C.R. Math. Rep. Acad. Sci. Canada

Vol. XII, No. 5 **October 1990 octobre**

CONTENTS

FERNIQUE TYPE INEQUALITIES FOR NOT NECESSARILY GAUSSIAN PROCESSES

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Abstract: We present inequalities and moduli of continuity results for Banach space valued stochastic processes which are not assumed to be Gaussian.

1. Introduction. When studying sample path continuity of Gaussian and sub-Gaussian processes, inequalities of Fernique play an important role (cf., e.g., Fernique (1975), Jain and Marcus (1978)). For the sake of quoting one of them from Fernique (1975), let X be a separable Gaussian process on $[0,1]$. With the covariance function r of X one associates the $\mathbf{R}^+$ valued function ϕ on $[0,1]$, defined by

$$\phi(h) = \sup_{|s-t|\le h} (r(s,s) - 2r(s,t) + r(t,t))^{1/2}.$$

THEOREM A. *Assume that the integral* $\int_0^\infty \phi(e^{-x^2})dx$ *is finite. Then the sample paths of* X *are almost surely continuous, and with* $\theta \ge 2$, $x \ge (1+4\log 2)^{1/2}$ *we have also*

$$(1.1)\quad P\left\{ \sup_{a\le t\le b} |X(t)| \ge x\left(\sup_{(s,t)\in[0,1]^2} r^{1/2}(s,t) + (2+2^{1/2}) \int_1^\infty \phi\left(\frac{b-a}{2}\theta^{-u^2}\right)du \right)\right\} \le (5/2)\theta^2 \int_x^\infty e^{-u^2/2}du.$$

Our aim here is to generalize the inequality of (1.1) to Banach space valued stochastic processes for the sake of studying their moduli of continuity properties.

We will also apply our results to studying the continuity and moduli of continuity sample path properties of ℓ^p-valued Ornstein-Uhlenbeck processes for $1 \le p \le 2$.

Towards this end, let $\{Y(t),\ -\infty < t < \infty\} = \{X_k(t),\ -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, i.e., $X_k(\cdot)$ is a stationary, mean zero Gaussian process with

$$(1.2)\qquad EX_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|),\quad k = 1,2,\ldots,$$

where $\gamma_k \geq 0$, $\lambda_k > 0$.

A special case of a recent result of Fernique (1990) reads as follows: *for each* $x \in \mathbf{R}^+$ *let* $\lambda(x) = \sup\{\lambda_k : \gamma_k > \lambda_k x\}$. *Then* $Y(\cdot) \in \ell^p$, $2 \leq p < \infty$, *is a.s. continuous if and only if*

$$\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2} < \infty \quad \text{and} \quad \int_0^\infty (\log^+ \lambda(x))^{p/2} x^{p/2-1} dx < \infty. \tag{1.3}$$

Schmuland (1990) proved: *if* $\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2}(\log(\lambda_k \vee e))^{p/2} < \infty$, *for* $1 \leq p < 2$ *and* $\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2}(\log(\gamma_k \vee e))^{p/2} < \infty$ *for* $2 \leq p < \infty$, *then* $Y(\cdot) \in \ell^p$, $1 \leq p < \infty$, *has a.s. continuous sample paths.* We note that for proving this result, Schmuland does not assume the independence of the Ornstein-Uhlenbeck processes $X_k(\cdot)$.

2. Inequalities. The following results are extensions of those in Sections 2 and 3 in Csáki and Csörgő (1990). Proofs, as well as further details, will appear in a forthcoming paper by the authors.

THEOREM 2.1. *Let* $\mathcal{B}$ *be a separable Banach space with norm* $\|\ \|$ *and let* $\{\Gamma(t), -\infty < t < \infty\}$ *be a stochastic process with values in* $\mathcal{B}$. *Let* P *be the probability measure generated by* $\Gamma(\cdot)$. *Assume that* $\Gamma(\cdot)$ *is* P*-almost surely continuous with respect to* $\|\ \|$ *and that for* $|t| \leq t_0 \leq \infty$, $x \geq 1$ *and* $0 < h \leq h_0 \leq \infty$ *there exists a positive monotone non-decreasing function* $\phi(h)$ *such that*

$$P\{\|\Gamma(t+h) - \Gamma(t)\| \geq x\phi(h)\} \leq Kx^\alpha \exp(-\gamma x^\beta) \tag{2.1}$$

with some $K, \gamma, \beta > 0$ *and* $-\infty < \alpha < \infty$. *Then with* $R = 2^{2^r}$, r *a positive integer, we have*

$$\begin{aligned} P\Big\{ \sup_{0 \leq t \leq T-a} \sup_{0 < s \leq a} &\|\Gamma(t+s) - \Gamma(t)\| \\ &\geq x\Big(\phi(a + a/R) + (1 \vee 2^{1/\beta - 1})\beta\big(1 + (\tfrac{4}{\gamma})^{1/\beta}\big)\Big) \int_{2^{r/\beta}}^{\infty} \phi(ae^{-z^\beta})dz\Big\} \\ &\leq \{(TR/a + 1)(R+1)K + (2TKC_r/a)\}x^\alpha \exp(-\gamma x^\beta), \end{aligned} \tag{2.2}$$

for any $0 < T < t_0$, $0 < a \leq T$, $x \geq 1$ *and* $a + a/R < h_0$, *where* $C_r = C_r(\alpha, \beta, \gamma) = \sum_{j=0}^{\infty}\left(\frac{2^{r+j+1}}{\gamma} + 1\right)^{\alpha/\beta}(2/e)^{2^{r+j+1}}$, *provided* $\int_1^\infty \phi(e^{-z^\beta})dz$ *converges.*

On letting $a = T$ in (2.2) we obtain the following generalization of Theorem A.

COROLLARY 2.1. *Under the conditions of Theorem 2.1 and $y > 0$ we have*

$$P\left\{\sup_{0\le t\le T}\|\Gamma(t)\| \ge x(\phi(T+T/R) + (1\vee 2^{1/\beta-1})\beta(1+(\frac{4}{\gamma})^{1/\beta}))\int_{2^{r/\beta}}^{\infty}\phi(Te^{-z^{\beta}})dz + y\right\}$$

$$\text{(2.3)}\qquad \le \{(R+1)^2K + 2KC_r\}x^{\alpha}\exp(-\gamma x^{\beta}) + P\{\|\Gamma(0)\| > y\}.$$

The statement of (2.3) follows from Theorem 2.1 by noting that we have $\|\Gamma(t)\| \le \|\Gamma(t)-\Gamma(0)\| + \|\Gamma(0)\|$.

A weaker version of (2.3), requiring the convergence of $\int_0^\infty \phi(e^{-z^\beta})z^\beta dz$, was given by Kalinauskaité (1986).

As an application of Theorem 2.1 and Corollary 2.1 respectively, here we present only the following general estimation for the moduli of continuity of $\Gamma(\cdot)$.

THEOREM 2.2. *Let $\{\Gamma(t),\ -\infty < t < \infty\}$ and $\phi(\cdot)$ be as in Theorem 2.1, and assume also that $\phi(\cdot)$ is a regularly varying function at zero with a positive exponent. Then we have*

$$\text{(2.4)}\qquad \limsup_{h\downarrow 0}\sup_{0\le t\le 1-h}\sup_{0\le s\le h}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\phi(h)((1/\gamma)\log(1/h))^{1/\beta}} \le 1 \quad \text{a.s.}$$

and, for any t fixed,

$$\text{(2.5)}\qquad \limsup_{h\downarrow 0}\sup_{0\le s\le h}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\phi(h)((1/\gamma)\log\log(1/h))^{1/\beta}} \le 1 \quad \text{a.s.}$$

3. ℓ^p-valued Ornstein-Uhlenbeck processes. Let $Y(\cdot)$ be the infinite dimensional Ornstein-Uhlenbeck process introduced in Section 1.

We introduce the following notations:

$$\text{(3.1)}\qquad \sigma_k(h) = \{E(X_k(t+h)-X_k(t))^2\}^{1/2} = \{(2\gamma_k/\lambda_k)(1-\exp(-\lambda_k h))\}^{1/2},$$

$$\text{(3.2)}\qquad \sigma_{(p)}(h) = \sum_{k=1}^{\infty}\sigma_k^p(h) = \sum_{k=1}^{\infty}\{(2\gamma_k/\lambda_k)(1-\exp(-\lambda_k h))\}^{p/2}.$$

We note that $E\|Y(t)\|^p_{\ell^p} = \sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2}E|Z|^p$, where Z is a standard normal random variable. Hence $Y(t) \in \ell^p$, $1 \le p < \infty$, almost surely for every fixed t if and only if

(3.3) $$\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2} < \infty$$

First one shows that for $1 \le p < 2$ we have

(3.4) $$P\{\|Y(t+h) - Y(t)\|_{\ell^p} \ge x\sigma_{(p)}^{1/p}(h)\} \le K\exp(-\gamma x^p)$$

for all $x, \gamma > 0$ and some positive constant K.

Also, for $p = 2$ it was shown in Csáki-Csörgő (1990) that we have

(3.5) $$P\{\|Y(t+h) - Y(t)\|_{\ell^2} \ge x\sigma_{(2)}^{1/2}(h)\} \le K\exp(-(1-\varepsilon)x^2/2)$$

for all $x, \varepsilon > 0$ and some positive constant K.

Consequently, by (2.3), (3.4) and (3.5), with $1 \le p \le 2$ we obtain

(3.6) $$\begin{aligned} P\Big\{ \sup_{0\le t\le T} \|Y(t)\|_{\ell^p} &\ge x\big(\sigma_{(p)}^{1/p}(T + T/R) + (1 \vee 2^{1/p-1})p(1 + (\frac{4}{\gamma})^{1/p})\big) \\ &\quad \times \int_{2^{r/p}}^{\infty} \sigma_{(p)}^{1/p}(Te^{-z^p})dz + y\Big\} \\ &\le \{(R+1)^2K + 2KC_r\}\exp(-\gamma x^p) + P\{\|Y(0)\| > y\} \end{aligned}$$

for all $x \ge 1$, $y > 0$, where $\gamma > 0$ is arbitrary for $1 \le p < 2$, and for $p = 2$ we have $0 < \gamma < 1/2$ (cf. (3.5)).

As a result of (3.6) we can obtain the following sufficiency condition for the a.s. continuity of $Y(\cdot) \in \ell^p$, $1 \le p \le 2$.

THEOREM 3.1. *The convergence of the integral* $\int_1^{\infty} \sigma_{(p)}^{1/p}(e^{-z^p})dz$ *is a sufficient condition for the almost sure continuity of* $Y(\cdot) \in \ell^p$, $1 \le p \le 2$.

Obviously, the necessary and sufficient condition of Fernique (1990), quoted in Section 1 (cf. also Fernique (1989)), for the almost sure continuity of $Y(\cdot) \in \ell^p$ is superior to our sufficient one of Theorem 3.1. For $1 \le p < 2$ the sufficient condition of the latter is to be compared to that of Schmuland (1990). This we will do in our forthcoming paper.

Another application of (3.6) now combined with Theorem 3.1 yields the following local laws of the iterated logarithm.

THEOREM 3.2. *Assume that for some* $1 \le p \le 2$, $\sigma_{(p)}(h)$ *is a regularly varying function at zero with a positive exponent. Then we have for any fixed* t

$$\limsup_{h\downarrow 0} \sup_{0\le s\le h} \frac{\|Y(t)-Y(t+s)\|_{\ell^2}}{\sigma_{(2)}^{1/2}(h)(2\log\log(1/h))^{1/2}} \le 1 \quad \text{a.s., if} \quad p=2, \tag{3.7}$$

$$\limsup_{h\downarrow 0} \sup_{0\le s\le h} \frac{\|Y(t)-Y(t+s)\|_{\ell^p}}{\sigma_{(p)}^{1/p}(h)(\log\log(1/h))^{1/p}} = 0 \quad \text{a.s., if} \quad 1\le p<2. \tag{3.8}$$

Combining now (2.2), (3.4) and (3.5), with $1 \le p \le 2$ we get

$$\begin{aligned} P\Big\{ &\sup_{0\le t\le T-a} \sup_{0<s\le a} \|Y(t+s)-Y(t)\|_{\ell^p} \\ &\ge x\big(\sigma_{(p)}^{1/p}(a+a/R) + (1\vee 2^{1/p-1})p(1+(\tfrac{4}{\gamma})^{1/p})\big)\int_{2^{r/p}}^{\infty} \sigma_{(p)}^{1/p}(ae^{-z^p})dz\Big\} \\ &\le \{(TR/a+1)(R+1)K + (2TKC_r/a)\}\exp(-\gamma x^p) \end{aligned} \tag{3.9}$$

for all $x \ge 1$, where $\gamma > 0$ is arbitrary for $1 \le p < 2$, and for $p = 2$ we have $0 < \gamma < 1/2$ (cf. (3.5)).

The inequality of (3.9) when combined with Theorem 3.1 implies estimations for the moduli of continuity of $Y(\cdot) \in \ell^p$, $1 \le p \le 2$.

THEOREM 3.3. *Assume that for some* $1 \le p \le 2$, $\sigma_{(p)}(h)$ *is a regularly varying function at zero with a positive exponent. Then we have*

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{\ell^2}}{\sigma_{(2)}^{1/2}(h)(2\log(1/h))^{1/2}} \le 1 \quad \text{a.s., if} \quad p=2, \tag{3.10}$$

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{\ell^p}}{\sigma_{(p)}^{1/p}(h)(\log(1/h))^{1/p}} = 0 \quad \text{a.s., if} \quad 1\le p<2. \tag{3.11}$$

For a sharp version of (3.10) we refer to Csáki and Csörgő (1990, Theorem 4.1). A similar sharp version of (3.7) is also true. We intend to continue studying these, as well as refinements of (3.8) and (3.11), in our forthcoming paper.

ACKNOWLEDGEMENTS. This work was done while Endre Csáki was visiting at Carleton University. This research was financially supported by an NSERC Canada Grant of Miklós Csörgő and by Hungarian National Foundation for Scientific Research, Grant No. 1808.

The authors also wish to thank X. Fernique and B. Schmuland for sending their manuscripts [4] and [7], prior to their publication.

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Received August 3, 1990

THE STOKES PROBLEM AND THE MIXED BOUNDARY CONDITIONS

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Abstract. We consider the steady Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. As boundary conditions we require that the velocity and the normal stress components vanish on Γ_0 and Γ_1 respectively. We prove existence and uniqueness results for this problem.

