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LOCAL ALTERNATIVE RINGS AND FINITE ALTERNATIVE RIGHT CHAIN RINGS

BY

C. A. BAKER, N. D. LANE AND J. W. LORIMER

Presented by G. de B. Robinson, F.R.S.C.

Introduction. The Artin-Zorn theorem states that a finite alternative skew field is a field. Geometrically, this means that a finite Moufang projective plane is desarguesian. In this paper, we present some characterizations of local alternative rings, generalizing the associative case, and then use these results to prove that a finite alternative right chain ring is a projective Hjelmslev associative ring. Geometrically this means that a finite Moufang projective Klingenberg plane for which any two points have a joining line is a desarguesian projective Hjelmslev plane [3].

1. **Elementary properties of naring.** In this section, R is always a naring (a not necessarily associative ring with $1 \neq 0$). $U_r = \{a | \exists b, ab = 1\}$ ($U_l = \{a | \exists b, ba = 1\}$) is the set of right (left) invertible elements of R and $U = \{a | \exists b, ab = ba = 1\}$ is the set of invertible elements. $N_r = R \setminus U_r$, $N_l = R \setminus U_l$ and $N = R \setminus U$. A right (left) ideal I of R is *proper* if $I \neq R$.

1.1 *Every proper right (left) ideal of R is contained in $N_r(N_l)$ and hence in N .*

1.2 **DEFINITION.** A naring R is *local* if its set N of non-units forms an ideal.

As in the associative case, we have

1.3 **THEOREM.** *A naring R is local if and only if for each $a \in R$, a or $1-a$ is invertible, and $N \cdot U \subseteq N$ and $U \cdot N \subseteq N$.*

1.4 **PROPOSITION.** *A local naring R is of stable range two, i.e. $ax + y \in U$ implies $x + by \in U$ for some $b \in R$; cf. [7; p. 179 and Theorem 2].*

1.5 **DEFINITION.** An *alternative ring* is a naring R such that $a(ab) = a^2b$ and $(ab)b = ab^2$ for all $a, b \in R$. If $N = \{0\}$, then R is called an *alternative skew*

field. We recall that for an alternative ring R , the associator of a, b, c in R is $(a, b, c) = (ab)c - a(bc)$. Its elementary properties are given in [12] or [13; pp. 35,36].

1.6 DEFINITION. An *inversive ring* is a ring R such that $ab = 1$ implies $a(bc) = (ca)b = c$ for all $a, b, c \in R$. If $N = \{0\}$, then R is an *inversive skew field*.

1.7 Every alternative ring is inversive and every local inversive ring with $|R/N| \neq 2$ is a local alternative ring.

The first part is in [12] and the second is [6; Satz 3] with corrections from [1; Vol. 3, 22.3.6 and p. 601].

2. Local alternative rings. An extension of [13; Lemma 9, p. 205], obtained from the inversive laws, yields

2.1 PROPOSITION. Let R be an alternative ring and $a, b \in R$. If any two of a, b, ab, ba are invertible, then all of them are invertible.

Using 1.7 and 2.1, we can now prove the following results about U and N .

2.2 PROPOSITION. Let R be an alternative ring. Then

- (1) $N \cdot U \subseteq N$ and $U \cdot N \subseteq N$.
- (2) $ab \in U$ implies $a, b \in U$ or $a, b \in N$, for $a, b \in R$.
- (3) U is a multiplicative loop and $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in U$.
- (4) The following statements are equivalent.
 - (a) $R \cdot N \subseteq N$ and $N \cdot R \subseteq N$.
 - (b) $N \cdot N \subseteq N$.
 - (c) $ab \in U$ implies $a, b \in U$ for $a, b \in R$.
 - (d) $ab \in U$ implies $a \in U_r$ and $b \in U_l$, for $a, b \in R$; and $xy = 1$

implies $yx = 1$ for $x, y \in R$.

Now, for an alternative ring R , and $a \in R$, the principal right ideal generated by a is denoted by $\langle a \rangle R$. Using the results in [11] and [12], we can verify the following assertion.

2.3 PROPOSITION. Let R be an alternative ring and $a \in R$. The following statements are equivalent for a .

- (i)_a $1 + x$ is right invertible for all $x \in \langle a \rangle R$.
- (ii)_a $1 - x$ is right [left] invertible for all $x \in \langle a \rangle R$ [for all $x \in R \langle a \rangle$].
- (iii)_a $1 - x$ is invertible for all $x \in \langle a \rangle R$ [for all $x \in R \langle a \rangle$].
- (iv)_a $1 - x$ is invertible for all $x \in aR$ [for all $x \in Ra$].

The *Jacobson-Smiley radical*, $J(R)$, of an alternative ring R is the set of elements satisfying any one of the conditions in 2.3; cf. [13; p. 210]. By [13; Theorem 4, p. 207, Prop. 1, p. 199 and Theorem 5, p. 210], we have

2.4 PROPOSITION. *Let R be an alternative ring. Then $J(R)$ is the intersection of all maximal left [right] ideals and is the largest proper ideal of R consisting entirely of elements a such that $1-a$ is invertible.*

We can now present characterizations of local alternative rings generalizing the results for associative rings in [9; p. 75]. These results are used in the coordinatization of Moufang projective Klingenberg planes in [2], [3] and [4].

2.5 THEOREM. *Let R be an alternative ring with $J = J(R)$. The following statements are equivalent.*

- (1) R is local
- (2) For any $a \in R$, a or $1-a$ is right invertible.
- (3) For any $a \in R$, a or $1-a$ is invertible.
- (4) $J = N$.
- (5) R has a unique maximal right ideal and $\langle n \rangle R \neq R$ for each $n \in N_r$.
- (6) R/J is an alternative skew field.

Moreover, for any local alternative ring R , $U_r = U_l = U$ and $ab \in U$ implies $a, b \in U$ for all $a, b \in R$.

PROOF. By 1.3 and 2.2(1), we conclude that (1) and (3) are equivalent.

Hence (1) \Rightarrow (2).

(2) \Rightarrow (3). It suffices to show that $ab = 1$ implies $ba = 1$. Now $ab = 1$ yields $(a, b, c) = 0 = (b, a, c)$ for any $c \in R$, by [12]. Hence

$$(*) \quad (ba)c = b(ac) \text{ for any } c \in R.$$

We claim: $ba \in U_r$. If not, then (2) implies that $1 - ba \in U_r$. Hence

$1 = (1 - ba)s$ for some $s \in R$. Then, by (*) and 1.7, $a = a[(1 - ba)s] = a[s - (ba)s] = as - a[(ba)s] = as - a[b(as)] = as - as = 0$, in contradiction to $ab = 1$. Hence ba is right invertible and so $(ba)t = 1$ for some $t \in R$. By (*), $(ba)t = b(at)$. Then by 1.7, $a = a[(ba)t] = a[b(at)] = at$. As a consequence of (*) and the fact that $a = at$, we obtain $ba = b(at) = (ba)t = 1$.

(3) \Rightarrow (4). Assume R is local, so N is an ideal. By 1.1, $J \subseteq N$. Now take $a \in N \setminus J$. Then, by 2.3 (iv)a, there is an element $y \in R$ such that $1 - ay \in N$. By (3), $ay \in U$ and so $a \in U$, by 2.2(4); a contradiction. Hence $J = N$.

(4) \Rightarrow (5) Assume $J = N$. Then $J = N_{\mathbf{r}}$. Let M be a maximal right ideal of R . Then 2.4 and 1.1 imply $N_{\mathbf{r}} = J \subseteq M \subseteq N_{\mathbf{r}}$ and so $M = J$.

(5) \Rightarrow (6) Let M be the unique maximal right ideal of R . By 2.4, $J = M \subseteq N_{\mathbf{r}}$. Now $x \in N_{\mathbf{r}}$ implies $\langle x \rangle R \neq R$ and so $\langle x \rangle R \subseteq M$. Hence $x \in M$ and so $N_{\mathbf{r}} = M$. But $R/J = R/M$ has the descending chain condition and so by [7; p. 181 and Theorem 2], R is of stable range two and $N_{\mathbf{r}} = N$. Then $J = N$ and thus clearly $R/J = R/N$ is an alternative skew field.

(6) \Rightarrow (1). Assume R/J is an alternative skew field and hence local. By 1.4, R/J is of stable range two; and hence R is also of stable range two and $N = N_{\mathbf{r}}$, by [7; Theorem 2]. By 1.1, $J \subseteq N$. Conversely, if $n \in N \setminus J$, then $n+J$ is invertible in R/J and so there exists $m \in R$ such that $mn = 1-j$ with $j \in J$. By 2.4, $1-j$ is invertible. Since R is of stable range two, $mn + 0 \in U$ implies $n \in U$, a contradiction. Thus $N = J$ is an ideal.

The last assertion now follows from 1.4, 2.1 and 2.2(4).

3. Finite alternative right chain rings.

3.1 DEFINITION. R is an *alternative right (left) chain ring* if and only if R is an alternative ring and $a, b \in R$ implies $aR \subseteq bR$ or $bR \subseteq aR$ ($Ra \subseteq Rb$ or $Rb \subseteq Ra$). R is an *alternative chain ring* if R is both an alternative left and right chain ring.

We then obtain the following results.

3.2 THEOREM. Let R be an alternative ring.

(a) R is an alternative right chain ring if and only if $a, b \in N$ implies $aR \subseteq bR$ or $bR \subseteq aR$.

(b) If R is an alternative right chain ring then R is local.

PROOF. (a) follows easily from the inversive laws. (b) Take $a \in R$. Then there exists $x \in R$ so that $a = (1-a)x$ or $1-a = ay$ for some $y \in R$. Hence $1 = (1-a) + a = (1-a)(1+x)$ or $1 = a(1+y)$, and so a or $1-a$ is right invertible. Hence R is local by 2.5.

3.3 DEFINITION. R is an *projective Hjelmslev alternative ring* if and only if R is an alternative chain ring and satisfies the following conditions.

(H1) If $a \in N$, then there is an element $b \in R \setminus \{0\}$ such that $ab = 0$; i.e., every non-unit is a left zero divisor.

(H2) If $a \in N$, then there is an element $b \in R \setminus \{0\}$ such that $ba = 0$; i.e., every non-unit is a right zero divisor.

If R is an alternative right chain ring which satisfies (H1) and (H2), then R is called an *affine Hjelmslev alternative ring*. Clearly, an affine Hjelmslev alternative ring is local, by 3.2(b).

