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The Uniform Density of Sets of Integers and Fermat's Last Theorem

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Presented by P. Ribenboim, F.R.S.C.

1. Introduction. In this note we take a "density theory" approach to the problem of measuring certain sets of integers including the set of exponents for which Fermat's Last Theorem is true. It is proved that this last set has *uniform* density equal to one. This is a slightly stonger statement than has been previously made about this set. The method is particularly simple and transparent and is applied to finding the uniform densities of other sets of integers from Number Theory.

2. Density concepts. Let B be a set of positive integers and write $B(x, y)$ for the number of elements in $B \cap [x, y]$. The lower and upper uniform densities are defined as follows. Let

$$\beta_s = \liminf_{t \to \infty} B(t+1, t+s).$$

That is, β_s is the smallest number which occurs infinitely often as the number of elements of B which lie in an interval of length s. It is not hard to show that $\lim_{s \to \infty} \beta_s/s$ exists, and this is the *lower uniform density* of B, denoted by $\underline{u}(B)$:

$$\underline{u}(B) = \lim_{s \to \infty} \frac{1}{s} \liminf_{t \to \infty} B(t+1, t+s).$$

Similarly, with

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$$\beta^s = \limsup_{t \to \infty} B(t+1, t+s),$$

the *upper uniform density* of B is $\overline{u}(B) = \lim_{s \to \infty} \beta^s/s$, or

$$\overline{u}(B) = \lim_{s \to \infty} \frac{1}{s} \limsup_{t \to \infty} B(t+1, t+s).$$

If $\underline{u}(B) = \overline{u}(B) = u(B)$, then $u(B)$ is the *(natural) uniform density* of B.

With this notation the definitions of the *lower asymptotic density* $\underline{d}(B)$ and the *upper asymptotic density* $\overline{d}(B)$ are respectively

$$\underline{d}(B) = \liminf_{s \to \infty} \frac{1}{s} B(1, s).$$

and

$$\overline{d}(B) = \limsup_{s \to \infty} \frac{1}{s} B(1, s).$$

If $\underline{d}(B) = \overline{d}(B) = d(B)$, then $d(B)$ is the *(natural) asymptotic density* of B.

It's clear that

$$\underline{u}(B) \le \underline{d}(B) \le \overline{d}(B) \le \overline{u}(B)$$

for any set B, and it is easy to produce an example of a set B with $d(B) = 1$ and $\underline{u}(B) = 0$, and a set C with $d(C) = 0$ and $\overline{u}(C) = 1$. Thus the statement that a set has uniform density 1 (or uniform density 0) is in fact stronger than the corresponding statement about asymptotic density. As another example, let S be the set of square-free integers. It is well known that $d(S) = 6/\pi^2$ while it can easily be shown that $\underline{u}(S) = 0$ and $\overline{u}(S) = d(S)$.

We mention three important properties of these densities which can be proved without difficulty directly from the definitions: 1) If $A \subset B$, then $\delta(A) \le \delta(B)$ where δ stands for any of the density functions defined above. 2) For either upper density, $\overline{\delta}$, and any sets A and B, $\overline{\delta}(A \cup B) \le \overline{\delta}(A) + \overline{\delta}(B)$. 3) If B is a union of disjoint arithmetic progressions,

$$B = \bigcup_{i=1}^{n} \{a_i t + b_i : t = 0, 1, 2, \ldots\},$$

then $u(B)$ exists and equals $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$.

3. Three lemmas. Let $P = \{p_1 < p_2 < p_3 < \ldots\}$ be the set of prime numbers and let $N_k = \{x : x$ is not divisible by $p_1, p_2, \ldots, p_k\}$. More generally, if $Q = \{q_1, q_2, \ldots, q_k\}$ is a set of distinct primes, then let $N_Q = \{x : x$ is not divisible by $q_1, q_2, \ldots, q_k\}$. For any set S and integer n write S_n for the set of all $x \in S$ which are divisible by n. We begin with a well known computation.

Lemma 1. $u(N_Q) = (1 - \frac{1}{q_1})(1 - \frac{1}{q_2}) \cdots (1 - \frac{1}{q_k})$.

Proof. Let $R = q_1 q_2 \cdots q_k$. Evidently, N_Q is the disjoint union of the arithmetic progressions $\bigcup_a \{Rt + a\}$ where a ranges over the $\phi(R)$ elements of $[1, R]$ which are prime to R. Hence

$$u(N_Q) = \frac{\phi(R)}{R} = (1 - \frac{1}{q_1})(1 - \frac{1}{q_2}) \cdots (1 - \frac{1}{q_k}).$$

Lemma 2. Let $Q = \{q_1, q_2, q_3, \ldots\}$ be a set of primes for which

$$\sum 1/q_i = \infty. \qquad (1)$$

Let S be a set of positive integers and suppose that $u(S_q) = 0$ for each prime q in Q. Then $u(S) = 0$.

Proof. Let $Q_k = \{q_1, q_2, \ldots, q_k\}$. Then, for each k,

$$S \subset N_{Q_k} \cup S_{q_1} \cup S_{q_2} \cup \ldots \cup S_{q_k}.$$

Hence,

$$\bar{u}(S) \le \bar{u}(N_{Q_k}) + \bar{u}(S_{q_1}) + \ldots + \bar{u}(S_{q_k}) = \bar{u}(N_{Q_k})$$

$$= (1 - \frac{1}{q_1})(1 - \frac{1}{q_2}) \cdots (1 - \frac{1}{q_k}).$$

As the last product tends to zero as $k \to \infty$, the lemma is proved.

Lemma 3. $u(S) = 0$ if and only if there exists a set of primes Q with infinite reciprocal sum, such that for any sequence $(q_1, q_2, \ldots.)$ of distinct elements of Q, $u(S_{q_1 q_2 \cdots q_k}) = 0$ for some k.

Proof. If $u(S) = 0$, the conclusion is obvious with $Q = P$. On the other hand, if $\bar{u}(S) > 0$ and Q is any set of primes satisfying (1), then, using Lemma 2, we can find q_1 in Q such that $\bar{u}(S_{q_1}) > 0$. Again, since $(S_p)_q = S_{pq}$ for distinct primes p and q, and using Lemma 2, we can find q_2 in $Q - \{q_1\}$ such that $\bar{u}(S_{q_1 q_2}) > 0$. Continuing in this manner we construct the required infinite sequence such that $\bar{u}(S_{q_1 q_2 \cdots q_k}) > 0$ for all k.

4. Applications. Before moving on to Fermat's last theorem we prove striking properties concerning the number of prime factors of a "typical" integer (cf. [4] Sections 22.11, 22.12).

Theorem 1. $u(P) = 0$.

Proof. Take $S = P$ in Lemma 2. Each S_p is a singleton.

Theorem 2. Let $G^t = \{ x : x$ is the product of no more than t prime numbers (counting multiplicities)$\}$. Then $u(G^t) = 0$.

Proof. $G^1 = P \cup \{1\}$ and so $u(G^1) = 0$. Proceeding inductively,

$$(G^{t+1})_p = pG^t \text{ and so } u((G^{t+1})_p) = 0.$$

By Lemma 2 $u(G^{t+1}) = 0$. (Here we use $kA = \{ kx : x \in A \}$ and the fact that $\bar{u}(kA) = \frac{1}{k}\bar{u}(A)$.)

Using Lemma 3 we can prove the more difficult result presented in the next theorem.

Theorem 3. Let $H^t = \{ x : x$ has t or fewer prime divisors $\}$. Then $u(H^t) = 0$.

Proof. Fix t, take $Q = P$, and let $q_1, q_2, \ldots, q_{t+1}$ be any $t+1$ distinct primes. Clearly $(H^t)_{q_1 q_2 \cdots q_{t+1}}$ is empty and we may apply Lemma 3.

Finally, we prove that the set of exponents, F, for which Fermat's Last Theorem is *false* has uniform density zero. Faltings' Theorem [1] implies that for each odd prime p the equation $x^p + y^p = z^p$ has only *finitely many* primitive solutions, and recently Heath-Brown [5] and Granville [3] have shown independently as a corollary to Faltings' Theorem that the set T of exponents n for which $x^n + y^n = z^n$ has *no* primitive solution (and hence no solution at all in positive integers) has *natural asymptotic* density 1. Of course, the idea behind our proof is also the use of Faltings' Theorem for prime exponents p. Both Heath-Brown and Granville attribute this idea to Filaseta [2].

Theorem 4. Let F be the set of all n such that $x^n + y^n = z^n$ has a solution. Then $u(F) = 0$.

Proof. Fix any odd prime p. Then for each $n \geq 3$,

$$\left\{\begin{array}{l} p \text{ divides } n \\ a^n + b^n = c^n \\ (a, b, c) = 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (a^{n/p})^p + (b^{n/p})^p = (c^{n/p})^p \\ (a^{n/p}, b^{n/p}, c^{n/p}) = 1 \end{array}\right\}.$$

and so, by Faltings' Theorem, each odd prime p divides only finitely many elements of F (since $a^{n/p}$ must assume at most finitely many values with $a > 1$). Therefore F_p is finite and so, by Lemma 2, F has uniform density 0.

It is apparently still unknown whether or not Fermat's Last Theorem is true for an infinite set of prime exponents. Filaseta proved that for any $n \geq 3$, Fermat's Last Theorem is true for exponent kn for all large k. The proof of this can be gleened from the proof of Theorem 4. It is interesting to note that with this result we can easily construct a sequence of products of two primes, $q_1 q_2$, $q_3 q_4$, $q_5 q_6$, ... , such that Fermat's Last Theorem is true for each member of the sequence and each prime p equals exactly one q_i.

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THE SPACE OF ISOMETRIC IMMERSIONS IS NO MANIFOLD IN GENERAL

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Presented by P. Ribenboim, F.R.S.C.

Abstract: In [2] E. Binz tries to prove that the space of immersions isometric to a given one with values in some Euclidean space is a manifold. Unfortunately this proof is wrong. Nevertheless the constructions give some geometric insight, so we give a short summary of the ideas in terms of principle fibre bundles and associated bundles and show where the error occurs. Finally we construct an embedding of S^2 in R^3 such that the space of immersions isometric to it is not locally arcwise connected and hence cannot be a manifold.

Binz's construction

Let M be an m-dimensional manifold and $i : M \to R^n$ be a fixed immersion with values in Euclidean space. $di : TM \to R^n$ will denote the principle part of the tangential map $Ti = (i, di)$ with respect to the canonical splitting of TR^n. In [3] it is shown that given any immersion $j : M \to R^n$ isometric to i, i.e. inducing the same metric on M, we can find a map $S : M \to S0(n)$ such that $dj(x) = S(x)di(x)$. Clearly $S(x)$ is uniquely determined only up to $S0(im\ di(x)^{\perp})$, where $im\ di(x)$ denotes the image of the linear map $di(x)$. Hence any such j gives rise to a section of a bundle whose fibre over $x \in M$ is $S0(n)/S0(im\ di(x)^{\perp})$. To identify this bundle, we use the following construction: let $V_{m,n} = S0(n)/S0(n-m)$ denote the Stiefel-variety as a principal fibre bundle over the Grassmann-manifold. The canonical right action of the structure group $S0(m)$ defines a left action λ on $V_{m,n}$ by $\lambda(g,p) := p \cdot g^{-1}$. Let $V_{m,n}[V_{m,n}]$ denote the bundle over $G_{m,n}$ obtained by associating the Stiefel-bundle to itself via this action. The elements of $V_{m,n}[V_{m,n}]$ are pairs of orthonormal m-frames (V, W) modulo the equivalence relation $(V, W) \sim (V \cdot G, W \cdot G)$ for all $G \in S0(m)$, and therefore (V, W) may be interpreted as an isometry defined on the subspace generated by V with values in R^n, more precisely onto the subspace generated by W. Hence we can think of (V, W) as an element of $S0(n)/S0(V^{\perp})$. The projection of (V, W) to the basis of this bundle, i.e. to $G_{m,n}$, is simply the subspace generated by V. If we consider $im\ di$ as a map with values in $G_{m,n}$ (which is possible since i is an immersion), S can be interpreted as a section of the pull-back $(im\ di)^*(V_{m,n}[V_{m,n}])$. Conversely each section of this bundle can be composed pointwise with di to give a candidate for the differential of an immersion isometric to i provided that a certain integrability - condition is satisfied which is described in the following lemma (for a proof cf. [1]).

