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PRIME POWERS IN CONTINUED FRACTIONS RELATED TO THE
CLASS NUMBER ONE PROBLEM FOR REAL QUADRATIC FIELDS

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Abstract:

It is the purpose of this note to classify those $d \equiv 1 \pmod{4p}$, for any prime p , such that, in the continued fraction expansion of $(1 + \sqrt{d})/2 = w$ all of the Q_i 's are twice a power of p , where Q_i 's are defined by the recurrence relations: $P_0 = 1$; $Q_0 = 2$; $Q_i Q_{i+1} = d - P_{i+1}^2$; $P_{i+1} = a_i Q_i - P_i$ for $i \geq 0$ with $a_i = [(P_i + \sqrt{d})/Q_i]$ for $i \geq 0$. For the $p = 2$ case such forms were of interest to L. Bernstein in [1]-[2], although his interest differs from that herein. Moreover, C. Levesque et al. [3] had a similar interest.

We establish furthermore using the above classification, a lower bound (based on the continued fraction period of w) beyond which we know that the class number $h(d)$ of $Q(\sqrt{d})$ exceeds 1.

§1. Notation and Preliminaries.

Let $d \equiv 1 \pmod{4}$ be a square-free positive integer and w as above. Let $w = \langle a, \overline{a_1, a_2, \dots, a_k} \rangle = \langle a, \overline{a_1, a_2, \dots, a_{k-1}, 2a-1} \rangle$ denote the continued fraction expansion of w , hence with period k . Then $a_0 = a = [w]$, where $[w]$ denotes the greatest integer less than or equal to w . Some other standard facts are: $Q_{i+1} Q_i = d - P_{i+1}^2$ for $i \geq 0$; $a_i < a$ for $k > i > 0$; $P_i < \sqrt{d}$ for $i \geq 0$; $Q_i < 2\sqrt{d}$ for $i \geq 0$ and $-1 < (P_i - \sqrt{d})/Q_i < 0$ for $i > 0$. Moreover $a_i = a_{k-i}$ for $1 \leq i \leq k-1$.

The ring of integers of $K = Q(\sqrt{d})$ will be denoted \mathcal{O}_K . Principal ideals generated by an element a are denoted (a) , and the \mathcal{O}_K -primes are denoted by upper case script letters \mathcal{P} , \mathcal{R} , etc.

We refer the reader to [14] or [9]–[10] for details of the theory of reduced ideals used herein.

§2. The Classification.

Throughout this section d will be a positive square-free integer and p any prime with $d \equiv 1 \pmod{4p}$. The notation of §1 is in force throughout. The proofs of the following results are too lengthy to be included here so will appear elsewhere.

Theorem 2.1. If $d \equiv 1 \pmod{4p}$ then all $Q_i/2$ are powers of p if and only if $d = (2a-1)^2 + 4p^s$, $s \geq 0$, with $2a-1 = p^s a_1 + g$ where $0 \leq g < p^s$, $s \equiv 0 \pmod{(k-1)/2}$ for $k \geq 1$ odd and $a_1 g + 1 = \begin{cases} p^{2s/(k-1)} & \text{if } k > 1 \\ 1 & \text{if } k = 1 \end{cases}$.

Since we are interested in the connection with the class number one problem then we now look at what happens when the Q_k -primes above p are principal.

Theorem 2.2. If $d \equiv 1 \pmod{4p}$ and all the Q_k -primes above p are principal then $Q_i = 2p$ for some i with $0 < i < k$. If all $Q_j/2 = p^{s_j}$ for $s_j > 0$ and all j with $0 < j < k$ then:

- (i) $a = bp + c$ where $c \in \{0,1\}$, $b \geq 1$.
- (ii) If $c = 1$ then $P_i = 2a-1$ or $2a-3$; $d - (2a-1)^2 = 4p^s$ and $d - (2a-3)^2 = 4p^t$ where $t > s \geq 1$.
- (iii) If $c = 0$ then $P_i = 2a-2p+1$; $d - (2a-2p+1)^2 = 4p^t$; $d - (2a-1)^2 = 4p^s$ and $t > s \geq 1$.

Theorem 2.3. Let $d \equiv 1 \pmod{4p}$.

If $c = 1$ and all the Q_k -primes above p are principal, then all $Q_i/2$ are powers of p if and only if $d = (2a-1)^2 + 4p^s = (p^{s+1} - p^s + 1)^2 + 4p^s$ and $k = 2s + 1$. Moreover, $[\sqrt{d}] = 2a$ if and only if $p = 2$.

Theorem 2.4. Let $d \equiv 1 \pmod{4p}$.

If $c = 0$ and the \mathcal{O}_k -primes above p are principal, then all $Q_i/2$ are powers of p if and only if $d = (2a-1)^2 + 4p^s = (p^s + p - 1)^2 + 4p^s$ and $k = 2s + 1$; $s \geq 1$.

Moreover $[\sqrt{d}] = 2a$.

Remark 2.1. It can be shown that when d is prime and all $Q_i/2$ are powers of p then $k = 3$. Thus the classification of all forms $d \equiv 1 \pmod{4p}$ such that all $Q_i/2$ are powers of p (including therefore all primes) is $d = ((p^s(p^s-1)/g)+g)^2 + 4p^s$; $s > 0$, $0 < g < p^s$.

Remark 2.2. It can also be shown that when the \mathcal{O}_k -primes above p are principal then $Q_2 = Q_{k-2} = 2p$. However, it's possible that the \mathcal{O}_k -primes above p are principal while $h(d) > 1$ of course. For example, if $d = 10301$ then $k = 5$, $s = 2$, $p = 5$, $h(d) = 3$ and the \mathcal{O}_k -primes above 5 are principal since $Q_2 = 10$. We would like to know precisely those d with $h(d) = 1$ and all $Q_i/2$ as powers of p . For $p = 2$ we have:

Conjecture 2.1. If $d \equiv 1 \pmod{8}$ and all Q_i 's are powers of 2 then $h(d) = 1$ if and only if $d \in \{17, 41, 113, 353, 1217\}$.

Remark 2.3. Using the techniques of [5] and [7] we have been able to verify that Conjecture 2.1 holds for all except possibly one value remaining. For various reasons it appears to this author that verifying that the remaining value does not exist may be as difficult as solving Gauss' class number one problem for real quadratic fields.

Remark 2.4. The situation for $p > 2$ seems less clear than the above case in connection with $h(d) = 1$. However, the results of the next section shed some light on the situation.

§3. $h(d) = 1$.

As in §2, d is a positive square-free integer with $d \equiv 1 \pmod{4p}$, p any prime, throughout this section. We now achieve a lower bound on d (based on k) for which $h(d) > 1$.

The following two preliminary results are used in the proof of the main result, to appear elsewhere.

Lemma 3.1. Let $d - x^2 = 2^e p^f t$; $e \geq 2$, $f \geq 1$, $x > 0$, $\sqrt{d} - x < 2^{e-1} p^f$ and $\sqrt{d} + x > 2^{e-1} p^f$ then $\mathcal{J} = [t, (x + \sqrt{d})/2]$ is a reduced ideal in \mathcal{O}_k .

Lemma 3.2. Suppose that there exists a Q_i divisible by a prime other than 2 or p . Then there exists an odd prime $q \neq 2, p$ dividing Q_i for some i such that the \mathcal{O}_k -primes above q are principal.

The main result is:

Theorem 3.1. Let $d \equiv 1 \pmod{4p}$ such that not all $Q_i/2$ are powers of p . Thus if $d > 4p^{k-1}$ then $h(d) > 1$.

This improves [8, Theorem 2.1] for the case $d \equiv 1 \pmod{4p}$.

Remark 3.1. In [8] we were able to show that for a fixed period k , $h(d) = 1$ for at most finitely many d . What Theorem 3.1 does is to give an explicit bound for our $d \equiv 1 \pmod{4p}$ case.

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STABILITY OF HOMOMORPHISMS AND COMPLETENESS

by

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Abstract: Let G be an abelian group containing an element of infinite order and V be a normed vector space. It is proved that homomorphisms from G into V are stable if and only if V is complete.

DEFINITION 1- A group G has the property of the stability of homomorphisms (in short G is stable) with respect to a normed linear space V if, for every function $f:G \rightarrow V$ satisfying

$$\|f(xy) - f(x) - f(y)\| \leq K$$

for $x, y \in G$ and for some K , there exist a constant K' and a $\phi \in \text{Hom}(G, V)$ such that $\|f(x) - \phi(x)\| \leq K'$ for all $x \in G$.

If V is a Banach space, then this property does not depend on which Banach space V we take (see [1]). It is well known that all amenable groups are stable with respect to any Banach space and that free groups are not (see [1], [3], [4]).

A natural question is the role of completeness of the space of values V . The following example shows that without completeness we may lose stability, so in some sense stability does depend on V : let $f: \mathbb{Z} \rightarrow \mathbb{Q}$ be defined by $f(x) = [\lambda x]$, where λ is irrational, then $|f(x+y) - f(x) - f(y)| \leq 1$, but for each $\phi \in \text{Hom}(\mathbb{Z}, \mathbb{Q})$ the difference $f(x) - \phi(x)$ is unbounded.

The aim of this paper is to prove the following fact:
let A be an abelian group containing an element of infinite order, let $(V, \|\cdot\|)$ be a normed vector space and assume that A is stable with respect to V ; then V is a complete space.

We need some theorems about the structure of divisible groups; they can be found in the book Fuchs, Infinite Abelian Groups ([2]).

THEOREM 1- ([2], Th.21.2, p.100) *A divisible subgroup D of an abelian group A is a direct summand of A , $A=D \oplus C$ for some subgroup C of A . This C can be chosen so as to contain a preassigned subgroup B of A with $D \cap B = 0$.*

THEOREM 2- ([2], Th. 23.1, p.104) *Any divisible group D is a direct sum of quasicyclic and full rational groups.*

From Theorem 2 and its proof we get the following.

COROLLARY 1- *Any divisible group D with an element of infinite order has the decomposition $D \cong \mathbb{Q} \oplus E$.*

THEOREM 3- ([2], Th. 24.1, p.106) *Every abelian group can be embedded as a subgroup in a divisible group.*

Using these theorems we can prove the following.

