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This issue

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is dedicated by his colleagues to

Professor Gilbert de Beauregard Robinson
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1906-



Algebraist, author, cabinetmaker, local historian, managing editor, Canadian Journal of Mathematics 1948-77, indefatigable organizer, practiced photographer, president, Academy of Sciences, Royal Society of Canada 1961-62, research vice-president, University of Toronto 1965-71, symmetric group theorist and professor par excellence,

On the occasion of his retirement from the editorial board, and with grateful thanks for his vital contribution as production editor of this journal during its formative period 1979-1990.

Strong Stacks and Classifying Spaces

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Presented by J. Lambek, F.R.S.C.

Abstract

We introduce the notion of strong stack for a category \mathbb{C} in a Grothendieck topos \mathcal{E} , and prove that any category \mathbb{C} has a (representable) strong stack completion \mathbb{C}^* . Moreover, we show that there is a Quillen homotopy structure on the category of categories in \mathcal{E} , in which the weak equivalences are the internal categorical equivalences. Strong stacks are the fibrant objects in this structure. In the final section we discuss the connection between strong stacks and classifying spaces for groupoids in the category $S(\mathcal{E})$ of simplicial objects of \mathcal{E} .

1. Stacks

Let \mathcal{E} be a Grothendieck topos, and suppose $p: \mathcal{F} \rightarrow \mathcal{E}$ is a categorical fibration over \mathcal{E} .

Recall [2] that \mathcal{F} is a stack (for the canonical topology on \mathcal{E}) if

(1) For each set I , and each I -indexed family $\{X_i \mid i \in I\}$ of objects of \mathcal{E} , the canonical functor

$$\mathcal{F}(\sum_{i \in I} X_i) \rightarrow \prod_{i \in I} \mathcal{F}(X_i)$$

is an equivalence of categories, and

(2) For each epimorphism $q: X \twoheadrightarrow Y$, the canonical functor $\mathcal{F}(Y) \rightarrow \text{des}\mathcal{F}(X)$ is an equivalence of categories, where $\text{des}\mathcal{F}(X)$ is the category of objects of $\mathcal{F}(X)$ provided with descent data relative to the kernel pair of q .

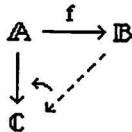
A category \mathbb{C} in \mathcal{E} is called a stack if its externalization $\mathcal{F}(X) = \text{hom}(X, \mathbb{C})$ is a stack.

Note that condition (1) is automatic in this case. A functor $f: \mathbb{A} \rightarrow \mathbb{B}$ between categories in \mathcal{E} is said to be a *weak equivalence* if it is internally full, faithful, and representative. f is a *strong equivalence* if there is a functor $g: \mathbb{B} \rightarrow \mathbb{A}$ together with isomorphisms $gf \simeq \text{id}_{\mathbb{A}}$ and $fg \simeq \text{id}_{\mathbb{B}}$. (See [2] for a discussion of different notions of equivalence for internal categories.)

Proposition 1 The following are equivalent for a category \mathbb{C} in \mathcal{E} .

(1) \mathbb{C} is a stack.

(2) Every diagram



with f a weak equivalence has a dotted filler making the resulting triangle commute up to isomorphism.

(3) Every diagram as above with f a weak equivalence injective on objects has a dotted filler up to isomorphism.

Indication of proof: Let $q: X \rightarrow Y$ be an epimorphism of \mathcal{E} , and denote by $\ker(q)$ the equivalence relation $X \times_Y X \rightrightarrows X$ on X determined by q . Then $\ker(q)$ is a category in \mathcal{E} and the canonical functor $\ker(q) \rightarrow \text{dis}Y$ is a weak equivalence, where $\text{dis}Y$ is the object Y considered as a discrete category of \mathcal{E} . Descent data on X with respect to q (and \mathbb{C}) is simply a functor $\ker(q) \rightarrow \mathbb{C}$. Thus, (2) implies (1). The proof that (1) implies (2) can be found in [1]. A cone construction applied to a weak equivalence $f: \mathcal{A} \rightarrow \mathcal{B}$ shows (3) implies (2). ■

If \mathbb{C} is a category in \mathcal{E} , let $\text{Iso}(\mathbb{C})$ denote the groupoid in \mathcal{E} whose objects are those of \mathbb{C} , and whose morphisms are the isomorphisms of \mathbb{C} . The following proposition follows immediately from the definition of stack as described in the proof of Proposition 1.

Proposition 2 \mathbb{C} is a stack iff $\text{Iso}(\mathbb{C})$ is.

Let $\mathbb{G} = (s, t): G_1 \rightarrow G_0 \times G_0$ be a groupoid in \mathcal{E} . A (right) \mathbb{G} -torsor is a non-empty object E (i.e. $E \rightarrow 1$ is surjective) over G_0 , equipped with a free (contravariant) action $a: E \times_{G_0} G_1 \rightarrow E$, which is transitive. A \mathbb{G} -torsor $E \rightarrow X$ over X is a \mathbb{G} -torsor in \mathcal{E}/X . The set of isomorphism classes of \mathbb{G} -torsors over X is denoted by $H^1(X, \mathbb{G})$.

Proposition 3 A groupoid \mathbb{G} is a stack iff every \mathbb{G} -torsor $p: E \rightarrow X$ over X has a section $s: X \rightarrow E$.

Let \mathbb{C} be a category in \mathcal{E} . A (representable) stack completion of \mathbb{C} is a weak equivalence $\mathbb{C} \rightarrow \mathbb{C}^*$, such that \mathbb{C}^* is a stack. It follows from Proposition 1 that stack completions are defined up to strong equivalence of categories. To obtain the stack completion of a category \mathbb{C} , it is enough to have the stack completion of its groupoid of isomorphisms. In fact, suppose $\text{Iso}(\mathbb{C}) \rightarrow \text{Iso}(\mathbb{C})^*$ is a stack completion of the groupoid $\text{Iso}(\mathbb{C})$. Taking the pushout

$$\begin{array}{ccc}
 \mathbb{C} & \longrightarrow & \mathbb{C}^* \\
 \uparrow & & \uparrow \\
 \text{Iso}(\mathbb{C}) & \longrightarrow & \text{Iso}(\mathbb{C})^*
 \end{array}$$

provides a stack completion $\mathbb{C} \rightarrow \mathbb{C}^*$ of \mathbb{C} .

2. Strong stacks

Definition 1 A category \mathbb{C} in \mathcal{E} is a *strong stack* if condition (3) of Proposition 1 holds on the nose. That is, if each diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{f} & \mathbb{B} \\ \downarrow & \dashrightarrow & \\ \mathbb{C} & & \end{array}$$

with f a weak equivalence injective on objects has a dotted filler making the resulting triangle commute.

Proposition 4 A category \mathbb{C} in \mathcal{E} is a strong stack iff $\text{Iso}(\mathbb{C})$ is.

Proposition 5 A groupoid \mathbb{G} is a strong stack iff for each X , and each \mathbb{G} -torsor $p:E \rightarrow X$ over X , every diagram

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{E} \\ \downarrow & \dashrightarrow & \downarrow p \\ \mathbb{B} & \longrightarrow & \mathbb{X} \end{array}$$

with $\mathbb{A} \hookrightarrow \mathbb{B}$ a monomorphism has a dotted filler making both triangles commute.

Definition 2 If \mathbb{C} is a category in \mathcal{E} , a (representable) *strong stack completion* of \mathbb{C} is a weak equivalence $\mathbb{C} \hookrightarrow \mathbb{C}^*$ injective on objects, such that \mathbb{C}^* is a strong stack.

The strong stack completion of a category \mathbb{C} can be obtained from the strong stack completion of its groupoid of isomorphisms $\text{Iso}(\mathbb{C})$ as above. For the case of groupoids we have

Theorem 1 Every groupoid in \mathcal{E} has a representable strong stack completion.

Indication of proof: As in [4] we establish the fact that there is a small set of generators for the weak equivalences of groupoids $\mathbb{A} \hookrightarrow \mathbb{B}$ injective on objects. Furthermore, such functors are stable under pushouts, so we can obtain the strong stack completion of a groupoid \mathbb{G} by iterated pushouts and ordinal colimits. ■

With Theorem 1 we have established the existence of strong stack completions for a arbitrary category \mathbb{C} in \mathcal{E} . Strong stacks are the fibrant objects for a Quillen homotopy structure[5]. In fact, we have

Theorem 2 There is a Quillen homotopy structure on the category of categories in \mathcal{E} , in which the weak equivalences are the categorical weak equivalences, the cofibrations are the functors injective on objects, and the fibrations have the right lifting property with respect to the cofibration weak equivalences.

For an example of a strong stack completion, let A be an abelian group in \mathcal{E} . To obtain the strong stack completion of A (considered as a groupoid in \mathcal{E} with one object), embed A in an injective I , and denote I/A by B . Let $p:I \rightarrow B$ be the quotient map. Let $s, t : B \times I \rightarrow B$ be, respectively, the first projection, and the map $t(x,y) = x+p(y)$.

Proposition 6 The groupoid $(s,t):B \times I \rightarrow B \times B$, where the composition is given by addition, is a strong stack completion of A .

3.Strong stacks and classifying spaces

Let $S(\mathcal{E})$ denote the category of simplicial objects in \mathcal{E} . Recall [4] that there is a Quillen homotopy structure on $S(\mathcal{E})$, in which the weak equivalences are maps $f: X \rightarrow Y$ inducing isomorphisms on the homotopy sheaves, the cofibrations are the monomorphisms, and the fibrations are maps having the right lifting property with respect to the cofibration weak equivalences, which we call *anodyne extensions*. For any groupoid $\mathbb{G} = (s,t): G_1 \rightarrow G_0 \times G_0$ in $S(\mathcal{E})$, let $B\mathbb{G}$ denote the diagonal complex of its nerve $N\mathbb{G}$, considered as a double simplicial object of \mathcal{E} .

Theorem 3 Let \mathbb{G} be a groupoid in \mathcal{E} . If $\mathbb{G} \hookrightarrow \mathbb{G}^*$ is a strong stack completion of \mathbb{G} , then G_0^* is weakly equivalent (in $S(\mathcal{E})$) to $B\mathbb{G}$.

Indication of proof: Proposition 5 says that each \mathbb{G}^* torsor is a trivial fibration (in $S(\mathcal{E})$) over its base. In particular, $t : G_1^* \rightarrow G_0^*$ (and therefore also s) is a trivial fibration. Thus, the inclusion $u : G_0^* \rightarrow G_1^*$ of the units of \mathbb{G}^* is a weak equivalence. In fact, u defines a functor $\text{dis}G_0^* \rightarrow \mathbb{G}^*$, which induces a mapping on nerves having a weak equivalence in each dimension. Applying B yields a weak equivalence $G_0^* \cong B\text{dis}G_0^* \rightarrow B\mathbb{G}^*$. But $B\mathbb{G} \rightarrow B\mathbb{G}^*$ is also a weak equivalence, giving the result. ■

Recall that the *homotopy category* of $S(\mathcal{E})$ is obtained by formally inverting the weak equivalences (or just the anodyne extensions) of $S(\mathcal{E})$.

Definition 3 A groupoid \mathbb{G} in $S(\mathcal{E})$ is said to be *amenable* if the functor $H^1(\cdot, \mathbb{G})$ inverts anodyne extensions, i.e. passes to the homotopy category.

Note that amenability is invariant under weak (categorical) equivalence of groupoids.

Proposition 7 If \mathbb{G} is an amenable strong stack, then G_0 is fibrant in the Quillen structure on $S(\mathcal{E})$.

If X and Y are objects of $S(\mathcal{E})$, and Y is fibrant, recall that the set of maps from X to Y in the homotopy category of $S(\mathcal{E})$ is in 1-1 correspondence with the set $[X, Y]$ of *homotopy classes* of maps from X to Y . (A homotopy is a mapping $X \times I \rightarrow Y$ with I the constant simplicial sheaf on the 1-simplex $\Delta[1]$ of S .)

Theorem 4 Let \mathbb{G} be an amenable groupoid in $S(\mathcal{E})$, and $\mathbb{G} \hookrightarrow \mathbb{G}^*$ a strong stack completion of \mathbb{G} . Then for any X in $S(\mathcal{E})$, $H^1(X, \mathbb{G}) \cong [X, G_0^*]$. That is, G_0^* is a classifying space for \mathbb{G} -torsors.

The proof parallels that of Proposition 1 in [4], which contains a full discussion of amenability and classifying spaces.

We remark that in the category of simplicial sets, a locally transitive groupoid \mathbb{G} is amenable, where locally transitive means that $(s, t) : G_1 \rightarrow G_0 \times G_0$ is a Kan fibration. Thus, Theorem 4 provides a classifying space for any locally transitive groupoid, e.g. any simplicial group.

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METRIQUES CONFORMEMENT PLATES ET EQUATIONS DE YANG-MILLS

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*Presented by G.F.D. DuBois, F.R.S.C.***1 - English abstract :**

Let (M, g) be a Riemannian compact manifold, and let us consider the differential equation $(E) \quad g\Delta f = \text{Hess}f - f\text{Ric}(g)$ where $\text{Hess}f$ denotes the second covariant derivative of f , $\Delta f = \text{trace}(\text{Hess}f)$ and $\text{Ric}(g)$ is the Ricci curvature tensor relative to g . $f \equiv 0$ is the trivial solution of the linear equation (E) .

This equation may have a non-trivial solution, when (M, g) is the product manifold $(S^1 \times S^{n-1}, dt^2 + g_0)$ where (S^{n-1}, g_0) is the standard $(n-1)$ -dimensional sphere with scalar curvature R_0 and (S^1, dt^2) is the circle of length l parametrized by t where

$$l = 2\pi \sqrt{\frac{n-1}{R_0}}.$$

R. Schoen [5] has analysed the solutions on the S^1 factor of the Yamabe equation on $S^1 \times S^{n-1}$ and exhibited among them a S^1 parameter families of inequivalent solutions which are not minimizing for the Yamabe problem. In this note we are interested in equivalent solutions of this equation, thus in a prescribed conformal class of metrics. In particular we prove that for $n = 4$ and $n = 6$ any warped metric can be deformed in $\mathcal{C}(g_t)$ to a 2-parameters analytical family with constant scalar curvature R , which lie within the conformal class $\mathcal{C}(g_t)$. To that end we use a classical result of S.T. Yau : any warped metric $dt^2 + f^2(t) g_0$ on $S^1 \times S^{n-1}$ must be conformal to a Riemannian metric product $d\theta^2 + g_0$ where θ is a S^1 parametrisation with length $(S^1) = \int_{S^1} \frac{dt}{f(t)}$. Therefore, the metrics have harmonic Riemannian curvature and non parallel Ricci tensor. After that, we exhibit an explicit

family of conformally flat metrics on \mathbf{R}^n for $n = 4, 6$ with constant scalar curvature. When $n = 4$, these results allow us to give a family of solutions of Yang-Mills equations.