1. INTRODUCTION. Let Ω be a bounded open set in $\mathbb{R}^2$ with a lipschitz-continuous boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ where Γ_0 is a smooth open subset of Γ and Γ_1 is the interior of $\Gamma - \Gamma_0$. We would like to associate with the Stokes equations:

$$(1.1) \qquad \begin{cases} -\nu\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla.\mathbf{u} = 0 & \text{in } \Omega \end{cases}$$

the mixed boundary conditions:

$$(1.2) \qquad \begin{cases} \mathbf{u}\big|_{\Gamma_0} = 0 \\ \left(\nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - pI\right).\mathbf{n}\big|_{\Gamma_1} = 0 \end{cases}$$

where $\mathbf{u}$ denotes the velocity field, p the pressure, $\mathbf{f}$ the given exterior force, $\nu > 0$ the constant kinematic viscosity and $\mathbf{n}$ denotes the unit outward normal to Γ_1.
The existence and uniqueness of the solution of this problem have been investigated by many authors in the case of the Dirichlet boundary conditions, i.e. when $\Gamma_1 = \emptyset$ [2], [3], [5].

In this paper, we give a proof of these results when $\Gamma_1 \neq \emptyset$.

2. NOTATIONS AND THEOREMS. In order to introduce a mixed weak formulation of (1.1)-(1.2), we define the spaces

$$V(\Omega,\Gamma_0) = \{\mathbf{v} \in (H^1(\Omega))^2 : \mathbf{v}\big|_{\Gamma_0} = 0\}$$

$$H = \left\{(\mathbf{v},q) \in V(\Omega,\Gamma_0) \times L^2(\Omega)\ ;\ L(\mathbf{v},q) = -\nu\Delta\mathbf{v} + \nabla q \in (L^{r'}(\Omega))^2\right\}$$

$$H^{\frac{1}{2}}_{00}(\Gamma_1) = \{v \in L^2(\Gamma_1);\ \exists w \in H^1(\Omega)\ :\ w\big|_{\Gamma_0} = 0\ ,\ w\big|_{\Gamma_1} = v\}$$

where r is such that $H^1(\Omega) \subset L^r(\Omega)$ and $\frac{1}{r'} + \frac{1}{r} = 1$, and we denote by $< .,. >$ and $<< .,. >>$ the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$ on the one end, $H^{\frac{1}{2}}_{00}(\Gamma_1)$ and its dual space $H^{\frac{1}{2}}_{00}(\Gamma_1)'$ on the other end.
We also define the continuous bilinear forms:

$$a(\mathbf{u},\mathbf{v}) = \frac{\nu}{2}\int_\Omega (\nabla\mathbf{u} + \nabla\mathbf{u}^T) : (\nabla\mathbf{v} + \nabla\mathbf{v}^T)\, dx, \quad \forall \mathbf{u},\mathbf{v} \in (H^1(\Omega))^2$$

$$b(\mathbf{v},q) = -\int_\Omega q\nabla.\mathbf{v}\, dx, \quad \forall\mathbf{v} \in (H^1(\Omega))^2, \forall q \in L^2(\Omega)$$

and we provide the Sobolev spaces $L^2(\Omega), H^1(\Omega)$, and $(H^1(\Omega))^2$ with the norms :

$$||q||_{0,\Omega} = \left\{\int_\Omega |\, q\,|^2\, dx\right\}^{\frac{1}{2}}, \forall q \in L^2(\Omega)$$

$$||v||_{1,\Omega} = \left\{||v||^2_{0,\Omega} + ||\nabla v||^2_{0,\Omega}\right\}^{\frac{1}{2}}, \forall v \in H^1(\Omega)$$

$$||\mathbf{v}||_{1,\Omega} = \left\{||v_1||^2_{1,\Omega} + ||v_2||^2_{1,\Omega}\right\}^{\frac{1}{2}}, \forall \mathbf{v} = (v_1,v_2) \in (H^1(\Omega))^2$$

We then have the following result :

Lemma 1.1. *Let* $\mathbf{f}$ *belong to* $(L^{r'}(\Omega))^2$. *Then the two following problems (a) and* *(b) are equivalent:*

(a) *mixed boundary Stokes problem : to find* $(\mathbf{u},p) \in H$ *such that*

$$(1.3) \qquad \begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla.\mathbf{u} = 0 & \text{in } \Omega \\ (\nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - pI).\mathbf{n}\big|_{\Gamma_1} = 0 \end{cases}$$

(b) *Variational mixed boundary Stokes problem : to find* $(\mathbf{u},p) \in V(\Omega,\Gamma_0) \times L^2(\Omega)$ *such that*

$$(1.4) \qquad \begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \langle \mathbf{f},\mathbf{v}\rangle, & \forall \mathbf{v} \in V(\Omega,\Gamma_0) \\ b(\mathbf{u},q) = 0, & \forall q \in L^2(\Omega) \end{cases}$$

PROOF: This lemma follows from the Green formula:

$$(1.5) \qquad \begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) =< L(\mathbf{u},p),\mathbf{v} > + << \gamma_1(\mathbf{u},p),\gamma_0(\mathbf{v}) >> \\ \forall(\mathbf{u},p) \in H, \quad \forall \mathbf{v} \in V(\Omega,\Gamma_0) \end{cases}$$

where

$$\begin{cases} \gamma_1(\mathbf{u},p) = (\nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - pI).\mathbf{n}\big|_{\Gamma_1} \\ \gamma_0(\mathbf{v}) = \mathbf{v}\big|_{\Gamma_1} \end{cases}$$

It is clear that (1.3) implies (1.4). Conversely (1.4) holds $\forall \mathbf{v} \in (H_0^1(\Omega))^2$ since $(H_0^1(\Omega))^2 \subset V(\Omega,\Gamma_0)$, and this implies that $L(\mathbf{u},p) = \mathbf{f}$ in the sense of distributions . Since $\mathbf{f}$ belongs to $(L^{r'}(\Omega))^2$, we deduce that $(\mathbf{u},p) \in H$. Now from (1.4) and the use of the Green formula (1.5) we deduce that:

$$<< \gamma_1(\mathbf{u},p),\gamma_0(\mathbf{v}) >>= 0, \quad \forall \mathbf{v} \in V(\Omega,\Gamma_0)$$

and by the surjectivity of the operator γ_0 [3],[5] this implies (1.3) ∎

In order to solve the variational problem (1.4) we have to prove the following lemma.

Lemma 1.2. *Let a be a given real number.*
Then there exists a unique pair $(\mathbf{u}_1, \lambda) \in V(\Omega, \Gamma_0) \times I\!R$ such that :

$$(1.6) \qquad \begin{cases} \displaystyle\int_\Omega \nabla\mathbf{u}_1 : \nabla\mathbf{v}\, dx = \lambda \int_\Omega \nabla.\mathbf{v}\, dx, \quad \forall \mathbf{v} \in V(\Omega, \Gamma_0) \\ \displaystyle\int_\Omega \nabla.\mathbf{u}_1\, dx = a \end{cases}$$

PROOF: We can view (1.6) as a mixed problem of the form

$$\begin{cases} A(\mathbf{u}_1, \mathbf{v}) + B(\mathbf{v}, \lambda) = 0, \quad \forall \mathbf{v} \in V(\Omega, \Gamma_0) \\ B(\mathbf{u}_1, \mu) = -a\mu, \quad \forall \mu \in I\!R \end{cases}$$

where $A(\mathbf{u}_1, \mathbf{v}) = \int_\Omega \nabla\mathbf{u}_1 : \nabla\mathbf{v}\, dx$ and $B(\mathbf{v}, \lambda) = -\lambda \int_\Omega \nabla.\mathbf{v}\, dx$.
The continuous bilinear forms $A(.,.)$ and $B(.,.)$ satisfy the two following conditions

$$A(\mathbf{v}, \mathbf{v}) \geq C_1 ||\mathbf{v}||^2_{1,\Omega}, \quad \forall \mathbf{v} \in V(\Omega, \Gamma_0)$$

$$\sup_{\mathbf{v}\in V(\Omega,\Gamma_0)} \frac{| B(\mathbf{v}, \mu) |}{||\mathbf{v}||_{1,\Omega}} \geq C_2 \mid \mu \mid, \quad \forall \mu \in I\!R$$

where the positive constants C_1 [4] and C_2 are such that:

$$||\mathbf{v}||_{1,\Omega} \leq C_1 ||\nabla\mathbf{v}||_{0,\Omega} \text{ and } C_2 = \sup_{\mathbf{v}\in V(\Omega,\Gamma_0)} \frac{| \int_\Omega \nabla.\mathbf{v}\, dx |}{||\mathbf{v}||_{1,\Omega}} > 0$$

Therefore, applying the classical results of Brezzi [2], Girault-Raviart [3], we obtain the existence of a unique pair $(\mathbf{u}_1, \lambda)$ solution of (1.6) ∎

As an application, we now prove:

Lemma 1.3. *There exists a constant $C > 0$ such that:*

$$(1.7) \qquad \sup_{\mathbf{v}\in V(\Omega,\Gamma_0)} \frac{| \int_\Omega q\nabla.\mathbf{v}\, dx |}{||\mathbf{v}||_{1,\Omega}} \geq C||q||_{0,\Omega}, \quad \forall q \in L^2(\Omega)$$

PROOF: Since $(H_0^1(\Omega))^2 \subset V(\Omega,\Gamma_0)$ we have

$$\sup_{\mathbf{v}\in V(\Omega,\Gamma_0)} \frac{|\int_\Omega q\nabla.\mathbf{v}\,dx|}{||\mathbf{v}||_{1,\Omega}} \geq \sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{|<\nabla q,\mathbf{v}>|}{||\mathbf{v}||_{1,\Omega}} = ||\nabla q||_{-1,\Omega}$$

Hence

$$\sup_{\mathbf{v}\in V(\Omega,\Gamma_0)} \frac{|\int_\Omega q\nabla.\mathbf{v}\,dx|}{||\mathbf{v}||_{1,\Omega}} \geq ||\nabla q||_{-1,\Omega} \tag{1.8}$$

Now let $a = -\int_\Omega 1\,d\Omega$.

By virtue of lemma1.2, there exists a unique function $\mathbf{u}_1 \in V(\Omega,\Gamma_0)$ such that

$$\int_\Omega (1+\nabla.\mathbf{u}_1)\,dx = 0$$

Therefore, since the divergence operator maps [3] $(H_0^1(\Omega))^2$ onto the space

$$L_0^2(\Omega) = \left\{q \in L^2(\Omega)\ ;\ \int_\Omega q\,dx = 0\right\}$$

there exists $\mathbf{u}_2 \in (H_0^1(\Omega))^2$ such that $\nabla.\mathbf{u}_2 = 1+\nabla.\mathbf{u}_1$. The function $\mathbf{u}_2-\mathbf{u}_1$ is then in $V(\Omega,\Gamma_0)$ and satisfies $\nabla.(\mathbf{u}_2-\mathbf{u}_1) = 1$. Hence

$$\sup_{\mathbf{v}\in V(\Omega,\Gamma_0)} \frac{|\int_\Omega q\nabla.\mathbf{v}\,dx|}{||\mathbf{v}||_{1,\Omega}} \geq \frac{|\int_\Omega q\nabla.(\mathbf{u}_2-\mathbf{u}_1)\,dx|}{||(\mathbf{u}_2-\mathbf{u}_1)||_{1,\Omega}} = \frac{|\int_\Omega q\,dx|}{||(\mathbf{u}_2-\mathbf{u}_1)||_{1,\Omega}} \tag{1.9}$$

and since

$$\left\{||\nabla q||_{-1,\Omega} + |\int_\Omega q\,dx|\right\}$$

is a norm in $L^2(\Omega)$ which is equivalent to the usual norm $||q||_{0,\Omega}$ [5], we deduce the desired result from (1.8) and (1.9) ∎

Let us state the main result of this section.

Theorem. *Let Ω be a bounded subset of $\mathbb{R}^2$, with a Lipschitz continuous boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ and let* $\mathbf{f}$ *be a given function in* $(L^{r'}(\Omega))^2$.
Then there exists one and only one pair of functions $(\mathbf{u}, p) \in V(\Omega, \Gamma_0) \times L^2(\Omega)$ *such that* (1.4) *holds* .

PROOF: The form $a(.,.)$ is $V(\Omega, \Gamma_0)$-elliptic and the form $b(.,.)$ satisfies the inf-sup condition (1.7). Therefore, thanks to the results of [2] , [3] , problem (1.4) has one and only one solution $(\mathbf{u}, p)$ in $V(\Omega, \Gamma_0) \times L^2(\Omega)$ ∎

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Received August 8, 1990

SUR UN PROBLÈME PÉRIODIQUE DE TYPE CARATHÉODORY-NIRENBERG

Par

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0.INTRODUCTION:

Dans cette note, nous nous intéressons à un théorème d'existence d'une solution périodique au sens de Carathéodory pour le problème :

(*) $$y''(t) = f(t,y(t),y'(t)) \qquad \text{p.p. dans } [0,1],$$

Une solution périodique au sens de Carathéodory de (*) est une fonction $y \in C^1$ telle que y' absolument continue qui vérifie $y(0) = y(1)$, $y'(0) = y'(1)$.

Nous nous proposons d'étendre certains résultats d'existence en remplaçant la continuité par une fonction de Carathéodory qui n'est pas nécessairement 1-périodique et sans condition de croissance sur la variable y'. Comme conséquence nous obtenons certaines extentions d'un théorème de type Nirenberg ainsi que l'équation de Linéard.

1. NOTATIONS ET RÉSULTATS PRÉLIMINAIRES.

Dans la suite, C^k dénotera l'ensemble des fonctions $u : [0,1] \longrightarrow \mathrm{R}$ continûment différentiables jusqu'à l'ordre k, muni de la norme $\|u\|_k = \max\{ \|u\|_0, \|u'\|_0, ..., \|u^{(k)}\|_0 \}$, où $\|u\|_0 = \sup\{ |u(t)| \mid t \in [0,1]\}$. L'ensemble des fonctions continues C^0 est noté C; $C_0 = \{ u \in C / u(0) = 0 \}$. Pour $k > 1$, on définit $C^{k,1} = \{ u \in C^{k-1} | \ u^{(k-1)}$ est absolument continue$\}$.