Lemma: *If $A : M \rightarrow Mat(n)$ is a matrix-valued C^1-function, the differential $\nabla(Adi)$ of the R^n- valued one-from Adi obtained by pointwise composition of A and di can be computed as follows:*

$$\nabla(Adi)(X,Y) = dA(X)di(Y) - dA(Y)di(X),$$

where dA again denotes the principle part of TA.

Since $Sdi = dj$, we conclude from this lemma that $dS(X)di(Y) - dS(Y)di(X) = 0$ for all vector-fields X, Y on M. Conversely if this condition is satisfied and $H^1(M) = 0$, we can find a map $j : M \rightarrow R^n$ uniquely determined up to a constant such that $Sdi = dj$. Clearly j is an immersion isometric to i if S has values in $SO(n)$. This leads to the somewhat strange situation that we have a linear equation on the non-linear space $(im\ di)^*(V_{m,n}[V_{m,n}])$. In order to show that this is a manifold we have to linearize the space without disturbing the linearity of the integrability-condition.

Clearly the most natural way to do this is to use the locally defined inverse of the exponential map $\exp : so(n) \rightarrow SO(n)$, which is possible although we must pay attention to the fact that we deal with certain quotients of $SO(n)$ instead of the whole group itself. The last goal is to compute how the integrability-condition is transformed: the lemma cited above and the chain-rule yield

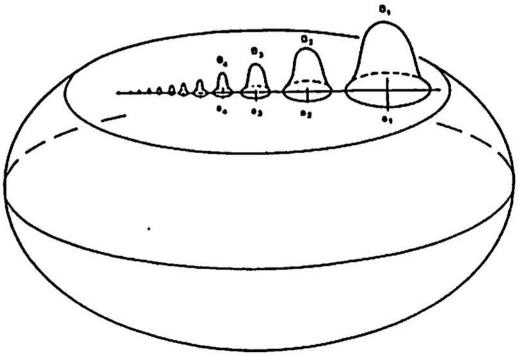
$$\begin{aligned}\nabla(\exp^{-1} \circ S \cdot di)(X,Y) &= d(\exp^{-1} \circ S)(X) \cdot di(Y) - d(\exp^{-1} \circ S)(Y) \cdot di(X) = \\ &= d\exp^{-1} \circ S \cdot dS(X) \cdot di(Y) - d\exp^{-1} \circ S \cdot dS(Y) \cdot di(X) = \\ &= d\exp^{-1} \circ S \cdot (dS(X)di(Y) - dS(Y)di(Y)) = \\ &= d\exp^{-1} \circ S \cdot \nabla(Sdi)(X,Y),\end{aligned}$$

which indicates that the integrability - conditions for Sdi and $\exp^{-1} \circ S \cdot di$ are equivalent. Unfortunately this computation is erroneous since it is not possible to get $d\exp^{-1} \circ S$ out of the bracket: being more careful we must have written $(\exp^{-1} \circ S) \cdot di$ which clearly is not the same as the meaningless expression $\exp^{-1} \circ (Sdi)$. But the lines above make use of this wrong associative law and hence are false. In fact the linearity of the integrability-condition is destroyed and it is transformed to a non-linear, quasilinear first-order differential equation [6].

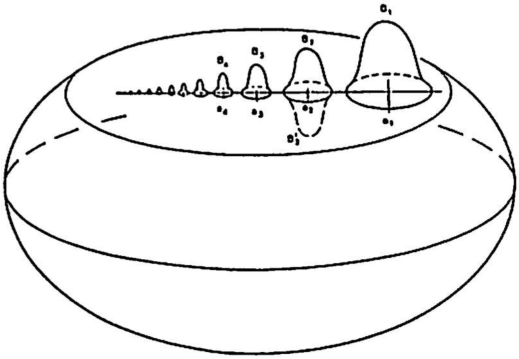
A counterexample

We will show that the space of isometric immersions is not a manifold in general by constructing an explicit counterexample. This example uses an embedding i of S^2 in R^3 such that the space of immersions which are isometric to i is not locally arcwise connected and hence cannot be a manifold. Since S^2 is a compact manifold, there are no discussions what topology we should use on the space of immersions, but the only topology that would allow the space of isometric immersions to be a manifold would be a very fine and nearly discrete topology. Our example will be a modification of a standard-example originally due to Cohn-Vossen, which can be found in [5].

To construct the embedding i of S^2 in R^3, take the standard embedding of S^2 and deform it so that it has a planar piece. Choose a convergent sequence $(a_n)_{n \in N}$ on this planar piece and pairwise disjoint C^∞-bumps B_n above every point a_n of the sequence. If the height and all derivatives of these bumps decrease rapidly enough, everything will fit together to give a C^∞-embedding i of the sphere the image of which looks approximately like the following picture:



Since every bump B_n is isometric to its mirror image B'_n in the plane, to each bump B_n corresponds a non-trivial isometric immersion i_n, i.e. an isometric immersion which is not simply the composition of the given one followed by a Euclidean motion:



This shows that there are isometric immersions arbitrarily close to the given immersion because there are arbitrarily small bumps. If the space of isometric immersions would be a manifold, it would be locally arcwise connected, hence it should be possible to join our immersion i to one of the immersions i_n by an arc of isometric immersions. But look at the top of the bump B_n: this is a region of positive curvature which should be connected to its own mirror image by a smooth homotopy in the space of isometric immersions. This is impossible simply because any surface of positive curvature has a natural orientation which has to be preserved during such a bending. So we see that this space of isometric immersions equiped with its natural topology as a subspace of the space of immersions cannot be a manifold.

Final remarks: Remember that a subgroup of a Lie-group need not be a Lie-group unless we refine its topology if necessary. Perhaps in our case the situation is the same: using a finer topology the above space becomes a manifold, but a very uninteresting one, namely a disconnected sum of countably many copies of the group of Euclidean motions in R^3. But note that the point where the proof breaks down has nothing to do with topology. Finally I must admit that I have no idea concerning the same question for analytic immersions; clearly the above example only is C^∞.

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On Fermat's Last Theorem

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To the memory of Taro Morishima

The Kummer-Mirimanoff congruences were devised in connection with the first case of Fermat's last theorem. In the present paper we shall derive some congruences related to the Kummer-Mirimanoff ones, using a certain identity. As a consequence of this, an easier and shorter proof for Benneton's theorem is given.

1. Introduction

Let p be an odd prime, B_m be the m-th Bernoulli number defined by $v/(e^v - 1) = \sum_{k=0}^{\infty} (B_k/k!)v^k$ and $f_n(v) = \sum_{i=1}^{p-1} i^{n-1}v^i$ for an integer n.

We consider the equation

$$x^p + y^p + z^p = 0, \quad p \nmid xyz, \tag{1}$$

where x, y and z are integers prime to each other.

It is well known that if (1) is satisfied, then

$$(K_1) \qquad f_{p-1}(t) \equiv 0 \pmod p,$$

$$(K_m) \qquad B_m f_{p-m}(t) \equiv 0 \pmod p, \; m = 2, 3, \ldots, p-2,$$

where $t \in H = \{-x/y, -y/x, -y/z, -z/y, -z/x, -x/z \pmod p\}$ (see e.g.[4]). Here we may assume that $t \not\equiv 0, 1 \pmod p$, because $x + y + z \equiv 0 \pmod p$ and $p \nmid xyz$.

It is also known that the system (K_m) $(m = 1, 2, \ldots, p-2)$

is equivalent to

$$(M_1 = K_1) \qquad f_{p-1}(t) \equiv 0 \pmod{p},$$

$$(M_g) \qquad f_g(t)f_{p-g}(t) \equiv 0 \pmod{p},\ g = 2, 3, \ldots, (p-1)/2.$$

The above (K_m) and (M_g) are the so-called Kummer-Mirimanoff congruences. As a relation between $B_i f_{p-i}(t)$ and $f_j(t)f_{p-j}(t)$, we can give the following congruence (cf.[1]):

$$\frac{1+t}{2} f_{p-1}(t) + \frac{1-t}{n+1} \sum_{i=2}^{p-2-n} \binom{p-1-n}{i} B_i f_{p-i}(t)$$
$$\equiv - \sum_{j=2}^{n+1} \binom{n}{j-1} f_j(t) f_{p-j}(t) \pmod{p},$$

where n is any integer with $1 \le n \le p - 4$.

On the other hand, Benneton [2] proved that (K_m) for $m = 1, 2, \ldots, p-2$ (hence (M_g) for $g = 1, 2, \ldots, (p-1)/2$) implies

$$(BG_r) \qquad \sum_{\substack{i=1 \\ \overline{ir}>p/2}}^{p-1} \frac{1}{i} t^i \equiv 0 \pmod{p},\ r = 1, 2, \ldots, p-1,$$

where $\overline{n}$ is an integer such that $n \equiv \overline{n} \pmod{p}$ and $0 \le \overline{n} \le p - 1$. Benneton's result, in precisely this form, appears in Granville's Ph.D. thesis ([3], Theorem L3-(h)).

In this paper we shall prove the following theorems:

Theorem 1. *If the equation* (1) *is satisfied, then for* $t \in H$

$$(A_k) \qquad \sum_{i=1}^{p-1} \frac{\varepsilon_{ik}}{i} t^i \equiv 0 \pmod{p},\ k = 1, 2, \ldots, p-1,$$

where $\varepsilon_n = 1$ *if* $\overline{n}$ *is odd, and* $\varepsilon_n = 0$ *if* $\overline{n}$ *is even*.

Theorem 2. *Let* $t \not\equiv 0, 1 \pmod{p}$. *If* 2 *is a primitive root* mod p, *then the system* (K_m) $(m = 1, 2, \ldots, p-2)$ *is equivalent to* (A_k) $(k = 1, 2, \ldots, p-1)$.

We note that (A_k) for $k = 1, 2, \ldots, p-1$ is equivalent to (BG_r) for $r = 1, 2, \ldots, p-1$. In fact, by noting that $\varepsilon_n = 1$ if and only if $\overline{n/2} > p/2$, we see that $\varepsilon_{ik} = 1$ if and only if $\overline{ir} > p/2$, where $r = k(p+1)/2$.

2. Proofs of the Theorems

Let $B'_n = \{(2^n - 1)/n\} B_n$ for $n \geq 1$ and denote by $[G(v)]_0^{(m)}$ the value of $d^m G(v)/dv^m$ at $v = 0$ for a differentiable function $G(v)$.