Finally, we can present our generalizations of the Artin-Zorn theorem [8; 6.20] by showing that finite alternative right chain rings are associative. Geometrically, this means that a finite Moufang projective Klingenberg plane for which any two points have a joining line is a desarguesian projective Hjelmslev plane [3].

3.4 THEOREM. *Let R be a finite naring. The following statements are equivalent.*

- (1) R is a projective Hjelmslev alternative ring.
- (2) R is an affine Hjelmslev alternative ring.
- (3) R is an alternative ring and $a, b \in N$ implies $a \in bR$ or $b \in aR$.
- (4) R is an alternative right [left] chain ring.
- (5) R is a local alternative ring with $J = aR$ [$J = Ra$] for some $a \in R$.
- (6) R is an associative right [left] chain ring.

PROOF. [(1) \Rightarrow (2)] is obvious and [(2) \Rightarrow (3)] follows from 3.2(a). By 3.2(a), (3) \Rightarrow (4).

(4) \Rightarrow (5): R is local by 3.2(b). Since R is finite and the set $\{xR \mid x \in R\}$ is a chain, we may assume $J = \{a_1, a_2, \dots, a_n\}$, where $a_1R \subseteq a_2R \subseteq \dots \subseteq a_nR$.

Hence $J \subseteq a_nR \subseteq J$, and so $J = a_nR$.

(5) \Rightarrow (6): By [6; Satz 4], R is associative. Since R is finite and hence artinian, J is nilpotent. Hence if $J \neq \{0\}$, then, by [10; Lemma 1.8(b)], $J^2 \neq J$ and $J = bR$ for all $b \in J \setminus J^2$, and so, by [5; Lemma 1], R is a chain ring.

(6) \Rightarrow (1): If $J = \{0\}$, then R is a finite skew field, and hence a field. Let $J \neq \{0\}$. By the corollary of [5; Lemma 1], R is a proper associative Hjelmslev ring.

Finally, we note that the structure of finite right chain rings and hence, from the result above, of finite alternative right chain rings is described in [5].

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, MOUNT ALLISON UNIVERSITY, SACKVILLE, NEW BRUNSWICK, CANADA, E0A 3C0.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, CANADA, L8S 4K1.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA, M5S 1A1.

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WHEN IS THE ASSOCIATED GRADED RING OF A
ONE-DIMENSIONAL GORENSTEIN RING, GORENSTEIN ?

Isabella Ramella

Presented by P. Ribenboim, F.R.S.C.

ABSTRACT. It is shown that if the associated graded ring $G(A)$, of a one-dimensional Gorenstein ring A , is reduced and the first difference of its Hilbert function is symmetric then $G(A)$ is Gorenstein.

We assume that (A, M) is a one-dimensional local Gorenstein ring with associated graded ring $G(A) = A/M \oplus (M/M^2) \oplus (M^2/M^3) \oplus \dots$ and residue field $A/M = k$. The ideal $G(M) = (M/M^2) \oplus (M^2/M^3) \oplus \dots$ is the maximal homogeneous ideal of $G(A)$. $H(n) = \dim_k(M^n/M^{n+1})$ ($n \in \mathbb{N}$) is the Hilbert function of A and $\Delta H(n) = H(n) - H(n-1)$ ($\Delta H(0) = 1$) is the difference function of $H(n)$. With $e = e(A)$ we denote the multiplicity of A (recall that $e = H(n)$, for $n \gg 0$) and with $\text{emdim } A = \dim_k(M/M^2) = H(1)$ the embedding dimension of A . We set:

$$\nu = \text{Min } \{n \mid H(n) = e\}.$$

If $G(A)$ is Gorenstein, then in [1, Theorem 3] it is proved that the first difference of the Hilbert function of A is symmetric, that is

$$\Delta H(n) = \Delta H(\nu - n), \quad 0 \leq n \leq \nu.$$

We show that if $G(A)$ is reduced the converse holds. This could give a hint for characterizing, in general (without the assumption $G(A)$ reduced), when $G(A)$ is Gorenstein, via Hilbert functions. We recall that various authors have shown that, under strong assumption on the ring A , $G(A)$ can be not Gorenstein. Only one general result [7, Theorem] is known so far: if $\text{emdim } A \leq 2$ or

$e \leq \text{semdim } A+1$ then $G(A)$ is Gorenstein.

THEOREM. If $G(A)$ is reduced and $\Delta H(n)=\Delta H(\nu-n)$, $0 \leq n \leq \nu$, then $G(A)$ is Gorenstein.

Proof. If $G(A)$ is reduced then the integral closure B of A coincides with the ring obtained by blowing up M in A , $U(M^n:M^n) = U\{b \in B \mid bM^n \subset M^n\}$, and $G(B)$ is the integral closure of $G(A)$ [6, Proposition 2.1]. We show first that:

$$\dim_k(B/A) = \dim_k(A/M^\nu)$$

$$\text{and} \quad \dim_k(G(B)/G(A)) = \dim_k(G(A)/G(M)^\nu).$$

In fact, on has:

$$\dim_k(A/M^\nu) = H(0) + \dots + H(\nu-1)$$

and

$$H(n) = H(\nu) - (\Delta H(n+1) + \dots + \Delta H(\nu)).$$

But $H(\nu) = e$ and then by assumption

$$H(n) = e - (\Delta H(0) + \dots + \Delta H(\nu-n-1)) = e - H(\nu-n-1).$$

Now, $B = M^\nu : M^\nu$ [3, Corollary 1.6], that is $M^\nu B = M^\nu$. Furthermore $\dim_k(B/M^\nu) = e_\nu$ [4, Theorem 12.4] and

$$\begin{aligned} 2\dim_k(A/M^\nu) &= 2[e_\nu - (H(\nu-1) + \dots + H(0))] = \\ &= 2e_\nu - 2\dim_k(A/M^\nu). \end{aligned}$$

Then $2\dim_k(A/M^\nu) = \dim_k(B/M^\nu)$ and

$$\dim_k(B/A) = \dim_k(B/M^\nu) - \dim_k(A/M^\nu) = \dim_k(A/M^\nu).$$

If we apply the same arguments as before to the rings $G(A)$, $G(B)$ and the ideal $G(M)^\nu$ we get also that $\dim_k(G(B)/G(A)) = \dim_k(G(A)/G(M)^\nu)$, as we wanted.

Let $I = \text{Ann}_A(B/A)$ be the conductor of A in B . If A is Gorenstein $\dim_k(B/A) = \dim_k(A/I)$ [4, Th.13.4], then $\dim_k(A/I) = \dim_k(A/M^\vee)$; now $M^\vee B \subset B$ implies $M^\vee \subset I$. Hence $I = M^\vee$. From this follows [5, Theorem 2.4] that $G(M)^\vee$ is the conductor of $G(A)$ in $G(B)$ and by the preceding $\dim_k(G(B)/G(A)) = \dim_k(G(A)/G(M)^\vee)$ that is $G(A)$ is Gorenstein [2, p.32].

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SPATIALLY PERIODIC STOKES FLOW STIRRED BY A ROTLET INTERIOR TO A CLOSED CORRUGATED BOUNDARY

D.W. Pravica

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Abstract : Spatially periodic solutions of the creeping flow equations are found for the stream function in which the motion is stirred by a two dimensional rotlet in the region interior to a closed corrugated boundary. Streamlines are given for different geometrical configurations. In some cases there is separation of the streamlines in the crevices of the corrugation.

§1. Introduction

Fluid motion in the presence of a periodic or corrugated boundary is of interest in connection with flow in a porous medium. The reader is referred to [1]-[7] for earlier work in this area.

The present paper considers the two dimensional fluid motion interior to a corrugated cylinder stirred by a rotlet or equivalently a circular cylinder of small radius λ rotating with angular velocity Ω . The corrugated boundary is modeled by the interior fluted column transformation in one case and by the inverse of the exterior fluted column in the second case. Exact explicit spatially periodic solutions of the Stokes equations are given for the stream function in the two flows. The main result of the paper is to demonstrate the sensitivity of the boundary geometry to the production of streamline separation in the two cases. The first case was reported earlier in [8] but the second case is a new analytical solution for spatially periodic flow. The results are of possible relevance to contamination at periodic boundaries induced by reverse flow.

§2. The Transformation

The exterior fluted column transformation is defined by

$$z = \zeta + \frac{\epsilon}{\zeta^{n-1}}, \quad z = x + iy, \quad \zeta = \rho e^{i\phi}, \quad (2.1)$$

where ϵ is a constant, and $n \geq 1$ is an integer. The mapping is conformal in the region $\rho \geq 1$, provided

$$0 \leq \epsilon(n-1) < 1. \quad (2.2)$$

The circle $|\zeta| = \rho = 1$, in the ζ - plane, maps into a closed curve \mathcal{C} which has n peaks lying between the circles $r = 1 \pm \epsilon$. Define analytic inversion by the transformation

$$z' = \frac{1}{z} \quad (2.3)$$

which represents geometric inversion plus a reflection in the real axis. The region exterior to $\rho = 1$ maps into the region interior to the inverted curve \mathcal{C}' of \mathcal{C} . The curve \mathcal{C}' is a closed corrugated boundary with n peaks lying between the circles $r' = 1/(1 \pm \epsilon)$.