(1.1) Proposition . Soient $u \in C^{2,1}$ et $t_0 \in (a,b)$. Si u atteint son maximum (resp. minimum) en t_0 alors pour tout V voisinage de t_0, l'ensemble $\{t \in V | \ u''(t) \le 0\}$ (resp. $\{ t \in V \mid u''(t) \ge 0 \}$) est de mesure non nulle.

Définition.- Une fonction $f : [0,1] \times \mathrm{R}^2 \longrightarrow \mathrm{R}$ est dite <u>de Carathéodory</u> si elle vérifie :

(a) $x \longrightarrow f(t,x)$ est continue p.p. tout $t \in [0,1]$;

(b) $t \longrightarrow f(t,x)$ est mesurable pour tout $x \in \mathrm{R}^2$;

(c) pour tout $K > 0$ il existe $h_K \in L^1$ tel que , si $\|x\| \le K$ alors $|f(t,x)| \le h_K(t)$ p.p. $t \in [0,1]$.

Définition. Etant donnés $M > 0$ et $f : [0,1] \times \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R}$ une fonction de Carathéodory, on définit alors $f_1 : [0,1] \times \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R}$ <u>la fonction associée à (f, M)</u> par:

$$f_1(t,y,p) = \begin{cases} \max(-M, f(t,y,p) - y) & \text{si } y > M; \\ f(t,y,p) - y & \text{si } -M \le y \le M; \\ \min(M, f(t,y,p) - y) & \text{si } y < -M; \end{cases}$$

et $N_{f_1} : C^1 \longrightarrow C_0$ l'opérateur de Carathéodory associé à (f, M) par:

$$N_{f_1}[u](t) = \int_0^t f_1(s,u(s),u'(s))ds \quad \text{pour tous } u \in C^1 \text{ et } t \in [0,1].$$

Remarque: La fonction f_1 n'est pas une fonction de Carathéodory; elle ne vérifie pas la condition (a).

(1.2) Proposition. Soit $f : [0,1] \times \mathsf{R} \times \mathsf{R} \longrightarrow \mathsf{R}$ une fonction de Carathéodory qui vérifie :

(H1) il existe $M > 0$, $f(t,M,0) \geq 0$ et $f(t,-M,0) \leq 0$ p.p. dans [0,1]. Alors,

(i) N_{f_1}, l'opérateur associé à (f, M), est continu;

(ii) N_{f_1} est complètement continu.

Pour une démonstration voir [4] et [7].

2. RESULTAT D'EXISTENCE PRINCIPAL.

(2.1) Théorème. Supposons que f est une fonction de Carathéodory qui vérifie les hypothèses .

(H1) il existe $M > 0$, $f(t,M,0) \geq 0$ et $f(t,-M,0) \leq 0$ p.p. dans [0,1],

(H2) il existe $M_1 > 0$ tel que pour tout $(t,y) \in [0,1]x[-M,M]$et $|p| > M_1$, il existe un voisinage UxV de $(t,(y,p))$ tel que l'une des conditions suivantes est toujours vérifiée :

(a) $f(u,v,q) > 0$ p.p. $u \in U$, pour tout $(v, q) \in V$

ou bien

(b) $f(u,v,q) < 0$ p.p. $u \in U$, pour tout $(v,q) \in V$.

(H3) Il existe $0 < \delta < 1$ tel que pour tout $y \in [-M,M]$ et $|p| > M_1$ on a

$$\text{sign}(f(t,y,p)) = \text{sign}(f(u,y,p)) \quad \text{p.p. } t \text{ dans } (0, \delta) \text{ et p.p. } u \text{ dans } (1 - \delta,1).$$

Alors le problème périodique :

(N) $\quad y''(t) = f(t, y(t), y'(t)) \quad$ p.p. dans [0,1]; $y(0) = y(1)$, $y'(0) = y'(1)$,

admet au moins une solution y au sens de Carathéodory dans $C^{2,1}$ telle que $-M \leq y(t) \leq M$ pour tout $t \in [0,1]$ (où M est la constante de (H1)).

Dans le cas où $f(t,y,p)$ est une fonction continue, on retrouve certains résultats d'existence d'une solution classique démontrés par NIRENBERG [9], GRANAS- GUENTHER- LEE [5], et [8].

Par exemple la fonction $f(t,y,p) = t^{-\alpha} p^n + y^{2m+1} + t$, $0 < \alpha < 1$ est une fonction qui vérifie les hypothèses (H1), (H2), et (H3) mais ne vérifie pas les hypothèses des théorèmes d'existence dans [5], [8], et [9].

Démonstration du théorème (2.1): Soit f une fonction de Carathéodory qui vérifie (H1), (H2), et (H3). Pour tout $\lambda \in [0,1]$, considérons la famille des problèmes :

$(N)_\lambda$ $\quad y''(t) = \lambda f(t,y(t),y'(t))$ p.p. dans $[0,1]$; $y(0) = y(1)$, $y'(0) = y'(1)$,

et la famille des problèmes associés :

$(N)_{1\lambda}$ $\quad y''(t) - y(t) = f_\lambda(t,y(t),y'(t))$ p.p. dans $[0,1]$; $y(0) = y(1)$, $y'(0) = y'(1)$,

où f_λ est la fonction associée à $(\lambda f, M)$, $(N)_{1\lambda}$ est appelé le problème associé à $(N)_\lambda$.
Notons que la fonction λf vérifie (H1) pour tout $\lambda \in [0,1]$.

(2.2) Lemme.

(i) Pour tout $\lambda \in [0,1]$ et toute solution y de $(N)_{1\lambda}$, on a la majoration $\|y\|_0 \le M$;

(ii) Toute solution y de $(N)_{1\lambda}$ est une solution y de $(N)_\lambda$. De plus, on a la majoration $\|y'\|_0 \le M_1$ pour tout $\lambda \in [0,1]$.

Preuve du lemme. (i) Soit y une solution de $(N)_{1\lambda}$, nous allons montrer que $\|y\|_0 \le M$. En effet, soit $b \in [0,1]$, tel que la fonction $t \longrightarrow y(t)$ atteint son maximum positif en b. Supposons que $y(b) > M$ et que $b \in (0,1)$, il existe alors un voisinage V de b tel que $y(t) > M$ pour tout $t \in V$. D'après la définition de f_1 on a $y''(t) - y(t) \ge -M$ p.p. dans V; par conséquent $y''(t) > 0$ p.p. dans V et on obtient une contradiction avec la proposition (1.1).
Supposons maintenant que $b = 1$ et que $y(1) > M$, il existe alors $a < 1$ tel que $y(t) > M$ pour tout $t \in [a,1]$. Par suite $y''(t) > 0$ p.p. dans $[a,1]$ et $y'(1) - y'(t) = \int_t^1 y''(s)ds > 0$ pour tout $t \in [a,1]$; puisque $y'(1) = 0$ on a $y'(t) < 0$ et on obtient que $y(1) < y(a)$, ce qui est une contradiction avec le fait que y atteint son maximum en 1. La démonstration est analogue si on suppose que $b = 0$ et $y(0) > M$.
Dans le cas où la fonction $t \longrightarrow y(t)$ atteint son minimum négatif en b telle que $y(b) < -M$ et $b \in (0,1)$, il existe un voisinage V de b tel que $y(t) < -M$ pour tout $t \in V$. D'après la définition de f_1 on a : $y''(t) - y(t) \le M$ p.p. dans V; par conséquent $y''(t) < 0$ p.p. dans V et on obtient une contradiction avec la proposition (1.1). Les autres cas se traitent d'une façon analogue et on obtient que $-M \le y(t) \le M$ pour tout $t \in [0,1]$.
(ii) Soit y une solution de $(N)_{1\lambda}$, $\lambda \in (0,1]$, d'après la définition de f_λ et (ii) on a :

$$y''(t) = \lambda f(t,y(t),y'(t)) \quad \text{p.p. dans } [0,1],$$

et donc y est une solution du problème $(N)_\lambda$. Soit maintenant $b \in [0,1]$, tel que la fonction $t \longrightarrow y'(t)$ atteint son maximum positif en b. Supposons que $y'(b) > M_1$, $b \in (0,1)$, alors $|y(b)| < M$, sinon y(b) atteindra son extrémum en b ce implique que $y'(b) = 0$. Donc d'après (H2) et la continuité de la fonction $(t,y(t),y'(t))$, il existe un voisinage $(b - \delta, b + \delta)$ de b dans (0,1) tel que l'une des alternatives suivantes soit vérifiée :

(a) $y''(t) > 0$ p.p. dans $(b, b + \delta)$ ce qui implique la contradiction $y'(b + \delta) > y'(b)$;

ou bien

(b) $y''(t) < 0$ p.p. dans $(b - \delta, b)$ ce qui implique la contradiction $y'(b - \delta) > y'(b)$.

Maintenant, si $b = 0$ alors y' atteint également son maximum en 1, d'après (H3) et la continuité de la fonction $(t,y(t),y'(t))$, il existe deux intervalles $(0, \delta)$, $(1-\delta, 1)$ tels que l'une des alternatives suivantes soit vérifiée :

(a) $y''(t) > 0$ p.p. dans $(0, \delta)$ ce qui implique la contradiction $y'(\delta) > y'(0)$;

ou bien

(b) $y''(t) < 0$ p.p. dans $(1 - \delta, 1)$ ce qui implique la contradiction $y'(1-\delta) > y'(1)$.

Les autres cas se traitent d'une façon similaire. Si $\lambda = 0$ alors les seules solutions sont les fonctions constantes comprises entre $-M$ et M, et la démonstration du lemme est complète.

Fin de la démonstration du théorème (2.1)

Considérons $C^1_b = \{u \in C^1 \mid u(0) = u(1), u'(0) = u'(1)\}$, $C_0 = \{u \in C \mid u(0) = 0\}$, l'opérateur linéaire bijectif continu $L : C^1_b \longrightarrow C_0$ et l'opérateur $N_{f\lambda} : C^1_b \longrightarrow C_0$ définis par :

$$L[u](t) = u'(t) - u'(0) - \int_0^t u(s)ds \quad \text{et} \quad N_{f\lambda}[u](t) = \int_0^t f_\lambda(s, u(s), u'(s))ds,$$

pour tous $u \in C^1_b$, $t \in [0,1]$ et $\lambda \in [0,1]$.

L'opérateur $N_{f\lambda}$ est l'opérateur de Carathéodory associé à $(\lambda f, M)$; donc, d'après la proposition (1.2) $N_{f\lambda}$ est continu et complètement continu . De plus, le problème $(N)_{1\lambda}$ est équivalent au problème de point fixe de l'opérateur complètement continu $L^{-1}N_{f\lambda} : C^1_b \longrightarrow C^1_b$. Posons alors $r = 1+\max\{M, M_1\}$ où M et M_1 sont les constantes de (H1) et (H2) respectivement, et définissons $K_r = \{u \in C^1_b \mid \|u\|_1 \leq r\}$. Par le choix de r et (ii) du lemme (2.2), la famille d'opérateurs $L^{-1}N_{f\lambda}$ est sans point fixe sur la frontière de K_r pour tout $\lambda \in [0,1]$. Par conséquent $L^{-1}N_{f\lambda} : [0,1]\text{x}K_r \longrightarrow C^1_b$ est une homotopie compacte sans point fixe sur la frontière de K_r entre $L^{-1}N_{f1}$ et $L^{-1}N_{f0}$. De plus, $\tau L^{-1}N_{f0} : [0,1]\text{x}K_r \longrightarrow C^1_b$ est une homotopie compacte sans point fixe sur la frontière de K_r entre $L^{-1}N_{f0}$ et la fonction constante nulle qui est dans K_r. Par le théorème de transversalité topologique (voir [2],[3]) $L^{-1}N_{f1}$ a un point fixe qui est solution du problème (N) et la démonstration du théorème est complète.

3. PROBLÈME PÉRIODIQUE DE TYPE NIRENBERG.

Dans cette partie nous discutons certains résultats d'existence de type Nirenberg pour le problème périodique où $f(t,y,p) = y\alpha(t,y,p) - \beta(t,y,p)$.

(3.1) Théorème - Soient $\alpha(t,y,p)$ et $\beta(t,y,p)$ deux fonctions de Carathéodory qui vérifient les hypothèses:

(i) Il existe $M > 0$ tel que $(\beta(t,y,0)/y\,\alpha(t,y,0) \leq 1$ et $\alpha(t,y,0) > 0)$ si $|y| = M$,

(ii) Il existe $M_1 > 0$ telle que $\alpha(t,y,p) > 0$ p.p. dans $[0,1]$, $y \in [-M,M]$ et $|p| > M_1$,

(iii) $\beta(t,y,p)/ \alpha(t,y,p) \longrightarrow C$ quand $|p| \longrightarrow \infty$ uniformément pour presque tout t dans $[0,1]$ et pour tout $y \in [-M,M]$.

Si $|C| > M$ alors le problème :

(N) $$y''(t) = y(t)\alpha(t,y(t),y'(t)) - \beta(t,y(t),y'(t)) \quad \text{p.p. dans } [0,1];$$

admet au moins une solution y périodique au sens de Carathéodory dans $C^{2,1}$.

Démonstration : Il est facile de voir que la condition (i) implique l'hypothèse (H1). Il suffit maintenant de vérifier (H2), et (H3). En effet, supposons que $C > 0$, il existe $\varepsilon > 0$ tel que $M < C - \varepsilon$. D'après (iii), il existe M_2 tel que $C - \varepsilon < \beta(t,y,p)/ \alpha(t,y,p)$ pour presque tout $t \in [0,1]$, $y \in (-M,M)$ et $|p| > M_2$. Donc, $y - \beta(t,y,p)/ \alpha(t,y,p) < M - (C - \varepsilon) < 0$ pour tout $(t,y) \in [0,1]x[-M,M]$ et $|p| > M_2$. Si on prend $M_3 = \max (M_1,M_2)$, la fonction $f(t,y,p) = y\alpha(t,y,p) - \beta(t,y,p) > 0$ p.p. dans $[0,1]$, $y \in (-M,M)$ et $|p| > M_3$. Si $C < 0$ la démonstration est similaire, et la démonstration est complète.

Remarque : La condition (ii) peut être remplacée par la condition :

(ii)' Il existe $M_1 > 0$ telle que $\alpha(t,y,p) < 0$ p.p. dans $[0,1]$, $y \in (-M,M)$ et $|p| > M_1$.

