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ON THE CHARACTERISTIC FUNCTIONS OF THE KOTZ TYPE DISTRIBUTIONS

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Presented by D.A.S. Fraser, F.R.S.C.

ABSTRACT : The Kotz type distributions form an important class of elliptically symmetric distributions. These distributions are studied in the book of Fang, Kotz and Ng [1, Chapter 3.2]. In the particular case when the parameter s equals 1 Iyengar and Tong [2, pp. 19-20] determined explicitly the characteristic functions of these probability laws. In the present note a general formula for the characteristic functions, valid for any $s > \frac{1}{2}$, is derived by a different method.

MAIN RESULT. The density function of the k -variate Kotz type distributions of parameters $\underline{\mu}, \underline{\Sigma}, N, r$ and s is given by

$$f_k(\underline{x} : \underline{\mu}, \underline{\Sigma}, N, r, s) = \frac{s\Gamma(\frac{k}{2})}{\pi^{\frac{k}{2}}\Gamma(\frac{2N+k-s}{2})} \cdot r^{\frac{2N+k-s}{2}} \cdot |\underline{\Sigma}|^{-1/2} \cdot$$

$$[(\underline{x} - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{x} - \underline{\mu})]^{N-1} \cdot \exp \left[-r[(\underline{x} - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{x} - \underline{\mu})]^s \right],$$

where $\underline{\mu} \in \mathbb{R}^k$, the symmetric $k \times k$ matrix $\underline{\Sigma} > 0$, $r > 0$, $s > 0$, $2N + k > 2$, $k \in \{1, 2, 3, \dots\}$.

In order to calculate the characteristic function

$$\begin{aligned} \varphi_{X;k}(\underline{t} : \underline{\mu}, \underline{\Sigma}, N, r, s) &= E[e^{i\underline{t}'\underline{X}}] \\ &= \int_{\mathbb{R}^k} e^{i\underline{t}'\underline{x}} f_k(\underline{x} : \underline{\mu}, \underline{\Sigma}, N, r, s) d\underline{x} \end{aligned}$$

we study-motivated by [1, p.32] - the expression

$$\psi_k(\underline{t}'\underline{\Sigma}\underline{t} : N, r, s) := \varphi_{X;k}(\underline{t} : \underline{\mu}, \underline{\Sigma}, N, r, s) e^{-i\underline{t}'\underline{\mu}}.$$

We see that it is sufficient to investigate the particular case $\mu = 0$ and $\Sigma = I_k$, since the knowledge of ψ_k yields $\varphi_{X;k}$ in general. For any elliptically symmetric probability law with these parameters, thus for any spherical distribution, the corresponding random vector admits the representation $\tilde{X} \sim R\tilde{U}^{(k)}$, where $\tilde{U}^{(k)}$ is the uniform base of the distribution and R is independent of $\tilde{U}^{(k)}$ [1, p. 30, f. (2.7)]. The theorem of Schoenberg [1, p. 29, f. (2.5)] and results on the characteristic function of $\tilde{U}^{(k)}$ [1, p. 70, f. (3.2)] imply in general that

$$\begin{aligned}\psi_k(|\tilde{t}|^2) &= \int_0^\infty e^{it'U^{(k)}r} dF_R(r) \\ &= \frac{\Gamma(\frac{k}{2})}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (|\tilde{t}|^2)^{\ell}}{(2\ell)!} \frac{\Gamma(\frac{2\ell+1}{2})}{\Gamma(\frac{k+2\ell}{2})} \int_0^\infty r^{2\ell} dF_R(r)\end{aligned}$$

(where F_R is the distribution function of R), provided that $E[R^{2\ell}] < \infty$ for $\ell = 1, 2, \dots$ and the series of the right hand side is convergent. Now for the particular case of the Kotz type distribution we have [1, p.77]

$$E[R^{2\ell}] = r^{-\frac{1}{2}} \left[\Gamma\left(\frac{2N+k-2}{2s}\right) \right]^{-1} \Gamma\left(\frac{2N+k+2\ell-2}{2s}\right).$$

Substituting this in the preceding formula for ψ_k we find

$$\psi_k(|\tilde{t}|^2) = \frac{\Gamma(\frac{k}{2})}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \left\{ \frac{(-1)^\ell (|\tilde{t}|^2)^{\ell}}{(2\ell)!} r^{-\frac{1}{2}} \frac{\Gamma(\frac{2\ell+1}{2})}{\Gamma(\frac{k+2\ell}{2})} \frac{\Gamma(\frac{2N+k+2\ell-2}{2s})}{\Gamma(\frac{2N+k-2}{2s})} \right\}.$$

It follows that the characteristic function of the Kotz type distribution with parameters μ, Σ, N, r and $s > \frac{1}{2}$ is given by

$$\begin{aligned}\varphi_{X;k}(\tilde{t} : \mu, \Sigma, N, r, s) &= \frac{1}{\sqrt{\pi}} e^{it'\mu} \sum_{\nu=0}^{\infty} \left\{ \frac{\left(\frac{-t'\Sigma t}{4r^2/s}\right)^\nu}{(2\nu)!} \cdot \frac{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{2N+k+2\nu-2}{2s})}{\Gamma(\nu + \frac{k}{2})\Gamma(\frac{2N+k-2}{2s})} \right\} \\ &= e^{it'\mu} \sum_{\nu=0}^{\infty} \left\{ \frac{\left(\frac{-t'\Sigma t}{4r^2/s}\right)^\nu}{\nu!} \cdot \frac{\Gamma(\frac{k}{2})}{\Gamma(\nu + \frac{k}{2})} \frac{\Gamma(\frac{2N+k+2\nu-2}{2s})}{\Gamma(\frac{2N+k-2}{2s})} \right\}.\end{aligned}$$

REMARKS. a) The above result is not applicable for $0 < s < \frac{1}{2}$ due to the fact that the series diverges in these cases for $|t| \neq 0$ although all moments of the distribution exist. See [3, p. 277, Ex. 18] for other examples of such non-analytic characteristic functions.

b) The above result applies also for non-integer N with $2N + k > 2$ and not only for positive integers N as the result of Iyengar and Tong.

c) Our result is compatible with the findings of Iyengar and Tong [2, (27) & (31)]. To show this it is sufficient to prove that (*)

$$\sum_{\nu=0}^{\rho} \binom{\rho}{\nu} \frac{\Gamma(N + \frac{k}{2} + \nu - 1)}{\Gamma(\frac{k}{2} + \nu)} (-1)^{\rho-\nu} = \begin{cases} \frac{(N-1)!}{(N-1-\rho)!} \frac{\Gamma(N + \frac{k}{2} - 1)}{\Gamma(\frac{k}{2} + \rho)} & [\rho = 0, 1, \dots, N-1], \\ 0 & [\rho = N, N+1, \dots]. \end{cases}$$

Proof for $\rho = 0, 1, \dots, N-1$: For $N = 1$ and $\rho = 0$ and $k \in \{1, 2, \dots\}$ both sides of (*) give 1. Suppose now the relation (*) has been established for $N = 1, \dots, N_0$, $\rho = 0, 1, \dots, N-1$ and $k \in \{1, 2, \dots\}$. We want to show that it holds also for $N = N_0 + 1$, $\rho = 0, 1, \dots, N_0$ and $k \in \{1, 2, \dots\}$.

$$\begin{aligned} \sum_{\nu=0}^{\rho} \binom{\rho}{\nu} \frac{\Gamma(N_0 + \frac{k}{2} + \nu)}{\Gamma(\frac{k}{2} + \nu)} (-1)^{\rho-\nu} &= \frac{(N_0 - 1)!}{(N_0 - 1 - \rho)!} \frac{\Gamma(N_0 + \frac{k}{2})}{\Gamma(\frac{k}{2} + \rho)} \\ &+ \sum_{\nu=0}^{\rho} \binom{\rho}{\nu} \nu \frac{\Gamma(N_0 + \frac{k}{2} + \nu - 1)}{\Gamma(\frac{k}{2} + \nu)} (-1)^{\rho-\nu} \\ &= \frac{(N_0 - 1)!}{(N_0 - 1 - \rho)!} \frac{\Gamma(N_0 + \frac{k}{2})}{\Gamma(\frac{k}{2} + \rho)} + \rho \sum_{\tau=0}^{\rho-1} \binom{\rho-1}{\tau} \\ &\cdot \frac{\Gamma(N_0 + (\frac{k}{2} + 1) + \tau - 1)}{\Gamma(\frac{k}{2} + 1 + \tau)} (-1)^{\rho-1-\tau} \\ &= \frac{(N_0 - 1)!}{(N_0 - 1 - \rho)!} \frac{\Gamma(N_0 + \frac{k}{2})}{\Gamma(\frac{k}{2} + \rho)} [1 + \rho \frac{1}{N_0 - \rho}] \\ &= \frac{N_0!}{(N_0 - \rho)!} \frac{\Gamma(N_0 + \frac{k}{2})}{\Gamma(\frac{k}{2} + \rho)}. \end{aligned}$$

Proof for $\rho = N, N+1, \dots$: For $N = 1$ and $\rho, k \in \{1, 2, \dots\}$ both sides of (*) are 0. Suppose now the relation (*) has been established for $N = 1, \dots, N_0$; $\rho = N, N+1, \dots$ and $k \in \{1, 2, \dots\}$. We want to show that it holds also for $N = N_0 + 1$, $\rho = N_0 + 1, N_0 + 2, \dots$ and $k \in \{1, 2, \dots\}$.

$$\begin{aligned} \sum_{\nu=0}^{\rho} \binom{\rho}{\nu} \frac{\Gamma(N_0 + \frac{k}{2} + \nu)}{\Gamma(\frac{k}{2} + \nu)} (-1)^{\rho-\nu} &= (N_0 + \frac{k}{2} - 1) \cdot 0 + \rho \sum_{r=0}^{\rho-1} \binom{\rho-1}{r} \\ &\quad \cdot \frac{\Gamma(N_0 + (\frac{k}{2} + 1) + r - 1)}{\Gamma((\frac{k}{2} + 1) + r)} (-1)^{\rho-1-r} \\ &= \rho \cdot 0 = 0. \end{aligned}$$

d) From the main result follows easily

$$E[\tilde{X}] = \mu, \quad D[\tilde{X}] = E[(\tilde{X} - \mu)(\tilde{X} - \mu)'] = \frac{1}{kr^{1/s}} \frac{\Gamma(\frac{2N+k}{2s})}{\Gamma(\frac{2N+k-2}{2s})} \Sigma$$

and

$$E[(X_i - \mu_i)(X_j - \mu_j)(X_\ell - \mu_\ell)] = 0 \quad [i, j, \ell \in \{1, \dots, k\}].$$

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UNIVERSAL APPEARANCE OF PERIODIC ORBITS IN THE LORENZ EQUATIONS

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ABSTRACT. Numerical techniques are used to discover stable, periodic orbits in the Lorenz equations. A detailed recording of these periodic attractors leads to a bifurcation diagram showing a striking agreement with the qualitative changes occurring in the quadratic equation. This agreement goes far beyond a simple period-doubling cascade.

RÉSUMÉ. Une liste numérique détaillée des attracteurs stables des équations de Lorenz donne un diagramme des bifurcations qui ressemble fortement à celui de l'itération d'un polynôme du second degré. Cette ressemblance dépasse la simple cascade du dédoublement de la période.

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Keywords: bifurcations, periodic solutions, Lorenz equations.

1. Introduction. The Lorenz equations [3] are a three-dimensional system of coupled ordinary nonlinear differential equations with defining vector field $f = (f_1, f_2, f_3)$:

$$\dot{x} = f_1(x, y, z) = \sigma(y - x)$$

$$\dot{y} = f_2(x, y, z) = rx - y - xz$$

$$\dot{z} = f_3(x, y, z) = xy - bz.$$

These equations have been studied extensively for Lorenz' original parameter values $\sigma = 10$, $b = 8/3$, $0 \leq r < \infty$ [3], [8], [5], and it is well known that for large enough r -values the equations have a single stable symmetric periodic orbit which, together with the three unstable stationary points, makes up the whole of the nonwandering set [8]. With $\sigma = 10$ and $b = 8/3$ throughout this paper, this orbit is shown at $r = 350$ in Fig. 1 [8].

We introduce the symbols x^+ and x^- to refer to a local maximum in $z(t)$ of a given periodic orbit with corresponding positive x -coordinate, resp. to a local maximum in $z(t)$ with corresponding negative x -coordinate, and consequently we denote this orbit an (x^+x^-) -orbit. This orbit, when followed with decreasing r in parameter space, first undergoes

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a pitchfork bifurcation at $r^* \approx 312.974$ and then remains unstable all the way down to $r \approx 13.92656$ where it bifurcates out of phase space in a homoclinic explosion [8]. At $r = r^*$ two asymmetric, but symmetrically located, stable periodic (x^+x^-) -orbits branch off and leave the symmetric (x^+x^-) -orbit unstable, see Fig. 2 [8] for one of them at $r = 260$. As r is decreased further, each of these asymmetric (x^+x^-) -orbits then undergoes the well known period-doubling scenario with limit point $r_\infty \approx 215.364$ [8].

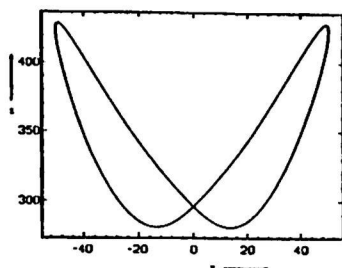


FIGURE 1. (left) Symmetric hyperbolic and (asymptotically) stable (x^+x^-) -orbit at $r = 350$, [8].

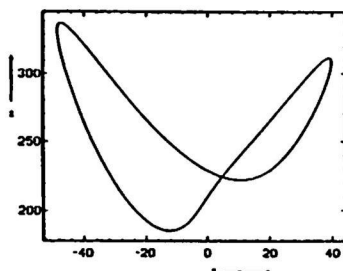


FIGURE 2. (right) Asymmetric hyperbolic and (asymptotically) stable (x^+x^-) -orbit at $r = 260$, [8].

2. Universal appearance of stable periodic orbits.

In order to find a periodic orbit in phase space, to follow it across parameter space and to determine its stability, I considered the Poincaré map around an assumed periodic orbit. This map, developed up to second order terms, with an integrated Adams-Bashforth method to solve the occurring differential equations and an IMSL subroutine to solve the resulting three-dimensional nonlinear system, provided a fairly well working algorithm for that purpose. For more details see [5], [6], [8].

The following table shows asymmetric stable periodic $(x^+x^-)^p$ -orbits with $p \leq 7$ and their corresponding r -values in comparison with the superstable p -periodic orbits of the quadratic map $a_{i+1} = h_\mu := 1 - \mu a_i^2$ [7].