Proof of Theorem 1. Let k be an integer such that $1 \leq k \leq p-1$. Observe the identity

$$\omega_{k,t}(v) = \mu_{k,t}(v) + \nu_{k,t}(v), \tag{2}$$

where

$$\omega_{k,t}(v) = \frac{e^v}{e^v + 1} \sum_{i=1}^{p-1} i^{-1} (te^{kv})^i,$$

$$\mu_{k,t}(v) = \sum_{i=1}^{p-1} \{ \sum_{j=0}^{ik} (-1)^j e^{(ik-j)v} \} i^{-1} t^i$$

and

$$\nu_{k,t}(v) = - f_0((-1)^k t) \frac{1}{e^v + 1}.$$

Since $[e^v/(e^v + 1)]_0^{(0)} = 1/2$, $[e^v/(e^v + 1)]_0^{(n)} = -[1/(e^v + 1)]_0^{(n)} = B'_{n+1}$ for $n \geq 1$ and $[\sum_{i=1}^{p-1} i^{-1} (te^{kv})^i]_0^{(n)} = k^n f_n(t)$ for $n \geq 0$, we may give

$$[\omega_{k,t}(v)]_0^{(p-1)} = \frac{1}{2} k^{p-1} f_{p-1}(t) + \sum_{i=1}^{p-1} \binom{p-1}{i} B'_{i+1} \{k^{p-1-i} f_{p-1-i}(t)\}.$$

Also we have

$$[\mu_{k,t}(v)]_0^{(p-1)} = \sum_{i=1}^{p-1} S'_{p-1}(ik) i^{-1} t^i$$

and

$$[\nu_{k,t}(v)]_0^{(p-1)} = f_0((-1)^k t) B'_p,$$

where $S'_j(n) = \sum_{i=0}^{n-1}(-1)^i(n-i)^j$ for $n \geq 1$.

Therefore we can deduce from (2) that

$$\frac{1}{2}k^{p-1}f_{p-1}(t) + \sum_{i=1}^{p-1}\binom{p-1}{i}k^{p-1-i}\{B'_{i+1}f_{p-1-i}(t)\}$$
$$= \sum_{i=1}^{p-1} S'_{p-1}(ik)i^{-1}t^i + f_0((-1)^k t)B'_p.$$

Here $B'_p = 0$, B'_{p-1} is p-integral and $f_1(t) = (t^p - t)/(t-1) \equiv 0 \pmod p$. Also, since

$$S'_{p-1}(ik) \equiv 1 \pmod p \text{ if } \overline{ik} \text{ is odd,}$$
$$\equiv 0 \pmod p \text{ if } \overline{ik} \text{ is even,}$$

we have

$$\frac{1}{2}k^{p-1}f_{p-1}(t) + \sum_{i=1}^{p-3}\binom{p-1}{i}k^{p-1-i}\{B'_{i+1}f_{p-1-i}(t)\}$$
$$\equiv \sum_{i=1}^{p-1}\frac{\varepsilon_{ik}}{i}t^i \pmod p. \tag{3}$$

The congruence (K_m) implies $B'_m f_{p-m}(t) \equiv 0 \pmod p$ for each $m = 2, 3, \ldots, p-2$, which induces the congruences indicated in the theorem. This completes the proof.

Proof of Theorem 2. It suffices to verify that if 2 is a primitive root mod p, then (K_m) $(m = 1, 2, \ldots, p-2)$ are deduced from (A_k) $(k = 1, 2, \ldots, p-1)$. Let $D = (a_{ij})$ be a matrix of order $p-2$ with $a_{ij} = i^j$ $(1 \leq i, j \leq p-2)$, then $\det D \not\equiv 0 \pmod p$. Also $B'_m f_{p-m}(t) \equiv 0 \pmod p$ implies (K_m) in this case. Hence, from (3) the result clearly follows.

We shall give some notes on the system (A_k). It is easily seen that $\varepsilon_{ik} = 1$ if and only if $\varepsilon_{i(p-k)} = 0$. Hence (K_1) may be deduced from (A_k) and (A_{p-k}). And also (K_1) and (A_k) give (A_{p-k}) for each $k = 1, 2, \ldots, (p-1)/2$. So the system (A_k) $(k = 1, 2,$

..., p - 1) includes at most (p + 1)/2 (= 1 + (p - 1)/2) independent congruences. On the other hand, we see that $\varepsilon_{ik} = 1$ if and only if $\varepsilon_{(p-i)k} = 0$. This shows that the left hand side of (A_k) has exactly (p - 1)/2 terms. To cite instances,

$$(A_1) \qquad \sum_{\substack{0<i<p \\ i:\text{odd}}} \frac{1}{i} t^i \equiv 0 \pmod{p},$$

$$(A_2) \qquad \sum_{p/2<i<p} \frac{1}{i} t^i \equiv 0 \pmod{p},$$

$$(A_3) \qquad \sum_{\substack{0<i<p/3 \\ i:\text{odd}}} \frac{1}{i} t^i + \sum_{\substack{p/3<i<2p/3 \\ i:\text{even}}} \frac{1}{i} t^i + \sum_{\substack{2p/3<i<p \\ i:\text{odd}}} \frac{1}{i} t^i \equiv 0 \pmod{p}.$$

Incidentally, since $2\varepsilon_n = 1 - (-1)^{\overline{n}}$, we may state that the system (A_k) for k = 1, 2, ..., (p + 1)/2 is equivalent to

$$(T_s) \qquad \sum_{i=1}^{p-1} \frac{(-1)^{\overline{is}}}{i} t^i \equiv 0 \pmod{p}, \quad s = 0, 1, \ldots, (p - 1)/2.$$

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ABSOLUTELY FLAT MONOIDS ARE AMALGAMATION BASES

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The title means that if S and T are overmonoids of an absolutely flat U (i.e. every U-set is flat) then they can both be embedded over U in a common overmonoid. Combined with the recent proof of the absolute flatness of inverse semigroups [2], this yields an alternate proof that inverse semigroups are amalgamation bases [5]. The result depends on a description of the monoid amalgam in terms of tensor products, implicitly contained in [4], whose interest perhaps merits our making it explicit also for amalgamated products in which the factors may not be embedded: i.e. for general monoid pushouts.

The result is attained as a consequence of exhibiting the pushout of S and T over U as a direct limit, in the category of bi-U-sets, of alternating tensor products of copies of S and T reduced by the congruence which identifies the different copies of S and T.

A monoid morphism from U can be used to equip the codomain M with a bi-U structure: define the action of every u as multiplication by its image. Then an extension of the morphism to an overmonoid S of U − this is a bi-U-set under bi-multiplication by U − becomes a bi-U-morphism to M. If S and T are overmonoids on which morphisms to M agreeing on U are defined, then the map from $S \times T$, which sends every pair to the product of its images in M, factors through the tensor product $S \otimes T$ (all tensor products are taken over U); more generally, the map from every alternating product $\ldots S \times T \ldots$ to the product of the images in M factors through $\ldots S \otimes T \ldots$. There are bi-U-maps between these alternating tensor products − map the shorter product on those elements of the longer which have 1 in the additional places − which commute with the maps to the product of images in M: this system of tensor products constitutes a direct bi-U system. However its limit is not yet the sought for monoid amalgam. Observe that ...s ⊗ 1 ⊗ s'... has the same product of images in M as does ...ss' ⊗ 1 ⊗ 1... − one should also identify these pairs in every product ...S ⊗ T ⊗ S... . These are just the reductions needed to make the composite bi-U-maps ...S → ...S ⊗ T → ...S ⊗ T ⊗ S right S-maps. The identifications are compatible with the bi-action of U and with the bi-U system maps − i.e. the images of identified pairs are identified − and so reducing modulo the identifications yields another direct bi-U system. It is this system of alternating tensor products, reduced modulo these identifications (as well as the dual ones which identify ...t ⊗ 1 ⊗ t'... with ...tt' ⊗ 1 ⊗ 1...), whose direct limit is a monoid, hence the sought for amalgam. Indeed, taking the tensor product of two finite alternating tensor products yields again such an alternating product (of length the sum of their lengths, or the sum less one, depending on whether the contiguous end factors are different or the same): this defines an associative multiplication on the disjoint union of the alternating tensor products − in particular, an action of S and T on the union; it is compatible with the identifications, hence passes to the reduced products. This entails that the action of an alternating composite of s ε S and t ε T depends on the tuple via its image in the reduced tensor product; since 1 acts as the identity, the action effected by the image is unchanged under system map and so induces an action of the limit on itself. This may be construed as a binary composition, the associativity of (reduced) tensor product ensuring associativity for this composition; and since 1 acts as identity, the limit is a monoid – thus the monoid amalgam of S and T.

It remains to note the injectivity of the system maps when S and T are monoid extensions of an absolutely flat U. Starting with the right S-set P, U-embedded in the right U-set Q, one has $P \otimes S$ embedded in $Q \otimes S$ since S is left U-flat, $Q = Q \otimes 1 = Q \otimes U$ embedded in $Q \otimes S$ since Q is right U-flat. The identification of $p \otimes s$ with $ps \otimes 1$ is right S-compatible and so its transitive closure is a right S-congruence on the S-subset $P \otimes S$; it may be extended with the identity on the complement to such a congruence on $Q \otimes S$; and modulo this, Q remains embedded. This shows that when an alternating reduced product P, ending in S, is embedded in the next one Q, ending in T, then Q will in turn be embedded in the following one again ending in S: by induction, all the system maps are embeddings. Thus each of the reduced products – in particular S and T – are embedded in the limit.

The complete article was accepted for publication in the Journal of Pure and Applied Algebra in 1983 and has been held up since without explanation (offered or obtainable). In the interim two related treatments have appeared: In [6] the argument of [4] is similarly recast using U-sets, tensor products and direct limits; a more complete translation is effected in [8], which also exhibits the monoid pushout as the above direct limit of reduced bi-U-sets (although in a somewhat less intrinsic formulation).

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POLYMORPHIC LINEAR LOGIC AND TOPOS MODELS

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Presented by P. Ribenboim, F.R.S.C.

Abstract

We give a definition of a "linear fibration", which is a hyperdoctrine model of polymorphic linear logic, and show how to internalise the fibration, generating topos models. This gives a constructive set theoretical context for the logic of Petri nets, as recently developed by N. Martí-Oliet and J. Meseguer. Also, we sketch how this can be further extended to include the exponential operator ! . In this context, the topos model we construct can be embedded in the model constructed by A.M. Pitts.

0 Introduction

In [4], it is shown how to enrich the logic of Petri nets with *gedanken* states and processes, by embedding it into linear logic. Recently, Martí-Oliet and Meseguer have asked how to extend this even further to include polymorphism. It turns out that the process is fairly straightforward, and for maximal impact (so as to include constructive set theory), can be internalised to give topos models of polymorphic linear logic. Here we give the necessary definitions, and sketch the outline of the construction.

These notes should be thought of as a sequel to [7], extending the categorical semantics of linear logic to include polymorphism. As there, we (sketchily) describe the interpretation of polymorphic λ-calculus as the (indexed) Kleisli category induced by the !-cotriple. However, once in this context, we have another well-known model, *viz.* the internal full subcategory of a presheaf topos as constructed by A.M. Pitts [5]. Our construction does not give fullness, perhaps fortunately, but it will turn out that our model does embed faithfully into Pitts' model. (In essence his model consists of certain families, and ours corresponds to the "constant" families.)

1 Definitions

1.1 Linear fibrations

We begin with a definition:

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Definition 1 *A linear fibration* (**L**, **S**) *consists of*

1. *a category* **S** *with finite products, a distinguished object* U*, and exponentiation* U^A *for arbitrary* A,
2. *an indexed category* **L** *over* **S** *satisfying*
   - *(a)* $Obj(\mathbf{L}(A)) = Hom_{\mathbf{S}}(A, U)$ *for all objects* A *of* **S**, *and* $f^* = \mathbf{L}(f)$ *acts by composition on objects, for all morphisms* f *of* **S**,
   - *(b)* $\mathbf{L}(A)$ *is a linear category (i.e. $*$-autonomous with finite products—see [4, 7] for details), and* f^* *is a linear functor, for all* A, f,
   - *(c)* **L** *is weakly complete: i.e. for every object* C *of* **S** *there is an (indexed) right adjoint* Π_C *to the (indexed) functor* κ_C.

