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CONTENTS

Algebraic Extensions of Fields of Generalized Power Series

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Abstract: This paper deals with algebraic extensions of fields of generalized power series, in particular, with the determination of the algebraic closure.

1. Let R be a commutative ring with unit element. Let $(S, \leq)$ be an ordered commutative additive monoid; assume that the order is strict: if $s, s', t \in S$ and $s < s'$ then $s + t < s' + t$.

Let $A = [[R^{S\leq}]]$ be the ring of generalized power series with coefficients in R and exponents in S; its elements are the mappings $f : S \to R$, having support $\mathrm{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ which is artinian (it has no infinite descending chain) and narrow (it has no infinite subset of pairwise order-incomparable elements); the operation of addition is pointwise, and the operation of multiplication is well-defined by $(f * g)(s) = \sum_{t+u=s} f(t)g(u)$ for every $s \in S$. For details, see [Ri 1], [Ri 2].

An order $\leq$ on S is *subtotal* if for any $s, t \in S$ there exists an integer $k \geq 1$, such that $ks \leq kt$ or $kt \leq ks$.

If $\leq$ is a subtotal compatible order on S, let $\leq'$ be defined by: $s \leq' t$ if there exists $k \geq 1$ such that $ks \leq kt$. Then $\leq'$ is a compatible total order on S, said to be associated to $\leq$.

It was shown in [El-Ri] that $A = [[R^{S,\leq}]]$ is a field if and only if R is a field, S is a subtotally ordered group. In this situation, if $f \in A$, $f \neq 0$, let $\pi'(f)$ be the

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first element of supp(f) in the total order $\leq'$. Then π' defines a valuation on the field A.

Rayner [Ra 1] showed that the valued field (A, π') is henselian.

2. We announce the following results; the proofs will appear elsewhere. Throughout, $A = [[R^{S,\leq}]]$ is assumed to be a field.

(2.1). If $m \geq 2$ and $A = A^m$, then $R = R^m$ and $S = m \cdot S$. Conversely, if the characteristic of R does not divide m, if $R = R^m$ and $S = m \cdot S$, then $A = A^m$.

(2.2). Let char(R) $= p$. Then A is a perfect field if and only if R is perfect and S is p-divisible.

(2.3). Assume that A is a perfect field. Then A has no proper solvable extension if and only if R has no proper solvable extension and S is divisible.

(2.4). Let $F \mid R$ be a field extension of finite degree, let $\{\theta_1, \ldots, \theta_n\}$ be a basis of the R-vector space F. Then $\{\theta_1 e, \ldots, \theta_n e\}$ is a basis of the A-vector space $B = [[F^{S,\leq}]]$.

(2.5). If θ is algebraic over R, then θe is algebraic over A and the minimal polynomial of θ over R is also the minimal polynomial over θe over A; moreover, $A[\theta e] \cong [[R[\theta]^{S,\leq}]]$.

However if $R^* \mid R$ is an algebraic extension of infinite degree, then $A^* = [[R^{*S,\leq}]]$ is not an algebraic extension of A.

(2.6). Let $(S', \leq)$ be a subtotally ordered group; so it is necessarily torsion-free. Let S be a subgroup of S', endowed with the restriciton of the order $\leq$, and assume that S'/S is finite.

Let R be a field and $R' \mid R$ an algebraic extension of finite degree. Then $[[R'^{S',\leq}]]$ is an algebraic extension of finite degree of $A = [[R^{S,\leq}]]$.

3. In this section we announce results about algebraically closed fields of generalized power series; the special case where $(S, \leq)$ is totally ordered has been known (see for example, [Be], [Ra 1], [Ra 2]).

(3.1). If A is algebraically closed then R is algebraically closed and S is a divisible group.

(3.2). Let R be an algebraically closed field and let $(S, \leq)$ be a subtotally ordered divisible group. If $(S, \leq)$ is narrow or char$(R) = 0$, then A is algebraically closed.

(3.3). Let char$(R) = 0$ and let $R^{\bullet}$ be an algebraic closure of R. If $(S, \leq)$ is a subtotally ordered divisible group, then $A^{\bullet} = \bigcup_{F} [[F^{S,\leq}]]$ (where $R \subset F \subset R^{\bullet}, F \mid R$ extension of finite degree) is an algebraic closure of A.

(3.4). Let char$(R) = 0$ and let $R^{\bullet}$ be an algebraic closure of R. Let S be a torsion-free group and let $\leq$ be a compatible subtotal order on $\mathbb{Q}S$. Then:

i) $A^{\bullet} = \bigcup_{F} \bigcup_{n=1}^{\infty} [[F^{\frac{1}{n}S,\leq}]]$ (where $\leq$ is the restriction of $\leq$ to $\frac{1}{n}S$, $R \subset F \subset R^{\bullet}$, $F \mid R$ a finite extension) is an algebraically closed field.

ii) If, moreover, $S/n \cdot S$ is finite for every $n \geq 1$, then A^* is an algebraic closure of A.

(3.5). Let $\mathrm{char}(R) = p$, and assume that R is a perfect field; let R^* be its algebraic closure. Let $(S, \leq)$ be a totally ordered p-divisible group. Then:

i) the field $A^* = \bigcup_{F}\bigcup_{p \nmid n} [[{}^{\frac{1}{n}S,\leq}]]$ (for $R \subset F \subset R^*$, $F \mid R$ finite extension) is an algebraically closed field.

ii) If, moreover, $S/n \cdot S$ is a finite group for every $n \geq 1$, then A^* is an algebraic closure of A.

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Géométrie Algébrique - Algebraic Geometry

UNE INVOLUTION SUR LA VARIÉTÉ DES MODULES DES FIBRÉS INSTANTONS DE CLASSES DE CHERN $c_1 = 0$, $c_2 = 4$

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Presented by P. Ribenboim, F.R.S.C.

Résumé :

On décrit une involution sur un ouvert de Zariski de la variété des fibrés instantons de classes de Chern $c_1 = 0, c_2 = 4$.

AN INVOLUTION ON THE VARIETY OF INSTANTON BUNDLES WITH CHERN CLASSES $c_1 = 0$, $c_2 = 4$

Abstract :

We describe an involution on the variety of instanton bundles with Chern classes $c_1 = 0$, $c_2 = 4$.

Un fibré instanton E est un fibré vectoriel algébrique de rang deux sur $\mathbb{P}^3(\mathbb{C})$ vérifiant : $c_1(E) = 0$, $h^1(E(-2)) = 0$ et $h^0(E) = 0$. L'espace des modules des fibrés instantons de classes de chern $c_1 = 0, c_2 = 4$ est une variété quasi-projective irréductible ([2]) et lisse ([5]) de dimension 29. Pour un tel fibré, on a $h^0(E(i)) = 0$ pour $i \leq 0$ et donc $h^3(E(i)) = 0$ pour $i \geq -4$ par dualité de Serre. Le théorème de Riemann-Roch s'écrit alors :

$$h^2(E(i)) - h^1(E(i)) = 1/3(i+1)(1+2)(i+3) - c_2(i+2) \quad \text{pour} \quad -4 \leq i \leq 0.$$

On a donc $h^2(E(-2)) = h^1(E(-2)) = 0$. On peut obtenir un fibré de ce type en partant d'une courbe elliptique lisse connexe de degré huit assez générale et en effectuant la construction de Serre [4] Ex 4.3.3.

On supposera en particulier que la courbe n'est pas tracée sur une surface cubique et qu'elle n'a pas de quintisécante. Soit J_X le faisceau d'idéaux définissant la courbe dans $\mathbb{P}^3$. On a une suite exacte

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow E(2) \longrightarrow J_X(4) \longrightarrow 0.$$

Le fibré ainsi construit est stable et on a $h^1(E(-2)) = 0$, $h^1(E(-1)) = 4$, $h^1(E) = 6$, $h^1(E(1)) = 4$, $h^1(E(2)) = h^1(J_X(4))$. Ce dernier entier est nul car X ne contient pas six points alignés ([1]). Les fibrés ainsi obtenus constituent un ouvert de Zariski U de dimension 29 de l'espace des modules ([4], 3.4.3, 3.4.4). Cet ouvert est strictement contenu dans l'espace des modules (Prendre X sur une surface cubique).

PROPOSITION.- *A tout fibré* E *de* U, *on associe de façon naturelle un fibré* E' *de* U. *On a une suite exacte*

$$0 \longrightarrow E'(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^4 \longrightarrow E(2) \longrightarrow 0 .$$

Une section de E'(2) *est liée à une section de* E(2) *par deux surfaces quartiques. L'application* $E \longrightarrow E'$ *est une involution sans point fixe. Les fibrés* E *et* E' *ont les mêmes plans sauteurs.*

Démonstration :

Soit X une courbe elliptique lisse connexe de degré huit de $\mathbb{P}^3$, section d'un fibré E(2). On suppose que la courbe n'est pas tracée sur une surface cubique et ne contient pas cinq points alignés. On a $h^0(J_X(4)) = 3$ [1]. On va montrer que $J_X(4)$ est engendré par ses sections globales. Il s'agit de montrer que l'intersection de trois surfaces quartiques linéairement indépendantes contenant X est réduite à X. Pour cela, on utilise le résultat suivant : ([3] p. 155). Si une composante connexe de l'intersection de trois surfaces de $\mathbb{P}^3$ est une courbe X, alors l'équivalence de X dans l'intersection est $(n_1 + n_2 + n_3 - 4) \deg X + 2 - 2g$. (Les n_i sont les degrés des surfaces et g le genre de X).

On a donc le diagramme commutatif suivant :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^3}) \otimes \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & H^0(E(2)) \otimes \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & H^0(J_X(4)) \otimes \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & E(2) & \longrightarrow & (J_X(4)) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

On vérifie alors que la flèche verticale du milieu est surjective. Le fibré E(2) est engendré par ses sections globales. On note E' (-2) le noyau de la flèche verticale centrale. On vérifie immédiatement que E' est un instanton de classe de Chern $c_2 = 4$. En prenant les noyaux des flèches verticales du diagramme précédent, on obtient le diagramme suivant :

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E'(-2) & \longrightarrow & E'(-2) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}^4_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}^3_{\mathbb{P}^3} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & E(2) & \longrightarrow & J_X(4) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Supposons maintenant que X est liée à une courbe Y par deux surfaces quartiques. En utilisant le "mapping cone" ([7]) on obtient une résolution de Y :

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow E'(2) \longrightarrow J_Y(4) \longrightarrow 0$$

Si les quartiques sont générales alors Y est lisse, connexe et on a $h^0(J_Y(4)) = 3$, $h^0(J_Y(3)) = 0$ et $J_Y(4)$ engendré par ses sections car $E'(2)$ l'est. Donc Y n'a pas de 5-sécante et $E' \in U$.

Montrons que l'involution est sans point fixe. Supposons par l'absurde qu'il existe un fibré E tel que $E = E'$. On a alors les deux suites exactes :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & E(2) & \longrightarrow & J_X(4) & \longrightarrow & 0 \quad \text{et} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E(-2) & \longrightarrow & \mathcal{O}^3_{\mathbb{P}^3} & \longrightarrow & J_X(4) & \longrightarrow & 0
\end{array}$$

Si Q_1 et Q_2 sont deux surfaces quartiques assez générales contenant X, on peut construire le diagramme commutatif suivant :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-4) & \longrightarrow & E(-2) & \longrightarrow & \mathcal{J}_X & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^2 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^3 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_Z(4) & \longrightarrow & \mathcal{J}_X(4) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

La courbe X étant elliptique on a $\omega_X = \mathcal{O}_X$ où ω_X est le faiseau canonique. La dernière ligne signifie alors que la courbe Z est obtenue en considérant une structure de multiplicité deux sur X. En d'autres termes, les deux surfaces quartiques sont tangentes le long de X. Ceci est absurde [7] Proposition 4.1.

Considérons maintenant un plan H de $\mathbb{P}^3$. Ce plan est un plan sauteur pour le fibré si $H^0(H,E_H) \neq 0$. En considérant la suite exacte $0 \longrightarrow \mathcal{O}_H \longrightarrow E_H(2) \longrightarrow \mathcal{J}_{X\cap H}(2) \longrightarrow 0$, on voit que H est un plan sauteur si et seulement si les huit points de $X \cap H$ sont situés sur une conique de H. Considérons la correspondance d'incidence $F \subset \mathbb{P}^3 \times \mathbb{P}^3_*$. On note p et q les projections sur les facteurs. La résolution de F est

$$0 \longrightarrow \mathcal{O}(-1,-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}_3^*} \longrightarrow \mathcal{O}_F \longrightarrow 0.$$

Après tensorisation par $p^* E(-3)$ et image directe par q, on obtient :

$$R^1q_*(p^*E(-3)) \longrightarrow H^2(E(-4)) \otimes \mathcal{O}(-1) \longrightarrow H^2(E(-3)) \otimes \mathcal{O} \longrightarrow R^2q_*(p^*E(-3)).$$

Par dualité de Serre, on voit que les plans sauteurs constituent le support de $R^2q_*(p^*E(-3))$. Les dimensions des espaces vectoriels de cohomologie sont $h^2(E(-4)) = 6$, $h^2(E(-3)) = 4$. On a donc en général vingt plans sauteurs (Formule de Porteous). Après tensorisation par $p^*E(1)$ de la résolution de F, on obtient :

$$0 \longrightarrow q_* p^*E(1) \longrightarrow H^1(E) \otimes \mathcal{O}(-1) \xrightarrow{u} H^1(E(1)) \otimes \mathcal{O} \longrightarrow R^1 q_* p^*E(1) \longrightarrow 0.$$

Le faisceau $q_*p^*E(1)$ est stable réflexif de rang deux. Le support de $R^1q_*E(1)$ est formé des plans H tels que $H^1(H, E_H(1)) \neq 0$. La suite exacte $0 \longrightarrow \mathcal{O}_H \longrightarrow E_H(2) \longrightarrow J_{X\cap H}(4) \longrightarrow 0$, montre que $H^1(E_H(1)) = H^1(J_{X\cap H}(3))$. Donc H est un point du support de $R^1q_*p^*E(1)$ si et seulement si les huit points de $X \cap H$ n'imposent pas des conditions indépendantes aux cubiques de H. D'après ([1] Proposition) ceci signifie que les huit points sont sur une conique ou bien cinq d'entre eux sont alignés. Cette dernière possibilité est exclue. Le support de $R^1q_*p^*E(1)$ est donc formé par les points de $\mathbb{P}_3^*$ correspondant aux plans sauteurs. Le complexe d' Eagon-Northcott construit sur u s'écrit :

$$\begin{array}{ccccccccccc} 0 \longrightarrow \mathcal{O}^4(-6) \longrightarrow & \mathcal{O}^6(-5) & & \longrightarrow & & \mathcal{O}^6(-1) & \longrightarrow \mathcal{O}^4 \longrightarrow R^1 q_* p^* E(1) \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & q_* p^* E(1) & & & \\ & & \nearrow & & \searrow & & \\ & 0 & & & & 0 & \end{array}$$

Le polynôme de Chern de $q_*p^*E(1)$ est donc $c_t(q_*p^*E(1)) = 1 - 6t + 5t^2 + 20t^3$. Les vingt points correspondant aux plans sauteurs sont les points où $q_*p^*E(1)$ n'est pas localement libre.