2 - Soient (S^{n-1}, g_o) la sphère standard de dimension $n - 1$, R désignant la courbure scalaire et (S^1, dt^2) le cercle paramétré par t , dont la longueur est

$$l = \int_{S^1} dt = 2\pi \sqrt{\frac{n-1}{R}}.$$

Soient $g_t = dt^2 + g_o$ la métrique produit sur $S^1 \times S^{n-1}$, et dont la courbure scalaire est égale à R . Une métrique \tilde{g} est conformément équivalente à g_t , s'il existe une fonction $\varphi \in C^\infty(S^1 \times S^{n-1})$ strictement positive telle que : $\tilde{g} = \varphi^{\frac{4}{n-2}} g_t$. Si \tilde{R} désigne la courbure scalaire associée à \tilde{g} , alors \tilde{R} et φ sont reliées par l'équation différentielle :

$$4 \frac{n-1}{n-2} \Delta \varphi + R\varphi = \tilde{R} \varphi^{\frac{n+2}{n-2}},$$

Δ étant l'opérateur de Laplace sur la variété $(S^1 \times S^{n-1}, g_t)$.

On s'intéresse aux métriques de la classe conforme de g_t notée $\mathcal{C}(g_t)$, qui sont à courbure scalaire constante. On peut supposer à homothétie près que cette constante est encore égale à R .

Ceci revient à déterminer les solutions C^∞ strictement positives de l'équation :

$$4 \frac{n-1}{n-2} \Delta \varphi + R\varphi = R\varphi^{\frac{n+2}{n-2}}.$$

Si φ n'est fonction que du facteur S^1 , alors l'équation à résoudre est :

$$(2-1) \quad -4 \frac{n-1}{n-2} \varphi'' + R\varphi = R\varphi^{\frac{n+2}{n-2}}$$

où φ est une fonction de $x = d_{S^1}(P_o, P)$, la distance sur S^1 à un point fixe. Ainsi pour toute solution φ de l'équation de (2-1), la métrique $g = \varphi^{\frac{4}{n-2}} g_t$ appartient à $\mathcal{C}(g_t)$.

Théorème (2-2) : Soit (S^1, dt^2) le cercle de longueur $l = 2\pi \sqrt{\frac{n-1}{R}}$, où R est la courbure scalaire de la sphère standard (S^{n-1}, g_o) , et soit $f(t)$ une fonction sur S^1 vérifiant la condition (L) $l = \int_{S^1} \frac{dt}{f(t)}$. Alors pour $n = 4$ ou $n = 6$, il existe une déformation dans $\mathcal{C}(g_t)$ de la métrique "tordue" $g = dt^2 + f^2(t)g_o$ sur $S^1 \times S^{n-1}$ en une famille analytique à deux paramètres $(g_{\alpha, \beta})$ de métriques à courbure scalaire constante satisfaisant $g_{\alpha, 0} \equiv g$.

Démonstration : Pour $n = 4$ et $n = 6$ l'équation (2-1) devient respectivement

$$(2-3) \quad -6\varphi'' + R\varphi = R\varphi^3,$$

$$(2-4) \quad -5\varphi'' + R\varphi = R\varphi^2.$$

Ces équations différentielles ordinaires qui admettent $\varphi \equiv 1$ pour solution triviale peuvent être résolues explicitement au moyen des fonctions elliptiques :

a) La solution générale de l'équation (2-3) peut s'exprimer de la manière suivante

$$\varphi_{\alpha,\beta}(x) = \sqrt{\frac{2}{2-\beta^2}} dn \left[\sqrt{\frac{R}{6(2-\beta^2)}}(x+\alpha) \right]$$

où α et β sont des réels avec $0 \leq \beta < 1$ auquel cas la solution est périodique de période réelle

$$T = 2\sqrt{\frac{6(2-\beta^2)}{R}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\beta^2\sin^2\theta}}$$

$dn(x, \beta)$ étant la fonction de Jacobi satisfaisant l'équation différentielle :

$$\left(\frac{d}{dx}dn\right)^2 = (1-dn^2)(dn^2 + \beta^2 - 1)$$

avec $dn(0, \beta) = 1$, dont la période réelle est $2 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\beta^2\sin^2\theta}}$, [4].

Par ailleurs, cette solution générale qui est telle que $\varphi_{\alpha,0} \equiv 1$ peut s'écrire :

$$\varphi_{\alpha,\beta}(x) = \sqrt{\frac{2}{2-\beta^2}} \left[1 - \frac{\beta^2 R}{12(2-\beta^2)}(x+\alpha)^2 + \frac{\beta^2 R^2(4+\beta^2)}{3.6.4!(2-\beta^2)^2}(x+\alpha)^4 + \dots \right]$$

Ainsi $\varphi_{\alpha,\beta}(-\alpha) = \sqrt{\frac{2}{2-\beta^2}}$ et on montre que $\varphi_{\alpha,\beta}$ est une fonction C^∞ strictement positive.

Remarque : Notons que pour $\beta = 1$, cette famille dégénère en une courbe de solutions non périodiques. En effet : $\varphi_{\alpha,1}(x) = \frac{\sqrt{2}}{\cosh(\sqrt{\frac{R}{6}}(x+\alpha))}$

Précisons que lorsque $\beta \rightarrow 1$, la période a pour comportement

$$T(\beta) \sim 2\sqrt{\frac{6(2-\beta^2)}{R}} \text{Log}\left(\frac{4}{\sqrt{1-\beta^2}}\right)$$

b) On peut exprimer la solution générale de l'équation (2-4) de la manière suivante

$$\varphi(x) = \frac{1}{2} - \frac{30\alpha^2}{R} \wp(\alpha x + \omega')$$

où \wp est la fonction de Weierstrass relative à $g_2 = \frac{R}{300\alpha^2}$ et $g_3 = 4\wp^3(\omega') - \frac{R}{300\alpha^2}\wp(\omega')$, qui vérifient la condition $g_2^3 - 27g_3^2 > 0$, $2\omega'$ étant alors la période imaginaire.

Cette famille de solutions dépend de deux paramètres, en l'occurrence α et ω' . Le fait que la solution triviale $\varphi \equiv 1$ appartienne à cette famille sera plus explicite en exprimant ces solutions autrement, grâce à une méthode qu'on a mise en évidence, une première fois dans [2]. En effet, cette famille peut s'écrire

$$\varphi_{\alpha,\beta}(x) = a_0 + \sum_{n \geq 1} a_n \left(\sin \frac{\pi x}{\omega} \right)^n$$

où les coefficients sont fonctions de a_0 et a_1 , avec $a_0 = 1 + \beta$, auquel cas $\varphi_{\alpha,0}(x) \equiv 1$.

Remarque : Lorsque $g_3 = -\left(\frac{R}{30\alpha}\right)^3$, autrement dit quand la période devient infinie, ces solutions périodiques dégèrent en une famille à un paramètre :

$$\varphi_\alpha(u) = \frac{1}{2} - \frac{\alpha^2}{2} - \frac{3\alpha^2}{2 \sinh^2\left(\alpha x \sqrt{\frac{l}{30} + i\frac{\pi}{2}}\right)}$$

(2-5) - Remarques générales :

a) Du fait que la classe conforme d'une métrique produit sur $S^1 \times S^{n-1}$ dépend de la longueur l du cercle, on a alors une famille à un paramètre $\{g_l\}_{l \in \mathbb{R}}$ de métriques produits à courbure scalaire constante, mais non conformément équivalentes entre elles.

b) Sur la sphère canonique (S^n, g_0) , $n \geq 3$, la classe conforme de la métrique standard contient aussi des familles à courbure scalaire constante $R = (n-1)n\alpha^2$. Par exemple $g_t = (\sqrt{1+t^2} + t \cos \alpha x)^{-2} g_0$ où x désigne la distance à un point fixe de la sphère. Mais, d'après Obata, il existe un difféomorphisme conforme entre (S^n, g_t) et la sphère canonique.

c) Pour une dimension quelconque $n \geq 3$, on peut mettre en évidence une famille de métriques à un paramètre sur le produit "tordu" $(S^1 \times S^{n-1}, dt^2 + f^2 g_0)$ où la longueur de S^1 est $l = \int_{S^1} \frac{dt}{f(t)}$.

En effet, les fonctions $\varphi_\lambda(x) = \left[\sqrt{\frac{n}{n-2}} \frac{1}{\cosh(\alpha x + \lambda)} \right]^{\frac{n-2}{2}}$ sont des solutions C^∞ positives de l'équation (2-1) avec $R = (n-1)(n-2)\alpha^2$. En fait, ces fonctions sont des dégénérés de la famille des solutions périodiques comme on l'a vu pour $n = 4$ ou 6 , où φ_λ n'est autre que les $\varphi_{\alpha,1}$, établies par le théorème (2-2). Schoen [5] a montré que lorsqu'on fait varier la longueur l du cercle, en gardant R constante, c'est-à-dire en faisant varier la

classe conforme $\mathcal{C}(g_l)$, la solution de période l approche une des fonctions φ_λ quand $l \rightarrow \infty$. Ainsi, l'invariant de Yamabe $\sigma(S^1 \times S^{n-1})$ n'est pas atteint par une métrique C^∞ . De plus, sous la contrainte $\text{vol}(S^1 \times S^{n-1}, g_{\alpha\beta}) = 1$, on obtient alors une famille de métriques non minimisantes. Il est probable que le théorème (2-2) soit vrai pour tout $n \geq 3$.

3 - Dans cette partie on s'intéresse aux métriques conformément plates sur \mathbb{R}^n , $n \geq 3$; celles-ci s'écrivent $g = \varphi^{\frac{4}{n-2}} g_o$, g_o étant la métrique euclidienne. Si $R(x)$ désigne la courbure scalaire, alors φ doit satisfaire l'équation différentielle

$$4 \frac{n-1}{n-2} \Delta \varphi + R(x) \varphi^{\frac{n+2}{n-2}} = 0$$

où Δ est l'opérateur de Laplace euclidien.

Si la courbure scalaire est une constante positive, alors notre problème se ramène à l'étude des solutions de l'équation différentielle :

$$(3-1) \quad \Delta \varphi + \varphi^{\frac{n+2}{n-2}} = 0 \quad \text{sur } \mathbb{R}^n.$$

Pour $n = 4$, la résolution de (3-1) permet de déterminer des solutions des équations de Yang-Mills euclidiennes grâce à l'Ansatz de t'Hooft qui donne l'expression d'un champ de Yang-Mills en fonction d'une solution de (3-1)

$$A_\mu = i \sigma_{\mu\alpha} \partial_\alpha \text{Log} \varphi(x)$$

où $\sigma_{\mu\alpha}$ sont définis par les matrices de Pauli.

On établit le résultat suivant :

Théorème (3-2) : Pour $n = 4$ et 6, l'équation (3-1) sur \mathbb{R}^n admet une famille de solutions radiales symétriques dépendant analytiquement de deux paramètres $\{u_s, t\}$ telle que :

$$u_{0,t}(r) = \left(\frac{t \sqrt{n(n-2)}}{t^2 + r^2} \right)^{\frac{n-2}{2}}$$

Toutes ces solutions admettent deux singularités pour $s \neq 0$, à l'origine et à l'infini.

Démonstration : Soit la famille $\{\varphi_{\alpha,\beta}\}$ de fonctions mises en évidence lors de la

démonstration du théorème (2-2). Alors, la fonction :

$$u(r) = \left(\frac{n-2}{2r}\right)^{\frac{n-2}{2}} \varphi\left(-\frac{\text{Log}r}{\alpha}\right),$$

est une solution radiale symétrique de l'équation (3-1) pour $n = 4$ et 6 . Pour cela, on vérifie aisément que $\varphi(t) = \left(\frac{n-2}{2}e^{\alpha t}\right)^{\frac{n-2}{2}} \cdot u(e^{-\alpha t})$ est une solution de l'équation différentielle ordinaire :

$$4\frac{n-1}{n-2}\varphi'' - R\varphi + R\varphi^{\frac{n+2}{n-2}} = 0 \quad \text{avec } R = (n-1)(n-2)\alpha^2$$

si et seulement si u est une solution radiale symétrique de l'équation (3-1).

Par ailleurs, on a vu que la famille des $\varphi_{\alpha,\beta}$ est périodique pour $\alpha \neq 1$ et que $\varphi_{1,\beta}(x) = \left(\sqrt{\frac{n}{n-2} \frac{1}{\cosh(\alpha x + \beta)}}\right)^{\frac{n-2}{2}}$, d'où la fonction $u(r)$ correspondante

$$u_{1,\beta}(r) = \left(\frac{e^\beta \sqrt{n(n-2)}}{e^{2\beta} + r^2}\right)^{\frac{n-2}{2}}.$$

Remarque (3-3) :

Cette méthode de résolution de l'équation (3-1) pour $n = 4$, au moyen des fonctions de Jacobi, a été exploitée dans un contexte différent par des physiciens, notamment De Alfaro et Cervero [1].

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QUELQUES PROPRIETES DES SERIES

FORMELLES GENERALISEES

Ali BENHISSI

Presented by P. Ribenboim, F.R.S.C.

0 - Introduction :

Soient K un corps commutatif et G un groupe abélien totalement ordonné. On désigne par $K((T^G))$ le corps des séries formelles généralisées à coefficients dans K et à supports bien ordonnés dans G .

Ce travail est constitué de deux parties.

Dans la première partie, on donne une description des ordres de $K((T^G))$ en fonction de ceux de K et de certain type de sous-groupes de G . On montre qu'ils sont tous compatibles avec la valuation naturelle et qu'ils font de $K((T^G))$ un espace topologique complet.

Dans la deuxième partie, on donne une condition nécessaire et suffisante pour que $K((T^G))$ soit n -pythagoricien.

I - Séries formelles généralisées sur un corps ordonné :

Si K est un corps commutatif, on note $X(K)$ l'ensemble des ordres de K . Si G est un groupe abélien totalement ordonné, on note $C(G)$ l'ensemble des sous-groupes S de G vérifiant :

$S' + S' \subset S$, où S' est le complémentaire de S dans G .

Remarques :

1) $G \in C(G)$

2) Si $S \in C(G)$, alors $2G \subset S$.

Théorème :

L'application $\phi : X (K ((T^G))) \rightarrow C(G) \times X(K)$ est une bijection

$$\langle \rightarrow (\{ \alpha \in G/T^\alpha > 0 \} , \langle_K)$$

Corollaire 1 :

$$|X (K((T^G)))| = |C(G)| \cdot |X (K)|$$

Exemples :

1) Si $G = \mathbb{Z}\alpha$: un groupe cyclique infini, alors $C(G) = \{G, 2\alpha \mathbb{Z}\}$.

En particulier : $C(\mathbb{Z}) = \{ \mathbb{Z} , 2\mathbb{Z} \}$ et $|X (K((T)))| = 2 |X(K)|$.

2) Si G est 2 - divisible, alors $C(G) = \{G\}$. Dans ce cas $|X (K((T^G)))| = |X(K)|$.

Par exemple : $G = \mathbb{Q} ; \mathbb{R}$

Corollaire 2 :

Si (K, \langle) est un corps ordonné, le nombre d'extensions de \langle à $K((T^G))$ est égal à $|C(G)|$.

En particulier, tous les ordres de K ont le même nombre d'extensions à $K((T^G))$.

Plus précisément, si $S \in C(G)$, alors le cône positif P de $K((T^G))$ associé à

$$S \text{ est } P = \left\{ f(T) = a T^\alpha + \dots / \begin{cases} a \geq 0 \\ \alpha \in S \end{cases} \text{ ou } \begin{cases} a < 0 \\ \alpha \notin S \end{cases} \right\}$$

Pour $G = \mathbb{Z}$, le nombre d'extensions de \langle à $K((T))$ est 2, une pour laquelle $T > 0$ et l'autre pour laquelle $T < 0$.

