } \delta \in (0, 1/2).$$

Define

$$I^*(q, c) = \int_0^1 (t(1-t))^{-1} \exp(-c(t(1-t))^{-1} q^2(t)) dt.$$

For discussions concerning weight functions q and the integral $I^*(q, c)$, we refer to Csörgő, Csörgő, Horváth and Mason (1986) and Csörgő, Shao and Szyszkowicz (1990).

Csörgő and Horváth (1988) studied processes

$$(1.5) \quad Z_k = \sum_{i=1}^k \sum_{j=k+1}^n h(X_i, X_j), \quad 1 \leq k < n,$$

where h is a symmetric or an antisymmetric kernel. Using $\|\cdot/q\|$ -metrics, they obtained results under the null hypothesis H_0 , while Szyszkowicz (1991) studied such processes under the class of contiguous alternatives H_1 . In both papers the condition $E|h(X_1, X_2)|^\nu < \infty$ for some $\nu > 2$ is assumed. Here we improve on these results by assuming two moments only.

The description of contiguous alternatives as given in (1.1)–(1.3) was used by Khmaladze and Parjanadze (1986). They consider the asymptotic behaviour of partial sums based on the normalized sequential ranks ξ_1, \dots, ξ_n , namely $n^{-1/2} \sum_{1 \leq i \leq k} a(\xi_i)$, $1 \leq k \leq n$, with

$$(1.6) \quad a \in L^2(0, 1), \text{ where } \int_0^1 a(u) du = 0 \text{ and } \int_0^1 a^2(u) du = 1.$$

Assuming some mild, appropriate conditions on a , they obtain the limiting distribution of this process under H_0 as well as under the class of contiguous alternatives of H_1 . Here we present "bridge" like generalizations of their results, using also weight functions.

Proofs, as well as further details, will appear in a forthcoming paper by the author.

2. Antisymmetric Kernel. In this section we assume that h is an antisymmetric kernel, i.e., $h(x, y) = -h(y, x)$. Given H_0 , X_1, \dots, X_n are i.i.d.r.v's and we have $Eh(X_1, X_2) = 0$. Let $\tilde{h}(t) = Eh(X_1, t)$ and we assume

$$(2.1) \quad Eh^2(X_1, X_2) < \infty \quad \text{and} \quad 0 < \sigma^2 = E\tilde{h}^2(X_1).$$

Here we consider $U_k = Z_k$, where Z_k is defined by (1.5). It is shown in Csörgő and Horváth (1988) that U_k , which itself is not a U -statistic, can be expressed as a linear combination of U -statistics. Hence the study of U_k can be based on the projection of U -statistics on the basic observations (cf. e.g., Serfling, 1980).

We will assume throughout, without loss of generality, that all random variables and stochastic processes introduced so far, and later on, are defined on the same probability space.

THEOREM 2.1. *We assume that H_0 holds, $h(x, y) = -h(y, x)$ and (2.1) is satisfied. Then we can define a sequence of Brownian bridges $\{B_n(t), 0 \leq t \leq 1\}$ such that we have, as $n \rightarrow \infty$*

(a) *Let $q \in Q^*$. Then*

$$\sup_{0 < t < 1} \left| \frac{n^{-3/2}}{\sigma} U_{[(n+1)t]} - B_n(t) \right| / q(t) = o_P(1)$$

if and only if $I^(q, c) < \infty$ for all $c > 0$.*

(b) *Let $q \in Q^*$. Then*

$$\sup_{0 < t < 1} \left| \frac{n^{-3/2}}{\sigma} U_{[(n+1)t]} - B_n(t) \right| / q(t) = O_P(1)$$

if and only if $I^(q, c) < \infty$ for some $c > 0$.*

(c) *Let $q \in Q^*$. Then*

$$\sup_{0 < t < 1} \left| \frac{n^{-3/2}}{\sigma} U_{[(n+1)t]} \right| / q(t) \xrightarrow{D} \sup_{0 < t < 1} |B(t)| / q(t)$$

if and only if $I^(q, c)$ for some $c > 0$, where B is a Brownian bridge.*

REMARK 2.1. We emphasize that (b) and (c) do not follow from (a), nor does (c) from (b).

REMARK 2.2. Taking $h(x, y) = x - y$ gives U_k equal to $T_k = n(S_k - k/nS_n)$, $1 \leq k < n$, where $S_k = \sum_{i=1}^k X_i$, or equivalently to $T_k = k(n - k) \left(\frac{1}{k} \sum_{i=1}^k X_i - \frac{1}{n-k} \sum_{i=k+1}^n X_i \right)$.

Consequently Theorem 2.1 can be restated for these partial sum processes assuming only the existence of two moments for the summands. Part (a) of this version was proved by O'Reilly (1974) for continuous $q \in Q^*$ under the assumption of $E|X_1|^3 < \infty$.

Another example which has gained special attention in the literature is $h(x, y) = \text{sign}(x - y)$. Pettitt (1979) studied the process $U_k = \sum_{1 \leq i \leq k} \sum_{k+1 \leq j \leq n} \text{sign}(X_i - X_j)$. Theorem

2.1 indicates what kind of weight functions are possible for the process U_k under H_0 . We note also that U_k is equivalent to $R_k = 2 \sum_{1 \leq i \leq k} R_i^n - k(n + 1)$, where R_i^n are the ranks of X_1, \dots, X_n . Hence we can conclude similarly weighted results under H_0 , and also under the sequence of contiguous alternatives H_1 , for cumulatives of ranks as well.

Considering H_1 we have the following result.

THEOREM 2.2. *Assume that $h(x, y) = -h(y, x)$ and under H_0 (2.1) holds. Then under contiguous alternatives H_1 we have*

(a) Let $q \in Q^*$. If $I^*(q, c) < \infty$ for all $c > 0$, then

$$(2.4) \quad \left(\frac{n^{-3/2}}{\sigma} U_{[(n+1)t]} \right) / q(t) \xrightarrow{\mathcal{D}} (B(t) + b(t)) / q(t) \quad \text{in } D[0, 1],$$

where

$$b(t) = \frac{1}{\sigma} \int_0^t \int_0^1 g(s, u) \bar{h}(F^{-1}(u)) du ds + \frac{t}{\sigma} \int_0^1 \int_0^1 g(s, u) \bar{h}(F^{-1}(u)) du ds$$

and B is a Brownian bridge.