THEOREM 4- *Let A be an abelian group and let $\alpha \in A$ be an element of infinite order. If Z_α is the subgroup generated by α , then A is isomorphic to a subgroup of a group of the form $\mathbb{Q} \oplus E$, with $Z_\alpha \cong \mathbb{Z} \subset \mathbb{Q}$.*

Proof- By Theorem 3, A can be embedded as a subgroup in a divisible group D . Let T be the torsion group of D : since T is divisible, by Theorem 1 we have $D = T \oplus F$ for some subgroup F of D . Since $\alpha \in A \subset D$ is of infinite order, therefore $\alpha \notin T$ and $Z_\alpha \cap T = 0$. Again by Theorem 1 we can assume $Z_\alpha \subset F$. Choose now in F a maximal independent system containing α . By the proof of Theorem 2 we have

$$F \cong \bigoplus_{j \in J} \mathbb{Q}_{j_0} \quad \text{and} \quad Z_\alpha \cong \mathbb{Z} \subset \mathbb{Q}_{j_0} \quad \text{for some } j_0 \in J.$$

$$\text{Thus } D \cong T \oplus \left(\bigoplus_{j \in J} \mathbb{Q}_{j_0} \right) \cong \mathbb{Q}_{j_0} \oplus E \quad \text{and} \quad Z_\alpha \cong \mathbb{Z} \subset \mathbb{Q}_{j_0}. \quad \blacksquare$$

The next step consists of proving that, if we have a function

$f: \mathbb{Z} \rightarrow V$ (V normed vector space) such that for every $m, n \in \mathbb{Z}$ $\|f(m+n) - f(m) - f(n)\| \leq K$, then for any abelian group A with an element α of infinite order, we can construct a function $g: A \rightarrow V$ such that $g(n\alpha) = f(n)$ and $\|g(x+y) - g(x) - g(y)\| \leq K$ for every $x, y \in A$.

THEOREM 5- Let $f: \mathbb{Z} \rightarrow V$ (V normed vector space) satisfy the inequality

$$(1) \quad \|f(m+n) - f(m) - f(n)\| \leq K, \quad m, n \in \mathbb{Z}.$$

There exists an extension k of f , $k: \mathbb{R} \rightarrow V$, such that

$$(2) \quad \|k(x+y) - k(x) - k(y)\| \leq K, \quad x, y \in \mathbb{R}.$$

Proof- For every $x \in \mathbb{R}$ with $n \leq x < n+1$ (i.e. $n = [x]$), define

$$k(x) := (n+1-x)f(n) + (x-n)f(n+1).$$

Obviously $k|_{\mathbb{Z}} = f$. We have to show that (2) holds. Let $x, y \in \mathbb{R}$ and $n \leq x < n+1$, $m \leq y < m+1$, then $n+m \leq x+y < n+m+2$. We must distinguish two cases:

A) $x+y < n+m+1$:

$$\begin{aligned} k(x+y) - k(x) - k(y) &= (m+n+1-x-y)f(m+n) - (n+1-x)f(n) - \\ &- (m+1-y)f(m) + (x+y-m-n)f(m+n+1) - (x-n)f(n+1) - (y-m)f(m+1) = \\ &= (m+n+1-x-y)\{f(m+n) - f(n) - f(m)\} + (m-y)f(n) + (n-x)f(m) + \\ &+ (x+y-m-n)f(m+n+1) - (x-n)f(n+1) - (y-m)f(m+1) = \\ &= (m+n+1-x-y)\{f(m+n) - f(m) - f(n)\} + (x-n)\{f(m+n+1) - f(n+1) - \\ &- f(m)\} + (y-m)\{f(m+n+1) - f(m+1) - f(n)\} ; \end{aligned}$$

so we obtain, by (1)

$$\|k(x+y) - k(x) - k(y)\| \leq (m+n+1-x-y)\|f(m+n) - f(m) - f(n)\| + \\ + (x-n)\|f(m+n+1) - f(n+1) - f(m)\| + (y-m)\|f(m+n+1) - f(m+1) - f(n)\| \leq K$$

B) $x+y \geq n+m+1$:

$$\begin{aligned} k(x+y) - k(x) - k(y) &= (m+n+2-x-y)f(m+n+1) - (x-n)f(n+1) - \\ &- (y-m)f(m+1) + (x+y-m-n-1)f(m+n+2) - (n+1-x)f(n) - \\ &- (m+1-y)f(m) = (x+y-m-n-1)\{f(m+n+2) - f(n+1) - f(m+1)\} - \\ &- (m+1-y)f(n+1) - (n+1-x)f(m+1) + (m+n+2-x-y)f(m+n+1) - \end{aligned}$$

$$\begin{aligned}
& - (n+1-x)f(n) - (m+1-y)f(m) = \\
& = (x+y-m-n-1)\{f(m+n+2) - f(n+1) - f(m+1)\} + \\
& + (m+1-y)\{f(m+n+1) - f(n+1) - f(m)\} + \\
& + (n+1-x)\{f(m+n+1) - f(n) - f(m+1)\} , \\
& \text{so as in A) we get } \|k(x+y) - k(x) - k(y)\| \leq K. \quad \blacksquare
\end{aligned}$$

REMARK-If G is a subgroup of the reals containing \mathbb{Z} , then we get, by restricting the function k constructed in Theorem 5 to G , a function from G into V satisfying property (2) and extending f .

THEOREM 6-Let $f: \mathbb{Z} \rightarrow V$ (V a normed vector space) be such that for each $m, n \in \mathbb{Z}$ $\|f(m+n) - f(m) - f(n)\| \leq K$. Let A be an abelian group with an element α of infinite order. Then there exists a function $g: A \rightarrow V$ such that for each $x, y \in A$

$$\|g(x+y) - g(x) - g(y)\| \leq K \text{ and } g(n\alpha) = f(n).$$

Proof-By Theorem 4, the group A is isomorphic to a subgroup of the divisible group $D = \mathbb{Q} \otimes E$, with $\mathbb{Z} \otimes \alpha \cong \mathbb{Z} \subset \mathbb{Q}$. Now define a function $k: D \rightarrow V$ in the following way:

if $x \in D$, then x can be written uniquely as $x = q + r$, where $q \in \mathbb{Q}$ and $r \in E$; we define $k(x) = k(q)$, where $k(q)$ is defined on \mathbb{Q} starting from f as described in Theorem 5.

Obviously k satisfies the required properties on D and defining $g = k|_A$ we get the assumption. \blacksquare

THEOREM 7-Let $(V, \|\cdot\|)$ be a normed vector space. Assume that for every function $f: \mathbb{Z} \rightarrow V$ such that $\|f(m+n) - f(m) - f(n)\| \leq K$ for each $m, n \in \mathbb{Z}$, there exist a homomorphism $\phi \in \text{Hom}(\mathbb{Z}, V)$ and a constant K' such that $\|f(n) - \phi(n)\| \leq K'$ for each $n \in \mathbb{Z}$. Then V is complete.

Proof-Let $\{v_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in V : to prove its convergence we shall show that it has a convergent subsequence. The original sequence has a subsequence $\{v_{i_j}\}_{j \in \mathbb{N}}$ such that for every $i \in \mathbb{N}$ $\|v_{i_j} - v_{i_{j+1}}\| \leq 2^{-i}$. Now we construct the

following function $f: \mathbb{Z} \rightarrow V$:

define $f(n) := nv_n$ for $n \geq 0$ and $f(n) := -f(-n)$ for $n < 0$.

We have to prove that the Cauchy difference of f is bounded.

Let $n, m \geq 0$, then

$$\begin{aligned} \|f(n+m) - f(n) - f(m)\| &= \|(n+m)v_{n+m} - nv_n - mv_m\| = \\ &= \|n(v_{n+m} - v_n) + m(v_{n+m} - v_m)\| \leq n\|v_{n+m} - v_n\| + m\|v_{n+m} - v_m\| \leq \\ &\leq n \sum_{i=n}^{n+m-1} 2^{-i} + m \sum_{i=m}^{n+m-1} 2^{-i} = n2^{-n}(1-2^{-m}) + m2^{-m}(1-2^{-n}) \leq 1. \end{aligned}$$

If $n, m < 0$, we have, as before, that the norm of the Cauchy difference is less than 1.

Assume now $n > 0$, $m < 0$ and $n+m \geq 0$; define $r = -m$, so we have, as above,

$$\begin{aligned} \|f(n+m) - f(n) - f(m)\| &= \|f(n-r) - f(n) + f(r)\| = \\ &= \|(n-r)v_{n-r} - nv_n + rv_r\| = \|(n-r)(v_{n-r} - v_n) - rv_n + rv_r\| = \\ &= \|(n-r)(v_{n-r} - v_n) - m(v_n - v_r)\| \leq (n-r)\|v_n - v_{n-r}\| + r\|v_n - v_r\| \leq 1. \end{aligned}$$

If $n+m < 0$, working as above we get the same result.

Having proved that f has bounded Cauchy difference, we know that there exist $\phi \in \text{Hom}(\mathbb{Z}, V)$ and K' such that $\|f(n) - \phi(n)\| \leq K'$, $n \in \mathbb{Z}$. In particular if $n \geq 0$ this means $\|nv_n - n\phi(1)\| \leq K'$ and, dividing by n , $\|v_n - \phi(1)\| \leq \frac{K'}{n}$, so $v_n \rightarrow \phi(1)$ in V , thus V is complete. ■

THEOREM 8- Let A be an abelian group with an element α of infinite order. Assume that, for every function $f: A \rightarrow V$, ($(V, \|\cdot\|)$ is a normed vector space) with $\|f(x+y) - f(x) - f(y)\| \leq K$ ($x, y \in A$), there exist some $\phi \in \text{Hom}(A, V)$ and some constant K' such that $\|f(x) - \phi(x)\| \leq K'$. Then V is complete.

Proof-The proof follows immediately from Theorem 6 and Theorem 7. ■

It should be noted that the assumption of commutativity of the group A has been used only to prove that A is isomorphic to a subgroup of $G \otimes E$; our result remains true for any group with an element α of infinite order which is isomorphic to a subgroup of a group of the form $G \otimes E$, where $\mathbb{Z}_\alpha \subset G \subset \mathbb{R}$.

We make a final remark concerning the case where the group A (commutative or not) has no element of infinite order. Let $f: A \rightarrow V$ (where V is a normed vector space) be a function such that $\|f(x+y) - f(x) - f(y)\| \leq K$ for all $x, y \in A$; the set $\{2^n x; n \in \mathbb{N}\}$ is finite for every $x \in A$, so for every $x \in A$, $\lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$ exists and equals 0. By Proposition 1 in [1] it follows that the function f is bounded by K .

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CHARACTERIZATION OF A CLASS OF MEANS

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Introduction. Let I be a real interval. A function $M : I \times I \rightarrow R$ is called a (two variable) *quasiarithmetic mean* if there exists a strictly monotonic and continuous function $\phi : I \rightarrow R$ such that

$$M(x, y) = \phi^{-1} \left(\frac{\phi(x) + \phi(y)}{2} \right) \quad \text{for all } x, y \in I.$$

This property is characterized by the following result of Aczél [1]:

Theorem 1. *A function $M : I \times I \rightarrow R$ is a quasiarithmetic mean if and only if it has the following properties:*

- (i) *it is reflexive, i.e. $M(x, x) = x$ (for all $x \in I$);*
- (ii) *it is symmetric;*
- (iii) *it is continuous and strictly increasing in both variables;*
- (iv) *it satisfies the equation of bisymmetry, i.e.*

$$M(M(x, y), M(u, v)) = M(M(x, u), M(y, v)) \quad \text{for all } x, y, u, v \in I.$$

Result. In this note, motivated by the above theorem, we deal with the characterization of two variable means $M : I \times I \rightarrow R$ represented in the form

$$M(x, y) = \frac{p(x)x + p(y)y}{p(x) + p(y)}, \quad x, y \in I, \quad (1)$$

where $p : I \rightarrow]0, \infty[$ is an arbitrary function. A function M in this class is called an *arithmetic mean*

with weight function.