(3.2) Théorème - Soient $\alpha, \beta : [0,1] \times R^+ \times R \longrightarrow R$ deux fonctions de Carathéodory qui vérifient les hypothèses du théorème (3.1) et l'hypothèse:

(H) Pour tout $(t,y) \in (0,1)x(0,M)$, il existe un voisinage UxV de $(t,y,0)$ dans $(0,1)x(0,M)x\ R$ tel que $\alpha(u,q,p) > 0$ et $\beta(t,q,p) > 0$ p.p. dans U, (q,p) dans V.

Si $|C| > M$ alors le problème périodique :

(N) $$y''(t) = y(t)\alpha(t,y(t),y'(t)) - \beta(t,y(t),y'(t)) \quad \text{p.p. dans } [0,1];\ y(0) = y(1)\ ,\ y'(0) = y'(1)$$

admet au moins une solution positive y dans $C^{2,1}$.

Démonstration. Définissons $\alpha^*, \beta^* : [0,1] \times R \times R \longrightarrow R$ par

$$\alpha^* (t,y,p) = \alpha(t,|y|,p) \quad \text{et} \quad \beta^* (t,y,p) = \beta(t,|y|,p)$$

$f(t,y,p) = y\ \alpha^*(t,y,p) - \beta^*(t,y,p)$ vérifie les hypothèses du théorème (2.1). Par conséquent le problème périodique :

(N)' $y''(t) = y(t)\alpha(t,y(t),y'(t)) - \beta(t,y(t),y'(t))$ p.p. dans $[0,1]$; $y(0) =y(1)$, $y'(0) = y'(1)$

admet une solution y. De plus $y \geq 0$, sinon , d'après les conditions au bord y atteint son maximum en $t_0 \in (0,1)$ tel que $y(t_0) < 0$. D'après (b) et la continuité de y, il existe un voisinage V de t_0 tel que $y''(t) < 0$ p.p. dans , ce qui contredit la proposition (1.1). On en déduit que y est une solution positive de (N).

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Received August 13, 1990

Universal Bundles in Simplicial Sheaves

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Abstract

Let $\mathcal{E}$ be a Grothendieck topos, and denote by $S(\mathcal{E})$ the category of simplicial objects in $\mathcal{E}$. For a group G in $S(\mathcal{E})$, we study the question of the existence of a universal principal G–bundle in $S(\mathcal{E})$. We find that such a bundle exists if G is *amenable,* meaning that principal G–bundles extend uniquely along anodyne extensions of $S(\mathcal{E})$. Our principal theorem states that for any group G there is an anodyne homomorphism $G \hookrightarrow \tilde{G}$ with $\tilde{G}$ amenable. Finally, we introduce the concept of *pseudo-torsor,* which corrects for the lack of representability of $H^1(\ ,G)$.

1- Amenable groups

We begin with a brief review of the classical theory [3]. Given a group G in the category of simplicial sets S, there is a contractible space E, on which G acts freely, yielding a universal principal G-bundle $E \longrightarrow E/G = B$. E can be chosen to be a Kan complex, in which case B is also. The universality of $E \longrightarrow B$ takes the following form: for any simplicial set X, the map $f \longmapsto f^*(E)$ from hom(X,B) to the set of principal G-bundles over X induces a 1-1 correspondence between the set [X,B] of homotopy classes of maps, and the set $H^1(X,G)$ of isomorphism classes of principal G-bundles over X.

Replacing the category of sets by an arbitrary Grothendieck topos $\mathcal{E}$, we consider the analogous problem for the category of simplicial sheaves $S(\mathcal{E})$.

First, recall that if G is a group in any topos, a *G-torsor over X* is an object E provided with a free (right) G-action $\theta: E \times G \longrightarrow E$ (free means the map $(\theta, \pi): E \times G \longrightarrow E \times E$ is injective). $E \longrightarrow X$ is the projection of E onto the quotient $E/G = X$ of E by the action of G. In the topos of simplicial sets S, a G–torsor over X is simply a principal G–bundle over X. If G is a group in the topos $S(\mathcal{E})$, by a principal G-bundle over X, we will always mean a G–torsor over X in the above sense.

As to the homotopy theory of $S(\mathcal{E})$, recall [2] that there is a Quillen homotopy structure [4] on $S(\mathcal{E})$, in which the weak equivalences are maps $f: X \longrightarrow Y$ inducing isomorphisms on the homotopy sheaves, the cofibrations are the monomorphisms, and the fibrations are maps having the right lifting property with respect to the cofibration weak equivalences. We call a cofibration weak equivalence an *anodyne extension*.

We can recognize weak equivalences in a different way. Namely, if $p: \mathbb{B} \longrightarrow \mathcal{E}$ is a surjective geometric morphism with $\mathbb{B}$ boolean [1], then the axiom of choice is satisfied in $\mathbb{B}$, and the logic is classical. The morphism p induces a surjective geometric morphism, also called p, $S(\mathbb{B}) \longrightarrow S(\mathcal{E})$, and the theory of $S(\mathbb{B})$ is the same as S. A map $f: X \longrightarrow Y$ is a weak equivalence iff $p^*(f)$ is, so that geometric constructions yielding weak equivalences in S, yield weak equivalences in $S(\mathcal{E})$. We call a map f a *Kan fibration* if $p^*(f)$ is a Kan fibration. This notion is strictly weaker than that of fibration in $S(\mathcal{E})$.

Formally inverting the weak equivalences yields the *homotopy category* of $S(\mathcal{E})$. (We obtain the same category by inverting only the anodyne extensions.) If X and Y are simplicial sheaves, we denote by ho(X,Y) the set of morphisms from X to Y in the homotopy category. If Y is a fibrant simplicial sheaf, then ho(X,Y) is in 1-1 correspondence with the set [X,Y] of *homotopy classes* of maps from X to Y. (A homotopy is a mapping $X \times I \longrightarrow Y$ with I the constant simplicial sheaf on the 1-simplex $\Delta[1]$ of S.)

Now, given a group G in $S(\mathcal{E})$, we want to know if there is a universal G-torsor $E \longrightarrow B$. Equivalently, we can ask whether the functor $H^1(\,, G)$ can be represented in the homotopy category. An obvious necessary condition is that $H^1(\,, G)$ should invert anodyne extensions. More precisely, if $A \hookrightarrow B$ is an anodyne extension and $T \longrightarrow A$ is a G-torsor over A, then there should exist a G-torsor $S \longrightarrow B$, whose restriction to A is isomorphic to T. Moreover, S should be unique up to isomorphism.

<u>Definition 1</u> A group G in $S(\mathcal{E})$, whose torsors have the above property is said to be *amenable*.

Example A simplicialy discrete group is amenable.

Proposition 1 If a group G is amenable then there exists a universal G-torsor $E \longrightarrow B$.

Indication of proof: For any groupoid $\mathbb{G} = (s,t)\colon G_1 \longrightarrow G_0 \times G_0$ in $S(\mathcal{E})$, let $B\mathbb{G}$ denote the diagonal complex of its nerve $N\mathbb{G}$, considered as a double simplicial sheaf. Let $E\mathbb{G}$ denote $B\mathbb{G}'$, where $\mathbb{G}'$ is the groupoid of arrows of $\mathbb{G}$ (a morphism $h\colon f \longrightarrow g$ is an arrow h such that $hf = g$). The codomain t is a functor $\mathbb{G}' \longrightarrow \mathbb{G}$ inducing a mapping $E\mathbb{G} \longrightarrow B\mathbb{G}$. The domain s defines a functor $\mathbb{G}' \longrightarrow G_0$, where G_0 is considered as a discrete groupoid, and induces a mapping $E\mathbb{G} \longrightarrow G_0$, which is a weak equivalence. Moreover, $\mathbb{G}$ acts freely on the right of $\mathbb{G}'$, inducing a free $\mathbb{G}$-action on $E\mathbb{G}$.

When $\mathbb{G}$ is the group G, let $BG \hookrightarrow B$ be an anodyne extension with B fibrant. Since G is amenable, the torsor $EG \longrightarrow BG$ extends to a G-torsor $E \longrightarrow B$ with E contractible. An easy argument shows that $E \longrightarrow B$ is the required universal G-torsor.∎

A *graph with units* A in $S(\mathcal{E})$ is a diagram

$$A_0 \xrightarrow{\;u\;} A_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A_0$$

in $S(\mathcal{E})$ with $d^0u = d^1u = id_{A_0}$. An *anodyne extension of graphs* is a morphism of graphs $k : A \hookrightarrow B$ with k_0 and k_1 anodyne extensions of $S(\mathcal{E})$. If A is a graph with units in $S(\mathcal{E})$, let $F(A)$ denote the free groupoid on A.

Definition 2 A groupoid $\mathbb{G}$ in $S(\mathcal{E})$ is said to be *fibrant* if, for any anodyne extension of graphs $A \hookrightarrow B$, each diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & \mathbb{G} \\ \big\downarrow & \nearrow & \\ F(B) & & \end{array}$$

has a dotted filler.

Proposition 2 A groupoid $\mathbb{G}$ is fibrant iff G_0 is fibrant and $(s,t)\colon G_1 \longrightarrow G_0 \times G_0$ is a fibration of $S(\mathcal{E})$.

Call a groupoid $\mathbb{G}$ *locally transitive* if $(s,t)\colon G_1 \longrightarrow G_0 \times G_0$ is a Kan fibration.

Theorem 1 A locally transitive groupoid $\mathbb{G}$ in $S(\mathcal{E})$ has a completion $a\colon \mathbb{G} \hookrightarrow \bar{\mathbb{G}}$ such that a_0 and a_1 are anodyne extensions, and $\bar{\mathbb{G}}$ is fibrant.

Indication of proof: There is a small set of generators for the anodyne extensions of graphs, and pushouts of diagrams of the form

$$\begin{array}{ccc} F(A) & \longrightarrow & \mathbb{G} \\ \big\downarrow & & \\ F(B) & & \end{array}$$

with $A \hookrightarrow B$ anodyne and $\mathbb{G}$ locally transitive yield anodyne extensions of groupoids. Thus, we can obtain the desired completion by repeated pushouts and ordinal colimits.■

Theorem 2 For any group G in $S(\mathcal{E})$, there is an anodyne homomorphism $G \hookrightarrow \tilde{G}$ with $\tilde{G}$ amenable.

Indication of proof: Let $\mathbb{G}$ denote the locally transitive groupoid $ISO_G(EG, EG) \longrightarrow BG \times BG$, and let $\mathbb{G} \hookrightarrow \bar{\mathbb{G}}$ be a fibrant completion of $\mathbb{G}$. The vertex group $\tilde{G}$ of $\bar{\mathbb{G}}$ corresponding to the single vertex of BG is amenable and provides the desired anodyne homomorphism $G \hookrightarrow \tilde{G}$.■

Theorem 3 An abelian group in $S(\mathcal{E})$ has an abelian amenable completion.

Indication of proof: The category of abelian groups in $S(\mathcal{E})$ has a Quillen structure, in which the cofibrations are monomorphisms, the weak equivalences are homomorphisms which are weak equivalences in $S(\mathcal{E})$, and the fibrations have the right lifting property with respect to the cofibration weak equivalences. These fibrations are also fibrations in $S(\mathcal{E})$.

Let A be an abelian group in $S(\mathcal{E})$. Taking the cone on A yields a monomorphism $A \hookrightarrow E(A)$ with $E(A)$ contractible. Dividing by A gives an exact sequence $A \hookrightarrow E(A) \longrightarrow B(A)$, in which $E(A) \longrightarrow B(A)$ is an A-torsor. In the above Quillen structure, let $B(A) \hookrightarrow B$ be an anodyne homomorphism with B fibrant, and factor $E(A) \longrightarrow B(A) \hookrightarrow B$ as

$$\begin{array}{ccc} E(A) & \hookrightarrow & E \\ \downarrow & & \downarrow \\ B(A) & \hookrightarrow & B \end{array}$$

where $E(A) \hookrightarrow E$ is an anodyne homomorphism, and $E \longrightarrow B$ is a fibration. Let $\tilde{A}$ be the kernel of $E \longrightarrow B$. Then $A \hookrightarrow \tilde{A}$ is anodyne, and $\tilde{A}$ is fibrant and amenable. $E \longrightarrow B$ is a universal $\tilde{A}$-torsor.■

Remark This process yields a representation theorem for sheaf cohomology in terms of torsors:

$$H^n(\mathcal{E},\pi) \simeq H^1(\tilde{K}(\pi, n-1)),$$

where $\tilde{K}(\pi, n-1)$ is an abelian amenable completion of the abelian group $K(\pi, n-1)$.

2. Pseudo torsors

In this section we define a relaxed notion of torsor (or bundle), whose equivalence classes can be represented in the homotopy category for an arbitrary group G.

Definition 3 Let G be a group in $S(\mathcal{E})$. A *G-pseudo torsor over X* is a simplicial sheaf T over X, on which G acts freely, and is such that the map $T \longrightarrow X$ factors as

$$\begin{array}{ccc} & & T \\ & \swarrow & \downarrow \\ T/G & & \downarrow \\ & \searrow & \downarrow \\ & & X \end{array}$$

with $T/G \longrightarrow X$ a weak equivalence. A *morphism* of pseudo torsors is a commutative triangle

$$\begin{array}{ccc} T_1 & \longrightarrow & T_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

where the map $T_1 \longrightarrow T_2$ respects the action of G; it is always a weak equivalence.

The set of components of the category of G-pseudo torsors is denoted by $\tilde{H}^1(X,G)$. We remark that two pseudo torsors T_1 and T_2 over X are in the same component iff there is a third pseudo torsor T_3 over X and a pair of mappings $T_1 \longrightarrow T_3 \longleftarrow T_2$.

Let $G \hookrightarrow \tilde{G}$ be an amenable completion of G.

Theorem 4 $\tilde{H}^1(X,G) \simeq H^1(X,\tilde{G})$.