Montrons maintenant que E et E' ont les mêmes plans sauteurs. Soit H un plan sauteur de E. Les points de $X \cap H$ sont situés sur une conique. On peut trouver une quartique de H contenant les huit points et n'admettant pas la conique comme composante.

On a donc une résolution :

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-6) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow J_{X\cap H} \longrightarrow 0.$$

Considérons deux surfaces quartiques générales contenant X. La liée Y de X est une section de $E'(2)$. Les groupes de points $X \cap H$ et $Y \cap H$ sont alors liés par les traces sur H des deux surfaces quartiques. Le "mapping cone" montre alors que $Y \cap H$ est contenu dans une conique. Par conséquent H est un plan sauteur pour E'.

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ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

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ABSTRACT.

A subclass $A_n(\alpha,\beta)$ of certain analytic functions in the unit disk is introduced. The object of the present paper is to derive some properties of functions belonging to the class $A_n(\alpha,\beta)$.

I. INTRODUCTION.

Let A_n be the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N} = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z: |z| < 1\}$.

For functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = z + \sum_{k=n+1}^{\infty} a_{k,j} z^k , \tag{1.2}$$

we denote by $f_1 * f_2(z)$ the convolution (or the Hadamard product) of $f_1(z)$ and $f_2(z)$, that is,

$$f_1 * f_2(z) = z + \sum_{k=n+1}^{\infty} a_{k,1} a_{k,2} z^k . \tag{1.3}$$

With the aid of the convolutions, for a function $f(z)$ belonging to A_n, we define the following operator D^α by

$$D^\alpha f(z) = \frac{z}{(1-z)^{1+\alpha}} * f(z) \qquad (\alpha > -1). \tag{1.4}$$

The operator D^α was introduced by Ruscheweyh [4], so we call the operator D^α the Ruscheweyh derivative of order α.

It is well-known that

$$z(D^\alpha f(z))' = (\alpha + 1)D^{\alpha+1} f(z) - \alpha D^\alpha f(z) \qquad \text{(cf. [1])}. \tag{1.5}$$

Using the Ruscheweyh derivatives, we define

DEFINITION. A function $f(z)$ in A_n is said to be a member of the class $A_n(\alpha,\beta)$ if it satisfies

$$\left| \frac{D^{\alpha+1}f(z)}{z} - 1 \right| < 1 - \beta \tag{1.6}$$

for some α ($\alpha \geq 0$), β ($0 \leq \beta < 1$), and for all $z \in U$.

2. SOME PROPERTIES.

In order to derive our result, we need the following lemma due to Miller and Mocanu [3] (also, due to Jack [2]).

LEMMA. *Let the function*

$$w(z) = b_n z^n + b_{n+1} z^{n+1} + \dots \qquad (n \in N)$$

be regular in the unit disk U *with* $w(z) \not\equiv 0$ $(z \in U)$. *If* $z_0 = r_0 e^{i\theta_0}$ $(r_0 < 1)$ *and*

$$|w(z_0)| = \max_{|z| \leq r_0} |w(z)|,$$

then

$$z_0 w'(z_0) = m w(z_0),$$

where m *is real and* $m \geq n \geq 1$.

With the help of the above lemma, we prove

THEOREM I. *If* $f(z) \in A_n(\alpha,\beta)$, *then*

$$\frac{D^{\alpha}f(z)}{z} \prec 1 + \frac{(1+\alpha)(1-\beta)z}{n+\alpha+1} \qquad (z \in U), \tag{2.1}$$

where the symbol "$\prec$" *means subordination*.

PROOF. The result is clear for $f(z) \equiv z$ $(z \in U)$. Therefore, we prove for $f(z) \not\equiv z$ $(z \in U)$. Defining the function $w(z)$ by

$$\frac{D^{\alpha}f(z)}{z} = 1 + \frac{(1+\alpha)(1-\beta)w(z)}{n+\alpha+1}, \tag{2.2}$$

we see that $w(z) \not\equiv 0$ ($z \in \mathcal{U}$) and $w(z) = b_n z^n + b_{n+1} z^{n+1} + \dots$ is regular in the unit disk $\mathcal{U}$. By using (1.5), we have

$$\frac{D^{\alpha+1} f(z)}{z} = 1 + \frac{(1+\alpha)(1-\beta)}{n+\alpha+1}\left\{ w(z) + \frac{zw'(z)}{\alpha+1} \right\}. \tag{2.3}$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then the Lemma gives that

$$z_0 w'(z_0) = m w(z_0),$$

where m is real and $m \geq n \geq 1$. It follows that

$$\left| \frac{D^{\alpha+1} f(z_0)}{z_0} - 1 \right| = \frac{(1+\alpha)(1-\beta)}{n+\alpha+1} \left| w(z_0) + \frac{z_0 w'(z_0)}{\alpha+1} \right| \tag{2.4}$$

$$= \frac{(1-\beta)(m+\alpha+1)}{n+\alpha+1}$$

$$\geq 1 - \beta.$$

This contradicts the condition $f(z) \in A_n(\alpha,\beta)$. Thus, we have that $|w(z)| < 1$ for all $z \in \mathcal{U}$, which completes the assertion of Theorem 1.

Taking $\alpha = 0$ in Theorem 1, we have

COROLLARY 1. *If* $f(z) \in A_n$ *satisfies*

$$|f'(z) - 1| < 1 - \beta \quad (z \in \mathcal{U}), \tag{2.5}$$

then

$$\frac{f(z)}{z} \prec 1 + \frac{(1-\beta)z}{n+1} \quad (z \in \mathcal{U}). \tag{2.6}$$

Next, we prove

THEOREM 2. *If* $f(z) \in A_n(\alpha,\beta)$, *then*

$$\operatorname{Re}\left\{ e^{i\gamma} \frac{D^{\alpha} f(z)}{z} \right\} > 0 \quad (z \in \mathcal{U}), \tag{2.7}$$

where

$$(2.8)\qquad |\gamma| \leq \frac{\pi}{2} - \sin^{-1}\left(\frac{(1+\alpha)(1-\beta)}{n+\alpha+1}\right).$$

The bound of $|\gamma|$ is best possible and is achieved for the function $f(z)$ defined by

$$(2.9)\qquad f(z) = z + \frac{(1+\alpha)(1-\beta)}{(n+\alpha+1)C(\alpha,n+1)} z^{n+1},$$

where

$$(2.10)\qquad C(\alpha,n+1) = \frac{\prod_{j=2}^{n+1}(j+\alpha-1)}{n!}.$$

PROOF. In view of Theorem 1, we see that

$$(2.11)\qquad \left|\frac{D^{\alpha}f(z)}{z} - 1\right| < \frac{(1+\alpha)(1-\beta)}{n+\alpha+1} \qquad (z \in U).$$

It follows from (2.11) that

$$\mathrm{Re}\left\{e^{i\gamma}\frac{D^{\alpha}f(z)}{z}\right\} > 0 \qquad (z \in U)$$

for

$$|\gamma| \leq \frac{\pi}{2} - \sin^{-1}\left(\frac{(1+\alpha)(1-\beta)}{n+\alpha+1}\right).$$

Finally, the bound of $|\gamma|$ is best possible for the function $f(z) \in A_n(\alpha,\beta)$ defined by

$$(2.12)\qquad \frac{D^{\alpha}f(z)}{z} = 1 + \frac{(1+\alpha)(1-\beta)}{n+\alpha+1} z^n.$$

It is easy to see that (2.12) leads to

$$(2.13)\qquad \frac{1}{(1-z)^{1+\alpha}} * \frac{f(z)}{z} = 1 + \frac{(1+\alpha)(1-\beta)}{n+\alpha+1} z^n.$$

Noting that

$$(2.14)\qquad \frac{1}{(1-z)^{1+\alpha}} = 1 + \sum_{n=2}^{\infty} C(\alpha,n+1)z^n,$$

we see that the bound of $|\gamma|$ is best possible for the function f(z) defined by (2.9).

COROLLARY 2. *If* $f(z) \in A_n$ *satisfies the condition* (2.5), *then*

$$\text{(2.15)} \qquad \text{Re}\left\{e^{i\gamma}\frac{f(z)}{z}\right\} > 0 \qquad (z \in U),$$

where

$$\text{(2.16)} \qquad |\gamma| \leq \frac{\pi}{2} - \sin^{-1}\left(\frac{1-\beta}{n+1}\right).$$

The bound of $|\gamma|$ *is best possible for the function* f(z) *defined by*

$$\text{(2.17)} \qquad f(z) = z + \frac{1-\beta}{n+1}z^{n+1}.$$

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NOUVELLE MAJORATION DE LA NORME DES FACTEURS D'UN POLYNOME

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Presented by P.G. Rooney, F.R.S.C.

Résumé. *Le résultat principal de cet article est la majoration de la norme des facteurs d'un polynôme suivante :*

$$|P|\, M(Q) \leq 2^{-q} \frac{(p+q)^{p+q}}{p^p\, q^q} |PQ| .$$

1.L'objet de ce paragraphe est de démontrer un lemme qui majore la norme d'un polynôme en fonction du maximum de sa valeur sur un cercle de rayon ρ (on définit dans cette partie la norme du polynôme P par $|P| = \max\{\, |P(z)|\,,\ |z|=1 \,\}$).

Ce lemme va permettre d'améliorer un résultat de M.Mignotte qui majore la norme d'un facteur d'un polynôme en fonction de la norme de ce polynôme (cf. [4]).

Lemme. Soit $P \in \mathbb{C}[X]$ de degré p.
On suppose que toutes les racines de P, α_i ($1 \leq i \leq p$), sont dans le disque de rayon r ($r \geq 0$).
Alors

$$\max\{\,|P(z)|,|z|=r\,\} \leq \Big(\prod_{i=1}^{p} \frac{r+|\alpha_i|}{\rho+|\alpha_i|}\Big) \max\{\,|P(z)|,|z|=\rho\,\} ,$$

pour $\rho \geq r$.

preuve. Soit $z_0 \in \mathbb{C}$, $|z_0| = r$, tel que $\max\{\,|P(z)|\,,\ |z|=r\,\} = |P(z_0)|$.
On a $\max\{\,|P(z)|,|z|=\rho\,\} \geq |P((\rho/r)\, z_0)|$.
Or, en écrivant la valeur de P en $(\rho/r)z_0$, on obtient ,

$$|P((\rho/r)\, z_0)| = \Big(\prod_{i=1}^{p} \frac{|(\rho/r)z_0 - \alpha_i|}{|z_0 - \alpha_i|}\Big) \max\{\,|P(z)|,|z|=r\,\} .$$

Nous allons maintenant montrer que

$$\frac{|(\rho/r)\, z_0 - \gamma e^{i\theta}|}{|z_0 - \gamma e^{i\theta}|} \geq \frac{\rho + \gamma}{r + \gamma} \quad ,$$

pour tout θ et tout γ , avec $0 \leq \gamma \leq r$.

Il suffit en fait de vérifier que

$$\frac{|\rho + \gamma e^{i\theta}|}{|r + \gamma e^{i\theta}|} \geq \frac{\rho + \gamma}{r + \gamma} \quad .$$

Or ,

$$\frac{|\rho + \gamma e^{i\theta}|}{|r + \gamma e^{i\theta}|} = \sqrt{\frac{\rho^2 + 2\rho\gamma\cos\theta + \gamma^2}{r^2 + 2r\gamma\cos\theta + \gamma^2}} \quad ,$$

et la fonction

$$f(x) = \frac{\rho^2 + 2\rho\gamma x + \gamma^2}{r^2 + 2r\gamma x + \gamma^2}$$

est décroissante pour x compris entre -1 et 1 .

La fonction f atteint donc son minimum sur l'intervalle $[-1, 1]$ en 1, où elle vaut $[(\rho+\gamma)/(r+\gamma)]^2$, et le lemme est démontré .

<u>Corollaire 1.</u> Soit $P \in \mathbb{C}[X]$ de degré p .
On suppose que toutes les racines de P sont dans le disque unité .
Alors

$$|P| \leq \left(\frac{2}{\rho + 1}\right)^p \max\{\,|P(z)|, |z| = \rho\,\} \quad ,$$

pour $\rho \geq 1$.

<u>preuve.</u> Il suffit d'utiliser le lemme avec $r = 1$ et de vérifier que

$$\frac{1 + \gamma}{\rho + \gamma} \leq \frac{2}{\rho + 1} \quad ,$$

pour tout γ compris entre 0 et 1 .

On peut alors, grâce à ce résultat, démontrer le théorème suivant.