Theorem 2. A function $M : I \times I \rightarrow R$ is an arithmetic mean with weight function if and only if

(i) it has the mean value property, that is,

$$\min\{x, y\} < M(x, y) < \max\{x, y\} \quad \text{if } x, y \in I, x \neq y; \quad (2)$$

(ii) and satisfies

$$\begin{aligned} (x - M(x, z))(y - M(x, y))(z - M(z, y)) \\ = -(x - M(x, y))(y - M(z, y))(z - M(x, z)) \end{aligned} \quad (3)$$

for all $x, y, z \in I$.

Proof. "If part": (1) obviously implies the mean value property of M , as well as (3), in the case where at least two of the x, y, z coincide. If x, y, z are three different elements from I then (1) yields

$$\frac{p(x)}{p(y)} = \frac{y - M(x, y)}{M(x, y) - x}, \quad \frac{p(x)}{p(z)} = \frac{z - M(x, z)}{M(x, z) - x}, \quad \frac{p(z)}{p(y)} = \frac{y - M(z, y)}{M(z, y) - z}$$

and (3) then follows from

$$\frac{p(x)}{p(y)} = \frac{p(x)}{p(z)} = \frac{p(z)}{p(y)}.$$

"Only if part": Now we suppose that (2) and (3) are fulfilled. Let x, y, z be three different elements of I . Because of (2) we observe that all three differences $x - M(x, z)$ etc., occurring in (3), are different from zero. We interchange y and z in (3) and multiply the resulting equation by (3). After cancellation we obtain

$$(y - M(y, z))(z - M(z, y)) = (y - M(z, y))(z - M(y, z)),$$

which implies $M(x, y) = M(y, z)$. So M is a symmetric function. With $x = y = z$ we obtain from (3)

$$M(x, x) = x, \quad x \in I. \quad (4)$$

Now we fix an element z in I and we define $p : I \rightarrow]0, \infty[$ by

$$p(x) = \begin{cases} \frac{z - M(x, z)}{M(x, z) - x} & (x \in I, x \neq z), \\ 1 & (x = z). \end{cases} \quad (5)$$

(The positivity of the values of p is a consequence of (2)). Then it follows from (4), (5) and the symmetry of M that (1) holds in the cases $x = y$, $x = z$ and $y = z$. It remains to verify (1) for $x \neq y$, both being different from z . Therefore (3), (5) and the symmetry of M imply

$$\frac{y - M(x, y)}{M(x, y) - x} = \frac{p(x)}{p(y)},$$

which gives (1) in this case.

Remarks. (i) In the proof of Theorem 2 we did not use that I is an interval (which is important for Theorem 1), it is enough to assume that I has at least three different elements. (ii) The functional equation (3) has solutions which are not of the form (1), e.g.

$$M(x, y) = x, \quad M(x, y) = y, \quad M(x, y) = \max\{x, y\}, \quad M(x, y) = \min\{x, y\}.$$

Problem. A function $M : I \times I \rightarrow R$ is called a *quasiarithmetic mean with weight function* (see Bajraktarević [2]) if there exist two functions, a continuous strictly monotonic function $\phi : I \rightarrow R$ and a positive valued function p on I such that for all $x, y \in I$

$$M(x, y) = \phi^{-1} \left(\frac{p(x)\phi(x) + p(y)\phi(y)}{p(x) + p(y)} \right).$$

Taking $p(x) = 1$ or $\phi(x) = x$, it is clear that both the quasiarithmetic means and the arithmetic means with weight function belong to this class. This motivates the problem of characterization of this class of means.

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ON K_2 AND \mathbb{Z}_p -EXTENSIONS OF $\mathbb{Q}(\zeta_{p^r})$

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ABSTRACT. Suppose K is an abelian number field containing a primitive p th root of unity ζ_p with an odd prime p , and \tilde{K} is the compositum of all \mathbb{Z}_p -extensions of K . Define $A_K = \{ a \in K^* \mid K(\sqrt[p]{a}) \subset \tilde{K} \}$ and $C_K = \{ c \in K^* \mid (c, \zeta_p) \text{ is the identity element of } K_2(K) \}$. Then A_K/K^{*p} and C_K/K^{*p} are vectorspaces over the field $\mathbb{Z}/p\mathbb{Z}$ both having the same dimension p^{r-1} by a theorem of Brumer [11] and a theorem of Tate [10]. Coates [2] raised the question whether $A_K = C_K$, and Greenberg [4] proved that $A_K \neq C_K$ for $K = \mathbb{Q}(\sqrt[257]{7}, \sqrt{-3})$ and that $A_K = C_K$ for $K = \mathbb{Q}(\zeta_{p^n})$ and n sufficiently large. Let U be the group of p -units of K . The purpose of this note is to announce the following result: If r is sufficiently large then $A_K/K^{*p} = U/U^p = C_K/K^{*p}$ for $K = \mathbb{Q}(\zeta_{p^r})$. The proof will appear in [7].

1. By Matsumoto's theorem Milnor's K -group $K_2(K)$ of a field K may be described in terms of generators (a,b) with $a, b \in K^* = K \setminus \{0\}$ and relations $(a_1 a_2, b) = (a_1, b) + (a_2, b)$, $(a, b_1 b_2) = (a, b_1) + (a, b_2)$, $(a, 1-a) = 0$ für $a \neq 1$, see [8, §5, § 11].

Suppose K is a field containing a primitive p th root of unity ζ_p and set ${}_p K_2(K) = \{ x \in K_2(K) \mid p x = 0 \}$. Then the map

$$\varphi: K^* \rightarrow {}_p K_2(K), \quad c \mapsto (c, \zeta_p),$$

induces a surjective homomorphism $\bar{\varphi}: K^*/K^{*p} \rightarrow {}_p K_2(K)$ by a Theorem of Suslin [9, Th. 1.8], which has been proven before by Tate [10, Th. 6.1] for global fields. It is a difficult task to compute the kernel of $\bar{\varphi}$, even for global fields, see [3] and [4]. The case of arbitrary fields reduces to the case of global fields by a result of Suslin [9, Th. 3.5].

We now assume that $K := \mathbb{Q}(\zeta_{p^r})$ for some primitive p^r th root of unity ζ_{p^r} with an odd prime p and fixed $r > 0$. Let

$$K_{\infty} = \bigcup_{n=0} K_n \text{ with } K_n = \mathbb{Q}(\zeta_{p^{n+r}}) \text{ for all } n \geq 0$$

be the cyclotomic Z_p -extension of $K = K_0$, and let σ be a topological generator of the Galois group $\Gamma \cong Z_p$ of K_∞ over K satisfying

$$\sigma(\zeta) = \zeta^{1+p^r}$$

for all p -power roots of unity ζ . The group

$$\mathfrak{X} := K_\infty^* \otimes_{Z_p} (Q_p/Z_p)$$

carries a Γ -module structure $\Gamma \times \mathfrak{X} \rightarrow \mathfrak{X}$, $(\sigma, x) \mapsto \sigma(x)$, with Γ acting on the first factor. We now consider two new Γ -modules $\mathfrak{X}(1)$ and $\mathfrak{X}(-1)$ which have the same underlying set as \mathfrak{X} , but $\mathfrak{X}(1)$ has the following new action of Γ

$$\sigma \cdot x := (1+p^r) \sigma(x) \quad \text{for all } x \in \mathfrak{X},$$

and $\mathfrak{X}(-1)$ has the new action of Γ , defined by

$$\sigma \cdot x := (1+p^r)^{-1} \sigma(x) \quad \text{for all } x \in \mathfrak{X},$$

where $(1+p^r)^{-1}$ denotes the multiplicative inverse of $1+p^r$ in Z_p . Setting $C = \ker(\phi) = \{c \in K^* \mid c \otimes (\zeta_p) = 0\}$ one has by [4, Sec. 5]:

$$C = \{c \in K^* \mid c \otimes (\rho^{-1} \bmod Z_p)\} \cong \text{the maximal divisible subgroup of } \mathfrak{X}(1)^\Gamma.$$

Setting $A = \{a \in K^* \mid K(\sqrt[p]{a}) \subset \tilde{K}\}$ with \tilde{K} being the compositum of all Z_p -extensions of K in some fixed algebraic closure of $K = Q(\zeta_p)$, one has by [4, p. 1236/37]:

$$A = \{a \in K^* \mid a \otimes (\rho^{-1} \bmod Z_p)\} \cong \text{the maximal divisible subgroup of } \mathfrak{X}(-1)^\Gamma.$$

Let E_n be the group of p -units of K_n , that is the group of units in the ring $\mathcal{O}_{K_n}[p^{-1}]$ with \mathcal{O}_{K_n} being the ring of integers of K_n .

If Vandiver's conjecture holds for p , that is if p does not divide the class number of $Q(\zeta_p + \zeta_p^{-1})$, then we obtain from [3, p. 193] and [4, p. 1241] that $A/K^*P = E_0/E_0^P = C/K^*P$. We shall see in Corollary 3 below that this is true for an arbitrary prime p , provided that r is sufficiently large.

Defining for $u \in K_n^*$ and $n > 0$

$$\alpha_n(u) = \prod_{i=0}^{n-1} \sigma^i(u)^{(1+p^r)^{p^{n-1-i}}}$$

and

$$\gamma_n(u) = \prod_{i=0}^{n-1} \sigma^{-i}(u)^{(1+p^r)^{n-1-i}}.$$

one obtains by [7, Satz 1.1]:

$$\alpha_n(u) \otimes (p^{-n-r} \bmod \mathbb{Z}_p) \in \mathfrak{K}(-1)^\Gamma \quad \text{and} \quad \gamma_n(u) \otimes (p^{-n-r} \bmod \mathbb{Z}_p) \in \mathfrak{K}(1)^\Gamma.$$

We now consider sequences $(u_n \in K_n^*)_{n \geq 0}$ satisfying

$$(1) \quad N_{K_{n+1}/K_n}(u_{n+1}) = u_n \quad \text{for all } n \geq 0,$$

where $N_{K_{n+1}/K_n}: K_{n+1}^* \rightarrow K_n^*$, $u \mapsto \prod_{j=0}^{p-1} \sigma^j p^n(u)$, denotes the norm map.