(b) Let $q : (0, 1) \rightarrow (0, \infty)$ be such that (1.4) holds. Then (2.4) holds in $D[\delta, 1 - \delta]$ for any $0 < \delta < 1/2$.

3. Symmetric Kernel When considering a symmetric kernel i.e., $h(x, y) = h(y, x)$, similar type of results hold, however the limiting process is different (cf. Csörgő and Horváth, 1988). Let $Eh(X_1, X_2) = \theta$, and $\bar{h}(t) = E\{h(X_1, t) - \theta\}$. Given H_0 we assume

$$(3.1) \quad Eh^2(X_1, X_2) < \infty \quad \text{and} \quad 0 < \sigma^2 = E\bar{h}^2(X_1).$$

Here we consider $U_k = Z_k - k(n - k)\theta$, $1 \leq k < n$.

Let Γ denote the Gaussian process defined by

$$\Gamma(t) = (1 - t)W(t) + t\{W(1) - W(t)\}, \quad 0 \leq t \leq 1,$$

where $\{W(t), 0 \leq t \leq 1\}$ is a Wiener process.

THEOREM 3.1. We assume that H_0 holds, $h(x, y) = h(y, x)$ and (3.1) is satisfied. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \leq t \leq 1\}$, where for each $n \geq 1$

$$\{\Gamma_n(t), 0 \leq t \leq 1\} \xrightarrow{\mathcal{D}} \{\Gamma(t), 0 \leq t \leq 1\},$$

and such that (a), (b) and (c) of Theorem 2.1 hold with the sequence $\{B_n\}$ of Brownian bridges replaced by the sequence $\{\Gamma_n\}$, and Brownian bridge B replaced by the Gaussian process Γ .

THEOREM 3.2. We assume that $h(x, y) = h(y, x)$ and under H_0 (3.1) holds. Then under contiguous alternatives H_1 we have

(a) Let $q \in Q^*$. If $I(q, c) < \infty$ for all $c > 0$, then

$$(3.2) \quad \left(\frac{n^{-3/2}}{\sigma} U_{[(n+1)t]} \right) / q(t) \xrightarrow{\mathcal{D}} (\Gamma(t) + b_\Gamma(t)) / q(t) \quad \text{in } D[0, 1]$$

where

$$b_T(t) = \frac{(1-t)}{\sigma} \int_0^t \int_0^1 g(s,u) \tilde{h}(F^{-1}(u)) \, dud s \\ + \frac{t}{\sigma} \left(\int_0^1 \int_0^1 g(s,u) \tilde{h}(F^{-1}(u)) \, dud s - \int_0^t \int_0^1 g(s,u) \tilde{h}(F^{-1}(u)) \, dud s \right).$$

(b) Let $q : (0, 1) \rightarrow (0, \infty)$ be such that (1.4) holds. Then (3.2) holds in $D[\delta, 1 - \delta]$ for any $0 < \delta < 1/2$.

Contiguous alternatives of H_1 as parametrized in (1.1)-(1.3) accommodate the notion of a changepoint by assuming that there exists $\lambda \in (0, 1)$ such that $g(t, u) = 1\{t \geq \lambda\}g(u)$ for some square integrable function g .

Further discussions of the so-called changepoint problem as well as detailed proofs of related results will be published in a forthcoming paper by the author.

4. Sequential ranks. Let X_1, X_2, \dots be independent continuously distributed random variables with unknown distribution functions. For each $n \geq 1$, let ξ_1, \dots, ξ_n denote the normalized sequential ranks of the first n of the random variables X_1, X_2, \dots , i.e.

$$\xi_k = k^{-1} \sum_{1 \leq i \leq k} 1\{X_i < X_k\}, \quad k = 1, \dots, n.$$

We introduce

$$\mathcal{Y}_n(t) = \begin{cases} n^{-1/2} \left(\sum_{i=1}^{\lfloor (n+1)t \rfloor} a(\xi_i) - \frac{\lfloor (n+1)t \rfloor}{n} \sum_{i=1}^n a(\xi_i) \right), & 0 \leq t < 1 \\ 0, & t = 1. \end{cases}$$

THEOREM 4.1. Let H_0 hold. We assume that function a is as in (1.6) and

$$(4.1) \quad a(u) \text{ has a uniformly bounded derivative on } [0, 1].$$

Then Theorem 2.1 holds with $\frac{n-2t/2}{\sigma} \mathcal{U}_{\lfloor (n+1)t \rfloor}$ replaced by $\mathcal{Y}_n(t)$.

THEOREM 4.2. Assume that function a is such that (1.6) and (4.1) hold. Let B denote a Brownian bridge. Then, under the alternative H_1 , we have

(a) Let $q \in \mathcal{Q}^*$. If $I^*(q, c) < \infty$ for all $c > 0$, then

$$(4.2) \quad \mathcal{Y}_n(t)/q(t) \xrightarrow{\mathcal{D}} (B(t) + R(t) - tR(1))/q(t) \quad \text{in } D[0, 1],$$

where

$$R(t) = - \int_0^t \frac{A(s)}{s} \, ds + A(t) \quad \text{and} \quad A(t) = \int_0^t \int_0^1 g(s,u) a(u) \, ds.$$

(b) Let $q : (0, 1) \rightarrow (0, \infty)$ be such that $\inf_{\delta \leq t \leq 1-\delta} q(t) > 0$ for all $\delta \in (0, 1/2)$. Then (4.2) holds in $D[\delta, 1 - \delta]$ for any $\delta \in (0, 1/2)$.