<u>Théorème.</u> Soient P et Q dans $\mathbb{C}[X]$ de degrés p et q $(p>q)$.
Alors,

$$(1) \qquad |P|\, M(Q) \leq 2^{-q} \frac{(p+q)^{p+q}}{p^p q^q} |PQ| .$$

<u>preuve.</u> 1) On suppose que $F = PQ$ a toutes ses racines dans le disque unité.
D'après le corollaire 1, on a :

$$|P| \leq \left(\frac{2}{\rho+1}\right)^p \max\{|P(z)|, |z|=\rho\},$$

d'où l'on déduit que

$$|P| \leq \left(\frac{2}{\rho+1}\right)^p \frac{\max\{|F(z)|, |z|=\rho\}}{\min\{|Q(z)|, |z|=\rho\}} .$$

Or d'après le principe du maximum,

$$\max\{|F(z)|, |z|=\rho\} \leq |F|\, \rho^{p+q} .$$

De plus, les racines de Q se trouvant dans le disque unité, le coefficient dominant de Q est égal à la mesure de Mahler de Q, $M(Q)$, et on a

$$\min\{|Q(z)|, |z|=\rho\} \geq M(Q)\,(\rho-1)^q .$$

D'où

$$|P| \leq \left(\frac{2}{\rho+1}\right)^p \frac{\rho^{p+q}\, |F|}{(\rho-1)^q M(Q)} .$$

On obtient l'inégalité du théorème en posant $\rho = (p+q)/(p-q)$.

2) Si $F = PQ$ n'a pas toutes ses racines dans le disque unité, alors on transforme F, P et Q en F^*, P^* et Q^* en les multipliant par les facteurs de Blaschke

$$\left(B(\alpha, z) = \frac{\bar{\alpha} z - 1}{z - \alpha} \right)$$

de leurs racines de module plus grand que un.

Alors $F^* = P^* Q^*$, $|P^*| = |P|$, $|Q^*| = |Q|$, $M(Q^*) = M(Q)$, et on conclut avec ce qui précède.

2. Dans [2] on montre l'inégalité suivante ,

$$(2) \qquad L(P) \leq 2 \binom{[p/2]+q+1}{[p/2]} \|PQ\|_2 \ ,$$

pour des polynômes P et Q à coefficients complexes de degrés respectifs p et q , en supposant que les modules des coefficients extrémaux de Q soient supérieurs à un (L (P) est égal à la somme des modules des coefficients de P et $\| P Q \|_2$ est égal à la racine carré de la somme des carrés des modules des coefficients de P Q) .

L'interêt de ces inégalités est de permettre une majoration de la norme d'un facteur d'un polynôme irréductible ou quadratfrei à coefficients entiers en fonction de la norme du polynôme , qui ne soit pas exponentielle avec le degré du facteur , au moins lorsque la mesure du polynôme n'est pas trop grande , et on obtient alors une majoration de la norme de P meilleure que l'inégalité classique (cf. [5])

$$(3) \qquad L(P) \leq 2^p \|PQ\|_2 \ ,$$

y compris lorsque p est plus petit que q (cf. [4] et [1]) .

On utilise pour cela une majoration de la norme du facteur de degré p d'un polynôme de degré p + q dans le cas particulier où q est environ égal à $\sqrt{p}$.

Or dans ce cas particulier (1) n'est pas meilleure que (2) qui offre en outre l'avantage de majorer le module de chacun des coefficients du facteur et pas simplement sa norme (cf. corollaire 2 de [2]) .

En revanche (1) est meilleure que (2) et (3) lorsque $p > q > p^{\alpha}$ ($\alpha > 1/2$) , au moins lorsque p est assez grand .

<u>Remarque .</u> On peut également grâce au lemme précédent démontrer les inégalités

$$|P| \, L(Q) \leq \frac{(p+q)^{p+q}}{p^p q^q} |PQ| \qquad (p>q) \ ,$$

qui est meilleure que $L(P)L(Q) \leq 2^{p+q} \|PQ\|_2$ (cf. [4]) , ainsi que $|P| \, M(Q) \leq 2^p \, |PQ|$.

3. A.Granville a posé la question suivante (cf. [3]) :

- quel est le plus petit γ tel que l'on ait $|P| \leq \gamma^n \, |PQ|$, où n est le degré de P Q (en supposant que le coefficient dominant de Q est de module supérieur ou égal à 1) ?

Il a montré que l'on pouvait prendre $\gamma \leq (1 + \sqrt{5})/2$; grâce au théorème démontré en **1**, on peut améliorer sensiblement ce résultat .

Corollaire 2. Soient P et F deux polynômes a coefficients complexes tels que P divise F. Si l'on suppose que le coefficient dominant de P est majoré par le coefficient dominant de F en module, alors

$$|P| \leq (3/2)^n |F| \qquad (n = \text{degré de } F).$$

preuve. On reprend les notations du théorème en posant $n = p + q$ et $p = \lambda n$ $(1 \geq \lambda > 1/2)$.

On a alors

$$|P| \leq \left[\frac{1}{(2(1-\lambda))^{1-\lambda} \lambda^{\lambda}} \right]^n |F| .$$

Or le maximum de la fonction de λ présente sous l'exposant n est atteint pour $\lambda = 2/3$ et vaut $3/2$.

Le corollaire est donc vrai lorsque $\lambda > 1/2$. Pour montrer qu'il est vrai dans tous les cas il suffit de considérer la majoration $|P|\, M(F/P) \leq 2^p |F|$.

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CO-EXISTING PERIODIC ATTRACTORS IN THE LORENZ EQUATIONS

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ABSTRACT. Numerical techniques are used to locate co-existing stable periodic attractors in the Lorenz equations. Two such attractors are described which co-exist with the well-known stable symmetric orbit. These orbits were discovered as a result of a systematic search since they have very small basins of attraction and parameter ranges of existence. It is conjectured that there are many more such periodic attractors, and that it is typical in the study of dynamical systems that extensive numerical investigations reveal only the dominant stable periodic attractors whereas, in fact, there are other co-existing attractors.

RÉSUMÉ. On précise numériquement la position de deux attracteurs périodiques coexistant avec l'orbite stable et symétrique bien connue pour les équations de Lorenz au moyen d'une recherche systématique. En effet, ces attracteurs admettent des bassins d'attraction très exigus et n'existent que pour des valeurs très restreintes des paramètres. On conjecture l'existence de plusieurs autres attracteurs périodiques stables: il semble qu'une ample recherche numérique ne révèle souvent que l'orbite stable dominante alors que d'autres attracteurs coexistent avec elle.

Subject-classification: AMS(MOS): 34C25, 58F14, 58F22.
Keywords: bifurcations, periodic solutions, Lorenz equations.

1. Introduction. The Lorenz equations [1] are a three-dimensional system of coupled nonlinear ordinary differential equations with defining vector field $\mathbf{f} = (f_1, f_2, f_3)$:

$$\begin{aligned}\dot{x} &= f_1(x,y,z) = \sigma(y-x)\\ \dot{y} &= f_2(x,y,z) = rx - y - xz\\ \dot{z} &= f_3(x,y,z) = xy - bz.\end{aligned}$$

These equations have been studied extensively for Lorenz' original parameter values $\sigma = 10$, $b = 8/3$, $0 < r < \infty$ [2], [3]. It is well known that for large enough values of r, the equations have a single stable symmetric periodic orbit which, together with the three

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unstable stationary points, makes up the whole of the nonwandering set [3]. It is not known how large r needs to be for this analysis to hold, but numerical experiments show that the symmetric orbit becomes stable at $r \approx 313$ and, to the best of our knowledge, remains stable for all larger values of r. Taking $\sigma = 10$ and $b = 8/3$ throughout this paper as well, this orbit is shown at $r = 350$ in Fig. 1 [3].

In [3] it was also predicted that part of the underlying analysis could not apply for values of r less than approximately 500, since there are good reasons to believe that other stable periodic behaviour co-exists with this symmetric stable orbit in some of the parameter range $313 < r < 500$. None of this extra periodic behaviour was previously observed, despite the huge number of numerical experiments that were performed on the Lorenz equations in the last 27 years. (Numerous stable periodic orbits are known for lower values of r but these are not treated here.)

The purpose of this paper is to report on the existence of two parameter ranges with $r > 313$ where additional stable periodic behaviour occurs and to explain how the underlying co-existing stable periodic orbits were found. These results should act as a warning to numerical experimentalists who are tempted to report that they have found all attractors in a particular system of nonlinear differential equations.

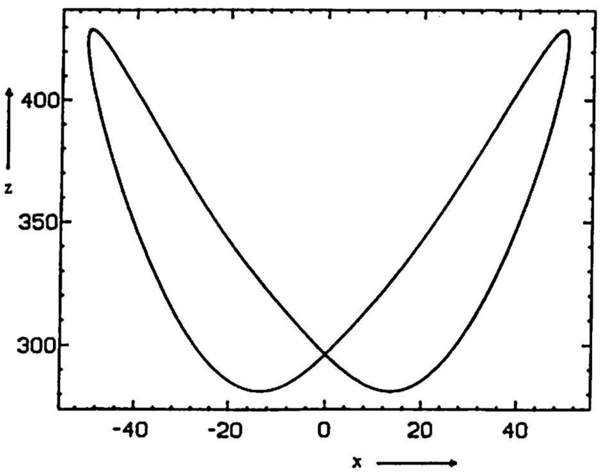

FIGURE 1. SYMMETRIC STABLE PERIODIC ORBIT AT $r = 350$, [3].

2. Co-existing orbits.

2.1) Method used to find stable periodic orbits

Suppose $\mathbf{x}^* := (x^*, y^*, z^*) \in \mathbf{R}^3$ is a point on a periodic orbit $\theta := \phi(t, \mathbf{x}^*)$ of period t^* satisfying $f_3(\mathbf{x}^*) \neq 0$. Then $\mathbf{x}^*$ is an element of the Poincaré return plane $E(z^*) := \{(x, y, z) \in \mathbf{R}^3 / z = z^*\}$. Moreover let $\mathbf{x}_0 := (x_s, y_s, z^*)$ near to $\mathbf{x}^*$ and t_0 near to t^* be suitable starting values for an iterative procedure. With $\underline{\mathbf{x}} := \phi(t_0, \mathbf{x}_0)$, $d\mathbf{x} = (dx_0, dy_0, 0) := \mathbf{x}^* - \mathbf{x}_0$ and $d\tau_0 := t^* - t_0$, a Taylor expansion up to second order terms gives:

$$\begin{aligned} \mathbf{x}^* &= \mathbf{x}_0 + d\mathbf{x} = \phi(t^*, \mathbf{x}^*) = \phi(t_0 + d\tau_0, \mathbf{x}_0 + d\mathbf{x}) \\ &\approx \underline{\mathbf{x}} + f(\underline{\mathbf{x}}) + D_{0,1}\phi(t_0, \mathbf{x}_0)d\mathbf{x} + \frac{1}{2}D_{2,0}\phi(t_0, \mathbf{x}_0)d\tau_0^2 \\ &\quad + D_{1,1}\phi(t_0, \mathbf{x}_0)d\tau_0 d\mathbf{x} + \frac{1}{2}D_{0,2}\phi(t_0, \mathbf{x}_0)d\mathbf{x}^2. \end{aligned} \tag{1}$$

Here the quantities $D_{0,1}$, $D_{2,0}$, $D_{1,1}$ and $D_{0,2}$ can be computed by solving an additional set of differential equations together with the original Lorenz model, see [2]. Neglecting higher order terms we now consider (1) as an exact three-dimensional nonlinear equation, use Newton's method to solve it for $(dx_0, dy_0, d\tau_0)$ and then apply the same procedure iteratively to $\mathbf{x}_{n+1} := \mathbf{x}_n + (dx_n, dy_n, 0)$ and $t_{n+1} := t_n + d\tau_n$. A fixed point $\mathbf{x}^* = (x^*, y^*, z^*) := \lim_{n\to\infty} \mathbf{x}_n$ then corresponds to a point of a periodic orbit in $E(z^*)$ of period $t^* := \lim_{n\to\infty} t_n$, and a similar technique serves to follow this fixed point in r-space. Finally the Floquet multipliers are the eigenvalues of the linearized Poincaré map $P : E(z^*) \to E(z^*)$ around $\mathbf{x}^*$ and are easily computed once a fixed point $\mathbf{x}^*$ and corresponding period t^* have been found, since $P_{linearized}$ corresponds to the following matrix:

$$\begin{bmatrix} m_{11} - m_{31}f_1(\mathbf{x}^*)/f_3(\mathbf{x}^*) & m_{12} - m_{32}f_1(\mathbf{x}^*)/f_3(\mathbf{x}^*) \\ m_{21} - m_{31}f_2(\mathbf{x}^*)/f_3(\mathbf{x}^*) & m_{22} - m_{32}f_2(\mathbf{x}^*)/f_3(\mathbf{x}^*) \end{bmatrix},$$

where the quantities $m_{ik} := m_{ik}(t^*)$ satisfy the additional differential equations:

$$\begin{aligned} \dot{m}_{11} &= \sigma(m_{21} - m_{11}) \\ \dot{m}_{12} &= \sigma(m_{22} - m_{12}) \\ \dot{m}_{21} &= (r - z)m_{11} - m_{21} - xm_{31} \\ \dot{m}_{22} &= (r - z)m_{12} - m_{22} - xm_{32} \\ \dot{m}_{31} &= ym_{11} + xm_{21} - bm_{31} \\ \dot{m}_{32} &= ym_{12} + xm_{22} - bm_{32} \end{aligned}$$

with initial conditions $m_{ik}(0) = \delta_{ik}$, $x(0) = x^*$, $y(0) = y^*$ and $z(0) = z^*$. Again, for more details see [2].

2.2) Numerical results

The first of the stable periodic orbits to be described here was discovered at a reasonably low value of r ($r < 200$), where it was unstable. This orbit was then followed with both increasing and decreasing r and Fig. 2 shows the orbit at $r = 328.08$ while Fig. 3 shows it at $r = 84.25$. In the former case, the computed Floquet multipliers imply its stability. Hence this periodic attractor co-exists with the stable symmetric orbit described in the introduction. In the latter, the orbit is highly unstable, its period is relatively large, and it is clear from the calculations that the orbit is approaching the saddle equilibrium point at the origin.