Equation (1) implies $u_n \in E_n$ for all $n \geq 0$. [7, Lemma 2.1].

PROPOSITION 1 [7, Satz 1.2]. *Suppose $(u_n \in E_n)_{n \geq 0}$ is a sequence satisfying (1). Then $\alpha_n(u_n) \otimes (p^{-n-r} \bmod \mathbb{Z}_p)$ is divisible in $\mathfrak{K}(-1)^\Gamma$, and $\gamma_n(u_n) \otimes (p^{-n-r} \bmod \mathbb{Z}_p)$ is divisible in $\mathfrak{K}(1)^\Gamma$ for all $n \geq 0$. In particular: $u_0 \in A \cap C$.*

Let r_2 be number of complex places of $K = \mathbb{Q}(\zeta_{p^r})$. Since $\dim_{\mathbb{Z}/p\mathbb{Z}}(C/K^*P) = p^{r_2+1} = \dim_{\mathbb{Z}/p\mathbb{Z}}(A/K^*P)$, (by [10] and [11]), Dirichlet's unit theorem and Proposition 1 yield the following

COROLLARY 1 [7, Korollar 1.3]. *If the p^{r-1} th power of any unit of K is a starting point u_0 of a sequence $(u_n \in E_n)_{n \geq 0}$ satisfying (1), then $A/K^*P = E_0/E_0^P = C/K^*P$.*

The following Proposition 2 directly follows from results of Iwasawa [5], Coates [1], and [6]. (Recall that $K = \mathbb{Q}(\zeta_{p^r})$).

PROPOSITION 2. *The following five conditions are equivalent.*

- (i) Each unit of K is a starting point u_0 of a sequence $(u_n \in E_n)_{n \geq 0}$ satisfying (1).
- (ii) The canonical homomorphism $\mathfrak{D}_0 \rightarrow \mathfrak{D}_m$ is injective for all $m \geq 0$, where \mathfrak{D}_m denotes the p -Sylow subgroup of the ideal class group of K_m .

- (iii) The canonical homomorphism $\mathfrak{A}_n \rightarrow \mathfrak{A}_m$ is injective for all $n > 0$ and all $m < n$.
- (iv) Each \mathbb{Z}_p -ring extension of $R = \mathbb{Q}_K[p^{-1}]$ has a normal basis over R .
- (v) The canonical homomorphism $K_2(K)(p) \rightarrow K_2(K_\infty)(p)$, with $K_2(K)(p)$ being the p -primary component of $K_2(K)$, is injective.

It is a classical open problem whether condition (iii) of Proposition 2 is true (for $K = \mathbb{Q}(\zeta_p)$). Proposition 2 and Corollary 1 imply the following

COROLLARY 2. *If one of the five conditions in Proposition 2 holds then $A/K^*P = E_0/E_0^P = C/K^*P$.*

2. As before, let $K_\infty = \bigcup_{n \geq 0} K_n$ with $K_n = \mathbb{Q}(\zeta_{p^{n+r}})$ be the cyclotomic \mathbb{Z}_p -extension of $K = K_0$, and let E_n be the group of p -units of K_n .

We now consider for each $n \geq 0$ the \mathbb{Z} -module

$$N_n = \{ u_n \in E_n \mid \exists \text{ a sequence } (u_m \in E_m)_{m \geq n} \text{ with } N_{K_m/K_n}(u_m) = u_n \forall m \geq n \}.$$

Using results of Iwasawa [5] one proves the following

PROPOSITION 3 [7, SATZ 2.2]. *The group E_n/N_n is a finite p -group for each $n \geq 0$, and there is an integer $s \geq 0$, independent on n , such that $|E_n/N_n| \leq p^s$ for all $n \geq 0$.*

Proposition 3 and Corollary 1 yield the following

COROLLARY 3. *If $r > s$ then $A/K^*P = E_0/E_0^P = C/K^*P$.*

If $r \leq s$, it still remains an open problem whether $A = C$ or not. It is shown in [7]: If $r = 1$, hence $K = \mathbb{Q}(\zeta_p)$, and if $s \geq 1$, then one can always construct a basis of A/K^*P and a basis of C/K^*P over $\mathbb{Z}/p\mathbb{Z}$ by using Proposition 1 and sequences which satisfy (1). This implies by [7, Kor. 3.7]: If the exponent of E_0/N_0 equals p and if $E_0/N_0 = E_1/N_1$ then $A = C$.

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HOW TO MAKE VECTOR BUNDLES ALGEBRAIC

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Abstract. We consider the problem, first studied by K. Lonsted, of representing vector bundles over a topological space as finitely generated projective modules over a ring having some nice algebraic properties.

Given a ring A with identity, we let $\text{Proj}(A)$ denote the monoid of isomorphism classes of finitely generated projective A -modules.

Let X be a compact topological space. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} (quaternions). It is well-known that the monoid $\text{VB}_{\mathbb{F}}(X)$ of isomorphism classes of continuous \mathbb{F} -vector bundles over X is canonically isomorphic to the monoid $\text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{C}(X))$, where $\mathcal{C}(X)$ is the ring of continuous functions from X to \mathbb{R} [1] or [3]. Under certain restrictions on X , one can find a Noetherian subring A of $\mathcal{C}(X)$ such that the homomorphism

$$\text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} A) \rightarrow \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{C}(X)),$$

induced by the inclusion $A \rightarrow \mathcal{C}(X)$, is bijective. This has been done initially by K. Lonsted, with X being a compact C^∞ manifold [11] or a finite CW complex [12].

Lonsted's results have been then given more elementary proofs and generalized by Swan [14] and Carral [6]. In some instances, one can pick A to be a localization of a finitely generated \mathbb{R} -algebra (cf. [11, 14, 6]). If X is a compact C^∞ manifold, then the best

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result in this direction is due to Benedetti and Tognoli [2] (cf. also Theorem 1 below). It seems, however, that their result is not widely known. For example, it is not mentioned in Swan's survey article [15].

In this note we show that the result of Benedetti and Tognoli allows one to obtain immediately a Lonsted-type theorem for compact Euclidean neighborhood retracts (cf. Theorem 2). Moreover, this construction produces a ring Λ which can be easily described and is most natural for the problem in question.

Let X be a subset of \mathbb{R}^n . A function $f: X \rightarrow \mathbb{R}$ is said to be a *polynomial function* (resp. a *regular function*) if there exists a polynomial F (resp. there exist polynomials F and G) in $\mathbb{R}[T_1, \dots, T_n]$ such that $f(x) = F(x)$ (resp. $G(x) \neq 0$ and $f(x) = F(x)/G(x)$) for x in X . Denote by $\mathcal{P}(X)$ and $\mathcal{R}(X)$ the ring of polynomial functions and regular functions on X , respectively. Obviously, $\mathcal{P}(X)$ is a finitely generated \mathbb{R} -algebra and $\mathcal{R}(X)$ is canonically isomorphic to the localization of $\mathcal{P}(X)$ with respect to $\{f \in \mathcal{P}(X) \mid f^{-1}(0) = \emptyset\}$. Denote by

$$\varphi_{X, \mathbb{R}} : \text{Proj}(\mathbb{F}_{\mathbb{R}} \mathcal{R}(X)) \rightarrow \text{Proj}(\mathbb{F}_{\mathbb{R}} \mathcal{P}(X))$$

the homomorphism induced by the inclusion $\mathcal{R}(X) \rightarrow \mathcal{P}(X)$. If X is compact, then the homomorphism $\varphi_{X, \mathbb{R}}$ is injective [14, Theorem 2.2]. In general, $\varphi_{X, \mathbb{R}}$ is not surjective even for very "nice" compact, connected, nonsingular algebraic subsets X of \mathbb{R}^n (cf. [4, 5]).

Benedetti and Tognoli [2, Theorem 4.2] (cf. also [10, Theorem 12]) proved the following.

Theorem 1. *Let M be a compact C^∞ submanifold of \mathbb{R}^n with $2\dim M + 1 \leq n$. Then there exists a C^∞ diffeomorphism $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$, arbitrarily close in the strong C^∞*

topology to the identity mapping of \mathbb{R}^n , such that $X = h(M)$ is a nonsingular algebraic subset of \mathbb{R}^n and the homomorphism

$$\varphi_{X,\mathbb{F}} : \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{A}(X)) \rightarrow \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{S}(X))$$

is bijective, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} .

We should mention that Theorem 1 is formulated in [2] and [10] in the (equivalent) language of strongly algebraic vector bundles and proved for $\mathbb{F} = \mathbb{R}$. However, the proof can be modified in an obvious way to cover the cases $\mathbb{F} = \mathbb{C}$ or \mathbb{H} . The reader may consult [3, Chapter 12] for the exposition of the theory of strongly algebraic vector bundles and their relations with projective modules. The strong C^∞ topology is defined in [8].

If in Theorem 1, the submanifold M is replaced by a compact neighborhood retract in \mathbb{R}^n , then a slightly weaker statement still holds true. It will be convenient to identify \mathbb{R}^n with the subset $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$, where 0 is the origin in \mathbb{R}^k .

Theorem 2. *Let K be a compact neighborhood retract in \mathbb{R}^n . Then there exists a C^∞ diffeomorphism $h : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$, arbitrarily close in the strong C^∞ topology to the identity mapping of \mathbb{R}^{2n+1} , such that for $X = h(K)$ the homomorphism*

$$\varphi_{X,\mathbb{F}} : \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{A}(X)) \rightarrow \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{S}(X))$$

is bijective, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} .

Proof. Pick a bounded neighborhood U of K in \mathbb{R}^n and a retraction $r : U \rightarrow K$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ function with $f^{-1}(0) = K$ and $f(x) > 0$ for x in $\mathbb{R}^n \setminus K$ (cf. [16], p. 93, Lemma 2.4)]. Clearly, we may assume that $f(x) > 1$ for $\|x\|$ sufficiently large. Hence if c is a sufficiently small positive real number, then $W = f^{-1}([0, c])$ is contained in U . By Sard's theorem [8], we may assume that c is a regular value of f . Then W is a compact submanifold (with boundary) of \mathbb{R}^n of dimension n . Let M be the double of W , that is, M is obtained from $(W \times \{0\}) \cup (W \times \{1\})$ by identifying the points $(x, 0)$ and $(x, 1)$ for x in ∂W . Let $\pi : M \rightarrow W$ be the mapping which sends the point of M represented by $(x, 0)$ or $(x, 1)$ into x . Clearly, $r_M = r \circ \pi$ is a retraction of M onto K . Moreover, by a standard embedding theorem, we may assume that M is a C^∞ submanifold of \mathbb{R}^{2n+1} containing K .