§3. The Flow Problem

The flow problem to be considered in this section is the motion interior to the closed curve \mathcal{C}' in the z' - plane which is stirred by a rotlet at $z' = 0$. In the limit of zero Reynolds numbers the creeping flow equations describing the flow are

$$\underline{q}' = \text{curl}'(-\psi' \hat{k}) = -\frac{\partial \psi'}{\partial y'} \hat{i} + \frac{\partial \psi'}{\partial x'} \hat{j}, \quad (3.1)$$

where \underline{q}' is the fluid velocity and ψ' is the stream function which will satisfy the biharmonic equation given by

$$\nabla'^4 \psi' = 0, \quad (3.2)$$

and the boundary conditions

$$\psi' = \frac{\partial \psi'}{\partial \rho} = 0, \quad \text{at } \rho = 1. \quad (3.3)$$

In the z - plane the stream function ψ also satisfies the biharmonic equation and the same boundary conditions, but the forcing term is chosen to be

$$\psi_0 = \left(\rho^2 + \frac{2\epsilon}{\rho^{n-2}} \cos n\phi + \frac{\epsilon^2}{\rho^{2n-2}} \right) \log \rho. \quad (3.4)$$

It is a known result that a solution of (3.2) is given by

$$\begin{aligned} \psi' &\equiv \frac{\psi}{x^2 + y^2} = \frac{\psi}{\left[\rho^2 + \frac{2\epsilon}{\rho^{n-2}} \cos n\phi + \frac{\epsilon^2}{\rho^{2n-2}} \right]} \\ &= \log \rho - \frac{1}{2} \frac{(1 - \epsilon^2)}{[1 + (n-1)\epsilon^2]} \\ &+ \frac{1}{\left[\rho^2 + \frac{2\epsilon}{\rho^{n-2}} \cos n\phi + \frac{\epsilon^2}{\rho^{2n-2}} \right]} \left\{ - \frac{n\epsilon^2}{[1 + (n-1)\epsilon^2]} \left[\rho^2 + \frac{\epsilon}{\rho^{n-2}} \cos n\phi \right] \right. \\ &\quad \left. + \frac{\epsilon \cos n\phi}{\rho^n} + \frac{1}{2} \frac{1 - \epsilon^4 + 2n\epsilon^2}{1 + (n-1)\epsilon^2} \right\}, \end{aligned} \quad (3.5)$$

which represents flow forced by a rotlet. The transition to separated flow for the $n = 3$ case is shown in Fig. 1. This phenomena also occurs for all $n \geq 3$.

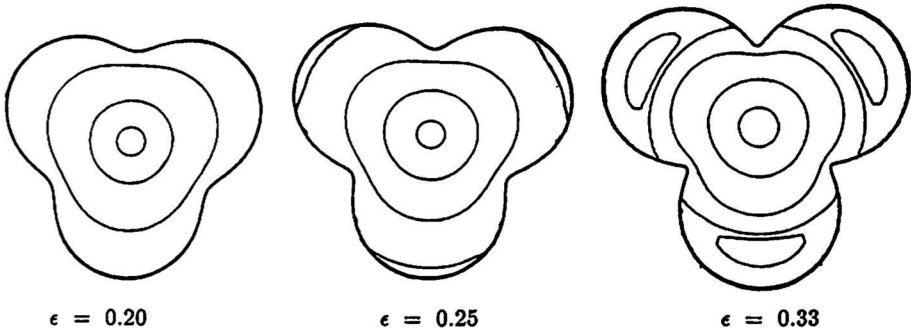


Fig. 1. : Some solutions for $n = 3$, and $|\epsilon| < 0.5$.

The boundary vorticity is found to be

$$(\omega')_{\rho=1} = (1 + 2\epsilon \cos(n\phi) + \epsilon^2) \left(4\epsilon \cos(n\phi) + 2 \frac{1 - n^2\epsilon^2 - (n-1)^2\epsilon^4}{1 + (n-1)\epsilon^2} \right). \quad (3.6)$$

In the vicinity of $z' = 0$, when $n \geq 3$,

$$\psi' \sim \log r' - \frac{1 + 2n\epsilon^2 - \epsilon^4}{2[1 + (n-1)\epsilon^2]} r'^2 \quad \text{as } r' \rightarrow 0, \quad (3.7)$$

which corresponds to the motion of a circular cylinder of small radius λ (< 1) rotating with constant angular velocity Ω' . If the outer boundary is replaced by a circular cylinder of radius $(1 + \epsilon)^{-1}$ the stream function ψ_1 for the flow is given by

$$\psi_1 = \log r_1 - \frac{1}{2}(1 + \epsilon)^2 r_1^2 \quad (3.8)$$

and the angular velocity of the boundary $r_1 = \lambda$ will be denoted by Ω_1 . In both cases the torques on the boundaries are the same and the ratio

$$\frac{\Omega'}{\Omega_1} = \frac{1 - \lambda^2 \frac{1 + 2n\epsilon^2 - \epsilon^4}{1 + (n-1)\epsilon^2}}{1 - \lambda^2(1 + \epsilon)^2} > 1, \quad (3.9)$$

for all $n \geq 3$ and $\epsilon > 0$. This equation indicates that for a given torque there is a small increase in angular velocity produced by the corrugations.

§4. Comparison with Solution of [8]

In [8] the (interior) fluted column transformation,

$$z = \zeta + \epsilon\zeta^{n+1}, \quad z = x + iy, \quad \zeta = \rho e^{i\phi}, \quad (4.1)$$

is considered. It is analytic in the interior of $|\zeta| = 1$, when $n \geq 1$ is an integer and

$$0 \leq \epsilon(n+1) < 1. \quad (4.2)$$

The solution for Stokes flow with a rotlet at the origin and no-slip boundary conditions is

$$\psi_{int} = \log \rho - \frac{\rho^2 - \epsilon^2 \rho^{2n+2}}{2[1 - (n+1)\epsilon^2]}. \quad (4.3)$$

No separation is present since the stream function is a function of ρ only. Hence the streamlines coincide with the fluted columns $\rho = \text{constant}$, as shown in Fig. 2.

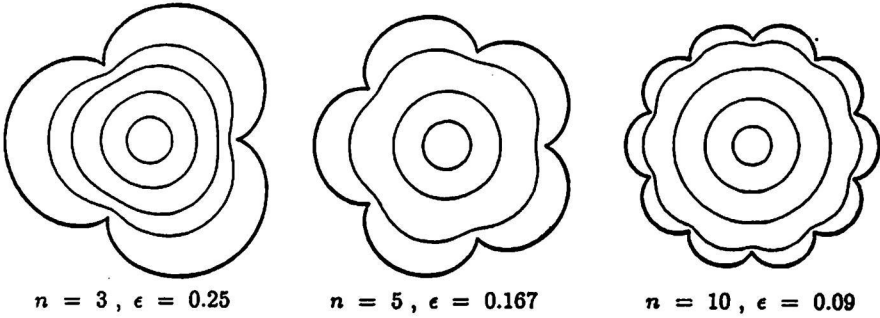


Fig. 2. : Some solutions from [8].

However, in the case of the exterior fluted column, for points on C' furthest from the rotlet, the vorticity is $\omega' = 2$ for $\epsilon = 0$, and $\omega' < 0$ for $\epsilon = 1/n$ ($n \geq 3$). Hence separation always occurs for ϵ sufficiently large, and satisfying (2.2) (see Fig. 3.).

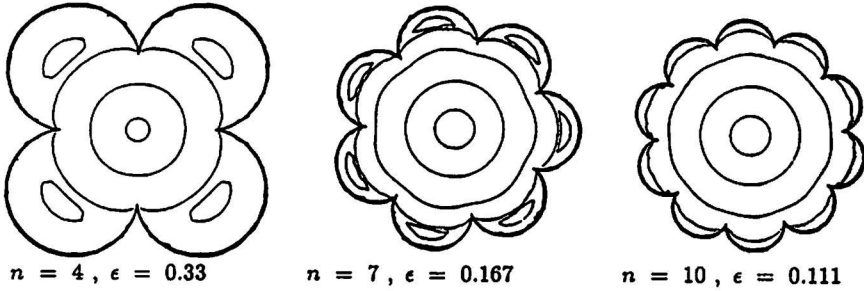


Fig. 3. : Some solutions for $\epsilon \simeq 1/(n-1)$.

In particular, separation is present if and only if $\epsilon > \epsilon_0$, where ϵ_0 can be expressed by

$$\epsilon_0 = \frac{1}{n+1} + \mathcal{O}\left(\frac{1}{n^3}\right). \tag{4.4}$$

The term $\mathcal{O}(1/n^3)$ is negative, so separation begins when ϵ is slightly less than $1/(n+1)$.

To measure the interaction of fluid flow with a distorted boundary, consider the curvatures at the furthest points on the boundaries; for a curve $z(\phi)$ in the complex plane

$$K = \frac{\text{Im}(\bar{z}\ddot{z})}{|z\dot{z}|^3}. \tag{4.5}$$

By taking $\epsilon = 1/(n+1)$ it is found that

$$\frac{n(n+5)}{4(n+1)} = K_{int}^{max} < K_{ext}^{max} = \frac{n+2}{4}, \quad (4.6)$$

which gives evidence for the existence of a critical curvature at which separation appears.

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Department of Mathematics
 University of Toronto
 Toronto, Ontario
 Canada M5S 1A1

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ON SOME ANALYTIC FUNCTIONS OF CLASS $P_k(\alpha)$

KHALIDA INAYAT NOOR

Presented by T. Bloom, F.R.S.C.

ABSTRACT: Let $P(\alpha)$, $0 < \alpha < 1$, be the class of analytic functions p in the unit disc E such that, for $z \in E$, $\operatorname{Re} p(z) > \alpha$. The class $P_k(\alpha)$; $k > 2$, $0 < \alpha < 1$, is defined as follows: An analytic function $h \in P_k(\alpha)$ if $h(z) = (\frac{k}{4} + \frac{1}{2}) p_1(z) - (\frac{k}{4} - \frac{1}{2}) p_2(z)$ with $p_1, p_2 \in P(\alpha)$. The radii of starlikeness and convexity of order α of a function f are determined, when f is restricted to certain classes of analytic functions related to $P_k(\alpha)$ under generalized Libera operators.

Keywords and Phrases: Starlike, convex, convolution, bounded boundary rotation, radius.

1980 Mathematics Subject Classification: 30A32, 30A34, 30C45.

1. INTRODUCTION

A number of important classes of univalent functions (e.g. convex, starlike) are related through their derivatives to functions with positive real part. Convex and starlike functions of order α are defined by requiring the related functions to have real part greater than α . We replace functions with real part greater than α by certain weighted differences of such functions and obtain some new classes of functions.

For $0 < \alpha < 1$, let $P(\alpha)$ be the class of functions p , analytic in E with $p(0) = 1$, such that $\operatorname{Re} p(z) > \alpha$ for $z \in E$. Also $C(\alpha)$ and $S^*(\alpha)$, $0 < \alpha < 1$, denote the classes of convex and starlike functions of order α respectively. A function f , analytic in E and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

is starlike of order α if $\frac{zf'(z)}{f(z)} \in P(\alpha)$ and is convex of order α , if $\frac{(zf'(z))'}{f'(z)} \in P(\alpha)$ for $z \in E$.

Let $P_k(\alpha)$, $k > 2$, $0 < \alpha < 1$ be the class of functions h , analytic in E , such that

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad (1.2)$$

where $p_1, p_2 \in P(\alpha)$.