The point of this definition is that it provides the hyperdoctrine formulation of what higher-order polymorphic linear logic is, analagous to the hyperdoctrine models defined in [6]. The usual syntactic presentation can be shown equivalent to this, *via* an equivalence-of-categories result; we leave that as an exercise.

It should also be mentioned that we have given the definition for higher-order linear logic, not merely second-order linear logic. If the reader only wants the second-order case, all that needs to be done is to drop the requirement in (1) that **S** have exponents U^A; furthermore, one may as well assume that **S** has the natural numbers as objects (where n represents U^n), or just take the base category as presented in [5]. In this case, we need only require in (2c) that Π_U exists, all other such product-functors being given by iteration in the obvious way.

1.2 Topos models

Next we define a topos model for polymorphic linear logic.

Definition 2 *A topos model of polymorphic linear logic is given by an elementary topos* **E** *together with an internal category* **U** *of* **E**. *Furthermore,* **U** *must have (internally in* **E***) finite products (including the terminal object) and* A*-indexed products (for any object* A *of the sub-cartesian closed category of* **E** *generated by* U_0*—this understood internally*[1]*), and must be (internally) a $*$-autonomous category.*

Most of these notions are straightforward, and should not cause any bother. The correct notion of the involution has proved a trifle subtle (see [4]), and so we'll sketch some details here, outlining what it means to say that an internal symmetric closed monoidal category **U** has an involution $(\text{-})^\perp : \mathbf{U}^{op} \to \mathbf{U}$. (The notion of the dual internal category $\mathbf{U}^{op}$ is standard.)

First, we need morphisms (of **E**)

$$(\text{-})^\perp : U_0 \to U_0 \quad \text{and} \quad s : U_0 \times U_0 \to U_1$$

so that $d_0 \circ s = `\multimap'$ and $d_1 \circ s = `\multimap' \circ (\text{-})^\perp \times (\text{-})^\perp \circ \sigma$, where σ is the "switch coordinates" isomorphism, and where $`\multimap' : U_0 \times U_0 \to U_0$ is the morphism giving the internal hom on

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[1] This is the appropriate notion for a "pure" view of the logic—if one wished to include other constants explicitly, this could be done by adding them to the generating set for the sub-ccc.

objects. It is a straightforward exercise to mimic the construction of a contravariant functor from this data, as given in [4]. The crucial point is that one of the properties of an (internal) symmetric monoidal closed category is that there be an isomorphism $(\text{-})^{\parallel} : U_1 \rightarrow U_1$ and a pullback square as follows:

$$\begin{tikzcd}
U_1 \arrow[r, "\langle d_0, d_1\rangle"] \arrow[d, "(\text{-})^{\parallel}"'] & U_0 \times U_0 \arrow[d, "(\text{`I'}, \text{`}\multimap\text{'})"] \\
U_1 \arrow[r, "\langle d_0, d_1\rangle"'] & U_0 \times U_0
\end{tikzcd}$$

Next we need a pair of (inverse iso) morphisms $d : U_0 \rightarrow U_1$ and $d^{-1} : U_0 \rightarrow U_1$ (*sic*), (plus the appropriate equations for domains, codomains, and the identity composites), and the equations

$$s \cdot s = d^{-1}\text{`}\multimap\text{'}d : U_0 \times U_0 \rightarrow U_1$$

$$((\text{-})^{\perp} \circ d) \cdot (d \circ (\text{-})^{\perp}) = \text{`}id\text{'} \circ (\text{-})^{\perp} : U_0 \rightarrow U_1$$

(We leave it to the reader to decipher the notation, with the hint that $f \cdot g$ is the internal composition in **U**, and $f \circ g$ is composition in **E**.)

2 Constructing topos models

At this point we use a simple trick, that is part of the legacy of work done in the 1970's on indexed categories; (the details have appeared recently in [1]).

Given a linear fibration $(\mathbf{L}, \mathbf{S})$, we can construct an internal $*$-autonomous category **U** in the presheaf topos $\mathbf{E} =_{def} \mathrm{Sets}^{\mathbf{S}^{op}}$. The object of objects is $U_0 = Obj\,\mathbf{L}(\text{-}) = Hom_{\mathbf{S}}(\text{-}, U)$ and the object of morphisms is $U_1 = M\varphi\mathbf{L}(\text{-})$. The rest of the structure is defined fibrewise, in the obvious manner. For example, the internal composition $\gamma : U_2 \dot{\rightarrow} U_1$ is the natural transformation that, at an object A of **S**, sends a composable pair $\langle f, g\rangle$ (of morphisms of $\mathbf{L}(A)$) to their composite (in $\mathbf{L}(A)$). Similarly, $(\text{-})^{\perp} : U_0 \dot{\rightarrow} U_0$ is the natural transformation that, at A, sends an object X to $X^{\perp}$ (in $\mathbf{L}(A)$).

In this model, **U** has A-indexed products for any representable object A (*i.e.* for any object of **S**, *via* the Yoneda embedding).

Given an internal category, the standard way to "externalise" this, to obtain a fibration, is just to take the *Hom* functor $Hom_{\mathbf{E}}(\text{-}, \mathbf{U}) : \mathbf{E}^{op} \rightarrow \mathbf{Cat}$. In our case, however, we can cut this back to **S** again, *via* the Yoneda embedding $H : \mathbf{S} \rightarrow \mathbf{E}$, and so obtain a fibration $\mathbf{L}'$ over **S**. Then $\mathbf{L}'(A) = Hom_{\mathbf{E}}(HA, \mathbf{U})$ is the category whose objects are natural transformations $HA \dot{\rightarrow} HU$, or equivalently (by the Yoneda Lemma) morphisms $A \rightarrow U$ in **S**, and whose morphisms are natural transformations $HA \dot{\rightarrow} M\varphi\mathbf{L}(\text{-})$, or equivalently, morphisms of $\mathbf{L}(A)$. Thus we see that $(\mathbf{L}, \mathbf{S})$ is equivalent to $(\mathbf{L}', \mathbf{S})$.

So we finally end up with the

Theorem 1 *Suppose* $(\mathbf{L}, \mathbf{S})$ *is a linear fibration. Then* $\mathrm{Sets}^{\mathbf{S}^{op}}$ *is a topos model of polymorphic linear logic, with internal category* **U** *as constructed above. Furthermore the "externalisation" of this internal model is, when restricted* via *the Yoneda embedding to the original base* **S**, *equivalent to the original fibration.*

Remark There is another well-known method of constructing internal categories from fibrations, *viz.* the "presheaves on the Grothendieck construction" method of A.M. Pitts [5]. It is perhaps worth pointing out that that approach does not seem to work in this setting—the cartesian closed structure of the fibres of the fibration being essential to the process. If one tries to replace the cartesian structure by the monoidal structure of linear logic, one quickly finds that at several points one wants the tensor to be a real product (*e.g.* to have projections), or the unit to be a terminal. In fact, in the category $Gr(\mathbf{L})$ of "elements" of $\mathbf{L}$, one cannot even make (the canonical image of) U into (the object of objects of) an internal category. However, once we pass to the cartesian closed structure generated by the exponentials (section 3), we shall find that our model embeds into Pitts' model.

3 The exponential operator !

3.1 Girard fibrations

Of course, although the term *linear logic* ought to refer only to the logic of the additive and multiplicative connectives, it is usually used to include also the exponential operators ! and ?. In [7] it was pointed out that ! has the structure of a cotriple (*a.k.a.* comonad), and in the fibrational context (of quantification), the cotriple must be indexed in the obvious way. Since the quantification considered in [7] was not polymorphic, but only predicate, it might be worth sketching some of the details here.

Definition 3 *A Girard fibration* $(\mathbf{G},\mathbf{S})$ *is a linear fibration for which*

1. *each fibre* $\mathbf{G}(A)$ *is a Girard category, in the sense of [7], via an indexed cotriple !, and*
2. *each "inverse image functor"* $f^\bullet$ *preserves this structure ("on the nose").*

To elaborate, this means that we have a cotriple $!_A : \mathbf{G}(A) \to \mathbf{G}(A)$ (for each A), so that $!X$ carries a comonoid structure in $\mathbf{G}(A)$, for each object X of $\mathbf{G}(A)$:

$$\mathrm{I}_A \xleftarrow{\epsilon'_A(X)} !_A X \xrightarrow{\delta'_A(X)} !_A X \otimes_A !_A X$$

which is the image *via* natural (and indexed) isomorphisms $\mathrm{I}_A \xrightarrow{\sim} !_A 1_A$ and $!_A X \otimes_A !_A Y \xrightarrow{\sim} !_A(X \times_A Y)$ of the canonical cartesian comonoid structure; *i.e.* the following diagram commutes (omitting subscripts):

$$\begin{tikzcd}
\mathrm{I} \arrow[dd, "\wr"'] & & !X \otimes !X \arrow[dd, "\wr"] \\
& !X \arrow[ul, "\epsilon'"] \arrow[ur, "\delta'"'] \arrow[dl, "!X"] \arrow[dr, "!\Delta"'] & \\
!1 & & !(X \times X)
\end{tikzcd}$$

By saying that each f^* preserves this structure, we imply that f^* commutes with these morphisms, and in particular with !; *i.e.* $f^*(!_A X) = !_A f^*(X)$. (We ignore the question of "up to coherent iso" in these notes.)

As in [7], from a Girard fibration $(\mathbf{G}, \mathbf{S})$ we can construct the Kleisli category (indexed, of course) $(\mathbf{G}_!, \mathbf{S})$; each fibre $\mathbf{G}_!(A) = (\mathbf{G}(A))_!$. As in [7], these fibres are cartesian closed, and so we end up with a hyperdoctrine model of polymorphic λ-calculus. (The quantifier Π_C remains the same, as in [7].)

3.2 Topos models

To get the notion of topos model of the full Girard logic, one need only add to the notion of a topos model from section 1.2 the internalisation of the above notions. So we must have an internal cotriple on $\mathbf{U}$, given by an internal functor $! : \mathbf{U} \to \mathbf{U}$ and internal natural transformations $\epsilon : ! \stackrel{\cdot}{\to} id$ and $\delta : ! \stackrel{\cdot}{\to} !!$, together with internal comonoid structure given by internal $\epsilon' : ! \stackrel{\cdot}{\to}$ 'I' and $\delta' : ! \stackrel{\cdot}{\to} ! \otimes !$ plus appropriate isos, commutative diagrams and the lot.

It is an easy exercise to see that this data allows the internal construction of the (internal) Kleisli category $\mathbf{U}_!$, and then it is easy to see that $\mathbf{U}_!$ is cartesian closed, as in [7], and is an internal topos model of polymorphic λ-calculus..

We shall further leave it to the reader to verify that the process of constructing topos models is coherent with respect to these notions: that starting with a Girard fibration, we can construct a topos model of full Girard logic, and so (*via* the Kleisli construction) a topos model of polymorphic λ-calculus, or we can arrive at this topos model by first using the Kleisli construction on the fibration.

3.3 The connection with Pitts' model

In this context, we have in front of us two topos models of polymorphic λ-caluclus, *viz.* the model $\mathbf{U}_!$ in the topos $\mathbf{E} = \mathbf{Sets}^{\mathbf{S}^{op}}$ and the model constructed by A.M. Pitts in [5], which we shall denote $\mathbf{U}'$ in the topos $\mathbf{E}' = \mathbf{Sets}^{Gr(\mathbf{G}_!)^{op}}$. (The latter is an internal full subcategory, the former is not. For the details of the construction of Pitts' model, see [5].) What is the connection between these?