By Theorem 1, there exists a C^∞ diffeomorphism $h : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$, arbitrarily close in the strong C^∞ topology to the identity mapping of \mathbb{R}^{2n+1} , such that for $Y = h(M)$, the homomorphism

$$\varphi_{Y, \mathbb{R}} : \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{R}(Y)) \rightarrow \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{S}(Y))$$

is bijective.

Obviously, if $X = h(K)$, then the mapping $r_Y : Y \rightarrow X$, defined by $r_Y(y) = h(r_M(h^{-1}(y)))$ for y in Y , is a retraction. Consider the following commutative diagram

$$\begin{array}{ccc} \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{R}(Y)) & \xrightarrow{\varphi_{Y, \mathbb{R}}} & \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{S}(Y)) \\ \alpha_{\mathbb{F}} \downarrow & & \downarrow \beta_{\mathbb{F}} \\ \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{R}(X)) & \xrightarrow{\varphi_{X, \mathbb{F}}} & \text{Proj}(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{S}(X)), \end{array}$$

where $\alpha_{\mathbb{F}}$ and $\beta_{\mathbb{F}}$ are induced by the ring homomorphisms $\mathcal{R}(Y) \rightarrow \mathcal{R}(X)$ and $\mathcal{G}(Y) \rightarrow \mathcal{G}(X)$ sending functions $g: Y \rightarrow \mathbb{R}$ into $g|_X$. Observe that $\beta_{\mathbb{F}}$ is surjective. Indeed, this follows immediately, since $\beta_{\mathbb{F}}$ corresponds to the homomorphism $VB_{\mathbb{F}}(Y) \rightarrow VB_{\mathbb{F}}(X)$ induced by the restriction of vector bundles (cf. the beginning of this note) and r_Y is a retraction. Now it is obvious that $\varphi_{X,\mathbb{F}}$ is surjective and therefore bijective (it is injective, X being compact). Hence the proof is complete.

Remark. It is well-known that every compact triangulable subset of \mathbb{R}^n is a neighborhood retract (cf. [9, p.30]). The class of triangulable spaces is very large. It contains, in particular, subanalytic sets [7].

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INDEX THEORY AND QUANTIZATION OF BOUNDARY VALUE PROBLEMS

Dedicated to the memory of Marshall Stone.

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Abstract. The second quantization functor associates to each skew-symmetric operator in one-particle space a derivation δ of the algebra which is based on the given commutation relations. Operator extensions in one-particle space are given by abstract boundary conditions, and correspond to extensions of the associated δ . We characterize the spatial theory of δ (in the Fock representation) by an index which generalizes the one studied earlier by Powers and Arveson in connection with the spatial cohomological obstruction for semigroups of endomorphisms of $\mathcal{B}(\mathcal{H})$. It is well known that such semigroups are generated by derivations, but derivations associated to *two-sided* boundary conditions generally do *not* generate semigroups. We show that the known index theory for semigroups generalizes to the quantization of arbitrary boundary conditions in one-particle space.

1. Introduction

In this paper, we consider the second quantization of the familiar boundary value problem for vector fields on smooth manifolds M with boundary. From the theory of boundary value problems, we know that the operator theoretic information is codified in a Krein space $[K]$, given by a certain indefinite quadratic form $\langle \cdot, \cdot \rangle$. In addition, a measure μ on M is given, and we have the positive-definite quadratic form determined by the Hilbert-space inner product on $L^2(M, \mu)$. We shall describe an index theory for the second quantization of the familiar L^2 -boundary value problem for the initially given vector field X on M . Our model allows an interpretation of particle production scattering theory. It is given in terms of the fundamental form $\langle \cdot, \cdot \rangle$ associated to X , and the corresponding Krein space.

In a purely operator theoretic setting, we generalize the index theory of Powers and

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Arveson, see [P1-2], [A1-3], [PP], and [PR]. It was shown (for semigroups of endomorphisms, which would correspond to a one-sided boundary value problem), in [PP], that the index of Powers, and that of Arveson, give the same number. This number is the dimension of a certain Hilbert space. In the case of general boundary conditions, including two-sided ones, as well as the known one-sided ones, associated to semigroups of isometries, we define an index in terms of the boundary value Krein space, and our index agrees with the Powers-Arveson index in the special case of endomorphisms associated to semigroups of isometries.

The interest in semigroups of isometries in Hilbert space dates back at least to the Douglas paper [D] which describes the universal Toeplitz algebra as the C^* -isomorphism class of such a semigroup of isometries (see also [BC] and [M]). We are motivated by this, as well as by the more recent developments of Powers [P1-2], Powers and Robinson [PR], and Arveson [A1-3] where an index is introduced for semigroups of endomorphisms to reflect a certain cohomological obstruction. This cohomological obstruction is *not* present, of course, in the familiar case of automorphisms, as opposed to endomorphisms, by virtue of Wigner's theorem [W]; see also [BR], Section 3.2.

2. Derivations

Let \mathcal{H} be a complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators in \mathcal{H} . Let δ be a *derivation* defined on a given domain $\mathcal{D}(\delta) \subset \mathcal{B}(\mathcal{H})$. The defining conditions on δ are linearity, Leibniz' rule

$$\delta(AB) = \delta(A)B + A\delta(B), \quad A, B \in \mathcal{D}(\delta), \quad (2.1)$$

and the Hermitian property

$$\delta(A^*) = \delta(A)^*, \quad A \in \mathcal{D}(\delta). \quad (2.2)$$

Let d be a closed skew-symmetric operator in \mathcal{H} with dense domain $\mathcal{D}(d) \subset \mathcal{H}$, and let d^* denote the adjoint operator with domain $\mathcal{D}(d^*)$. From the skew-symmetry,

$$(d\varphi, \psi) + (\varphi, d\psi) = 0, \quad \varphi, \psi \in \mathcal{D}(d), \quad (2.3)$$

we conclude $\mathcal{D}(d) \subset \mathcal{D}(d^*)$. Moreover, the quadratic form $\langle \cdot, \cdot \rangle$, given by

$$\langle \varphi, \psi \rangle := \frac{1}{2}((d^*\varphi, \psi) + (\varphi, d^*\psi)), \quad \varphi, \psi \in \mathcal{D}(d^*), \quad (2.4)$$

passes to the quotient $Q(d) := \mathcal{D}(d^*)/\mathcal{D}(d)$, and the pair $(Q(d), \langle \cdot, \cdot \rangle)$ is a Krein space. For differential operators, this is the space of boundary values, see, e.g., [DS], [St], [II], and [K].

We say that d implements δ if $A\varphi \in \mathcal{D}(d)$ for all $A \in \mathcal{D}(\delta)$ and all $\varphi \in \mathcal{D}(d)$, and, further,

$$\delta(A)\varphi = dA\varphi - Ad\varphi, \quad A \in \mathcal{D}(\delta), \quad \varphi \in \mathcal{D}(d). \tag{2.5}$$

It follows that each operator A in $\mathcal{D}(\delta)$ leaves $\mathcal{D}(d^*)$ invariant and passes to the quotient $Q(d)$, to yield a representation, denoted $\pi(A)$, $A \in \mathcal{D}(\delta)$. Let $\varphi \mapsto [\varphi] : \mathcal{D}(d^*) \rightarrow \mathcal{D}(d^*)/\mathcal{D}(d) =: Q(d)$ denote the natural quotient mapping. Then the representation $\pi = \pi_{\delta, d}$ (introduced first in a special case by Powers) is given by

$$\pi(A)[\varphi] := [A\varphi], \quad \varphi \in \mathcal{D}(d^*), \quad A \in \mathcal{D}(\delta). \tag{2.6}$$

Let \mathcal{D}_{\pm} denote the deficiency spaces given by

$$\mathcal{D}_{\pm} = \{\psi^{\pm} : d^*\psi^{\pm} = \pm\psi^{\pm}\}. \tag{2.7}$$

By von Neumann's theorem [DS], we have the natural isomorphism

$$Q(d) \simeq \mathcal{D}_{+} + \mathcal{D}_{-} \tag{2.8}$$

determined by the (Stone-von Neumann) decomposition

$$\mathcal{D}(d^*) = \mathcal{D}(d) + \mathcal{D}_{+} + \mathcal{D}_{-}. \tag{2.9}$$

For $A \in \mathcal{D}(\delta)$, let $\pi(A) = (\pi_{ij}(A))_{1 \leq i, j \leq 2}$ be the block-operator-matrix form, corresponding to the decomposition (2.8) of $Q(d)$, with entries,

$$\begin{aligned} \pi_{11}(A) : \mathcal{D}_{+} &\longrightarrow \mathcal{D}_{+} & \pi_{12}(A) : \mathcal{D}_{-} &\longrightarrow \mathcal{D}_{+} \\ \pi_{21}(A) : \mathcal{D}_{+} &\longrightarrow \mathcal{D}_{-} & \pi_{22}(A) : \mathcal{D}_{-} &\longrightarrow \mathcal{D}_{-}. \end{aligned}$$

We shall need the

Lemma 2.1. The representation π of $\mathcal{D}(\delta)$ on $Q(d)$ has bounded operator components, i.e., $\pi_{ij}(A) \in \mathcal{B}(\mathcal{K})$ for all $A \in \mathcal{D}(\delta)$. Furthermore,

$$\langle \pi(A)\varphi, \psi \rangle = \langle \varphi, \pi(A^*)\psi \rangle, \quad A \in \mathcal{D}(\delta), \quad \varphi, \psi \in \mathcal{D}(d^*). \tag{2.10}$$

The space \mathcal{V} of all operators $V : \mathcal{D}(d) \rightarrow Q(d)$ satisfying

$$VA = \pi(A)V, \quad A \in \mathcal{D}(\delta), \quad (2.11)$$

counts the "number of times" the representation π "contains" the (irreducible) identity representation of $\mathcal{D}(\delta)$ acting on \mathcal{K} . The operators in \mathcal{V} form a Krein space, and we identify it below for the quantized boundary value problem. The Krein dimension of \mathcal{V} will be called the V -index.

Theorem 2.2. There is a unique scalar valued form $\langle \cdot, \cdot \rangle$ on \mathcal{V} such that

$$\langle V\varphi, W\psi \rangle = \langle V, W \rangle (\varphi, \psi)$$

holds for all $V, W \in \mathcal{V}$, and $\varphi, \psi \in \mathcal{D}(\delta)$; this form is nondegenerate on \mathcal{V} .

The variety of all skew-symmetric operators d as specified above, satisfying (2.5), will be identified too as a covariant vector bundle of Krein spaces. The corresponding Krein dimension will be called the D -index.

Note that, when $\mathcal{D}_- = 0$, then the Krein space $Q(d)$ coincides with the Hilbert space \mathcal{D}_+ , and the Krein space dimension is a Hilbert space dimension. When $\mathcal{D}_- \neq 0$, this is not the case. In fact, the representation π can be shown in examples not to be equivalent to a Hermitian representation in any Hilbert space, i.e., the representation π of $\mathcal{D}(\delta)$ on $Q(d)$ is *not* unitarizable in the sense of [CP].