Remarques :

1) Le corps $K((T^G))$ n'est archimédien pour aucun de ses ordres : soient $\alpha < \beta$ deux éléments de G . Alors $T^{2\alpha}, T^{2\beta} > 0$ et pour tout $n \in \mathbb{N}^*$, on a : $n T^{2\beta} < T^{2\alpha}$.

2) Un corps ordonné est dit complet pour les coupures si toute partie bornée admet une borne supérieure. C'est le cas de \mathbb{R} .

Cette propriété n'est pas vraie pour le corps $K((T^G))$, d'après la remarque 1 et la proposition 1.22 du [2].

Considérons $\mathbb{R}((T))$ muni du prolongement de l'ordre usuel de \mathbb{R} pour lequel $T > 0$. La partie

$B = \{1 + nT / n \in \mathbb{N}^*\}$ est majorée par toutes les séries de la forme : $aT^{-m} + \dots$ avec $a > 0$ et $m \in \mathbb{N}^*$, et n'admet pas de borne supérieure dans $\mathbb{R}((T))$.

Proposition :

Tous les ordres de $K((T^G))$ sont compatibles avec sa valuation naturelle v ,

i.e. si $0 \leq f(T) \leq g(T)$ dans $K((T^G))$, alors $v(f) \geq v(g)$.

Rappels :

1) Soit v une valuation de groupe G sur un corps k .

Pour $x_0 \in k$ et $\alpha \in G$, on pose : $B(x_0, \alpha) = \{x \in k / v(x - x_0) > \alpha\}$.

La topologie \mathcal{T}_v de base $\{B(x_0, \alpha) / x_0 \in k, \alpha \in G\}$ est dite de v .

2) Soit (k, \leq) un corps ordonné.

Pour tout $x \in k$, on note $|x| = \max\{x, -x\}$.

Pour $x_0 \in k$ et $a \in k^+ = \{x \in k / x > 0\}$, on pose : $D(x_0, a) = \{x \in k / |x - x_0| < a\}$.

La topologie \mathcal{T}_{\leq} de base $\{D(x_0, a) / x_0 \in k, a \in k^+\}$ est dite de \leq .

Lemme :

Soit (k, v, \leq) un corps valué et ordonné. On suppose que :

\leq est compatible avec v ; i.e si $0 \leq a \leq b$, alors $v(a) \geq v(b)$.

Alors toute suite convergente (resp. de Cauchy) pour \mathcal{T}_{\leq} est convergente (resp. de Cauchy) pour \mathcal{T}_v et a la même limite, et réciproquement.

Proposition :

Le corps $K((T^G))$ est complet pour toutes les topologies définies par ses différents ordres.

Rappel :

Un corps ordonné (R, \leq) est dit clôture réelle du corps ordonné (k, \leq) si :

- i) R est ordonné maximal.
- ii) L'extension R/k est algébrique.
- iii) \leq est la restriction de \leq' à k .

Soient K_0 un corps ordonné de clôture algébrique réelle \tilde{K} et G un groupe abélien totalement ordonné d'enveloppe divisible $\Delta(G)$.

Soit $P(G)$ la plus petite famille-corps de $\Delta(G)$ contenant les parties bien ordonnées de G .

Pour la définition des familles corps, on peut voir [3].

On munit le corps $\tilde{K}((T^{\Delta(G)}))$ d'un prolongement quelconque de l'ordre de \tilde{K} . Alors :

Proposition :

Le corps : $R = \bigcup_{\substack{K_0 \subset K \subset \tilde{K} \\ \text{fini}}} K(P(G))$ est une clôture algébrique réelle de $K((T^G))$.

Corollaire :

Le corps : $\bigcup_{\substack{K_0 \subset K \subset K \\ \text{fini}}} K((T^{1/\infty}))$ est une clôture algébrique réelle de $K((T))$.

II - Séries formelles généralisées sur un corps n-pythagoricien.**Proposition 1 :**

Soient K un corps de caractéristique différente de 2, G un groupe abélien totalement ordonné et M l'idéal maximal de la valuation naturelle de $K((T^G))$.

Alors $1+M$ est un groupe multiplicatif 2- divisible.

Remarque 1 :

$\forall n \in \mathbb{N}^*$, $\forall x \in M$, $\exists y \in M$ tel que : $(1+y)^{2^n} = 1+x$.

Définition :

Soient K un corps et n un entier naturel non nul.

On dit que K est n-pythagoricien si $K^{2^n} + K^{2^n} = K^{2^n}$.

On dit que K est ∞ -pythagoricien s'il est n-pythagoricien pour tout $n \in \mathbb{N}^*$.

Remarque 2 :

Soient K un corps et $n \in \mathbb{N}^*$. Alors : $K^2 = K \Leftrightarrow K^{2^n} = K$.

Proposition 2 :

Soient K un corps de caractéristique différente de 2, G un groupe abélien totalement ordonné et $n \in \mathbb{N}^*$. Alors :

$$K((T^G)) \text{ est } n\text{-pythagoricien} \Leftrightarrow \begin{cases} \text{i) } K \text{ est } n\text{-pythagoricien et ordonnable.} \\ \text{ou} \\ \text{ii) } K^2 = K \text{ et } G \text{ est 2-divisible.} \end{cases}$$

Corollaire :

Soient K un corps de caractéristique différente de 2 et G un groupe totalement ordonné. Alors :

$$K((T^G)) \text{ est } \infty\text{-pythagoricien} \Leftrightarrow \begin{cases} \text{i) } K \text{ est } \infty\text{-pythagoricien et ordonnable.} \\ \text{ou} \\ \text{ii) } K^2 = K \text{ et } G \text{ est 2-divisible.} \end{cases}$$

Définition

Un corps K est dit n -strict- pythagoricien si :

K est n - pythagoricien, et

$$(*) \forall a \in -K^{2^n}, \text{ on a : } K^{2^n} + K^{2^n} a \subset \bigcup_{i:0}^{2^n-1} K^{2^n} a^i$$

Proposition 3 :

Soient K un corps de caractéristique différente de 2 et G un groupe abélien totalement ordonné.

Alors : $K((T^G))$ satisfait (*) si et seulement si K satisfait (*).

Corollaire 1

Soient K un corps de caractéristique différente de 2, G un groupe abélien totalement ordonné et n un entier naturel non nul. Alors :

$$K((T^G)) \text{ est } n\text{-strict-pythagoricien} \Leftrightarrow \begin{cases} K \text{ est } n\text{-strict pythagoricien et ordonnable} \\ \text{ou} \\ K^2 = K \text{ et } G \text{ est 2-divisible.} \end{cases}$$

$$K((T^G)) \text{ est } \infty\text{-strict-pythagorien} \Leftrightarrow \begin{cases} K \text{ est } \infty\text{-strict pythagorien et ordonnable} \\ \text{ou} \\ K^2 = K \text{ et } G \text{ est 2-divisible.} \end{cases}$$

Corollaire 2 :

Soient K un corps de caractéristique impaire, G un groupe abélien totalement ordonné et

$n \in \mathbb{N}^*$. Alors les propriétés suivantes sont équivalentes :

- i) K est n -pythagorien
- ii) K est n -strict-pythagorien
- iii) K est ∞ -pythagorien
- iv) K est ∞ -strict-pythagorien.
- v) $K^2 = K$ et G est 2-divisible.

Remarque :

Des généralisations de certains résultats de la deuxième partie apparaîtront dans [1] et [4].

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SOME ORDERS OF AUSLANDER TYPE ≥ 2

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Abstract. We consider generalized Bäckström orders defined over a complete Dedekind domain and classify those in terms of combinatorial data, which have the property that their Auslander orders are of finite lattice type. Without proof we characterize those orders having the property that their fourth respectively fifth iterated Auslander order exist.

Let A be an artin algebra which has only finitely many isomorphism classes of indecomposable modules, and let G be a finitely generated A -module having the property that up to isomorphism each indecomposable A -module occurs in G as a direct summand. Such a module G is called an additive generator for the category of A -modules. The endomorphism ring $\text{End}_A(G)$ of G is called the Auslander algebra of A . We denote it by $\mathcal{A}(A)$, and clearly, it is determined up to Morita equivalence only. If $\mathcal{A}(A)$ again is of finite representation type, we can continue and form its Auslander algebra $\mathcal{A}^2(A) = \mathcal{A}(\mathcal{A}(A))$ and so on. These types of algebras already were studied by Auslander in [A] and characterized by their homological properties.

A similar procedure can be carried out for classical orders. Namely if R is a complete Dedekind domain and Λ an R -order in a semisimple algebra over the quotient field of R , having only finitely many isomorphism classes of indecomposable lattices, one can form the Auslander order $\mathcal{A}(\Lambda)$ of Λ as being the endomorphism ring of an additive generator for the category of Λ -lattices. This was done by Auslander and Roggenkamp in [AR], where these Auslander orders also were characterized by their homological properties.

In this paper we consider the question how often one can in both cases continue to form Auslander algebras, respectively Auslander orders as described above. Thus our main interest concerns the Auslander type of a classical order Λ , where we say that Λ has infinite Auslander type if $\mathcal{A}^n(\Lambda)$ exists for all $n \in \mathbb{N}$, and Λ has Auslander type i for some $i \geq 0$ if $\mathcal{A}^i(\Lambda)$ exists and is of infinite lattice type. Similarly, the Auslander type of an artin algebra is defined.

In section 1 we recall a result due to Auslander which allows it to compare the artin situation with the situation for orders and to use known results about the Auslander type of artin algebras [IPTZ],[TZ] for considerations about the Auslander type of orders.

Generalized Bäckström orders of finite lattice type can easily be described and classified by means of combinatorial data only [R1],[R2]. It was a natural problem to characterize those generalized Bäckström orders which have an Auslander order of finite lattice type. This is

carried out in section 2.

Finally, we mention without proof a lattice theoretic characterization of those orders whose fourth or fifth Auslander order exist. In our terminology, they have Auslander type ≥ 4 . It turns out that these orders are from the point of view of their integral representation theory rather close to the orders of infinite Auslander type.

1. Auslander algebras and Auslander orders

We start recalling a result of Auslander which connects the category of lattices of an Auslander order in the sense of [AR] to the category of modules over the Auslander algebra of an artin algebra over a uniserial commutative ring.

Let R be a complete Dedekind domain with quotient field K , and let Λ be an R -order in a semisimple finite dimensional K -algebra. We view Λ as an R -category with objects the indecomposable projective Λ -modules. For \mathcal{K} an R -category we denote by $\text{mod } \mathcal{K}$ the additive category of finitely generated additive functors from \mathcal{K} into the category of abelian groups and by $\underline{\text{mod}} \mathcal{K}$ its quotient modulo the subcategory of projective objects, that are the representable functors. Similarly, we denote by $\text{latt } \mathcal{K}$, respectively $\underline{\text{latt}} \mathcal{K}$ the full subcategory of $\text{mod } \mathcal{K}$ consisting of those functors which take torsionfree R -modules as values, respectively its quotient modulo the full subcategory of representable functors. We now assume that Λ has finite lattice type, let G be an additive generator for the category of Λ -lattices, and let $\mathcal{A}(\Lambda) = \text{End}_{\Lambda}(G)$ be the Auslander order of Λ . Let $P(G,G) \subset \mathcal{A}(\Lambda)$ be the endomorphisms of G factoring through projective Λ -lattices and put $\Sigma = \Sigma(\Lambda) = \mathcal{A}(\Lambda)/P(G,G)$. Then Σ is an artin algebra over $R/(\text{Rad}R)^V$ for some $v \in \mathbb{N}$.

With this notation, the following result due to Auslander holds.

Theorem 1. There exists an equivalence of categories $\text{mod } \Sigma \cong \underline{\text{latt}} \mathcal{A}(\Lambda)$

Proof. [M] Let F be a finitely generated additive functor on $\text{latt } \Lambda$ which vanishes on Λ . Choose a minimal epimorphism from a representable functor onto F with kernel ΩF , thus getting an exact sequence of the form

$$0 \longrightarrow \Omega F \longrightarrow \text{Hom}_{\Lambda}(-, M) \longrightarrow F \longrightarrow 0$$

for $M \in \text{latt } \Lambda$. Then it is routine to verify that $F \mapsto \Omega F$ induces an equivalence from $\text{mod } \underline{\text{latt}} \Lambda$ to $\underline{\text{latt}} \text{latt } \Lambda$. Moreover, $F \mapsto F(G)$ induces an equivalence from $\text{mod } \underline{\text{latt}} \Lambda$ to $\text{mod } \Sigma$, and for a given $L \in \text{latt } \text{latt } \Lambda$, $L \mapsto L(G)$ induces an equivalence between $\text{latt } \text{latt } \Lambda$ and $\text{latt } \mathcal{A}(\Lambda)$, hence an equivalence from $\text{latt } \text{latt } \Lambda$ to $\text{latt } \mathcal{A}(\Lambda)$. Putting together these equivalences yields the desired equivalence. #

The equivalences

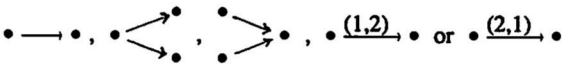
$$\begin{aligned} \text{lattlatt}(\mathcal{A}(\Lambda)) &\cong \text{modlatt}\mathcal{A}(\Lambda) \cong \text{mod}(\text{lattlatt}\Lambda) \\ &\cong \text{modmod}\Sigma \cong \text{mod}(\mathcal{A}(\Sigma)), \end{aligned}$$

can be iterated –as long as all the expressions make sense– and yield for $i \geq 0$ equivalences

$$\text{latt}(\mathcal{A}^{i+1}(\Lambda)) \cong \text{mod}\mathcal{A}^i(\Sigma).$$

Consequently an R -order Λ has Auslander type $i+1$ if and only if Σ has Auslander type i . Therefore, by results of [TZ] an R -order Λ has Auslander type > 5 if and only if it has infinite Auslander type, equivalently Σ is semisimple.

By [M],[W], this holds if and only if Λ is a Bäckström order with associated graph a disjoint union of oriented Dynkin diagrams [RR] of the form



These orders were called M -orders in [W].

2. Generalized Bäckström orders of Auslander type ≥ 2

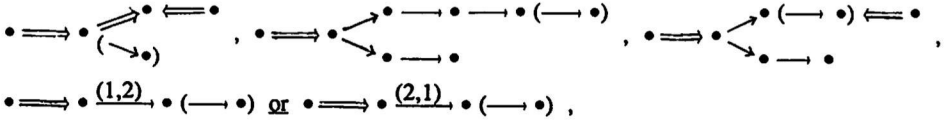
We now assume that Λ is a generalized Bäckström order, that is, there exists a hereditary overorder Γ of Λ in the same algebra such that $\text{Rad}\Gamma \subset \Lambda$, and each indecomposable direct summand of $\text{Rad}\Lambda$ is projective over Λ , or it is a Γ -lattice. Let \mathcal{A} be the residue class field of R , and let $\mathcal{D}_{\Lambda, \Gamma}$ be the \mathcal{A} -algebra given by the upper triangular ring

$$\mathcal{D}_{\Lambda, \Gamma} = \begin{bmatrix} \Gamma/\text{Rad}\Gamma & \Gamma/\text{Rad}\Gamma \\ 0 & \Lambda/\text{Rad}\Gamma \end{bmatrix}.$$

Then it is known from [R2],[R3] that $\mathcal{D}_{\Lambda, \Gamma}$ is a hereditary \mathcal{A} -algebra, and this property characterizes generalized Bäckström orders among the class of orders Λ which are subhereditary with respect to Γ , meaning that $\text{Rad}\Gamma$ is contained in Λ .