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REFERENCES

- [1] CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. (1986). Weighted empirical and quantile processes. *Ann. Probab.* 14 31-85.
- [2] CSÖRGŐ, M. and HORVÁTH, L. (1988). Invariance principles for changepoint problems. *J. Multivariate Anal.* 27 151-168.
- [3] CSÖRGŐ, M., SHAO, Q. and SZYSZKOWICZ, B. (1990). A note on local and global functions of a Wiener process and some Rényi-type statistics. *Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 154*, Carleton University - University of Ottawa, Ottawa. To appear in: *Studia Sci. Math. Hung.*
- [4] GREENWOOD, P.E. and SHIRYAYEV, A.N. (1985). *Contiguity and the Statistical Invariance Principle*. Gordon and Breach Science Publishers.
- [5] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Test*. Academic Press, New York.
- [6] KHMALADZE, E.V. and PARJANADZE, A.M. (1986). Functional limit theorems for linear statistics of sequential ranks. *Probab. Theory Related Fields* 73 322-334.
- [7] LE CAM, L. (1960). Locally asymptotically normal families of distributions. *University of California Publications in Statistics* 3, No. 2 37-98.
- [8] LE CAM, L. (1986). *Asymptotic Methods in Statistical Decision Theory*. Springer-Verlag, New York.
- [9] OOSTERHOFF, J. and VAN ZWET, W.R. (1975). A note on contiguity and Hellinger distance. In: J. Jurechkova, ed., *Contributions to Statistics*. Reidel, Dordrecht 157-166.
- [10] O'REILLY, N. (1974). On the weak convergence of empirical processes in sup-norm metrics. *Ann. Probab.* 2 642-651.
- [11] PETTITT, A.N. (1979). A non-parametric approach to the changepoint problem. *Appl. Statist.* 28 126-135.
- [12] ROUSSAS, G.G. (1972). *Contiguity of Probability Measures: Some Applications in Statistics*. Cambridge University Press.
- [13] SERFLING, R.J. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley, New York.
- [14] SZYSZKOWICZ, B. (1991). Changepoint problem and contiguous alternatives. *Statist. and Probab. Letters* 11 299-308.

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**THE PROBABILITY OF INVERTIBILITY FOR A MATRIX OVER A FINITE
COMMUTATIVE RING**

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Abstract. Each finite commutative ring R is uniquely expressed as a product of local rings (R_i, M_i) . It is proved that the probability that an $n \times n$ matrix over R be invertible is $\prod(1 - q_i^{-1}) \cdots (1 - q_i^{-n})$, where $q_i = |R_i/M_i|$. Asymptotics of this formula are also investigated.

Probabilistic interpretations have often been given for enumeration results in number theory and algebra (cf. [5, Exer. 21(a), p. 38], [1, Theorem 3.4]). We do so here for the enumeration of invertible $n \times n$ matrices over a finite commutative ring R (with unit). This enumeration is carried out in the spirit of Koblitz, who sketched the case $R = \mathbf{Z}/m\mathbf{Z}$ in [2, Exers. 16-20, pp. 77-78]. Our main probabilistic result is Theorem 2, with Corollary 3 recording the case of $R = \mathbf{Z}/m\mathbf{Z}$. Proposition 5 studies the asymptotics of the formula in Theorem 2. This leads to quantitative explanations in Remark 6(ii) as to why a "large" matrix over a "large" finite field is "typically" invertible, and why this need not be the case over a "large" finite commutative ring.

In what follows, n and m always denote positive integers, with $m \geq 2$; $M_n(A)$, the ring of $n \times n$ matrices over a ring A ; $GL_n(A)$, the group of

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invertible $n \times n$ matrices over A ; and F_q , the finite field with q elements. It is convenient to write $b(n, A) = |GL_n(A)|$ and $a(n, A) = b(n, A)/|A|^{n^2}$. If for each (i, j) , each element of A is equally likely as the (i, j) entry of an element of $M_n(A)$, then $a(n, A)$ is the probability that an $n \times n$ matrix over A will be invertible.

LEMMA 1. (i) $b(n, F_q) = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$ and $a(n, F_q) = (1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})$.

(ii) If $A = \prod A_i$ for finitely many rings A_i , then $b(n, A) = \prod b(n, A_i)$ and $a(n, A) = \prod a(n, A_i)$.

Proof. (i) The first assertion is well-known (cf. [3, p. 456]). The second assertion then follows from the above remarks, since $(q^n - q^i)/q^n = 1 - q^{-(n-i)}$.

(ii) Since $M_n(B) \cong B \otimes_{\mathbb{Z}} M_n(\mathbb{Z})$ for each ring B , there is a ring-isomorphism $M_n(A) \cong \prod M_n(A_i)$. This induces a group-isomorphism $GL_n(A) \cong \prod GL_n(A_i)$. Equating cardinalities gives the first assertion. Then

$$a(n, A) = b(n, A)/(\prod |A_i|)^{n^2} = (\prod b(n, A_i))/(\prod |A_i|^{n^2}) = \prod (b(n, A_i)/|A_i|^{n^2}) = \prod a(n, A_i). \quad \square$$

THEOREM 2. Let R be a finite commutative ring, necessarily uniquely expressed as a product of (finitely many finite) local rings (R_i, M_i) ; put $q_i = |R_i/M_i|$ for each i . Then

$$a(n, R) = \prod_i (1 - q_i^{-1}) \cdots (1 - q_i^{-n}) \text{ for each positive integer } n.$$

Proof. The decomposition $R = \prod R_i$, with R_i local, follows from the theory of artinian rings [6, Theorem 3, p. 205]. By Lemma 1, we may assume (R, M) local, and we need only prove that $a(n, R) = a(n, K)$, where $K = R/M$. Consider the surjective ring-homomorphism $f: M_n(R) \rightarrow M_n(K)$ that reduces each entry of a matrix modulo M . This induces a surjective group-homomorphism $g: GL_n(R) \rightarrow GL_n(K)$. (Indeed, if $f(N)$ is invertible, then $\det(N) \in R \setminus M$, the set of units of R , and so N is invertible.) Moreover, $\ker(g) = \{(r_{ij}) \in M_n(R) : r_{ij} - \delta_{ij} \in M \text{ for all } i, j\}$. (Notice that any such (r_{ij}) is invertible since its determinant is a unit.) Thus, $|\ker(g)| = |M|^{n^2}$. Hence, by the First Isomorphism Theorem and Lagrange's Theorem, we have $b(n, R)/|M|^{n^2} = b(n, K)$. It follows that $a(n, R) = b(n, K)|M|^{n^2}/|R|^{n^2} = b(n, K)/|R/M|^{n^2} = a(n, K)$. \square

COROLLARY 3. If $m = \prod p_i^{e_i}$ is the prime-power decomposition of m , then $a(n, \mathbb{Z}/m\mathbb{Z}) = \prod_i (1 - p_i^{-1}) \cdots (1 - p_i^{-n})$.