With the coefficients c_1 and c_2 chosen to fit a linear interpolation $r = c_1 - \mu c_2$ exactly at at $\mu = 1.3107026$ and $\mu = 2$, we found $c_1 := 241.34749$ and $c_2 := 19.186267$ and the last column emphasizes this relation

$$r \approx c_1 - \mu c_2 \quad (2)$$

TABLE 1. Asymmetric hyperbolic and (asymptotically) stable $(x^+x^-)^p$ -orbits in the Lorenz equations in comparison with the superstable p -periodic orbits in the quadratic map.

p	r	r - Seq.	μ	$(c_1 - \mu c_2)/r$
1	260.0	—	0.5	—
2	222.0	$\uparrow\downarrow$	1.0	1.0007
4	216.2	$\uparrow\downarrow\uparrow\downarrow$	1.3107026	1.0000
r_{∞}, μ_{∞}	215.364	—	1.40115519	0.9958
6	214.06	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$	1.4760146	0.9952
r'_1, μ'_1	213.2314569	—	1.54368901	0.9930
7	212.4634	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.5748891	0.9937
5	211.62	$\uparrow\downarrow\uparrow\downarrow$	1.6254137	0.9931
7	210.746	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.6740661	0.9928
3	209.425	$\uparrow\downarrow\uparrow$	1.7548777	0.9917
6	208.96	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$	1.7728929	0.9922
7	207.6806	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.8323150	0.9928
5	207.1192	$\uparrow\downarrow\uparrow\downarrow$	1.8607825	0.9929
7	206.65	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.8848036	0.9929
6	206.249	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$	1.9072801	0.9928
7	205.9158	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.9271477	0.9925
4	205.526	$\uparrow\downarrow\uparrow\downarrow$	1.9407998	0.9931
7	205.09722	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.9537059	0.9941
6	204.655	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$	1.9667732	0.9949
7	204.3663	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.9771796	0.9953
5	204.124	$\uparrow\downarrow\uparrow\downarrow$	1.9854243	0.9957
7	203.71547	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.9918142	0.9971
6	203.537	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$	1.9963761	0.9976
7	203.2734	$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$	1.9990957	0.9986
r_c^1, μ_c^1	202.9749563	—	2.0000000	1.0000

over the whole μ -range. The third column compares consecutive maxima z_i and z_{i+1} in $z(t)$ and consecutive maxima a_i and a_{i+1} of the underlying periodic solutions of both systems, starting with the absolute maximum in z_i , resp. a_i . For example " $\uparrow\downarrow$ " means $z_2 < z_1$, $z_2 < z_3$ and $z_3 < z_1$. See Figs. 3, 4, 5 and 6 for some of the orbits listed in this table.

The three-dimensional continuous Lorenz equations exhibit exactly the same behaviour as does the one-dimensional discrete quadratic map. Neither is a p -periodic orbit missing nor did I find one too many, but they also appear in the same universal order [1], [7], r and μ being related by an almost straight line. Even more, for all orbits the numerical procedure indicates a period-doubling bifurcation when r is decreased only slightly further. It is tempting to look for more similarities, and indeed r_{∞} , r'_1 and r_c^1 serve as additional examples of agreement between both systems. While r_{∞} (resp. μ_{∞} [1], [7]) is the limit

point of the first period-doubling, r'_1 (resp. μ'_1 [1], [7]) is the reverse-bifurcation value (or boundary-crisis value [2]) where the unstable manifold of the origin (resp. the orbit $0 \rightarrow 1 \rightarrow 1 - \mu \rightarrow \dots$) is included in the stable manifold of the unstable asymmetric (x^+x^-) -orbit that was stable at $r = 260$ (resp. the formerly stable fixed point of the quadratic map [1], [7]), see Fig. 7 for this heteroclinic orbit.

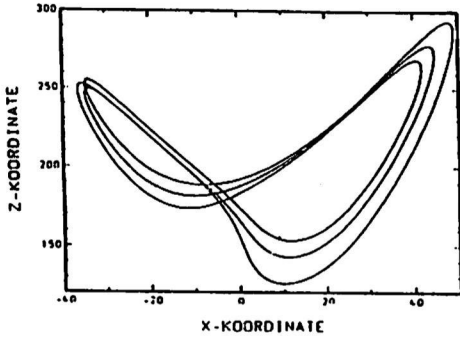


FIGURE 3. (left) Asymmetric hyperbolic and (asymptotically) stable $(x^+x^-)^3$ -orbit at $r = 209.425$.

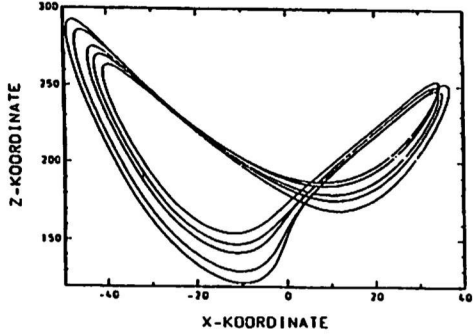


FIGURE 4. (right) Asymmetric hyperbolic and (asymptotically) stable $(x^+x^-)^5$ -orbit at $r = 207.1192$.

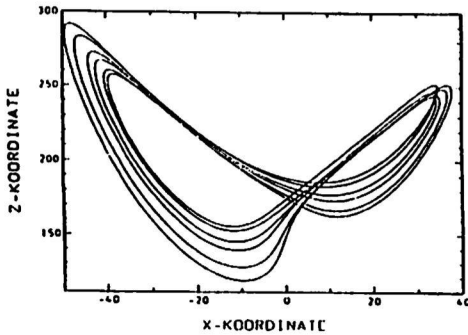


FIGURE 5. (left) Asymmetric hyperbolic and (asymptotically) stable $(x^+x^-)^6$ -orbit at $r = 204.655$.

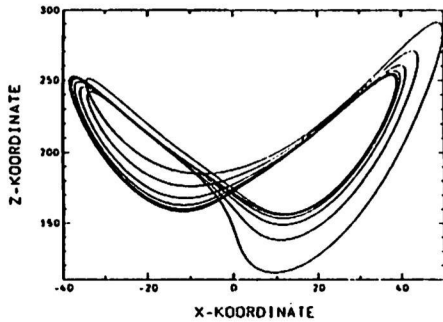


FIGURE 6. (right) Asymmetric hyperbolic and (asymptotically) stable $(x^+x^-)^7$ -orbit at $r = 203.2734$.

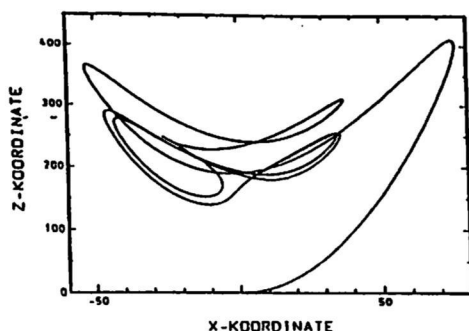


FIGURE 7. (left) Heteroclinic orbit from the origin to the formerly stable asymmetric (x^+x^-) -orbit at $r \approx 213.2314563$.

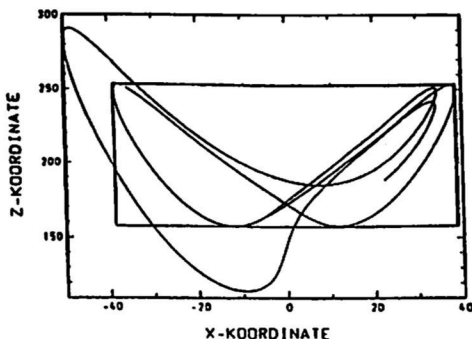


FIGURE 8. (right) Heteroclinic orbit from the origin to the formerly stable symmetric (x^+x^-) -orbit at ≈ 202.974956335 after a short transition time.

At last, r_c^1 (resp. $\mu_c^1 = 2$) closes the diagram. Again similar to the discrete iteration of $a_{i+1} = 1 - \mu a_i^2$, the unstable manifold of the origin (resp. the orbit $0 \rightarrow 1 \rightarrow 1 - \mu \rightarrow \dots$ [1], [7]) is here included in the stable manifold of that unstable periodic orbit which is required as unstable partner in the pitchfork bifurcation at $r^* = 312.974$. In case of the Lorenz equations this is the symmetric (x^+x^-) -orbit, shown in Fig. 1 for $r = 350$ [8]. This heteroclinic connection is shown in Fig. 8 after a short transition time.

3. Conjectures and Conclusion.

If we "translate" further phenomena from the quadratic map to the Lorenz equations, we should not be surprised to find also interior-crisis bifurcation values [2]. For example, we expect an $r = r_c^3$ in the interval $(207.6806, 208.96)$ where the unstable manifold of the origin is included in the stable manifold of the unstable $(x^+x^-)^3$ -orbit which is required as the unstable partner in a saddle-node bifurcation close to $r = 209.43$ involving the above listed stable $(x^+x^-)^3$ -orbit. Since $\mu_c^3 = 1.7903275$ [2], we expect by (2) that $r_c^3 \approx (c_1 - \mu_c^3 c_2)/0.992375$. Hence $r_c^3 \approx 208.5883$. By using (2) again, another boundary-crisis bifurcation value can be assumed close to $r_c^2 \approx (c_1 - 1.43035763 c_2)/0.9955 = 214.8712$. Here the unstable manifold of the origin will be included in the stable manifold of the unstable $(x^+x^-)^2$ -orbit which is stable at $r = 222$ and which is the once doubled (x^+x^-) -orbit shown in Fig. 2. For r_c^1 we still might compute with (2) that $r_c^1 \approx (c_1 - 1.40741 c_2)/0.9957506 = 215.25927$. Here the unstable manifold of the origin will be included in the stable manifold of the unstable $(x^+x^-)^4$ -orbit that was stable at $r = 216.2$ and which is the once more doubled $(x^+x^-)^2$ -orbit that was stable at $r = 222$. Going a last step further and assuming

infinitely many of these bifurcation values, for example a sequence $r'_k \rightarrow r_\infty$ as is the case for $\mu'_k \rightarrow \mu_\infty$ [1], and remembering that the famous strange Lorenz attractor [3], [8] at first evolves when the unstable manifold of the origin is included in the stable manifolds of the simple unstable x^+ - and x^- -orbits [8], we have good reasons to conjecture the existence of infinitely many new strange attractors, any two of which being separated by periodic behaviour.

Altogether these results show that, in case of the Lorenz equations for the above parameter range, one can successfully extract a one-dimensional return map $z_{i+1} = g_r(z_i)$ to describe essential characteristics of the full three-dimensional flow. Here z_i are the local maxima of $z(t)$ with the same sign in their corresponding x -coordinates. Furthermore this map $g_r(z)$ exhibits a single maximum in z as does the quadratic map $h_\mu(a)$ have in a . Not surprisingly it thus allows the same bifurcation pattern to arise. Finally these results suggest that whenever a reduction of the full nonlinear flow towards a general "one-hump" map is possible, one may conjecture similar transitions when the underlying parameters are changed.

At least the above table clearly describes the solutions' character in much more detail and precision than "noisy periodicity" [4] does.

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Passage d'une mesure d'approximation à une mesure de transcendance

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Résumé : Etant donné un nombre transcendant ω , dont on connaît une mesure d'approximation, on établit une mesure de transcendance de ω . Nous améliorons les résultats antérieurs quant à la dépendance en le degré.

Une mesure d'approximation d'un nombre complexe ω est une minoration de $|\omega - \xi|$ pour tout nombre algébrique ξ , en fonction du degré de ξ et de sa mesure de Mahler $M(\xi)$. Tandis qu'une mesure de transcendance de ω est une minoration de $|P(\omega)|$ pour tout polynôme $P \in \mathbb{Z}[X]$, en fonction du degré de P et de sa hauteur $H(P)$.

Les méthodes classiques de transcendance, comme celles de Gel'fond-Schneider ou de Baker, permettent d'obtenir des mesures d'approximation. Pour en déduire des mesures de transcendance, il existe des lemmes de passage comme le lemme 5 de [Ci], le lemme 2.3 de [W] ... Les lemmes existants ont l'inconvénient d'introduire des termes "trop gros". Par exemple, le lemme 5 de [Ci] fait apparaître $\deg(P) + \log H(P)$ là où on attend l'expression $\log(\deg(P) \cdot H(P))$, tandis que le lemme 2.3 de [W] comporte le terme $(\deg(P))^2$, ce qui exclut toute mesure en $\deg(P)$ ou en $\deg(P) \cdot \log(\deg(P))$.

On propose ici une amélioration de ce dernier résultat avec précisément le remplacement du terme $(\deg(P))^2$ par $\deg(P) \cdot \log(\deg(P))$. On trouvera une application de cette inégalité dans [D].

La démonstration suit les lignes de [W, lemme 2.3], l'amélioration provenant uniquement du lemme ci-dessous. L'idée de ce lemme revient à G.V. Chudnovsky [Ch, lemme 1.12]. Cependant, Chudnovsky utilisait une des notions de semi-discriminant qu'il a introduites alors que nous avons seulement besoin d'estimer la norme d'une dérivée.

Lemme . — Soient $P \in \mathbb{Z}[X]$ de degré $D \geq 1$, $z \in \mathbb{C}$ et ξ une racine de P à distance minimale de z . Alors, si s est la multiplicité de ξ comme racine de P , on a

$$|z - \xi|^s \leq \left(\binom{D+1}{s+1} H(P) \right)^{d-1} (2M(\xi))^{D-s} |P(z)|,$$

où $M(\xi)$ est la mesure de Mahler de ξ , d son degré et $H(P)$ la hauteur de P .

Démonstration.

Soit a le coefficient dominant du polynôme minimal de ξ et soient $\xi = \xi_1, \dots, \xi_d$ ses conjugués. Alors, le nombre $a^{D-s} \prod_{i=1}^d \frac{P^{(s)}(\xi_i)}{s!}$ est un entier rationnel non nul. Donc,

$$1 \leq |a|^{D-s} \prod_{i=1}^d \left| \frac{P^{(s)}(\xi_i)}{s!} \right|.$$

Majorons d'abord $\left| \frac{P^{(s)}(\xi)}{s!} \right|$. Si t est une racine quelconque de P , $|z - \xi| \leq |z - t|$ par hypothèse, ce qui implique $|t - \xi| \leq 2|z - t|$. D'autre part, si b désigne le coefficient dominant de P , on a la formule

$$\frac{P^{(s)}(\xi)}{s!} = b \prod_{t \neq \xi} (\xi - t).$$

Ainsi,
$$\left| \frac{P^{(s)}(\xi)}{s!} \right| \leq 2^{D-s} |P(z)| |z - \xi|^{-s}.$$

Pour $i \geq 2$, on majore trivialement :

$$\begin{aligned} \left| \frac{P^{(s)}(\xi_i)}{s!} \right| &\leq \sum_{k=s}^D \binom{k}{s} H(P) \max\{1, |\xi_i|\}^{D-s} \\ &= \binom{D+1}{s+1} H(P) \max\{1, |\xi_i|\}^{D-s}. \end{aligned}$$

D'où l'inégalité du lemme.

En utilisant les inégalités $\left(\frac{D+1}{s+1} \right) \leq D \left(\frac{D+1}{2} \right)^s$, $M(\xi)^s \leq M(P)$ et $s \leq D$,

on en déduit facilement le résultat suivant.

Corollaire I . — Sous les hypothèses du lemme, on a

$$|z - \xi|^s \leq (H(P)M(P)D(D+1))^{D-1} |P(z)|.$$

La majoration bien connue $M(P) \leq (D+1)^{1/2} H(P)$ fournit l'estimation qui suit.

Corollaire 2. — Dans les conditions du lemme, on a aussi

$$|z - \xi|^{18} \leq (H(P)^2 D(D+1)^{3/2})^{D-1} |P(z)|.$$

De ce corollaire, on déduit aussitôt le théorème ci-dessous.

Théorème. — Soit ω un nombre transcendant. Soit ψ une application de $N^* \times [2, \infty[$ dans R_+ qui vérifie les deux propriétés suivantes :

(H1) pour tout $n, s_1 \leq s_2$ implique $\psi(n, s_1) \leq \psi(n, s_2)$,

(H2) pour tout $s, n_1 \leq n_2$ implique $\frac{\psi(n_1, s)}{n_1} \leq \frac{\psi(n_2, s)}{n_2}$.

On suppose que pour tout nombre algébrique ξ on a

$$\text{Log} |\omega - \xi| \geq -\psi(\deg(\xi), \text{Log } M) \text{ si } M \geq \max\{2, M(\xi)\}.$$

Alors, pour tout polynôme $P \in \mathbb{Z}[X]$, de degré $D \geq 1$, de hauteur $\leq H$ (avec $H \geq 2$)

on a

$$\text{Log} |P(\omega)| \geq -\psi(D, \text{Log } H + \text{Log } D) - D(2 \text{Log } H + \frac{5}{2} \text{Log } D).$$

Un exemple.