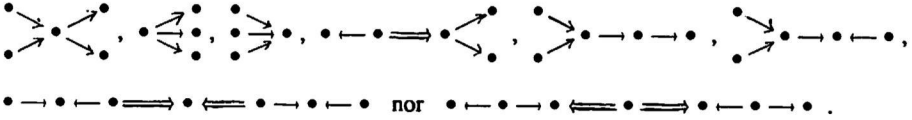
The graph $\mathcal{G}(\Lambda, \Gamma)$ associated to Λ with respect to Γ is defined as being the underlying valued quiver of the \mathcal{A} -species associated to $\mathcal{D}_{\Lambda, \Gamma}$ [G]. Recall from [R1], [R2] that if Λ is of finite lattice type, then the Auslander–Reiten quiver $\text{ARq}(\Lambda)$ of Λ is determined by $\mathcal{G}(\Lambda, \Gamma)$ together with a permutation on the set of sources of $\mathcal{G}(\Lambda, \Gamma)$. Even it is true that $\mathcal{G}(\Lambda, \Gamma)$ determines already the full subquiver of $\text{ARq}(\Lambda)$ consisting of the nonprojective vertices, which we denote by $\underline{\text{ARq}}(\Lambda)$. In order to classify the generalized Bäckström orders of Auslander type ≥ 2 , one only has to determine those valued quivers $\mathcal{G}(\Lambda, \Gamma)$ which produce a quiver $\underline{\text{ARq}}(\Lambda)$ of finite representation type.

Theorem 2. Let Λ be a generalized Bäckström order with associated graph $\mathcal{G} = \mathcal{G}(\Lambda, \Gamma)$ for a hereditary overorder Γ of Λ . Then Λ has Auslander type ≥ 2 if and only if \mathcal{G} is a disjoint union of valued quivers of the following form:


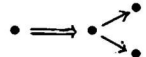



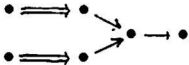
where all arrows $\bullet \rightarrow \bullet$ carry the trivial valuation $(1,1)$, $\bullet \rightrightarrows \bullet$ denotes a graph of length ℓ of the form $\bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet$ with $\ell+1$ vertices for $\ell \geq 0$, and where the arrows in brackets are optional.

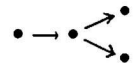
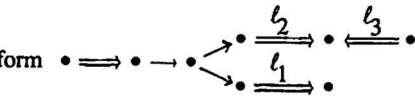
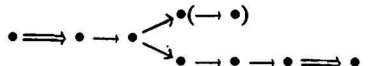
Proof. Let \mathcal{C} be a connected component of \mathcal{Q} . First, \mathcal{C} has at most one arrow with a nontrivial valuation, and it must be $(1,2)$ or $(2,1)$. Moreover, \mathcal{C} cannot contain a full subquiver of the form




Namely otherwise one easily checks that $\text{ARq}(\Lambda)$ would contain a full subquiver of infinite representation type.

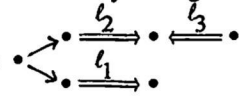
Having excluded these quivers as subquivers of \mathcal{C} , it is also easily checked that \mathcal{C} has at most one ramification vertex, that is a vertex where three or more arrows meet. Furthermore, by the construction of $\text{ARq}(\Lambda)$ from \mathcal{Q} , one sees easily that at each source of \mathcal{C}  there can be added an arbitrary tail of the form $\bullet \rightrightarrows \bullet$, getting  without changing the shape of $\text{ARq}(\Lambda)$.

If now \mathcal{C} contains a ramification vertex of the form , then \mathcal{C} must be of the form .

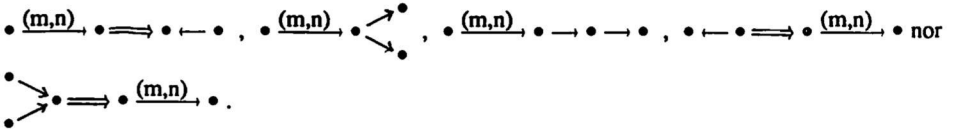
If \mathcal{C} contains a ramification vertex of the form , then \mathcal{C} must be of the form , $\bullet \xrightarrow{\ell} \bullet$ meaning that $\bullet \rightrightarrows \bullet$ has length ℓ . If $\ell_1=0$, then ℓ_2 and ℓ_3 can be arbitrary. If $\ell_1 \geq 2$, then ℓ_3 must be zero and ℓ_2 must be ≤ 1 , forcing that \mathcal{C} is of the form . If $\ell_1=1$, then ℓ_2 must be 0,1,2 or 3.

In the case where $\ell_2 \leq 1$, ℓ_3 can be arbitrary, whereas if $\ell_2 \geq 2$, then ℓ_3 must be zero.

If \mathcal{C} has no ramification vertex, then it either has no source with two arrows going out, and \mathcal{C} is of the form , or \mathcal{C} has a source with two arrows going out, and because this

source could be made into a ramification vertex by adding an arrow ending there without changing $\text{ARq}(\Lambda)$, \mathcal{G} must be of the form  with l_1, l_2 and l_3 satisfying the same conditions as above. Altogether this verifies our list in the Theorem if there are no nontrivial valuations on the arrows of \mathcal{G} .

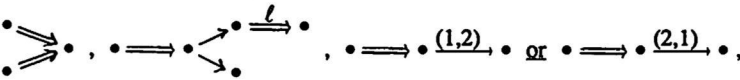
If otherwise \mathcal{G} contains a valued arrow $\bullet \xrightarrow{(m,n)} \bullet$ with $mn=2$, then one proceeds similarly as above by showing first that \mathcal{G} does not contain any subquiver of the form



In order to verify the sufficiency of the conditions, one easily determines $\text{ARq}(\Lambda)$ explicitly and checks that $\text{ARq}(\Lambda)$ is of finite representation type in all cases, e.g. using the Bongartz–Happel–Vossieck list [HV]. #

From this classification one can easily determine the Auslander type of the algebras $\Sigma(\Lambda)$ for Λ generalized Bäckström of Auslander type ≥ 2 . This gives the next result.

Theorem 3. Let Λ be a generalized Bäckström order of Auslander type ≥ 3 with associated graph \mathcal{G} as above. Then \mathcal{G} is a disjoint union of connected components of the form



where $l=0,1,2$ or 3 .

Moreover, Λ has infinite Auslander type if and only if for all components of the second form one has $l=0$. Otherwise, the Auslander type of Λ is 3 or 5. More precisely, if \mathcal{G} contains a connected component of the second form with $l=2$ or $l=3$, then Λ has Auslander type 3, whereas it has Auslander type 5 if always $l \leq 1$, and a component with $l=1$ actually occurs. #

3. Orders of Auslander type ≥ 4

Remember that a Λ -lattice is called bijective if it is both projective and injective in the category of Λ -lattices.

The following result is due to Drozd, Kirichenko and Roiter [DKR]. If Λ is a nonmaximal connected R -order, and B is an indecomposable bijective Λ -lattice, then there exists a unique minimal overorder Λ' of Λ in the same algebra such that B is not a Λ' -lattice, and all indecomposable Λ -lattices which are not isomorphic to B are also Λ' -lattices. More

generally, we call an overorder Λ' of Λ in the same algebra DKR-equivalent to Λ , if there exist indecomposable bijective Λ -lattices B_1, \dots, B_n which are not Λ' -lattices, and all indecomposable Λ -lattices not isomorphic to one of the B_i are also Λ' -lattices.

We have seen above using [TZ] that if Λ has Auslander type bigger than 5, it already has infinite Auslander type and therefore must be an M-order. As a generalization of this, we mention the following result whose proof will appear elsewhere.

Theorem 4. An R-order Λ has Auslander type ≥ 4 if and only if there is an M-order Λ' in the same algebra which is DKR-equivalent to Λ . #

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SUMS OF FK -SPACES*

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Presented by P.G. Rooney, F.R.S.C.

Abstract. For an FK -space X , we let the distinguished subsets $S(X), W(X), F(X)$ and $B(X)$ of X be the set of all sectionally convergent sequences, sectionally weakly convergent sequences, sectionally weakly Cauchy sequences and sectionally bounded sequences, respectively. In this paper, we discuss some inclusion relations among the sums of these subsets and the sums of FK -spaces. Thus we can answer some questions raised earlier by R. DeVos in [3].

We let X, Y, \dots be FK -spaces containing φ , the set of all finite sequences, so that $x^{[n]} = (x_1, x_2, \dots, x_n, 0, 0, \dots) \in X, Y, \dots$ for each sequence $x = (x_1, x_2, \dots)$.

Define the distinguished subsets in a usual way,

$$\begin{aligned} S(X) &= \{x \in X \mid x^{[n]} \rightarrow x \text{ in } X\}, \\ W(X) &= \{x \in X \mid x^{[n]} \rightarrow x \text{ weakly in } X\}, \\ F(X) &= \{x \in X \mid (x^{[n]})_n \text{ is weakly Cauchy in } X\}, \\ B(X) &= \{x \in X \mid (x^{[n]})_n \text{ is bounded in } X\}, \\ F^+(X) &= \{x \in \omega \mid (x^{[n]})_n \text{ is weakly Cauchy in } X\}, \\ B^+(X) &= \{x \in \omega \mid (x^{[n]})_n \text{ is bounded in } X\}, \end{aligned}$$

Moreover, the symbols $\omega, \ell^\infty, c, c_0, \ell, c_A, c_A^0$ (A a matrix) have the usual meanings, see [5].

If p and q are paranorms defining the FK -topology for X and Y respectively, then

$$r(z) = \inf \{p(x) + q(y) \mid z = x + y, x \in X, y \in Y\}$$

defines a paranorm on $X + Y$ giving an FK -topology for $X + Y$, see [4].

Theorem (DeVos [3])

(1) $X \subset Y$ implies $E(X) \subset E(Y)$ for $E = S, W, F, B, F^+, B^+$.

Moreover, if X is closed in Y then $E(X) = E(Y)$ for $E = S, W, F^+$ or B^+ , and $E(X) = E(Y) \cap X$ for $E = F, B$.

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(2) Let X^n be an FK-space for each $n \in \mathbb{N}$, and let $Y = \bigcap_n X^n$, which is an FK-space. then

$$E(Y) = \bigcap_n E(X^n) \text{ for } E = S, W, F \text{ or } B.$$

(3) Let $Z = X + Y$. Then

$$E(Z) \supset E(X) + E(Y) \text{ for } E = S, W, F, B, F^+ \text{ or } B^+.$$

We shall show that the inclusion can be strict in (3) above.

Example

Let $X = c_A$ and $Y = c_B$, where

$$A = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & & & \\ 1 & 2^{-2} & & & \\ 1 & 2^{-2} & 3^{-2} & & \\ 1 & 2^{-2} & 3^{-2} & 4^{-2} & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Then X, Y and $Z = X + Y$ are BK-spaces with norms defined by

$$\|x\|_X = \|Ax\|_\infty = \sup_n \{|x_n - x_{n-1}|\}, \quad \text{where } x_0 := 0,$$

$$\|y\|_Y = \|By\|_\infty = \sup_n \left\{ \left| \sum_{k=1}^n x_k k^{-2} \right| \right\},$$

$$\|z\|_Z = \inf_{z \in Z} \{ \|x\|_X + \|y\|_Y \mid z = x + y, x \in X, y \in Y \},$$

respectively.

The set $E(X) + E(Y)$ is strictly included in $E(X + Y)$ for $E = B^+, F^+, B, F, W, S$.

More exactly we have

E	$E(X)$	$E(Y)$	$E(X) + E(Y)$	$E(X + Y)$
B^+	ℓ^∞	v^γ	v^γ	$\supset X + v^\gamma$
F^+	ℓ^∞	Y	Y	$(X_0 + Y)^{\gamma\gamma}$
B	$\ell^\infty \cap X$	Y	Y	$X + Y$
F	$\ell^\infty \cap X$	Y	Y	$X_0 + Y$
W	$\ell^\infty \cap X_0$	Y	Y	$X_0 + Y$
S	c_0	Y	Y	$X_0 + Y$

Here $v = (1, 2^{-2}, 3^{-2}, \dots)$ and $X_0 = \{x \in X : Ax \in c_0\}$.

Proof.

$E(X)$:

(a) $\|x^{[n]}\|_X = \sup\{|x_1|, |x_2 - x_1|, \dots, |x_n - x_{n-1}|, |x_n|\}$ so that

$$\sup\{\|x^{[n]}\|_X : n \in \mathbb{N}\} < \infty \iff x \in \ell^\infty.$$

(b) $f \in X' \iff \exists \mu \in \mathbb{C}, t \in \ell : f(x) = \mu \lim_A x + t(Ax)$ for $x \in X$; thus

$$\sum_{k=1}^m f(e^k)x_k = \sum_{k=1}^m (t_k - t_{k+1})x_k = \sum_{k=1}^m t_k(x_k - x_{k-1}) - t_{m+1}x_m.$$

This shows

$$\sum_{k=1}^{\infty} f(e^k)x_k \text{ converges for all } f \in X' \iff \lim_{m \rightarrow \infty} t_{m+1}x_m \text{ exists for all } t \in \ell.$$

Hence we get

$$\begin{aligned} x \in F^+(X) &\iff x \in \ell^\infty; \text{ also} \\ x \in F(X) &\iff x \in \ell^\infty \cap X, \\ x \in W(X) &\iff x \in \ell^\infty \cap X \text{ and } \lim_A x = 0 \\ &\iff x \in X_0 \cap \ell^\infty. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \|x - x^{[n]}\|_X &= \sup\{|x_n|, |x_{n+1} - x_n|, |x_{n+2} - x_{n+1}|, \dots\} \rightarrow 0 \\ &\iff |x_n| \rightarrow 0 \text{ and } (Ax)_n \rightarrow 0 \\ &\iff x \in c_0. \end{aligned}$$

$E(Y)$:

(a) Since Y is AK , we have $S(Y) = W(Y) = F(Y) = B(Y) = Y$. Also, $\|y^{[n]}\|_Y = \sup\{|\sum_{k=1}^m y_k k^{-2}| : m \in \mathbb{N}\}$, hence

$$\sup_n \|y^{[n]}\|_Y = \sup\{|\sum_{k=1}^m y_k k^{-2}| : m \in \mathbb{N}\}$$

and $y \in B^+(Y) \iff y \in v^\gamma$.

(b) Since Y is AK , $\bar{\varphi}$ ($= Y$) is AK , and by [5, 10.4.11] we have $F^+(Y) = (\bar{\varphi})^{\beta\beta} = Y^{\beta\beta} = v^{\beta\beta} = v^\beta = Y$.