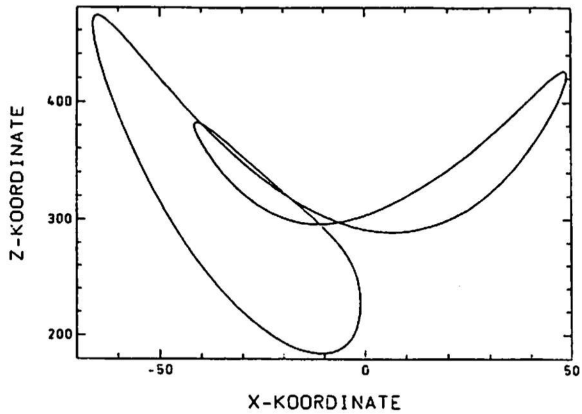

FIGURE 2. CO-EXISTING STABLE PERIODIC ORBIT AT $r = 328.08$.

The newly discovered orbit is stable only in a parameter range given by (approximately) $327.56 < r < 328.082$, and it has a very small basin of attraction. At $r \approx 327.56$ the orbit loses its stability in a period-doubling bifurcation, and at $r \approx 328.082$ it disappears in a saddle-node (or tangent) bifurcation involving an unstable periodic partner. This unstable periodic orbit was also found and it was followed for decreasing values of r down to $r \approx 205.15$. Again its period thereby eventually increased and it also seemed to approach the origin.

By very similar means another stable periodic orbit was found to co-exist with the well-known symmetric large-r orbit described in the introduction. In this case, the new orbit is also symmetric, and it is stable around $r = 337.77$. This orbit is shown in Fig. 4.

At a slightly larger value of r the orbit is destroyed in a saddle-node bifurcation involving an unstable symmetric periodic partner, and at a slightly lower value of r it loses its stability in a symmetry breaking bifurcation. Both symmetric periodic orbits involved in

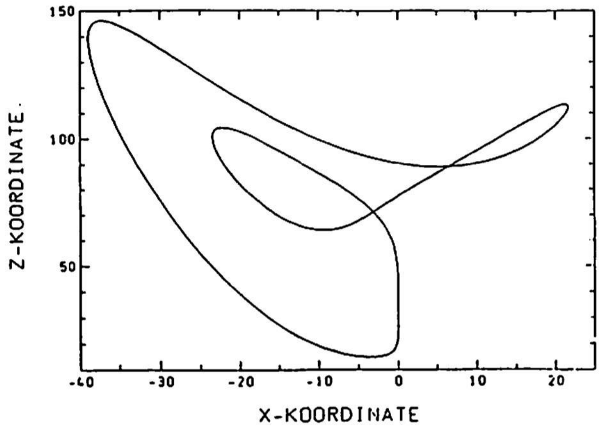

FIGURE 3. UNSTABLE PERIODIC ORBIT AT $r = 84.25$.

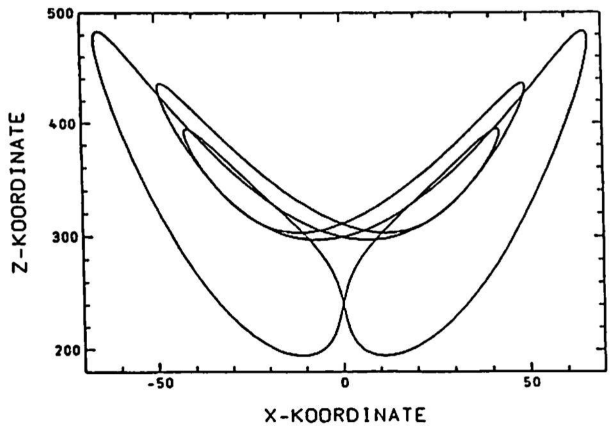

FIGURE 4. SYMMETRIC STABLE PERIODIC ORBIT AT $r = 337.77$.

the saddle-node bifurcation were followed with decreasing values of r, and again in both cases their period thereby eventually increased and they finally seemed to approach the origin.

TABLE 1. A point and the periods of the co-existing stable periodic orbits.

	x	y	z	Period	r
Orbit of Fig. 2	−21.69371	26.58114	328.47281	0.6126326	328.08
Orbit of Fig. 4	−29.29055	24.76479	360.21371	1.2047584	337.77

2.3) Bifurcations producing the newly discovered orbits

As each of the orbits described above was followed with decreasing r, their periods increased, they approached the saddle-point at the origin more closely and they became more unstable (one of the Floquet multipliers increases in absolute value). Eventually, in each case, the used numerical method was unable to follow the orbits any further. Nonetheless, it seems clear that the orbits were produced in homoclinic explosions as described in [3].

In the case of the orbit shown in Fig. 2 and Fig. 3, the relevant homoclinic explosion almost surely occurs at $r \approx 64.57$, and its unstable partner which is eventually involved in the saddle-node bifurcation with this orbit is almost surely produced in a homoclinic explosion at $r \approx 124.01$. These values agree well with the values given in [3] where Sparrow predicted that these orbits would exist and annihilate each other in a saddle-node bifurcation at a larger value of r, though the prediction about which of the orbits would eventually become stable was the wrong way around.

The symmetric orbit shown in Fig. 4 and its saddle-node bifurcation partner were almost surely produced in the same homoclinic explosions (at $r \approx 124.01$ and at $r \approx 64.57$ respectively). In this case, unlike the case just described, the orbit which eventually becomes stable is created at the larger value of r.

The fact that both pairs of orbits are produced in the same homoclinic explosions is not surprising. A careful study of the sequence of homoclinic explosions occurring in the system as r varies suggests that these are the only two bifurcations which could produce orbits with the appropriate topologies, and the theory tells us that we do expect such orbits to be part of the strange invariant set which is created at these bifurcations [3].

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CONFORMAL AND CURVATURE SYMMETRIES OF CONTACT METRIC MANIFOLDS

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Presented by P. Ribenboim, F.R.S.C.

ABSTRACT. We first propose a result regarding conformal Killing vector fields on contact metric manifolds. Next we prove that a vector field which leaves the curvature tensor of a K–contact metric manifold invariant is Killing. Finally we characterize a contact metric (but not K–contact) manifold whose curvature tensor is invariant by the characteristic vector field.

1. INTRODUCTION

Let (φ, ξ, η, g) be the contact structure of a contact metric manifold M^{2n+1} with the fundamental 2–form $\Phi = g(\cdot, \varphi\cdot)$ and the self–adjoint tensor $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ ($\mathcal{L}$ denoting Lie–differentiation). Tanno [5] proved that a conformal Killing vector field V on a contact metric manifold is Killing if V leaves invariant any one of the tensors η, ξ, φ and Φ. We obtain the following result.

PROPOSITION 1. *If a conformal Killing vector field* V *on a contact metric manifold is either orthogonal to* ξ, *or collinear with* ξ, *then* V *is Killing.*

Next we consider the curvature symmetry defined by the existence of a vector field V on a K–contact manifold (i.e. a contact metric manifold with ξ Killing) such that the curvature tensor R is invariant by V. In fact, we prove

THEOREM 1. *Let* V *be a vector field on a* K*–contact manifold such that* $\mathcal{L}_V R = 0$. *Then,* V *is Killing.*

Blair [3] proved that a contact metric manifold M^{2n+1} is K–contact if and only if the Ricci curvature $\mathrm{Ric}(\xi, \xi)$ in the direction of ξ is equal to $2n$. Now it is easy to verify that a contact metric manifold is K–contact iff ξ is affine Killing $(\mathcal{L}_\xi\nabla = 0)$. Also if a contact metric manifold is K–contact then obviously R is invariant by ξ $(\mathcal{L}_\xi R = 0)$;

however, the converse need not be true. To illustrate this we prove

THEOREM 2. *Let* $\mathcal{L}_\xi R = 0$ *on a contact metric (but not* K*–contact) manifold* M^{2n+1}. *Then* M^3 *is flat. For* $n > 1$, *the curvature operator* $R(X, Y)$ *annihilates* h *and the contact subbundle on* M^{2n+1} *is orthogonally decomposed into the eigenspaces* [0], [1] *and* [–1] *of* h *corresponding to the eigenvalues* 0, 1 *and* –1; *moreover the sectional curvature* K *satisfies:* $K(\xi, [0]) = 1$, $K(\xi, [\pm 1]) = K([0], [\pm 1]) = K([+1], [-1]) = 0$. *Furthermore for* $n > 1$, *if in addition to* $\mathcal{L}_\xi R = 0$ *we have* $\mathrm{Ric}(\xi, \xi) < 2$ *then* M^{2n+1} *is locally the Riemannian product of a flat* (n+1)*–dimensional manifold and an* n*–dimensional manifold of positive constant curvature* 4.

Note that the proof of Theorem 2 makes use of a result (see Theorem B in [3]) of Blair, which is better stated in view of the proof of the theorem in a previous paper [2] as:

THEOREM (Blair). *Let* M^{2n+1} *be a contact metric manifold and suppose that* $R(X, Y)\xi = 0$ *for all vector fields* X *and* Y. *Then* M^3 *is flat and for* $n > 1$, M^{2n+1} *is locally the Riemannian product of a flat* (n+1)*–dimensional manifold and an* n*–dimensional manifold of positive constant curvature* 4.

I would like to thank Professor David E. Blair for pointing out the fact that his theorem as mentioned in [3] could be stated better as above and for his valuable suggestions in general.

2. CONTACT MANIFOLDS

By a contact manifold we mean an odd–dimensional C^∞–manifold M^{2n+1} together with a global 1–form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η it is well known that there exists a unique vector field ξ (called the characteristic vector field) on M^{2n+1} satisfying $\eta(\xi) = 1$ and $(d\eta)(\xi, X) = 0$. A Riemannian metric g is said to be an associated metric if there exists a tensor field φ of type (1,1) such that $(d\eta)(X, Y) = g(X, \varphi Y)$, $\eta(X) = g(X, \xi)$ and $\varphi^2 = -I + \eta \otimes \xi$. The structure (φ, ξ, η, g) is called a contact metric structure and M^{2n+1} with this structure is called a contact metric manifold. For details we refer to [1]. Given a contact metric structure (φ, ξ, η, g) we define a tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. h is a self–adjoint operator which anti–commutes with φ (see e.g. [1], p. 54) and hence if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector φX. Clearly $h\xi = 0$. It is also easy to see that ξ is Killing with respect to g if and only if $h = 0$; a contact metric manifold for which ξ is Killing is said to be K–contact. It is also known that a contact metric manifold is K–contact if and only if the sectional curvature of a plane section containing ξ is equal to 1. Letting ∇ denote the Riemannian connection of g we have the following formulas for a general contact metric manifold (see [1], p. 66, [4] and [3]):

$$\nabla_X \xi = -\varphi X - \varphi h X \tag{1}$$

$$(\mathcal{L}_\xi g)(X, Y) = 2g(h\varphi X, Y) \tag{2}$$

$$R(\xi, X)\xi - \varphi R(\xi, \varphi X)\xi = 2(h^2 + \varphi^2)X \tag{3}$$

$$(\nabla_\xi h)X = \varphi\{X - h^2 X + R(\xi, X)\xi\} \tag{4}$$

$$\mathrm{Ric}(\xi, \xi) = 2n - \mathrm{tr}\, h^2 . \tag{5}$$

For a K–contact manifold $(h = 0)$ we know that

$$\text{(a)}\ R(\xi, X)\xi = \eta(X)\xi - X . \quad \text{(b)}\ \mathrm{Ric}(\xi, X) = 2n\eta(X) . \tag{6}$$

3. PROOFS OF THE RESULTS

PROOF OF PROPOSITION 1. As V is conformal Killing, $\mathcal{L}_V g = \sigma g$ for some function σ on M^{2n+1}. Then, as the integral curves of ξ are geodesics ($\nabla_\xi \xi = 0$, can be seen by setting $X = \xi$ in (1)), we have $\sigma = (\mathcal{L}_V g)(\xi, \xi) = 2g(\nabla_\xi V, \xi) = 2\nabla_\xi(g(V, \xi))$. So if V is orthogonal to ξ then $\sigma = 0$ and hence $\mathcal{L}_V g = 0$. If $V = \alpha\xi$ for some function α then $g(V, \xi) = \alpha$ and hence as noted before $\sigma = 2\xi\alpha$. Taking div of $V = \alpha\xi$ we find $\mathrm{div}\, V = \xi\alpha$ (because h being symmetric [2], $\mathrm{tr}\cdot\varphi h = 0$). So $2\,\mathrm{div}\, V = 2\xi\alpha = \sigma$.

Moreover, contracting $(\mathcal{L}_V g)(X, Y) = \sigma g(X, Y)$ yields $2 \operatorname{div} V = \sigma(2n+1)$. Consequently $\sigma = \sigma(2n+1)$ whence $\sigma = 0$. This completes the proof.

PROOF OF THEOREM 1. Lie–differentiating the identity: $g(R(W, X)Y, Z) + g(R(W, X)Z, Y) = 0$ along V and using the hypothesis $\mathcal{L}_V R = 0$ we find

$$(\mathcal{L}_V g)(R(W, X)Y, Z) + (\mathcal{L}_V g)(R(W, X)Z, Y) = 0 . \tag{7}$$

Substituting $W = Y = Z = \xi$ and using (6) we have

$$(\mathcal{L}_V g)(X, \xi) = \eta(X)(\mathcal{L}_V g)(\xi, \xi) . \tag{8}$$

Next we set $W = Y = \xi$ in (7), and using (6) and (8) we obtain $\mathcal{L}_V g = (\mathcal{L}_V g)(\xi, \xi)g$. Now 6(b) $\Rightarrow$ $\operatorname{Ric}(\xi, \xi) = 2n$. Applying $\mathcal{L}_V$ on it gives $\operatorname{Ric}(\mathcal{L}_V \xi, \xi) = 0$ (because $\mathcal{L}_V R = 0 \Rightarrow \mathcal{L}_V \operatorname{Ric} = 0$). This together with 6(b) $\Rightarrow$ $g(\mathcal{L}_V \xi, \xi) = 0$ whence $(\mathcal{L}_V g)(\xi, \xi) = 0$. Hence V is Killing, completing the proof.