Definition 1.1

A function f , analytic in E and given by (1.1), is said to belong to the class $R_k(\alpha)$; $k > 2$, $0 < \alpha < 1$, if and only if $\frac{zf'(z)}{f(z)} \in P_k(\alpha)$.

Clearly $R_2(\alpha) \equiv S^*(\alpha)$ and $R_k(0) = U_k$, the class of functions of bounded radius rotation, see [1].

Similarly, an analytic function f , given by (1.1) belongs to $V_k(\alpha)$ for $z \in E$ if and only if $\frac{(zf'(z))'}{f'(z)} \in P_k(\alpha)$.

It is obvious that

$$f \in V_k(\alpha), \text{ if and only if, } zf' \in R_k(\alpha) \quad (1.3)$$

It may be noted that $V_2(\alpha) \equiv C(\alpha)$ and $V_k(0) \equiv V_k$, the class of functions of bounded boundary rotation first discussed by Paatero, see [1].

Let f and g be analytic in E with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then the convolution (or the Hadamard product)

of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (1.4)$$

2. SOME PRELIMINARY RESULTS

Lemma 2.1

Let ϕ be convex and $g \in C(\alpha)$ or $S^*(\alpha)$. Then $\phi * g$ also belongs to the same class.

This result follows from Corollary 2 and theorem 8 in [4].

Lemma 2.2

Let $h \in P_k(\alpha)$. Then $h \in P(\alpha)$ for $|z| < r_0$, where

$$r_0 = \frac{k - \sqrt{k^2 - 4}}{2} \quad (2.1)$$

Proof

From Corollary 1 in [2] it can be deduced that, if $p \in P(\alpha)$, we have

$$\frac{1-(1-2\alpha)r}{1+r} < \operatorname{Re} p(z) < |\operatorname{Re} p(z)| < |p(z)| < \frac{1+(1-2\alpha)r}{1-r} \quad (2.2)$$

Using (1.2) and (2.2), we have

$$\begin{aligned} \operatorname{Re}[h(z)-\alpha] &> \left[\left(\frac{k}{4} + \frac{1}{2} \right) \frac{1-(1-2\alpha)r}{1+r} - \left(\frac{k}{4} - \frac{1}{2} \right) \frac{1+(1-2\alpha)r}{1-r} \right] - \alpha \\ &= \frac{(1-\alpha)}{(1-r^2)} [r^2 - kr + 1] \end{aligned}$$

Hence $h \in P(\alpha)$ for $|z| < r_0$, where r_0 is defined by (2.1).

Lemma 2.3

Let $f \in V_k(\alpha)$. Then $f \in C(\alpha)$ for $|z| < r_0$, where r_0 is given by (2.1).

The proof follows immediately from the definition of the class $V_k(\alpha)$ and the relation (1.2).

3. MAIN RESULTS

Theorem 3.1

Let $f \in V_k(\alpha)$. Then

$$I_c(f(z)) = F_c(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt, \quad \text{Re } c > 0, \quad (3.1)$$

is in $C(\alpha)$, for $|z| < r_0$ where r_0 is given by (2.1).

The case $c=1$ gives us Libera's Integral operator, see [1].

Proof

Let

$$h_c(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \quad \text{Re } c > 0.$$

The function $h_c(z)$ is convex, see [3]. Now

$$F_c(z) = (h_c * f)(z),$$

and the result follows by using Lemma 2.1 and Lemma 2.3.

Theorem 3.2

Let $f \in R_k(\alpha)$ and $I_c(f(z)) = F_c(z)$ be defined by (3.1). Then $F_c \in S^*(\alpha)$ for $|z| < r_0$.

Proof

Since $f \in R_k(\alpha)$, we have

$$h(z) = \frac{zf'(z)}{f(z)} \in P_k(\alpha) \quad \text{for } z \in E.$$

Now, from Lemma 2.2, it follows that $F_c \in S^*(\alpha)$ for $|z| < r_0$ where r_0 is defined by (2.1).

$$\text{Let } h_c(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \quad \text{Re } c > 0.$$

$$\text{Then } F_c(z) = (h_c * f)(z),$$

where h_c is convex and $f \in S^*(\alpha)$ for $|z| < r_0$. We obtain the required result by using Lemma 2.1 with $\phi = h_c$.

Theorem 3.3

Let β and m be any positive integers and $f \in R_k(\alpha)$. Then the function F , defined by

$$(F(z))^\beta = \frac{\beta+m}{z^m} \int_0^z t^{m-1} (f(t))^\beta dt \quad (3.2)$$

belongs to $S^*(\alpha)$ for $|z| < r_0$, where r_0 is given by (2.1).

Proof

$$\text{Let } J(z) = \int_0^z t^{m-1} (f(t))^\beta dt$$

Then

$$(F(z))^\beta = \frac{\beta+m}{z^m} J(z),$$

and

$$\beta \frac{zF'(z)}{F(z)} = \frac{zJ'(z)}{J(z)} - m$$

or

$$\frac{zF'(z)}{F(z)} = \frac{1}{\beta} \frac{zJ'(z) - m J(z)}{J(z)} = \frac{N(z)}{D(z)},$$

$N(0) = 0 = D(z)$, and $D(z)$ is a $(m+\beta-1)$ -valent starlike function

for $|z| < r_0$.

Also

$$\begin{aligned} \frac{N'(z)}{D'(z)} &= \frac{1}{\beta} \left\{ \frac{(zJ'(z))' - mJ'(z)}{J'(z)} \right\} \\ &= \frac{zF'(z)}{f(z)} \end{aligned}$$

Since $f \in R_k(\alpha)$, it follows that $\frac{N'(z)}{D'(z)} \in P(\alpha)$ for $|z| < r_0$. Hence, by a result of Bernardi, $\frac{N(z)}{D(z)} \in P(\alpha)$ for $|z| < r_0$, see [1]. This proves our result.

Theorem 3.4

Let β and m be positive integers and $f \in V_k(\alpha)$. Let F be defined by (3.2). Then $F \in C(\alpha)$ for $|z| < r_0$ where r_0 is given by (2.1).

Proof

The assertion of Theorem 3.4 readily follows by using the relation (1.3) and Theorem 3.3.

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Dr. Khalida Inayat Noor
King Saud University,
Mathematics Department,
P.O. Box 2455, Riyadh 11451,
Saudi Arabia

THE SCATTERING OF KELVIN WAVES BY A CHANNEL DISCONTINUITY

V.T. Buchwald

Presented by G.F.D. Duff, F.R.S.C.

An asymptotic matching technique similar to that of Buchwald (1971) is used to calculate the transmission and reflection coefficients of Kelvin waves in a narrow channel which are incident on a sudden discontinuity in channel width. The coefficients of the first two Poincaré wave modes are also computed. The results are of interest in understanding the properties of oscillations of straits such as the Bass Strait in south east Australia.

Following Buchwald (1971), the equations of motion of waves of frequency $\omega > 0$ in shallow water of uniform depth h rotating about a vertical axis with angular velocity $\frac{1}{2}f$, and assuming a time factor $\exp(i\omega t)$, are

$$hk^2u = -i\omega\xi_x - f\xi_y; \quad hk^2v = f\xi_x - i\omega\xi_y; \quad (1)$$

and

$$(\nabla^2 - k^2)\xi = 0, \quad (2)$$

where (u, v) are the particle velocities, ξ is the surface elevation, and $k^2 = (f^2 - \omega^2)/\alpha^2$, where $\alpha = (gh)^{\frac{1}{2}}$ is the shallow water wave velocity. For the geometry in figure 1, assume that there are incident and transmitted Kelvin waves as in the diagram.

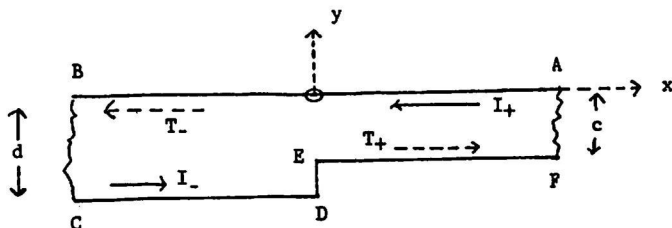


Figure 1

For $x > 0$, with $\mathcal{X} = (i\omega x + fy)/\alpha$,

$$\xi^+ = I_+ e^{\mathcal{X}} + T_+ e^{-\mathcal{X}} + \sum_{n=1}^{\infty} A_n^+ \left[i\omega s_n^+ \cos s_n^+(y+c) - f\mu_n^+ \sin s_n^+(y+c) \right] e^{-\mu_n^+ x}, \quad (3)$$

where $s_n^+ = n\pi/c$, and $\mu_n^+ = (s_n^{+2} + k^2)^{\frac{1}{2}}$. It may be verified that ξ^+ satisfies (2) with $v = 0$ at $y = 0, -c$. The coefficient I_+ of the incident Kelvin wave is assumed known and the scattered Kelvin wave coefficient T_+ as well as the coefficients A_n^+ of the Poincaré wave spectrum are to be determined. Note that for $k^2 > 0$ the Poincaré waves decay exponentially with increasing x . Similarly, for $x < 0$,

$$\xi^- = I_- e^{-\mathcal{X}} + T_- e^{\mathcal{X}} + \sum_{n=1}^{\infty} A_n^- \left[i\omega s_n^- \cos s_n^-(y+d) + f\mu_n^- \sin s_n^-(y+d) \right] e^{\mu_n^- x}, \quad (4)$$

where $s_n^- = n\pi/d$, and $\mu_n^- = (k^2 + s_n^{-2})^{\frac{1}{2}}$. In (4), I_- is assumed known and the other coefficients are to be determined.

In general the unknown scattering coefficients can only be determined numerically, but for $\delta = d \max(|f|, \omega)/\alpha \ll 1$ we may adopt an asymptotic matching technique similar to that used by Buchwald (1971). We consider a matching region close to the discontinuity, within which $\xi = \xi_M + O(\delta^2)$, where ξ_M satisfies

$$\nabla^2 \xi_M = 0, \quad (5)$$

and then ξ_M may be expressed in the form

$$\xi_M = i\omega\phi + f\psi, \quad (6)$$

where ϕ, ψ are conjugate harmonic functions which satisfy the Cauchy-Riemann relations $\phi_x = \psi_y$; $\phi_y = -\psi_x$. It follows that lines $\psi = \text{const.}$ are stream lines, so that $\psi = \text{const.}$ on the boundaries in figure 1.