Soit $Q(X) = X^d + aX - 1$, a et d entiers ≥ 2 . Alors

$$Q(a^{-1}) = a^{-d}, Q(a^{-1} - h) < a^{-d} - ah = 0 \text{ si } h = a^{-d-1},$$

donc Q a une racine $\xi = a^{-1} - \theta h$, avec $0 < \theta < 1$. Posons $R(X) = (aX - 1)Q(X)$

alors $R'(\xi) = (a\xi - 1)Q'(\xi) = -a^2\theta h$. Ainsi, si a est assez grand, on trouve

$$|R'(\xi)| < 2a^{-d+1} = 2H^{-(d-1)/2},$$

où H désigne la hauteur du polynôme R . Et si z est assez proche de ξ , on a

$$\text{Log} |R(z)| \leq \text{Log} |\omega - \xi| - \frac{d-1}{2} \text{Log } H + 1.$$

Cet exemple montre que le lemme ci-dessus ne peut pas être beaucoup amélioré en ce qui concerne le terme de la forme $D \text{Log } H$. Problème : qu'en est-il du terme $D \text{Log } D$?

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Quantum Central Extensions

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Abstract q -analogues of the Witt algebra, the Virasoro algebra and the simplest affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$ are constructed. Another kind of q -analogue of the enveloping algebra of the usual $\mathfrak{sl}(2)$ is also presented and its irreducible representations are classified.

1. The purpose of this paper is to give one possible solution to some questions proposed by Y.I. Manin in [3]. In section 2, we will construct the quantum Witt algebra W_q from the operators called J -derivations on the Laurent ring $\mathbb{C}[t, t^{-1}]$. It turns out that W_q satisfies the quantum Jacobi identity. Other objects which satisfy the quantum Jacobi identity are the quantum Virasoro algebra V_q and the quantum affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_q(2)$. All these examples are deformations of familiar objects and become those objects when $q = 1$. In section 3, we will give a theorem which asserts that V_q and $\widehat{\mathfrak{sl}}_q(2)$ are the quantum universal central extensions of W_q and $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{sl}_q(2)$, respectively. In section 4, we will classify the irreducible representations of a family of the q -analogues of the enveloping algebra of the usual $\mathfrak{sl}(2)$. Details will appear in a later paper.

Notation:

$[q] := (q^m - q^{-m}) / (q - q^{-1})$ where $m \in \mathbb{Z}$, $q \in \mathbb{C} \setminus \{0\}$ and $q^2 \neq 1$.

For any $\mathbb{Z}$ -graded vector spaces $V = \bigoplus_{n \in \mathbb{Z}} V_n$, $J \in \text{Hom}(V, V)$ is defined by $J(v_n) = q^n v_n$ for $n \in \mathbb{Z}$ and $v_n \in V_n$. $\sigma := (J + J^{-1})/2$.

2. Definition. 1. Let A be a $\mathbb{Z}$ -graded algebra. A linear map $d: A \rightarrow A$ is called a J -derivation if $d(ab) = d(a)J(b) + J^{-1}(a)d(b)$, for all $a, b \in A$. Define a family of linear operators on $\mathbb{C}[t, t^{-1}]$ by $d_n := -t^{n+1}(d/dt)$ and $d/dt: t^m \mapsto [m]t^{m-1}$ for all $m, n \in \mathbb{Z}$.

Proposition 1. With the notation above, we have

(1). $D := \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n$ is the $\mathbb{C}$ -vector space formed by all J -derivations of $\mathbb{C}[t, t^{-1}]$.

(2). The bracket $[,]$ on D defined by

$$[d_m, d_n] := J d_m J^{-1} d_n J - J d_n J^{-1} d_m J = [m-n] d_{m+n}$$

satisfies the following quantum Jacobi identity:

$$([x, y], \alpha(z)) + ([y, z], \alpha(x)) + ([z, x], \alpha(y)) = 0, \quad \text{where } x, y, z \in D \quad (2.1)$$

Definition 2. A $\mathbb{Z}$ -graded $\mathbb{C}$ -vector space L is called a quantum Lie algebra if there is a skew-symmetric bilinear map $[,] : L \times L \rightarrow L$ which satisfies the quantum Jacobi identity (2.1) for all $x, y, z \in L$ and preserves $\mathbb{Z}$ -grading, i.e. $[L_m, L_n] \subseteq L_{m+n}$ for $m, n \in \mathbb{Z}$.

Example 1 Quantum Witt Algebra. $W_q := \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n$ with bracket $[,]$ defined by $[d_m, d_n] := [m-n] d_{m+n}$, for $m, n \in \mathbb{Z}$. Its subalgebra $\mathbb{C} d_{-n} \oplus \mathbb{C} d_0 \oplus \mathbb{C} d_n$ is denoted by $sl_q^n(2)$. Note that $sl_q^n(2)$ has the $\mathbb{Z}$ -grading:

$$\deg(d_i) := i, \quad i = 0, \pm n.$$

Remark. W_q is simple iff q is not a root of unity.

Example 2 Quantum Virasoro Algebra. $V_q := \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n \oplus \mathbb{C} c$ with the following bracket:

$$[d_m, d_n] := [m-n] d_{m+n} + \frac{[m-1][m][m+1]}{[2][3](q^m + q^{-m})} \delta_{m+n,0} c, \quad [c, d_m] := 0.$$

where $m, n \in \mathbb{Z}$ and q is not a root of unity.

Example 3. If A is an associative algebra and $L = \bigoplus_{n \in \mathbb{Z}} L_n$ a quantum Lie algebra, then $A \otimes_{\mathbb{C}} L$ becomes a quantum Lie algebra under the bracket $[,]$ defined by

$$[a \otimes x_m, b \otimes y_n] := ab \otimes [x_m, y_n], \quad a, b \in A, x_m \in L_m \text{ and } y_n \in L_n$$

and the $\mathbb{Z}$ -grading: $(A \otimes L)_n := A \otimes L_n$, for $n \in \mathbb{Z}$.

Example 4. For a fixed positive integer n and $q^n \notin \{\pm 1, \pm i\}$, the simplest quantum Kac-Moody algebra $sl_q^{\hat{n}}(2)$ is defined as follows:

$$\begin{aligned} \widehat{sl}_q^n(2) &:= \mathbb{C}[t, t^{-1}] \otimes sl_q^n(2) \otimes \mathbb{C}c \\ [u \partial_0, v \partial_0] &:= -\text{Res}(u(dv)), \quad [u \partial_0, v \partial_{\pm n}] := [n]u v \partial_{\pm n} \\ [u \partial_n, v \partial_{-n}] &:= [2n]u v \partial_0 + \frac{2[2n]}{[n](q^n + q^{-n})} \text{Res}(u(dv)) \\ [u \partial_n, v \partial_n] &:= [u \partial_{-n}, v \partial_{-n}] := [c, \widehat{sl}_q^n(2)] := 0 \end{aligned}$$

and the $\mathbb{Z}$ -grading:

$$(sl_q^n(2))_m = \begin{cases} 0 & \text{if } m \neq 0, \pm n \\ \mathbb{C}[t, t^{-1}] \otimes C d_m & \text{if } m = \pm n \\ \mathbb{C}[t, t^{-1}] \otimes C d_0 + \mathbb{C}c & \text{if } m = 0 \end{cases}$$

where $u, v \in \mathbb{C}[t, t^{-1}]$, Res and d are the usual residue and derivation, respectively.

3. A linear map $f: L = \bigoplus_{n \in \mathbb{Z}} L_n \rightarrow L' = \bigoplus_{n \in \mathbb{Z}} L'_n$ (L, L' quantum Lie algebras) is called a homomorphism if $f([x, y]) = [f(x), f(y)]$ and $f(L_n) \subseteq L'_n$.

An exact sequence of quantum Lie algebras

$$0 \longrightarrow C \xrightarrow{i} E \xrightarrow{\pi} L \longrightarrow 0 \quad (3.1)$$

is said to be a quantum central extension of L by C if C is in the center of E . A morphism of the central extension (3.1) to another central extension

$$0 \longrightarrow C' \xrightarrow{i'} E' \xrightarrow{\pi'} L \longrightarrow 0 \quad (3.2)$$

is a pair (ψ, ϕ) of quantum Lie algebra homomorphisms such that the diagram

$$\begin{array}{ccccc} C & \xrightarrow{i} & E & \xrightarrow{\pi} & L \\ \psi \downarrow & & \downarrow \phi & & \parallel \\ C' & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & L \end{array}$$

is commutative. A central extension (3.1) of L is said to be universal if, given any central extension (3.2) of L , there is a morphism from (3.1) to (3.2). Two quantum central extensions E and

E' of L by C are equivalent if there is an isomorphism $\phi: E \cong E'$ making the following diagram commute:

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & C & \xrightarrow{\quad 1 \quad} & E & \xrightarrow{\quad \pi \quad} & L \xrightarrow{\quad} 0 \\ & & \parallel & & \downarrow \phi & & \parallel \\ 0 & \xrightarrow{\quad} & C' & \xrightarrow{\quad 1' \quad} & E' & \xrightarrow{\quad \pi' \quad} & L \xrightarrow{\quad} 0 \end{array}$$

Let V be a $\mathbb{Z}$ -graded vector space and L a quantum Lie algebra. A skew-symmetric bilinear map $\alpha: L \times L \rightarrow V$ is said to be a quantum 2-cocycle of L with coefficients in V if α satisfies the following conditions (where $m, n \in \mathbb{Z}$ and $x, y, z \in L$):

$$(1). \quad \alpha(L_m, L_n) \subseteq V_{m+n}. \quad (3.3)$$

$$(2). \quad \alpha([x, y], \alpha(z)) + \alpha([y, z], \alpha(x)) + \alpha([z, x], \alpha(y)) = 0$$

Let $Z_q^2(L, V)$ be the space of all quantum 2-cocycles of L with coefficients in V and $B_q^2(L, V)$ the space of all skew-symmetric bilinear maps $\alpha: L \times L \rightarrow V$ satisfying (3.3) and $\alpha(x, y) = f([x, y])$,

for all $x, y \in L$ and for some ($\mathbb{Z}$ -grading preserving) $f \in \text{Hom}(L, V)$. It is clear that $B_q^2(L, V) \subseteq Z_q^2(L, V)$. The quotient $H_q^2(L, V) := Z_q^2(L, V) / B_q^2(L, V)$ is called the quantum second cohomology group of L with coefficients in V . The equivalence class containing the central extension (3.1) is defined to be the set of all central extensions of L by C which are equivalent to (3.1). Let $CE_q(L, C)$ denote the set of all equivalence classes of central extensions of L by C . Then we have

Proposition 2. If C is abelian, then there is a bijection between $H_q^2(L, C)$ and $CE_q(L, C)$.

Proposition 3. Let L be a quantum Lie algebra. If $[L, L] = L$, then L has a universal quantum central extension.

Theorem 4. (1) If q is not a root of unity, then V_q is the universal quantum central extension of W_q .

(2) If $q^n \in \{\pm 1, \pm i\}$, then $sl_q^n(2)$ is the universal quantum central extension of $\mathbb{C}[t, t^{-1}] \otimes sl_q^n(2)$.

4. Let n be a fixed positive integer. Because of Proposition 1, we define the q -analogue of the enveloping algebra of the usual $\mathfrak{sl}(2)$ as the associative algebra $U(\mathfrak{sl}_q^n(2))$ generated by $\{J^{\pm 1}, d_0, d_{\pm n}\}$ with the following relations:

$$\begin{aligned} JJ^{-1} &= J^{-1}J = 1, & Jd_kJ^{-1} &= q^k d_k \\ d_n d_{-n} &= q^{-2n} d_{-n} d_n + q^{-n} [2n] d_0 J^{-1} \\ d_0 d_k &= q^k d_k d_0 - [k] d_k J^{-1}, & \text{where } k &= 0, \pm n. \end{aligned}$$

The following theorem can be obtained by the standard argument in Lie algebras.

Theorem 5. Let q be not a root of unity. Then

(1) $\forall r \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$ and $k \geq 0$, there exists a irreducible $U(\mathfrak{sl}_q^n(2))$ -module $V(r, k) := \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \dots \oplus \mathbb{C}v_k$ with

$$\begin{aligned} J(v_s) &= r^{-1} q^{-sn} v_s, & d_{-n}(v_s) &= r v_{s+1} \\ d_0(v_s) &= ([sn] - [n][2n]^{-1} [kn] q^{(k-s+1)n}) r v_s \end{aligned}$$

$$d_n(v_s) = [sn][s-k+1]n! q^{kn} r v_{s-1}, \text{ where } v_{-1} = v_{k+1} = 0, s=0, 1, \dots, k.$$

(2) If V is a nonzero finite dimensional irreducible $U(\mathfrak{sl}_q^n(2))$ -module, then $V = V(r, k)$ for some $r \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$ and $k \geq 0$.

(3) $V(r, k) \cong V(r', k)$ (as $U(\mathfrak{sl}_q^n(2))$ -module) iff $r = r'$.

(4) $c := [2n] d_0^2 J^{-2} - [n] J^{-1} d_{-n} d_n J^{-1} - [n] J^{-1} d_n d_{-n} J^{-1}$ is in the center of $U(\mathfrak{sl}_q^n(2))$.

(5) $c = [n]^2 [2n]^{-1} [kn][k+2]n! q^{2kn} r^4$ on $V(r, k)$.

Remark. As above, we also can introduce the q -analogues of the enveloping algebras of Witt algebra and Virasoro algebra. We are going to discuss their representations in another paper.

Finally, if q is not a root of unity and $L = \bigoplus_{n \in \mathbb{Z}} L_n$ is a quantum Lie algebra, then L_0 is a Lie algebra and every L_n is a L_0 -module. Hence, classifying finite dimensional simple quantum Lie algebra will provide us with an opportunity to use the representation

theory of Lie algebras and the techniques employed in the classification of finite dimensional simple Lie super algebras at the same time.

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ON FERMAT'S EQUATION IN THE FIRST CASE

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1. Introduction

We assume that x, y, z are natural numbers and that p is an odd prime such that

$$(1) \quad x^p + y^p = z^p, \quad (x, y, z) = 1, \quad p \nmid xyz.$$

Then by the well-known relations of Barlow and Abel (see [4], p. 53) there are the positive integers a, b, c, x_1, y_1, z_1 such that

$$(2) \quad x + y = c^p, \quad z = cx_1, \quad (c, x_1) = 1,$$

$$(3) \quad z - x = b^p, \quad y = by_1, \quad (b, y_1) = 1,$$

$$(4) \quad z - y = a^p, \quad x = ax_1, \quad (a, x_1) = 1.$$

The prime factors of the numbers x_1, y_1, z_1 are of the form $2np^2 + 1$ [4, p. 58]. Let c_0 be the largest factor of c such that no prime factor of c is of the form $2np + 1$ (notice that $p \nmid c_0$ by (1)). Respectively, we define a_0 and b_0 for a and b . We remark that all the numbers a, b, c, x_1, y_1, z_1 are pairwise relatively prime and $a - c$ and $b - c$ cannot be equal zero.

The purpose of this paper is to give some information on the numbers a, b, c, a_0, b_0, c_0 and also to give some properties of the potential solutions of (1).

2. Results

Theorem 1. The numbers a, b, c, a_0, b_0, c_0 satisfy the condition

$$(5) \quad p^2 a_0 b_0 c_0 \mid a + b - c.$$

Proof. Let q be a prime factor of c_0 with $q^r \parallel c_0$ (i.e., $q^r \mid c_0$, but $q^{r+1} \nmid c_0$). We have to prove $q^r \mid a + b - c$.