Proof. By the Chinese Remainder Theorem, $\mathbb{Z}/m\mathbb{Z} \cong \prod \mathbb{Z}/p_i^{e_i}$; and the residue field of $\mathbb{Z}/p_i^{e_i}$ is \mathbb{F}_{p_i} . Thus, Theorem 2 yields the assertion. \square

REMARK 4. (i) It is important to note that $|R|$ does not determine $a(n, R)$. We illustrate this with the four commutative rings of order 4. Using Lemma 1 and Corollary 3, we have $a(2, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = 9/64$, $a(2, \mathbb{Z}/4\mathbb{Z}) = 3/8 = a(2, (\mathbb{Z}/2\mathbb{Z})[X]/(X^2))$, and $a(2, \mathbb{F}_4) = 45/64$.

(ii) From the definition of $a(-, -)$ or Corollary 3, note that $a(1, \mathbb{Z}/m\mathbb{Z}) = \varphi(m)/m$. Although the asymptotics of φ are not as pleasant as for τ (cf. [5, Exer. 11(c), p.33]), satisfactory asymptotics for $a(-, -)$ can be

established: see Proposition 5 and Remark 6. There, we study a number $L(R)$, which is intuitively the probability that a "large" matrix over a finite commutative ring R be invertible.

PROPOSITION 5. Let R, R_i, q_i be as in Theorem 2. Then $L(R) = \lim_{n \rightarrow \infty} a(n, R)$ exists, equals $\prod_i L(R_i)$, and is strictly between 0 and 1. Moreover, $L(R_i) \sim \exp(-1/q_i)$, in the sense that $\exp((-2q_i - 3)/2(q_i^2 - 1)) \geq L(R_i) \geq \exp((-2q_i^3 + q_i^2 + 2q_i - 2)/2(q_i^2 - 1)(q_i - 1)^2)$.

Proof. (Sketch) Applying standard limit theorems to Lemma 1(ii), we may suppose (R, M) local, with residue field of order q . It suffices to establish the displayed inequality, with $R_i = R, q_i = q$. (The fact that q is a prime-power will not be needed here; we use only that $2 \leq q$.) View $L(R)$ as the infinite product $\prod_{1 \leq n} (1 - q^{-n}) = \prod (1 + \alpha_n)$, where $\alpha_n = -q^{-n}$. By a standard result in advanced calculus [4, Theorem III, p. 487], this infinite product converges (and so $L(R)$ exists) because the geometric series $\sum \alpha_n^2$ converges absolutely. By the proof of [4, Theorem II, p. 486], to establish the displayed inequality, it is enough to find suitable upper and lower bounds for $\sum \ln(1 + \alpha_n)$. This may be done as in the proof of [4, Theorem III, p. 487] by expressing $\ln(1 + \alpha_n)$ as $\alpha_n - \beta_n$ and then using Taylor's Formula with Remainder to bound β_n above by $1/2q^{2n}(1 - q^{-1})^2$ and below by $1/2q^{2n}$. The associated calculations with geometric series are left to the reader. \square

REMARK 6. (i) Despite Proposition 5, another limit associated with

$a(-, -)$ does not exist. Specifically, $\lim_{m \rightarrow \infty} a(n, \mathbb{Z}/m\mathbb{Z})$ does not exist. To see this, let m approach ∞ through a subsequence $\{m_k\}$. If $m_k = p^k$ for some fixed prime p , then the sequence $\{a(n, \mathbb{Z}/m_k\mathbb{Z})\}$ is constant and the limit in question is $(1 - p^{-1}) \cdots (1 - p^{-n})$. If, instead, $m_k = p_k$ (which denotes the k -th prime), then the limit is 1. Moreover, the limit can actually be 0. For simplicity, we show this when $n = 1$. Consider $m_k = p_1 p_2 \cdots p_k$. Then $\lim_{k \rightarrow \infty} a(1, \mathbb{Z}/m_k\mathbb{Z}) = \prod (1 - p_k^{-1})$, and this infinite product diverges to zero by [4, Theorem III(i)]. (To apply the cited result, note that $\sum -p_k^{-1}$ diverges and $\sum p_k^{-2}$ converges.)

(ii) If one wishes simpler bounds for $L(R_i)$ than those given in Proposition 5, one may use the upper bound $\exp(-1/q_i)$ and the lower bound $\exp(-2/q_i)$. Since $\lim_{q \rightarrow \infty} \exp(-1/q) = 1$, the bounds in Proposition 5 explain why the typical "large" matrix over \mathbb{F}_q is invertible, for "large" q . Indeed, one has the following. If $0 < \epsilon < 1$, then $L(\mathbb{F}_q) > 1 - \epsilon$ for all (prime-powers) $q > -2/\ln(1 - \epsilon)$.

Moreover, the bounds in Proposition 5 seem quite good even for a "small" q , such as 5. Indeed, the upper bound for $L(\mathbb{Z}/5\mathbb{Z})$ given by Proposition 5 is 0.7627..., and the lower bound is 0.7538... . Double-precision computer calculation of the product of the first 1000 factors of the infinite product indicates that the actual value, $L(\mathbb{Z}/5\mathbb{Z}) = \prod_{1 \leq n} (1 - 5^{-n})$, is approximately 0.7603... .

On the other hand, the bounds in Proposition 5 also explain why the typical "large" matrix over a "large" finite commutative ring need not be invertible. Indeed, by analyzing infinite products that are "dominated" by the infinite product that was shown to diverge to zero in (i), one can

obtain the following result. If $\epsilon > 0$ and p_j denotes the j -th prime, then $L(\mathbb{Z}/p_1 p_2 \cdots p_k \mathbb{Z}) < \epsilon$ for all sufficiently large k .