The special case $\mathcal{D}_- = 0$ reduces to the analysis of Powers and Arveson of E_0 -semigroups of endomorphisms $\{\alpha_t : t \in \mathbb{R}_+\}$ of $\mathcal{B}(\mathcal{K})$. Let us recall the defining properties of an E_0 -semigroup α :

- (i) α_t is strongly continuous $\mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{K})$ relative to the w^* -topology,
- (ii) $\alpha_t(AB) = \alpha_t(A)\alpha_t(B)$, $t \in \mathbb{R}_+$, $A, B \in \mathcal{B}(\mathcal{K})$,
- (iii) $\alpha_t(A^*) = (\alpha_t(A))^*$, $t \in \mathbb{R}_+$, $A \in \mathcal{B}(\mathcal{K})$,
- (iv) $\alpha_{s+t}(A) = \alpha_s(\alpha_t(A))$, $s, t \in \mathbb{R}_+$, $A \in \mathcal{B}(\mathcal{K})$.

Powers and Arveson studied the possibility of choosing operators $\{U_t : t \in \mathbb{R}_+\}$ satisfying

$$U_t A = \alpha_t(A)U_t, \quad \text{for all } t \in \mathbb{R}_+, \quad A \in \mathcal{B}(\mathcal{K}), \quad (2.12)$$

and Arveson showed that the manifold of $\{U_t\}$'s may be given the structure of a Hilbert bundle in a canonical way. Its dimension is the Arveson-index, and is denoted $\text{ind}(\alpha)$. In

[A1], Arveson proved the additivity property, $\text{ind}(\alpha \otimes \beta) = \text{ind}(\alpha) + \text{ind}(\beta)$, suggested by analogy to the corresponding functorial property of the Fredholm index. It will be convenient for us to call $\text{ind}(\alpha)$ the U-index.

When formula (2.12) is differentiated at $t = 0$, it follows that a solution U yields a pair (δ, d) as specified above, where the derivation δ is the infinitesimal generator, i.e.,

$$\delta(A) = \lim_{t \rightarrow 0^+} t^{-1}(\alpha_t(A) - A) \quad (2.13)$$

with $\mathcal{D}(\delta)$ consisting of those $A \in \mathcal{B}(\mathcal{K})$ where the w^* -limit (2.13) can be calculated. It was noted in [PP] that the \mathcal{D} -space for d must vanish in this case. The main result of [PP] may be restated as follows:

$$(D\text{-index}) = (V\text{-index}) = (U\text{-index}).$$

We note that E_0 -semigroups correspond to a quantization of one-sided boundary conditions; *but* that more general boundary conditions, such as two-sided conditions on the real line, e.g., Dirichlet on the unit-interval, quantize to derivations of an infinite-particle Fock space; and that the corresponding derivations do *not* generate semigroups. It follows that the $(U\text{-index}) = (\text{Arveson index} = \text{ind}(\alpha))$ *cannot* be defined in this context. However, the V -index may be defined (in terms of genuine Krein spaces) and we have

Theorem 2.3. Let the pair (δ, d) be obtained in the anti-symmetric Fock space by canonical second quantization of a given skew-symmetric operator d_1 in one-particle space, and let Q_1 denote the boundary Krein space of d_1 . For $h \in Q_1$, and $A \in \mathcal{D}(\delta)$, define $V_h(A) = [Ah]$. Then the mapping $h \rightarrow V_h$ is an isometric isomorphism between the two Krein spaces Q_1 and \mathcal{V} (relative to the respective forms), and each V_h is bounded.

There is evidence which suggests that $D\text{-index} = V\text{-index}$, also, in our setting corresponding to more general boundary conditions, but we have not been able to establish this. (We have the functor going in one direction, but we have, so far, been unable to show that it is an isomorphism between the two categories.)

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INDEX AND SECOND QUANTIZATION

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Abstract. In this paper we study the derivations obtained by second quantization of skew-symmetric operators d_1 in one-particle space. We sketch how to find covariant skew-symmetric operators in Fock space which implement the derivation obtained by extending d_1 to the CAR-algebra.

1. Introduction

One of the principal motivations for Powers' work [P1-2] on index theory for E_0 -semigroups was to obtain more insight into the extension problem for quasi-free derivations on the CAR (canonical anticommutation relations) algebra. It was hoped that the index would serve as a measure of the obstruction for a quasi-free derivation to admit an extension which is an infinitesimal generator of a C^* -dynamical system. There is an exact correlation between the index and the extendibility problem in the situation studied by Powers, which we have described above (see [P1], [J2]), and there is evidence that the V -index (details below) is a correct indication of the obstruction in the more general setting as well (see [JP1], [Pr]). To make these ideas more precise, we give a brief description of the CAR algebra and quasi-free derivations. More detailed accounts may be found in [PS] or [BR].

Let \mathfrak{h} be a complex Hilbert space, and let $\mathfrak{A} := \mathfrak{A}(\mathfrak{h})$ denote the C^* -algebra obtained as the completion of the polynomial $*$ -algebra in the (annihilation) operators $a(h)$ which satisfy the relations

$$\begin{aligned}
 a^*(h)a(k) + a(k)a^*(h) &= (k,h)I \\
 a(h)a(k) + a(k)a(h) &= 0, \quad h,k \in \mathfrak{h}.
 \end{aligned}
 \tag{1.1}$$

Let $\mathfrak{F} = \Gamma(\mathfrak{h})$ be the antisymmetric Fock space, $\mathfrak{F} = \sum_{n=0}^{\infty} \oplus \mathfrak{F}_n$, where

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$$\mathcal{H}_0 := \mathbb{C}\Omega, \quad \mathcal{H}_1 := \mathfrak{h}, \quad \mathcal{H}_n := \bigwedge_1^n \mathfrak{h}, \dots, \quad (1.2)$$

with Ω the (unit Fock) vacuum vector.

Let d_1 be a given skew-symmetric operator on \mathfrak{h} with dense domain $\mathcal{D}(d_1)$. Applying the second quantization functor to d_1 yields a quasi-free derivation $\delta := \delta_{d_1}$, defined on the annihilation operators $a(h)$, $h \in \mathcal{D}(d_1)$, by

$$\delta(a(h)) := a(d_1 h), \quad (1.3)$$

and extended to the polynomial $*$ -algebra in the operators $a(h)$, $h \in \mathcal{D}(d_1)$, by the Leibniz rule. δ is a closable derivation. A longstanding conjecture of Powers asserts that δ has a generator extension on \mathfrak{A} if and only if d_1 has equal deficiency indices; moreover, if d_1 has equal deficiency indices, then each generator extension of δ corresponds to a skew-adjoint extension of d_1 . To date, this conjecture has been verified only in a few cases. These include all of the cases where the deficiency indices are $(n,0)$ or $(0,n)$ with $n > 0$, [J2], and some instances where d_1 has indices $(1,1)$, [JP1], [Pr].

We note as well that d_1 gives rise to a skew-symmetric operator $d := \Gamma_{\mathfrak{A}}(d_1)$ on Fock space, defined on wedge product vectors by

$$d(h_1 \wedge \dots \wedge h_n) = \sum_i h_1 \wedge \dots \wedge (d_1 h_i) \wedge \dots \wedge h_n, \quad (1.4)$$

or equivalently, $d(A\Omega) = \delta(A)\Omega$, $A \in \mathcal{D}(\delta)$. Then $\mathcal{D}(d)$ is a module over $\mathcal{D}(\delta)$ and

$$\delta(A)\varphi = dA\varphi - Ad\varphi, \quad A \in \mathcal{D}(\delta), \quad \varphi \in \mathcal{D}(d). \quad (1.5)$$

Since the Fock state, $\omega(A) := (\Omega, A\Omega)$, is pure, the corresponding representation of \mathfrak{A} (i.e., the GNS-construction) is irreducible, and therefore generates $\mathcal{B}(\mathcal{H})$. It is faithful since \mathfrak{A} is simple. It follows that δ extends to a derivation (also denoted δ) with w^* -dense domain in $\mathcal{B}(\mathcal{H})$.

Make the quotient construction in the respective spaces \mathfrak{h} and \mathcal{H} to get associated Krein spaces $Q_1 := Q(d_1, \mathfrak{h}) = \mathcal{D}(d_1^*)/\mathcal{D}(d_1)$, and $Q := Q(d, \mathcal{H}) = \mathcal{D}(d^*)/\mathcal{D}(d)$, and let π be the representation of $\mathcal{D}(\delta)$ on Q which is given by $\pi(A)[\varphi] = [A\varphi]$, all $A \in \mathcal{D}(\delta)$, $\varphi \in \mathcal{D}(d^*)$.

2. Existence and Uniqueness of Krein Representations

Definition 2.1. A Krein representation is a linear mapping, $f \mapsto V_f$, from Q_1 into all linear operators, $V : \mathcal{F} \rightarrow Q$, satisfying

$$VA = \pi(A)V \quad \text{for all } A \in \mathcal{D}(\delta), \quad (2.1a)$$

$$\langle V_f \varphi, V_g \psi \rangle = \langle f, g \rangle (\varphi, \psi)_{\mathcal{F}} \quad \text{for all } f, g \in Q_1, \quad \varphi, \psi \in \mathcal{F} \quad (2.1b)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ denotes the Hilbert inner product in Fock space \mathcal{F} .

We will be able to compute the V -index [JP3] for the pair (δ, d) associated to an arbitrarily given skew-symmetric operator d_1 in one-particle space \mathfrak{h} , and we show that the V -index equals the dimension of the Krein space $Q_1 = Q(d_1, \mathfrak{h})$.

The result is based on the following:

Theorem 2.2. There is one and, up to automorphisms of Q_1 , only one Krein representation V associated to a given skew-symmetric operator d_1 in one-particle space.

The details of proof (rather lengthy) will be given in [JP2], but we shall describe here the functor from d_1 to V by giving a formula for the Krein representation V . The uniqueness part of the conclusion then states that, up to automorphisms of Q_1 , every such representation must necessarily be the explicitly given one.

If V is a given Krein representation, i.e., assumed to satisfy (2.1 a–b), then the automorphisms of Q_1 clearly act on V . The uniqueness assertion states that this is the full extent of the possible nonuniqueness. An automorphism u of Q_1 is required, by definition, to preserve the indefinite form $\langle \cdot, \cdot \rangle$ (defining the Krein space Q_1), i.e.,

$$\langle uf, ug \rangle = \langle f, g \rangle, \quad \text{all } f, g \in Q_1. \quad (2.2)$$

For a given Krein representation V , it is immediate that $f \mapsto V_{u(f)}$ is one as well. (If the given operator d_1 in one-particle space has deficiency indices $(1,1)$, then the group of all automorphisms u satisfying (2.2) is the familiar reductive Lie group $U(1,1)$.)