$E(X) + E(Y)$:

The assertions follow from $\ell^\infty \subset Y$.

$E(X + Y)$:

(a) We first show $X \subset B(X + Y)$: Let $z \in X$, then

$$\begin{aligned} z^{[n]} &= (z_1, z_2, \dots, z_{n-1}, z_n, 0, 0, \dots) \\ &= x^n + y^n, \quad \text{where} \\ x^n &= (z_1, z_2, \dots, z_{n-1}, z_n, z_n, z_n, \dots), \\ y^n &= (0, 0, \dots, 0, 0, -z_n, -z_n, \dots). \end{aligned}$$

We have

$$\begin{aligned} \|x^n\|_X &= \sup\{|z_1|, |z_2 - z_1|, \dots, |z_n - z_{n-1}|, 0, 0, \dots\} \\ &= \sup\{|(Az)_1|, |(Az)_2|, \dots, |(Az)_n|, 0, 0, \dots\} \\ &\leq \|z\|_X < \infty \quad \text{and} \\ \|y^n\|_Y &= \sup\{|z_n| \sum_{k=n+1}^m k^{-2} : m \geq n+1\}. \end{aligned}$$

Now, for $x \in X$, $z_n = \sum_{k=1}^n \xi_k$ with some $\xi = (\xi_k) \in c$, whence $z_n = \mathcal{O}(n)$. Thus

$$\|y^n\|_Y = \mathcal{O}(n) \sum_{k=n+1}^{\infty} k^{-2} = \mathcal{O}(1) \text{ for } z \in X.$$

Next, we show $X_0 + Y \subset S(X + Y)$: Similarly as before, we get

$$\begin{aligned} z - z^{[n]} &= (0, \dots, 0, z_{n+1}, z_{n+2}, \dots) \\ &= w^n + y^n, \quad \text{where} \\ w^n &= (0, \dots, 0, z_{n+1} - z_n, z_{n+2} - z_n, \dots), \\ y^n &= (0, \dots, 0, z_n, z_n, \dots) \text{ as above.} \\ \|w^n\|_X &= \sup\{|z_{n+1} - z_n|, |z_{n+2} - z_n|, \dots\} \\ &= \sup\{|(Az)_{n+1}|, |(Az)_{n+2}|, \dots\} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ if } z \in X_0; \\ \|y^n\|_Y &= |z_n| \sum_{k=n+1}^{\infty} k^{-2} \\ &= o(n)\mathcal{O}(n^{-1}) = o(1), \end{aligned}$$

since $z_n = \sum_{k=1}^n \xi_k = o(n)$ for $\xi = (\xi_k) \in c_0$.

Now we show $u := (1, 2, 3, \dots) \notin F(X + Y)$: Let the matrix H be defined by

$$(Hz)_n = \frac{1}{\log(n+1)} \sum_{k=1}^n z_k k^{-2}.$$

Then $c_H \supset X + Y$. In fact, let $z = x + y$ with $x \in X$ and $y \in Y$. Then $x_n = \sum_{k=1}^n \xi_k$ and $y_n = n^2(\eta_n - \eta_{n+1})$ with some $\xi = (\xi_k) \in c$ and $\eta = (\eta_n) \in c$. This leads to

$$\begin{aligned}(Hz)_n &= \frac{1}{\log(n+1)} \left(\sum_{k=1}^n k^{-2} \sum_{\nu=1}^k \xi_\nu + \sum_{k=1}^n k^{-2} k^2 (\eta_k - \eta_{k+1}) \right) \\ &= \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \sigma_k + \frac{1}{\log(n+1)} (\eta_1 - \eta_{n+1}), \quad \text{where} \\ \sigma_k &= \frac{1}{k} \sum_{\nu=1}^k \xi_\nu.\end{aligned}$$

If $\xi \in c$, then $\sigma \in c$ (with $\lim \sigma = \lim \xi$) and hence $\lim_H z = \lim_n \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \sigma_k = \lim \sigma = \lim \xi$. Moreover, this shows

$$\lim_H|_X = \lim_A \quad \text{and} \quad \lim_H|_Y = 0.$$

In particular, $\lim_H u = \lim_A u = 1$ and $\lim_H z = 0$ for $z \in X_0 + Y$, whence $u \notin F(X + Y)$.

Last, we observe that $S = X_0 + Y = (\lim_H|_{X+Y})^\perp$ is closed. Hence $\overline{\varphi} = S$ is AK , and by [5, 10.4.11] we get $F^+(X + Y) = (\overline{\varphi})^{\beta\beta} = (X_0 + Y)^{\beta\beta}$.

From $X_0 + X \neq Y$ (e.g. $x = \left(\frac{n+1}{\log(n+1)}\right)_n \in X_0 \setminus Y$) we conclude that

$$E(X) + E(Y) \neq E(X + Y) \quad \text{for} \quad E = B^+, F^+, B, F, W, S.$$

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SUR L'INÉGALITÉ D'ALLAN SINCLAIR POUR
LES OPÉRATEURS HERMITIENS

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Summary. Given a hermitian element h in a Banach algebra, a well-known result of Allan Sinclair [4] states that $\rho(h + i\alpha 1) = \|h + i\alpha 1\|$ for all complex numbers α . F.F.Bonsall and J.Duncan [2, Theorem 25.6] have given a complicated example of a normal element, in a subalgebra of the algebra of continuous functions on the closed unit disc, for which the spectral radius is different from the norm. In this paper we give a much more easy example of a normal element a in a finite-dimensional algebra such that $\rho(a) < \|a\|$.

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Soit A une algèbre de Banach complexe unitaire et soit $x \in A$, on dénote par

$$V(x) = \{f(x) : f \in A', \|f\| = f(1) = 1\}.$$

Un élément h dans une algèbre de Banach est dit hermitien si $V(h)$ est réel. Le lemme 5.2 de [1] affirme que h est hermitien si et seulement si $\|e^{ith}\| = 1$, pour tout t réel. Un élément $a \in A$ est dit normal (au sens général, dû à G.Lumer) si $a = h + ik$, où h, k sont des éléments hermitiens qui commutent dans A . Il est clair que ces notions généralisent les notions classiques d'opérateurs hermitiens et normaux sur un espace de Hilbert. A.M.Sinclair a démontré dans [4] que pour un élément hermitien h dans A , on a

$$\rho(h + \alpha 1) = \|h + \alpha 1\|, \text{ pour tout } \alpha \in \mathbb{C} \tag{1}$$

où $\rho(h)$ désigne le rayon spectral de h et $\|\cdot\|$ la norme d'algèbre de A . Une question naturelle se pose: peut-on généraliser ce résultat aux cas des éléments normaux ?

En considérant la sous-algèbre B des fonctions continues sur le disque fermé, vérifiant les conditions du théorème 25.2 de [2], F.F.Bonsall et J.Duncan ont donné un exemple d'élément normal dans B qui ne vérifie pas (1). Dans ce travail nous proposons un exemple simple d'opérateur normal a dans une algèbre de Banach de dimension finie tel que $\rho(a) < \|a\|$.

Soit a un élément d'une algèbre de Banach, on dénote par R_a et L_a les opérateurs de multiplication respectivement à droite et à gauche, et par C_a la dérivation intérieure $R_a - L_a$.

Lemme 1. Si h est hermitien dans A , alors R_h et L_h sont hermitiens dans l'algèbre $L(A)$ munie de la norme d'opérateurs.

Démonstration : Si $x \in A$ alors $\|e^{itR_h}x\| \leq \|x\| \|e^{itR_h}\| = \|x\|$, donc $\|e^{itR_h}\| \leq 1$, quel que soit t réel. Mais $1 \leq \|e^{itR_h}\| \|e^{-itR_h}\|$ implique que nécessairement $\|e^{itR_h}\| = 1$, pour tout t réel. Ainsi R_h est hermitien d'après le lemme 5.2 de [1]. Le raisonnement est identique pour L_h .

Lemme 2. Si a est normal dans A alors C_a est normal dans $L(A)$.

Démonstration: Soit $a = h + ik$ où h, k sont hermitiens, tels que $hk = kh$, alors $C_a = C_h + iC_k$, avec $C_h C_k = C_k C_h$, du lemme 1 on déduit le résultat.

Lemme 3. $Sp(C_a) \subset \{\lambda - \mu : \lambda, \mu \in Sp a\}$.

Démonstration: (voir W.Rudin [3], Corollaire du théorème 11.23).

D'après J.G.Stampfli ([5], Corollaire 1), on a le résultat suivant.

Lemme 4. Soit T un opérateur normal (au sens traditionnel) dans l'algèbre des opérateurs linéaires bornés sur un espace de Hilbert, alors $\|C_T\|$ est égal à deux fois le rayon du plus petit disque contenant le spectre de T .

Notre résultat principal s'énonce de la façon suivante

Théorème: Dans l'algèbre $L(M_3(\mathbb{C}))$, munie de la norme d'opérateurs, il existe un opérateur normal a (au sens général, dû à G.Lumer) tel que $\rho(a) < \|a\|$.

Démonstration. Considérons l'opérateur normal T défini par:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \bar{\alpha} \end{pmatrix} \quad \text{où } \alpha = e^{i\frac{2\pi}{3}}.$$

Du lemme 2 on déduit que C_T est normal au sens général dans $L(M_3(\mathbb{C}))$. De plus $Sp(C_T) \subset \{0, 1 - \alpha, 1 - \bar{\alpha}, \alpha - \bar{\alpha}, \alpha - 1, \bar{\alpha} - 1, \bar{\alpha} - \alpha\}$, donc $\rho(C_T) \leq \sqrt{3}$. D'autre part, d'après le lemme 4, on a $\|C_T\| = 2$. Ainsi $\rho(C_T) < \|C_T\|$.

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ON THE THETA SERIES OF POSITIVE QUATERNARY QUADRATIC FORMS

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Abstract. The purpose of this note is to describe results regarding the occurrence of genera containing inequivalent positive definite integral quaternary quadratic forms with identical theta series, including a new example possessing this property.

§1. Introduction. It has recently been proved by Bayer and Nart [1] that two positive definite integral quadratic forms which have identical theta series, and which satisfy additional obviously necessary conditions, must lie in the same genus. That inequivalent forms within a genus can have identical theta series was first shown by Witt for forms of rank 16 [11], and later for forms of rank 12 by Kneser [6] and rank 8 by Kitaoka [5]. By contrast, it was conjectured that such examples no longer exist when the rank is reduced to 4 [4]. The recent discovery by Schiemann [10] of a genus containing two inequivalent positive quaternary forms of discriminant 1729 having identical theta series has shown this conjecture to be false. In the final section of this paper, we exhibit another such pair.

The theta series θ_f associated to the quadratic form f lies in the space $\mathcal{M}_k(\Gamma_0(N), \chi)$ of modular forms of weight k , level N and character χ , where k is half the rank of f , N is the level of f , and χ is a particular character modulo N determined by the discriminant D of f . For forms f and g in the same genus, $\theta_f - \theta_g$ is known to lie in the subspace $\mathcal{J}_k(\Gamma_0(N), \chi)$ of

$\mathcal{H}_k(\Gamma_0(N), \chi)$ consisting of all cusp forms. Thus, to find a genus containing inequivalent forms with identical theta series, it suffices to find one which contains more than one class but for which the corresponding space $\mathcal{J}_k(\Gamma_0(N), \chi)$ has dimension zero. The examples in ranks 8, 12 and 16 referred to above all arise in this way. That no similar examples occur in this way in rank 4 is shown by the following result, the proof of which will be outlined in the next two sections of the paper.

Theorem. Let f be a positive definite integral quaternary quadratic form of level N and character χ . If $\dim \mathcal{J}_2(\Gamma_0(N), \chi) = 0$, then f lies in a genus of one class.

§2. Possible levels and discriminants. Denote the dimension of $\mathcal{J}_2(\Gamma_0(N), \chi)$ by $d(N, \chi)$. As the characters χ associated to positive quaternary quadratic forms have the property that $\chi(-1) = 1$, $d(N, \chi)$ can be computed from [2; Theorem 1]. A straightforward analysis of the terms in the formula given there shows that, in a search for those N and χ for which $d(N, \chi) = 0$, potential values of N are restricted as follows.

Lemma 1. Let $N \in \mathbb{N}$ and let χ_0 denote the trivial character modulo N . If $d(N, \chi_0) > 1$, then $d(N, \chi) > 0$ for all characters χ modulo N for which $\chi(-1) = 1$.

Lemma 2. Let $N \in \mathbb{N}$. Then $d(N, \chi_0) \leq 1$ if and only if $N \leq 21$ or $N = 24, 25, 27, 32, 36$ or 49 .

For brevity, in the remainder of the paper the word "form" will refer always to a positive definite integral quaternary quadratic form. Now suppose D occurs as the discriminant of such

a form of level N and character χ for which $d(N, \chi) = 0$. As $N|D|N^4$ (see, e.g., [8]), the prime factors of D are restricted by Lemmas 1 and 2. If $D = D_0 D_1^2$ with D_0 square free, then χ is a quadratic character modulo $4D_0$ which is completely determined by D . Direct computation now yields a complete list of the possible D and the corresponding values of N . The resulting list contains 124 discriminants, the largest of which is $2^{19} = 524,288$.

§3. Enumeration and analysis of genera. Having thus restricted to a finite list of discriminants, it is now a finite problem to produce representative forms from all genera of forms of level N and character χ for which $d(N, \chi) = 0$. For this purpose, our starting point was the use of the second author's table [7], which contains representatives of all equivalence classes of positive quaternary forms of discriminant not exceeding 1732. However, since many of the discriminants which need to be investigated here exceed the scope of this table, it was necessary to find an algorithmic procedure for generating forms having these larger discriminants from those of smallest discriminants. To do this, we adapted a method due to Pall [9; §12], the use of which is summarized in the following lemma.

Lemma 3. Let f be a form with associated matrix F and discriminant D . If p is a prime such that $D = p^2 D'$, where D' is itself the discriminant of a form, then f is equivalent to a form having an associated matrix of the type $P^t A P$, where P ranges through the $p^3 + p^2 + p + 1$ matrices of the types

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & a & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} p & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are non-negative integers less than p , and A ranges through matrices associated to representatives of the distinct equivalence classes of forms of discriminant D' .

For our application, it is useful to further note that if the form f has level N and the form of discriminant D' from which it is derived in this manner has level N' , then $N' \leq N$.

A computer algorithm based on the use of Lemma 3 was implemented to generate representative forms from all remaining genera of level N and character χ for which $d(N, \chi) = 0$. It then remains only to check that each such genus contains a single class. As the forms in [7] are separated into genera, this can be verified by inspection for the forms within the range of this table. For the remaining forms, the order of the group of automorphs was first computed and it was then checked via the Minkowski-Siegel mass formula that the genus and class coincide in all cases. The proof of the theorem stated in §1 is now complete.