REFERENCES

- [1] D. E. Dobbs and M. J. Lancaster, The expected dimension of the sum of vector subspaces, Bull. Austral. Math. Soc., to appear.
- [2] N. Koblitz, A course in number theory and cryptography, Graduate Texts in Math. 114, Springer - Verlag, New York, 1987.
- [3] R. Lidl and H. Niederreiter, Finite fields, Encyclopedia of Math. and its Appl., vol. 20, Addison - Wesley, Reading, 1983.
- [4] J. M. H. Olmsted, Advanced calculus, Appleton - Century - Crofts, New York, 1961.
- [5] I. M. Vinogradov, Elements of number theory, Dover, New York, 1954.
- [6] O. Zariski and P. Samuel, Commutative Algebra, vol. I, Van Nostrand, Princeton, 1958.

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**On the classification of C^* -algebras of real rank zero:
inductive limits of matrix algebras over graphs**

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Presented by G.A. Elliott, F.R.S.C.

Abstract

In this paper a K -theoretic classification is given of the real rank zero C^* -algebras that can be expressed as inductive limits of sequences of finite direct sums of matrix algebras over finite connected graphs (possibly with multiple vertices). The special case that the graphs are circles is due to Elliott.

Statement of the Result

The problem we are interested in is to classify certain C^* -algebras by their K -theory data. Given a C^* -algebra A , there are two abelian groups associated with it. Namely, $K_0(A)$, which is often an ordered group, and $K_1(A)$ [1]. The complete invariant in the present classification will be the graded group $K_* = K_0 \oplus K_1$, together with the distinguished subset consisting of the pairs $([e], [u])$, where e is a projection in the algebra and u is a partial unitary with support (and range) e [7]. Following [7], we shall call this set the graded dimension range.

The first result of this kind was the classification of inductive limits of sequences of finite direct sums of matrix algebras (called AF algebras), in which case K_1 is trivial [6]. The group K_0 , together with the dimension range, is a complete invariant. In [7], this classification was extended. It was shown that if one replaces the matrix algebras by matrix algebras over the unit circle and restricts the limit C^* -algebras to have real rank zero (see Definition 2), then K_* , as a graded group, together with the graded dimension range, is a complete invariant. This important work suggested the problem of classifying all separable nuclear C^* -algebras of real rank zero and of stable rank one (i.e., such that the invertible elements are dense). (Cf. [8].)

In this paper, the C^* -algebras considered are those real rank zero C^* -algebras that can be expressed as inductive limits of sequences of finite direct sums of basic building blocks defined as follows:

Definition 1 Let X be a finite connected graph and denote by $C(X)$ the C^* -algebra of complex-valued continuous functions on X . A C^* -algebra A will

be said to be a basic building block if A is isomorphic to a sub- C^* -algebra of $C(X) \otimes M_n$ of the following form:

$$\{f \in C(X) \otimes M_n \mid f \text{ has block diagonal form at each vertex of } X\}.$$

for some positive integer n . The number of blocks at each vertex should be the multiplicity of that vertex.

Example The following is an example of basic building block:

$$A = \{f \in M_2(C[0, 1]) \mid f(0) \text{ is diagonal}\}.$$

Definition 2 A unital C^* -algebra is said to be of real rank zero if the set of invertible selfadjoint elements is dense in the set of all selfadjoint elements [5].

The main result of this paper is the following:

Theorem 1 *Let A and B be two C^* -algebras of real rank zero. Suppose that A and B are the inductive limits of sequences (A_i) and (B_i) of finite direct sums of basic building blocks. Suppose that the graded groups $K_*(A)$ and $K_*(B)$ are isomorphic, in a way preserving the graded dimension range. Then A and B are isomorphic.*

The work was clearly motivated by the question described above and the result generalized the classification theorems in [6] and [7] (referred to above), which correspond to the cases of points and circles, respectively.

One can show that the main result of [2] can be extended to this setting. As an application, a simple C^* -algebra with unique trace arising as an inductive limit of the basic building blocks considered in this paper must be of real rank zero and therefore comes under the classification theorem. (This remark is used in [4].)

Another reason to consider C^* -algebras of this kind is to study group actions. In this setting, one will be able to construct interesting actions other than inner actions: see [3], [9] and [4].

Sketch of the Proof of Theorem 1

For simplicity, we shall sketch the proof only for graphs in which each vertex has multiplicity one. For the general case, see [10].

The proof can be divided into four steps.

Step 1 One can replace arbitrary graphs in Theorem 1 by graphs of the following special form: $S^1 \vee \dots \vee S^1$, i.e., finitely many circles joined at one point.

This follows from Theorem 2 below, together with Theorem 2.2 and Theorem 2.1 of [7]. This reduction is crucial in this work since we can only prove Theorem 3 and Theorem 4 in this case. The proof of Theorem 2 uses the following pairing lemma.

Suppose that ϕ is a unital C^* -algebra homomorphism between two basic building blocks $A = M_n \circ C(X)$ and $B = M_m \circ C(Y)$. For each $y \in Y$, the evaluation map τ_y from B to M_m given by

$$\tau_y(g) = g(y) \text{ for } g \in B$$

is a C^* -algebra homomorphism and hence gives rise to a representation $\tau_y \phi: A \rightarrow M_m$. There are then $k = m/n$ points $(x_i)_{i=1}^k$ in X and a unitary $u \in M_m$ such that

$$(\tau_y \phi)(f) = \text{Ad}(u) \begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_k) \end{pmatrix} \text{ for } f \in A.$$

Lemma 1 *Let A be a C^* -algebra of real rank zero, arising as the inductive limit of a sequence (A_i) of basic building blocks. For any i and any $\epsilon > 0$, there exists $j > i$ such that for any two points in the spectrum of A_j , the corresponding two families of points in the spectrum of A_i can be matched up to be within ϵ point by point.*

With a slight modification, both of the statement and of the proof, Lemma 1 still holds if each A_i is a finite direct sum of basic building blocks. This is the version used in proving Theorem 2 below. In fact, the lemma is still true if instead of basic building blocks one has matrix algebras over arbitrary compact metric spaces (see [10]).