Indication of proof: A direct argument shows $\tilde{H}^1(X,G) \simeq \mathrm{ho}(X, BG)$. The map $BG \longrightarrow B\tilde{G}$ is a weak equivalence. Let $B\tilde{G} \hookrightarrow B$ be anodyne with B fibrant. By Proposition 1 we have:

$$\tilde{H}^1(X,G) \simeq \mathrm{ho}(X, BG) \simeq \mathrm{ho}(X, B\tilde{G}) \simeq [X, B] \simeq H^1(X,\tilde{G}).$$

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Received August 27, 1990

Sur l'équation de Pillai et la différence entre deux produits de puissances de nombres entiers

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Summary. We give an estimate from below for the distance between two products of powers of rational integers. As an example, we show that for positive integers x, y, m, n with $y \geq 2$, $x^m > y^n$ and $m/\log y < 1.37 \cdot 10^{12}$, we have $x^m - y^n > x^{m/2}$.

Résumé. Soient k et n deux entiers positifs avec $1 \leq k \leq n$, et soient $a_1, \ldots, a_n$, $h_1, \ldots, h_n$ des entiers positifs, avec

$$a_1^{h_1} \cdots a_k^{h_k} \neq a_{k+1}^{h_{k+1}} \cdots a_n^{h_n}.$$

On cherche à minorer la distance entre ces deux nombres. Les seules estimations effectives non triviales valables pour n quelconque utilisaient jusqu'ici la méthode de Baker. Nous proposons une nouvelle minoration provenant d'une méthode de transcendance différente, et nous en donnons une application à l'équation de Pillai $x^m - y^n = k$.

1. Enoncé du résultat.

Soient n un entier ≥ 2, $a_1, \ldots, a_n$ des nombres entiers positifs multiplicativement indépendants, et $b_1, \ldots, b_n$ des nombres entiers rationnels non tous nuls ; ainsi

$$a_1^{b_1} \cdots a_n^{b_n} \neq 1.$$

Nous ordonnons les nombres $a_i^{|b_i|}$ de telle sorte que $a_n^{|b_n|}$ soit le plus grand, et nous appelons a_0 le plus petit des a_i avec $1 \leq i \leq n-1$:

$$\max_{1\leq i\leq n-1} |b_i| \log a_i \leq |b_n| \log a_n, \qquad a_0 = \min_{1\leq i\leq n-1} a_i.$$

Nous posons également

$$M = \max\{n^{4n}e^{20n+10}\,;\ |b_n|/\log a_0\}.$$

<u>Théorème</u>. <u>Sous ces hypothèses</u>, <u>on a</u>

$$\left|a_1^{b_1}\cdots a_n^{b_n} - 1\right| > \exp\{-C_0(n)\log M \log a_1 \cdots \log a_n\},$$

<u>avec</u>

$$C_0(n) < e^{3n+9}n^{4n+5}.$$

Les meilleures minorations générales et explicites connues jusqu'à maintenant étaient celles de [BGMMS]. Notre énoncé est légérement plus précis car il ne fait pas intervenir de terme parasite en $\log\log a$ avec $a = \max_{1\leq i\leq n} a_i$.

La condition $M \geq n^{4n}e^{20n+10}$ n'est pas vraiment restrictive : en comparant notre estimation avec la minoration triviale, nous allons montrer qu'il n'y a pas de restriction à supposer $|b_n|/\log a_0$ assez grand.

En effet, $\left|a_1^{b_1}\cdots a_n^{b_n} - 1\right|$ étant un nombre rationnel non nul, il est minoré par l'inverse d'un dénominateur :

$$\left|a_1^{b_1}\cdots a_n^{b_n} - 1\right| \geq \exp\{-n|b_n|\log a_n\}$$

grâce à notre condition $a_i^{|b_i|} \leq a_n^{|b_n|}$. Par conséquent, pour établir la minoration du théorème il n'y a pas de restriction à supposer

$$(1) \qquad n|b_n| \geq C_0(n)\log M \log a_1 \cdots \log a_{n-1}.$$

Le membre de droite de (1) est au moins $C_0(n)\log M \log a_0$; en minorant $\log M$ par $e^3 n$, tant que notre constante $C_0(n)$ sera supérieure à $n^{4n+5}e^{3n}$, on pourra supposer sans perte de généralité

$$|b_n|/\log a_0 \geq n^{4n+5}e^{3n+3}.$$

De plus, si $n \geq 3$, le membre de droite de (1) est plus grand que $n(\log a_0)^2$, donc si on remplaçait $|b_n|/\log a_0$ par $|b_n|$ dans la définition de M on ne perdrait même pas un facteur 2 dans l'estimation finale.

2. Démonstration.

Nous montrons d'abord comment se ramener aux notations de [W2] (avec un passage aux logarithmes), puis nous indiquons comment conclure en reprenant les arguments de [W2]. Pour terminer nous explicitons le calcul de la constante $C_0(n)$.

a) Dès que $\left|a_1^{b_1}\cdots a_n^{b_n}-1\right| \leq 1/3$, on a

$$|b_1\log\alpha_1+\cdots+b_n\log\alpha_n| \leq \frac{3}{2}\left|a_1^{b_1}\cdots a_n^{b_n}-1\right|$$

(voir par exemple [PW] lemme 2.3). Le problème est donc ramené à la minoration d'une combinaison linéaire de logarithmes de nombres algébriques.

b) Pour nous ramener aux notations de [W2], nous posons

$$\log A_i = e(n+1)\log a_i, \qquad E=e, \qquad D=1,$$

$$\epsilon_1 = 0.23, \qquad \epsilon_2 = 2\cdot 10^{-5}, \qquad \epsilon_3 = 1/76,$$

$$G=(1+\epsilon_1)\log M, \qquad Z = 10+2\log n,$$

$$c_3 = \frac{n(2n+3)^2}{n+1}\cdot(1+\epsilon_3),$$

$$c_2 = n(n+1)^2\left(\frac{3n+1}{2}\right)^{n-1} c_3^{n+1}\cdot(1+\epsilon_2)$$

$$= \frac{2n^{n+2}(3n+1)^{n-1}(2n+3)^{2n+2}}{2^n(n+1)^{n-1}}\cdot(1+\epsilon_3)^{n+1}\cdot(1+\epsilon_2),$$

$$c_1 = c_4 = \frac{3n+1}{2n} c_2.$$

On va voir que le théorème est vrai avec

$$(2) \qquad C_0(n) = (5 + \log n) e^n (n+1)^n (1 + \epsilon_1)(5n + 13) c_2.$$

Vérifions d'abord que l'on a

$$G \geq \max\{2(n+1)Z\,;\, 2 + \log(|b_n|/\log a_0)\}.$$

En effet, pour $n \geq 2$, on vérifie $4\log n + 10 < \epsilon_1(4n\log n + 20n + 10)$, donc $(1+\epsilon_1)\log M > 4(n+1)(5+\log n)$. D'autre part on a $\log M \geq 50 + 8\log 2 > 2/\epsilon_1$, donc $2 + \log M < (1+\epsilon_1)\log M$. La minoration annoncée de G est bien vérifiée.

La principale différence avec [W2] est l'absence d'un facteur 2 dans la définition de c_3 ; cela permet de gagner un coefficient 2^{n+1} sur ce que nous donnerait une application brutale des résultats de [W2]. Cette amélioration est due au fait que les a_i sont réels positifs (dans [W2] on travaille avec des nombres algébriques complexes). Pour justifier ce raffinement, on note d'abord que, dans le lemme 2.1 de [W1], quand les u_{ij} sont réels, on peut remplacer l'exposant 2ϱ de l'hypothèse par ϱ. Il en résulte que, dans l'hypothèse (2.8) du corollaire 2.7 de [W1], on peut omettre le facteur 2 à gauche, quand on suppose que les fonctions analytiques considérées ont un développement de Taylor à l'origine à coefficients réels. Ainsi, quand les nombres $\log \alpha_j$ de [W2] sont réels, on peut omettre le facteur 2 à gauche de (5.8) et de (6.4). La seule correction à apporter est dans la minoration (8.2) des c_i au paragraphe 8 de [W2] (II) : dans le cas $n = 2$ on a

$$c_3 > 33, \quad c_2 > 2 \cdot 10^6, \quad c_1 = c_4 > 3.9 \cdot 10^6.$$

Cela permet de vérifier, pour $n \geq 2$,

$$c_3 \geq 8n^2 + 1, \quad c_2 > 1.2 \cdot 10^5 n^4, \quad c_1 = c_4 > 2 \cdot 10^5 n^4.$$

On remplace finalement la condition (8.6) de [W2] par la majoration $C(n) \leq (5n+13)c_2/2$.

c) Il ne reste plus qu'à calculer numériquement les valeurs de C_0 données par (2) :

$n =$	2	3	4	5
$C_0(n) <$	$2.45 \cdot 10^{10}$	$4.65 \cdot 10^{15}$	$2.21 \cdot 10^{21}$	$2.22 \cdot 10^{27}$

$n =$	6	7	8	9
$C_0(n) <$	$4.22 \cdot 10^{33}$	$1.39 \cdot 10^{40}$	$7.43 \cdot 10^{46}$	$6.11 \cdot 10^{53}$

et

$$C_0(n) < e^{3n+6.6} n^{4n+5} \qquad \text{pour } n \geq 10.$$

3. <u>Application à l'équation de Pillai.</u>

Le cas $n = 2$ du théorème fournit l'énoncé suivant :

<u>Corollaire</u>. <u>Soient</u> x, y, m, n, k <u>des entiers positifs satisfaisant</u> $x^m - y^n = k$. <u>On suppose</u>

$$x^m \geq k^2 \qquad \underline{\text{et}} \qquad y \geq 2.$$

<u>Alors on a</u>

$$m < C \log y \qquad \underline{\text{avec}} \qquad C = 1.37 \cdot 10^{12}.$$

<u>Remarque</u>. L'existence d'une constante absolue C vérifiant la conclusion du corollaire résulte déjà du théorème 2.2 de [PW], mais la constante n'y est pas explicite. Toutes les autres estimations (notamment celles de [BGMMS]) font intervenir un terme logarithmique supplémentaire. D'un point de vue numérique, l'estimation que l'on déduit de la section 9b de [MW], à savoir

$$m \leq 9600(\log y)\bigl(8 + \log \max\{m, n\}\bigr)^2,$$

(9600 apparaît comme majorant de $2(\log 3/\log 2)^2 1905$) est meilleure tant que m et n ne sont pas trop grands :

$$\max\{m,n\} \leq 10^{5184}.$$

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Received August 27, 1990

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REAL RANK ZERO FAMILIES OF C*-ALGEBRAS.

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This paper is devoted to a short proof of the following result about real rank zero [1], and families of C*-algebras [2]. Suppose that X is a compact topological space, and that $A(x)$ $(x \in X)$ is a family of C*-algebras over X. Then the family A has real rank zero if and only if both X has real rank zero and each $A(x)$ has real rank zero.

To begin, we give a brief summary of the definitions we shall need.

Definition 1 : Continuous fields of Banach spaces.

Let X be a topological space, and $\prod_{x\in X} E(x)$ be a product of Banach spaces. Then we consider a vector subspace $\Gamma \subseteq \prod_{x\in X} E(x)$ with the following properties:

(a) For all $s \in \Gamma$, the function $x \mapsto |s(x)|$ is continuous on X.

(b) For all $x \in X$, $\Gamma(x)$ is dense in $E(x)$.

(c) Let $r \in \prod_{x\in X} E(x)$, and suppose that for all $x \in X$ and $\epsilon > 0$, there is an element $s \in \Gamma$ so that $|s(y) - r(y)| < \epsilon$ for all y in some neighbourhood of x in X. Then $r \in \Gamma$.

Proposition 2.

If $s \in \Gamma$, and $f : X \to \mathbb{C}$ is continuous, then $f.s \in \Gamma$.

Proof. [2;10.1.9].

Definition 3 : Continuous fields of C*-algebras.

If each $E(x)$ is a C*-algebra, and if Γ is closed under pointwise $*$ and multiplication, we say that $\mathcal{E} = (X, E, \Gamma)$ is a continuous field of C*-algebras over X.

In this case, we can form a C*-algebra by setting $||s|| = \sup_{x\in X} |s(x)|$, for $s \in \Gamma$, and taking those elements of Γ of finite norm.

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Definition 4 : Real rank zero topological spaces.

We shall say that a topological space X has real rank zero, if for any function $f : X \longrightarrow \mathbb{R}$, and any $\epsilon > 0$, there is a function $g : X \longrightarrow \mathbb{R}$ which is uniformly within ϵ of f, and with $1/g$ bounded on X.

Proposition 5.

Suppose that the unital C*-algebra A is generated by a continuous field of C*-algebras $A(x)$ over a topological space X. Then the set of $x \in X$ with the associated algebra $A(x) = 0$ is both open and closed.

Proof. If the identity in A is I, it follows that $I(x)$ is the identity in $A(x)$. The norm of the identity in a C*-algebra is either 0 or 1, and can only be 0 of the algebra is $\{0\}$. Thus the set of $x \in X$ with the associated algebra $A(x) = 0$ is contained in the open set $\{x \in X \mid |I(x)| < \frac{1}{3}\}$, and its complement is contained in the open set $\{x \in X \mid |I(x)| > \frac{2}{3}\}$. □

We shall use the last result to assume that a unital C*-algebra A, which is generated by a continuous field of C*-algebras $A(x)$ over a topological space X, has $A(x) \neq 0$ for all $x \in X$.

Theorem 6.

Suppose that the unital C*-algebra A is generated by a continuous field Γ of C*-algebras $A(x)$ over a compact topological space X. Then if A has real rank zero, so has X.

Proof. Take a continuous function $f : X \longrightarrow \mathbb{R}$, and an $\epsilon > 0$. Then the field $f.I \in \Gamma$, and so corresponds to an element $b \in A$. Then $b(x)$ has spectrum in $A(x)$ consisting of one point, $f(x)$. By the real rank zero property, there is an invertible element c of A within ϵ of b. The spectrum of $c(x)$ is then contained in an interval of radius ϵ about $f(x)$. Since c is invertible, there is an interval $(-\delta, \delta)$ in the complement of the spectrum of c, and thus also in the complement of the spectrum of $c(x)$ for all $x \in X$.

Now take a continuous function $k^+ : \mathbb{R} \longrightarrow \mathbb{R}$ which has $k^+(t) = t$ on the interval $[\delta, \infty)$, and is zero on the interval $(-\infty, -\delta]$. Then $k^+(c) \in A$, and $k^+(c)(x)$ has norm either $\geq \delta$, or zero. Thus the set $U^+ = \{x \in X \mid |k^+(c)(x)| > 0\}$ is open and closed, as its complement U^-. If $x \in U^+$, we see that

$$\big|f(x) - |k^+(c)(x)|\big| < \epsilon \ ,$$

since $|k^+(c)(x)|$ is the biggest element of the spectrum of $c(x)$.