§4. An example. In this section, we describe a pair of positive quaternary forms, found through the use of the algorithm based on Lemma 3, which are in the same genus, in distinct classes, and which have the same theta series. Here we will describe a form by listing its coefficients in the sequence $(a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{13}, a_{23}, a_{14}, a_{24}, a_{34})$, where a_{ij} is the coefficient of $x_i x_j$. We begin with the forms

$$f_1 = (2, 4, 5, 5, 2, 0, 3, 1, 1, 5) \quad \text{and} \quad f_2 = (2, 4, 4, 5, 0, 1, 1, 1, -4, 2)$$

of discriminant 1729 which Schiemann [10] proved to satisfy these conditions. Descending from the maximal lattices associated with

these forms via transformations of determinant 7, we generated 114 classes of sublattices of discriminant $84721 = 7^2 \cdot 1729$ and level $N = 12103 = 7 \cdot 1729$. By a computer search of theta coefficients, the forms

$$g_1 = (6, 8, 10, 15, -4, 0, 5, 1, 4, 8) \text{ and } g_2 = (6, 8, 10, 15, -1, 6, 1, 3, -7, 2)$$

in the same genus, in distinct classes, and corresponding to sublattices of f_1 and f_2 , respectively, were isolated from this list of 114. Using a well-tested computer program, g_1 and g_2 were found to have the same numbers of representations of n through $n = 2615$. Following Schiemann's use of a theorem of Hecke [3], the computations $\mu_0(12103) = 15680$ and

$$\frac{2\mu_0(12103)}{12} + 1 < 2615$$

imply that two such forms with the same first 2615 theta coefficients will have identical theta series. Thus g_1 and g_2 have the requisite properties.

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A Unified Möbius Inversion Formula

by

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Presented by J.H.H. Chalk, F.R.S.C

Abstract Classical Möbius Inversion formulae are subsumed in a single formula.

1. Let A be a subset of the real numbers \mathbf{R} and let

$$(1) \quad f: A \rightarrow \mathbf{C}$$

be a function on A with values in the complex numbers \mathbf{C} . If \mathbf{N} denotes the natural numbers, let " \circ " denote an operation on the product set $A \times \mathbf{N}$ of the type:

$$(2) \quad \circ: \begin{cases} A \times \mathbf{N} \rightarrow \mathbf{R} \\ (x, n) \mapsto \circ(x, n) := x \circ n, \end{cases}$$

which satisfies, in the first instance, the two axioms:

$$A1. \quad (x \circ n) \circ m = x \circ (nm), \quad \forall (x, n, m) \in A \times \mathbf{N} \times \mathbf{N},$$

$$A2. \quad x \circ 1 = x, \quad \forall x \in A.$$

Further, let " \sim " denote a relation from \mathbf{N} to A , i.e.,

$$(3) \quad \sim \subseteq \mathbf{N} \times A.$$

On writing this in the form

$$(4) \quad n \sim x : \Leftrightarrow (n, x) \in \sim, \quad \forall n \in \mathbf{N}, \quad x \in A$$

the relation also satisfies two axioms:

$$B1. \quad nm \sim x \Rightarrow n \sim x, \quad \forall (x, n, m) \in A \times \mathbf{N} \times \mathbf{N},$$

$$B2. \quad 1 \sim x, \quad \forall x \in A.$$

There are, moreover, two 'mixed' axioms:

$$C1. \quad n \sim x \Rightarrow x \circ n \in A, \quad \forall n \in \mathbf{N}, \quad x \in A,$$

$$C2. \quad n \sim (x \circ m) \Leftrightarrow nm \sim x, \quad \forall (x, n, m) \in A \times \mathbf{N} \times \mathbf{N}.$$

By C1 we can introduce the Möbius formulae

$$(5) \quad g(x) := \sum_{\substack{n \in \mathbf{N} \\ n \sim x}} f(x \circ n), \quad \forall x \in A, \quad n \in \mathbf{N} \quad \text{and}$$

$$(6) \quad f(x) = \sum_{\substack{n \in \mathbf{N} \\ n \sim x}} \mu(n)g(x \circ n), \quad \forall x \in A, \quad n \in \mathbf{N},$$

where $\mu(n)$ is the Möbius function (for some special cases, see below or [1] pp. 234-6). Note that C1 ensures that $g : A \rightarrow \mathbf{C}$ is indeed a well-defined function, while C2 is required for the proof that (5) \Rightarrow (6). In fact, (5) and (6) are inversions in the sense that, conversely, (6) \Rightarrow (5). Of course, if $\text{card} \{n \in \mathbf{N}, n \sim x\} = \infty$ there are convergence considerations and we shall offer sufficient conditions after the formal proof.

2. Proof that (5) \Leftrightarrow (6).

To obtain (6) from (5), we show that

$$\sum_{\substack{n \in \mathbf{N} \\ n \sim x}} \mu(n)g(x \circ n) = \sum_{\substack{n \in \mathbf{N} \\ n \sim x}} \mu(n) \left\{ \sum_{\substack{m \in \mathbf{N} \\ m \sim x \circ n}} f\{(x \circ n) \circ m\} \right\} = f(x).$$

By C2 and A1, the repeated sum is

$$\sum_{\substack{n \in \mathbf{N} \\ n \sim x}} \mu(n) \left\{ \sum_{\substack{m \in \mathbf{N} \\ nm \sim x}} f(x \circ nm) \right\}$$

and, by B1, this can be written as a double sum:

$$\sum_{\substack{n, m \in \mathbf{N} \\ nm \sim x}} \mu(n)f(x \circ nm)$$

On collecting terms with $nm = t$, we obtain

$$\sum_{\substack{t \in \mathbb{N} \\ t \sim 1}} \sum_{d|t} \mu(d) f(x \circ t) = \sum_{\substack{t \in \mathbb{N} \\ t \sim 1}} f(x \circ t) \left\{ \sum_{d|t} \mu(d) \right\}$$

and by the well-known property of $\mu(n)$, this becomes

$$\begin{aligned} \sum_{\substack{t \in \mathbb{N} \\ t \sim 1}} f(x \circ t) &= f(x \circ 1), \text{ by B2} \\ &= f(x), \text{ by A2,} \end{aligned}$$

as required. This argument can be repeated on

$$\sum_{\substack{n \in \mathbb{N} \\ n \sim 1}} f(x \circ n) = \sum_{\substack{n \in \mathbb{N} \\ n \sim 1}} \left\{ \sum_{\substack{m \in \mathbb{N} \\ m \sim 1 \circ n}} \mu(m) g((x \circ m) \circ n) \right\};$$

the main steps being

$$\sum_{\substack{n \in \mathbb{N} \\ n \sim 1}} \left\{ \sum_{\substack{m \in \mathbb{N} \\ m \sim 1 \circ n}} \mu(m) g(x \circ nm) \right\} = \sum_{\substack{n, m \in \mathbb{N} \\ nm \sim 1}} \mu(m) g(x \circ nm) = g(x)$$

and so (6) \Rightarrow (5).

3. Remarks.

If $\text{card} \{n \in \mathbb{N}, n \sim x\} = \infty$, it is now clear that a sufficient condition to justify the manipulation of the infinite series is the absolute convergence of the double series

$$\begin{aligned} \sum_{\substack{n, m \in \mathbb{N} \\ nm \sim 1}} f(x \circ nm) \quad \text{or, simply that of the series} \\ \sum_{\substack{t \in \mathbb{N} \\ t \sim 1}} \tau(t) f(x \circ t), \quad \text{where } \tau(t) = \sum_{d|t} 1. \end{aligned}$$

Examples (special cases satisfying all six axioms).

(i) $A = \mathbb{N}, \quad n \sim \Leftrightarrow n|m, \quad n \circ m = \frac{n}{m}$

$$g(x) = \sum_{n|z} f\left(\frac{x}{n}\right) \Leftrightarrow f(x) = \sum_{n|z} \mu(n) g\left(\frac{x}{n}\right)$$

$$(ii) \quad A = [0, \infty), \quad n \sim x \Leftrightarrow n \leq x, \quad x \circ n = \frac{x}{n}$$

$$g(x) = \sum_{n \leq x} f\left(\frac{x}{n}\right) \Leftrightarrow f(x) = \sum_{n \leq x} \mu(n)g\left(\frac{x}{n}\right)$$

$$(iii) \quad (\text{cf., remarks above})$$

$$A = \mathbf{R}, \quad n \sim x, \quad \forall n \in \mathbf{N}, \quad x \in \mathbf{R}, \quad x \circ n = nx$$

$$g(x) = \sum_{n=1}^{\infty} f\left(\frac{x}{n}\right) \Leftrightarrow f(x) = \sum_{n=1}^{\infty} \mu(n)g\left(\frac{x}{n}\right)$$

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**ALGORITHME POUR CALCULER LES POINTS DE HEEGNER
DE $Y_0(N)$.**

par

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Presented by J. Friedlander, F.R.S.C.

Résumé : Dans ce travail, on donne un algorithme pour calculer les points de Heegner de $Y_0(N)$, de discriminant D , avec D fondamental, impair, $(D, N) = 1$ et tel que le groupe des classes d'idéaux du corps quadratique imaginaire $\mathbf{Q}(\sqrt{D})$ est cyclique.

**AN ALGORITHM FOR COMPUTING THE HEEGNER POINTS
ON $Y_0(N)$.**

Abstract : In this work, we give an algorithm for computing the Heegner points on $Y_0(N)$ of discriminant D , with D fundamental, odd, relatively prime to N and such that the ideal class group of the imaginary quadratic field $\mathbf{Q}(\sqrt{D})$ is cyclic.

1. Position du problème.

Soit $Y_0(N)$ la courbe $H/\Gamma_0(N)$, où H est le demi-plan de Poincaré et $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) / c \equiv 0 \pmod{N} \right\}$, N étant un entier ≥ 1 . Un point de Heegner de $Y_0(N)$ est un point z de H , défini modulo $\Gamma_0(N)$, quadratique, tel que les nombres z et Nz ont le même discriminant [1].

Pour un autre point de vue, un point de Heegner de $Y_0(N)$ peut être interprété comme un couple de courbes elliptiques liées par une isogénie α cyclique d'ordre N , $(E \xrightarrow{\alpha} E')$ telles que E et E' ont la même multiplication complexe (voir [1] et [2]).

Soit D un discriminant fondamental, impair et $(D, N) = 1$. Alors, les points de Heegner de $Y_0(N)$ de discriminant D existent si et seulement si D est congru à un carré (mod $4N$). Dans ce cas, il existe $2^s h(D)$ points de Heegner de $Y_0(N)$, où s est le nombre de diviseurs premiers distincts de N , et $h(D)$ est le nombre des classes d'idéaux du corps quadratique $K = \mathbf{Q}(\sqrt{D})$, [2]. D'après Gross et Zagier

[2], on a la correspondance :

$$\left\{ \begin{array}{l} \text{Les points de Heegner de } Y_0(N) \\ \text{de discriminant } D \end{array} \right\} \leftrightarrow \{\text{Les paires } (\mathcal{A}, \eta)\}$$

où $\mathcal{A} \in cl_K$ le groupe des classes d'idéaux de K et η un idéal entier primitif de \mathcal{O}_K l'anneau des entiers de K de norme N ("primitif" signifie que $\mathcal{O}_K/\eta \simeq \mathbb{Z}/N\mathbb{Z}$ ou ce qui revient à dire que η n'est pas divisible comme idéal par tout entier > 1).

On démontre aussi dans [2], une correspondance entre les idéaux primitifs $\eta \subset \mathcal{O}_K$ de norme N et les solutions β de la congruence $\beta^2 \equiv D \pmod{4N}$, $\beta \in \mathbb{Z}/2N\mathbb{Z}$. Cette correspondance est donnée par :

$$\eta = \left(N, \frac{\beta + \sqrt{D}}{2} \right) = \mathbb{Z}N + \mathbb{Z} \frac{\beta + \sqrt{D}}{2} \leftarrow \beta.$$

On suppose que D est congru à un carré modulo $4N$. Alors pour calculer les points de Heegner de $Y_0(N)$ de discriminant D (définis modulo $\Gamma_0(N)$), en utilisant les correspondances précédentes, on procède de la façon suivante : étant donné une solution β de la congruence $\beta^2 \equiv D \pmod{4N}$, $\beta \in \mathbb{Z}/2N\mathbb{Z}$, on cherche un système complet de représentants des classes d'idéaux de K : $a_1, a_2, \dots, a_{h(D)}$ tels que pour tout $i = 1, 2, \dots, h(D)$, l'idéal a_i peut s'écrire :

$$a_i = A_i \mathbb{Z} + \frac{B_i + \sqrt{D}}{2} \mathbb{Z}$$

avec A_i est la norme de a_i ($N_{K/\mathbb{Q}}(a_i) = A_i$)

$$A_i \equiv 0 \pmod{N}$$

et $B_i \equiv \beta \pmod{2N}$

ou ce qui revient à dire à chercher les a_i tels que a_i soit primitif et $a_i \subset \eta$, où $\eta = \left(N, \frac{\beta + \sqrt{D}}{2} \right)$, l'idéal primitif de norme N correspondant à β . Dans la suite, on dira que les a_i vérifient les conditions de Heegner relativement à β (ou η).

Etant donné les idéaux a_i , les points de Heegner correspondant à β sont :

$$\frac{B_i + \sqrt{D}}{2A_i} \pmod{\Gamma_0(N)}, \quad 1 \leq i \leq h(D).$$

Pour de plus amples renseignements, voir [2] et [3].

Remarques :

D et N étant comme précédemment.

1) Si $h(D) = 1$, alors les points de Heegner de $Y_0(N)$ de discriminant D sont :

$$\frac{\beta + \sqrt{D}}{2N} \pmod{\Gamma_0(N)},$$

où les β sont les solutions de $\beta^2 \equiv D \pmod{4N}$, $\beta \in \mathbf{Z}/2N\mathbf{Z}$.

Ainsi, pour calculer les points de Heegner, il suffit de trouver les entiers β .

2) Si $N = 1$, pour calculer les points de Heegner, il suffit de chercher des représentants primitifs des classes d'idéaux : $a_1, a_2, \dots, a_{h(D)}$. En effet, on démontre facilement que pour tout idéal entier primitif a de norme A , il existe un entier B tel que $a = AZ + \frac{B + \sqrt{D}}{2}Z$; et que si pour tout $i = 1, \dots, h(D)$ l'idéal a_i s'écrit : $a_i = A_iZ + \frac{B_i + \sqrt{D}}{2}Z$; alors les points de Heegner seront : $\frac{B_i + \sqrt{D}}{2A_i} \pmod{\Gamma_0(N)}$, $1 \leq i \leq h(D)$.

Soit D un discriminant fondamental, impair tel que le groupe des classes d'idéaux $cl_{\mathbf{Q}(\sqrt{D})}$ est cyclique. L'objet de ce travail est de donner un algorithme permettant le calcul des idéaux a_i vérifiant les conditions de Heegner relativement à une solution β de $\beta^2 \equiv D \pmod{4N}$, $\beta \in \mathbf{Z}/2N\mathbf{Z}$, (si cette congruence admet des solutions), quand l'entier N décrit un ensemble $\{2, 3, \dots, C\}$ avec $(N, D) = 1$. (C étant un entier arbitraire).

Dans toute la suite, D désignera un discriminant vérifiant les conditions ci-dessus, K le corps quadratique imaginaire $\mathbf{Q}(\sqrt{D})$, \mathcal{O}_K l'anneau des entiers de K , cl_K le groupe des classes d'idéaux de K et $h(D)$ le nombre des classes d'idéaux. Alors, on démontre facilement le lemme suivant :

Lemme : Si $h(D) > 1$, alors il existe un nombre premier p et un entier a tels que :

$$cl_K = \{\{\mathcal{P}^i\}, i = 1, \dots, h(D)\}$$

où $\mathcal{P} = pZ + \frac{a + \sqrt{D}}{2}Z$ (\mathcal{P}^i désigne la classe de l'idéal \mathcal{P}^i).