The Schwartz-Christoffel mapping

$$\frac{dz}{d\tau} = \frac{c}{\pi\tau} \left(\frac{\tau-1}{\tau-a} \right)^{\frac{1}{2}}; \quad \pi z = c \cosh^{-1} \left\{ \frac{\tau-\beta_1}{\beta_2} \right\} - d \cosh^{-1} \left\{ \frac{\beta_1-a/\tau}{\beta_2} \right\} - ic; \quad (7)$$

where $a = c^2/d^2 < 1$, $\beta_1 = \frac{1}{2}(1+a)$, $\beta_2 = \frac{1}{2}(1+a)$, transforms the channel interior in the $z = x + iy$ plane onto the upper half of the complex τ plane, as in figure 2.

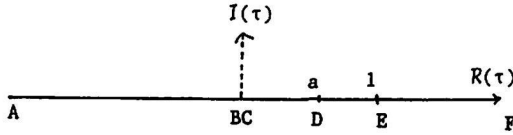


Figure 2

In (7) $\cosh^{-1} \rho = \log[\rho + (\rho^2 - 1)^{\frac{1}{2}}]$, and we choose the branch for which $(\rho^2 - 1)^{\frac{1}{2}} \sim \rho$, as $\rho \rightarrow \infty$. From (7) it may be shown that, as $\tau \rightarrow 0$,

$$z = -id - E + \frac{d}{\pi} [\log \tau + a_1 \tau + a_2 \tau^2] + O(\tau^3), \quad (8)$$

where $a_1 = (1-a)/2a$, $a_2 = (1-a)/(3+a)/16a^2$, $\pi E = d \log(\frac{2a}{\beta_2}) - c \cosh^{-1}(\frac{\beta_1}{\beta_2})$.

The expression in (8) may be inverted by iteration to yield

$$\log \tau = \Phi_1 - a_1 e^{\Phi_1} + (a_1^2 - a_2) e^{2\Phi_1} + O(e^{3\Phi_1}), \quad (9)$$

for $\Phi_1 = (z + id + E)\pi/d$, as $x \rightarrow -\infty$. Similarly, as $\tau \rightarrow \infty$,

$$z = -ic + F + \frac{c}{\pi} [\log \tau + b_1 \tau^{-1} + b_2 \tau^{-2}] + O(\tau^{-3}), \quad (10)$$

where $b_1 = (1-a)/2$, $b_2 = (1-a)(1+3a)/16$, and $\pi F = c \log(\frac{2}{\beta_2}) - d \cosh^{-1}(\frac{\beta_1}{\beta_2})$,

whence

$$\log \tau = \Phi_2 - b_1 e^{-\Phi_2} - (b_2 + b_1^2) e^{-2\Phi_2} + O(e^{-3\Phi_2}), \quad (11)$$

for $\Phi_2 = (z + ic - F)\pi/c$, as $x \rightarrow \infty$.

It may now be shown that the real and imaginary parts of $\log \tau$ satisfy the appropriate boundary conditions, and, as in Buchwald (1971), we may assume that, in (6),

$$\xi_M = X + i\omega Y \log |\tau| + fY \arg \tau, \quad (12)$$

where X, Y are constants. Hence, using (9), as $\tau \rightarrow 0$, $x \rightarrow -\infty$,

$$\begin{aligned} \xi_M = & X + i\omega\pi E/d + f\pi Y + \alpha\pi\mathcal{X}/d \\ & - a_1 Y \left[i\omega \cos s_1^-(y+d) + f \sin s_1^-(y+d) \right] e^{s_1^-(z+E)} + \dots \\ & + (a_1^2 - a_2) Y \left[i\omega \cos s_2^-(y+d) + f \sin s_2^-(y+d) \right] e^{s_2^-(z+E)} + \dots \end{aligned} \quad (13)$$

Comparing ξ_M in (13) with (4) up to $O(\delta)$, and noting that $\mathcal{X} = O(\delta)$, we obtain the set of equations, on comparing like terms,

$$T_- + I_- = X + i\omega\pi E/d + f\pi; \quad T_- - I_- = \alpha\pi Y/d \quad \text{and} \quad (14a)$$

$$A_1^- s_1^- = -a_1 Y e^{s_1^- E}; \quad A_2^- s_2^- = (a_1^2 - a_2) Y e^{s_2^- E}. \quad (14b)$$

Similarly, as $\tau \rightarrow \infty$, $x \rightarrow \infty$, and, using (11)

$$\begin{aligned} \xi_M = & X - i\omega\pi F/c + f\pi Y + \alpha Y \pi \mathcal{X}/c - b_1 Y \left[i\omega \cos s_1^+(y+d) - f \sin s_1^+(y+d) \right] e^{-s_1^+(z-F)} \\ & - (b_2 + b_1^2) Y \left[i\omega \cos s_2^+(y+d) - f \sin s_2^+(y+d) \right] e^{-s_2^+(z-F)} + \dots \end{aligned} \quad (15)$$

Term by term comparison with (3) up to $O(\delta)$ yields

$$I_+ + T_+ = X - \omega\pi Y F/c + f\pi Y; \quad I_+ - T_+ = \alpha\pi Y/c; \quad (16a)$$

$$s_1^+ A_1^+ = -b_1 Y e^{-s_1^+ F}, \quad s_2^+ A_2^+ = -(b_2 + b_1^2) Y e^{-s_2^+ F}. \quad (16b)$$

We may now solve (14a) and (16a) simultaneously to yield

$$(1 - i\delta_i)T_+ = 2DI_- - (D - C + i\delta_1)I_+, \quad (17)$$

$$(1 - i\delta_i)T_- = (D - C - i\delta_1)I_- + 2CI_+, \quad (18)$$

$$\pi\alpha Y(1 - i\delta_i) = 2cD(I_+ + I_-), \quad (18)$$

where $C = c/(c+d)$, $D = d/(c+d)$, $\delta_1 = \frac{\omega cd}{\alpha(c+d)}G$, and

$$G = Ed^{-1} + Fc^{-1}. \quad (20)$$

Substitution of Y from (19) into (14b) and (16b) yields the first two Poincaré wave coefficients on either side of the discontinuity. In principle higher Poincaré mode coefficients can be calculated (with increasing difficulty) by continuing the series in (8) and (10) and then inverting by step by step iteration, as in (9) and (11). However, the higher modes decrease in amplitude as $|x| \rightarrow \infty$, and for $x < -E$ in $x < 0$, and $x > F$ in $x > 0$. Graphs of the magnitudes of Ed^{-1} and Fc^{-1} are given in figure 3 for $0 < \sqrt{a} < 1$. Note that $F < 0$ for all \sqrt{a} , and hence the Poincaré wave expansion in $x > 0$ is valid for all $x > 0$. The maximum value of Ed^{-1} is 0.067 for $\sqrt{a} = 0.83$, so that the range of values of x for which the expansion is convergent for negative x depends on \sqrt{a} , but in any case always converges for $x < -0.067d$.

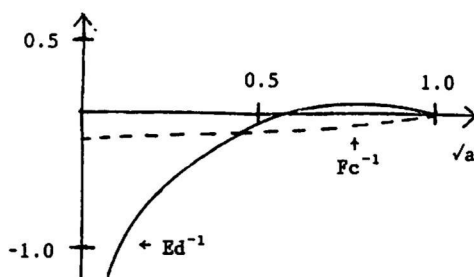


Figure 3

Note that $\delta_1 = O(\delta)$, and $\delta_1 \rightarrow 0$ as $\omega \rightarrow 0$. The results as $\omega \rightarrow 0$ should be compared with those in Middleton and Viera (1990) for a more complicated geometry appertaining to the Bass Strait.

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Department of Mathematics
University of Toronto
Toronto, Ontario
Canada, M5S 1A1

Permanent address:
School of Mathematics
University of New South Wales
Kensington NSW 2033
Australia

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On Derivative Estimates for the Navier Stokes Equations in R^3

G.F.D. Duff, F.R.S.C.

Abstract. Estimates are given showing inclusion in mixed spaces $L^{p_2}(0, T; L^{p_1}(R^3))$ of derivatives of all orders of solutions of the Navier Stokes equations, with $1 < p_1 < 2$, $p_2 > 0$.

Introduction. The possible and presumed presence of singularities that develop over time in solutions of the three dimensional Navier Stokes equations has posed longstanding problems in the study of regularity and uniqueness of such solutions. Recently it has been possible to show that derivatives with respect to space and time coordinates satisfy mixed Lebesgue estimates of the form $D_t^r D_x^s u(x, t) \in L^{p_2}(0, T; L^{p_1}(\Omega))$ where $\frac{3}{p_1} + \frac{1}{p_2} = 2r + s + 1$, $p_1 \geq 2$, $\Omega \subseteq R^3$ [3]. Here it will be shown how these results can be extended to cover the range $1 < p_1 < 2$ in the particular case $\Omega = R^3$ for which the techniques take their simplest form. For all derivatives beyond the gradient, however, certain limiting results are not obtained for $1 \leq p_1 < 2$, for which other methods will be required.

1. **The initial value problem.** The Navier Stokes equations in R^3 are

$$u_{i,t} + u_k u_{i,k} = -p_{,it} + \nu \Delta u_i, \quad u_{i,i} = 0,$$

where $u_i(x, t)$, ($x \in R^3$, $t > 0$) denotes the velocity vector, p the pressure and ν the constant viscosity and Δ is the Laplacian operator. Here $i, k = 1, 2, 3$, the Einstein summation convention is used for repeated indices, and commas denote partial derivatives.

For simplicity we restrict attention to the case of nonzero initial data only and postulate existence of a suitable solution with

$$u_i(x, 0) = u_i(x) \in L^2(R^3).$$

The classical energy integrals then yield

$$\|u\|_2^2 + 2\nu \int_0^t \|\nabla u\|_2^2 d\tau \leq \|u_0\|_2^2,$$

where

$$\|u\|_p^p = \int_{R^3} |u|^p d_3x, \quad p \geq 1,$$

and it follows that $u \in L^\infty(0, \infty; L^2(R^3))$ and $\nabla u \in L^2(0, \infty; L^2(R^3))$ the latter implying $u \in L^2(0, \infty; L^6(R^3))$ by Sobolev's inequality. The resulting inclusions for u , ∇u and the higher space and time derivatives are shown in the index diagram (Figure 1) for $1 \leq p_1 \leq \infty$, $0 < p_2 \leq \infty$, in which the spaces to which any function belongs form a convex set. The higher derivatives $D_i^r D_x^s u$, where $r, s_1, s_2, s_3 = 0, 1, 2, 3 \dots$, have been shown to belong to $L^{\frac{2}{1+r+\frac{1}{2}s}}(0, T, L^2(R^3))$, as mentioned above.