In the same manner, we take a function $k^- : \mathbb{R} \to \mathbb{R}$ which is zero on the interval $[\delta, \infty)$, and has $k^-(t) = t$ on the interval $(-\infty, -\delta]$. Then if $x \in U^-$, we see that $\big|f(x) + |k^-(c)(x)|\big| < \epsilon$. Thus the continuous function h defined by

$$h(x) = \begin{cases} |k^+(c)(x)| & x \in U^+ \\ -|k^-(c)(x)| & x \in U^- \end{cases},$$

is uniformly within ϵ of f, and $|h(x)| \geq \delta$ for all $x \in X$. □

Proposition 7.

Suppose that the unital C*-algebra A is generated by a continuous field of C*-algebras $A(x)$ over a compact topological space X. Then if there is an $x_0 \in X$ with $A(x_0)$ not real rank zero, then A is not real rank zero.

Proof. Take $a_0 \in A(x_0)$ not in the closure of the invertible elements. By the density axiom 1(b) for a family of algebras, there is an element $a \in A$, so that $a(x_0)$ is sufficiently close to a_0 that it is also not in the closure of the invertible elements in $A(x_0)$. Then a is not in the closure of the invertible elements in A. □

Theorem 8.

Suppose that the unital C*-algebra A is generated by a continuous field of C*-algebras $A(x)$ over a compact topological space X. Then if X has real rank zero, and for every $x \in X$. $A(x)$ has real rank zero, then A has real rank zero.

Proof. Suppose that there is an Hermitian $a \in A(x)$ which is a distance $> \epsilon$ from any invertible elements. Since every $A(x)$ is real rank zero, for every $x \in X$, there is an Hermitian $b_x \in A$ with $b_x(x)$ invertible, and $|b_x(x) - a(x)| < \epsilon/2$. Suppose that the spectrum of $b_x(x)$ has no points within the interval $(-\delta_x, \delta_x)$.

Now we construct a closed and open set U_x containing x so that, for all $y \in U_x$, the spectrum of $b_x(y)$ has no points within the interval $(-\delta_x/2, \delta_x/2)$. This is done by taking a function $f : \mathbb{R} \to [0,1]$ with $f(t) = 0$ for all $|t| > \delta_x$, and $f(t) = 1$ for all $|t| < \delta_x/2$. Then $g(y) = 2\big|f(b_x(y))\big| - 1$ is a continuous function on X, and there is another continuous function $h : X \to \mathbb{R}$ which is never zero, and which is uniformly within 1/2 of g. Then $U_x = h^{-1}(-\infty, 0)$ is both closed and open, and has the required property.

Next we find a closed and open set $W_x \subseteq U_x$ containing x so that, for all $y \in W_x$, $|b_x(y) - a(y)| < \epsilon$. To do this, take a function $f_1 : \mathbb{R} \to [0,1]$ with $f_1(t) = 0$ for all $|t| < \epsilon/2$, and $f_1(t) = 1$ for all $|t| \geq \epsilon$. Then $g_1(y) = 2|f_1(b_x(y) - a(y))| - 1$ is a continuous function on X, and there is another continuous function $h_1 : X \to \mathbb{R}$ which is never zero, and which is uniformly within 1/2 of g_1. Then $W_x = U_x \cap h_1^{-1}(-\infty, 0)$ is both closed and open, and has the required property.

Now take a finite cover $W_1, \ldots, W_n$ of the W_x, corresponding to $x_1, \ldots, x_n$. Define a disjoint closed and open cover $V_1, \ldots, V_n$ of X recursively as follows: $V_1 = W_1$, $V_{k+1} = W_{k+1} \setminus (V_1 \cup \cdots \cup V_k)$. Then we define $c \in A$ by

$$c(y) = b_{x_i}(y) \qquad y \in V_i \ .$$

This is in A by axiom 1(c), and is invertible by the following lemma:

Lemma 9.

If an Hermitian $c \in A$ is such that there is a number $\delta > 0$ with the spectrum of $c(x)$ containing no points of $(-\delta, \delta)$ for all $x \in X$, then c is invertible.

Proof. Take a continuous function $s : \mathbb{R} \to \mathbb{R}$ which is equal to the multiplicative inverse function for all $t \notin (-\delta, \delta)$. Then $s(c)$ is the inverse of c, and is in A. □

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Received July 5, 1990

THE INDEX OF A TOWER OF SEMI-SIMPLE ALGEBRAS

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Abstract: Towers of finite dimensional semi-simple algebras and their inductive limits have been subject of a number of studies in the theory of C^*-algebras, in particular in the language of AF-algebras. A significant part of the theory is purely algebraic and in this brief account, we wish to outline a general approach to the theory of pairs of semi-simple algebras, the Jones fundamental construction of a tower and its index. The index is expressed in terms of the largest real part of the eigenvalues of a Coxeter transformation associated with the associated (weighted) valued graph. In this way, a "discrete" nature of the set of all possible values for this index seems to be clarified.

Throughout the paper, $A \subseteq B$ will denote a k-pair, i.e. a pair of finite dimensional semi-simple k-algebras with unital embedding. Recall that semi-simple k-algebras are finite products of full matrix rings over division k-algebras; those which are products of division k-algebras will be called basic.

THEOREM 1. *Let $A \subseteq B$ be a k-pair and*

$$A = \prod_{i=1}^{a} \mathrm{Mat}(x_i, F_i), \quad B = \prod_{j=1}^{b} \mathrm{Mat}(y_j, G_j).$$

Then there are (finite dimensional) F_i-spaces X_i and F_i-G_j-bimodules ${}_{F_i}({}_iM_j)_{G_j}$ such that

$$A \simeq \mathrm{End}\, X_F \quad \text{and} \quad B \simeq \mathrm{End}\,(X_F \otimes {}_FM_G),$$

where $X_F = \bigoplus_{i=1}^{a} X_i$ and ${}_FM_G = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} {}_iM_j$ with canonical operations by the basic k-algebras

$$F = F_1 \times F_2 \times \ldots \times F_a \quad \text{and} \quad G = G_1 \times G_2 \times \ldots \times G_b.$$

Furthermore,

$$\mathrm{End}\, B_A \simeq \mathrm{End}\,(X_F \otimes {}_FM_G \otimes {}_GM_F^*),$$

where ${}_GM_F^* = \mathrm{Hom}_G({}_FM_G, {}_GG_G)$. Moreover, writing

$$\mathrm{End}\, B_A = \prod_{\ell=1}^{a} \mathrm{Mat}(z_\ell, F_\ell),$$

we have

$$y_j = \sum_{i=1}^{a} x_i u_{ij} \quad \text{and} \quad z_\ell = \sum_{i=1}^{a} a_i \sum_{j=1}^{b} u_{ij} v_{j\ell},$$

where $u_{ij} = \dim({}_iM_j)_{G_j}$ and $v_{ij} = \dim_{F_i}({}_iM_j)$.

As a consequence of Theorem 1, one obtains the following assertions.

THEOREM 2. *There is a bijective correspondence between the pairs $A \subseteq B$ of finite dimensional semi-simple k-algebras and the pairs $({}_FM_G, X_F)$ of finite dimensional F-G-bimodules and finite dimensional F-vector spaces over basic semi-simple k-algebras F and G. Given $({}_FM_G, X_F)$, the corresponding pair is*

$$\operatorname{End}(X_F) \subseteq \operatorname{End}(X_F \otimes {}_FM_G).$$

THEOREM 3. *Given a pair $A \subseteq B$, let $B \subseteq C$ be the pair with $C = \operatorname{End}(B_A)$. If $({}_FM_G, X_F)$ defines $A \subseteq B$ then*

$$(\operatorname{Hom}_G({}_FM_G, {}_GG_G), X_F \otimes {}_FM_G)$$

defines $B \subseteq C$.

In particular, to a pair $A \subseteq B$ one can attach a the weighted valued graph of the corresponding k-species $(F, G, {}_FM_G)$ (see e.g. [DR1]). Write $U = (u_{ij})$, $V = (v_{ij})$.

Given a k-pair $A \subseteq B$, the fundamental construction of V.F.R. Jones [J1] defines the tower of finite dimensional semi-simple k-algebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_p \subseteq A_{p+1} \subseteq \ldots,$$

where $A_0 = A$, $A_1 = B$ and $A_{p+1} = \operatorname{End}(A_p)_{A_{p-1}}$ for $p \geq 1$.

Let ${}_FM_G$ be the F-G-bimodule associated with the k-pair $A_0 = A \subseteq B = A_1$ as in Theorem 1. Then, in view of Theorem 3, the bimodule associated with the k-pair $A_p \subseteq A_{p+1}$ is either ${}_FM_G$ if p is even or ${}_GM_F^* = \operatorname{Hom}_G({}_FM_G, {}_GG_G)$ if p is odd.

Let us attach to the k-pair $A \subseteq B$ the *tower transformation* t_{AB} of the a-dimensional real space $\mathbb{R}^{(a)}$ defined by

$$t_{AB}(\mathrm{x}) = \mathrm{x}UV^{tr}$$

and recall [GHJ] that the *index* of $A \subseteq B$ is

$$[B : A] = \limsup_{p \to \infty} |t_{AB}^p(\mathrm{x})|^{1/p},$$

where the norm $|\ldots|$ denotes the sum of the (positive) coordinates of $t_{AB}^p(\mathrm{x})$. Thus, the Perron–Frobenius theory yields that $[B : A]$ is the largest real eigenvalue of t_{AB}.

We attach to a given k-pair $A \subseteq B$, the Coxeter transformation c_{AB} of the real space

$$\mathbf{R}^{(a+b)} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbf{R}^{(a)},\ \mathbf{y} \in \mathbf{R}^{(b)}\}$$

given by the matrix

$$\begin{pmatrix} -I & -U \\ V^{tr} & V^{tr}U - I \end{pmatrix} = \begin{pmatrix} -I & 0 \\ V^{tr} & I \end{pmatrix} \begin{pmatrix} I & U \\ 0 & -I \end{pmatrix}$$

(see [Bo]). It is well-known that the eigenvalues of c_{AB} and t_{AB} are closely related: if λ is an eigenvalue of c_{AB}, then $2 + \lambda + \lambda^{-1}$ is an eigenvalue of t_{AB}, and all eigenvalues of t_{AB} are obtained this way.

Thus, the theorem characterizing the representation type of a hereditary tensor algebra [DR1] and the basic properties of the eigenvalues of Coxeter transformations clarify the link between the values of $[B : A]$ and the representation type of $\mathcal{A}(A, B) = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$, answering a question raised by Jones in [J2].

THEOREM 4. *Let $A \subseteq B$ be a (connected) pair of finite-dimensional semi-simple k-algebras, and $[B : A]$ its index. Then*

(i) $[B : A] < 4$ if and only if $\mathcal{A}(A, B)$ is of finite representation type, and, in this case,

$$[B : A] = 4\cos^2 \tfrac{\pi}{n} \text{ for some } n \geq 3.$$

(ii) $[B : A] = 4$ if and only if $\mathcal{A}(A, B)$ is of tame representation type.

(iii) $[B : A] > 4$ if and only if $\mathcal{A}(A, B)$ is of wild representation type, and, in this case, $[B : A] \geq \rho_0$, where $\rho_0 = 4.02641794918695985 99\ldots$ is the largest real root of the polynomial

$$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1.$$

Let us also point out that Zhang [Z] has established that $\lambda_* = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{23}{108}}}$ is the least accumulation point of the largest real eigenvalues of the graphs which are neither Dynkin nor Euclidean. Thus, the value $\rho_* = 2 + \rho_* + \rho_*^{-1} = 4.079595623\ldots$ is the corresponding accumulation point in the set of the values of the index.

A paper with complete proofs of the statements will appear elsewhere.

REMARK. The above results were presented at the Tsukuba International Conference on Representations of Algebras on August 16, 1990. Subsequently, we have learnt at the ICM in Kyoto that S. Popa has also derived the existence of the gaps for the index of pairs of multimatrix algebras; in particular, he established the gap $[4, \rho_0]$ which coincides with the gap obtained for general pairs of semi-simple algebras.

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Received: Sept. 10, 1990

PYTHAGOREAN AND EUCLIDEAN ORDERED VALUED FIELDS

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Presented by P. *Ribenboim*, F.R.S.C.

Abstract. Necessary and sufficient conditions are given for an ordered valued field (F, v) to be Pythagorean (resp. Euclidean). These results are applied to relative Henselizations of (F, v), and relative Neumann-completions of (F, v) in immediate, valued formal power series field extensions $(F^\wedge, v^\wedge)$ of (F, v).

0. **Introduction.** Call a group $(\Gamma, \cdot, 1)$ <u>divisible by</u> $n \in N$ if for all $\gamma \in \Gamma$, there is a $\delta \in \Gamma$ with $\delta^n = \gamma$. For subsets E of an ordered group $(G, <, +, 0)$, let $E^* = \{g \in E: g \neq 0\}$, $E^> = \{g \in E: g > 0\}$, and let $\Sigma(E) = \{\sum_{i=1}^n x_i: n \in N$, and $x_1, \ldots, x_n \in E\}$. For subsets S of a field F, let $S^2 = \{x^2: x \in S\}$. Recall that F is <u>Pythagorean</u> if $\Sigma(F^2) = F^2$. Henceforth assume that (F, v) is an ordered valued field (i.e., F is an ordered field and v is an o-valuation of F [5, p. 39]). Let A be its valuation ring, M its maximal ideal, U its units, let λ be its place, K its residue class field, and let $(G, <, +, 0)$ be its value group ordered such that $v(x + y) \leq$ Max.$\{v(x), v(y)\}$, for all $x, y \in F$. Let $P = \{x \in F: x \geq 0\}$, and recall that F is <u>Euclidean</u> if $P = F^2$. A <u>coefficient field</u> of (F, v) is a subfield L of A, for which $\lambda(L) = K$.

1. Two lemmas.

LEMMA 0. F is Euclidean iff (a) K is Euclidean, (b) G is divisible by 2, and (c) $(1 + M, \cdot, 1)$ is divisible by 2.