2. Description de l'algorithme.

Les données : D , $h(D)$ et les entiers p et a du lemme précédent (on suppose que $h(D) > 1$).

Pour $2 \leq N \leq C$ tel que $(D, N) = 1$ et D est congru à un carré modulo $4N$, soient $(\pm B_{0,i})_{1 \leq i \leq 2^{s-1}}$ (s étant le nombre de diviseurs premiers distincts de N) les solutions dans $\mathbb{Z}/2N\mathbb{Z}$ de la congruence : $B^2 \equiv D \pmod{4N}$, avec $1 \leq B_{0,i} < 2N$ (on verra plus loin le choix des $B_{0,i}$). Alors, les idéaux primitifs de norme N sont :

$$\eta_{0,i} = \left(N, \frac{B_{0,i} + \sqrt{D}}{2}\right) \quad \text{et} \quad \eta'_{0,i} = \left(N, \frac{-B_{0,i} + \sqrt{D}}{2}\right) \quad i = 1, \dots, 2^{s-1}.$$

Pour construire des représentants primitifs des classes d'idéaux de K , vérifiant les conditions de Heegner par rapport à un idéal primitif de norme N , on est amené à envisager les deux cas suivants :

1^{er} cas : p ne divise pas DN , ou p divise N .

Dans ce cas, on démontre facilement que p se décompose :

$$(p) = \mathcal{P}\bar{\mathcal{P}} \quad \text{où} \quad \mathcal{P} = \left(p, \frac{a + \sqrt{D}}{2}\right) \quad \bar{\mathcal{P}} = \left(p, \frac{-a + \sqrt{D}}{2}\right)$$

Pour $1 \leq i \leq 2^{s-1}$ fixé, on pose :

$$\eta_{j,i} = \eta_{0,i}\mathcal{P}^j \quad \text{et} \quad \eta'_{j,i} = \eta'_{0,i}\bar{\mathcal{P}}^j \\ j = 0, 1, \dots, h(D) - 1.$$

Alors, on a : $cl_K = \{[\eta_{j,i}], 0 \leq j < h(D)\} = \{[\eta'_{j,i}], 0 \leq j < h(D)\}$.

On va voir maintenant les conditions qu'il faut poser sur $B_{0,i}$ pour que les idéaux $\eta_{j,i}$ et $\eta'_{j,i}$ soient primitifs : comme $\eta_{0,i}$, $\eta'_{0,i}$, \mathcal{P} et $\bar{\mathcal{P}}$ sont primitifs, donc $\eta_{j,i}$ (resp. $\eta'_{j,i}$) est non primitif ssi $\bar{\mathcal{P}}$ divise $\eta_{0,i}$ (resp. \mathcal{P} divise $\eta'_{0,i}$). Par conséquent, si p ne divise pas DN , alors $\eta_{j,i}$ et $\eta'_{j,i}$ sont primitifs (donc on peut choisir $B_{0,i}$ d'une façon arbitraire). Si p divise N , on choisit $B_{0,i}$ tel que $B_{0,i} \equiv a \pmod{2p}$. Ce qui entraîne que \mathcal{P} divise $\eta_{0,i}$ et $\bar{\mathcal{P}}$ divise $\eta'_{0,i}$. Ainsi les idéaux $\eta_{j,i}$ et $\eta'_{j,i}$ sont primitifs.

D'où, en choisissant pour tout $1 \leq i \leq 2^{s-1}$, les $B_{0,i}$ telles que $B_{0,i} \equiv a \pmod{2p}$ (resp. d'une manière arbitraire) dans le cas où p divise N (resp. p ne divise pas ND), les idéaux $\eta_{j,i}$ et $\eta'_{j,i}$ sont primitifs, et donc ils peuvent s'écrire sous la forme suivante :

$$\eta_{j,i} = \left(Np^j, \frac{B_{j,i} + \sqrt{D}}{2}\right), \eta'_{j,i} = \left(Np^j, \frac{-B_{j,i} + \sqrt{D}}{2}\right)$$

où

$$\begin{cases} B_{j,i}^2 \equiv D \pmod{4Np^j}, & B_{j,i} \in \mathbb{Z}/2Np^j\mathbb{Z} \\ B_{j,i} \equiv B_{0,i} \pmod{2N}, & B_{j,i} \equiv a \pmod{2p} \\ 1 \leq j \leq h(D) - 1 \\ 1 \leq i \leq 2^{s-1} \end{cases}$$

Ce qui entraîne que les points de Heegner de $Y_0(N)$ de discriminant D sont :

$$\frac{\pm B_{j,i} + \sqrt{D}}{2Np^j} \pmod{\Gamma_0(N)}, \quad 1 \leq i \leq 2^{s-1}, \quad 0 \leq j \leq h(D) - 1.$$

Par conséquent, trouver les points de Heegner, revient à trouver les nombres $(B_{j,i})_{\substack{0 \leq j \leq h(D)-1 \\ 1 \leq i \leq 2^{s-1}}}$.

On vérifie facilement que pour trouver $B_{j+1,i}$, connaissant $B_{j,i}$, il suffit de trouver un entier $k = 0, \pm 1, \dots$, ou $\pm p$ tel que :

$$\begin{cases} (B_{j,i} + 2Np^j k)^2 \equiv D \pmod{4Np^j} \\ B_{j,i} + 2Np^j k \equiv a \pmod{2p}. \end{cases}$$

Dans ce cas, on peut prendre $B_{j+1,i} = B_{j,i} + 2Np^j k$ (un tel k existe).

2^{ème} cas : p divise D .

Dans ce cas $h(D) = 2, p \neq 2$ et $(p) = \mathcal{P}^2$ où $\mathcal{P} = (p, \frac{p+\sqrt{D}}{2})$. Pour $1 \leq i \leq 2^{s-1}$ fixé, posons :

$$\eta_{1,i} = \eta_{0,i}\mathcal{P} \text{ et } \eta'_{1,i} = \eta'_{0,i}\mathcal{P}.$$

Alors, on a : $cl_K = \{[\eta_{j,i}], j = 0, 1\} = \{[\eta'_{j,i}], j = 0, 1\}$.

Puisque $\eta_{0,i}, \eta'_{0,i}$ et \mathcal{P} sont primitifs et p ne divise pas N , alors $\eta_{1,i}$ et $\eta'_{1,i}$ sont primitifs, et donc ils peuvent s'écrire :

$$\begin{aligned} \eta_{1,i} &= (pN, \frac{B_{1,i} + \sqrt{D}}{2}), & \eta'_{1,i} &= (pN, \frac{-B_{1,i} + \sqrt{D}}{2}) \\ \begin{cases} B_{1,i}^2 \equiv D \pmod{4Np}, & B_{1,i} \in \mathbb{Z}/2Np\mathbb{Z} \\ B_{1,i} \equiv B_{0,i} \pmod{2N}, & B_{1,i} \equiv 0 \pmod{p} \end{cases} \end{aligned}$$

Ce qui entraîne que les points de Heegner sont :

$$\frac{\pm B_{j,i} + \sqrt{D}}{2Np^j} \pmod{\Gamma_0(N)}, \quad 1 \leq i \leq 2^{s-1}, \quad j = 0, 1.$$

On vérifie facilement qu'on peut prendre $B_{1,i} = B_{0,i} + 2Nk$ où $k = 0, \pm 1, \dots$, ou $\pm p$ tel que : $B_{0,i} + 2Nk \equiv 0 \pmod{p}$ et $(B_{0,i} + 2Nk)^2 \equiv D \pmod{4Np}$.

D'où on a la proposition suivante :

Proposition : Soit D un discriminant fondamental, impair tel que $h(D) > 1$ et $cl_K = \{\mathcal{P}^i, i = 1, \dots, h(D)\}$ où $\mathcal{P} = (p, \frac{a+\sqrt{D}}{2})$ avec p un nombre premier et a un entier tel que $a^2 \equiv D \pmod{4p}$.

Soit N un entier > 1 tel que $(N, D) = 1$ et D est un carré modulo $4N$. Soient $(\pm B_{0,i})_{1 \leq i \leq 2^{s-1}}$ les solutions dans $\mathbb{Z}/2N\mathbb{Z}$ de la congruence $B^2 \equiv D \pmod{4N}$, on prendra $B_{0,i} \equiv a \pmod{2p}$ si p divise N , (s étant le nombre de diviseurs premiers distincts de N).

Alors les points de Heegner sont :

$$\frac{\pm B_{j,i} + \sqrt{D}}{2Np^j} \pmod{\Gamma_0(N)}, \quad 1 \leq i \leq 2^{s-1}, \quad 0 \leq j \leq h(D) - 1$$

où $(B_{j,i})_{\substack{1 \leq i \leq 2^{s-1} \\ 1 \leq j \leq h(D)-1}}$ vérifie :

$$\begin{cases} B_{j,i}^2 \equiv D \pmod{4Np^j}, & B_{j,i} \in \mathbb{Z}/2Np^j\mathbb{Z} \\ B_{j,i} \equiv B_{0,i} \pmod{2N}, & B_{j,i} \equiv a \pmod{2p}. \end{cases}$$

Pour $1 \leq i \leq 2^{s-1}$ et $0 \leq j \leq h(D) - 2$, on peut prendre $B_{j+1,i} = B_{j,i} + 2Np^j k$, où $k = 0, \pm 1, \dots$, ou $\pm p$ tel que : $B_{j,i} + 2Np^j k \equiv a \pmod{2p}$ et $(B_{j,i} + 2Np^j k)^2 \equiv D \pmod{4Np^{j+1}}$.

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LOW REYNOLD S NUMBER VISCOUS JETS

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Presented by G.F.D. Duff, F.R.S.C.

1. Introduction. In showing that there exist unbounded domains $\Omega \subset \mathbb{R}^3$ such that the steady Navier-Stokes problem

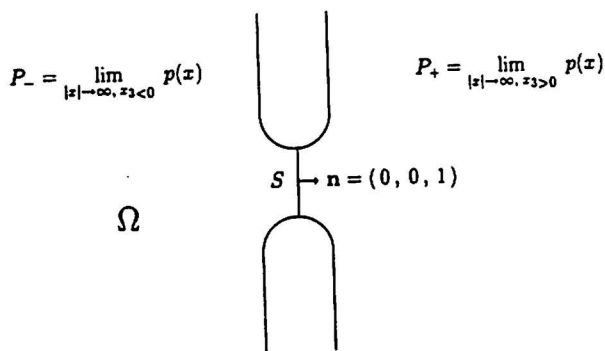
$$(1) \quad \Delta u - \nabla p = \lambda u \cdot \nabla u, \quad \nabla \cdot u = 0, \quad u|_{\partial\Omega} = 0, \quad u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

has nontrivial solutions, Heywood [5] drew attention to the problem of a steady jet through an aperture in a wall. If a fluid occupies the two half spaces on either side of an infinite wall, as well as a region removed from the wall by a hole that connects the two half spaces, then there exist nontrivial solutions with either a prescribed net flux through the hole

$$(2) \quad \int_S u \cdot n \, ds = F,$$

or alternatively with a prescribed pressure drop

$$(3) \quad P_- - P_+ = P$$



Various existence theorems for these problems, based on methods of functional analysis and on an a-priori bound for the solution's Dirichlet integral $\|\nabla u\|^2 = \int_{\Omega} |\nabla u|^2 dx$ were given in [5], [6], [8] and [10]. Although these solutions belong to $C^\infty(\bar{\Omega})$ if $\partial\Omega \in C^\infty$, little is known about their decay properties as $|x| \rightarrow \infty$. Because of this, the uniqueness of these solutions, which is expected in the case of small data, has not been proven.

Inspired by the works of Odqvist [9] and Finn [3] for the interior and exterior problems, we have constructed a solution, in the case of small data, by potential theoretic methods. The solution that is obtained has the spatial decay $u \sim r^{-2}$ at infinity, and is unique among all such solutions, and even among all solutions which decay like r^{-1} at infinity and have a finite Dirichlet integral.

2. Integral Equation. Our plan is to first obtain a solution $\{u_0, p_0\}$ of the Stokes equations

$$(4) \quad \Delta u - \nabla p = 0, \quad \nabla \cdot u = 0, \quad u \Big|_{\partial\Omega} = 0, \quad u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty,$$

satisfying either (2) or (3), and then to seek the solution u of the corresponding nonlinear problem as a solution of the integral equation

$$(5) \quad u(x) = \lambda \int_{\Omega} u(y) \cdot \nabla u(y) G(x, y) dy + u_0(x),$$

with associated pressure

$$(6) \quad p(x) = \lambda \int_{\Omega} u(y) \cdot \nabla u(y) g(x, y) dy + p_0(x),$$

where $\{G, g\}$ is a Green's tensor for the Stokes equations satisfying either the flux condition

$$(7) \quad \int_S G \cdot n ds = 0,$$

or the pressure condition

$$(8) \quad \lim_{|y| \rightarrow \infty, y_3 < 0} g(x, y) = \lim_{|y| \rightarrow \infty, y_3 > 0} g(x, y) = 0.$$

3. Existence of $\{u_0, p_0\}$ and $\{G, g\}$. The Stokes problems (4), (2) and (4), (3) were proven in [5] to be uniquely solvable, for solutions with a finite Dirichlet integral. Thus, we are assured of the existence of $\{u_0, p_0\}$. Later, in section 4, u_0 will be shown to decay like r^{-2} at infinity.

Let the fundamental tensor for the Stokes equations be denoted by $\{ E, e \}$. The Green's tensor $\{ G, g \}$, for either problem, is of the form $G = E - A, g = e - \alpha$, where $\{ A, \alpha \}$ satisfies

$$\Delta_y A + \nabla_y \alpha = 0, \quad \nabla \cdot A = 0, \quad A \Big|_{\partial\Omega} = E, \quad A(x, y) \rightarrow 0 \text{ as } |y| \rightarrow \infty.$$

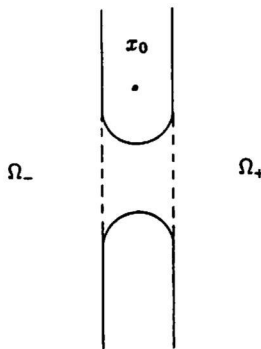
We construct $\{ A, \alpha \}$ as follows. Let $\{ E^\gamma, e^\gamma \}$ be the truncated fundamental tensor of Fujita [4]. It is solenoidal, equals E for $|x - y| < \gamma$, and vanishes for $|x - y| > 2\gamma$. For fixed $x \in \Omega$, and sufficiently small γ , the tensor $\hat{E}^\gamma \equiv E - E^\gamma$ equals E on $\partial\Omega$, and has a finite Dirichlet integral. If we let $A = \hat{E}^\gamma - W$ and $\alpha = e - e^\gamma - w$, where $\{ W, w \}$ is the unique solution of

$$(9) \quad \begin{aligned} \Delta_y W + \nabla_y w &= \Delta_y \hat{E}^\gamma + \nabla_y (e - e^\gamma), & \nabla \cdot W &= 0, \\ W \Big|_{\partial\Omega} &= 0, & W(x, y) &\rightarrow 0 \text{ as } |y| \rightarrow \infty, & \int_S W \cdot n \, ds &= 0, \end{aligned}$$

then we obtain $\{ G, g \}$ satisfying (7). To obtain $\{ G, g \}$ satisfying (8), we take $\{ W, w \}$ to be the unique solution of (9) with the flux condition $\int_S W \cdot n \, ds = 0$ replaced by the pressure condition $\lim_{|y| \rightarrow \infty} w(x, y) = 0$.