By addition it follows from (3) and (4) that

$$a^p + b^p = 2z - (x + y) = z - (x + y - z).$$

Now $p|a^p + b^p$, since $p|x + y - z$, but $p \nmid z$. Because $c|x$ and $c|x + y$, we find by the above equation that $q^r|a^p + b^p$. But it is well-known that every prime factor of $(a^p + b^p)/(a + b)$ has the form $2np + 1$, whence $q^r|a + b$. The same method applying to the expressions $c^p - a^p$ and $c^p - b^p$ gives the corresponding results for b_0 and a_0 .

The result $p^2|a + b - c$ follows, for instance easily from the well-known congruences

$$x + y - z = ax_1 + by_1 - cz_1 \equiv 0 \pmod{p^3},$$

$$x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{p^2}.$$

Since the numbers p^2, a_0, b_0, c_0 are pairwise relatively prime, (5) follows.

Theorem 2. If in (1) $x < y$, then

$$(6) \quad x > p(z - y) > p(a_0b_0c_0p^2 + 1)^p.$$

Proof. Since $z - y < z - x < x + y$, we have, by (2), (3), (4), $a < b < c$. This and the inequality $a + b > c$ ([3], see [4], p. 228 and also [1], pp. 45-47, where a simple proof for this result is given) yield

$$(7) \quad 0 < a + b - c < a.$$

By Theorem 1 it follows from this that $p^2a_0b_0c_0 + 1 \leq a$, which by combining with (4) gives the latter estimate (6). The former estimate (6) has been proved in [1, pp. 43-44].

Our knowledge of the size and of the factors of the integers a_0, b_0, c_0 seems to be very minimal. Clearly 2 always belongs to these factors and 3 and 5 for many p (cf. [3], p. 69). We know a little more of the factors of $a + b - c$ [1, p. 55], namely, that $30p^2|a + b - c$. Thus, for instance, $x > p(30p^2 + 1)^p$. This result, however, is much weaker than some estimates presented in [2, 4].

Let

$$(8) \quad a = a_0\tau_0, \quad b = b_0\sigma_0, \quad c = c_0\delta_0$$

be the given factorizations of a, b, c . It follows from the definition of a_0, b_0, c_0 that

every prime factor of r_0, s_0, t_0 is of the form $2np + 1$.

Assume now that q is a factor of xyz . Since $p \nmid xyz$, we have by Furtwängler's well-known criterion (see [4], p. 169) $q^{p-1} \equiv 1 \pmod{p^2}$. Thus, for instance,

$$a_0^{p-1} \equiv b_0^{p-1} \equiv c_0^{p-1} \equiv 1 \pmod{p^2}.$$

For r_0, s_0, t_0 we easily obtain

$$r_0 \equiv s_0 \equiv t_0 \equiv 1 \pmod{p^2}.$$

In fact, on the one hand $r_0^p \equiv 1 \pmod{p^2}$, since $r_0 \equiv 1 \pmod{p}$ and on the other hand $r_0^p \equiv r_0 \pmod{p^2}$ by Furtwängler's result. Therefore $r_0 \equiv 1 \pmod{p^2}$. Similarly, we get $s_0 \equiv 1, t_0 \equiv 1 \pmod{p^2}$. Now by (8) $a + b - c \equiv a_0 + b_0 - c_0 \pmod{p^2}$. Using (5) and the fact that exactly one of a_0, b_0, c_0 is even, we have

$$(9) \quad a_0 + b_0 - c_0 \equiv 0 \pmod{2p^2}.$$

The following denotes that r_0, s_0, t_0 are relatively large.

Theorem 3. The numbers r_0, s_0, t_0 satisfy the estimates

$$r_0 \geq p^2 b_0 c_0 + 1, \quad s_0 \geq p^2 a_0 c_0 + 1, \quad t_0 \geq p^2 a_0 b_0 + 1$$

and if in (1) $x < y$, then $x > p(a_0 b_0 c_0 p^2 + a_0)^p$.

Proof. Combining (5) and (7) we get $p^2 a_0 b_0 c_0 \leq a + b - c$, which by (7) gives further $p^2 a_0 b_0 c_0 < a$. Since $a = r_0 a_0$, this implies $r_0 \geq p^2 b_0 c_0 + 1$. Similarly, we obtain the estimates for s_0 and t_0 .

Now by (4),

$$x - y = (a_0 r_0)^p \geq (a_0 b_0 c_0 p^2 + a_0)^p$$

and finally from this and from (6) we have the last estimate of our theorem.

Theorem 3 contains the following result presented earlier in [1, Satz XII].

Corollary. Every one of the numbers r_0, s_0, t_0 and so also every one of a, b, c has at least one prime factor of the form $2np^2 + 1$. Thus r_0, s_0, t_0 are not less than $4p^2 + 1$.

Proof. Obviously all the numbers r_0, s_0, t_0 are larger than 1 and so each contains at least one prime factor. This is of the form $2np + 1$ according to the definition (8) of r_0, s_0, t_0 . It has been proved above by Furtwängler's criterion that this factor has the form $2np^2 + 1$.

Clearly, $2p^2 + 1 \equiv 0 \pmod{3}$, whence the smallest prime of the form $2np^2 + 1$ is greater or equal than $4p^2 + 1$. This completes the proof.

Theorem 4. If two of the numbers a_0, b_0, c_0 are congruent $\pmod{p}$ (for instance, only one is greater than 1), then $x \equiv y$, $z \equiv 2x \pmod{p^3}$ and $2^p \equiv 2 \pmod{p^4}$.

Proof. If $a_0 \equiv c_0 \pmod{p}$, then $x \equiv z \pmod{p}$ and so by (1) $y \equiv 0 \pmod{p}$, contrary to the assumption $p \nmid xyz$. Similarly, the case $b_0 \equiv c_0 \pmod{p}$ is impossible. Thus $x \equiv y \pmod{p}$ and as well-known [4, p. 58]

$$(x - y)\left(\frac{x^p - y^p}{x - y} - 1\right) = x^p - x - (y^p - y) \equiv 0 \pmod{p^3}.$$

Here the quotient is divisible by p and so $x \equiv y \pmod{p^3}$. By $x + y - z \equiv 0 \pmod{p^3}$, we have further $2x \equiv z \pmod{p^3}$. Now $x^p \equiv y^p$, $2^p x^p \equiv z^p \pmod{p^4}$ and therefore

$$0 = x^p + y^p - z^p \equiv -(2^p - 2)x^p \pmod{p^4},$$

from which the last congruence of the Theorem at once follows (cf. [5]). This finishes the proof.

Theorem 5. If at least one of the numbers a_0, b_0, c_0 is greater than p^2 , then

$$x > \frac{1}{2}p^{2p-2}(4p^2 + 1)^p > (2p^2)^{2p-1} > p^{4p}.$$

If all these numbers are smaller than p^2 , then

$$(10) \quad a_0 + b_0 = c_0.$$

Proof. By (8) we conclude that such a number of a, b, c , the first factor of which is greater than p^2 , is by virtue of the Corollary greater than $p^2(4p^2 + 1)$. Because $a < b < c$, this holds also for c .

From (1), we have

$$x^p = (z - y)(z^{p-1} + z^{p-2}y + \dots + y^{p-1}) > (z - y)z^{p-1}$$

and $z > \frac{1}{2}(x + y) = \frac{1}{2}c^p$. By these estimates and using (4) and (2) we further find $x^p > (\frac{1}{2}ac^{p-1})^p$. Now the estimate of x presented in the Theorem follows immediately from this, since $c > p^2(4p^2 + 1)$ and, by the Corollary, $a \geq r_0 \geq 4p^2 + 1$.

If all the numbers a_0, b_0, c_0 are smaller than p^2 , then

$$-p^2 < a_0 + b_0 - c_0 < 2p^2.$$

Moreover, $a_0 + b_0 - c_0$ is divisible by $2p^2$ because of (9). Consequently, (10) follows, which completes the proof.

Apparently, (10) seems to be a very strict condition. Can, for instance, $a_0 b_0 c_0$ be equal to 2, 6 or 30? For $a_0 b_0 c_0 = 2$ the only possibility is $(a_0, b_0, c_0) = (1, 1, 3)$. For 6 we get $(1, 2, 3)$ and $(2, 1, 3)$ and for 30 $(1, 5, 6)$, $(5, 1, 6)$, $(2, 3, 5)$ and $(3, 2, 5)$. The Theorem 4 concerns among other things the first case $a_0 b_0 c_0 = 2$, and supports the conjecture that this case is not possible.

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UN MODELE D'ACCORD ENTRE OBSERVATEURS
SUR UNE ECHELLE NOMINALE

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Presented by D.A.S. Fraser, F.R.S.C.

Abstract

By an appropriate hypothesis testing in a simulated example, one can show that Cohen's kappa can sometimes take values whose interpretation may be difficult. A new statistics measuring interobserver agreement is derived which does not involve pairwise considerations. This statistics is compared with kappa's measure and can be interpreted as a sample variance. Some properties relevant to Partition's Theory show that its values are ordered according to adequate agreement criteria.

1. Introduction et notations

Dans une approche nouvelle, on désire étudier la concordance entre d observateurs évaluant n sujets sur une échelle qualitative nominale, d'un point de vue global sans faire intervenir la notion de couple d'observateurs ou de corrélation. On indiquera quels liens existent entre le coefficient kappa introduit par [Co] et le nouveau coefficient proposé, en se plaçant dans le cas non pondéré. On fera appel à la notion de partition reprenant ainsi l'approche de [Tr].

On suppose que les évaluations (ou catégories) sont les éléments d'un ensemble fini d'évaluations E dans N sans structure de cardinal k . Soit $\{i_1, \dots, i_{m_i}\}$, $i=1, n$, le sous-ensemble des m_i évaluations de E données au sujet i . On note z_q^i , $q=1, m_i$, le nombre d'observateurs ayant porté l'évaluation i_q sur le sujet i . La suite $p^i = (z_1^i, \dots, z_{m_i}^i)$ est un élément de l'ensemble D des partitions de l'entier d . On posera $D^* = D \setminus \{d\}$.

2. Les limites de l'interprétation des valeurs de la mesure kappa

{LK} réalisent des tests de comparaison entre différentes statistiques kappa définies sur différents tableaux de données et pour différents systèmes de pondérations; ou des tests de signification sur ces mêmes statistiques. Les interprétations en termes de degrés comparés d'accord interobservateur sont évidemment liées à l'interprétation des différentes valeurs possibles du coefficient $\kappa = (\pi_0 - \pi_c) / (1 - \pi_c)$. Voilà pourquoi les auteurs, cités plus haut, nous donnent quelques "points de repères" sur la correspondance entre certaines plages d'estimations de κ et "l'intensité de l'accord".

A titre d'illustration, reprenons le test de l'hypothèse paramétrique $\kappa = 0$ en appliquant les formules de [Sc] au cas où $\text{Card}(E) = k \geq 4$, k pair, $d \geq 4$, d pair, $2n$ divisible par k ; chaque sujet a reçu deux évaluations distinctes, chacune de la part de la moitié des observateurs; enfin, $\forall q_1, q_2, q_1 \neq q_2, \sum_i z_{q_1}^i = \sum_i z_{q_2}^i$. D'où:

$$\hat{\kappa} = \frac{k(d-2) - 2(d-1)}{2(d-1)(k-1)} \geq \frac{d-3}{3(d-1)}$$

$$\text{s.e.}(\hat{\kappa}) = \frac{\sqrt{2}}{\sqrt{nd(d-1)(k-1)}} \leq \frac{1}{(d-1)\sqrt{n}}$$

$$\hat{\kappa} / \text{s.e.}(\hat{\kappa}) \geq \frac{d-3}{3} \sqrt{n}$$

Par conséquent, la statistique du test de $\kappa = 0$ peut prendre des valeurs arbitrairement grandes avec d ou n . Le rejet de $\kappa = 0$, pour un seuil de signification arbitrairement grand, peut donc se produire dans une situation caractéristique de désaccord. Ajoutons enfin que si, par exemple, $k=d=15$, on obtient $\hat{\kappa} = .426$, valeur reflétant une intensité de l'accord "moyenne" selon [LK], alors que l'on pourrait aussi parler d'absence d'accord.

3. La mesure kappa exprimée comme un écart à une loi marginale

On a les estimations usuelles de π_0 et π_c :

$$\hat{\pi}_0 = \frac{1}{n} \sum_i \frac{1}{d(d-1)} \sum_q z_q^i (z_q^i - 1)$$

$$\hat{\pi}_c = \sum_q \left(\frac{1}{nd} \sum_i z_q^i \right)^2$$

D'où:

$$\hat{\kappa} = \frac{\hat{\pi}_0 - \hat{\pi}_c}{1 - \hat{\pi}_c} = \frac{d}{n(d-1)(1-\hat{\pi}_c)} \sum_i \sum_q \left[\frac{z_q^i}{d} - \frac{\bar{z}_q}{d} \right]^2 - \frac{1}{d-1}$$

où $\bar{z}_q = (1/n) \sum_i z_q^i$.

$\hat{\kappa}$ est donc une fonction croissante des écarts des différentes proportions multinomiales z_q^i/d d'une ligne du tableau (z_q^i) , aux proportions marginales $\bar{z}_q/d$ lorsque celles-ci sont fixées. La forme d'une telle expression prise par $\hat{\kappa}$ est tout à fait cohérente avec l'hypothèse de l'absence d'accord formulée par [F1]: les évaluations sont attribuées suivant des probabilités invariantes d'un sujet à l'autre, estimées par $\bar{z}_q/d$.

4. Notion de désaccord et critères du degré d'accord

L'illustration énoncée plus haut, souligne l'avantage de définir le désaccord sur un sujet donné, comme la présence d'une répartition des observateurs en groupes d'égale importance numérique: la somme des écarts des z_q^i à d/m_i ($p^i \in D^*$), évaluée par $\sum_q (z_q^i - d/m_i)^2$ va donc être un 1^{er} critère du degré de l'accord. Le 2^{ème} critère qui s'impose est le nombre m_i . Comme synthèse des deux critères précédents, le critère de la variance des composantes de p^i ,

$$\text{Var}(p^i) = (1/m_i) \sum_{q=1}^{m_i} (z_q^i - d/m_i)^2$$

répond à de bonnes propriétés pour mesurer le degré de l'accord. En effet, on a les résultats généraux suivants sur l'ensemble D :

Proposition 1. Soit $p=(z_1, \dots, z_m) \in D$. $\text{Var}(p)$ atteint sa valeur minimum si et seulement si $\forall q \in \{1, \dots, m\}, z_q = E p$.

Proposition 2. Soient $p=(z_1, \dots, z_{q'}, \dots, z_{q''}, \dots, z_m) \in D^*$ et $p'=(z_1, \dots, z_{q'}-1, \dots, z_{q''}+1, \dots, z_m)$ telle que $1 \leq q' < q'' \leq m$ et $2 \leq z_{q'} \leq z_{q''} \leq d-1$. Alors: $\text{Var}(p) < \text{Var}(p')$.

Proposition 3. Soient $p=(1^{m-1}, d-m+1) \in D^*$ et $p'=(1^{m-2}, d-m+2)$ telle que $m \geq 3$ et $d \geq 3$. Alors: $\text{Var}(p) < \text{Var}(p')$.

Corollaire. $\text{Var}(p)$ atteint sa valeur maximum si et seulement si $p=(1, d-1)$.

En conclusion, on admettra donc que, si p^i et $p^{i'}$, éléments de D^* , sont des partitions observées sur les sujets i et i' , l'intensité de l'accord a été plus grande dans le cas de i' que dans le cas de i si $\text{Var}(p^i) < \text{Var}(p^{i'})$.