Before stating Theorem 2, we need the following

Definition 3 Let X be a finite connected graph with k edges, L_1, L_2, \dots, L_k . For $\epsilon > 0$, a finite subset F of a basic building block A isomorphic to $M_n(C(X))$ is said to be approximately constant, to within ϵ , if there exists a unitary u in $\oplus_{i=1}^k M_n(C(L_i))$ such that the image of each element of F , under the natural embedding of A into $\oplus_{i=1}^k M_n(C(L_i))$ combined with $\text{Ad}u$, when considered as a matrix-valued function on the disjoint union of the L_i , is approximately the same at each point, to within ϵ .

The definition in [7] for a circle can be viewed as a special case, by considering the circle as a graph with a single vertex.

Theorem 2 Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$ be a sequence of finite direct sums of basic building blocks. and suppose that the C^* -algebra inductive limit is of real rank zero. For each i , each finite subset $F \subset A_i$, and each $\epsilon > 0$, there exist $j > i$, a sub- C^* -algebra $A_j^0 \subset A_j$ isomorphic to a finite direct sum of basic building blocks, and a map ϕ from A_i to A_j^0 such that the component of $\phi(F)$ in each basic building block of A_j^0 is approximately constant, to within ϵ , and such that the following diagram approximately commutes, to within ϵ , on F :

$$\begin{array}{ccc} A_i & \rightarrow & A_j \\ & \searrow & \uparrow \\ & & A_j^0 \end{array}$$

Furthermore, the spectrum of each basic building block of A_j^0 may be required to have the special form $S^1 \vee S^1 \vee \dots \vee S^1$.

Step 2 The isomorphism $K_*(A) \rightarrow K_*(B)$ can be lifted to an intertwining of subsequences of the sequences $K_*(A_i)$ and $K_*(B_i)$.

More precisely, after passing to subsequences of A_i and B_i and changing notation, one has a commutative diagram

$$\begin{array}{ccccccc} K_*A_1 & \rightarrow & K_*A_2 & \rightarrow & \dots & \rightarrow & K_*A \\ \downarrow & \nearrow & \downarrow & & & & \parallel \\ K_*B_1 & \rightarrow & K_*B_2 & \rightarrow & \dots & \rightarrow & K_*B \end{array}$$

where each map preserves both the graded group structure and the graded dimension range. This is due to the special nature of A_i and B_i . For example, an application of Bott periodicity shows that K_*A_i and K_*B_i , as groups, are finite direct sums of copies of \mathbb{Z} .

Step 3 The maps intertwining the sequences K_*A_i and K_*B_i can be lifted, individually, to C^* -algebra homomorphisms. In other words, one has a not necessarily commutative diagram

$$\begin{array}{ccccccc} A_1 & \rightarrow & A_2 & \rightarrow & \dots & \rightarrow & A \\ \downarrow & \nearrow & \downarrow & & & & \\ B_1 & \rightarrow & B_2 & \rightarrow & \dots & \rightarrow & B \end{array}$$

This follows from

Theorem 3 Let A and B be finite direct sums of basic building blocks. Suppose that the spectrum of each basic building block of A or B has the special

form considered in Step 1. For any map $\phi: K_*A \rightarrow K_*B$ preserving the graded group structure and graded dimension range, there is a C^* -algebra homomorphism $\psi: A \rightarrow B$ such that $K_*\psi = \phi$.

Step 4 Let $C_1 \rightarrow C_2 \rightarrow \dots$ be a sequence of finite direct sums of basic building blocks, and suppose that the C^* -algebra inductive limit is of real rank zero. We shall say that the sequence is special if for each i , each finite subset F of C_i , and each $\epsilon > 0$, there exists $j > i$ such that the image of F in each basic building block of C_j is approximately constant, to within ϵ . (This property is stronger than that guaranteed by Theorem 2.)

Applying Theorem 2 above together with Theorems 2.2 and 2.1 of [7], we may replace the sequences A_i and B_i at the very beginning so that they are both special. After passing again to subsequences of A_i and B_i (and changing notation), it is now possible to change each of the homomorphisms $A_i \rightarrow B_i$ and $B_i \rightarrow A_{i+1}$ obtained in Step 3 by an inner automorphism in such a way that the diagram becomes an approximate intertwining in the sense of Theorem 2.1 of [7]. This follows from Theorem 2.2 of [7] together with

Theorem 4 Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$ be a special sequence of finite direct sums of basic building blocks, each of which has the special form of Step 1. Let i be fixed, and let ϕ and ψ be two unital C^* -algebra homomorphisms from A_i to A_{i+1} such that $K_*\phi = K_*\psi$. Let F be a finite subset of A_i , and let $\epsilon > 0$ be such that the set of components of the elements of F in each basic building block of A_i is approximately constant to within ϵ . It follows that there exists $j > i$ such that ϕ and ψ composed with the map from A_{i+1} to A_j are approximately unitary equivalent on F , to within ϵ .

It follows from Theorem 2.1 of [7] that A and B are isomorphic.

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References

1. B. Blackadar, *K-Theory for Operator Algebras*. Mathematical Sciences Research Institute Publications 5, Springer-Verlag, New York, 1986.
2. B. Blackadar, O. Bratteli, G. A. Elliott, and A. Kumjian, Reduction of real

rank in inductive limits of C^* -algebras. *Math. Ann.*, to appear.

3. O. Bratteli, G. A. Elliott, D. E. Evans, and A. Kishimoto. Finite group actions on AF algebras obtained by folding the interval. *K-Theory*, to appear.

4. O. Bratteli, A. Kishimoto, Non-commutative spheres III: Irrational rotations. preprint.

5. L. G. Brown and G. K. Pedersen. C^* -algebras of real rank zero. *J. Funct. Anal.*, **99** (1991), 131-149.

6. G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra* **38** (1976), 29-44.

7. G. A. Elliott, On the classification of C^* -algebras of real rank zero, preprint.

8. G. A. Elliott. Dimension groups with torsion. *International J. of Math.* **1** (1990), 361-384.

9. D. E. Evans and A. Kishimoto. Compact group actions on UHF algebras obtained by folding the interval. *J. Funct. Anal.* **98** (1991), 346-360.

10. H. Su, On the classification of C^* -algebras of real rank zero: inductive limits of matrix algebras over non-Hausdorff graphs. preprint.

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L^2 -COHOMOLOGY AND HECKE OPERATORS*Salahoddin Shokranian***Presented by J. Lambek, F.R.S.C.***Abstract**

In this paper the spectral decomposition of the L^2 -cohomology representation of Hecke operators for a twisted group is studied. The result is used to define a L^2 -Lefschetz number for a twisted group.