2. Estimates for u . Multiplying the Navier Stokes equations by $|u_i|^a \text{sgn } u_i$ and integrating by parts, one finds

$$\frac{1}{1+a} \frac{d}{dt} \int_{R^3} |u_i|^{1+a} dx + \frac{4\nu a}{(1+a)^2} \int_{R^3} (\nabla |u_i|^{\frac{1+a}{2}})^2 dx = - \int_{R^3} |u_i|^a \text{sgn } u_i p_i dx$$

Since $\Delta p = -(u_k u_i)_{,ik}$ and consequently

$$p(x) = \frac{1}{4\pi} \int_{R^3} \frac{1}{r} (u_k u_i)_{,ik} dy, \quad r = |x - y|,$$

it follows that

$$\begin{aligned} p_{,j}(x) &= \frac{1}{4\pi} \int_{R^3} \frac{x_j - y_j}{r^3} (u_k u_i)_{,ik} dy \\ &= \frac{1}{4\pi} \int_{R^3} \left\{ \frac{\delta_{ij}}{r^3} - 3 \frac{(x_i - y_i)(x_j - y_j)}{r^5} \right\} (u_k u_i)_{,k} dy. \end{aligned}$$

As this principal value kernel is Calderon-Zygmund, one finds

$$\|p_{,j}\|_r \leq C \|u \nabla u\|_r \leq C \|u\|_p \|\nabla u\|_q$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < 1$. Thus if $\frac{1+a}{p_1} + \frac{1}{q_1} = 1$ then

$$\left| \int_{R^3} |u_i|^a p_i dx \right| \leq C \|u_i\|_{p_1}^{1+a} \|\nabla u\|_{q_1}$$

so that with $p_1 = \frac{10}{3}$, $q_1 = 2$, the most favourable choice, this expression is integrable over $R^3 \times (0, \infty)$ if $a = \frac{2}{3}$. Hence

$$\|u\|_{1+a}^{1+a} + \frac{4\nu a}{1+a} \int_0^t \|\nabla |u|^{\frac{1+a}{2}}\|_2^2 d\tau \leq \|u_0\|_{1+a}^{1+a} + K_a$$

holds for all positive t , and as $t \rightarrow \infty$, with $a = \frac{2}{3}$.

Assuming $u_0 \in L^{1+a}(R^3)$ we then find for $a = \frac{2}{3}$ that $u \in L^\infty(0, \infty; L^{1+a}(R^3))$ and $\nabla|u|^{\frac{1+a}{2}} \in L^2(0, \infty; L^2(R^3))$ so that $|u|^{\frac{1+a}{2}} \in L^2(0, \infty; L^6(R^3))$ and $u \in L^{1+a}(0, \infty; L^{3+3a}(R^3))$. This permits a bootstrap process for decreasing a which has a natural goal $a = \frac{1}{5}$ when the interpolating u -line $\frac{3}{p_1} + \frac{2}{p_2} = \frac{3}{1+a}$, $1+a \leq p_2 \leq \infty$, meets the diagonal at $p_1 = p_2 = 2$. We then have $\|u\|_2 \|\nabla u\|_2 \in L^1(0, \infty)$ so the above inclusions hold for $a > 0$, as well as all intermediate values of a , provided only that $\|u_0\|_{1+a}$ is finite.

Theorem 1. *If $u_0 \in L^p(R^3)$ where $1 < p \leq \frac{5}{3}$, then $u \in L^\infty(0, \infty; L^p(R^3))$.*

3. Estimates for ∇u . The vorticity $\omega_i = \text{curl } u_i$ satisfies $\omega_{i,i} = 0$ and

$$\omega_{i,t} + u_k \omega_{i,k} = \omega_k u_{i,k} + \nu \Delta \omega_i.$$

Multiplying by $\text{sgn } \omega_i$ and integrating, yields

$$\frac{d}{dt} \int_{R^3} |\omega_i| dx + \int_{R^3} u_k |\omega_i|_{,k} dx = \int_{R^3} \text{sgn } \omega_i \omega_k u_{i,k} dx + \nu \int_{R^3} \Delta |\omega_i| dx.$$

The second integral on the left vanishes after integration by parts, while

$$\int_{R^3} \Delta |\omega_i| dx = \sum_j \int_{\delta\Omega_j} \frac{\partial |\omega_i|}{\partial n} ds \leq 0$$

where Ω_j denote the nodal domains of ω_i on R^3 , that is, ω_i vanishes on $\delta\Omega_j$. Since n denotes the outward normal, the semiboundedness property follows. Thus

$$\frac{d}{dt} \int_{R^3} |\omega_i| dx \leq \int_{R^3} |\omega_k| |u_{i,k}| dx \leq \int_{R^3} |\nabla u|^2 dx$$

and after integration over $(0, T)$ we find

$$\int_{R^3} |\omega_i| dx \leq \int_{R^3} |\omega_{i0}| dx + \int_0^T \|\nabla u\|_2^2 dt.$$

Theorem 2. *If $\omega_{i0} \in L^1(R^3)$ then $\omega_i \in L^\infty(0, \infty; L^1(R^3))$.*

A similar result for ∇u was recently found using compensated compactness [2].

4. Estimates for Δu and higher derivatives.

The demonstration given by Ladyzhenskaya for the solution of the linear Cauchy problem [4, Chap. 4, §5]

$$u_{i,t} - \nu \Delta u_i = -\nabla_i p + F_i(x, t)$$

where $F_i(x, t) \in L^r(R^3 \times (0, T))$, $r > 1$, can be extended to mixed spaces $L^{r_1}(0, T; L^{r_2}(R^3))$ where $r_1 > 1$, $r_2 > 1$, by Corollary I of [5] for which the Marcinkiewicz multiplier condition is the same. With $F_i(x, t) = u_k \nabla_k u_i$ we find $\frac{3}{r_1} + \frac{2}{r_2} = 4$ and conclude that $\Delta u_i, u_{i,t}, p_i \in L^{p_1}(0, T; L^{p_2}(R^3))$ for

$$\frac{3}{p_1} + \frac{2}{p_2} = 4, \quad 1 < p_1 < \frac{3}{2}.$$

A related result can now be found for the higher space and time derivatives of u , by means of interpolation on the orders of space derivatives in the first instance.

Interpolation Lemma. Let $0 < s < \ell$ and $1 \leq p_1 \leq \infty, 1 \leq q_1 \leq \infty, 0 < p_2 \leq \infty, 0 < q_2 < \infty$ and let

$$\frac{1}{r_i} = \left(1 - \frac{s}{\ell}\right) \frac{1}{p_i} + \frac{s}{\ell} \frac{1}{q_i}, \quad i = 1, 2.$$

Then

$$\|D_s^t f\|_{r_1, r_2} \leq C \|f\|_{p_1, p_2}^{1-\frac{t}{\ell}} \|D_s^t f\|_{q_1, q_2}^{\frac{t}{\ell}};$$

$\|f\|_{p_1, p_2}$ is the norm in $L^{p_1}(0, T; L^{p_2}(R^3))$.

This formula relating orders $s = s_1 + s_2 + s_3$ of space derivatives to geometric interpolation on a line

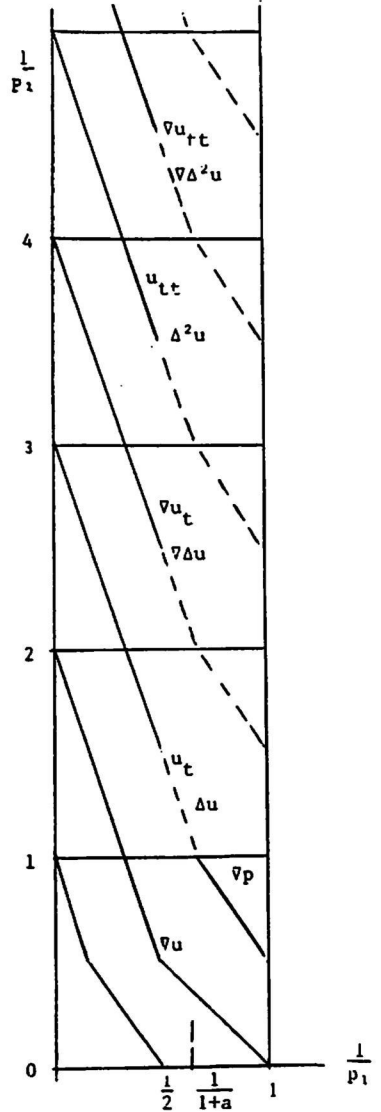


Figure 1
Index diagram for $L^{p_1}(0, T; L^{p_2}(R^3))$, $1 < p_1 \leq \infty$

segment in the index diagram follows from the basic multiplicative inequality of [1, vol. 1, p. 311] as the extreme case $\theta = s/\ell$. The fractional exponents for p_2 and q_2 are possible because no time derivatives are used in this interpolation.

Applying the formula to $f = D_x^2 u$ we find $\frac{1}{q_1} = \frac{1}{2}$, $\frac{1}{q_2} = \ell + \frac{3}{2}$ and that $\|D_x^{\ell+2} u\|_{r_1, r_2}$ falls on the line segment

$$\frac{3}{r_1} + \frac{2}{r_2} = 4 + s\left(2 + \frac{1}{2\ell}\right) \rightarrow 4 + 2s$$

as $\ell \rightarrow \infty$. For large ℓ , the segments approach the limiting position indicated by $4 + 2s$ on the right side, and we conclude that $D_x^{\ell+2} u$ lies in every space above this open line segment, where $1 < r_1 < \frac{3}{2}$ as shown in Figure 1.