5. Une nouvelle mesure de concordance

Remarque 1. Par convention, on pourra caractériser l'accord parfait à l'aide de la quantité $d^2/4$ qui représente la variance entre les éléments du couple $(0, d)$.

Remarque 2. L'échelle des valeurs de $\text{Var}(p)$, lorsque p décrit D^* , est conservée si l'on ajoute à chacune de ces variances une quantité inférieure à l'écart minimum entre ces différentes variances: $\forall m \geq 2, d^2/(4m^{12d})$ est une quantité adéquate.

En résumé, la statistique $(1/m_i) \sum_{q=1}^{m_i} (z_q^i - d/m_i)^2 + d^2/(4m_i^{12d})$ mesure bien l'accord interobservateur sur le sujet i lorsque $m_i \geq 2$ puisqu'elle préserve l'ordre des valeurs de $(1/m_i) \sum_{q=1}^{m_i} (z_q^i - d/m_i)^2$. Elle atteint

son maximum lorsque $m_i=1$ (accord parfait). Elle approxime la variance de $(z_1^i, \dots, z_m^i)$ lorsque $m_i \geq 2$, et est égale à la variance de $(0, d)$ lorsque $m_i=1$. Elle est comprise entre $d^2/(4d^{12d})$ et $d^2/4$. On définit donc, par normalisation, comme nouvelle mesure de concordance, la moyenne statistique sur les sujets:

$$V = \frac{4}{n} \sum_{i=1}^n \frac{1}{m_i} \sum_{q=1}^{m_i} \left[\frac{z_q^i}{d} - \frac{1}{m_i} \right]^2 + \frac{1}{n} \sum_{i=1}^n m_i^{-12d}$$

V est une fonction croissante des écarts des proportions multinomiales z_q^i/d ($z_q^i \neq 0$) d'une ligne du tableau des z_q^i à une distribution uniforme sur les évaluations prises en compte.

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QUELQUES PROPRIETES DE LA CLOTURE SEPARABLE D'UN ANNEAU CONNEXE

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RESUME

Nous présentons quelques résultats dans le cadre de la théorie de Galois des anneaux commutatifs. Nous avons développé quelques aspects sur la clôture séparable d'un anneau connexe tel que le passage au quotient de et la permanence de certaines propriétés. Ces résultats figurent dans la thèse que j'ai soutenue à l'Université Paul Sabatier à Toulouse. Ces résultats servent entre autre à développer certains points sur les anneaux Henséliens.

1-INTRODUCTION

Les anneaux en question sont commutatifs à élément unité, et les homomorphismes d'anneaux sont unitaires. Un anneau A est dit connexe si ses seuls idempotents sont 0 et 1. Une A -algèbre R est dite extension de A si l'homomorphisme de A dans R : $a \mapsto a.1$ est injectif. On identifie A à son image dans R .

1-1-Définition

Une A -algèbre R est dite séparable si l'homomorphisme ϕ de $R \otimes_A R$ dans R défini par $\phi: \sum r_i \otimes r_i' \mapsto \sum r_i r_i'$ fait de R un $R \otimes_A R$ -module projectif. Une A -algèbre R est dite fortement séparable, si elle est projective et séparable. ([1]).

R est alors une extension finie de A , i.e. le A -module R est de type fini. On note $\text{Gal}(R/A)$ le groupe des automorphismes de A -algèbres de R . Si G est un sous groupe de $\text{Gal}(R/A)$, R^G désigne la sous A -algèbre des éléments de R invariants par ceux de G .

Soit A un anneau connexe. Une A -algèbre R est dite localement fortement séparable si toute partie finie de R est contenue dans une sous-algèbre fortement séparable. $R = \bigcup_{i \in I} R_i$, (R_i/A) est fortement séparable pour tout $i \in I$. R est donc entière et fidèlement plate sur A .

1-2-Définition

Soit R un anneau connexe et G un groupe fini d'automorphismes de R et soit $A = R^G$. Si la A -algèbre R est séparable on dit que R est une extension galoisienne de A . (R est alors fortement séparable).

Une extension connexe non nécessairement finie (R/A) où $R = \bigcup_{i \in I} R_i$, est galoisienne si (R_i/A) l'est pour tout $i \in I$. On dit que R est une clôture séparable de A si pour toute R -algèbre fortement séparable connexe Γ on a: $\Gamma = R$.

On note $\Omega_j(A)$ cette clôture. Soit $G = \text{Gal}(\Omega_j(A)/A)$ et $G_i = \text{Gal}(R_i/A)$. Alors on a: $G = \varprojlim G_i$, G n'est pas vide et c'est un groupe topologique totalement discontinu et $\Omega_j(A)^G = A$. (I.D.M.)

II-PROPRIETES DES IDEAUX MAXIMAUX DANS UNE CLOTURE SEPRABLE

On établit les liens entre la séparabilité et la non ramification et on démontre :

II-1-Proposition .

Soit A un anneau local d'idéal maximal m , $k = A/m$ le corps résiduel de A et $\Omega_j(A)$ la clôture séparable de A . Alors la A -algèbre $\Omega_j(A)$ est non ramifiée. [Proposition 1.5.11.(T)]

Soit S une extension galoisienne finie d'un anneau connexe A et q et q' deux idéaux maximaux de S au dessus d'un même idéal maximal de A .

Alors il existe $\sigma \in \text{Gal}(S/A)$ tel que $q' = \sigma(q)$. Ref[ChapVAlgCTh.2.p42 (B)]. Puis on passe aux idéaux maximaux et en tenant compte du fait que $G = \varprojlim G_i$ et que $R = \bigcup_{i \in I} R_i$, on démontre :

II-2-Proposition

Soit A un anneau connexe $\Omega_j(A)$ sa clôture séparable. Soient q et q' deux idéaux maximaux de $\Omega_j(A)$ au dessus d'un même idéal maximal de A .

Alors il existe $\sigma \in \text{Gal}(\Omega_j(A)/A)$ tel que $q' = \sigma(q)$. Si A est local Soient q et q' deux idéaux maximaux de $\Omega_j(A)$. Alors il existe $\sigma \in \text{Gal}(\Omega_j(A)/A)$ tel que $q' = \sigma(q)$. [Proposition 1.5.13.(T)]

En considérant les propriétés topologiques de G on démontre:

II-3- Proposition.

Soit A un anneau local, $\Omega_j(A)$ la clôture séparable de A , $G = \text{Gal}(\Omega_j(A)/A)$; q , un idéal maximal de $\Omega_j(A)$, $H = G^Z(q)$ le groupe de décomposition de q par rapport à A c.a.d. $H = \{ \sigma \in G / \sigma(q) = q \}$ alors H est un sous groupe fermé de G . [Proposition 1.5.15.(T)].

Il en découle :

II-4-Corollaire.

Soit A un anneau local $\Omega_j(A)$ la clôture séparable de A et q un idéal maximal de $\Omega_j(A)$. Soit H le groupe de décomposition de q et $\Omega_j(A)^H$ l'anneau de décomposition de q .

Alors $\Omega_j(A)$ a un seul idéal maximal au dessus de $q \cap \Omega_j(A)^H$

[Corollaire I.5.16.(T)].

III-CLOTURE SEPARABLE D'UN ANNEAU LOCAL INTEGRALEMENT CLOS.

On étudie d'abord les anneaux locaux intégralement clos et on démontre :

III-1-Lemme .

Soit A un anneau local intégralement clos de corps des fractions K . Une A -algèbre fortement séparable connexe R est un anneau intégralement clos de corps des fractions L extension séparable de K . [Proposition II.1.1.(T)]

On en Dédult :

III-2-Proposition .

Soit A un anneau local intégralement clos de corps des fractions K . La clôture séparable $\Omega_j(A)$ de A est un sous anneau intégralement clos de $\Omega_s(A)$ la fermeture intégrale de A dans $\Omega_j(K)$ la clôture algébrique séparable du corps K .

Si A n'est pas un corps, l'inclusion de $\Omega_j(A)$ dans $\Omega_s(A)$ est stricte.

[Proposition II.1.2.(T)]

IV-CLOTURE SEPARABLE D'UN QUOTIENT D'UN ANNEAU LOCAL INTEGRALEMENT CLOS

Un anneau local A peut s'exprimer sous la forme $A=B/b$ où B est un anneau local intégralement clos. On étudie les extensions galoisiennes dans le cas d'un anneau local intégralement clos et on passe au quotient. Toutefois ce passage peut faire apparaître des idempotents et alors on utilise la notion d'extension galoisienne finie sous sa forme la plus générale présentée dans (CH) ; et on démontre en faisant intervenir la notion d'élément primitif :

IV-1-Proposition (de relèvement)

Soit A un anneau local, $\mathfrak{a}$ un idéal de A distinct de A et $\bar{A}$ une $\bar{A}$ -algèbre fortement séparable connexe, où $\bar{A} = A/\mathfrak{a}$. On suppose que $\bar{A}$ admet un élément primitif ce qui est réalisé en particulier dans l'un des deux cas suivants : I) le corps résiduel k de A est infini. II) l'idéal $\mathfrak{a}$ est l'idéal maximal $\mathfrak{m}$ de A .

Alors : (1) Il existe une A -algèbre fortement séparable T telle que la $\bar{A}$ -algèbre $\bar{T} = T/\mathfrak{a}T$ soit isomorphe à $\bar{A}$. (2) Si $\bar{A}$ est galoisienne sur $\bar{k}$, on peut choisir T galoisienne sur A . [Proposition II.2.3(T)].

IV-2-Proposition

Soit A un anneau local, $\mathfrak{a}$ un idéal de A distinct de A , et $\bar{A}$ l'anneau quotient $A/\mathfrak{a}$. (1) Il existe un homomorphisme canonique ϕ de $\bar{A}$ -algèbre de $\overline{\Omega_j(A)} = (\Omega_j(A))/\mathfrak{a}\Omega_j(A)$ dans $\Omega_j(\bar{A})$. (2) Si toute $\bar{A}$ -algèbre fortement séparable admet un élément primitif, en particulier si le corps résiduel k de A est infini ou si l'idéal $\mathfrak{a}$ est l'idéal maximal $\mathfrak{m}$ de A l'homomorphisme ϕ de (1) est surjectif. [Proposition II.2.4 (T)]

IV-3-Corollaire .

Soit A un anneau local d'idéal maximal $\mathfrak{m}$ et $k = A/\mathfrak{m}$. Soit $\mathfrak{q}$ un idéal maximal de $\Omega_j(A)$. Alors la clôture séparable $\Omega_j(k)$ de k est isomorphe à $\Omega_j(A)/\mathfrak{q}$. [Corollaire II.2.5.(T)] .

IV-4-Théorème .

Soit A un anneau local intégralement clos, $\mathfrak{a}$ un idéal de A distinct de A , ϕ un homomorphisme de $\bar{A}$ -algèbre de $\overline{\Omega_j(A)}$ dans $\Omega_j(\bar{A})$ comme dans la proposition [Prop.VI.2.]. On suppose que toute $\bar{A}$ -algèbre fortement séparable connexe admet un élément primitif ce qui permet de supposer que ϕ est surjectif. U étant un élément maximal de l'ensemble des sous algèbres localement fortement séparables de $\Omega_j(A)$ d'image connexe dans $\Omega_j(\bar{A})$ par ϕ . Alors la restriction de ϕ à U est un isomorphisme de U sur $\Omega_j(\bar{A})$ [Théorème II.2.7.(T)] .

V- CLOTURE SEPARABLE D'UN ANNEAU DE POLYNOMES D'UN ANNEAU LOCAL INTEGRALEMENT CLOS.

La clôture séparable est définie pour des anneaux connexes c'est pourquoi on s'est limité à certains anneaux.

V-1-Lemme .

Soit A un anneau local intégralement clos et soit S une A -algèbre fortement séparable connexe. Alors la $A[t]$ -algèbre $S[t]$ est fortement séparable et elle est connexe. [Lemme II.3.1].

V-2-Corollaire .

Soit A un anneau local intégralement clos on a l'inclusion : $\Omega_j(A[t]) \supset (\Omega_j(A))[t]$ [Corollaire II.3.2(T)].

V-3-Corollaire

Soit A un anneau local intégralement clos et S une A -algèbre fortement séparable connexe. Alors $S[t_1, \dots, t_n]$ est une $A[t_1, \dots, t_n]$ -algèbre fortement séparable connexe et on a : $(\Omega_j(A))[t_1, \dots, t_n]$ est inclus dans $\Omega_j(A[t_1, \dots, t_n])$. [Corollaire II.3.3.(T)]

VI. CLOTRE SEPARABLE D'UN ANNEAU DE SERIES FORMELLES

Soit k un corps et $A = k[[t]]$. Alors l'anneau A est local intégralement clos et hensélien.

VI-1-Proposition .

Pour qu'une A -algèbre S soit fortement séparable et connexe il faut il suffit qu'elle soit isomorphe à $k'[[t]]$ où k' est une extension algébrique séparable de k . [Proposition .II.4.1(T)]

VI-2-Corollaire

Soit k un corps. Alors la clôture séparable $\Omega_j(k[[t]])$ de $k[[t]]$ est égale à $(\Omega_j(k))[[t]]$. Elle est isomorphe à $\Omega_j(k) \otimes_k k[[t]]$. Les groupes de Galois de $\Omega_j(k[[t]])$ et $\Omega_j(k)$ respectivement sur $k[[t]]$ et sur k sont isomorphes. [Corollaire II.4.2(T)].

Remarque Ce résultat reste vrai pour $A[[t]]$ si A/m est infini, ($A[[t]]$ est alors local intégralement clos et de corps résiduel infini.)

VI-3- Corollaire.

Soit k un corps. Alors la clôture séparable $\Omega_j(k[t_1, \dots, t_n])$ de $k[t_1, \dots, t_n]$ est égale à $(\Omega_j(k))[t_1, \dots, t_n]$. Elle est isomorphe à $\Omega_j(k) \otimes_k k[t_1, \dots, t_n]$. Les groupes de Galois de $\Omega_j(k[t_1, \dots, t_n])$ et $\Omega_j(k)$ respectivement sur $k[t_1, \dots, t_n]$ et sur k sont isomorphes. (Corollaire II.4.3 (T))

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A note on the equation $x^2 - 1 = y^2(z^2 - 1)$

by Maurice Mignotte, Strasbourg

Presented by P. Ribenboim, F.R.S.C.

Abstract.— We consider the equation $x^2 - 1 = y^2(z^2 - 1)$, and prove that the greatest prime factor of y can not be very small when both z is large enough and y is large enough with respect to z .

I. Introduction

This paper was motivated by a very recent work of E. Thomas [Th] who considered a family of cubic equations. In both cases the units of the order we work in are known, and linear forms in logarithms can be applied and explicit estimates can be given.

The result we prove is the following.

THEOREM. — *Consider the equation in positive rational integers*

$$(*) \quad x^2 - 1 = y^2(z^2 - 1), \quad z \geq 2, y > 0.$$

Let $\eta > 0$ be a fixed number. Then, for z large enough, namely for

$$(i) \quad \text{Log } z > \max \{20, \exp(e^{77}(1 + \eta)/\eta^2)\},$$

and for y large enough with respect to z , i.e. for

$$(ii) \quad y \geq \exp(\{\text{Log}(2z)\}^{2+\eta}),$$

we have

$$(**) \quad P(y) > \frac{\eta}{2.1(2+\eta)} \text{Log Log } y,$$

where, as usual, $P(y)$ denotes the greatest prime factor of y .

II. Proof

Let $\eta > 0$ be a fixed number. Consider positive integers x , y and z which satisfy conditions (*), (i) and (ii) above.