PROOF OF THEOREM 2. As in the proof of Theorem 1, Lie–differentiating the identity $g(R(W, X)Y, Z) + g(R(W, X)Z, Y) = 0$ along ξ and using $\mathcal{L}_\xi R = 0$ we have

$$(\mathcal{L}_\xi g)(R(W, X)Y, Z) + (\mathcal{L}_\xi g)(R(W, X)Z, Y) = 0 \tag{9}$$

Use of (2) in (9) and the substitution $Y = \xi$ provides

$$h(R(X, Y)\xi) = 0 . \tag{10}$$

Equation (10) implies $g(R(\xi, X)\xi, h\varphi Y) = 0$ and $g(\varphi R(\xi, \varphi X)\xi, h\varphi Y) = 0$. Using these two equations in (3) we obtain

$$h^3 = h . \tag{11}$$

This shows that h has only three eigenvalues 0, 1 and -1 and accordingly the contact subbundle is decomposed orthogonally into the eigenspaces [0], [1] and [-1]. Now we return to formula (4), apply h on it and use (10) and (11) so as to have

$$h(\nabla_\xi h) = 0 . \tag{12}$$

Next, taking the Lie–derivative of (3) along ξ and using (10) we get $(\mathcal{L}_\xi h^2)X$

$= -\varphi R(\xi, hX)\xi$. But $g(\varphi R(\xi, hX)\xi, Y) = -g(R(\xi, hX)\xi, \varphi Y) = -g(R(\xi, \varphi Y)\xi, hX)$ $= -g(hR(\xi, \varphi Y)\xi, X) = 0$. Therefore we obtain $\mathcal{L}_\xi h^2 = 0$. This together with (12) implies $(\nabla_\xi h)h = 0$. Combining the last equation with (12) we find $\nabla_\xi h^2 = 0$ and so $(\nabla_\xi h^2)hX = 0$. Therefore $\nabla_\xi h^3 X - h^2(\nabla_\xi h)X - h^3\nabla_\xi X = 0$ which reduces with the aid of (11) and (12) to

$$\nabla_\xi h = 0. \tag{13}$$

Substituting this result in (3) we obtain

$$R(\xi, X)\xi = (h^2 + \varphi^2)X. \tag{14}$$

Now $(\mathcal{L}_\xi h)X = (\nabla_\xi h)X - \nabla_{hX}\xi + h\nabla_X\xi = 2\varphi(h + h^2)X$, using (1) and (13). That is

$$\mathcal{L}_\xi h = 2\varphi(h + h^2). \tag{15}$$

Let us recall (9) and transform it using (2) as

$$\varphi hR(W, X)Z + R(W, X)h\varphi Z = 0. \tag{16}$$

Replacing Z by φZ in (16) we have $\varphi hR(W, X)\varphi Z = R(W, X)hZ$. Lie–differentiating this along ξ and using (15) we find

$$hR(W, X)\varphi Z - \varphi hR(W, X)hZ + R(W, X)\varphi(h + h^2)Z = 0.$$

Now replacing Z by φZ in the above equation and using (10) and (16) gives

$$h(R(X, Y)Z) = R(X, Y)hZ. \tag{17}$$

Hence $R(X, Y)h = 0$. At this point we first consider the 3–dimensional case $(n = 1)$. Since h has eigenvalues 0, 1 and −1 we observe in view of the continuity of h and its eigenvalues that either h vanishes everywhere which is not possible by hypothesis (M^3 is not K–contact), or h has eigenvalues 0, 1 and −1 everywhere on M^3. Hence $h^2 = I - \eta \otimes \xi$ and $\mathrm{rank}(h) = 2$ on M^3. Thus equation (10) implies $R(X, Y)\xi = 0$ for arbitrary vector fields X, Y. Hence Blair's theorem (mentioned in the introduction) applies and M^3 is flat.

Next we consider the case $n > 1$. We note from equations (14) and (17) that $K(\xi, [0]) = 1$, $K(\xi, [\pm 1]) = 0$, $K([0], [\pm 1]) = K([+1], [-1]) = 0$. To prove the last part, we have by hypothesis $Ric(\xi, \xi) < 2$ that $\sum_{i=1}^{n} \lambda_i^2 > n-1$ where $\pm\lambda_i$ are the eigenvalues of h other than the eigenvalue 0 corresponding to the eigenvector ξ. As we have found, λ_i could be one of the values 0, 1 and -1. This shows that $\sum_{i=1}^{n} \lambda_i^2 = n$ and hence we conclude that $\lambda_i^2 = 1$ for all i. Therefore $rank(h) = 2n$ everywhere and so (10) implies $R(X, Y)\xi = 0$ for all vector fields X and Y on M^{2n+1}. Again Blair's theorem applies and we conclude that M^{2n+1} is locally the Riemannian product of a flat (n+1)–dimensional manifold and an n–dimensional manifold of constant curvature 4. This completes the proof.

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A MATHEMATICAL MODEL FOR CONVECTIVE STABILITY PROBLEMS IN A TALL VERTICAL ANNULUS

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Presented by G.F.D. Duff, F.R.S.C.

ABSTRACT. This report investigates the stability of a convective fluid motion caused by an internal heat generation in a tall vertical annulus. It also considers the influence of a uniform longitudinal, resp. non-uniform radial, magnetic field in the case the channel walls are kept at different constant temperatures.

RÉSUMÉ. On étudie la stabilité de la convection d'un fluide sous génération thermique interne dans un long cylindre annulaire. On considère aussi l'effet d'un champ magnétique longitudinal uniforme, resp. radial non uniforme, dans le cas où les parois du canal sont à des températures différentes constantes.

Subject-classification: AMS(MOS): 65L10, 65L15, 76E15, 76E25.
Keywords: stability of fluid flow, magnetohydrodynamics, pseudospectral method.

1. Introduction. Much attention, both theoretical and experimental, has been given to the convection of incompressible fluids in vertical channels in view of numerous applications, like the description of processes taking place in the Earth's mantle, the cooling of nuclear reactors, etc. An extensive bibliographical survey of convective stability problems can be found in [1].

A method of solution to convective stability problems in tubular domains is presented in this report, (a) for a convective flow caused by an internal heat generation, and (b) for a convective flow due to different constant wall temperatures in a magnetic field. These problems will be referred to as Problem 1 and Problem 2, respectively. Two kinds of magnetic fields are considered in Problem 2: an uniform longitudinal field

$$\mathbf{B} = (0, 0, B_0),$$

where B_0 is a constant, and a non-uniform radial field

$$\mathbf{B} = (\frac{1}{r}, 0, 0).$$

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2. Mathematical formulation. Convective stability problems are described by the Oberbeck-Boussinesq equations

$$\frac{\partial \mathbf{v}}{\partial t} + G(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \Delta \mathbf{v} + T\mathbf{k}, \tag{1}$$

$$\frac{\partial T}{\partial t} + G\mathbf{v}\cdot(\nabla T) = \frac{1}{\mathrm{Pr}}\Delta T, \tag{2}$$

$$\mathrm{div}\ \mathbf{v} = 0, \tag{3}$$

where $\mathbf{v}$, T and p are, respectively, the velocity, temperature and pressure of the fluid; G is the Grasshof number, Pr is the Prandtl number and $\mathbf{k} = (0,0,1)$ is the vertical vector with respect to the cylindrical polar coordinates (r,φ,z). Equations (1)–(3) have the following steady solution:

$$\mathbf{v} = (0,0,v_0(r)), \qquad T = T_0(r), \qquad p = p_0(z), \tag{4}$$

with appropriate boundary conditions for the functions $v_0(r)$ and $T_0(r)$. Furthermore, the fluid flux through any channel cross-section must be zero:

$$\int_{r_1}^{r_2} r v_0(r)dr = 0, \tag{5}$$

where r_1 and r_2 are the radii of the inner and outer cylinders, respectively. The term $1/\mathrm{Pr}$ is to be added to the right-hand side of equation (2) for Problem 1 while for Problem 2 the additional equations for the magnetic field $\mathbf{B}$ are to be added to system (1)–(3) together with an extra term on the right-hand side for equation (2), (see [1]).

The stability of the flow (4) is investigated by the method of small perturbations. The perturbed motion $\mathbf{v}_0+\mathbf{v}$, T_0+T, p_0+p, where $\mathbf{v}$, T and p are small unsteady perturbations, is linearized in the neighbourhood of the basic flow (4) and the solution of the linearized equations (1)–(3) is sought in the form

$$\begin{aligned}\mathbf{v}(r,\varphi,z,t) &= \mathbf{w}(r)\exp(-\lambda t + im\varphi + ikz),\\ p(r,\varphi,z,t) &= q(r)\exp(-\lambda t + im\varphi + ikz),\\ T(r,\varphi,z,t) &= \theta(r)\exp(-\lambda t + im\varphi + ikz).\end{aligned}$$

For example, the amplitude equations for Problem 1 in the axisymmetric case ($m = 0$) are

$$\begin{aligned}&\varphi^{iv} - \frac{2}{r}\varphi''' + \frac{3}{r^2}\varphi'' - \frac{3}{r^3}\varphi' - 2k^2\varphi'' + \frac{2k^2}{r}\varphi' + k^4\varphi\\ &\quad = ikG[v_0(\varphi'' - \frac{\varphi'}{r} - k^2\varphi) + \varphi(\frac{v_0'}{r} - v_0'')] - r\theta' - \lambda(\varphi'' - \frac{\varphi'}{r} - k^2\varphi),\end{aligned} \tag{6}$$

$$\theta'' + \frac{1}{r}\theta' - k^2\theta = ikGPr(v_0\theta - \frac{T_0'}{r}\varphi) - \lambda \mathrm{Pr}\theta, \tag{7}$$

where $\varphi(r)$ is the amplitude of the stream function $\psi(r,z,t)$ defined by the equations and boundary conditions

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r},$$

$$\varphi(r_j) = 0, \qquad \varphi'(r_j) = 0, \qquad \theta(r_j) = 0, \qquad j = 1,2. \tag{8}$$

In [2], the numerical solution of convective stability problems, like the boundary value problems (6)–(8), uses variants of Runge-Kutta methods and orthogonalization at each step of the integration. This technique is very efficient provided one has a good initial guess for the desired eigenvalue.

In [3], Orszag uses the excellent approximation properties of Chebyshev polynomials to solve hydrodynamic stability problems. Today there exist many ways of discretizing boundary value problems by means of orthogonal polynomials, (see, for example, [4] and [5]).

In the problem at hand it proves convenient to use Chebyshev polynomials. The zeros of the highest degree Chebyshev polynomial $T_n(x)$ are used as collocation points. A high resolution is necessary near the walls since boundary layers are formed near the channel walls in the case of a radial magnetic field. As the zeros of T_n are packed near the ends of the interval $(-1,1)$, this high resolution may be achieved near the boundaries of the channel with moderate values of n. Numerical results [6] also indicate that the highest degree n is not sensitive to variations in the Prandtl number.

The discretization procedure leads to a generalized eigenvalue problem

$$(A - \lambda B)\mathbf{u} = 0, \tag{9}$$

where A and B are general complex-valued matrices. The numerical solution to (9) can be found by the LZ or QZ-algorithms.

3. Computational results. Results are summarized for Problems 1 and 2 respectively.

Problem 1. For low Prandtl numbers, instability is produced by the interaction of flows moving in opposite directions. The computed critical values of the Grasshof number G for the small Prandtl-number approximation as a function of $R = r_1/r_2$ are given in Fig. 1 for the axisymmetric ($m = 0$) and asymmetric ($m = 1$) perturbations.

From a mathematical point of view, the small Prandtl-number approximation means that one neglects the term $r\theta'$ in equation (6) and that equation (7) is not considered at all. It is obvious from Fig. 1 that for small values of R ($0.05 < R < 0.29$) the critical Grasshof number corresponds to an asymmetric disturbance, while, in the range $0.29 < R < 1$, the

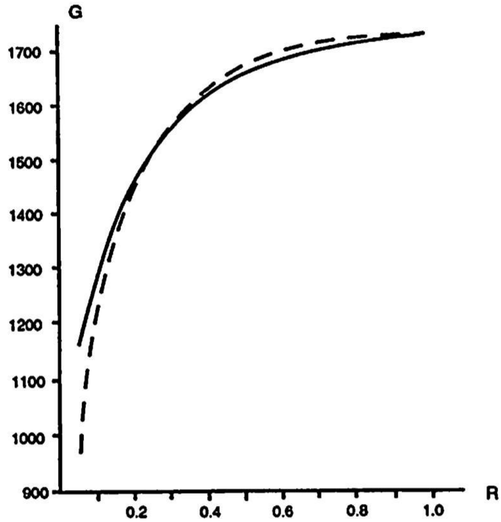


Figure 1. *The dependence of the critical Grasshof number, G, upon the distance, R, between the cylinders. The solid (resp. dashed) line represents the symmetric (resp. asymmetric) shear mode.*

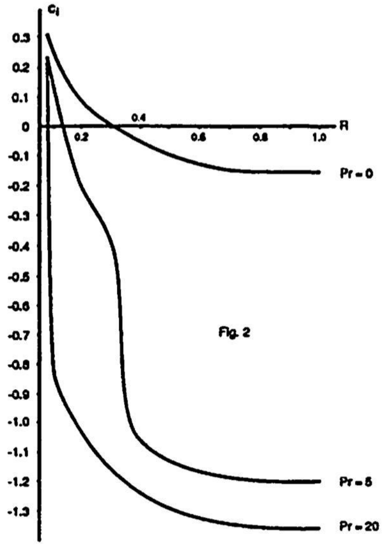
instability is caused by axisymmetric perturbations. The critical values of the Grasshof number increases as the gap between the cylinders decreases.

For high Prandtl numbers instability occurs in the form of thermal running waves for a large interval of values of the ratio of the inner to the outer radii. The computational results for axisymmetric and asymmetric modes are presented in Figs. 2 and 3.