PROOF. Assume F is Euclidean. For $k \in K^{>}$ there is an $a \in A$ with $\lambda(a) = k$. Since λ preserves $\leq$, $a > 0$. Since F is Euclidean, there exists $b \in F$ with $b^2 = a$. Since $0 \geq v(a) = 2v(b)$, $b \in A$. $(\lambda(b))^2 = k$, proving (a). For $g \in G$, there is an $x \in F^{>}$ with $v(x) = g$. Since F is Euclidean, there is a $y \in F$ with $y^2 = x$. $g = 2v(y)$, proving (b). Let $m \in M$. Since F is Euclidean, there exists a $z \in F^{>}$ with $z^2 = 1 + m$. Since $2v(z) = v(1 + m) = v(1) = 0$, $z \in U^{>}$. Since $\lambda(z)^2 = \lambda(1 + m) = 1$ and $z > 0$, $\lambda(z) = 1$. There is a $\mu \in M$ with $z = 1 + \mu$, proving (c). Now assume (a), (b), and (c). Let $x \in F^{>}$. By (b), there is an $h \in G$ with $2h = v(x)$. There is a $y \in F$ with $v(y) = h$; hence $v(y^2) = v(x)$, and $x/y^2 \in U^{>}$. Let $k = \lambda(x/y^2)$. Since $x/y^2 \in A^{>}$, and since λ preserves $\leq$, $k \in K^{>}$. By (a), there is a $c \in K$ with $c^2 = k$. There is an $a \in A$ with $\lambda(a) = c$. Since $\lambda(x/y^2a^2) = \lambda(x/y^2)\lambda(1/a^2) = k/c^2 = 1$, there is an $m \in M$ with $x/y^2a^2 = 1 + m$. By (c), $1 + m$ has a square root in $\Gamma + M$. Let $z = ya\sqrt{(1 + m)} \in F$; then $x = y^2a^2(\sqrt{(1 + m)})^2 = z^2$, proving F is Euclidean. □

For $x, y \in F^*$, let $g = v(x)$, and $h = v(y)$. Without loss of generality, we may let $g \geq h$. Let $z = y/x$; then $x^2 + y^2 = x^2(1 + z^2)$.

.0) (i) F is Pythagorean iff (ii) $1 + M^2$ and $1 + U^2$ are subsets of F^2.

PROOF. Clearly, (i) implies (ii). Assume (ii). If $g > h$, $z \in M$. If $g = h$, $z \in U$. In both cases $x^2 + y^2 \in F^2$, proving (i). □

LEMMA 1. Assume that (F, v) has a coefficient field L. F is Pythagorean iff (α) K is Pythagorean, and (β) Π, defined to be $\{1 + 2\ell m/(1 + \ell^2) + m^2/(1 + \ell^2) : \ell \in L \text{ and } m \in M\}$, is a subset of F^2.

PROOF. Assume (α) and (β). Since $g \geq h$, $z \in A$; thus there are $\ell \in L$ and $m \in M$ with $z = \ell + m$. Hence $1 + z^2 = 1 + \ell^2 + 2\ell m + m^2$. By (α), there is a $\delta \in L$ with $\delta^2 = 1 + \ell^2$. Since -1 is not in K^2, $\delta \neq 0$. $1 + z^2 = \delta^2 + 2\ell m + m^2 = \delta^2(1 + 2\ell m/(1 + \ell^2) + m^2/(1 + \ell^2))$. By (β), there exists $f \in F$ with $1 + 2\ell m/(1 + \ell^2) + m^2/(1 + \ell^2) = f^2$. Hence $x^2 + y^2 = (x\delta f)^2 \in F^2$, proving that F is Pythagorean. Now assume that F is a Pythagorean field. Let $\ell, \rho \in L$. $\ell^2 + \rho^2 = f^2$, for some $f \in F$. $\lambda(\ell)^2 + \lambda(\rho)^2 = \lambda(f)^2$, proving (α). For $\ell \in L$, $m \in M$, let $z = \ell + m$. $1 + z^2 = 1 + \ell^2 + 2\ell m + m^2$. By (α), there is a $\delta \in L$ with $\delta^2 = 1 + \ell^2$. Since -1 is not in K^2, $\delta \neq 0$. Then $\delta^2(1 + 2\ell m/(1 + \ell^2) + m^2/(1 + \ell^2)) = \delta^2 + 2\ell m + m^2 = 1 + z^2$. Since F is Pythagorean, there exists $f \in F$ with $f^2 = 1/\delta^2 + z^2/\delta^2$. Hence $(1 + 2\ell m/(1 + \ell^2) + m^2/(1 + \ell^2)) = f^2$, proving (β). □

2. Applications. Let $\mathrm{Hen}(F, v) = \{f \in A[X] : f \text{ is monic}, f(0) \in M, \text{ and } f'(0) \in U\}$. Call (F, v) <u>Henselian relative to a subset</u> H <u>of</u> Hen(F, v) if $f \in H$ implies there exists $m \in M$ with $f(m) = 0$. (F, v)

is Henselian iff it is Henselian relative to Hen(F, v) [3, 16.6]. Let $(F^{\sim}, v^{\sim})$ be the Henselization of (F, v). Call the smallest valued subfield of $(F^{\sim}, v^{\sim})$ containing (F, v) that is Henselian relative to H *the Henselization of* (F, v) *relative to* H. For subsets S of M, define $H_2(S)$ to be $\{(1 + X)^2 - (1 + \mu) : \mu \in S\}$.

COROLLARY 0. F is Euclidean iff K is Euclidean, G is divisible by 2, and (F, v) is Henselian relative to $H_2(M)$.

COROLLARY 1. Assume that (F, v) has a coefficient field. F is Pythagorean iff K is Pythagorean and (F, v) is Henselian relative to $H_2(\Pi - 1)$.

By a slight extension of Theorem 2 of [6], there exists an order-preserving monomorphism Ψ of (F, v) into the valued formal power series field $(F^{\wedge}, v^{\wedge}) = (H(G, K, \rho), v^{\wedge})$ [5, pp. 49-64] (ρ being a positive factor set of G in K); with $(F^{\wedge}, v^{\wedge})$ an immediate extension of $(F_0, v_0) = (\Psi(F), v^{\wedge}|\Psi(F))$, and $v = v^{\wedge} \circ \Psi$. Elements of the ring K[[X]] of formal Taylor series with coefficients in K, may be evaluated over $M^{\wedge}$ [4, 4.7]. (F_0, v_0) is *Neumann-complete relative to a subset* Ξ *of* $K[[X]] \times 2^M$ if for all $(f, M(f))$ in Ξ, $f(M(f))$ is a subset of F_0 [1]. For subsets S of M, define $\nabla_2(S)$ to be $\{\sum_{j=0}^{\infty} \binom{1/2}{j} \cdot X^j, S\}$. From Lemma 0, Lemma 1, and an extension of [4, 4.91], obtained by using a slight extension of Theorem 7.30 of [2], we have the following.

COROLLARY 3. F_0 is Euclidean iff K is Euclidean, G is divisible by 2, and (F_0, v_0) is Neumann-complete relative to $\nabla_2(M_0)$.

COROLLARY 4. Assume that K is a subfield of F_0. F_0 is Pythagorean iff K is Pythagorean and (F_0, v_0) is Neumann-complete relative to $\nabla_2(\Pi_0 - 1)$.

Let $\nabla_2(M_0)(F_0)^0 = F_0$; let $\nabla_2(M_0)(F_0)^n$ be the field generated by $\nabla_2(M_0)(F_0)^{n-1}$ and $\{\sum_{j=0}^{\infty} \binom{1/2}{j}\cdot m^j$: $m\in\nabla_2(M_0)(F_0)^{n-1}$ and $v\hat{}(m) < 0\}$ in $F\hat{}$; and let $\nabla_2(M_0)(F_0)^{\omega}$ be the union of $(\nabla_2(M_0)(F_0)^n)_{n<\omega}$.

COROLLARY 5. Assume that K is a Euclidean subfield of F_0, and that G is divisible by 2. $\nabla_2(M_0)(F_0)^{\omega}$ is the smallest Euclidean subfield of $F\hat{}$ that contains F_0.

Let $\nabla_2(\Pi_0 - 1)(F_0)^0 = F_0$; let $\nabla_2(\Pi_0 - 1)(F_0)^n$ be the subfield of $F\hat{}$ generated by $\nabla_2(\Pi_0 - 1)(F_0)^{n-1}$ and $\{\sum_{j=0}^{\infty} \binom{1/2}{j}\cdot\mu^j$: $\mu = 2\ell m/(1 + \ell^2) + m^2/(1 + \ell^2)$, $\ell\in L$, $m\in\nabla_2(\Pi_0 - 1)(F_0)^{n-1}$, and $v(m) < 0\}$; and let $\nabla_2(\Pi_0 - 1)(F_0)^{\omega}$ be the union of $(\nabla_2(\Pi_0 - 1)(F_0)^n)_{n<\omega}$.

COROLLARY 6. Assume that K is a Pythagorean subfield of F_0. $\nabla_2(\Pi_0 - 1)(F_0)^{\omega}$ is the smallest Pythagorean subfield of $F\hat{}$ that contains F_0.

3. Remark. By [2, 7.41], an extension of Neumann's Theorem [4], each $f(X_1, \ldots, X_n)$ in $K[[X_1, \ldots, X_n]]$ may be evaluated over $M_0{}^n$. The degree of dependence of such constructions on the embedding Ψ has been considered, and will be reported on soon.

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Received August 20, 1990

SERIES FORMELLES GENERALISEES SUR UN CORPS PYTHAGORICIEN

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Résumé :

Dans tout l'article K désigne un corps commutatif, G un groupe abélien totalement ordonné et $K((T^G))$ le corps des séries formelles généralisées à coefficients dans K et à exposants dans G.

Dans ce travail on donne une condition nécessaire et suffisante pour que $K((T^G))$ soit pythagoricien et on explicite une clôture pythagoricienne de ce corps.

Si $G = \mathbb{Z}$, les résultats sont connus dans [2] et [6].

Proposition. 1:

Le corps $K((T^G))$ est ordonné maximal si et seulement si le corps K est ordonné maximal et le groupe G est divisible.

Exemple:

Le corps $\mathbb{R}((T^{\mathbb{Q}}))$ est ordonné maximal.

Lemme. 1.

Si la caractéristique de K est différente de 2, Alors pour toute série x de $K((T^G))$ de valuation strictement positive, la série 1+x admet une racine carrée dans $K((T^G))$.

Proposition 2 :

Soit $(K,<)$ un corps ordonné. On met sur $K((T^G))$ l'ordre défini par Neumann [4].

Notons P l'ensemble des éléments positifs de K et $\mathcal{P}$ ceux de $K((T^G))$.

Alors : $[K((T^G))]^2 = \mathcal{P}$ si et seulement si $K^2 = P$ et G est 2-divisible.

Définition :

Le corps K est dit pythagoricien si $K^2 + K^2 = K^2$.

Exemples :

1) Soient K un corps ordonné maximal, donc pythagoricien et G un groupe divisible. D'après la proposition (1), le corps $K((T^G))$ est ordonné maximal, donc pythagoricien.

2) Soient K un corps ordonné dont l'ensemble des éléments positifs est égal à K^2 et G un groupe 2-divisible.

D'après la proposition (2), l'ensemble des éléments positifs du corps ordonné $K((T^G))$ est $[K((T^G))]^2$. Donc $K((T^G))$ est pythagoricien.

Ces exemples seront généralisés à un groupe G quelconque.

Remarque :

En caractéristique 2, tout corps est pythagoricien.

On suppose dans la suite que la caractéristique de K est différente de 2.

Proposition 3 :

$[K((T^G))]^2 = K((T^G))$ si et seulement si $K^2 = K$ et G est 2-divisible.

Théorème 1 :

$K((T^G))$ est pythagoricien $\Leftrightarrow$ $\begin{cases} \text{i) } K^2 = K \text{ et } G \text{ est 2-divisible,} \\ \quad \text{ou} \\ \text{ii) } K \text{ est pythagoricien et ordonnable.} \end{cases}$

Corollaire 1:

Si K est de caractéristique impaire, les trois propriétés suivantes sont équivalentes :

i) $K((T^G))$ est pythagoricien.

ii) $[K((T^G))]^2 = K((T^G))$.

iii) $K^2 = K$ et G est 2-divisible.

Corollaire 2 [2, cor.19] :

$K((T))$ est pythagoricien si et seulement si K est pythagoricien et ordonnable.

Corollaire 3 :

Le corps des séries de Puiseux $K((T^{1/\infty})) = \bigcup_{n:1}^{\infty} K((T^{1/n}))$ est pythagoricien si et seulement si K est pythagoricien.

Théorème 2 :

Soient K un corps de caractéristique différente de 2 et de clôture pythagoricienne P et G un groupe abélien totalement ordonné.

Dans le cas où K n'est pas ordonnable, on suppose de plus que G est 2-divisible.

Alors : $\widetilde{K} = \bigcup_{\underbrace{K \subset L \subset P}_{\text{fini}}} L((T^G))$ est une clôture pythagoricienne de $K((T^G))$.

Remarques :

1) Dans le cas où K n'est pas ordonnable, l'hypothèse G est 2-divisible est impérative dans le théorème.

En effet : $-1 \in \Sigma K^2 \subset \Sigma P^2 = P^2$. Soit $i \in P$ tel que : $i^2 = -1$.

Soit $\alpha \in G$ non 2-divisible. la série : $\left(\frac{1+T^\alpha}{2}\right)^2 + \left(\frac{i-iT^\alpha}{2}\right)^2 = T^\alpha$ ne peut pas être un carré dans $\widetilde{K}$.

2) Avec les notations et les hypothèses du théorème (2), on suppose de plus que K n'est pas pythagoricien. D'après [6], l'extension P/K est séparable

de degré infini on peut donc trouver une suite $(x_n)_{n \in \mathbb{N}}$ d'éléments de P tels que : $\lim_{n \to +\infty} \deg_K x_n = +\infty$.

Soit $\alpha \in G^+ - \{0\}$. La série $f(T) = \sum_{n:0}^{\infty} x_n T^{n\alpha}$ de $P((T^G))$ n'appartient pas à $\widetilde{K}$.

Donc $P((T^G))$ contient strictement $\widetilde{K}$.

Corollaire :

Soient p un entier premier impair, $\overline{\mathbb{F}}_p$ la clôture algébrique du corps premier $\mathbb{F}_p$, K un sous-corps de $\overline{\mathbb{F}}_p$ et G un groupe 2-divisible.