Writing equation (*) in the Pellian form $x^2 - Dy^2 = 1$ with $D = z^2 - 1$, we deduce (see for example [Na]) that there exists a positive integer m such that

$$y = \frac{\varepsilon^m - \varepsilon^{-m}}{2\sqrt{z^2 - 1}},$$

where $\varepsilon = z + \sqrt{z^2 - 1}$ is a fundamental unit of the order $\mathbb{Z}[\sqrt{z^2 - 1}]$.

Since (ii) holds, $m \text{Log } \varepsilon > (\text{Log}(2z))^{2+\eta}$ and

$$(1) \quad m > (\text{Log}(2z))^{1+\eta}.$$

We write the factorization of y in a product of prime numbers as $y = p_1^{r_1} \cdots p_s^{r_s}$, with $p_1 < \dots < p_s = P$.

Put

$$\Lambda = r_1 \text{Log } p_1 + \dots + r_s \text{Log } p_s - m \text{Log } \varepsilon + \text{Log}(2\sqrt{z^2 - 1}).$$

It is easy to get an upper bound for Λ :

$$\text{Log } |\Lambda| = \text{Log} |\text{Log}(1 - \varepsilon^{-2m})| \leq \text{Log}(2\varepsilon^{-2m}) < -2m \text{Log } z.$$

Using the result of M. Waldschmidt given in [Wa] we obtain the following lower bound

$$\begin{aligned} \text{Log } |\Lambda| \geq & -2^{8(s+2)} (s+2)^{2(s+2)} 2^{s+4} (\text{Log } P)^s (\text{Log}(2z))^2 \\ & \times \text{Log}(2\varepsilon \text{Log } z) \left(\text{Log } B + \text{Log} \{2\varepsilon \text{Log}(2z)\} \right), \end{aligned}$$

where $B = \max \{r_1, \dots, r_s, m\}$.

The estimate $y < \varepsilon^m$ implies $r_i \leq m(\log \varepsilon / \log 2)$ for $i = 1, \dots, s$; thus $r_i < \frac{1}{\log 2} m \log(2z)$ and, for $\log z > 20$,

$$\log |\Lambda| \geq -c_0 (s+2)^4 (2^9(s+2)^2 \log P)^s (\log z)^2 (\log \log z) \log(m \log z).$$

where $c_0 = 3.26 \times 10^6$.

Comparing the upper bound and the lower bound of $\log |\Lambda|$ and using (1), we get

$$\frac{m^{\eta/(1+\eta)}}{(\log m)^2} \leq 2(c_0 + 1)(s+2)^4 (2^9(s+2)^2 \log P)^s.$$

For $\log m \geq 2 \times 10^6 (1+\eta)^2 / \eta^2$ — which is true by inequalities (1) and (i) — this implies

$$\xi \log m \leq s \log (2^9(s+2)^2 \log P) + \log (2(c_0 + 1)(s+2)^4),$$

where $\xi = 0.999 \eta / (1 + \eta)$.

Suppose first that $\log P \leq 2 \log(s+2)$. Then,

$$\begin{aligned} \xi \log m &\leq 2s \log s \left(\frac{\log(s+2)}{\log s} + \frac{\log(2^{10} \log(s+2))}{2 \log s} + \frac{\log((c_0 + 1)(s+2)^4)}{2s \log s} \right) \\ &\leq 2.09 s \log s, \end{aligned}$$

for $\log z > \exp\{e^{77}(1+\eta)/\eta^2\}$.

In this case, the well-known inequality $p_s \geq s \log s$ (see [R-S]) implies

$$P > \frac{\eta}{2.1(1+\eta)} \log m.$$

When $\log P > 2 \log(s+2)$, it is easy to verify that the previous lower bound is still true.

The conclusion follows from the obvious estimate $y < \varepsilon^m$ which, by (1), implies

$$\text{Log Log } y < \text{Log } m + \text{Log Log } \varepsilon < \text{Log } m + \text{Log Log } (2z) < 2 \text{Log } m.$$

III. A variant

If we consider the special case of equation (*) when y has only one prime factor, for example when y is a power of 2, arguing in the same way than above, one can prove the following result.

PROPOSITION. — *There exist effectively computable absolute constants c_1 and c_2 such that if positive rational integers x , k and z satisfy*

$$x^2 - 1 = 2^{2k} (z^2 - 1), \quad z > c_1,$$

then

$$k < c_2 \cdot \text{Log } z \cdot (\text{Log Log } z)^2.$$

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EMPIRICAL TYPE PROCESSES AND CONTIGUITY

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Abstract: We consider two-time parameter empirical type processes based on observations, ranks and sequential ranks. We describe some weighted asymptotic behaviour of these processes in a unified way under the null hypothesis of observations being identically distributed, as well as under a class of contiguous alternatives.

1. Introduction. Let $X_1, X_2, \dots$ be independent random variables. We wish to test the null hypothesis

$$H_0 : X_i, i \leq i \leq n, \text{ have the same distribution } F,$$

versus the alternative hypothesis

$$H_1 : X_i, i \leq i \leq n, \text{ have the respective distribution functions } F_{in},$$

where we assume that all F_{in} are absolutely continuous with respect to the distribution function F and

$$(1.1) \quad \left[\frac{dF_{in}}{dF} (F^{-1}(u)) \right]^{1/2} = 1 + \frac{1}{2n^{1/2}} h_n(t, u), \quad \frac{i-1}{n} < t \leq \frac{i}{n},$$

where $F^{-1}(u) = \inf \{x : F(x) \geq u\}$, $0 \leq u \leq 1$, $F^{-1}(0) = F^{-1}(0^+)$. We assume also that there exists a function $h \in L^2[0, 1]^2$ such that

$$(1.2) \quad \int_0^1 h(t, u) du = 0 \quad \text{for almost all } t \in [0, 1]$$

and

$$(1.3) \quad \int_0^1 \int_0^1 \left[h_n(t, u) - h\left(\frac{[nt]}{n}, u\right) \right]^2 dudt \rightarrow 0, \quad n \rightarrow \infty.$$

It is known that the sequence of direct products $F_{1n} \times \dots \times F_{nn}$, $n = 1, 2, \dots$ is contiguous (for the notion of contiguity see eg. Le Cam, 1960, 1986; Greenwood and Shirayev, 1985; Hájek and Šidak, 1986; Roussas, 1972) to the sequence $F \times \dots \times F$ (cf. Oosterhoff and Van Zwet, 1975).

Let $R_{1n}, \dots, R_{nn}$ denote the normalized ranks

$$R_{kn} = n^{-1} \sum_{i=1}^n \mathbf{1}\{X_i \leq X_k\}, \quad k = 1, \dots, n,$$

and let $\xi_1, \dots, \xi_n$ be the normalized sequential ranks

$$\xi_k = k^{-1} \sum_{i=1}^k 1\{X_i < X_k\}, \quad k = 1, \dots, n,$$

of the first n of the random variables $X_1, X_2, \dots$.

Define the two-time parameter empirical processes

$$(1.4) \quad \alpha_n(s, t) = n^{-1/2} \sum_{i=1}^{[nt]} (1\{F(X_i) \leq s\} - s),$$

$$(1.5) \quad \beta_n(s, t) = n^{-1/2} \sum_{i=1}^{[nt]} (1\{R_{in} \leq s\} - s),$$

$$(1.6) \quad \gamma_n(s, t) = n^{-1/2} \sum_{i=1}^{[nt]} (1\{\xi_i \leq s\} - s),$$

where $0 \leq s, t \leq 1$, and F is assumed to be continuous. Such empirical processes were studied by Pardzhanadze and Khmaladze (1986) under the class of contiguous alternatives H_1 .

We introduce the "bridge" like (tied down at $t = 0$ and $t = 1$) empirical processes

$$\alpha_n(s, t) - t\alpha_n(s, 1)$$

as well as

$$\gamma_n(s, t) - t\gamma_n(s, 1)$$

and note also that $\beta_n(s, t)$ has such a structure already. We present asymptotic results for weighted versions of these processes under the null hypothesis H_0 and under the sequence of contiguous alternatives of H_1 . For proofs of all the presented results we refer to a forthcoming paper by the author.

2. Weighted Empirical Processes Based on Observations. We will assume throughout, without loss of generality, that all random variables and stochastic processes introduced so far and later on, are defined on the same probability space.

We introduce

$$\hat{\alpha}_n(s, t) = \begin{cases} n^{-1/2} \left(\sum_{i=1}^{[(n+1)t]} 1\{F(X_i) \leq s\} - \frac{[(n+1)t]}{n} \sum_{i=1}^n 1\{F(X_i) \leq s\} \right) & , 0 \leq t < 1, 0 \leq s \leq 1 \\ 0 & , t = 1, 0 \leq s \leq 1. \end{cases}$$

We will call q a positive function on $(0, 1)$ if $q : (0, 1) \rightarrow (0, \infty)$ is such that

$$(2.1) \quad \inf_{\delta \leq t \leq 1-\delta} q(t) > 0 \quad \text{for all } 0 < \delta < 1/2.$$

Let $\{\Gamma(s, t); 0 \leq s \leq 1, 0 \leq t \leq 1\}$ denote a Gaussian process with mean zero and covariance function

$$(2.2) \quad E\Gamma(s_1, t_1)\Gamma(s_2, t_2) = (s_1 \wedge s_2 - s_1 s_2)(t_1 \wedge t_2 - t_1 t_2).$$

THEOREM 2.1. *We assume that H_0 holds. Then, there exists a Kiefer process $K(\cdot, \cdot)$ such that with the sequence of stochastic processes $\Gamma_n(\cdot, \cdot)$*

$$\begin{aligned} \{\Gamma_n(s, t), 0 \leq s \leq 1, 0 \leq t \leq 1\} &= \{n^{-1/2}(K(s, nt) - tK(s, n)), 0 \leq s \leq 1, 0 \leq t \leq 1\} \\ &\stackrel{D}{=} \{\Gamma(s, t), 0 \leq s \leq 1, 0 \leq t \leq 1\} \text{ for each } n \geq 1, \end{aligned}$$

and with a weight function q which is positive on $(0, 1)$ we have, as $n \rightarrow \infty$

(a) if

$$(2.3) \quad \lim_{t \downarrow 0} (t(1-t) \log \log 1/t(1-t))^{1/2}/q(t) = \lim_{t \uparrow 1} (t(1-t) \log \log 1/t(1-t))^{1/2}/q(t) = 0,$$

then

$$\sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\hat{\alpha}_n(s, t) - \Gamma_n(s, t)|/q(t) = o_P(1),$$

(b) if

$$(2.4) \quad \begin{cases} \lim_{t \downarrow 0} (t(1-t) \log \log 1/t(1-t))^{1/2}/q(t) < \infty \\ \text{and} \\ \lim_{t \uparrow 1} (t(1-t) \log \log 1/t(1-t))^{1/2}/q(t) < \infty, \end{cases}$$

then

$$\sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\hat{\alpha}_n(s, t) - \Gamma_n(s, t)|/q(t) = O_P(1),$$

(c) if (2.4) holds then we have also

$$\sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\hat{\alpha}_n(s, t)|/q(t) \xrightarrow{D} \sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\Gamma(s, t)|/q(t).$$

REMARK 2.1. We emphasize that (b) and (c) do not follow from (a), nor does (c) from (b).

The proof of Theorem 2.1 is based on a similar weighted approximation theorem for the two-time parameter empirical process of (1.4). Proofs of both theorems as well as further details will appear in the mentioned forthcoming paper by the author.

3. Ranks. We define

$$\hat{\beta}_n(s, t) = \begin{cases} n^{-1/2} \sum_{i=1}^{[(n+1)t]} (1\{R_{in} \leq s\} - \frac{[ns]}{n}) & , 0 \leq s \leq 1, 0 \leq t < 1 \\ 0 & , 0 \leq s \leq 1, t = 1. \end{cases}$$

THEOREM 3.1. We assume that H_0 holds. Then Theorem 2.1 holds true with the same Kiefer process $K(\cdot, \cdot)$ and $\hat{\alpha}_n(s, t)$ replaced by $\hat{\beta}_n(s, t)$.

The proof of Theorem 3.1 is based on the fact that

$$\tilde{\beta}_n(s, t) = n^{-1/2} \sum_{i=1}^{[nt]} (1\{R_{in} \leq s\} - \frac{[ns]}{n}) = n^{-1/2} \sum_{i=1}^{[nt]} (1\{F(X_i) \leq U_n(s)\} - \frac{[ns]}{n}),$$

where $U_n(s)$ is defined by

$$(3.1) \quad U_n(s) = \begin{cases} 0 & \text{for } s \in [0, 1/n) \\ U_{k:n} = F(X_{k:n}) & \text{for } s \in [k/n, (k+1)/n), k = 1, \dots, n-1 \\ U_{n:n} & \text{for } s = 1, \end{cases}$$

and $X_{1:n} \leq \dots \leq X_{n:n}$ denote the order statistics of $X_1, \dots, X_n$. We note also that the process $\tilde{\beta}_n(s, t)$ already has a "bridge" type structure as it is, namely we have

$$\tilde{\beta}_n(s, t) = \alpha_n(U_n(s), t) - \frac{[nt]}{n} \alpha_n(U_n(s), 1).$$

4. Sequential Ranks. We introduce

$$\hat{\gamma}_n(s, t) = \begin{cases} n^{-1/2} \left(\sum_{i=1}^{[(n+1)t]} 1\{\xi_i \leq s\} - \frac{[(n+1)t]}{n} \sum_{i=1}^n 1\{\xi_i \leq s\} \right) & , 0 \leq t \leq 1, 0 \leq s \leq 1 \\ 0 & , t = 1, 0 \leq s \leq 1. \end{cases}$$

Let $\{\Gamma(s, t), 0 \leq s \leq 1, 0 \leq t \leq 1\}$ denote again the Gaussian process with mean zero and covariance function (2.2).

THEOREM 4.1. We assume that H_0 holds. Then there exists a sequence of stochastic processes $\Gamma_n, n \geq 1$,

$$\{\Gamma_n(s, t), 0 \leq s \leq 1, 0 \leq t \leq 1\} \stackrel{P}{\Rightarrow} \{\Gamma(s, t), 0 \leq s \leq 1, 0 \leq t \leq 1\}$$

such that with a weight function q which is positive on $(0, 1)$, we have as $n \rightarrow \infty$

(a) if (2.3) holds, then

$$\sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\hat{\gamma}_n(s, t) - \Gamma_n(s, t)|/q(t) = o_P(1),$$

(b) if (2.4) holds, then

$$\sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\hat{\gamma}_n(s, t) - \Gamma_n(s, t)|/q(t) = O_P(1),$$

(c) if (2.4) holds, then

$$\sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\hat{\gamma}_n(s, t)|/q(t) \xrightarrow{P} \sup_{0 < t < 1} \sup_{0 \leq s \leq 1} |\Gamma(s, t)|/q(t).$$

5. Contiguous Alternatives. Let

$$(5.1) \quad d(s, t) = c(s, t) - tc(s, 1),$$

where

$$c(s, t) = \int_0^t \int_0^s h(\tau, u) du d\tau$$

and $h(\cdot, \cdot)$ is the function defined by (1.1)-(1.3).

Let also

$$(5.2) \quad e(s, t) = r(s, t) - tr(s, 1),$$

where

$$r(s, t) = - \int_0^t \frac{c(s, \tau)}{\tau} d\tau.$$

THEOREM 5.1. We assume that H_1 holds.