The normalized critical speed, $c_i = \lambda_i/(kGv_{0\,\mathrm{max}})$, is measured in the units of the base flow (4), where λ_i is the imaginary part of the purely imaginary eigenvalue associated with c_i and $v_{0\,\mathrm{max}}$ is the norm of the maximum nondimensional velocity of the base flow.

The main observation here is that only the axisymmetric disturbances lead to instability at high Prandtl numbers, at least in the interval $0.05 < R < 1$, since the neutral curves for $\mathrm{Pr} = 5$ and $\mathrm{Pr} = 20$ in the asymmetric case lie above the corresponding curves in the axisymmetric case.

The transition from the shear mode to the buoyant mode as the Prandtl number increases can be found by a close analysis of Figs. 2 and 3 while taking into account the important role that the ratio, R, of the inner to outer radii plays in this transition at high Prandtl numbers. In fact, for each Prandtl number there exists, in some sense, a critical value, R^*, of R above which the wave speed of the perturbations increases considerably and concommittently the critical Grasshof number decreases abruptly. It is obvious from these figures that $R^* \approx 0.1$ for Pr = 20 and $R^* \approx 0.3$ for Pr = 5. Therefore the interval of values of R for which instability is associated with thermal running waves increases when the Prandtl number increases.



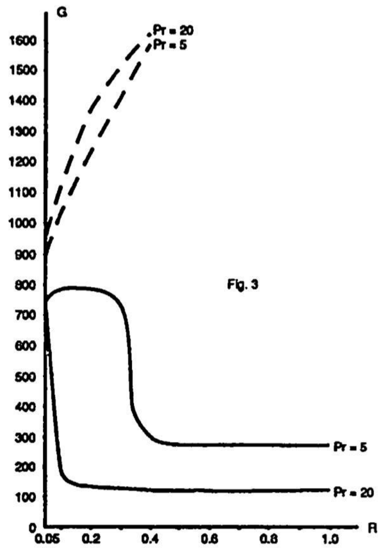


Figure 2. (left) *The dependence of the critical wave speed,* c_i, *upon the distance,* R, *between the cylinders.*

Figure 3. (right) *The dependence of the critical Grasshof number,* G, *upon the distance,* R, *between the cylinders for different values of the Prandtl number. The solid (resp. dashed) lines represent the symmetric (resp. asymmetric) modes.*

Fig. 3 shows that for the range of R: $0.4 < R < 1$ for Pr = 5 and $0.1 < R < 1$ for Pr = 20, the critical Grasshof number varies very little; hence the curvature is not important and the critical Grasshof numbers correspond approximately to the case of a planar vertical slot.

Problem 2. Since the longitudinal field does not interact with the base flow, its influence on the stability is relatively weak. It is found that the critical Grasshof numbers increase linearly with the Hartmann number, the latter being a dimensionless parameter which is proportional to the strength of the applied magnetic field. The radial field interacts not only with the perturbations, but also with the base flow; thus it modifies the latter considerably. Computations show a strong increase in stability for wide gaps between the cylinders, (see Fig. 4).

For small gaps, however, the base flow does not vary much with changes in the Hartmann number. Thus the critical Grasshof numbers increase more slowly, for each fixed value of the Hartmann number, when the gap width between the cylinders decreases. It is found that for sufficiently large Hartmann numbers, the critical wave speeds change sign, i.e. the perturbations begin to travel in the direction of gravity.

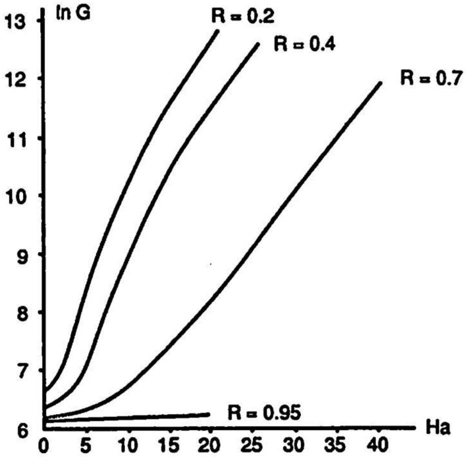


Figure 4. *The critical Grasshof number, G, as a function of the Hartmann number,* Ha, *for* Pr = 0 *and various values of R.*

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THE POORMAN'S TRANSFORMATION ON HILBERT SPACE

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Abstract. A generalization of the Fourier transform is defined, which maps summable sequences to integrable periodic functions. We examine its Hilbert space extension and its inverse, both of which are in general unbounded operators.

Introduction. Given a complex-valued, integrable function f defined on the circle Π of complex numbers of modulus one, a bounded transformation T_f from the Banach space of summable sequences $\{\xi_n : n \in \mathbf{Z}\}$ to the Banach space of integrable functions h on the circle is defined by the formula

$$h(z) = (T_f\xi)(z) = \sum_n f(z^n)\xi_n, \qquad z \in \Pi.$$

This transformation has applications in signal processing and is named the Poorman's transform since for certain choices of the function f, it gives an approximation to the usual Fourier transform that is inexpensive to compute. (cf. [Lam]). Notice that when f is the identity function $f(z) = z$, the operator T_z is exactly the Fourier transform, while the cosine, sine and Hartley transforms are obtained by taking the real and imaginary parts of z, or the sum of the two, (cf. [Brace]), so T_f is a generalization of these well-known transforms.

When f is square-integrable, the transform T_f may be considered to be a densely-defined Hilbert space operator from $l^2(\mathbf{Z})$ into $L^2(\Pi)$ with dense domain $l^1(\mathbf{Z})$. To compute the norm of this operator, it is convenient to use usual Fourier transform T_z to define a densely-defined operator S_f from $l^2(\mathbf{Z})$ into itself, via the equation $S_f = T_z^{-1}T_f$. With $\{\xi_n : n \in \mathbf{Z}\}$ a sequence in $l^1(\mathbf{Z})$ and $\{\eta_k : k \in \mathbf{Z}\}$ its image in $l^2(\mathbf{Z})$, $\eta = S_f\xi$, an easy computation with

Fourier coefficients shows that

$$\eta_0 = f(1)\xi_0 + a_0 \sum_{n\neq 0} \xi_n$$

$$\eta_k = \sum_{mn=k} a_m \xi_n, \qquad k \neq 0,$$

where $\{a_m : m \in \mathbf{Z}\}$ are the Fourier coefficients of the function f, with $f(z) = \sum_m a_m z^m$.

By examining the equation for η_0, it is clear that the Hilbert space operator S_f is unbounded if $a_0 \neq 0$; moreover, the value $f(1)$ plays a distinguished role. On the other hand, the equation for the remaining η_k is simply a convolution of the a_m with the ξ_n over the multiplicative semigroup of non-zero integers. Ignoring the zero-th terms, it is appropriate to consider a, ξ, and η as functions on the multiplicative group $\mathbf{Q}^\circ$ of non-zero rationals with support on the semigroup $\mathbf{Z}^\circ$ of non-zero integers, related by convolution. Thus the following theorem is not surprising.

Theorem 1. Let f be a square-integrable function on the circle Π with Fourier coefficients $\{a_m : m \in \mathbf{Z}\}$ and T_f, S_f the densely defined operators on $l^2(\mathbf{Z})$ described above. If $a_0 \neq 0$ then T_f and S_f are unbounded. Otherwise,

$$\|T_f\| = \|S_f\| = \max(\quad |f(1)|, \quad \|\widehat{a}\|_\infty \quad),$$

where $\|\widehat{a}\|_\infty$ is the essential supremum norm of the square-integrable function $\widehat{a}$ defined on the dual group of $\mathbf{Q}^\circ$ as the Fourier transform of the sequence a considered as a square-summable function on $\mathbf{Q}^\circ$.

Proof. For $a_0 = 0$, the key idea is to note that the convolution operation $\eta = a * \xi$ can be extended to functions ξ in the Hilbert space $l^2(\mathbf{Q}^\circ)$, in which case the usual group algebra results indicate the norm is equal to the essential supremum of the Fourier transform of a, as a function on the group $\mathbf{Q}^\circ$. Compressing the operator to the invariant subspace $l^2(\mathbf{Z}^\circ)$ does not change the norm, since any ξ in $l^2(\mathbf{Q}^\circ)$ on which the convolution is large can be unitarily shifted within epsilon into $l^2(\mathbf{Z}^\circ)$, on which the convolution is still large. ∎

Example. The hope was that the essential supremum for $\widehat{a}$ would be related to the essential supremum of f, since after all the function f is also a Fourier transform its coefficients $\{a_m\}$, but over a different group. However, this is not the case: f can be bounded with $\widehat{a}$ essentially unbounded. This is precisely what happens with some of the most interesting choices of f for applications, namely when f is a piecewise constant approximation to the complex exponential function. For example, take f to be the function on Π defined by

$$f(z) = \begin{cases} 1, & arg(z) \in [0, \pi/2) \\ i, & arg(z) \in [\pi/2, \pi) \\ -1, & arg(z) \in [\pi, 3\pi/2) \\ -i, & arg(z) \in [3\pi/2, 2\pi). \end{cases}$$

The Fourier coefficients of f are easily computed to be

$$a_m = \begin{cases} 2(i-1)/\pi m, & m = 1 \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Noting that the expansion of the non-zero rational $r = (-1)^{n_0} 2^{n_1} 3^{n_2} 5^{n_3} \ldots$ as a product of primes identifies $\mathbf{Q}^\circ$ with the group $\mathbf{Z}_2 \oplus (\bigoplus_1^\infty \mathbf{Z})$, we see that the dual group is the product $\mathbf{Z}_2 \oplus (\prod_1^\infty \Pi)$, which is topologically two copies on a infinite product of tori. By 44.43 and 44.53 of [H&R], $\widehat{a}$ can be computed almost everywhere as a limit, with

$$\widehat{a}(k_0, z_1, z_2, \ldots) = \lim_{m \to \infty} \sum_r a_r (-1)^{k_0 n_0} z_1^{n_1} z_2^{n_2} z_3^{n_3} \ldots z_m^{n_m},$$

where the sum is over rationals of the form $r = (-1)^{n_0} 2^{n_1} 3^{n_2} 5^{n_3} \ldots p_m^{n_m}$, with p_m the m-th prime. Since a_r is zero except when r is an odd integer, and the $\pm$ signs cancel for $k_0 = 1$, the sum simplifies considerably to

$$\widehat{a}(1, z_1, z_2, \ldots) = \lim_{m \to \infty} \frac{2(i-1)}{\pi} \sum_{n_2, n_3, \ldots n_m \geq 0} \frac{z_2^{n_2} z_3^{n_3} \ldots z_m^{n_m}}{3^{n_2} 5^{n_3} \ldots p_m^{n_m}}$$

$$= \frac{2(i-1)}{\pi} \prod_{j \geq 2} \frac{1}{1 - z_j/p_j}$$

Although this infinite product converges for almost all tuples $\{z_1, z_2, \ldots\}$, it is not absolutely convergent since the sum of reciprocals of all primes is divergent. In particular, it is not hard to see that the product is essentially unbounded in a neighbourhood of the point $\{1, z_1, z_2, \ldots\} = \{1, 1, 1, \ldots\}$. Indeed, simply integrate $|\widehat{a}|$ on a neighbour of the form

$$U_m = 1 \times I_\epsilon \times I_\epsilon \times \ldots \times \Pi \times \Pi \ldots,$$

where I_ϵ is a small interval in Π centered at the point $z = 1$, which is repeated m times in the above product; an easy calculation shows

$$\frac{1}{\mu(U_m)} \int_{U_m} |\widehat{a}| d\mu \quad \approx \quad \prod_{j=2}^{j=m} (1 + \frac{1}{2p_j}) \times \prod_{j=m+1}^{\infty} (1 - \frac{1}{4p_j^2}),$$

which diverges as m approaches infinity. Thus $\|\widehat{a}\|_\infty$ is infinite and the operators T_f and S_f are unbounded on Hilbert space.

Inverses. It was pointed out in Theorem 1 of [Lam] that the operator T_f and hence S_f have trivial kernel if and only if neither $f(\overline{z}) = f(z)$ nor $f(\overline{z}) = -f(z)$ holds true for almost all z, and $f(1) \neq 0$. Equivalently, the kernel is trivial if and only if there exists m and m' with $a_m \neq a_{-m}$, $a_{m'} \neq -a_{-m'}$, and $f(1) \neq 0$. This can easily be seen by noting the convolution $\eta = a * \xi$ may be rewritten for $k \neq 0$ as

$$(n_k + n_{-k}) = \sum_{\substack{mn=k \\ m,n>0}} (a_m + a_{-m})(\xi_n + \xi_{-n}),$$

$$(n_k - n_{-k}) = \sum_{\substack{mn=k \\ m,n>0}} (a_m - a_{-m})(\xi_n - \xi_{-n}),$$

while the zero-th terms are given by $\eta_0 = f(1)\xi_0$. What is also clear is that the convolution can be inverted if and only if the lowest order terms in the above sums are non-zero. Namely, if we denote by $e = \{e_k : k \in \mathbf{Z}\}$ the identity convolution sequence, (that is the sequence with $e_1 = 1$, $e_k = 0$ otherwise), then for any sequence $a = \{a_m : m \neq 0\}$ of complex numbers, there exists a sequence b with $e = a * b$ if and only if $a_1 \neq \pm a_{-1}$. Even with a a finite sequence, the inverse sequence b may be unbounded; nevertheless b acts as a well-defined convolution operator on $l^2(\mathbf{Z}^\circ)$ with dense domain, since the convolution sums always involve only finitely many terms, and acts as the formal inverse to convolution by a. Thus we have the following.

Theorem 2. Let f be a square-integrable function on the circle Π with Fourier coefficients $\{a_m : m \in \mathbf{Z}\}$, $a_0 = 0$. Then the Hilbert space operators T_f and S_f have (unbounded) inverses if and only if $a_1 \neq \pm a_{-1}$ and $f(1) \neq 0$. Moreover, when the inverses exist, the inverse for S_f decomposes as a direct sum $f(1)^{-1}I \oplus b$ where I is the identity operator on the subspace $l^2(0)$ and b is a convolution operator acting on $l^2(\mathbf{Z}^\circ)$, coming from the inverse sequence for a.