First note that there is a geometric morphism between the toposes induced by the "projection" functor $p : Gr(\mathbf{G}_!) \to \mathbf{S}$ which sends an object $(A, X : A \to U)$ to A, and a morphism (α, f) to α. (There is another geometric morphism in the reverse direction induced by the right adjoint T to p, which embeds $\mathbf{S}$ in $Gr(\mathbf{G}_!)$ *via* terminals.) The functor p^* then carries the internal category $\mathbf{U}_!$ to an internal category (which I shall denote here by $\mathbf{U}$—the context should help avoid confusion) in $\mathbf{E}'$.

It is perhaps not too surprising that each of these internal categories (in $\mathbf{E}'$) has the same object of objects: $U_0 = U'_0 = HTU$, the (representable) functor which sends $(A, X : A \to U)$ to $Hom_{\mathbf{S}}(A, U) = Obj\,\mathbf{G}_!(A)$. However, $\mathbf{U}$ has "fewer" morphisms than (the full subcategory) $\mathbf{U}'$: U_1 sends $(A, X : A \to U)$ to $M\varphi\mathbf{G}_!(A)$, and U'_1 sends $(A, X : A \to U)$ to $Hom_{Gr(\mathbf{G}_!)}((A, X : A \to U), (U^2, `\Rightarrow' : U^2 \to U))$. (These may be thought of as "families" of $\mathbf{G}_!(A)$-morphisms, indexed by X. From this point of view, $U_1(X)$ consists of the "constant" families.) Recall that the morphism '$\Rightarrow$': $U^2 \to U$ (in $\mathbf{S}$) is the composite '$\multimap$' $\circ$ ('!' $\times\, \mathrm{id}_U$); *i.e.* internally '$u \Rightarrow v$' = '$!u \multimap v$'.

Then, there is a faithful internal functor (preserving the polymorphic λ-calculus structure) $K : \mathbf{U} \to \mathbf{U}'$, which is identity on objects:

$K_0 : U_0 \overset{\cdot}{\to} U_0'$ is the identity natural transformation, and

$K_1 : U_1 \overset{\cdot}{\to} U_1'$ is the natural transformation that, at an object $(A, X : A \to U)$, sends a morphism $f : Z \to Y$ (of $\mathbf{G}_1(A)$) to $(\langle Z, Y\rangle, `f')$, where $`f' : X \to \top \to (Z \Rightarrow Y)$ is the canonical "name" of f.

To check that K is a functor as claimed will be left as an exercise.

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ON THE MINIMAL FREE RESOLUTIONS OF FINITE SETS OF POINTS IN P^n

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Presented by P. Ribenboim, F.R.S.C.

Abstract. There is apparently a great interest for the minimal free resolutions of finite sets of points and projective varieties (see, e.g.[1],[2],[3],[7],[9],[10],[11]). For instance, the minimal free resolution of $n+2$ points spanning P^n yields a starting point for a structure theory for projective varieties of "degree = = codimension + 2" (see [10]). The aim of this paper is to generalize this free resolution. Moreover, we describe some applications. We will discuss here the basic ideas of the proofs (the complete details will be contained in [12]).

Notation. We denote by X a finite set of distinct points in $P^n = P^n_K$, with $n > 2$ and K any algebraically closed field, and by card X the cardinality of X (we will assume card $X > n+1$). Let $I(X)$ be the defining ideal of X in the polynomial ring $R = K[X_o,\ldots,X_n]$ and let $A = R/I(X)$ be the homogeneous coordinate ring of X. Then the Hilbert function $\{\mathrm{rank}_K A_t\}_{t>o}$ will be denoted by $\{h_X(t)\}_{t>o}$. Also we say that X is in *generic* position, if $h_X(t) = \min\{\mathrm{card}\, X, \binom{n+t}{n}\}$ for any $t > o$. Finally, we say that X is in *general* position, if no $n+1$ points of X lie on a hyperplane.

We start with a "key-result" of geometrical flavour.

LEMMA : *Let* X *be a finite set of points in* P^n *and let* t *be any positive integer. We set* $h_X(t) =: \mathrm{card}\, X - \delta$ $(\delta>0)$.

If there is a subset $X'_t =: X' \subset X$ *such that* :

(i) $h_{X'}(t) = \mathrm{card}\, X' - \delta$, *and*

(ii) $h_V(t) = \mathrm{card}\, V$, *for all subsets* V *of* X' *with* card V = card $X' - \delta$, *then* X' *is unique*.

Moreover, if $\delta = 1$, *then such a set* X' *always exists* *(which is not true, in general, if* $\delta>2$).

Proof: Clearly we may assume $\delta \geq 1$. Now, in order to prove the unicity of X' , it is enough to show that, if X'' is any subset of X satisfying both (i) and (ii), then $X' \subset X''$.

Suppose not. Then we start from the general relation:

$$(*) \qquad h_{X' \cup X''}(t) = h_{X'}(t) + h_{X''}(t) - h_{X' \cap X''}(t) \quad ,$$

and we distinguish two cases :

(1) $\operatorname{card}(X' \cap X'') \leq \operatorname{card} X' - \delta$, in which case, if V denotes a subset of X' containing $X' \cap X''$, with $\operatorname{card} V = \operatorname{card} X' - \delta$, we get (by (ii)): $h_V(t) = \operatorname{card} V$, hence $h_{X' \cap X''}(t) \geq \operatorname{card}(X' \cap X'')$. Also, in view of (i), it follows from (*) that:

$$\begin{aligned} h_{X' \cup X''}(t) &\leq \operatorname{card} X' - \delta + \operatorname{card} X'' - \delta - \operatorname{card}(X' \cap X'') = \\ &= \operatorname{card}(X' \cup X'') - 2\delta \quad , \end{aligned}$$

which gives a contradiction, since $h_X(t) = \operatorname{card} X - \delta$.

(2) $\operatorname{card}(X' \cap X'') > \operatorname{card} X' - \delta$. In this case, we put : $X' \cap X'' = V \cup \{P_1, \ldots, P_\gamma\}$, where $\operatorname{card} V = \operatorname{card} X' - \delta$ and $P_1, \ldots, P_\gamma \notin V$ $(1 \leq \gamma < \delta)$. It follows from (ii) that

$$h_{X' \cap X''}(t) \geq h_V(t) = \operatorname{card} V = \operatorname{card}(X' \cap X'') - \gamma \quad .$$

Therefore, in view of (i), we get from (*) :

$$\begin{aligned} h_{X' \cup X''}(t) &\leq \operatorname{card} X' - \delta + \operatorname{card} X'' - \delta - \operatorname{card}(X' \cap X'') + \gamma \\ &\leq \operatorname{card}(X' \cup X'') - 2\delta + \gamma \leq \operatorname{card}(X' \cup X'') - \delta - 1 \; , \end{aligned}$$

which gives a contradiction again, since $h_X(t) = \operatorname{card} X - \delta$.

This completes the proof of the unicity of X' .

Also, if $\delta = 1$, such a set X' certainly exists: enough to take a subset X' of X , minimal with respect to the property that $h_{X'}(t) = \operatorname{card} X' - 1$. But, if $\delta > 1$, then X' may not exist. For example, take a set X of six points in $\mathbb{P}^3$, say $X = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$, with P_1, P_2, P_3 lying on a line L and Q_1, Q_2, Q_3 outside L and moreover such that each plane joining L with Q_i $(i=1,2,3)$ does not contain any point Q_j with $j \neq i$. Then $h_X(1) = 6 - 2 = 4$; yet, there is no subset X' of X satisfying both (i) and (ii) for $t = 1$, which ends the proof.

REMARK: We point out that, whenever the set X'_t considered in the Lemma exists, we get new "invariants" attached to X, say $\alpha_{t,m}(X) =: h_{X'_t}(m)$ for $m \geqslant 1$, since X'_t actually depends on X and t. For example, the integer $\alpha_{t,1}(X) - 1$ gives the dimension of the subspace of $\mathbf{P}^n$ spanned by X'_t. Also this integer turns out to be very useful for the study of finite sets of points in $\mathbf{P}^n$, as is shown by our next theorem. In particular, if $\alpha_{t,1}(X) = 2$ (and $\delta \geqslant 1$), then X contains at least $t+1$ collinear points (the converse being true, if $\delta = 1$). We will exploit the properties of these new "invariants" in a future paper.

We are now able to state our main result.

THEOREM: *With the notation as above, let* $X \subset \mathbf{P}^n$ *be a set of* $\binom{n+d-1}{n} + 1$ *points in generic position* $(n,d \geqslant 2)$. *Then* :

(1) X *has a minimal free resolution of the form* :

$$(+)\quad \begin{aligned} 0 &\to R^{\beta_n}(-d-n) \oplus R^{\alpha_n}(-d-n+1) \to \ldots \to R^{\beta_p}(-d-p) \oplus R^{\alpha_p}(-d-p+1) \to \\ &\to R^{\alpha_{p-1}}(-d-p+2) \to \ldots \to R^{\alpha_1}(-d) \to R \to A \to 0 \end{aligned}$$

for some integer p $(1 \leqslant p \leqslant n)$, *where for* $i = 1,\ldots,n$:

$$\alpha_i = \binom{d-1+n}{d-1+i}\binom{i-2+d}{d-1} - \binom{n}{i-1} + \binom{n-p}{i-p-1}, \qquad \beta_i = \binom{n-p}{i-p}.$$

(2) *we get* (+), *if and only if*, $p = \alpha_{d-1,1}(X) - 1$.

Sketch of the proof: In view of the Lemma and the Remark above, the integer $p = \alpha_{d-1,1}(X) - 1$ is well defined. Now the theorem says that the minimal free resolution of X only depends on this new "invariant" p. In fact, since X is in generic position, we get: $[\operatorname{Tor}_i(K,A)]_j = 0$ for $j \neq d+i$, $d+i-1$ (see, e.g. [11] or [12]). Therefore the β_i's can be computed by a suitable extension of the techniques used in [9] and [4], which is done in [12]. Finally, the α_i's are obtained using the Hilbert function of X.

Before stating some consequences of our Theorem, it is worth observing (see [12]) that it is not difficult to construct examples where the "invariant" p considered above assumes all possible values $(1 < p < n)$.

COROLLARY 1 : *Let* $X \subset P^n$ *be a set of* $\binom{n+d-1}{n}+1$ *points in generic position* $(n,d > 2)$. *Consider the following two conditions* :

(i) X *has a minimal free resolution of the form* :

$$0 \to R^{\beta_n}(-d-n) \oplus R^{\alpha_n}(-d-n+1) \to R^{\alpha_{n-1}}(-d-n+2) \to \dots \to$$

$$\to R^{\alpha_1}(-d) \to R \to A \to 0 \quad , \quad \textit{where}$$

$$\alpha_i = \binom{d-1+n}{d-1+i}\binom{i-2+d}{d-1} - \binom{n}{i-1} \quad , \quad \beta_i = 0 \quad \textit{for} \quad i=1,\dots,n-1 \quad ,$$

and $\alpha_n = \binom{d-2+n}{d-1}$, $\beta_n = 1$.

(ii) *no* $d+1$ *points of* X *lie on a hyperplane* .

Then (ii) ⇒ (i) , *the converse being true, in general, only for* $n = 2$.

Proof: (ii) ⇒ (i) . Suppose not; then it follows from our Theorem that the set $X' = X'_{d-1}$ (which is uniquely determined, in view of the Lemma above) spans a subspace of P^n of dimension $< n-1$. Therefore, at least $h_{X'}(d-1)+1$ points of X lie on a hyperplane: contradiction .

(i) ⇒ (ii) . In fact, if $n = 2$, this implication is clear, in view of the Theorem and the Remark above. Yet, for $n > 2$, the same need not be true (see [12]) .

We note that the implication (ii) ⇒ (i) in the corollary above is proved also in [11] . The next statement follows directly from our Theorem (see also the Remark) and moreover provides an answer, in the particular case here studied, to Problem (B) stated in [5] .