(a) Let q be positive on $(0, 1)$ and such that (2.3) holds. Then we have, as $n \rightarrow \infty$

$$(5.3) \quad \hat{\alpha}_n(s, t)/q(t) \xrightarrow{P} (\Gamma(s, t) + d(s, t))/q(t),$$

$$(5.4) \quad \hat{\beta}_n(s, t)/q(t) \xrightarrow{P} (\Gamma(s, t) + d(s, t))/q(t),$$

$$(5.5) \quad \hat{\gamma}_n(s, t)/q(t) \xrightarrow{P} (\Gamma(s, t) + c(s, t))/q(t),$$

in $D(0, 1)^2$, where $\{\Gamma(s, t), 0 \leq s \leq 1, 0 \leq t \leq 1\}$ is a Gaussian process with mean zero and covariance as in (2.2).

(b) Let q be positive on $(0, 1)$. Then (5.3), (5.4) and (5.5) hold true in $D([0, 1] \times [\delta, 1 - \delta])$ for any $0 < \delta < 1/2$.

The proof of Theorem 5.1 is based on a similar weak convergence theorem under the alternative hypothesis H_1 for weighted two-time parameter empirical processes of (1.4) and (1.6). Proofs of these theorems will appear in a forthcoming paper by the author, where we also discuss these results in connection with the so-called changepoint problem. Contiguous alternatives of H_1 as parametrized in (1.1)-(1.3) accommodate the notation of a changepoint by assuming that there exists $\lambda \in (0, 1)$ such that $g(t, u) = 1\{t \geq \lambda\}g(u)$ for some square integrable function g . Csörgő and Horváth (1988) studied U -statistic type processes in connection with some changepoint problems. Introducing weight functions, they obtain results under the null hypothesis H_0 , while Szyszkowicz (1991) studies such processes under the class of contiguous alternatives of H_1 .

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A Generalization of Ramanujan's Best Theorem on Continued Fractions

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Abstract. A ${}_9F_8$ series contiguous relation derived by Wilson is used to generalize Ramanujan's best theorem on continued fractions.

1. Introduction. Chapter 12 of Ramanujan's second notebook [9] is almost entirely devoted to continued fractions. The entries have recently been examined by B. Berndt et al. [3]. A principal formula (nine other entries depend on it) involving gamma functions and which is described as his "crowning achievement" is

Entry 40. Let $P = \prod \Gamma((1 + \alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon)/2)$ (0, 2 or 4 minus signs) and $Q = \prod \Gamma((1 + \alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon)/2)$ (1 or 3 minus signs) where there are eight gamma functions in each product and the signs are chosen so that the argument of each gamma function contains one of the specified number of minus signs. Then, if one of the parameters $\beta, \gamma, \delta, \epsilon$ is equal to a nonzero integer,

$$(1.1) \quad \frac{P-Q}{P+Q} = \frac{8\alpha\beta\gamma\delta\epsilon}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \dots$$

$$\text{where} \quad \beta_n = 64\pi(\alpha^2 - n^2),$$

$$\alpha_n = (2n+1)\{2(1+\Sigma\alpha^4) - (\Sigma\alpha^2 - (2n^2+2n+1))^2 - (2n^2+2n+2)^2\},$$

with these products and sums ranging over the parameters $\alpha, \beta, \gamma, \delta, \epsilon$.

Watson [10] has given the only published proof of Entry 40 and, although the continued fraction terminates, called it the "best of the theorems on continued fractions to be found in Ramanujan's manuscript notebook". He also noted that (contrary to Ramanujan's original claim) (1.1) is not true when α as a nonzero integer causes the continued fraction to terminate. The formula is also not valid whenever the continued fraction fails to terminate, although it is known to converge in general to a symmetric meromorphic function of the parameters $\alpha, \beta, \gamma, \delta, \epsilon$ (cf. L. Jacobsen [5]). Only by obtaining a closed form for this meromorphic function can one fully understand Entry 40 (see Corollary 5 of Section 3).

In current work [4], [7], [8], [12] a connection is made between several of Ramanujan's Chapter 12 Entries (Entries 25, 33-36 and 39) and the general theory of hypergeometric orthonormal functions (cf. Askey and Wilson [1] and Wilson [11]). Here we extend this connection to include Entry 40 using the methods in [8]. This approach also yields the following missing companion to Entry 40.

Proposition 1. Let $P' = \prod \Gamma((3 + \alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon)/4)$ (0, 2 or 4 minus signs) and $Q' = \prod \Gamma((1 + \alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon)/4)$ (1 or 3 minus signs). Then, if one of the parameters $\beta, \gamma, \delta, \epsilon$ is an odd integer,

$$(1.2) \quad \frac{Q'}{P'} = \frac{-1}{\alpha_0} - \frac{2b_1}{\alpha_1} - \frac{b_2}{\alpha_2} - \frac{b_3}{\alpha_3} - \dots,$$

$$\begin{aligned}\text{where } b_n &= (\Pi((2n-1)^2 - \alpha^2))/(16)^3(2n-1)^2, \\ a_n &= \{2n^6 + n^4(5 - \Sigma\alpha^2)/4 + n^2(-26 + (1 + \Sigma\alpha^2)^2 - 2\Sigma\alpha^4)/64 - a_0\}/(4n^2 - 1), \\ a_0 &= \{2(1 - \Sigma\alpha^4) + (1 - \Sigma\alpha^2)^2 - 8\Pi\alpha\}/(16)^2\end{aligned}$$

with these products and sums ranging over the parameters $\alpha, \beta, \gamma, \delta, \epsilon$.

We call this a missing companion to Entry 40 because, if we formally replace α by $x + \epsilon$ and let $\epsilon \rightarrow \infty$ we obtain a generalization of Entry 39 [8, Theorem 4], whereas this same procedure applied to Entry 40, yields Entry 35 (cf. Watson [10]).

In what follows we obtain nonterminating versions of (1.1) and (1.2) by considering two solutions to a contiguous relation derived by Wilson [11] and applying Pincherles' theorem [6].

2. Contiguous Relation. Consider a terminating 2-balanced very well poised ${}_8F_8$ hypergeometric function

$$F = F(a; b, c, d, e, f, g, h) := {}_8F_8 \left(\begin{matrix} a, \frac{a}{2} + 1, b, c, d, e, f, g, h \\ \frac{a}{2}, 1 + a - b, 1 + a - c, \dots, 1 + a - h \end{matrix}; 1 \right),$$

where $h = -n$, $n = 0, 1, \dots$ and $g = n + s - 1$, $s = 3 + 3a - b - c - d - e - f$. Wilson [11, (4.51)] has derived the contiguous relation

$$\begin{aligned}(2.1) \quad & \frac{h(a-h)_2(g+b-a-1)(g+c-a-1)\dots(g+f-a-1)}{h-g+1} [F(g-, h+) - F] \\ & - \frac{-g(a-g)_2(h+b-a-1)(h+c-a-1)\dots(h+f-a-1)}{g-h+1} [F(g+, h-) - F] \\ & - (g-h)(g+h-a-1)bcdef F = 0\end{aligned}$$

where $(a)_m = \Gamma(a+m)/\Gamma(a)$ and $F(g \mp, h \pm)$ is F with g and h replaced by $g \mp 1$ and $h \pm 1$ respectively. Equation (2.1) yields the recurrence relation for Wilson's biorthogonal rational functions which satisfy the most general explicit hypergeometric orthogonality currently known (cf. Wilson [11, (3.9), (3.11), (4.55)]).

We may reexpress (2.1) by stating that the linear second order difference equation

$$\begin{aligned}(2.2) \quad & X_{n+1} - a_n X_n + b_n X_{n-1} = 0, \quad n \geq 0, \\ b_n &= n(n+s-2)(n+a-b)(n+a-c)\dots(n+a-f)(n+s+b-a-2) \\ & \times (n+s+c-a-2)\dots(n+s+f-a-2)/(2n+s-1)(2n+s-2)^2(2n+s-3), \\ a_n &= \left[\frac{(n+s-1)(n+s-1-a)(n+s-2-a)(n+a+1-b)(n+a+1-c)\dots(n+a+1-f)}{(2n+s)} \right. \\ & + \frac{n(n+a)(n+a+1)(n+s+b-a-2)(n+s+c-a-2)\dots(n+s+f-a-2)}{(2n+s-2)} \\ & \left. + (2n+s-1)(s-a-2)bcdef/(2n+s-1)(n+s-a-2)(n+a+1) \right]\end{aligned}$$

has a solution

$$\begin{aligned}(2.3) \quad & X_n^{(1)} = \Gamma \left(\begin{matrix} n+s-1, n+s-a-1, n+a+1-b, n+a+1-c, \dots, n+a+1-f \\ 2n+s-1, n+a+1 \end{matrix} \right) \\ & \times F(a; b, c, d, e, f, n+s-1, -n).\end{aligned}$$

In (2.3) we use the multiple gamma notation

$$\Gamma \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \right) = \left(\prod_{i=1}^r \Gamma(a_i) \right) / \left(\prod_{i=1}^s \Gamma(b_i) \right).$$

Note that for the exceptional values $s = 1, 2$ the a_n, b_n and b_{n+1} in (2.2) and $X_n^{(1)}, X_{n-1}^{(1)}$ in (2.3) are indeterminate at $n = 0$. We resolve this indeterminacy by taking the limit $n \rightarrow 0$.

Since (2.2) is a second order equation we seek a second linearly independent solution. This is obtained in terms of another terminating ${}_9F_8$ in the cases $s = 1, 2, \dots$ by noticing that with the replacement

$$(2.4) \quad (a, b, c, d, e, f, n+s, -n) \rightarrow (1-a, 1-b, 1-c, 1-d, 1-e, 1-f, 2-n-s, 1+n)$$

we have

$$(2.5) \quad (a_n, b_n) \rightarrow (a_n, b_{n+1}).$$

Note that $b_n \rightarrow b_{n+1}$ is easy to check but the verification of $a_n \rightarrow a_n$ requires a lot of tedious algebra.

Applying the transformation (2.4) to (2.2) with $X_n = X_n^{(1)}$ and using (2.5) yields, after renormalization, a second solution

$$(2.6) \quad X_n^{(2)} = \Gamma \left(\begin{matrix} n+1, n+a, n+s+b-a-1, n+s+c-a-1, \dots, n+s+f-a-1 \\ 2n+s-1, n+s-a \end{matrix} \right) \times F(1-a; 1-b, 1-c, 1-d, 1-e, 1-f, 2-n-s, n+1), \quad s = 1, 2, \dots$$

where the termination of the ${}_9F_8$ is now due to the parameter $2-n-s$.

3. Asymptotics and Pincherle's Theorem. The large n asymptotics of (2.3) and (2.6) follows from Stirling's formula and Tannery's theorem. Thus

$$(3.1) \quad X_n^{(1)} \sim \frac{(\Gamma(n))^6}{\Gamma(2n)} 2^{1-s} n^{2s} W(a; b, c, d, e, f), \quad s > 1 + \operatorname{Re} a$$

$$(3.2) \quad X_n^{(2)} \sim \frac{(\Gamma(n))^6}{\Gamma(2n)} 2^{1-s} n^{2s} W(1-a; 1-b, 1-c, 1-d, 1-e, 1-f), \quad s < 2 + \operatorname{Re} a$$

where $W(a; b, c, d, e, f) := {}_7F_8 \left(\begin{matrix} a, \frac{a}{2} + 1, b, c, d, e, f \\ \frac{a}{2}, 1+a-b, 1+a-c, \dots, 1+a-f; 1 \end{matrix} \right).$

Let $W_1 := W(a; b, c, d, e, f)$, $s > 1 + \operatorname{Re} a$ and its analytic continuation otherwise, $W_2 := W(1-a; 1-b, 1-c, 1-d, 1-e, 1-f)$, $s < 2 + \operatorname{Re} a$ and its analytic continuation otherwise and

$$(3.3) \quad X_n^{(3)} := W_2 X_n^{(1)} - W_1 X_n^{(2)}.$$

It follows from (3.1) and (3.2) that

$$(3.4) \quad \lim_{n \rightarrow \infty} X_n^{(3)} / X_n^{(1)} = 0, \quad s - 2 < \operatorname{Re} a < s - 1.$$

Thus $X_n^{(3)}$ is a minimal (subdominant) solution to (2.2) and we have

Theorem 2. Let $s = 1, 2, \dots$. Then

$$(3.5) \quad \frac{1}{a_0} - \frac{b_1}{a_1} - \frac{b_2}{a_2} - \dots = \lim_{n \rightarrow 0} \frac{(W_2 X_n^{(1)} - W_1 X_n^{(2)})}{b_n (W_2 X_{n-1}^{(1)} - W_1 X_{n-1}^{(2)})}.$$

Proof. Equation (3.5) follows from (3.4) and Pincherle's theorem [6] if $s - 2 < \operatorname{Re} a < s - 1$. The limit on the right takes care of the exceptional values $s = 1, 2$ which are otherwise indeterminate. For other values of $\operatorname{Re} a$ the result follows by analytic continuation since both the left and right sides of (3.5) are meromorphic functions of the parameter a . For the left side of (3.5) this is true because of the parabola theorem [6] since from (2.2) we have $-b_n/a_n a_{n-1} = -\frac{1}{4} + 3/16n^2 + O(1/n^3)$ (see also Jacobsen [5]). For the right side of (3.5) we have gamma functions and two ${}_4F_3$'s, each having an analytic continuation in terms of gamma functions and a pair of convergent ${}_4F_3$'s (cf. Bailey [2, 4.4(4)]). $\square$

For the exceptional values $s = 1$ and 2 , Theorem 2 is the nonterminating version of Proposition 1 and Entry 40 respectively. To derive the terminating version we need to express the ratio W_1/W_2 in terms of gamma functions when $b - a$ is an integer. Consider

$$\widetilde{W}(a; b, c, d, e, f) := \frac{W(a; b, c, d, e, f)}{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}$$

and

$$U(a; b, c, d, e, f) := \Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(f) \widetilde{W}(a; b, c, d, e, f).$$

Lemma 3. If $b - a$ is an integer then

$$(3.6) \quad U(a; b, c, d, e, f) = U(2b - a; b, b + c - a, b + d - a, b + e - a, b + f - a).$$

Proof. We apply the limit $-h, g \rightarrow \infty$ to Bailey [2, 6.8(3)]. $\square$

Lemma 4. If $s = 3 + 3a - b - c - d - e - f$ and $b - a$ are integer then

$$(3.7) \quad \frac{\widetilde{W}(a; b, c, d, e, f)}{\widetilde{W}(1-a; 1-b, 1-c, 1-d, 1-e, 1-f)} = (-1)^{s+b-a} \times \Gamma \left(\begin{matrix} b+c-a, b+d-a, b+e-a, b+f-a, 2-a, 1-b, c+d-a, c+e-a, c+f-a \\ 1+a, c, d, e, f, 2-s+a, 1+a-e-f, 1+a-d-f, 1+a-d-e \end{matrix} \right).$$

Proof. We apply the identity (3.6) to the numerator $\widetilde{W}$ and the transformation [2, 7.6(2)] to the denominator $\widetilde{W}$. This yields (3.7) after cancelling a common linear combination of ${}_4F_3$'s from the numerator and denominator using [2, 4.4(4)] and the relation

$$\lim_{n \rightarrow -n} (\Gamma(c))^{-1} {}_4F_3 \left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix} ; 1 \right) = \Gamma \left(\begin{matrix} a+n+1, b+n+1, c+n+1, d+n+1, f, g \\ a, b, c, d, f+n+1, g+n+1, 2+n \end{matrix} \right) \times {}_4F_3 \left(\begin{matrix} a+n+1, b+n+1, c+n+1, d+n+1 \\ 2+n, f+n+1, g+n+1 \end{matrix} ; 1 \right),$$

$n = -1, 0, 1, 2, \dots$ where $n = 3 - s + a - b$ or $s - 1 - a + b$. $\square$

We now give Theorem 2 more explicitly for the exceptional values $s = 1$ and 2 .