1. In the context of representation theory, Lefschetz numbers have been studied by several mathematicians for the purpose of calculating the multiplicity of an automorphic representation in the space of cohomology. For example, suppose that G is a connected reductive group over Q such that $G(\mathbb{R})$ the group of its real points, has a discrete series representation. In this case J. Arthur in his work [A] has shown that, L^2 -Lefschetz numbers for Hecke operators can be used to calculate the "sum" of multiplicities of an irreducible discrete series representation in a finite Langlands' packet [L]. The work in [A] is based on several fundamental results; the spectral decomposition of cohomology due to Borel and Casselman [BC], the theorem of Clozel and Delorme [CD] on the existence of Euler-Poincaré function, and an invariant trace formula [A1]. Indeed, the assumption that $G(\mathbb{R})$ has discrete series is necessary for the application of the first two results.

Our aim is to define and to study a L^2 -Lefschetz number for certain non-connected reductive groups which are important in arithmetic through their use in the base change theory. This will be done in two papers, beginning with this one.

2. In general we define a twisted group as being the group generated by a connected component of a disconnected algebraic group over a field. In

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a special case, let G be a connected reductive group over \mathbb{Q} , and ϵ an automorphism of G of finite order defined over \mathbb{Q} . Suppose that $\langle \epsilon \rangle$ is the group of order ℓ generated by ϵ . Denote by \mathbf{A} the ring of adèles of \mathbb{Q} . Then, define the ϵ -twisted group $G^\epsilon(\mathbf{A})$ to be the semi-direct product $G(\mathbf{A}) \rtimes \langle \epsilon \rangle$. There is a representation π of $G(\mathbf{A})$ called ϵ -stable, which can be extended to the non-connected group $G^\epsilon(\mathbf{A})$ (cf. [AC], for groups over local fields). Let \mathbf{A}_0 be the subring of finite adèles, then $\mathbf{A} = \mathbb{R} \times \mathbf{A}_0$, and $G(\mathbf{A}) = G(\mathbb{R})G(\mathbf{A}_0)$. For a group H let $\Pi(H)$ be the set of irreducible representations of H . Then, any $\pi \in \Pi(G(\mathbf{A}))$ has the tensor product decomposition $\pi = \pi_\infty \otimes \pi_0$, where $\pi_\infty \in \Pi(G(\mathbb{R}))$, $\pi_0 \in \Pi(G(\mathbf{A}_0))$.

3. Let (μ, V_μ) be a finite dimensional irreducible representation of G defined over \mathbb{Q} . Then μ can be considered as an irreducible representation of $G(\mathbb{R})$ or $G(\mathbb{C})$. Let A_G be the split component of the center of G . Define a quasi-character μ denoted $\xi_\mu(z)$, of $A_G(\mathbb{R})^0$ by

$$\mu(xz) = \xi_\mu(z)^{-1}\mu(x),$$

where $A_G(\mathbb{R})^0$ is the identity component of the real points of A_G , $z \in A_G(\mathbb{R})^0$, and $x \in G(\mathbb{R})$. There is a normal subgroup $G(\mathbf{A})^1$ of $G(\mathbf{A})$ which contains $G(\mathbb{Q})$ as a diagonal discrete subgroup, such that $G(\mathbf{A})$ is the direct product of $A_G(\mathbb{R})^0$ and $G(\mathbf{A})^1$ (cf. [A2]). Let $L(\mu)$ be the space of functions φ on $G(\mathbb{Q}) \backslash G(\mathbf{A})$ such that

$$\varphi(zx) = \xi_\mu(z)\varphi(x), \quad (z \in A_G(\mathbb{R})^0, x \in G(\mathbf{A})).$$

For $\varphi \in L(\mu)$, define

$$(R^\epsilon(\varphi))(x) = \varphi(\epsilon^{-1}(x)),$$

and

$$(R^\epsilon(f)\varphi)(x) = \int_{G(\mathbf{A})} f(y)\varphi(\epsilon^{-1}(xy))dy,$$

where $x, y \in G(\mathbf{A})$. Then by [Mü], the restriction of $R^\epsilon(f)$ to the discrete spectrum $L_d(\mu)$ of $L(\mu)$ under the regular representation of $G(\mathbf{A})$ is of trace class.

4. Fix a maximal compact subgroup $K(\mathbb{R})$ of $G(\mathbb{R})$, and set $K'(\mathbb{R}) =$

$K(\mathbb{R})A_G(\mathbb{R})^0$. Define a globally symmetric space with respect to a fixed left $G(\mathbb{R})$ -invariant metric by: $X = G(\mathbb{R})/K'(\mathbb{R})$. Let K_0 be an open compact subgroup of $G(\mathbb{A}_0)$. Then let

$$S_{K_0} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_0) / K_0.$$

We have

$$S = \varinjlim_{K_0} S_{K_0} = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A})).$$

Consider the local system

$$\tilde{V}_\mu = V_\mu(\mathcal{C}) \times_{G(\mathbb{Q})} (X \times G(\mathbb{A}_0)).$$

This is a sheaf, and a q -form with values in \tilde{V}_μ is a section of the bundle $\wedge^q T^*(S) \otimes \tilde{V}_\mu$. Observe that $G(\mathbb{A}_0)$ acts on the L^2 -complex $A_{(2)}^*(S, \tilde{V}_\mu)$ through its action on S , and this defines a L^2 -cohomology representation of $G(\mathbb{A}_0)$ by means of the operators

$$H_{(2)}^*(g, \tilde{V}_\mu) = \bigoplus_{q \geq 0} H_{(2)}^q(g, \tilde{V}_\mu),$$

where $g \in G(\mathbb{A}_0)$.

Lemma 1. The L^2 -cohomology representation of $G(\mathbb{A}_0)$ is smooth.