COROLLARY 2 : The ideal of $\binom{n+d-1}{n}+1$ points in generic position in $\mathbb{P}^n$ $(n,d \geq 2)$ is generated by forms of degree d, if and only if, no $d+1$ points of X lie on a line .

Finally, we mention that the techniques developed in [12] provide the minimal free resolution of $n+3$ points in $\mathbb{P}^n$ (see also [9],[4],[6]) , which are in general position :

PROPOSITION : Let $X \subset \mathbb{P}^n$ be a set of $n+3$ points in general position. Then X has the following minimal free resolution :

$$0 \to R^2(-n-2) \to R^n(-n-1) \oplus R^{\alpha_{n-1}}(-n) \to$$

$$\to R^{\alpha_{n-2}}(-n+1) \to \ldots \to R^{\alpha_1}(-2) \to R \to R/I(X) \to 0 \quad ,$$

where $\alpha_i = i\binom{n+1}{i+1} - 2\binom{n}{i-1}$ for $i = 1,\ldots,n-1$.

In connection with this last statement and also in view of the work done in [10], we want to state the following :

PROBLEM : To describe explicitly the minimal free resolution of $n+3$ points spanning $\mathbb{P}^n$ $(n \geq 2)$.

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NOTE ON A CLASS OF FUNCTIONS OF GODUNOVA AND LEVIN

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Presented by G.F.D. Duff, F.R.S.C.

Abstract. Some inequalities for functions of Godunova and Levin are given.

E.M.Wright [1] proved the following result:

Let I be an interval from R and $f:I\to R^+$ be either monotonic or a convex function. If $x_1,x_2,x_3\in I$, then

$$f(x_1)(x_1-x_2)(x_1-x_3)+f(x_2)(x_2-x_1)(x_2-x_3)+f(x_3)(x_3-x_1)(x_3-x_2)\geqq 0, \tag{1}$$

with equality if and only if $x_1=x_2=x_3$.

E.K.Godunova and V.I.Levin [2] introduced the following class of functions:

A function f, defined on I, is said to belong to class Q if it is nonnegative and satisfies the following inequality

$$f(\lambda x+(1-\lambda)y)\leqq\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda} \tag{2}$$

for all $x,y\in I$ and $\lambda\in(0,1)$.

They also noted that nonnegative monotonic and nonnegative convex functions belong to this class of functions, and gave, for example, the following results:

If $x_1,x_2,x_3\in I$ and $f\in Q$, then (1) is valid. If $x_1<x_2<x_3$, we have the following equivalent forms of (1):

$$f(x_2)\leqq\frac{x_3-x_1}{x_3-x_2}f(x_1)+\frac{x_3-x_1}{x_2-x_1}f(x_3), \tag{3}$$

and

$$\frac{f(x_1)}{x_3-x_2}+\frac{f(x_2)}{x_1-x_3}+\frac{f(x_3)}{x_2-x_1}\geqq 0. \tag{4}$$

The best known consequence of (1) is the well-known Schur inequality ($f(x)=x^t$, $t\in R$).

First, we shall note that inequalities (1-4) are equivalent. The following inequality is related to the well-known Jensen inequality for convex functions:

Theorem 1. If $f \in Q$, $x \in I^n$, $n \geqq 2$, w a positive n-tuple, then

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqq W_n \sum_{i=1}^{n} \frac{f(x_i)}{w_i} \qquad \left(W_n := \sum_{i=1}^{n} w_i\right). \tag{5}$$

Proof. The proof of (5) is by induction and the case $n = 2$ is just (2). Suppose then the result is valid for all k, $2 \leqq k \leqq n-1$.

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) = f\left(\frac{w_n}{W_n} x_n + \frac{W_{n-1}}{W_n}\cdot\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i x_i\right)$$

$$\leqq W_n\left(\frac{1}{W_{n-1}} f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i x_i\right) + \frac{1}{w_n} f(x_n)\right),$$

by the $n = 2$ case,

$$\leqq W_n\left(\sum_{i=1}^{n-1} \frac{f(x_i)}{w_i} + \frac{f(x_n)}{w_n}\right)$$

$$= W_n \sum_{i=1}^{n} \frac{f(x_i)}{w_i}, \text{ by the induction hypothesis.} \square$$

Theorem 2. Let w be a real n-tuple such that

$$w_1 > 0, \quad w_i < 0 \quad (i = 2,\ldots,n), \quad W_n > 0. \tag{6}$$

If $f \in Q$, $x \in I^n$, $n \geqq 2$, $\bar{x} = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in I$, then

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geqq W_n \sum_{i=1}^{n} \frac{f(x_i)}{w_i}. \tag{6}$$

Proof. This is a simple consequence of Theorem 1 if we use the substitutions:

$$w_1 \to W_n, \quad x_1 \to \bar{x}, \quad p_i \to -p_i, \; x_i \to x_i \quad (i=2,\ldots,n).$$

Now, let J be a finite nonempty set of positive integers. If we define the index set function F by

$$F(J) = \frac{1}{W_J} f\left(\frac{1}{W_J}\sum_{i\in J} w_i x_i\right) - \sum_{i\in J} \frac{f(x_i)}{w_i}$$

and if

$$W_J = \sum_{i\in J} w_i, \quad A_J(x;w) = \frac{1}{W_J}\sum_{i\in J} w_i x_i,$$

then the following theorem is valid:

Theorem 3. Let $f\in Q$, let J and K be finite nonempty sets of positive integers such that $J\cap K = \emptyset$, $w = (w_i)_{i\in J\cup K}$, and let $x = (x_i)_{i\in J\cup K}$ be real sequences such that $w_i \neq 0$, $x_i\in I$ $(i\in J\cup K)$, $W_{J\cup K} > 0$, $A_S(x;w)\in I$ $(S = J, K, J\cup K)$.

If $W_J > 0$ and $W_K > 0$, then

$$(7) \qquad F(J\cup K) \leqq F(J) + F(K).$$

If $W_J W_K < 0$, then the sense of (7) reverses.

Proof. This is a simple consequence of Theorems 1 and 2. We must put only

$$x_1 = A_J(x;w),\ w_1 = W_J,\ x_2 = A_K(x;w),\ w_2 = W_K.$$

COROLLARY 1. (a) If $w_i > 0$ $(i=1,\ldots,n)$, $I_k = \{1,\ldots,k\}$, then

$$(8) \qquad F(I_n) \leqq F(I_{n-1}) \leqq \ldots \leqq F(I_2) \leqq 0$$

and

$$(9) \quad F(I_n) \leqq \min_{1\leqq i<j\leqq n} \left((p_i+p_j)^{-1} f\left(\frac{p_i x_i + p_j x_j}{p_i+p_j}\right) - p_i^{-1} f(x_i) - p_j^{-1} f(x_j)\right).$$

(b) If $A_n(x;w)\in I$ and if (6) holds, then the reverse inequalities in (8) are valid, and

$$(10) \quad F(I_n) \geqq \max_{2\leqq i\leqq n} \left((p_1+p_i)^{-1} f\left(\frac{p_1 x_1 + p_i x_i}{p_1+p_i}\right) - p_1^{-1} f(x_1) - p_i^{-1} f(x_i)\right).$$

Finally, we shall prove the following conversion of Theorem 1:

Theorem 4. If $f\in Q$, $[m,M]\subset I$, $x\in [m,M]^n$, w a positive n-tuple, then

$$(11) \qquad \sum_{i=1}^{n} \frac{f(x_i)}{w_i} \leqq f(m)\sum_{i=1}^{n} \frac{M-m}{w_i(M-x_i)} + f(M)\sum_{i=1}^{n} \frac{M-m}{w_i(x_i-m)}.$$

Proof. From (3) we have, $1 \leqq i \leqq n$,

$$f(x_i) \leqq \frac{M-m}{M-x_i} f(m) + \frac{M-m}{x_i-m} f(M)$$

which clearly implies (11).

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A tiling of $\mathbf{R}^3$ by nearly congruent rhombi

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Presented by H.S.M. Coxeter, F.R.S.C.

In a recent paper [1] one of us raised the question of whether it is possible to tile $\mathbf{R}^3$ with congruent copies of the unit square $[0,1]^2$. In the context of the question, a collection of subsets of $\mathbf{R}^n$ is said to provide a tiling if each point of $\mathbf{R}^n$ belongs to one and only one member of the collection. As shown in [1] this strict, set-theoretic definition makes it impossible to tile $\mathbf{R}^2$ with unit squares or even with an assortment of homeomorphs of the unit square. On the other hand, the definition definitely does allow tilings of $\mathbf{R}^3$ by various homeomorphs of $[0,1]^2$ because it allows tilings with congruent copies of $[0,1)^3$ and this set is homeomorphic to $[0,1]^2 \times [0,1)$. A construction based directly on this observation can provide a tiling by an assortment of crinkled hexagonal disks, each arbitrarily close in shape to a fixed planar regular hexagon. A quite different idea gives a tiling by an assortment of rhombic tiles, each with edge length 1 but having acute angles dense in $(0,\frac{\pi}{2}]$. These constructions appear in [1] and the purpose of the present note is to give a refinement which combines their best features - nearly congruent tiles, as in the first construction, and planar ones, as in the second.

Theorem. For any $\epsilon > 0$ it is possible to tile $\mathbf{R}^3$ by an assortment of rhombic tiles with edge length 1 and acute angle restricted to lie in the interval $(\frac{\pi}{2} - \epsilon, \frac{\pi}{2}]$.

We prove this Theorem with a sequence of lemmas and remarks.

Lemma 1. For any $\epsilon > 0$ it is possible to tile the upper half-plane $U = \{(x,y) \in \mathbf{R}^2 : y > 0\}$ with closed unit line segments whose angle with the horizontal is restricted to lie in the interval $[0,\epsilon)$.

Proof. Consider the $\frac{\pi}{2}$-column $0 \leq x \leq 1,\ y > 0$ tiled with horizontal closed unit line segments; its leading edge $x=0$ and open foot $y=0,\ 0 \leq x \leq 1$ meet at its leading vertex (0,0) at an angle $\frac{\pi}{2}$. The effect of applying the affine mapping

$$(x, y) \rightarrow (x + y\ \cos\theta,\ y\ \sin\theta), \quad 0 < \theta < \frac{\pi}{2},$$

to this $\frac{\pi}{2}$-column is to transform it into a θ-column tiled with horizontal closed unit line segments; the leading edge and open foot of this θ-column meet at its leading vertex at an angle θ.

Assume $\varepsilon < \frac{\pi}{2}$ and let $\psi(x)$ be a function such as $\frac{\varepsilon}{2}(1 - \frac{x}{1+|x|})$ which is strictly decreasing on $\mathbb{R}$ and satisfies $\varepsilon > \psi(x) > 0$. At each integer point $(n,0)$ on the x-axis, attach the open wedge $\mathrm{W}(n)$ consisting of rays inclined to the x-axis at angles $\psi(n-1) > \theta > \psi(n)$. The space between these open wedges can be tiled satisfactorily by placing a $\psi(n)$-column with its open foot on the x-axis and its leading vertex at $(n,0)$, $n=0, \pm1, \pm2, \cdots$. This done, it remains to tile the open wedges.