4. Decay properties of u_0 and G . A solution of the Stokes equations in a half space, $x_3 > 0$, taking boundary values b with compact support on the bounding plane, can be found explicitly in the form of a double layer potential

$$(10) \quad u(x) = \int_{\xi_3=0} b(\xi) K(x, \xi) \, d\xi_1 \, d\xi_2, \quad p(x) = \int_{\xi_3=0} b(\xi) k(x, \xi) \, d\xi_1 \, d\xi_2.$$



Such solutions decay like $u(x) \sim r^{-2}$ and $\nabla u(x) \sim r^{-3}$ as $|x| \rightarrow \infty$. In particular, there exists a solution of this form which equals u_0 on the boundary of Ω_+ . A uniqueness

theorem given in [5] allows us to identify u_0 with this explicit solution and thus to conclude for it the same decay properties in Ω_+ . Similar decay properties are obtained in Ω_- .

Green's tensor $G_+(x, y) = E(x, y) - \int_{\xi_3=0} E(y, \xi) K(x, \xi) d\xi_1 d\xi_2$ for the half space $x_3 > 0$ satisfies $G_+ \sim |x - y|^{-1}$ and $\nabla G_+ \sim |x - y|^{-2}$. Since $G - G_+$ is a solution of the Stokes equations in the half space Ω_+ , with compactly supported boundary values, we have $G \sim |x - y|^{-1}$ and $\nabla G \sim |x - y|^{-2}$ by an argument like that above.

Through a somewhat lengthy calculation, utilizing a decomposition of Ω into a number of different parts, we obtain the fundamental inequality

$$(11) \quad \int_{\Omega} \frac{1}{|y - x_0|^4} |\nabla G(x, y)| dy \leq \frac{M}{|x - x_0|^2}$$

where $x_0 \in \bar{\Omega}^c$, and $M > 1$ is a constant. For convenience, we assume $\text{dist}(x_0, \partial\Omega) \geq 1$.

5. Perturbation series. The solution of the integral equation (5) is sought in the form

$$(12) \quad u(x) = \sum_{n=0}^{\infty} u_n(x) \lambda^n$$

where u_0 is the solution of (4) satisfying either (2) or (3). Substituting into (5) and integrating by parts, we obtain the recurrence relation

$$(13) \quad u_{n+1}(x) = - \sum_{k=0}^{\infty} \int_{\Omega} u_k(y) \cdot \nabla_y G(x, y) u_{n-k}(y) dy.$$

To prove the convergence of (12), consider first the prescribed flux problem (1), (2). Choosing a_0 such that $|x - x_0|^2 |w(x)| \leq a_0$, where w is the solution of (4) satisfying $\int_S w \cdot n ds = 1$, and then setting recursively $a_{n+1} = |F| M \sum_{k=0}^n a_k a_{n-k}$, we can show using (11) and (13) that the series $|F| \sum_{n=0}^{\infty} a_n \lambda^n$ dominates $|x - x_0|^2 \sum_{n=0}^{\infty} u_n(x) \lambda^n$. Provided $4 M a_0 |F| |\lambda| < 1$, the series $\sum_{n=0}^{\infty} a_n \lambda^n$ converges and is bounded by $2a_0$. This implies that (12) is convergent and that $u \sim r^{-2}$. By a similar series argument, we show that $\sum_{n=0}^{\infty} \nabla u_n(x) \lambda^n$ converges to ∇u . We conclude that u is a solution of (5), and by a potential theoretic bootstrapping argument that $u \in C^\infty(\bar{\Omega})$, and finally that u satisfies (1), (2).

The only change in this argument for the pressure drop problem (1), (3) is that in choosing a_0 , w is required to be the solution of (4) with associated pressure drop equal to 1, and P is substituted for F in the subsequent calculations.

Theorem 1 *The prescribed flux problem (1), (2) has a solution $u \in C^\infty(\bar{\Omega})$ satisfying*

$$(14) \quad |u(x)| \leq \frac{2 a_0 |F|}{|x - x_0|^2}, \quad \text{provided} \quad |\lambda| |F| < \frac{1}{4 a_0 M}$$

The solvability of the pressure drop problem (1), (3) is described by replacing F by P in (14). For both problems, $|\nabla u| \sim r^{-2}$. The solution of either problem is unique within the class of solutions which decay like r^{-1} and have finite Dirichlet integrals.

The proof of uniqueness is similar to that given for nonsteady problems in [5] and [6]. Let $\{\tilde{u}, \tilde{p}\}$ be a second solution which decays like r^{-1} and has a finite Dirichlet integral. The difference $\{w, q\} = \{u - \tilde{u}, p - \tilde{p}\}$ of the two solutions satisfies the equation $\Delta w - \nabla q = \lambda (w \cdot \nabla u + u \cdot \nabla w - w \cdot \nabla w)$. Multiplying by w , and integrating by parts over Ω , using the decay properties which have been established, one obtains

$$\|\nabla w\|^2 - (\nabla q, w) = \lambda (w \cdot \nabla u, w) = -\lambda (w \cdot \nabla w, u).$$

Using either (2) or (3), the second term on the left can be shown to vanish. Thus

$$\|\nabla w\|^2 \leq 4 a_0 |F| |\lambda| \|\nabla w\|^2,$$

which implies $\|\nabla w\| = 0$.

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A categorical theory of cumulative hierarchies of sets

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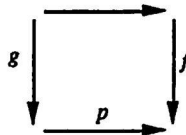
Presented by J. Lambek, F.R.S.C.

We begin by describing a theory of small sets and small maps, which opens the road for the development of a strict extension of topos theory in which the ordinals and the cumulative hierarchy of sets are representable. Categories with this representability property could be called elementary universes. The construction of sheaf categories inside such universes yields other universes. In an elementary universe, the cumulative hierarchy is a model of intuitionistic Zermelo-Fraenkel set theory (including a strong principle of transfinite induction).

Axioms for small maps. A class \mathcal{A} of maps in a pretopos \mathcal{T} is called a class of small maps if it satisfies the following axioms:

A1 Any isomorphism belongs to \mathcal{A} , and \mathcal{A} is closed under composition.

A2 In a pullback square



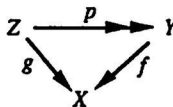
if f belongs to \mathcal{A} then so does g .

A3 In a pullback square as above, if g belongs to \mathcal{A} and p is epi, then f belongs to \mathcal{A} .

A4 The map $1 + 1 \rightarrow 1$ belongs to \mathcal{A} .

A5 If $Y \rightarrow X$ and $Y' \rightarrow X'$ belong to \mathcal{A} then so does their sum $Y + Y' \rightarrow X + X'$.

A6 In a diagram



if g belongs to \mathcal{A} and p is epi then f belongs to \mathcal{A} .

Remarks. (i) When the pretopos is a Grothendieck topos then axiom A5 should be stated for an arbitrary family $Y_i \rightarrow X_i$ ($i \in I$) of small maps.

(ii) A class of small maps defines a class of small objects (the small maps $X \rightarrow 1$). To specify a class of small maps, it of course suffices to describe a suitable class of small objects in every slice \mathcal{T}/B .

Examples. (a) The Kuratowski-finite objects (or maps) in a pretopos or in a Grothendieck topos is the smallest class of small maps. In any elementary topos (or in any arithmetic universe) the class of (internally) Kuratowski-finite objects is a class of small objects.

(b) If \mathcal{E} is the topos of sheaves over a given pretopos, then the class of quasi-coherent objects (or maps) in \mathcal{E} is a class of small objects.

(c) In any elementary topos \mathcal{E} , let us say that a map $f: B \rightarrow A$ is a choice map if $\pi_f: \mathcal{E}/B \rightarrow \mathcal{E}/A$ preserves epimorphisms. The class of choice maps is a class of small maps.

Representability. Let \mathcal{A} be a class of small maps in a pretopos \mathcal{T} . For any object $X \in \mathcal{T}$ we construct a functor

$$P^s(X): \mathcal{T}^{\text{op}} \rightarrow \mathbf{sets},$$

which assigns to an object I to \mathcal{T} the set of subobjects $R \rightarrow X \times I$ such that $R \rightarrow I$ belongs to \mathcal{A} .

By axiom A3, $P^s(X)$ is a contravariant functor of I ; for any arrow $f: J \rightarrow I$ in \mathcal{T} induces $f^\#: P^s(X)(I) \rightarrow P^s(X)(J)$ by pullback:

$$\begin{array}{ccccc} f^\#(R) & \xrightarrow{\quad} & R & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow & & \\ J & \xrightarrow{\quad f \quad} & I & & \end{array}$$

We say that the class \mathcal{A} is representable if for each object X the functor $P^s(X)$ is representable by an object $P_s(X)$ of \mathcal{T} .

Remarks. (1) A map $f: X \rightarrow Y$ defines a map (direct image)

$$f_! : P_s(X) \rightarrow P_s(Y),$$

so that $P_s(X)$ becomes a covariant functor of X . When f is a small map, we also have an inverse image map

$$f^! : P_s(Y) \rightarrow P_s(X),$$

so that $P_s(X)$ is a contravariant functor on small maps. These direct and inverse images together satisfy a Beck-Chevalley condition.

(2) $P_s(X)$ has the structure of a (sup) semi-lattice. This structure is preserved by direct image maps, and by inverse image maps along small maps.

(3) If a class \mathcal{A} of small maps in \mathcal{T} is representable, then so is the induced class in any slice \mathcal{T}/B .

(4) When \mathcal{A} is representable we have for every X a universal relation $R \rightarrow X \times P_s(X)$ such that $R \rightarrow P_s(X)$ is small. This relation R is the membership relation $[\varepsilon]_X$ between "elements" of X and subobjects of X . There is also a singleton map $\{\cdot\} : X \rightarrow P_s(X)$, corresponding to the diagonal $X \rightarrow X \times X$.

(5) Every map $f : B \rightarrow A$ in the class \mathcal{A} is a pullback of the map $[\varepsilon]_A \rightarrow P_s(A)$. This shows that the class \mathcal{A} is determined by P_s and the membership relations.

Small maps and sup-semi-lattices. For a poset L in a pretopos \mathcal{T} , we say that suprema in L exist along a map $f : B \rightarrow A$ when the map $L^{B/A} \rightarrow L$ of sheaves on \mathcal{T}/A has a left adjoint. (Here B/A is f as an object of \mathcal{T}/A , and we identify L with its pullback in \mathcal{T}/A .) When L is a (sup) semi-lattice then the collection of maps along which suprema exist is a class of small maps.

Furthermore, given a class \mathcal{A} of small maps, we say that a semi-lattice L is \mathcal{A} -complete when the suprema in L exist along any map belonging to \mathcal{A} .

Proposition. $P_s(X)$ is the free \mathcal{A} -complete semi-lattice on X . Correspondingly, the functor P_s can be given the structure of a monad $P_s P_s(X) \rightarrow P_s(X)$.

We remark that the class \mathcal{A} can be obtained from the monad P_s alone: a map $f : B \rightarrow A$ in \mathcal{T} belongs to \mathcal{A} iff suprema along it exist for every (sup-)semi-lattice of the form $P_s(X)$.

Let \mathcal{A} be a representable class of small maps in a pretopos \mathcal{T} . For an object A of \mathcal{T} , the \mathcal{A} -hierarchy on A is an object $V(A)$ (or $V_{\mathcal{A}}(A)$ if necessary) equipped with the structure of an \mathcal{A} -complete semi-lattice together with a "successor"-operation $s : V(A) \rightarrow V(A)$, which is moreover free on A as an object equipped with these operations. (Note: here the freeness condition should be formulated with arbitrary parameter objects.)

Definition. A transfinite arithmetic universe is a pretopos \mathcal{T} equipped with a representable class \mathcal{A} of small maps, for which the hierarchy $V(A)$ exists for every object A of \mathcal{T} .

Remarks. (1) An ε -relation on $V(A)$ can be defined by: $x \varepsilon y$ iff $x \leq s(y)$.

(2) When $A = \emptyset$, the object $V(\emptyset) = V$ is called the \mathcal{A} -hierarchy. The map $A \rightarrow V(A)$ is injective, and its image is the collection of "atoms" of $V(A)$. In the sequel we shall concentrate on V . (With small modifications, the results obtained apply also to $V(A)$.)

Theorem. *The map $r : P_s(V) \rightarrow V$ defined by the formula*

$$r(X) = \bigvee_{sx} s(x)$$

is an isomorphism of \mathcal{A} -complete semi-lattices.

Remarks. (1) The proof of this theorem uses a successor operation s' on $P_s(V)$ defined by the singleton map $\{\cdot\} : V \rightarrow P_s(V)$, as

$$s'(X) = \{ \bigvee_{sx} s(x) \}.$$

Since V is free there exists a unique map $i : V \rightarrow P_s(V)$ preserving \mathcal{A} -suprema and successors. We show that r and i are mutually inverse.

(2) It follows that V has the following elementary properties:

- (a) for all $x, y \in V : sx \leq sy$ implies $x = y$;
- (b) the map $p_2 : \{(x, y) \mid x \varepsilon y\} \rightarrow V$ is small, and $y = \bigvee_{sx} s(x)$.

(3) Under the above isomorphism between V and $P_s(V)$, suprema correspond to unions, and the successor s corresponds to the singleton.

(4) Consider the successor operation s' on V defined by the formula $s'(x) = xvs(x)$. There is a unique sup-preserving map $\text{rank} : V \rightarrow V$ such that $\text{rank}(s(x)) = s'(\text{rank}(x))$; this map is idempotent. Its image is the object O of ordinals. It is the free sup-lattice with a successor operation s satisfying the inequality $x \leq sx$. In addition we have that $x \leq y$ implies $sx \leq sy$. By freeness, one can immediately obtain familiar operations on ordinals and derive their basic properties.

Intuitionistic set theory. Let \mathfrak{U} be a transfinite universe, with universal quantification along any map, and a small natural number object (again free with respect to arbitrary parameters).

Consider the following additional axioms on the class \mathcal{A} of small maps in \mathfrak{U} :

A7 Monomorphisms are small. It follows that $P_s(X) = P(X)$ if X is small.

A8 For any object X in \mathfrak{U} and any small object Y in \mathfrak{U}/X , the object $P_s(Y/X)$ is again small in \mathfrak{U}/X . It follows that small maps are exponentiable.

A10 (Replacement). The functor $P_s(\cdot) : \mathfrak{U} \rightarrow \mathfrak{U}$ preserves epimorphisms.

Theorem. *If \mathfrak{U} satisfies the addition axioms A8-10 above, then the hierarchy V is a model of intuitionistic Zermelo-Fraenkel set theory.*

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