Alors $\bigcup_{n:0}^{\infty} K\mathbb{F}_{p^{2^n}}((T^G))$ est une clôture pythagoricienne de $K((T^G))$.

En particulier, le corps $\bigcup_{n:0}^{\infty} \mathbb{F}_{p^{m2^n}}((T^G))$ est une clôture pythagoricienne de $\mathbb{F}_{p^m}((T^G))$.

Lemme 2 :

Soient K un corps non ordonnable de caractéristique différente de 2 et de clôture pythagoricienne P, G un groupe abélien totalement ordonné et $\mathcal{P}$ la clôture pythagoricienne de $K((T^G))$.

Alors : $\forall \alpha \in G-\{0\}$, on a : $T^{\alpha/2^n} \in \mathcal{P}$.

Soient K un corps non ordonnable de clôture pythagoricienne P et de caractéristique différente de 2 et G un groupe non 2-divisible.

D'après [3,ch.8, § 3.1], on a : $P^2 = P$.

On note $G' = \left\{ \frac{\alpha}{2^n} / \alpha \in G,\ n \in \mathbb{N} \right\}$: c'est un groupe 2-divisible.

D'après le théorème (1), le corps $P((T^{G'}))$ est phthagoricien.

D'après le théorème (2), le sous-corps : $\widetilde{K} = \bigcup_{\substack{K \subset L \subset P \\ \text{fini}}} L((T^{G'}))$ est aussi pythagoricien.

Soient $\overline{K}$ une clôture algébrique de K contenant P, $\Delta(G)$ l'enveloppe divisible de G, $\mathfrak{B}(G)$ l'ensemble des parties bien ordonnées de G, et

$\mathcal{F}(G) = \left\{ X \subset \Delta(G) : \langle X \cup G \rangle / G \text{ de type fini},\ X = \frac{1}{d} A,\ d \in \mathbb{N}^*,\ A \in \mathfrak{B}(G) \right\}$.

Dans cette notation $\langle X \cup G \rangle$ désigne le sous groupe de $\Delta(G)$ engendré par

$X \cup G$ et $\langle X \cup G \rangle / G$ le groupe quotient.

Soit $\mathcal{P}(G)$ l'ensemble des éléments de $\mathcal{F}(G)$ pour les quels d est une puissance de 2. Les éléments de $\mathcal{P}(G)$ sont des parties de $G^{\cdot}$.

Dans [5] on montre que si L est un corps, alors l'ensemble :

$L(\mathcal{F}(G)) = \{f(T) \in L((T^{\Delta(G)})) \,/\, S(f) \in \mathcal{F}(G)\}$ est un sous-corps de $L((T^{\Delta(G)}))$, où $S(f)$ est le support de f.

Considérons l'ensemble : $L(\mathcal{P}(G)) = \{ f \in L(\mathcal{F}(G)) / S(f) \in \mathcal{P}(G) \}$ et le corps : $\hat{K} = \bigcup_{\underbrace{K \subset L \subset K}_{\text{fini}}} L(\mathcal{F}(G))$.

Si K est de caractéristique nulle, d'après [1, Prop.3], le corps $\hat{K}$ est une clôture algébrique de $K((T^G))$.

Si K est de caractéristique p non nulle, d'après [1, Prop.2], toute extension finie de $\hat{K}$ est de degré égal à une puissance de p. Dans les deux cas $\hat{K}$ est pythagoricien. Il en est de même pour $\tilde{K} \cap \hat{K}$.

Théorème 3:

Avec les notations et les hypothèses précédentes, le corps :

$$\tilde{K} \cap \hat{K} = \bigcup_{\underbrace{K \subset L \subset K}_{\text{fini}}} L(\mathcal{P}(G)) \text{ est une clôture pythagoricienne de } K((T^G)).$$

Corollaire 1:

Soient p un entier premier impair, $\overline{\mathbb{F}}_p$ une clôture algébrique du corps premier $\mathbb{F}_p$ et K un sous-corps de $\overline{\mathbb{F}}_p$. Soit G un groupe non 2-divisible.

Alors $\bigcup_{n:0}^{\infty} K\mathbb{F}_{p^{2^n}}(\mathcal{P}(G))$ est une clôture pythagoricienne de $K((T^G))$.

En particulier, une clôture pythagoricienne de $\mathbb{F}_{p^m}((T^G))$ est

$$\bigcup_{n:0}^{\infty} \mathbb{F}_{p^{m2^n}}(\mathcal{P}(G)).$$

Corollaire 2 [2, Prop.18] :

Soit K un corps non ordonnable de clôture pythagoricienne P.

Alors $\bigcup_{n=0}^{\infty} P.K((T^{\frac{1}{2^n}}))$ est une clôture pythagoricienne de K((T)).

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Reçu le 5 juin 1990

Squares in Fields of Generalized Power Series

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Abstract: The paper is devoted to the investigation of properties of fields of generalized power series, which are related to squares and reality.

1. Let R be a commutative ring with unit element, let $(S, \leq)$ be an ordered commutative additive monoid; assume that the order is strict: if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$. Let $A = [[R^{S,\leq}]]$ be the *ring of generalized power series with coefficients in R and exponents in S*. Its elements are the maps $f : S \to R$ with support $\operatorname{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ artinian (i.e. containing no infinite descending chain) and narrow (i.e., containing no infinite subset of pairwise order-incomparable elements). The addition is pointwise and the multiplication is well defined by $(f * g)(s) = \sum_{t+u=s} f(t)g(u)$, for every $s \in S$. For details, se [Ri 1], [Ri 2].

An order $\leq$ on S is *subtotal* if for every $s, t \in S$ there exists an integer $k \geq 1$ such that $ks \leq kt$ or $kt \leq ks$.

If $\leq$ is a subtotal order, let $\leq'$ be defined by $s \leq' t$ when there exists $k \geq 1$ such that $ks \leq kt$. Then $\leq'$ is a compatible total order on S, said to be *associated* to $\leq$. If $(S, \leq)$ is a subtotally ordered group, then necessarily S is torsion-free.

It was shown in [El-Ri] that A is a field if and only if R is a field and $(S, \leq)$ is a subtotally ordered group. In this situation, for every $f \in A$, $f \neq 0$, let $\pi'(f)$ be

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX the first element (in the total order $\leq'$ associated to $\leq$) of supp(f). Then π' is a valuation of the field A, called the *natural valuation.*

2. Let F be a field. I recall some well-known definitions and introduce notations.

$\boxed{m}_F$ =set of all sums of m squares in F.

$\square_F$ = set of sums of squares in F.

$\lambda(F)$ = the *level* of F = the smallest $n \geq 1$ such that $-1 \in \boxed{n}_F$, or ∞, if $-1 \notin \square_F$.

$PN(F)$ = the *Pythagoras number* of F = the smallest integer m such that $\square_F = \boxed{m}_F$.

F is a *real field* when $-1 \notin \square_F$.

F is a *pythagorean field* when $PN(F) = 1$.

$Py(F)$ = the pythagorean closure of F.

F is an *euclidean* field when $F = F^2 \cup (-F^2)$.

If F is a real field and $F^\bullet$ a real closure of F, let $Eu_{F^\bullet}(F)$ denote the euclidean closure of F in $F^\bullet$.

3. In this note, I announce the results stated below. The proofs will be published elsewhere [Ri 4]. It should be observed that it is not assumed that the order $\leq$ is total; for this particular case, some of the results were obtained independently by Alling [Al] and Benhissi [Be]. There are of course many instances of subtotally ordered groups which are not totally ordered.

(3.1). $\lambda(R) = \lambda(A)$. In particular, R is a real field if and only if A is a real field.

(3.2). If R is a real field, then $PN(A) = PN(R)$.

(3.3). If R is not a real field, then $\lambda(R) \leq PN(A) \leq 1+\lambda(R)$. If R has characteristic $\text{char}(R) \neq 2$ and $S \neq 2 \cdot S$, then $PN(A) = 1 + \lambda(R)$.

(3.4). Assume $\text{char}(R) \neq 2$.

i) If R is a real field, then A is a pythagorean field if and only if R is pythagorean.

ii) If R is not a real field, then the following conditions are equivalent:

a) A is pythagorean.

b) $A = A^2$.

c) $R = R^2$ and $S = 2 \cdot S$.

d) R is pythagorean and $S = 2 \cdot S$.

(3.5). Let $\text{char}(R) \neq 2$, let S be a torsion-free group, let $(\mathbb{Q}S, \leq)$ be subtotally ordered. For every $n \geq 1$, let $\leq$ still denote the restriction of $\leq$ to $\frac{1}{n}S$; let $A_n = [[R^{\frac{1}{n}S,\leq}]]$ (for $n \geq 1$) and $A^* = \bigcup_{n=1}^{\infty} A_n$. Then A^* is pythagorean if and only if R is pythagorean.

(3.6). If R is a real field or $S = 2 \cdot S$, then $Py(A) = \bigcup_F [[F^{S,\leq}]]$ (where F is any subfield of $Py(R)$, $[F : R] < \infty$).

(3.7). Assume that $\text{char}(R) \neq 2$ and R is not a real field; assume also that S is a torsion-free group which is not 2-divisible, $S/2 \cdot S$ is finite and $(\mathbb{Q}S, \leq)$ is subtotally ordered. Then (with previous notations) $Py(A) = \bigcup_F \bigcup_{n=1}^{\infty} [[F^{\frac{1}{n}S,\leq}]]$ (where F is any subfield of $Py(R)$ such that $[F:R] < \infty$).

(3.8). A is a real closed field if and only if R is real closed and S is divisible.

(3.9). Let R be a real field, R^* a real closure; let $(S, \leq$ be a subtotally ordered torsion-free divi8sible group. Then $A^* = \bigcup_F [[F^{S,\leq}]]$ (where $R \subset F \subset R^*$ and $[F:R] < \infty$) is a real closure of A.

(3.10). Let R be a real field, R^* a real closure of R; let S be a torsion-free group and let $\leq$ be a compatible subtotal order on $\mathbf{Q}S$. Then:

i) $A^* = \bigcup_F \bigcup_{n=1}^{\infty} [[F^{\frac{1}{n}S,\leq}]]$ (with $R \subset F \subset R^*, [F:R] < \infty$) is a real closed field.

ii) If moreover, S/nS is finite, for every $n \geq 1$, then A^* is a real closure of A.

(3.11). A is an euclidean field if and only if R is an euclidean field and $S = 2 \cdot S$.

Let R be a real field, R^* a real closure of R, let $(S, \leq)$ be a subtotally ordered divisible group, $A = [[R^{S,\leq}]]$ and A^* as in (3.9).

(3.12). $Eu_{A^*}(A) = \bigcup_F [[F^{S,\leq}]]$ (for $R \subset F \subset Eu_{R^*}(R), [F:R] < \infty$).

Let R be a real euclidean field, R^* a real closure of R, S a torsion-free group which is n-divisible, for every $n \geq 1$, and such that $S/2 \cdot S$ is finite. Let $\leq$ be a subtotal compatible order on $\mathbf{Q}S = \frac{1}{2^\infty}S$ and $A^* = \bigcup_F \bigcup_{k=1}^{\infty} [[F^{\frac{1}{2^k}S,\leq}]] \subseteq [[R^{*\mathbf{Q}S,\leq}]]$, where $R \subset F \subset R^*$ and $[F:R] < \infty$; by (3.10) A^* is a real closure of A.

(3.13). With above hypothesis and notations:

$$Eu_{A^*}(A) = \bigcup_{n=0}^{\infty} E_n,$$

$$\text{where } E_0 = A, \; E_{n+1} = E_n\left(e_{s/2^{n+1}}\right)_{s\in S}.$$

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Received September 18, 1990

A remark on the equation of Fermat

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Presented by P. Ribenboim, F.R.S.C.

Abstract. In this note we improve a well-known and old result of A. Grünert concerning the equation of Fermat.

1. Introduction

From the well-known and old result of A. Grünert ([2], see also [3], p. 226) it follows that if n is an integer, then $n \geq 2$ and $0 < x < y < z$ are integers satisfying the equation of Fermat

(1) $$x^n + y^n = z^n$$

then

(2) $$x > n$$

We prove the following theorem, which improves (2).

Theorem. *Let k be a positive integer. If $n > (2k+1)c$, where $c = \frac{\log 2}{2(1-\log 2)}$ and if the equation (1) has a solution in integers $0 < x < y < z$ then*

(3) $$x > \begin{cases} n+k & \text{if } z-y=1 \\ 2(n+k) & \text{if } z-y=2 \\ \dfrac{2^n}{n}(n+k) & \text{if } z-y>2 \end{cases}$$

2. Lemmas

Lemma 1. *Let k be a positive integer. If the equation (1) has a solution in positive integers $x < y < z$ for $n > (2k+1)c$, where $c = \frac{\log 2}{2(1-\log 2)}$ then*

$$z - y < \frac{x}{n+k}\,. \tag{4}$$

The proof of Lemma 1 follows immediately from the proof of Theorem 1 in [1].

Lemma 2. *If n, x, y, z are positive integers with $x < y < z$ and $(x, y, z) = 1$ such that (1) holds, then there exist $\delta \in \{0, 1\}$ and positive integers a, d with $d \mid n$ such that*

$$z - y = 2^{\delta} d^{-1} a^{n}\,. \tag{5}$$

The proof of Lemma 2 follows from a well-known result of Barlow and Abel (see [3], comp. [4], Chapter 11).

3. Proof of the Theorem

First we remark that from Lemma 1 it follows that

$$x > (z-y)(n+k) \tag{6}$$

and we get $x > n+k$ if $z - y = 1$ and $x > 2(n+k)$ if $z - y = 2$. Thus we can assume that $z - y > 2$. From (5) of Lemma 2 we find that $a > 1$. Indeed, if $a = 1$ then by (5) there follows

$$(z-y)d = 2^{\delta}\,. \tag{7}$$

Since $z - y > 2$ we have by (7) $2d < 2^{\delta}$ which is impossible because $\delta = 0$ or 1. From (5) for $a > 1$ we obtain

$$z - y \geq \frac{2^n}{d}\,. \tag{8}$$

Since $d \mid n$ thus $d \le n$ and by (8) we get

(9) $$z - y \ge \frac{2^n}{n} .$$

From (9) and (6) we now obtain

$$x > \frac{2^n}{n}(n+k)$$

and the proof of our Theorem is complete.

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Received August 8, 1990

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