A fan in the wedge $\mathrm{W}(n)$ with vertex $(n,0)$ is a collection of rays eminating from $(n,0)$ and inclined to the x-axis at a sequence of angles θ_m such as $\psi(n-1) - \frac{1}{2^m}(\psi(n-1) - \psi(n))$, $m = 1, 2, 3, \cdots$, chosen so that the rays lie in the wedge and accumulate at its upper arm. To tile the part of this wedge near $(n,0)$, begin by inserting a $(\theta_1 - \psi(n))$-column with its leading vertex at $(n,0)$ and its leading edge along the first ray of the fan so that its open foot rests on the lower arm of the wedge. Then for $m=2,3,4,\cdots$, continue by inserting a $(\theta_m - \theta_{m-1})$-column with its leading vertex at $(n,0)$ and its leading edge along the m^{th} ray of the fan so that its open foot rests on the preceding column. This procedure completes the tiling of the unit sector about the vertex of the wedge and uses segments inclined to the x-axis at angles less than $\psi(n-1) < \varepsilon$. It leaves untiled in $\mathrm{W}(n)$ a sequence of open wedges whose upper arms are inclined to the x-axis at angles less than $\psi(n-1)$. By repeating the procedure in these second generation wedges and others produced subsequently we obtain a satisfactory tiling of each $\mathrm{W}(n)$ and hence of the entire upper half-plane.

The tiling produced in Lemma 1 can be used to produce a tiling of the slab $-1 \le z \le 0$ with unit squares which will be incorporated later in the tiling of $\mathbb{R}^3$ by rhombi. At this stage, the application of the Lemma does not make use of the fact that the line segments employed in the tiling of U are almost parallel to the x-axis. In the first of three steps, we reflect the tiling of the half-plane $U = \{(x,y) \in \mathbf{R}^2 \ : \ y > 0\}$ in the line $y = -\frac{1}{2}$ to obtain a tiling of the half-plane $L = \{(x,y) \in \mathbf{R}^2 \ : \ y < -1\}$. Next, we complete the tiling of the two half-planes U and L to a tiling of $\mathbf{R}^2$ by filling the gap $-1 \leq y \leq 0$ between them with closed unit line segments perpendicular to the x-axis. Finally we produce a tiling of the slab with squares congruent to $[0,1]^2$ by taking the cartesian product of the tiled (x,y)-plane with the interval $-1 \leq z \leq 0$.

A copy of the portion of the tiled slab with $y > 0$ can be given a quarter turn about the x-axis to place it on top of the slab in the form of a $\frac{\pi}{2}$-wall, $0 \leq y \leq 1$, $z > 0$, tiled with squares that are almost parallel to the (x,y)-plane. In fact by using the full force of Lemma 1 we can say that a typical square in this wall has vertices

$$\begin{aligned} A &= (x,0,z) & B &= (x,1,z) \\ D &= (x+\cos\theta, 0, z+\sin\theta) & C &= (x+\cos\theta, 1, z+\sin\theta) \end{aligned}$$

for suitable x, z, and θ where $0 \leq \theta < \epsilon$.

The effect of applying the affine transformation

$$(x,y,z) \rightarrow (x, y + z\cos\phi, z\sin\phi), \quad 0 < \phi < \frac{\pi}{2},$$

to this $\frac{\pi}{2}$-wall is to transform it into a ϕ-wall tiled by rhombi with edge length 1.

Lemma 2. The rhombi which occur in the ϕ-wall above have acute angles that lie in the interval $(\frac{\pi}{2} - \epsilon, \frac{\pi}{2}]$.

Proof. The image of the typical square mentioned above has vertices

$$\begin{aligned} A' &= (x, z\cos\phi, z\sin\phi) & B' &= (x, 1+z\cos\phi, z\sin\phi) \\ D' &= (x+\cos\theta, (z+\sin\theta)\cos\phi, (z+\sin\theta)\sin\phi) & C' &= (x+\cos\theta, 1+(z+\sin\theta)\cos\phi, (z+\sin\theta)\sin\phi). \end{aligned}$$

If α' is the angle at the vertex A$'$ in this rhombus then

$$\cos\alpha' = A'B' \cdot A'D'$$

$$
\begin{aligned}
&= (0,1,0)\cdot(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)\\
&= \sin\theta\cos\phi.
\end{aligned}
$$

Since $\cos\alpha' \geq 0$, α' is an acute angle or a right-angle. On the other hand,

$$\cos\alpha' = \sin\theta\cos\phi \leq \sin\theta < \sin\varepsilon = \cos(\frac{\pi}{2} - \varepsilon)$$

and hence $\alpha' > \frac{\pi}{2} - \varepsilon$ as required.

Lemma 2 shows that the ϕ-walls just constructed are tiled by rhombi of the type required by the Theorem. But now these ϕ-walls appear in the (y, z)-plane exactly as the θ-columns of Lemma 1 appear in the (x, y)-plane and so a construction exactly analogous to that of Lemma 1 gives a satisfactory tiling of the half-space $z > 0$. When this is reflected in the plane $z = -\frac{1}{2}$ and the two tiled half-spaces combined with the tiled slab constructed earlier, we obtain a satisfactory tiling of $\mathbf{R}^3$ by rhombi. This completes the proof of the Theorem.

We remark that this construction makes even more tantalizing the question of whether $\mathbf{R}^3$ can be tiled by squares congruent to $[0,1]^2$.

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LOCALLY-INTEGRAL EXTENSION FOR LINEAR FUNCTIONALS

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Presented by J. Lambek, F.R.S.C.

ABSTRACT. For function vector lattices B and arbitrary nonnegative linear $I:B\longrightarrow\mathbb{R}$ in [3] an integral extension $\bar{I}:B_0\longrightarrow\overline{\mathbb{R}}$ of Lebesgue power has been introduced. Here a suitable extension $R_1(B,I)$ is obtained, using a local convergence in measure for sequences. Now Lebesgue's and Monotone convergence theorems for $R_1(B,I)$ are given, and integration with respect to finitely additive set functions of [8] and [5] are a special case.

INTRODUCTION. In this note we present some still unpublished results concerning integral extension for arbitrary nonnegative linear functional on function vector lattices, and finitely additive integration. For a ring Ω of sets from an arbitrary set X and $\mu:\Omega\longrightarrow[0,+\infty[$ only finitely additive, the space of Riemann - μ - -integrable functions $R_1(\mu,\overline{\mathbb{R}})$ were presented essentially by Loomis in [10], for Banach space-valued functions, have been introduced by Dunford-Schwartz in [5], and more generally by Günzler in [8]. The analogue extension process, without or weaker continuity conditions on the elementary integral, has been treated by Aummann [2], Loomis [10] and Gould [6].

In [3] has been generalized the process of a Daniell-Bourbaki integral, see [11], starting in this case with a nonnegative linear relations $+$, $\alpha.$, $=$, $\le$, $\vee$, $\wedge$. The triple (X,B,I) is called a Loomis system.

With this assumption, the main object of this note is to introduce an appropiate local "convergence in measure" for sequences, to obtain the space of Riemann - I - integrable functions $R_1(B,I)$. Here Lebesgue's and Monotone convergence theorems for $R_1(B,I)$ are given, and integration with respect to finitely additive set functions of [7], [8] and [5] are special cases.

We assume a Loomis system (X,B,I) and the following definitions and results of [3]:

A preliminary extension is defined by

$$B^+ := \{f \in \overline{\mathbb{R}}^X;\ f = \sup g,\ g \in B,\ g \le f\}$$

For any $f \in \overline{\mathbb{R}}^X$, $\quad I^+(f) := \sup\{I(g);\ g \in B,\ g \le f\}$, with $\sup \emptyset = -\infty$, $B^- := -B^+$, $I^-(f) := -I^+(-f)$.

Since I^+ is no additive on B^+ it is introduced the class:

$$B_+ := \{f \in B^+;\ I^+(f+g) = I^+(f) + I^+(g),\ \text{for any } g \in B^+\}$$

and $B_- := -B_+$; then I^+ is additive on B_+. I^+ and I^- both coincide on $B_+ \cap B_-$. Now, using the class B_+ and B_-, for each $f \in \overline{\mathbb{R}}^X$ the upper and lower integrals $\overline{I}$ and $\underline{I}$ are defined as usual:

$\overline{I}(f) := \inf\{I^+(g);\ g \in B_+,\ g \ge f\}$, with $\inf \emptyset = +\infty$, $\forall f \in \overline{\mathbb{R}}^X$ and $\underline{I}(f) := -\overline{I}(-f)$; and we have

$$I^+(f) \le \underline{I}(f) \le \overline{I}(f) \le I^-(f) \text{ for } f \in \overline{\mathbb{R}}^X,$$

and $\overline{I}(f+g) \le \overline{I}(f) + \overline{I}(g)$ for $f, g \in \overline{\mathbb{R}}^X$.

The class of <u>I-summable functions</u> it is defined by

$$B_0 := \{f \in \overline{\mathbb{R}}^X;\ \underline{I}(f) = \overline{I}(f) \in \mathbb{R}\}$$

If $f \in B_0$ then $I(f) := \underline{I}(f) = \overline{I}(f)$. I is linear on B_0 and $B_0 \cap \mathbb{R}^X$ is a vector space.

B is dense in B_0 with respect to $\|f\|_I := \overline{I}(|f|)$, and I/B_0 is the maximal extension of I/B in the sense of Aumann [2] with respect to the integral (semi) norm $\overline{I}$. The terminology and notations used in this note is similar to those of [3] and [4].

1. RIEMANN-I-INTEGRABILITY

Let (X,B,I) be a Loomis system.

DEFINITION 1.1. (*I^--convergence*) Let f, $(f_n)_n \subset \overline{\mathbb{R}}^X$, $(f_n)_n \longrightarrow f(I^-)$ means to each $h \in B$, $h \geq 0$, $\varepsilon > 0$, there exists $n_0(\varepsilon,h) \in \mathbb{N}$ such that $I^-(|f_n - f| \wedge h) \leq \varepsilon$ if $n \geq n_0$.
i.e., to each $h \in B$, $h \geq 0$, $\varepsilon > 0$, there exist $n_0(\varepsilon,h) \in \mathbb{N}$, $k_n \in B$ such that $|f_n - f| \wedge h \leq k_n$ and $I(k_n) \leq \varepsilon$, for all $n \geq n_0$.

DEFINITION 1.2. The set $R_1(B,I)$ of *I-integrable functions* is defined as the set of all $f \in \overline{\mathbb{R}}^X$, to which there exists a sequence $(h_n)_n \subset B$ which is an I-Cauchy sequence with respect to the I-integral-seminorm $\| \ \|_I$, and with $(h_n)_n \longrightarrow f(I^-)$.
Here $|h|:X \longrightarrow \mathbb{R}$ with $|h|(x) := |h(x)|$ belongs to B if $h \in B$, so $\|h\|_I := I(|h|)$ is defined, and so the concept of I-Cauchy sequence above.

The map $I:R_1(B,I) \longrightarrow \mathbb{R}$, with $I(f) := \lim I(h_n)$ as $n \to \infty$, is well defined. $R_1(B,I)$ is a vector lattice of extended real-valued functions, with the usual convention, and I is linear on it.

B is dense in $R_1(B,I)$ with respect to the seminorm $\|f\|_I := I(|f|)$ for all $f \in R_1(B,I)$.

Let $R_e^1(B,I) := \{f \in \mathbb{R}^X;\ \forall\ \varepsilon > 0,\ \exists\ h,\ g \in B,\ h \leq f \leq g$ and $I(g-h) < \varepsilon\}$ the class of *proper-Riemann-integrable functions* (see [10]).

PROPOSITION 1.3. i) In general $R_e^1(B,I) \subsetneq R_1(B,I)$.
ii) $R_e^1(B,I)$ is the closure of B with respect to I^-, i.e.,

$$f \in R_e^1(B,I) \Leftrightarrow \forall\ \varepsilon > 0,\ \exists\ h \in B;\ I^-(|f-h|) < \varepsilon$$

iii) $f \in R_e^1(B,I) \Leftrightarrow f \in R_1(B,I)$ and $|f| \leq h \in B$.

With ii) $R_e^1(B,I)$ is the "Aumann-closure" of B (see [2] p. 443) Property iii) is the classical characterization for B-bounded integrable functions (see [8] p. 259, ex. 124).