Corollary 5. If $s = 2$ then (3.5) can be rewritten as

$$\begin{aligned}
 (3.8) \quad \frac{1}{a_0} - \frac{b_1}{a_1} - \frac{b_2}{a_2} - \dots &= \frac{2}{\alpha\beta\gamma\delta\epsilon} \left(\frac{V_2 - V_1}{V_2 + V_1} \right), \\
 b_n &= \frac{(n^2 - \alpha^2)(n^2 - \beta^2)(n^2 - \gamma^2)(n^2 - \delta^2)(n^2 - \epsilon^2)}{4(2n+1)(2n-1)}, \\
 a_n &= [(2n^2 + 2n + 2)^2 + (\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 - 2n^2 - 2n - 1)^2 \\
 &\quad - 2(1 + \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4)]/16, \\
 V_1 &= V_1(\alpha, \beta, \gamma, \delta, \epsilon) = \Gamma\left(\frac{a}{2-a}\right) \widetilde{W}(a; a-\alpha, a-\beta, a-\gamma, a-\delta, a-\epsilon) \\
 V_2 &= V_1(-\alpha, -\beta, -\gamma, -\delta, -\epsilon), \\
 a &= (1 + \alpha + \beta + \gamma + \delta + \epsilon)/2.
 \end{aligned}$$

If one of the parameters $\beta, \gamma, \delta, \epsilon$ is a nonzero integer then (3.8) is equivalent to (1.1) with

$$(3.9) \quad \frac{V_2 - V_1}{V_2 + V_1} = \frac{P - Q}{P + Q}.$$

Proof. In Theorem 2 we put $s = 2$ and $\alpha = a - b, \beta = a - c, \gamma = a - d, \delta = a - e$ and $\epsilon = a - f$. Then a_n and b_n in (2.2) reduce after a lot of algebra to the expressions in (3.8) and the right side of (3.5) becomes the right side of (3.8) after applying the ${}_1F_0$ transformation [2, 7.5 (1)] to W_2 . If one of the parameters $\beta, \gamma, \delta, \epsilon$ is a nonzero integer we obtain (3.9) by using Lemma 4 and the reflection identity $\Gamma(1-z)\Gamma(z) = \pi/\sin \pi z$. The equivalence with Entry 40 now follows by recognizing that $\alpha_n = 16(2n+1)a_n$, $\beta_n = -(16)^2(2n-1)(2n+1)b_n$. An equivalence transformation for continued fractions [6] gives $\frac{1}{a_0} - \frac{b_1}{a_1} - \frac{b_2}{a_2} - \dots = \frac{1a}{a_0} + \frac{\beta_1}{a_1} + \frac{\beta_2}{a_2} - \dots$. $\square$

Corollary 6. If $s = 1$ then (3.5) can be rewritten as

$$(3.10) \quad \frac{1}{a_0} - \frac{b_1}{a_1} - \frac{b_2}{a_2} - \dots = 2 \left(a_0 + \Gamma \left(\frac{1+a, a-1}{-a, -a} \right) \frac{\widetilde{W}_1}{\widetilde{W}_2} \right)^{-1},$$

with

$$\begin{aligned}
 a &= (\alpha + \beta + \gamma + \delta + \epsilon - 1)/4 \\
 \widetilde{W}_1 &= \widetilde{W}(a; a + \frac{1}{2} - \frac{\alpha}{2}, a + \frac{1}{2} - \frac{\beta}{2}, a + \frac{1}{2} - \frac{\gamma}{2}, a + \frac{1}{2} - \frac{\delta}{2}, a + \frac{1}{2} - \frac{\epsilon}{2}), \\
 \widetilde{W}_2 &= \widetilde{W}(1-a; \frac{1}{2} - a + \frac{\alpha}{2}, \frac{1}{2} - a + \frac{\beta}{2}, \frac{1}{2} - a + \frac{\gamma}{2}, \frac{1}{2} - a + \frac{\delta}{2}, \frac{1}{2} - a + \frac{\epsilon}{2}),
 \end{aligned}$$

and the a_n, b_n as given in Proposition 1. If one of the parameters $\beta, \gamma, \delta, \epsilon$ is an odd integer then (3.10) is equivalent to (1.2) with

$$(3.11) \quad \Gamma \left(\frac{1+a, a-1}{-a, -a} \right) \frac{\widetilde{W}_1}{\widetilde{W}_2} = -\frac{P'}{Q'}.$$

Proof. In Theorem 2 we put $s = 1$ and $\alpha = 2(a-b)+1, \beta = 2(a-c)+1, \gamma = 2(a-d)+1, \delta = 2(a-e)+1, \epsilon = 2(a-f)+1$ to obtain (3.10). To express the a_n and b_n in the form given

in Proposition 1 requires a considerable amount of algebra which is made simpler by choosing special values for some parameters and making use of the symmetry. To obtain (3.11) we use Lemma 4 and the reflection identity for gamma functions. Equation (1.2) now follows from (3.10) and (3.11). $\square$

There are two further extensions of the above results which will be dealt with in future publications:

- 1) The associated case where we solve the recurrence (2.2) with the coefficients a_n, b_n replaced by $a_{n+\epsilon}, b_{n+\epsilon}$.
- 2) The q -analogue where (2.1) is replaced by a corresponding contiguous relation for $10\phi_9$ basic hypergeometric functions.

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CRITERIA FOR LIMIT INFERIOR OF SMALL INCREMENTS OF BANACH SPACE VALUED STOCHASTIC PROCESSES

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Abstract: We present criteria for the limit inferior behaviour of increments of Banach space valued stochastic processes in general, and for that of infinite series of independent Ornstein-Uhlenbeck processes in particular.

1. Introduction Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k , i.e., $X_k(\cdot)$ is a stationary, mean zero Gaussian process with

$$(1.1) \quad EX_k(s)X_k(t) = (\gamma_k/\lambda_k) \exp(-\lambda_k|t-s|), \quad k = 1, 2, \dots,$$

where $\gamma_k \geq 0$, $\lambda_k > 0$.

The process $Y(\cdot)$ was introduced by Dawson (1972), who established also some of its sample path continuity properties in ℓ^2 . For necessary and sufficient conditions for the continuity of $Y(\cdot) \in \ell^p$, $2 \leq p < \infty$, we refer to Fernique (1989, 1990). For moduli of continuity results for ℓ^p -valued Ornstein-Uhlenbeck processes, as well as for those of their ℓ^2 norm squared process, we refer to Csáki and Csörgő (1990a, b), Csáki, Csörgő and Shao (1991), Schmuland (1990) and to the references cited in these papers.

The infinite series of the independent Ornstein-Uhlenbeck coordinate processes of $Y(\cdot)$, namely the process $X(\cdot)$, defined by

$$(1.2) \quad \{X(t), -\infty < t < \infty\} = \left\{ \sum_{k=1}^{\infty} X_k(t), -\infty < t < \infty \right\},$$

was studied by Csáki, Csörgő, Lin and Révész (1990), who obtained the following Lévy type moduli of continuity for the process $X(\cdot)$ (cf. also Csáki, Csörgő and Shao (1991)).

For conditions guaranteeing the existence of $X(\cdot)$ as an almost surely continuous Gaussian process we refer to Csáki *et al.* (1990).

THEOREM A. Assume that $\{X(t), -\infty < t < \infty\}$ is almost surely continuous and let

$$(1.3) \quad \begin{aligned} \sigma^2(h) &= E(X(t+h) - X(t))^2 = E(X(h) - X(0))^2 \\ &= 2 \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} (1 - e^{-\lambda_k h}), \quad h > 0. \end{aligned}$$

Suppose that there exist positive constants α and c such that

$$(1.4) \quad \sigma(\alpha h) \leq c \alpha^\alpha \sigma(h) \quad \text{for all } 0 \leq \alpha, h \leq 1.$$

Then

$$(1.5) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} \frac{|X(t+s) - X(t)|}{\sigma(h) (2 \log \frac{1}{h})^{\frac{1}{2}}} = 1 \text{ a.s.}$$

$$(1.6) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq s \leq h} \frac{|X(s) - X(0)|}{\sigma(h) (2 \log \log \frac{1}{h})^{\frac{1}{2}}} = 1 \text{ a.s.}$$

We note that the result of (1.5) for $X(\cdot)$ parallels the Lévy modulus of continuity for a standard Wiener process (cf., e.g. Chapter 1 of Csörgö and Révész (1981)). While studying the problem of how small the increments of a Wiener process can be, Csörgö and Révész (1979) (cf. also Chapter 1 of Csörgö and Révész (1981)) established the following exact modulus of non-differentiability for a standard Wiener process $\{W(t), 0 \leq t < \infty\}$.

THEOREM B. We have

$$(1.7) \quad \lim_{h \rightarrow 0} \inf_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} \left(\frac{8 \log \frac{1}{h}}{\pi^2 h} \right)^{\frac{1}{2}} |W(t+s) - W(t)| = 1 \text{ a.s.}$$

This theorem implies the well-known result of Wiener (1923): *Almost all sample functions of a Wiener process (Brownian motion) are nowhere differentiable.*

Our aim here is to present similar results for the process $X(\cdot)$, as well as some criteria for the limit inferior behaviour of increments of general, Banach space valued processes.

2. On small increments and moduli of non-differentiability of infinite series of independent Ornstein-Uhlenbeck processes. Throughout this section we assume

that $\Gamma_0 = \sum_{k=1}^{\infty} \gamma_k / \lambda_k < \infty$. This condition ensures that $\sigma^2(h)$ of (1.3) is finite and that $\lim_{h \downarrow 0} \sigma(h) = 0$. The proofs, as well as further details, of the following results will appear in a forthcoming paper by the authors.

THEOREM 2.1 We have

$$(2.1) \quad \liminf_{h \rightarrow 0} \inf_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{(\log \frac{1}{h})^{\frac{1}{2}}}{\sigma(h)} |X(t+s) - X(t)| \geq \frac{1}{12\sqrt{3}} \text{ a.s.}$$

$$(2.2) \quad \liminf_{h \rightarrow 0} \frac{(\log \log \frac{1}{h})^{\frac{1}{2}}}{\sigma(h)} \sup_{0 \leq s \leq h} |X(t+s) - X(t)| \geq \frac{1}{12\sqrt{3}} \text{ a.s.}$$

for each t .

THEOREM 2.2 Assume that there exist positive constants θ_1 , θ_2 and α such that

$$(2.3) \quad \theta_1 h^\alpha \leq \sigma(h) \leq \theta_2 h^\alpha$$

for every sufficiently small h . Then

$$(2.4) \quad C_1 \leq \liminf_{h \rightarrow 0} \inf_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{(\log \frac{1}{h})^\alpha}{\sigma(h)} |X(t+s) - X(t)| \\ \leq \limsup_{h \rightarrow 0} \inf_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{(\log \frac{1}{h})^\alpha}{\sigma(h)} |X(t+s) - X(t)| \leq C_2 \text{ a.s.,}$$

where $C_1 = \frac{\theta_1}{\theta_2 4\sqrt{2} 8^\alpha}$, $C_2 = 12^\alpha \cdot 48 \frac{\theta_2}{\theta_1} \left(2 + \frac{\theta_2}{\theta_1 \sqrt{\alpha}}\right)$.

REMARK 2.1 On account of $\Gamma_0 < \infty$, it is easy to see that $\liminf_{h \rightarrow 0} \sigma^2(h)/h > 0$. Hence α in (2.3) must be not bigger than $1/2$. The condition (2.3) also implies

$$(2.5) \quad \lim_{h \rightarrow 0} h \left(\log \frac{1}{h} \right)^\alpha / \sigma(h) = 0.$$

Hence, in case of having (2.3), we conclude that almost all sample functions of the process $X(\cdot)$ are non-differentiable and their Newton quotients can be so big as $\sigma(h)/(h(\log \frac{1}{h})^\alpha)$ for all small enough $h > 0$. In particular, if we assume also that $\Gamma_1 = \sum_{k=1}^{\infty} \gamma_k < \infty$, then

$$\lim_{h \rightarrow 0} \sigma^2(h)/(2h\Gamma_1) = 1,$$

i.e., in this case, the condition (2.3) is satisfied with $\alpha = 1/2$ and $\theta_1 = \theta_2 = (2\Gamma_1)^{\frac{1}{2}}$, and almost all sample functions of the process $X(\cdot)$ are non-differentiable with their Newton quotients as big as $(h \log \frac{1}{h})^{-\frac{1}{2}}$ for all small enough $h > 0$, just like those of a Wiener process, according to Theorem B.

REMARK 2.2 Since $\liminf_{h \rightarrow 0} \sigma^2(h)/h > 0$, we have

$$(2.6) \quad \lim_{h \rightarrow 0} \left(h^2 \log \frac{1}{h} / \sigma^2(h) \right)^{\frac{1}{2}} = 0.$$

Hence (2.1) can be also interpreted in terms of non-differentiability of the sample functions of the process $X(\cdot)$ along the lines of Remark 2.1. The statement of (2.2) can be viewed as an analog of the local version of the law of the iterated logarithm of Chung (1948) for a Wiener process, the so-called other LIL (cf. Example 2 of Section 1.7 of Csörgő and Révész (1981)).

3. Criteria for limit inferior of increments of Banach space valued processes.

The key to Theorems 2.1 and 2.2 is our Theorem 3.1 below and its application to real-valued Gaussian processes.

Let $\mathcal{B}$ be a separable Banach space with norm $\|\cdot\|$, $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in $\mathcal{B}$, and P be the probability measure generated by $\Gamma(\cdot)$.

THEOREM 3.1 Let $\{a_T, T \geq 0\}$, $\{b_T, T \geq 0\}$ be non-negative functions and $\mu(s, t)$ be non-negative and non-increasing in s and non-decreasing in t . Assume

$$(3.1) \quad a_T + \frac{1}{a_T} \rightarrow \infty \text{ as } T \rightarrow \infty,$$

$$(3.2) \quad P \left\{ \sup_{0 \leq s \leq y} \|\Gamma(t+s) - \Gamma(t)\| \leq \mu(x, y) \right\} \leq K(1+x)^{-1-\varepsilon} \log^{-1-\varepsilon}(y+1/y)$$

for some $K > 0$, $\varepsilon > 0$, and for every $T > 0$, $0 \leq t \leq b_T$, $\frac{1}{2}a_T \leq y \leq 2a_T$ and $b_T/a_T \leq x \leq 2b_T/a_T$. Then we have

$$(3.3) \quad \liminf_{T \rightarrow \infty} \inf_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\mu(2b_T/a_T, \frac{1}{2}a_T)} \geq \frac{1}{2} \text{ a.s.,}$$

where $\log x = \ln \max(x, e)$ and $\ln$ is the natural logarithm.

Applying Theorem 3.1 to Gaussian processes with stationary increments, we can obtain the next result.

THEOREM 3.2 Let $\{G(t), -\infty < t < \infty\}$ be a real-valued Gaussian process with mean zero and stationary increments. Let

$$(3.4) \quad \nu^2(h) = E(G(t+h) - G(t))^2, \quad t \geq 0, h \geq 0.$$

Let a_T and b_T be non-negative functions on $[0, \infty)$, satisfying (3.1). Put $a^* = \sup_{T \geq 0} a_T$. Assume that $\nu^2(h)$ is non-decreasing and concave on $[0, a^*)$. Then

$$(3.5) \quad \liminf_{T \rightarrow \infty} \inf_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \frac{|G(t+s) - G(t)|}{\nu \left(a_T / \left(8 \left(\log \frac{b_T}{a_T} + \log \log \left(a_T + \frac{1}{a_T} \right) \right) \right) \right)} \geq \frac{1}{4\sqrt{2}} \quad \text{a.s.}$$

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