Proof. One needs to show that $H_{(2)}^*(S, \tilde{V}_\mu)$ is the union over open compact subgroups K_0 of the space of fixed vectors $H_{(2)}^*(S, \tilde{V}_\mu)^{K_0}$. But,

$$H_{(2)}^*(S, \tilde{V}_\mu) = \varinjlim_{K_0} H_{(2)}^*(S_{K_0}, \tilde{V}_\mu).$$

Let $\mathcal{H} = \mathcal{H}(G(\mathbb{A}_0), K_0)$ be the Hecke algebra of locally constant functions on $G(\mathbb{A}_0)$ which are K_0 -bi-invariant. For $h \in \mathcal{H}$, define the operator of finite rank

$$H_{(2)}^q(\epsilon, h, \tilde{V}_\mu) = \int_{G(\mathbb{A}_0)} h(g) H_{(2)}^q(\epsilon^{-1}(g), \tilde{V}_\mu) dg.$$

5. We say that a representation of $G(\mathbb{A})$ is automorphic if it occurs in the discrete spectrum of $L(\mu)$. Let π be an automorphic representation

of $G(\mathbf{A})$, and π_R its Archimedean component. Let $\mathcal{G} = \text{Lie } G(\mathbb{R})$, and $H^q(\mathcal{G}, K'(\mathbb{R}); \pi_R \otimes \mu)$ the q -th relative Lie algebra cohomology. Then, we say that π_R is a representation with cohomology if

$$H^*(\mathcal{G}, K'(\mathbb{R}); \pi_R \otimes \mu) = \bigoplus_{q \geq 0} H^q(\mathcal{G}, K'(\mathbb{R}); \pi_R \otimes \mu) \neq 0,$$

The space of representation with cohomology is denoted by $\Pi_{cl}(G(\mathbb{R}))$. According to ([BC], Theorem 4.5), the L^2 -cohomology $H^q_{(2)}(S_{K_0}, \tilde{V}_\mu)$ has the following spectral decomposition

$$\bigoplus_{\pi \in \Pi(G(\mathbf{A}))} m_{\text{disc}}(\pi)(H^q(\mathcal{G}, K'(\mathbb{R}); \pi_R \otimes \mu) \otimes V(\pi_0^{K_0})),$$

where $V(\pi_0^{K_0})$ is a certain finite dimensional subspace of the underlying space of π_0 , and $m_{\text{disc}}(\pi)$ is the multiplicity for which π occurs discretely as a direct sum in $L(\mu)$.

Now, suppose that π_R is an ϵ -stable representation in $\Pi_{cl}(G(\mathbb{R}))$. Define the ϵ -twisted Euler-Poincaré Characteristic of π_R by

$$\chi_\mu^\epsilon(\pi_R) = \sum_q (-1)^q \dim H^q(\mathcal{G}, K'(\mathbb{R}) : \pi_R \otimes \mu).$$

Then, define the ϵ -twisted Lefschetz number of triple (ϵ, μ, h) by

$$L(\epsilon, \mu, h) = \sum_q (-1)^q \text{tr}(H^q_{(2)}(\epsilon, h, \tilde{V}_\mu)).$$

Observe that $L(\epsilon, \mu, h)$ is well defined since $H^q_{(2)}(\epsilon, h, \tilde{V}_\mu)$ is of finite rank.

Lemma 2. Let $\Pi_{cl}(G(\mathbf{A}))$ be the space of irreducible representations π of $G(\mathbf{A})$ whose Archimedean component $\pi_R \in \Pi_{cl}(G(\mathbb{R}))$. Then, for any $h \in \mathcal{H}$, if π_R is ϵ -stable we have

$$L(\epsilon, \mu, h) = \sum_{\pi \in \Pi(G(\mathbf{A}))} m_{\text{disc}}(\pi) \cdot \chi_\mu^\epsilon(\pi_R) \cdot \text{tr} \pi_0(h, \epsilon),$$

where

$$\pi_0(h, \epsilon) = \int_{G(\mathbf{A}_0)} h(g) \pi_0(\epsilon^{-1}(g)) dg.$$

Proof. This is a direct consequence of the decomposition of L^2 -cohomology and the definition of $\chi_\mu^e(\pi_R)$. ■

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References

- [A] J. Arthur, *The L^2 -Lefschetz number of Hecke operators*, Invent. Math. 97 (1989) 257-290.
- [A1] _____, *The trace formula in invariant form*, Annals of Math. 114 (1981) 1-74.
- [A2] _____, *The Selberg trace formula for groups of F -rank one*, Annals of Math. 100 (1974) 326-385.
- [AC] _____ and L. Clozel, *Simple Algebras, Base Change, And The Advanced Theory Of The Trace Formula*, Annals of Math. Studies, Vol. 120 (1989) Princeton Univ. Press.
- [BC] A. Borel and W. Casselman, *L^2 -cohomology of locally symmetric manifolds of finite volume*, Duke Math. J. 50 (1983) 625-647.
- [CD] L. Clozel and P. Delorme, *Sur le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs réels*, C.R. Acad. Sc. Paris, t. 300, Series 1, n° 11, 1985.
- [L] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Representation Theory and Harmonic Analysis on Semisimple Lie Groups, Math. Surveys Vol. 31 Amer. Math. Soc. (1989) 101-170.
- [Mü] W. Müller, *Trace class conjecture in the theory of automorphic forms*, Annals of Math. 130 (1989) 473-529.

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Class Numbers Bounded Below by the Divisor Function

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CORRIGENDA

Page 120. On line 2 of Theorem 2.1, $2a - |r - 1|$ should be $(2a - |r - 1|)^2$. On line 3 of Applications I(2), (d) should be $h(d)$.

Page 121. On line 4 of Theorem 2.2(a) $(2\ell - |r-1|/4)$ should be $(2\ell - |r-1|/4)^2$. On line 1 of Theorem 2.2(b) insert "of $(2\ell - |r - 1|)^2$ " after "t" and on the same line replace " $\dots \equiv 0 \pmod{4}$ " by " $\dots \not\equiv 0 \pmod{4}$ ".

Fixed Point Subalgebras of the Rotation Algebra

by C. Farsi and N. Watling

[C.R. Math. Rep. Acad. Sci. Canada - Vol. XIII, No. 2, April 1990, pp. 75-80]

CORRIGENDUM

p. 79. On line 2, in the second row, second column of the upper table, replace the entry $\frac{q}{3}$ by $\frac{q-3}{3}$.

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