Since $u_t \in L^{p_1}(0, T; L^{p_2}(R^3))$ for $\frac{3}{p_1} + \frac{2}{p_2} = 4$, $1 < p_1 < \frac{3}{2}$, we can carry out the same interpolations for u_t as for $D_x^2 u$, and thus find the result for $D_t D_x^m u$, $m = 1, 2, \dots$. For $D_t^2 u$, we differentiate the Navier Stokes equations with respect to t , finding

$$D_t^2 u_i + D_t u_k \cdot u_{i,k} + u_k \cdot D_t u_{i,k} = -D_t p_{,i} + \nu \Delta D_t u_i.$$

Now

$$\begin{aligned} D_t p_{,i} &= \frac{1}{4\pi} \int_{R^3} \frac{x_i - y_i}{r^3} \{D_t u_{k,j} \cdot u_{j,k} + u_{k,j} \cdot D_t u_{j,k}\} dy \\ &= \frac{1}{4\pi} \int_{R^3} \left\{ \frac{r^2 \delta_{ij} - 3(x_i - y_i)(x_j - y_j)}{r^5} \right\} \{D_t u_k \cdot u_{j,k} + u_k D_t u_{j,k}\} dy. \end{aligned}$$

Since the integral has a Calderon-Zygmund principal value kernel, the L class of $D_t p_{,i}$ will be the same (for $p_1 > 1$) as those for $D_t u \cdot \nabla u$ and $u \cdot D_t \nabla u$. These two terms both include the segment $\frac{3}{p_1} + \frac{2}{p_2} = 8$, $1 < p_1 < \frac{3}{2}$, while $\Delta D_t u$ extends only to the open region above this segment. Consequently the latter inequality, namely $\frac{3}{p_1} + \frac{2}{p_2} > 8$ for $1 < p_1 < \frac{3}{2}$, holds for $D_t^2 u$. Again we may interpolate with x -derivatives as above, so finding $D_t^2 D_x^m u$ covers the region $\frac{3}{p_1} + \frac{2}{p_2} > 8 + 2m$ for $1 < p_1 < \frac{3}{2}$.

The higher order time derivatives $D_t^r D_x^m u$ can now be treated by induction on r . We can assume the inclusion in $\frac{3}{p_1} + \frac{2}{p_2} > 4j$ for $D_t^j u$ and all $j \leq r$, and then establish the result for $D_t^{r+1} u$, and later the result for $D_t^{r+1} D_x^m u$. Now

$$D_t^{r+1} u_i + \sum_{j=0}^r \binom{r}{j} D_t^j u_k \cdot D_t^{r-j} u_{i,k} = D_t^r p_{,i} + \nu \nabla D_t^r u_i$$

and

$$D_t^r p_i = \frac{1}{4\pi} \int_{R^3} \left\{ \frac{r^2 \delta_{ij} - 3(x_i - y_i)(x_j - y_j)}{r^5} \right\} \times \sum_{j=0}^r \binom{r}{j} \{ D_t^j u_k \cdot D_t^{r-j} u_{\ell,k} \} dy.$$

By the Calderon-Zygmund theorem, therefore, the pressure term is included in the same space as all of the products $D_t^j u_k \cdot D_t^{r-j} u_{\ell,k}$ for $j = 0 \dots, r$. It can be checked that these now comprise the $L^{p_2}(0, T; L^{p_1}(R^3))$ spaces in the set above $\frac{3}{p_1} + \frac{2}{p_2} \leq 4 + 4r$ for $1 < p_1 < \frac{3}{2}$. However the term $D_t^r \Delta u_i$, by the induction hypothesis, lies only in those spaces where $\frac{3}{p_1} + \frac{2}{p_2} > 4 + 4r$. Consequently $D_t^{r+1} u_i$ also lies in these spaces.

Interpolation for the space derivatives $D_t^{r+1} D_x^s u$ now follows the pattern described above, with the result that $D_t^{r+1} D_x^s u_i$ lies in the open set of spaces with $\frac{3}{p_1} + \frac{2}{p_2} > 4 + 4r + 2s$.

Induction on r then establishes

Theorem 3. For $1 < p_1 \leq \frac{3}{2}$, $2r + s \geq 2$ we have $D_t^r D_x^s u \in L^{p_2}(0, T; L^{p_1}(R^3))$ where $\frac{3}{p_1} + \frac{2}{p_2} > 4r + 2s$ while for $\frac{3}{2} < p_1 \leq 2$ we must have $\frac{3}{p_1} + \frac{1}{p_2} > 2r + s + 1$.

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Department of Mathematics
University of Toronto
Canada M5S 1A1

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On the K-theory of the symmetrized non-commutative torus

A. Kumjian*

Presented by G.A. Elliott, F.R.S.C.

Abstract

In this note the K-groups of the crossed product of the rotation algebra by the flip are computed. The crossed product is characterized as the universal C^* -algebra generated by three self-adjoint unitaries satisfying a certain commutation relation.

Introduction

Let A_θ denote the universal C^* -algebra generated by two unitaries, u and v , subject to the commutation relation:

$$uv = e^{2i\pi\theta}vu.$$

Let σ denote the involutive automorphism of A_θ for which $\sigma(u) = u^*$ and $\sigma(v) = v^*$. In this note we show that the crossed product, $B_\theta = A_\theta \rtimes_\sigma \mathbb{Z}_2$, is the universal C^* -algebra generated by three self-adjoint unitaries, v_i for $i = 1, 2, 3$, satisfying the relation:

$$v_1 v_2 v_3 = e^{2i\pi\theta} v_3 v_2 v_1,$$

and compute its K-theory using Natsume's exact sequence [Na]. One obtains:

$$K_0(B_\theta) = \mathbb{Z}^6$$

$$K_1(B_\theta) = 0.$$

This result was obtained in [BEEK2] for θ rational (the inclusion, $\mathbb{Z}^6 \subseteq K_0(B_\theta)$, was established for all θ). This note was inspired by George Elliott's observation (presented at a conference) of the existence of four self-adjoint unitaries in B_θ ; R. Nest first noted the relevance of these self-adjoint unitaries to the K-theory (see also [Ne]). I am indebted to Bruce Blackadar for a number of useful conversations during the course of this work.

Preliminaries

Let $G = \mathbb{Z}_2 * \mathbb{Z}_2$, with generators, x_1, x_2 ; it is easy to see that $G \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where words of even length form the normal subgroup isomorphic to \mathbb{Z} (the element $x_1 x_2$ generates), and the \mathbb{Z}_2 -action is given by inversion. One sees that $C = C^*(G)$ is the universal C^* -algebra generated by two self adjoint unitaries, v_1, v_2 , with a natural embedding of $C(\mathbb{T}) \cong C^*(\mathbb{Z})$ into it (the C^* -subalgebra generated by $v_1 v_2$). It is wellknown that C is isomorphic to the algebra of continuous functions from the unit interval to $M_2(\mathbb{C})$ which are diagonal at the endpoints. Moreover, $K_0(C)$ is the free abelian group generated by $[1], [\frac{1}{2}(v_1 + 1)]$, and $[\frac{1}{2}(v_2 + 1)]$, while $K_1(C) = 0$ (cf. [L] or [B1] 6.10.4).

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Results

Theorem: The following C^* -algebras are isomorphic:

i the universal C^* -algebra generated by three self-adjoint unitaries v_i , $i = 1, 2, 3$, subject to the condition:

$$v_1 v_2 v_3 = e^{2i\pi\theta} v_3 v_2 v_1 \quad (*)$$

ii the universal C^* -algebra generated by four self-adjoint unitaries v_i , $i = 1, 2, 3, 4$, subject to the condition:

$$v_1 v_2 = e^{i\pi\theta} v_4 v_3 \quad (\dagger)$$

iii B_θ

iv $C_{C(\mathbb{T})}^* C$

v $C(\mathbb{T}) \rtimes \mathbb{Z}_2 * \mathbb{Z}_2$.

Proof: (i \cong ii) A simple calculation shows that condition (*) is equivalent to the requirement that $e^{-i\pi\theta} v_1 v_2 v_3$ be a self-adjoint unitary; denote this element by v_4 . Condition (\dagger) is simply a restatement of this definition.

(i \cong iii) The C^* -algebra, B_θ , is (the universal such) generated by two unitaries, u, v , together with a self-adjoint unitary, w , satisfying the conditions:

$$uv = e^{2i\pi\theta} vu$$

$$uwu = u^*$$

$$wvw = v^*.$$

Put $u = v_1 v_2$, $v = v_3 v_2$, and $w = v_2$; this defines the desired isomorphism.

(ii \cong iv) Obvious.

(iii \cong v) Note that $A_\theta \cong C(\mathbb{T}) \rtimes \mathbb{Z}$ where the action on \mathbb{T} is translation by $e^{2i\pi\theta}$, and σ flips the generators of $C(\mathbb{T})$ and \mathbb{Z} . Hence,

$$B_\theta = A_\theta \rtimes_\sigma \mathbb{Z}_2 \cong C(\mathbb{T}) \rtimes \mathbb{Z} \rtimes \mathbb{Z}_2.$$

Remarks:

i A cyclic permutation of the generators, v_1, v_2, v_3 , induces an automorphism β of B_θ ; apply $\text{Ad } v_1$ to (*) to obtain the same expression but for a permutation of indices.

ii As above, let $\gamma \in \text{Aut}(B_\theta)$ be defined by the cyclic permutation of v_1, v_2, v_3, v_4 . Note that (\dagger) is equivalent to the condition:

$$v_1 v_2 v_3 v_4 = e^{i\pi\theta} 1;$$

applying $\text{Ad } v_1$ to this equation yields the desired permutation of indices.

K-theory computation

Lance adapted the methods of [PV] to compute the K-theory of the reduced C^* -algebra of the free product of countable amenable groups (see [L]). Natsume generalized this result in [Na] allowing for amalgamation over a common subgroup and an action on a C^* -algebra to obtain the following periodic six-term exact sequence:

$$\begin{array}{ccccccc}
 K_0(A \rtimes_r K) & & \xrightarrow{i_1^* - i_2^*} & & K_0(A \rtimes_r H_1) \oplus K_0(A \rtimes_r H_2) & & \xrightarrow{j_1^* + j_2^*} & & K_0(A \rtimes_r H_1 \star_K H_2) \\
 \uparrow & & & & & & & & \downarrow \\
 K_1(A \rtimes_r H_1 \star_K H_2) & & \xrightarrow{j_1^* + j_2^*} & & K_1(A \rtimes_r H_1) \oplus K_1(A \rtimes_r H_2) & & \xrightarrow{i_1^* - i_2^*} & & K_1(A \rtimes_r K)
 \end{array}$$

where $i_k : A \rtimes_r K \rightarrow A \rtimes_r H_k$ and $j_k : A \rtimes_r H_k \rightarrow A \rtimes_r H_1 \star_K H_2$ are the natural embeddings. In our computation K is trivial, $H_1 \cong H_2 \cong \mathbb{Z}_2$, and $A \cong C(\mathbb{T})$.