For the *existence* (of a Krein representation) we proceed as follows: Let d_1 be a given skew-symmetric operator in one-particle space \mathfrak{h} , and let Q_1 be the corresponding

Krein space $\mathcal{D}(d_1^*)/\mathcal{D}(d_1)$ where we recall $\mathcal{D}(d_1) \subset \mathcal{D}(d_1^*) \subset \mathfrak{h}$.

For $f \in Q_1$, define an operator V_f mapping the dense subspace $\mathcal{D}(\delta)\Omega$ of \mathcal{K} , to Q , by setting

$$V_f(A\Omega) = [Af], \tag{2.3}$$

where $[Af]$ denotes the equivalence class determined by Af in the space Q . Inspired by an observation of Powers, we can show that V_f is a bounded operator on $\mathcal{D}(\delta)\Omega$, so that we have a unique bounded extension, also denoted V_f , from \mathcal{K} to Q . Moreover, using a rather surprising result (details below and in [JP2]) about the nature of a core for the adjoint operator d^* of the skew-symmetric operator d on Fock space, we can show the following key result.

Theorem 2.3. Suppose $V : \mathcal{D}(d^*) \rightarrow Q$ is an intertwining operator for the representation π , i.e.,

$$V(A\varphi) = \pi(A)V\varphi \tag{2.4}$$

for all $\varphi \in \mathcal{D}(d^*)$. Then there is an $f \in Q_1$ such that V is the bounded operator V_f on $\mathcal{D}(d^*)$, satisfying the identities 2.1.

Corollary 2.4. Let d_1 be a given skew-symmetric operator on one-particle space \mathfrak{h} , and let Q_1 be the corresponding Krein space $\mathcal{D}(d_1^*)/\mathcal{D}(d_1)$. Let δ, d be the pair obtained (in Section 1) from d_1 by application of the quantization functor Γ_α . Then the V -index for the derivation δ coincides with the Krein index of Q_1 .

The proof of these results is based on the observation that for V in the space \mathcal{V} of intertwining operators, applying (2.1a) to the vacuum vector gives

$$V(A\Omega) = \pi(A)V\Omega, \quad A \in \mathcal{D}(\delta), \tag{2.5}$$

so that there is a vector F (namely, $V\Omega$) in the sum $\mathcal{D}_+ + \mathcal{D}_-$ of the deficiency spaces of d such that $V = V_F$. Writing $F = \sum_1^\infty F_n$, $F_n \in \mathcal{K}_n \cap (\mathcal{D}_+ + \mathcal{D}_-)$, and using the observation that π commutes with the gauge transformations, one sees that V_{F_n} is an intertwining operator for π for each n . For $n > 1$, and $h \in \mathcal{D}(d_1)$,

$$\begin{aligned} \pi(a(h))V_{F_n} \Omega &= \pi(a(h))[F_n] \\ &= V_{F_n}(a(h)\Omega) \\ &= 0. \end{aligned} \tag{2.6}$$

Using (2.6) and the core result for d^* alluded to above, we can show $\langle G, F_n \rangle = 0$ for all $G \in \mathcal{D}(d^*)$. Hence $V_{F_n} = 0$ for $n > 1$, and $V = V_{F_1}$ obtains.

3. Extensions of d

In computing the index of (δ, d) , we saw that vectors f in Q_1 satisfying $\langle f, f \rangle \neq 0$ are in one-to-one correspondence with skew-symmetric operators D_f satisfying $\mathcal{D}(D_f) \subset \mathcal{D}(d^*)$ and $d^* + D_f = \lambda I$ where $\lambda = \langle f, f \rangle$. We conclude by noting that subspaces of "zero-vectors" $Q_1(0) = \{f \in Q_1 : \langle f, f \rangle = 0\}$ are associated to a skew-symmetric extension \tilde{d} of d such that \tilde{d} satisfies (1.5), and moreover that \tilde{d} is maximal among all such skew-symmetric extensions of d if and only if the corresponding subspace is maximal. (In general, \tilde{d} will not be maximally skew-symmetric. There will be proper skew-symmetric extensions of \tilde{d} not satisfying (1.5).)

Theorem 3.1. Let the pair (δ, d) be as above, and let $f \rightarrow V_f$ be the Krein representation of Q_1 where Q_1 is the "boundary space" of the given operator d_1 in one-particle space. Let S be a linear subspace of Q_1 with $\langle f, h \rangle = 0, f, h \in S$.

Then the range of

$$V : S \otimes \mathcal{K} \longrightarrow Q \approx \mathcal{D}_+ + \mathcal{D}_-$$

is the graph of a partial isometry $\mathcal{D}_+ \rightarrow \mathcal{D}_-$ defining by inverse Cayley transform a skew-symmetric extension \tilde{d} of d in \mathcal{K} . Moreover, \tilde{d} satisfies (1.5), and it is maximal among all the skew-symmetric extensions satisfying (1.5) if and only if S is maximal in the set of linear spaces S , as specified, with respect to inclusion.

Corollary 3.2. (i) If the operator d_1 in one-particle space has deficiency indices n_{\pm} satisfying $n_+ n_- \neq 0$, then each \tilde{d} will have deficiency indices \tilde{n}_{\pm} also satisfying $\tilde{n}_+ \tilde{n}_- \neq 0$; in particular, \tilde{d} is not essentially skew-adjoint.

(ii) The two pairs (δ, d) and (δ, \tilde{d}) have the same V -index, which is given in
Theorem 3.1.

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Linear Operators in $\ell_\infty(B)$ with a Universality Property

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communicated by David Boyd, FRSC

ABSTRACT

It is shown that certain Banach spaces B have the following property. There are bounded linear operators $L_n : B \rightarrow B$ ($n = 0, 1, \dots$) such that if $f_n : X \rightarrow X$ are nonexpansive and, for some $z \in X$, the orbit of z under the semigroup generated by $\{f_n\}$ is bounded then there is an isometry Φ of X into $\ell_\infty(B)$ with $\Phi f_n = L_n \Phi$.

1. INTRODUCTION

1.1. Given a Banach space B , $\ell_\infty(B)$ denotes the Banach space consisting of all sequences $x = (x_0, x_1, \dots, x_n, \dots)$ with $x_n \in B$, $n = 0, 1, \dots$ and $\|x\|_\infty = \sup\{\|x_n\| : n = 0, 1, \dots\}$. In [2] we constructed a linear operator $P : \ell_\infty(B) \rightarrow \ell_\infty(B)$ with the following property. If X is a metric space which is isometrically embeddable into B , and $f : X \rightarrow X$ is a nonexpansive mapping having a bounded orbit, then there exists an isometry Φ sending X into $\ell_\infty(B)$ such that $\Phi(f(x)) = P(\Phi(x))$, ($x \in X$). It is this universality property which we wish to extend to a family

$\mathcal{L} = \{L_n : n = 0, 1, 2, \dots\}$ of linear operators. More precisely, if $\{f_n\}$ is a collection of nonexpansive mappings such that for some $z \in X$ $\{f(z) : f \in \mathcal{F}\}$, where \mathcal{F} is the semigroup generated by $\{f_n\}$, is bounded and X is isometric to a subset of B , then there exists an isometry Φ of X into $\ell_\infty(B)$ such that, for all $n = 0, 1, 2, \dots$

$$\Phi f_n = L_n \Phi . \quad (*)$$

Equivalently, the diagrams

$$\begin{array}{ccc}
 \ell_\infty(B) & \xrightarrow{L_n} & \ell_\infty(B) \\
 \Phi \downarrow & & \downarrow \Phi \\
 X & \xrightarrow{f_n} & X
 \end{array}$$

all commute.

1.2. An analogous result (in ℓ_2) was established in [1]. There, a family of bounded linear operators was constructed such that, for any separable metric space X and any countable family $\{f_n\}$ of continuous selfmaps of X , there is a homeomorphism h of X into ℓ_2 such that, with Φ replaced by h and $\ell_\infty(B)$ by ℓ_2 , the above commuting property (*) is satisfied.

1.3. The construction of the set \mathcal{L} is in many ways almost identical with that of \mathcal{P} in [1]. As was the case there, each L_n could be described as a generalized shift. Hence correspondences have to be set up between the coordinates $\{x_k\}$ of x and those of $L_n(X)$. Since the precise construction of these correspondences is described in detail in [1], we shall present them in this paper in a somewhat summary manner.

2. The set \mathcal{L} and the isometry Φ .

2.1. For any nonnegative integers k, m let $r(k; m)$ be defined by setting

$$r(k; m) = \frac{1}{2} (k+m)(k+m+1) + m ;$$

write $r(k)$ for $r(k; 0)$.

Given a Banach space B let $L_n : \ell_\infty(B) \rightarrow \ell_\infty(B)$ be defined by setting

$$L_n(x) = (x_{r(0;n)}, x_{r(1;n)}, \dots, x_{r(k;n)}, \dots)$$

for any $x = (x_0, x_1, x_2, \dots) \in \ell_\infty(B)$ and $n = (0, 1, 2, \dots)$. Because the coordinates of $L_n(x)$ are chosen from those of x it is clear that L_n is well defined and $\|L_n\| \leq 1$, $n = 0, 1, 2, \dots$.

2.2. Some further reindexing is needed for the definition of Φ . First we introduce integers $r(k; m_1, \dots, m_n)$. These are defined by setting

$r(k; m_1, m_2) = r(r(k; m_1); m_2)$ and, inductively,

$$r(k; m_1, \dots, m_{n-1}, m_n) = r(r(k; m_1, \dots, m_{n-1}); m_n) . \quad (1)$$

Finally, let

$$s(k; m_1, \dots, m_n) = r(k; 0, m_1, \dots, m_n) . \quad (2)$$

Here k, m_1, \dots, m_n are integers with $k \geq 0$ and $m_1, \dots, m_n \geq 1$. As shown in [1] every integer $p \geq 0$ is uniquely representable in the form (2).