By the usual Tauberian theorems, we also have the following.

Corollary. If f is a continuous function on Π with summable Fourier coefficients $\{a_m\}$, $a_0 = 0$, $a_1 \neq \pm a_{-1}$, and $\widehat{a} \neq 0$ on the dual group of $\mathbf{Q}^\circ$, then T_f and S_f are bounded, with bounded inverses $S_g T_z^{-1}$ and S_g respectively, for some continuous function g on Π with summable Fourier coefficients.

Other groups. There is an obvious extension of the Poorman's transform to general locally compact abelian groups, and indeed to $\mathbf{Z}^\circ$ modules. The norms can behave quite differently, however. For instance, T_f is bounded on the space $L^2(\mathbf{R}^N)$, for f the piecewise constant function considered in the example above.

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A completeness theorem for open maps

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A formal concept of open map in a Grothendieck topos is described by specifying closure conditions on the class of all open maps. For any topos $\mathcal{E}$ the topos $\mathcal{E}^2$ of arrows has a canonical class of open maps (the quasi-pullback squares). This canonical class in addition satisfies an axiom of replacement. We prove that any class of open maps in any topos $\mathcal{F}$ which satisfies this extra replacement axiom is obtained by transferring the canonical class of $\mathcal{E}^2$ along a geometric morphism $\mathcal{E}^2 \longrightarrow \mathcal{F}$. A similar result is true for pretoposes and for exact categories. As a consequence, we obtain corresponding results for classes of etale maps, thereby improving a result of E. Dubuc (cf. Dubuc (199?)).

Axioms for open maps. A class R of arrows in a topos is called a class of open maps if it satisfies the following axioms A1-6.

A1 Any isomorphism belongs to R, and R is closed under composition;

A2 ("stability") The pullback of an arrow in R along any other arrow again belongs to R;

A3 ("descent") An arrow belongs to R if its pullback along some epimorphism does;

A4 For any set I, the unique map $\Sigma_{i \in I} 1 \longrightarrow 1$ belongs to R;

A5 If $Y_i \longrightarrow X_i$ ($i \in I$) is a family of arrows in R, then their sum $\Sigma Y_i \longrightarrow \Sigma X_i$ belongs to R;

A6 ("quotients") If $f = g \circ p$ where p is epi and f belongs to R, then g also belongs to R;

A class R of arrows in a topos is called a class of etale maps if it satisfies the axioms A1-5 above, and the following two axioms A7, A8.

A7 If an arrow $f : Y \longrightarrow X$ belongs to R then so does its diagonal $\Delta f : Y \longrightarrow Y \times_X Y$;

A8 If $f = g \circ p$ as in

$$\begin{array}{ccc} Z & \xrightarrow{p} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

where p is epi and both f and Δg belong to R, then g also belongs to R.

A class of maps R is said to satisfy the replacement axiom

(A9) if for any maps $Y \xrightarrow{p} X \xrightarrow{f} I$ with p epi and $f \in R$, there exists a quasi-pullback square of the form

$$\begin{array}{ccccc} Z & \longrightarrow & Y & \twoheadrightarrow & X \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f} \\ J & & \xrightarrow{\quad \alpha \quad} & & I \end{array}$$

where α is epi and $g \in R$. (Recall that a square is called quasi-pullback if the evident arrow from its initial corner to the pullback is epi.)

Examples. (a) For any topos $\mathcal{E}$, the quasi-pullback squares form a class of open maps in the arrow category $\mathcal{E}^2$, and the pullback squares form a class of etale maps; both classes satisfy replacement.

(b) Say that an object Y in a topos $\mathcal{E}$ has path-lifting if the geometric morphism

$$(\mathcal{E}/Y)^{\mathrm{I}} \longrightarrow (\mathcal{E}/Y)\times_{\mathcal{E}}\mathcal{E}^{\mathrm{I}}$$

(which is always a local homeomorphism) is surjective. Similarly, say that Y has unique path-lifting if this geometric morphism is bijective. The class of arrows $Y \longrightarrow X$ such that Y has (unique) path-lifting as an object of $\mathcal{E}/X$ is a class of open maps (resp. etale maps). (Notice that an object Y in a presheaf topos $\mathcal{E} = \hat{\mathbb{C}}$ has path-lifting if Y transforms the arrows of $\mathbb{C}$ into epis, and unique path-lifting if it transforms them into isos. In particular, example (a) is a special case of this example. Moreover, when $\mathcal{E}$ is locally simply connected the objects Y having unique path lifting are the locally constant sheaves). In particular, there is a canonical class of open maps in the topos Sh(I) of sheaves over the unit interval.

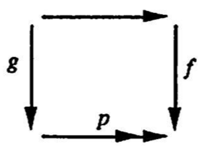
(c) The definition of path-lifting given above can be adapted to the case of a topos $\mathcal{E}$ over a base topos $\mathcal{S}$ by using the appropriate path space. The resulting class of open maps in $\mathcal{E}$ contains the maps $p^*(u)$ where $p\colon \mathcal{E} \longrightarrow \mathcal{S}$ is the structure maps. In particular, there is a canonical class of open maps in the topos $\mathcal{E} \times I$ as a topos over $\mathcal{E}$.

(d) Any set of maps in a topos generates a smallest class of open maps (or of etale maps). In particular, there is always a smallest class of open maps and one of etale maps. It is often the case that the smallest class of etale maps coincides with the class of maps having unique path-lifting, as before.

(e) Familiar notions of open (or flat) maps and etale maps from topology and algebraic geometry yield classes of open and etale maps in the corresponding gros toposes. In fact, if $\mathbb{C}$ is a site with pullbacks and R is a class of "open" maps in $\mathbb{C}$ stable under composition and under pullback, and covers in $\mathbb{C}$ by maps in R, then the generated class of open maps between sheaves can be explicitly described, and satisfies the additional replacement axiom.

Remarks. (a) Any class of open maps gives rise to a class of etale maps, namely the class of open maps $Y \longrightarrow X$ with open diagonal $Y \longrightarrow Y \times_X Y$.

(b) Conversely, a class R of etale maps satisfying replacement is contained in a smallest class of open maps O, namely those maps f which fit into a quasi-pullback square



with g etale. Then R is the class of etale maps given by O as in (a).

(c) In the particular case of an arrow-topos $\mathcal{E}^2$, the canonical classes of open maps and etale map are obtained from each other in this way.

If $f : \mathfrak{F} \longrightarrow \mathcal{E}$ is a geometric morphism and O is a class of open maps in $\mathfrak{F}$ then there is an <u>induced</u> class of open maps in $\mathcal{E}$, namely those maps whose inverse image under f is open in $\mathfrak{F}$.

Theorem. *Let O a class of open maps in a topos $\mathcal{E}$ satisfying replacement. Then there exists a topos $\mathfrak{F}$ and a map f: $\mathfrak{F}^2 \longrightarrow \mathcal{E}$ such that O is induced from the canonical class in $\mathfrak{F}^2$.*

If $\mathcal{E}$ is a separable topos and the class O is countably generated then $\mathfrak{F}$ can be chosen to be separable as well, from which one concludes that $\mathfrak{F}$ can be taken to be a coproduct Sets/I of copies of the topos of sets.

Remarks. (a) It follows from the theorem that a similar result is true with etale maps instead of open maps. In the special case where the class of etale maps is generated from the site as in example (e) above, this result was announced by E. Dubuc [1].

(b) It should be possible to obtain a similar result (with weaker assumptions) if we replace the topos $\mathfrak{F}^2$ by the topos $\mathfrak{F} \times I$ equipped with its canonical class of open maps (example (b)).

We also state the following consequence of the preceding theorem:

Corollary. *For a topos $\mathcal{E}$ and a class O as in the theorem, the full subcategory of $\mathcal{E}$ consisting of etale objects E (i.e., objects such that $E \longrightarrow 1$ and $\Delta : E \longrightarrow E \times E$ are open) is a topos.*

There should be an axiomatization of proper maps and compact maps, similar to the one for open maps and etale maps given above. However, it seems that the closure properties of proper and compact maps are not yet fully understood. A possible axiomatization uses the axioms A1-8 above, but with finite sums only in A4, together with an appropriate form of the Heine-Borel property as a replacement axiom. (Perhaps such an axiomatization should take place in the context of an already given class of open maps.)

If for a class of proper maps one takes as axioms A1-3, finite A4, A5, A6 and replacement as in A9 above, then a completeness theorem can be proved for pretopos, similar to the theorem above. Moreover with these axioms there is in any topos a smallest class of proper maps, namely the maps with Kuratowski finite fibers.

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SUR LES MONADES DE DEGRÉS SUPÉRIEURS

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Résumé. On montre que l'objet $\underline{C}^T$ des T-algèbres correspondant à un n-monade T (construction fondamentale de degré supérieur $n \geq 2$ dans [M]) sur $\underline{C}$ dans une 2-catégorie quelconque $\mathbb{C}$ est toujours isomorphe "au-dessus de $\underline{C}$" à celui des $\overline{T}$ - algèbres pour un "monade à gauche" (voir ci-bas) $\overline{T}$ sur $\underline{C}$ ce qui permet d'obtenir des conditions nécessaires et suffisantes pour que $\overline{T}$ soit un monade en utilisant un résultat essentiellement folklorique sur ces derniers: il suffit de généraliser l'isomorphisme entre la catégorie des M-ensembles, M un demi-groupe avec élément neutre à gauche e (ou monoïde à gauche) et celle des Me-ensembles.

Le présent article est un résumé de [B] où on trouvera toutes les démonstrations et des résultats complémentaires.

1. Monades de degrés supérieurs

Définitions. Un n-monade, $n \geq 2$, dans $\mathbb{C}$ est un triplet ordonné $\mathbf{T} = (T, \mu, \eta)$ où $T : \underline{C} \longrightarrow \underline{C}$, $\mu : T^n \longrightarrow T$, $\eta : I_{\underline{C}} \longrightarrow T^{n-1}$, vérifient

(1.1) $\mu . \eta T = \iota_T$ (1.2) $\mu . T\eta = \iota_T$ (1.3) $\mu . \mu T^{n-1} = \mu . T^{n-1} \mu$

où $I_{\underline{C}}$ est l'identité sur $\underline{C}$ et ι_T celle sur T. Si $n = 2$ on a affaire à un monade; si on oublie (1.2) on obtient un n-monade à gauche (n-m.a.g) et un monade à gauche (m.a.g) lorsque $n = 2$.

À titre d'exemple, tout monade T itéré n fois donne un n-monade $T^{(n)} = (T, \mu^{(n)}, \eta^{(n)})$ où $\mu^{(n)} = \mu . \mu T . \ldots . \mu T^{n-2}$ et $\eta^{(n)} = \eta T^{n-2} . \ldots . \eta T . \eta$ avec $T^o = I_{\underline{C}}$. De même, tout monoïde à gauche M engendre un m.a.g $T = M \times -$ sur Ens comme pour les monades.

Lemme. Soit $\mathbf{T} = (T, \mu, \eta)$ un n-monade (resp. à gauche) sur $\underline{C}$ dans $\mathbb{C}$ et $\overline{\mathbf{T}} = (\overline{T}, \overline{\mu}, \overline{\eta}) = (T^{n-1}, \mu T^{n-2}, \eta)$ alors $\overline{\mathbf{T}}$ est un m.a.g sur $\underline{C}$ et $\overline{\overline{\mathbf{T}}} = \overline{\mathbf{T}}$.

$\overline{\mathbf{T}}$ n'est en général pas un monade. En effet, si $\mathbf{T}$ est un monade et $\mathbf{T}' = \mathbf{T}^{(n)}$, alors $\overline{\mathbf{T}}'$ est un monade ssi $\mu^{(n-1)} : T^{n-1} \longrightarrow T$ est un isomorphisme. Si par ailleurs $\mathbf{T}$ est un monade (à gauche) alors évidemment $\overline{\mathbf{T}} = \mathbf{T}$.

Définition (i). Soit $\mathbf{T}$ et $\mathbf{T}'$ des m.a.g.(resp. monades) sur $\underline{C}$ et $\underline{C}'$ respectivement dans $\mathbb{C}$. Un morphisme de $\mathbf{T}'$ à $\mathbf{T}$ est un triplet ordonné $(\mathbf{T}', (X, \phi), \mathbf{T})$, noté $(X, \phi) : \mathbf{T}' \longrightarrow \mathbf{T}$, où $X : \underline{C}' \longrightarrow \underline{C}$ et $\phi : TX \longrightarrow XT'$ vérifient

(2.1) $\phi . \eta X = X\eta'$ (2.2) $\phi . \mu X = X\mu' . \phi T' . T\phi$ (2.3) $\phi . \nu X = \phi$

avec $\nu = \mu . T\eta$ (cette troisième condition, triviale pour les monades, sera analysée plus loin)

(ii) Soit $\mathbf{T}$ un n-monade (resp. à gauche) et $\mathbf{T}'$ un n'-monade (resp. à gauche) sur $\underline{C}$ et $\underline{C}'$ respectivement, dans $\mathbb{C}$. Un morphisme de $\mathbf{T}'$ à $\mathbf{T}$ est un triplet ordonné $(\mathbf{T}', (X, \phi), \mathbf{T})$ où $(X, \phi) : \overline{\mathbf{T}}' \longrightarrow \overline{\mathbf{T}}$ est un morphisme de m.a.g . (On retrouve donc la définition de [M] pour les n-monades dans CAT.)