For finitely additive integration (see below part 3), An der Heiden in [1] uses a characterization due in [8]. The following proposition generalizes those results.

PROPOSITION 1.4. Let $f \in \overline{\mathbb{R}}^X_+$, then $f \in R_1(B,I)$ if and only if $f \wedge h \in R^1_0(B,I)$ for all $h \in B$, $h \geq 0$, and $I^+(f) < +\infty$.

2. CONVERGENCE THEOREMS

Observe that to get convergence theorems in the finitely additive case, everywhere convergence is not sufficient; now as in Dunford-Schwartz [5] p. 101-104 one has to use a suitable localized convergence in measure (definition 1.1.).

Completions with respect to abstract "integral norms" have been treated in [12]; with I^--integral we have

THEOREM 2.1. ($R_1(B,I)$ is "closed" with respect to I^--convergence)

Let $f \in \overline{\mathbb{R}}^X$, $(f_n)_n \subset R_1(B,I)$ an I-Cauchy sequence with $(f_n)_n \longrightarrow f(I^-)$. Then, $f \in R_1(B,I)$ and $I(f) = \lim I(f_n)$ as $n \to \infty$.

In [12] his convergence theorems are applicable only if his condition 2 p. 124 holds, which is very restrictive, (for example, $C_0(X,\mathbb{R})$ does not satisfy 2).

Now, one has corresponding convergence theorems for $R_1(B,I)$.

THEOREM 2.2. (Monotone convergence theorem)

Let $f \in \overline{\mathbb{R}}^X$, $(f_n)_n \subset R_1(B,I)$, $f_n \leq f_{n+1}$, $n = 1,2,\ldots$ with $(f_n)_n \longrightarrow f(I^-)$, and $\beta := \sup\{I(f_n);\ n \in \mathbb{N}\} < +\infty$.

Then, $f \in R_1(B,I)$, and $\lim I(f_n) = I(f) = \beta$, as $n \to \infty$.

THEOREM 2.3. (Lebesgue's bounded convergence theorem)

Let $f \in \overline{\mathbb{R}}^X$, $(f_n)_n \subset R_1(B,I)$ such that $(f_n)_n \longrightarrow f(I^-)$ and $|f_n| \leq g \in R_1(B,I)$, $n=1,2,\ldots$ Then, $f \in R_1(B,I)$ and $I(|f_n - f|) \longrightarrow 0$, as $n \to \infty$.

By convergence theorems and since $R_1(B,I) \cap B^+ \subset B_0$, we generalize theorem p. 262 of [4]: $R_1(B,I) \subset B_0 + N_1(B,I)$, where $N_1(B,I) := \{f \in R_1(B,I);\ I(|f|) = 0\}$.

3. APPLICATIONS: RIEMANN-μ-INTEGRABILITY

Let Ω be a semiring of subsets of X and μ a nonnegative finitely additive measure on Ω. B_Ω denotes the set of all step-functions $S(\Omega,\mathbb{R})$, and $I_\mu(h) := \int h\, d\mu$, $h \in B_\Omega$, where $S(\Omega,\mathbb{R})$ contains all $h = \sum_1^n a_i\, \chi_{A_i}$, with $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $A_i \in \Omega$ and $\int h\, d\mu = \sum_1^n a_i\, \mu(A_i)$.

Now, starting from B_Ω and I_μ by using the above methods, we obtain an integral extension to the I_μ-integrable functions class $R_1(B\ ,I_\mu)$.

Also, in this situation one can define μ-local convergence, $(f_n)_n \longrightarrow f(\mu)$, [7] p. 172, and by the lemma [8] p. 70, A. 2.72, one gets that if $(f_n)_n$, $f \in \overline{\mathbb{R}}^X$, then $(f_n)_n \longrightarrow f(\mu)$ if and only if $(f_n)_n \longrightarrow f(I_\mu^-)$, (see [9] p. 9). Therefore we have that $R_1(B_\Omega,I_\mu) = R^1(\mu,\overline{\mathbb{R}})$ = abstract-Riemann-μ-integrable functions of Günzler in [7], [8].

By [8] p. 70, 199, the "μ-integrability" defined by Dunford - Schwartz in [5], is a special case of the "abstract-Riemann-μ-integrability", and only if $X \in \Omega$ (Ω algebra) and $\mu(X) < +\infty$, these concepts coincide (convergence in μ-measure, locally-μ-convergence and I_μ^--convergence are equivalents).

Finally, for δ-ring Ω and $\mu:\Omega \longrightarrow [0,\infty[$ σ-additive, one has $R_1(\mu,\mathbb{R}) = L_1(\mu,\mathbb{R})$ (= Lebesgue-μ-integrable functions, for A. 146, p. 265, [8]).

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SURFACES OF FINITE TYPE AND CONSTANT CURVATURE IN THE 3-SPHERE

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Presented by G. de B. Robinson, F.R.S.C.

ABSTRACT. We show that a compact surface of finite type and constant curvature in the 3-sphere is a totally umbilical surface or the product of two circles.

INTRODUCTION.

Finite type submanifolds were introduced by the first author in [2]. A submanifold M^n of a Euclidean space E^{n+p} is said to be *of finite type* if each component of its position vector field X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^n, *i.e.*, if

$$X = X_0 + \sum_{t=1}^{k} X_t$$

where X_0 is a constant vector and $\Delta X_t = \lambda_t X_t$ for $t = 1,\dots,k$. If in particular all eigenvalues $\{\lambda_1,\lambda_2,\dots,\lambda_k\}$ are mutually different, then M^n is said to be of *k-type*.

In this paper we study compact surfaces M of constant curvature in $S^3(1)$, embedded standardly in E^4, such that M is of finite type in E^4. We do this by looking at the closed geodesics of M, which are, in this particular situation, mapped onto closed curves of finite type in S^3. Then we use the classification of those curves in [3] to obtain the following classification theorem.

THEOREM. *Let $M(c)$ be a compact surface with constant curvature c in $S^3(1)$. If $M(c)$ is of finite type, then*

(1) $c \geq 1$ and $M(c)$ is totally umbilical in S^3, or

(2) $c = 0$ and $M = S^1(a) \times S^1(b)$ with $a^2 + b^2 = 1$.

Note that the surfaces in (1) are of 1-type and the surfaces in (2) are of 2-type unless $a = b$, in which case it is of 1-type.

PROOF OF THE THEOREM

If $M(c)$ is a compact surface of constant curvature c in S^3, then we know that either $c = 0$ or $c \geq 1$ and in the latter case $M(c)$ is totally umbilical, see for instance [6, p.139]. So we only have to consider the case $c = 0$. In this case $M(0)$ is isometric to a torus R^2/Λ, where Λ is a lattice, say

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[2] This work was done while the second author was visiting the Michigan State University. He would like to thank MSU, and in particular the staff of the department for their hospitality.

$$\Lambda = \{(2\pi nu,\ 2\pi mv + 2\pi nw) \mid n,m \in \mathbf{Z}\},$$

where u, v and w are real numbers with $u,v > 0$. If (x,y) are Euclidean coordinates on $\mathbf{R}^2$, then the Laplacian Δ is given by

$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}.$$

The eigenfunctions of Δ are given by

$$\{\cos\left(\frac{lv - kw}{uv}x + \frac{k}{v}y\right),\ \sin\left(\frac{lv - kw}{uv}x + \frac{k}{v}y\right) \mid k,l \in \mathbf{Z}\}$$

and the spectrum of Δ is given by

$$\mathrm{Spec}\,(\mathbf{R}^2/\Lambda) = \left\{\left(\frac{lv - kw}{uv}\right)^2 + \left(\frac{k}{v}\right)^2 \mid k,l \in \mathbf{Z}\right\}.$$

Now suppose that $M(0)$ is of finite type, say of k-type. Then the position vector field X of $M(0)$ in $\mathbf{E}^4$ can be written as

$$X = \sum_{(k,l)\in T}\left\{A_{kl}\cos\left(\frac{lv - kw}{uv}x + \frac{k}{v}y\right) + B_{kl}\sin\left(\frac{lv - kw}{uv}x + \frac{k}{v}y\right)\right\},$$

where T is a finite subset of $\mathbf{Z} \times \mathbf{Z}$ and A_{kl} and B_{kl} are constant vectors in $\mathbf{E}^4$.

Let $\gamma_\theta(t) = (t\cos\theta, t\sin\theta)$ be a closed geodesic of $M(0)$ - there are infinitely many values θ such that γ_θ is closed. Then

$$(X(\gamma_\theta))(t) = \sum_{(k,l)\in T}\left\{A_{kl}\cos\left(\left(\frac{lv - kw}{uv}\cos\theta + \frac{k}{v}\sin\theta\right)t\right) + B_{kl}\sin\left(\frac{lv - kw}{uv}\cos\theta + \frac{k}{v}\sin\theta\right)t\right)\right\}. \quad (1)$$

Hence $X(\gamma_\theta)$ is a curve of finite type. Indeed, as the Laplacian of a curve, parameterized by the arc length s, is $-\frac{\partial^2}{\partial s^2}$, we see from (1) that $X(\gamma_\theta)$ is written as a finite sum of eigenfunctions of $-\frac{\partial^2}{\partial s^2}$. From [3] we know that every closed curve of finite type in $S^3(1)$ is a W-curve (*i.e.* has constant Frenet curvature) and is in particular of 1- or 2-type. Hence the set

$$\left\{\left(\frac{lv - kw}{uv}\cos\theta + \frac{k}{v}\sin\theta\right)^2 \mid (k,l) \in T\right\}$$

contains at most two different elements. Using the fact that this holds for an infinite number of θ's, this implies that T contains at most two elements. Therefore $M(0)$ is of 1-type, and hence congruent to the Clifford torus $M = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ [5, Corollary 3], or of 2-type, and hence the product of two circles $S^1(a)$ and $S^1(b)$ with radii a and b satisfying $a^2 + b^2 = 1$ [1, Theorem 3]. This finishes the proof of the theorem. (Q.E.D.)

REMARK 1: The relation between the conditions "to be of finite type" and "to have finite type geodesics" is studied by J. Deprez in his doctoral thesis [4]. He obtains amongst other things that the two conditions are equivalent for irreducible symmetric spaces of compact type.

REMARK 2: During the proof of our theorem we restricted our attention to closed geodesics, in order to enable us to apply known results about closed curves of finite type. This was however not essential, as we now indicate briefly.

If γ is a curve of finite type in $\mathbf{E}^n$ with arc length s, then γ can be written as a finite sum of eigenfunctions of the Laplacian $\Delta = -\frac{\partial^2}{\partial s^2}$. In particular

$$\gamma(s) = A_0 + B_0 s + \sum_{t=1}^{k_1}(A_t \cos(p_t s) + B_t \sin(p_t s)) +, \sum_{t=1}^{k_2}(C_t e^{q_t s} + D_t e^{-q_t s}).$$

If $k_2 \neq 0$, then we can always suppose that $0 < q_1 < \ldots < q_{k_2}$, and that not both C_t and D_t are zero.When we work out the condition $\| \gamma \| = 1$, we obtain very easily, using the fact that $e^{as}\cos(bs)$ and $e^{as}\sin(bs)$ form a set of linearly independent functions, that $C_{k_2} = 0$ and $D_{k_2} = 0$.This is a contradiction and hence $k_2 = 0$.

If in addition it is given that γ lies in some hypersphere of $\mathbf{E}^n$, say centered at the origin, then a similar linear independency argument on the condition $\| \gamma \| = r$ gives us, by looking at the coefficient of the function s^2, that $B_0 = 0$. Hence γ can be written using only cosines and sines. This is sufficient to apply the techniques, used in [3] and obtain similar results for nonclosed curves of finite type.

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