Proposition:

$$K_0(B_\theta) \cong \mathbb{Z}^6$$

$$K_1(B_\theta) \cong 0.$$

Proof: Since $B_\theta \cong C(\mathbb{T}) \rtimes \mathbb{Z}_2 \star \mathbb{Z}_2$ we may invoke Natsume's exact sequence (note $\mathbb{Z}_2 \star \mathbb{Z}_2$ is amenable):

$$\begin{array}{ccccccc}
 \mathbb{Z} & \rightarrow & \mathbb{Z}^3 \oplus \mathbb{Z}^3 & \rightarrow & K_0(B_\theta) \\
 \uparrow & & & & \downarrow \\
 K_1(B_\theta) & \leftarrow & 0 \oplus 0 & \leftarrow & \mathbb{Z}
 \end{array}$$

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Department of Mathematics, University of Nevada, Reno NV 89557 (alex@tahoe.unr.edu)

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UNITARY DILATION OF SYMPLECTIC VECTOR SPACES

Palle E.T. Jorgensen

Presented by G.A. Elliott, F.R.S.C.

Abstract. Conformal quantum field theories on Riemann surfaces give rise to specific symplectic spaces which are not polarized into ordinary Heisenberg algebras. We give axioms for general symplectic spaces V , and identify necessary and sufficient conditions for V to have unitary representations which are obtained by induction from Fock-representations of extended Heisenberg algebras.

1. Introduction. Central extensions of Abelian Lie algebras are labeled by alternating bilinear forms, as is well known: If (\cdot, \cdot) is a given form on a vector space V over a field \mathcal{A} , then the central extension may be realized as $V \oplus \mathcal{A}c$ where c represents a basis element for the added dimension, and where the Lie bracket is defined by, $[x, c] = 0$, $x \in V$, and $[x, y] := (x, y)c$, $x, y \in V$. Such Lie algebras arise in conformal quantum field theories on Riemannian surfaces of higher genus, and more specifically in recent work ([KN], [JKL]) on chiral algebras constructed from global Laurent expansions [FK] on particular surfaces (realized as Schottky doubles). If the surface M is the double of N , and its mirror image \bar{N} , glued at the boundary ∂N , then N and \bar{N} are linked by an antiholomorphic involution, $\vartheta: M \rightarrow M$, such that ∂N is the set of fixed-points for ϑ . Pick P_0 in N , in general position, and let $P_\omega = \vartheta P_0$. Let V be the space of all meromorphic functions on M which are holomorphic on $M \setminus \{P_0, P_\omega\}$. The alternating form is defined by

$$(f, g) := \text{Res}_{P_0} (df \cdot g), \quad f, g \in V,$$

and it may be computed as $\frac{1}{2\pi} \int_C df \cdot g$, where C is any cycle separating the point P_0 from P_ω . This leads to the following decomposition, $V = V_+ \oplus V_0 \oplus V_-$ where

functions in V_+ are holomorphic on \bar{N} , functions in V_- holomorphic on N , and $V_0 := \{f \in V : \text{ord } f \leq 0 \text{ at both points } P_0 \text{ and } P_\infty\}$. Then the dimension of V_0 is g_0+1 where g_0 is the genus. It follows that V is a Heisenberg algebra (i.e., has a polarization) only when $g_0 = 0$. While the representation theory of the Heisenberg algebras is well known, it is not clear how to get representations when $\dim V_0 > 1$. Representations of V were constructed recently in [JKL] as follows: It was shown that a space Ω of meromorphic one-forms on M acquires the structure of a Heisenberg algebra with polarization (Ω_\pm) such that $dV_\pm \subset \Omega_\pm$. Then it was shown that an associated Fock-representation π of Ω induces representations ρ of V . The induction is specified by the identity

$$\rho(f) = \pi(df), \quad f \in V_+ + V_- . \quad (1)$$

It was shown further that Ω carries an alternating form $(\cdot, \cdot)_\Omega$ such that, $(df, dg)_\Omega = (f, g)$, $f, g \in V$. The time-reflection operator, $Tf(z) := \overline{i(\overline{\vartheta(z)})}$, satisfies $(Tf, f) \geq 0$, $f \in V_+$; and a similar operator exists on Ω , with $(T\omega_+, \omega_+) \geq 0$, $\omega_+ \in \Omega_+$, and $dT = Td$. The spaces V and Ω are called *symplectic spaces*. We say that Ω is a *Heisenberg algebra* since it is *polarized*. A linear operator, $d : V \rightarrow \Omega$, is said to be an *isomorphism* if it has the properties which we listed above in the particular example.

2. Extensions of symplectic spaces. We shall be concerned with an axiomatic theory of symplectic spaces where it is only assumed that a given vector space V carries an alternating form (\cdot, \cdot) having certain generic properties: A subspace W is said to be *isotropic* if $(x, y) = 0$, $x, y \in W$. It is assumed that V has a direct decomposition, as a vector space,

$$V = V_+ \oplus V_0 \oplus V_- , \quad (2)$$

such that both V_\pm are isotropic.

Motivated by (1), we address the following two *dilation* questions. Let the symplectic space V be given with specified triple decomposition.

(i) Is there a Heisenberg algebra Ω and an isomorphism, $d : V \rightarrow \Omega$, (generally not onto) such that some representation of Ω induces representations of V ?

(ii) If, further, V carries a time–reflection operator (satisfying the positivity condition as part of the definition), and, further,

$$T : V_+ \rightarrow V_-, \quad T : V_- \rightarrow V_+, \quad T : V_0 \rightarrow V_0, \quad \text{and } (Tx, Ty) = \overline{(y, x)}, \quad x, y \in V, \quad (3)$$

may a Heisenberg algebra Ω be chosen also to have a time–reflection operator such that, $dT = Td$, and such that the Fock–representation of Ω induces representations of V ?

Note that, following standard Osterwalder–Schrader theory [GJ], the positivity properties provide both V_+ and Ω_+ with Hilbert space inner products, as follows, $\langle x_+, y_+ \rangle := (Tx_+, y_+)$, $x_+, y_+ \in V_+$ (resp. Ω_+), and associated norms. We shall denote the corresponding completions by the same symbols. With the assumptions from (ii), it follows that d induces an isometry, $d : V_+ \rightarrow \Omega_+$, between the two Hilbert spaces, and a co–isometry, $d^* : \Omega_+ \rightarrow V_+$. Also note that the symmetric tensor algebra over Ω_+ , i.e., the symmetric Fock–space $S(\Omega_+)$, is then a Hilbert space. When the adjoint operator is defined relative to the Hilbert–inner product on $S(\Omega_+)$, then the Fock–representation π satisfies $\pi(\omega)^* = \pi(T\omega)$, $\omega \in \Omega$. In considering possible induced representations ρ , as in (1), we shall restrict attention to those which satisfy the Hermitian property, $\rho(x)^* = \rho(Tx)$, $x \in V$, with the adjoint defined relative to the Hilbert–inner product of $S(V_+)$.

In [J], we give necessary and sufficient conditions for an affirmative solution to problem (i). (There are examples of symplectic spaces V with triple decomposition (2) when the answer is "no".) The context for (i) is alternating bi–linear forms over a given field of characteristic zero. For question (ii), the field will be \mathbb{C} .

3. Existence of unitary dilations. Here, we report our theorem for question (ii).

Theorem. *Let V be a symplectic space over \mathbb{C} with triple–decomposition as in (2), and with positive time–reflection operator T such that the compatibility conditions (3) are satisfied. Then the following two conditions, (a) and (b), are equivalent:*

- (a) *Problem (ii) is affirmative, i.e., there is a Heisenberg algebra Ω (as specified)*

and an isomorphism, $d : V \rightarrow \Omega$ (generally not onto) such that the Fock-representation of Ω induces representations of V .

(b) For every vector $v \in V_0$, there is a constant (depending on v) such that

$$|(Tx_+, v)| \leq \text{const} \|x_+\| \text{ for all } x_+ \in V_+ \quad (4)$$

where $\|\cdot\|$ is the Hilbert-norm, $\|x_+\| := (Tx_+, x_+)^{1/2}$.

Note that if (4) holds, then there is, by the Riesz theorem, a unique operator, $\xi_+ : V_0 \rightarrow V_+$, such that $(Tx_+, v) = \langle x_+, \xi_+(v) \rangle$, $x \in V_+$, $v \in V_0$. If (a) holds, then we show in [J] that induced representations on V_0 are given in terms of the creation/annihilation operators as follows: There is an operator, $\omega_+ : V_0 \rightarrow \Omega_+$, and $\lambda \in V_0^*$, such that

$$\rho(v) = \pi(\omega_+(v)) + \lambda(v)1 + \pi(T\omega_+(Tv)), \quad v \in V_0,$$

and the connection between the two is specified by the formula, $\xi_+(v) = d^* \omega_+(v)$, $v \in V_0$.

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Department of Mathematics, The University of Iowa, Iowa City, IA 52242 U.S.A.

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Mailing Addresses

1. C.A. Baker Department of Mathematics and Computer Science
Mount Allison University
Sackville, New Brunswick, Canada E0A 3C0
2. V.T. Buchwald School of Mathematics
University of New South Wales
Kensington, N.S.W. 2033, Australia
3. G.F.D. Duff Department of Mathematics
University of Toronto
Toronto, Ontario, Canada M5S 1A1
4. P.E.T. Jorgensen Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242, U.S.A.
5. A. Kumjian Department of Mathematics
University of Nevada
Reno, Nevada 89557, U.S.A.
6. N.D. Lane Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada L8S 4K1
7. J.W. Lorimer Department of Mathematics
University of Toronto
Toronto, Ontario, Canada M5S 1A1
8. K.I. Noor Mathematics Department
King Saud University
P.O. Box 2455, Riyadh 11451, Saudi Arabia
9. D.W. Pravica Department of Mathematics
University of Toronto
Toronto, Ontario, Canada M5S 1A1
10. I. Ramella Dipartimento di Matematica e Applicazioni
Università di Napoli
Via Mezzocannone, 8 - I80134 Napoli, Italy