Ces monades généralisés et leurs morphismes sont les objets et 1-cellules de 2-catégories notées $\mathbb{M}_2$ (resp. $\mathbb{M}_{-2}$) pour les monades (resp. à gauche) et $\mathbb{M}_\omega$ (resp. $\mathbb{M}_{-\omega}$) pour les monades (resp. à gauche) de degrés supérieurs $n \geq 2$, avec composition "habituelle" des 1-cellules définie comme suit dans les quatre cas: soit $(X, \phi) : \mathbf{T}' \longrightarrow \mathbf{T}$ et $(X', \phi') :$

$T'' \longrightarrow T'$ alors $(X, \phi) . (X', \phi') = (XX', X\phi' . \phi X') : T'' \longrightarrow T$. Il s'ensuit que le morphisme identité sera $(I, \nu) : T \longrightarrow T$ avec $\nu = \bar{\mu} . \bar{T}\bar{\eta}$, d'où $\nu^2 = \nu$. Les 2-cellules $\sigma : (X, \phi) \longrightarrow (Y, \psi)$ entre 1-cellules de T' à T sont dans tous les cas les 2-cellules $\sigma : X \longrightarrow Y$ de $\mathbb{C}$ vérifiant $\sigma\, \bar{T}' . \phi = \psi . \bar{T}\, \sigma$ avec compositions celles de $\mathbb{C}$. Il suffit en fait de faire les vérifications pour $\mathbb{M}_{-2}$ vu les définitions de morphismes dans les autres cas et on notera $\mathbb{J}^i_j : \mathbb{M}_i \longrightarrow \mathbb{M}_j$ les 2-inclusions pleines évidentes, soit $(i, j) = (2, -2)$, $(2, \omega)$ $(-2, -\omega)$, $(\omega, -\omega)$,... Si T est un n-m.a.g alors $(I, \nu) : T \longrightarrow \bar{T}$, où $\nu = \bar{\mu} . \bar{T}\bar{\eta}$, est bien un isomorphisme dans $\mathbb{M}_{-\omega}$ (différent de l'identité $(I, \nu) : T \longrightarrow T$ de $\mathbb{M}_{-\omega}$) d'où on obtient une 2-équivalence $\mathbb{P} : \mathbb{M}_{-\omega} \longrightarrow \mathbb{M}_{-2}$ avec $\mathbb{J}^{-2}_{-\omega} . \mathbb{P}$ (resp. $\mathbb{P} . \mathbb{J}^{-2}_{-\omega}$) 2-naturellement équivalent (resp. égal) à l'identité. Le 2-foncteur $\mathbb{D}_j : \mathbb{C} \longrightarrow \mathbb{M}_j$, $j \in \{-\omega, -2, 2, \omega\}$ qui à $\underline{C}$ associe le monade identité sur $\underline{C}$, jouera un rôle important dans la suite.

2. La condition (2.3)

Dans l'interprétation ensembliste en terme de monoïdes à gauche les morphismes de m.a.g. deviennent des homomorphismes de demi-groupe (2.1) préservant l'élément neutre à gauche (2.2) dont l'image est un sous-monoïde du codomaine (2.3). Ce qui se généralise à $\mathbb{C}$ à condition de supposer que pour $(I_{\underline{C}} . \phi) : T' \longrightarrow T$ dans $\mathbb{M}_{-2}$, $\phi : T \longrightarrow T'$ possède une factorisation rétraction ϕ_2 (avec $\phi_2 . \phi_2' = \iota_{T''}$) suivie d'une section ϕ_1 (avec $\phi_1' . \phi_1 = \iota_{T''}$) par une 1-cellule quelconque T'' sur $\underline{C}$. En effet, ceci entraîne que $(I, \phi) = (I, \phi_1) . (I, \phi_2)$ est une décomposition épi-mono dans $\mathbb{M}_{-2}$ par le monade T'' où (nécessairement par (2.1), (2.2) et la condition imposée à ϕ) $\mu'' = \phi_1' . \mu' . (\phi_1 * \phi_1)$ et $\eta'' = \phi_2 . \eta$.

On retrouve ainsi le cas particulier connu d'un m.a.g. T sur $\underline{C}$ avec $\nu = \mu . T\eta$: $T \longrightarrow T$ scindé, i.e. $\nu = \nu_1 . \nu_2$ et $\nu_2 . \nu_1 = \iota_{T''}$: T'' est alors un monade sur $\underline{C}$ avec $\mu'' = \nu_2 . \mu . (\nu_1 * \nu_1)$, $\eta'' = \nu_2 . \eta$ et $(I, \nu_2) : T'' \longrightarrow T$ un isomorphisme de

$\mathbb{M}_{-2}$. Si réciproquement il existe un isomorphisme (I, ϕ_1) du mag T à un monade T" dans $\mathbb{M}_{-2}$ alors ν est évidemment scindé.

3. Adjoints de degrés supérieurs.

Définition. Une n-adjonction, $n \geq 2$, (morphisme adjoint de degré n dans [M]), est un triplet ordonné $(\eta, \varepsilon, F, U)$ où $U : \underline{B} \longrightarrow \underline{C}$, $F : \underline{C} \longrightarrow \underline{B}$, $\eta : I_{\underline{C}} \longrightarrow (U\,F)^{n-1}$ et $\varepsilon : (FU)^{n-1} \longrightarrow I_{\underline{B}}$ vérifient $U\varepsilon . \eta U = \iota_U$ et $\varepsilon F . F\eta = \iota_F$. Si $n = 2$ on a une adjonction, si on n'a que la première identité ce sera une n-quasi adjonction (à gauche) et une quasi adjonction (à gauche) quand $n = 2$.

Ainsi, comme pour les monades, toute adjonction itérée n fois sera une n-adjonction et ces adjoints généralisés forment à leur tour des 2-catégories $\mathbb{A}_j$, $j \in \{-\omega, -2, 2, \omega\}$ avec 1-cellules définies comme à l'habitude, à savoir pour chaque cas, $(H, K) : (\eta', \varepsilon', F', U') \longrightarrow (\eta, \varepsilon, F, U)$ où $H : \underline{C}' \longrightarrow \underline{C}$ et $K : \underline{B}' \longrightarrow B$ vérifient $UK = H\,U'$ et 2-cellules $(\alpha, \beta) : (H, K) \longrightarrow (H', K')$ où $\alpha : H \longrightarrow H'$ et $\beta : K \longrightarrow K'$ satisfont $U\beta = \alpha U'$. On a alors les 2-inclusions évidentes $\mathbb{K}_j^i$ et une 2-équivalence $\mathbb{Q} : \mathbb{A}_{-\omega} \longrightarrow \mathbb{A}_{-2}$ qui envoie $(\eta, \varepsilon, F, U)$ dans $(\eta, \varepsilon, \overline{F}, U)$ avec $\overline{F} = FT^{n-2}$ et ne modifie que les domaines et codomaine de (H, K) et (α, β). On a en fait $\mathbb{K}_{-\omega}^{-2} . \mathbb{Q}$ (resp. $\mathbb{Q} . \mathbb{K}_{-\omega}^{-2}$) 2-équivalent (resp. égal) à l'identité. On obtient aussi pour chaque j le 2-foncteur $\mathbb{S}_j : \mathbb{A}_j \longrightarrow \mathbb{M}_j$ qui à $(\eta, \varepsilon, F, U)$ associe $T = (UF, U\varepsilon F, \eta)$, à (H, K) le morphisme $(H, \tau) : T' \longrightarrow T$ où $\tau = U\varepsilon K \overline{F}' . \overline{T} H \eta'$ et de (α, β) ne retient que α . En particulier $\mathbb{S}_j (I_{\underline{C}}, I_{\underline{B}}) = (I_{\underline{C}}, \nu)$ et c'est la raison pour laquelle il a fallu introduire la condition (2.3) dans la définition de morphisme.

4. Les T-algèbres

Théorème: (Le symbole $\dashv$ dénote ci-bas des 2-adjonctions.) Soit $\mathbb{C}$ une 2-catégorie quelconque.

1) Si $(\exists\, \mathbb{L}_{-2})\ (\mathbb{D}_{-2} \dashv \mathbb{L}_{-2})$ alors $(\exists\, \mathbb{L}_j)\ (\mathbb{D}_j \dashv \mathbb{L}_j)$ pour $j = 2, -\omega, \omega$

2) Les énoncés suivants sont équivalents pour chaque $j \in \{-\omega, -2, 2, \omega\}$

(i) $(\exists\, \mathbb{D}_j)\ \mathbb{D}_j \dashv \mathbb{L}_j$ (ii) $(\exists\, \mathbb{R}_j)\ \mathbb{S}_j \dashv \mathbb{R}_j$

(iii) $T \in \mathbb{M}_j$ possède un générateur maximal $(\eta^T, \varepsilon^T, F^T, U^T) \in \mathbb{A}_j$, i.e. $\mathbb{S}_j\ (\eta^T, \varepsilon^T, F^T, U^T) = T$ et si $\sigma : (H, \tau) \longrightarrow (H', \tau') : T' \longrightarrow T$ dans $\mathbb{M}_j$ avec $\mathbb{S}_j\ (\eta', \varepsilon', F', U') = T'$ alors $\exists!\ \tilde{H}$ tel que $U^T \tilde{H} = HU'$ et $U^T \varepsilon^T \tilde{H} = H U' \varepsilon' . \tau U'$ (resp. $\tau = U^T \varepsilon^T \tilde{H}\ \bar{F}' . \bar{T} H \eta'$), idem pour $\tilde{H}'$, et $\exists!\ (\tilde{\sigma} : \tilde{H}' \longrightarrow \tilde{H})$ tel que $U^T \tilde{\sigma} = \sigma U'$.

La démonstration de ce théorème pour les m.a.g. est essentiellement la même que pour les monades et entraîne immédiatement les autres versions (voir [B]). En fait, on pose $\underline{C}^T = \mathbb{L}_{-2}(T)$, l'objet des T-algèbres et $(U^T, \phi^T) : I_{\underline{C}^T} \longrightarrow T$ la counité de $\mathbb{D}_{-2} \dashv \mathbb{L}_{-2}$ évaluée en T, ce qui entraîne $\mathbb{D}_2 \dashv \mathbb{L}_2$ et $\mathbb{D}_{-\omega} \dashv \mathbb{L}_{-\omega}$ en prenant $\mathbb{L}_{-\omega} = \mathbb{L}_{-2} . \mathbb{P}$, d'où $\mathbb{D}_\omega \dashv \mathbb{L}_\omega$. On a donc dans ces deux derniers cas $\underline{C}^T = \underline{C}^{\bar{T}}$ et $(U^T, \phi^T) = (U^{\bar{T}}, \phi^{\bar{T}})$. On posera évidemment $\mathbb{R}_j(T) = (\eta^T, \varepsilon^T, F^T, U^T)$ et $\mathbb{R}_j\ (H, \tau) = (H, \tilde{H})$ etc. (voir [B]). Pour les m.a.g. dans CAT, $\underline{C}^T$ sera définie comme pour les monades, ce qui permet de retrouver la construction de [M] pour n-monades, i.e. $\underline{C}^T = \underline{C}^{\bar{T}}$ et $\tilde{H}(B') = (HU'(B'), (H' U' \varepsilon' . \tau U')_B)$ etc.

5. Réduction aux monades

En utilisant $\mathbb{S}_\omega$ et $\mathbb{R}_\omega$, si T est un n-monade sur $\underline{C}$ dans $\mathbb{C}$ alors il existe un monade T' sur $\underline{C}$ dans $\mathbb{C}$ et un isomorphisme $(I, K) : (\eta^T, \varepsilon^T, F^T, U^T) \longrightarrow (\eta^{T'}, \varepsilon^{T'}, U^{T'}, F^{T'})$ dans $\mathbb{A}_\omega$, i.e. un isomorphisme $K : \underline{C}^T \longrightarrow \underline{C}^{T'}$ de $\mathbb{C}$ au-dessus de $\underline{C}$, ssi il existe un isomorphisme $(I, \tau) : T \longrightarrow T'$ dans $\mathbb{M}_\omega$, d'où dans $\mathbb{M}_{-\omega}$, mais on a toujours un isomorphisme $(I, \nu) : \bar{T} \longrightarrow T$ dans $\mathbb{M}_{-\omega}$ où $(I, \tau)\ (I, \nu) = (I, \tau)$:

$\overline{T} \longrightarrow T'$ et ce dernier sera un isomorphisme ssi c'en est un dans $\mathbb{M}_{-2}$ i.e. ssi $\nu = \overline{\mu} \,.\, \overline{T}\,\overline{\eta}$ est scindé par ce qu'on a vu précédemment. En fait la condition demeure la même si on remplace "isomorphe" par équivalent, i.e. une adjonction avec unité et counité inversibles et qu'on suppose T' sur $\underline{C}'$ quelconque dans $\mathbb{C}$. Finalement si T est un monade sur $\underline{C}$ dans $\mathbb{C}$ et $T' = T^{(n)}$, $n \geq 3$, alors $\underline{C}^{T'}$ est isomorphe à $\underline{C}^{T}$ au-dessus de $\underline{C}$ car $\nu = \mu^{(n)} T^{n-2} \,.\, T^{n-1} \eta^{(n)} : T^{n-1} \longrightarrow T^{n-1}$ est scindé, avec $\nu_1 = T\eta^{(n-1)}$ et $\nu_2 = \mu^{(n-1)} : T^{n-1} \longrightarrow T$ et en plus $T'' = (T, \nu_2 \,.\, \mu^{(n)} T^{n-2} \,.\, (\nu_1 * \nu_1), \nu_2 \,.\, \eta^{(n)}) = T$. On pourrait aussi vérifier directement que $(I, \mu^{(n-1)}) : T \longrightarrow T^{(n)}$ est un isomorphisme de $\mathbb{M}_\infty$ avec inverse $(I, T\eta^{(n-1)})$.

6. Le cas minimal

En passant à $\mathbb{C}^*$ obtenue en dualisant la structure horizontale on obtient les résultats duaux sur le générateur minimal d'un n-monade en considérant les monades à droite (i.e. (1.2) et (1.3)) avec même condition sur $\nu = \mu \,.\, \eta T$ pour obtenir un isomorphisme.

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Articles Index - Volume XII

Paging of Vol. XII

(1) 1-52 (2) 53-74 (3) 75-96
(4) 97-148 (5) 149-208 (6) 209-268