2.3. Proposition 1

Let B be a Banach space, X a subset of $\ell_\infty(B)$, and let $f_n : X \rightarrow X$, $n = 0, 1, 2, \dots$ be nonexpansive mappings such that $\{f(z) : f \in \mathcal{F}\}$ is bounded for some $z \in X$. (Here \mathcal{F} denotes the semigroup generated by $\{f_n\}$.) For arbitrary $x = (x_0, x_1, \dots, x_n, \dots) \in X$ let $\phi_{s(k)}(x) = x_k$ and, inductively on n ,

$$\phi_{s(k; m_1, \dots, m_{n-1}, m_n)}(x) = \phi_{s(k; m_1, \dots, m_{n-1})}(f_{m_n}(x)) . \quad (3)$$

Let $\Phi(x)$ be defined by

$$\Phi(x) = (\phi_0(x), \phi_1(x), \phi_2(x), \dots) .$$

Then Φ maps X into $\ell_\infty(B)$, preserves distances, and satisfies (*) ; i.e.

$$\Phi_n = L_n \Phi \quad (n = 0, 1, 2, \dots) .$$

Proof. Let $p = s(k; m_1, \dots, m_{n-1}, m_n)$. For any $x \in X$,

$$\phi_p(x) = \phi_{s(k; m_1, \dots, m_{n-1})}(f_{m_n}(x)) = \dots = \phi_{s(k)}(f_{m_1} f_{m_2} \dots f_{m_n}(x)) .$$

Thus

$$\begin{aligned} \|\phi_p(x)\| &\leq \|\phi_p(x) - \phi_p(z)\| + \|\phi_p(z)\| \\ &= \|f_{m_1} \dots f_{m_n}(x) - f_{m_1} \dots f_{m_n}(z)\| + \|f_{m_1} \dots f_{m_n}(z)\| \\ &\leq \|x - z\| + \sup\{\|f(z)\| : f \in \mathcal{F}\} \end{aligned}$$

showing that $\Phi(x) \in \ell_\infty(B)$; i.e. Φ is well defined. Similarly, we find

that $\|\phi_p(x) - \phi_p(x')\| \leq \|x-x'\|$, so that $\|\Phi(x) - \Phi(x')\|_\infty \leq \|x-x'\|$, $(x, x' \in X)$.

On the other hand, $\|\Phi(x) - \Phi(x')\|_\infty \geq \|\phi_{s(k)}(x) - \phi_{s(k)}(x')\|_\infty = \|x_k - x'_k\|$; and

this being true for all $k = 0, 1, \dots$ we have $\|\Phi(x) - \Phi(x')\|_\infty \geq \|x-x'\|$. Hence

$\|\Phi(x) - \Phi(x')\|_\infty = \|x-x'\|$ as claimed. Finally, with $p = s(k; m_1, \dots, m_j)$,

$$(\Phi(f_n(x)))_p = \phi_{s(k; m_1, \dots, m_j)}(f_n(x)) = \phi_{s(k; m_1, \dots, m_j, n)}(x) .$$

On the other hand,

$$\begin{aligned} (L_n(\Phi(x)))_p &= (\Phi(x))_{r(p; n)} = \phi_{r(p; n)}(x) \\ &= \phi_{r(r(k; m_1, \dots, m_j); n)}(x) \\ &= \phi_{r(k; 0, m_1, \dots, m_j, n)}(x) = \phi_{s(k; m_1, \dots, m_j, n)}(x) . \end{aligned}$$

This being true for all $x \in X$ and $p = 0, 1, 2, \dots$ it follows that $\Phi f_n = L_n \Phi$, concluding the proof of the proposition.

2.4. The hypothesis, that $X \subset \ell_\infty(B)$, can obviously be replaced by

$X \subset B$, as the mapping sending $x \in X$ to $(x, 0, 0, \dots)$ is an isometry into $\ell_\infty(B)$.

2.5. By a result of Kuratowski [3], for every metric space X there is an isometry Γ sending X into a certain Banach space B . If $\{f_n\}$ is a collection of nonexpansive selfmaps of X then, clearly, the induced selfmaps $\Gamma f_n \Gamma^{-1} : \Gamma[X] \rightarrow \Gamma[X]$ are nonexpansive too. Further, if \mathcal{F} is the

semigroup generated by $\{f_n\}$ and $\mathcal{F}(z)$, the orbit of $z \in X$ under \mathcal{F} , is bounded then so is the orbit of $\Gamma(z)$ under the induced semigroup $\Gamma\mathcal{F}\Gamma^{-1} = \{\Gamma f_n \Gamma^{-1} : f_n \in \mathcal{F}\}$. It follows that Proposition 1 applies, X there being replaced by $\Gamma[X]$ and f_n by $\Gamma f_n \Gamma^{-1}$. In view of these facts the proof of the following theorem is quite straightforward (and omitted).

Theorem I. Let X be a metric space and $f_n : X \rightarrow X$, $n = 0, 1, 2, \dots$, nonexpansive selfmappings of X such that, for some $z \in X$, the orbit $\mathcal{F}(z)$ of z under the semigroup \mathcal{F} generated by $\{f_n\}$ is bounded. Then an isometry Φ of X into $\ell_\infty(B)$ exists such that (*) is satisfied; i.e.

$$\Phi f_n = L_n \Phi \quad (n = 0, 1, 2, \dots).$$

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A NOTE ON HADAMARD'S INEQUALITIES

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*Presented by P. Ribenboim, F.R.S.C.***Abstract.** First we prove a new extension of the inequality

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

where $f : [a, b] \rightarrow \mathbb{R}$ is a convex function. Thereafter we present a rational approximation for the function $x \mapsto e^{-x}$ ($x \geq 0$).

The double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1)$$

which is valid for all convex functions $f : [a, b] \rightarrow \mathbb{R}$ is known in literature as Hadamard's inequalities. However, J. Hadamard was not the first who discovered them. As it was pointed out by D.S. Mitrinović and I.B. Lacković [1] the inequalities (1) are due to C. Hermite who published them in 1883, ten years before Hadamard.

In the past different proofs as well as interesting extensions and applications of "the important inequalities of Hermite" [1, p.230] were given; see [1]. A remarkable generalization of double-inequality (1) is due to T.S. Nanjundiah [2] who proved :

If $f : [a, b] \rightarrow \mathbb{R}$ is convex, then we have for $n = 0, 1, 2, \dots$:

$$\begin{aligned} \frac{1}{n+1} \sum_{\nu=1}^{n+1} f\left(a + \frac{\nu}{n+2}(b-a)\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \frac{1}{n+2} \left[f(a) + f(b) + \sum_{\nu=1}^n f\left(a + \frac{\nu}{n+1}(b-a)\right) \right] \end{aligned} \quad (2)$$

where the left-hand side and the right-hand side of (2) increase and decrease respectively to the common limit $\frac{1}{b-a} \int_a^b f(x) dx$.

Recently, J. Sándor [3] has discovered a noteworthy extension of the left-hand side of (1):

Let $f : [a, b] \rightarrow \mathbb{R}$ be a $2k$ -times differentiable function having a continuous $2k$ -th derivative ($k \geq 1$). If $f^{(2k)}(x) \geq 0$ for $x \in (a, b)$ then

$$\sum_{\nu=0}^{k-1} \frac{1}{(2\nu+1)!} \left(\frac{b-a}{2}\right)^{2\nu} f^{(2\nu)}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx. \quad (3)$$

If $f^{(2k)}(x) > 0$ for $x \in (a, b)$ then inequality (3) is strict.

It is natural to ask: Does there exist a counterpart of (3) which provides a generalization of the right-hand side of (1) ?

The following theorem gives an affirmative answer to this question.

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a $2k$ -times differentiable function ($k \geq 1$).

If $f^{(2k)}(x) \geq 0$ for $x \in (a, b)$ then

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{2} \sum_{\nu=0}^{2k-2} \frac{(b-a)^\nu}{(\nu+1)!} \left[f^{(\nu)}(a) + (-1)^\nu f^{(\nu)}(b) \right]. \quad (4)$$

If $f^{(2k)}(x) > 0$ for $x \in (a, b)$ then inequality (4) is strict.

Proof. From Taylor's formula we obtain

$$f(x) = \sum_{\nu=0}^{2k-2} \frac{f^{(\nu)}(a)}{\nu!} (x-a)^\nu + \frac{f^{(2k-1)}(x_0)}{(2k-1)!} (x-a)^{2k-1}$$

with $a < x_0 < x \leq b$. Since $f^{(2k)}(x) \geq 0$ for $x \in (a, b)$ we conclude that $f^{(2k-1)}$ is increasing on $[a, b]$. Hence we get

$$f(x) \leq \sum_{\nu=0}^{2k-2} \frac{f^{(\nu)}(a)}{\nu!} (x-a)^\nu + \frac{f^{(2k-1)}(x)}{(2k-1)!} (x-a)^{2k-1}$$

and integration yields

$$\int_a^b f(x) dx \leq \sum_{\nu=0}^{2k-2} \frac{f^{(\nu)}(a)}{(\nu+1)!} (b-a)^{\nu+1} + \frac{1}{(2k-1)!} \int_a^b f^{(2k-1)}(x) (x-a)^{2k-1} dx. \quad (5)$$

Setting

$$I_n = \int_a^b f^{(n)}(x) (x-a)^n dx, \quad 0 \leq n \in \mathbb{Z},$$

we obtain the formula

$$\frac{I_n}{n!} = \sum_{\nu=1}^n (-1)^{n+\nu} \frac{(b-a)^\nu}{\nu!} f^{(\nu-1)}(b) + (-1)^n \int_a^b f(x) dx \quad (6)$$

which can be proved easily by induction on n .

Putting $n = 2k - 1$ in (6) we get from (5) :

$$\int_a^b f(x) dx \leq \sum_{\nu=0}^{2k-2} \frac{f^{(\nu)}(a)}{(\nu+1)!} (b-a)^{\nu+1} + \sum_{\nu=0}^{2k-2} (-1)^\nu \frac{f^{(\nu)}(b)}{(\nu+1)!} (b-a)^{\nu+1} - \int_a^b f(x) dx$$

which is equivalent to inequality (4).

Simple modifications of the proof reveal: If $f^{(2k)}(x) > 0$ for $x \in (a, b)$ then inequality (4) is strict. ■

Finally we present lower and upper rational bounds for the exponential function which we could not localize in literature.

Theorem 2. For all positive real x and for all integers $n \geq 0$ we have

$$\frac{1 + \frac{1}{2} \sum_{\nu=0}^{2n} \frac{(-x)^{\nu+1}}{(\nu+1)!}}{1 + \frac{1}{2} \sum_{\nu=0}^{2n} \frac{x^{\nu+1}}{(\nu+1)!}} < e^{-x} < \frac{1 + \frac{1}{2} \sum_{\nu=0}^{2n+1} \frac{(-x)^{\nu+1}}{(\nu+1)!}}{1 + \frac{1}{2} \sum_{\nu=0}^{2n+1} \frac{x^{\nu+1}}{(\nu+1)!}}. \quad (7)$$

Proof. The left-hand side of (7) follows immediately from Theorem 1 by setting $f(t) = e^t$, $a = 0 < x = b$ and $k = n + 1$ in (4).

To prove the second inequality of (7) we have to show

$$g(x) = e^x \left[1 + \frac{1}{2} \sum_{\nu=0}^{2n+1} \frac{(-x)^{\nu+1}}{(\nu+1)!} \right] - \left[1 + \frac{1}{2} \sum_{\nu=0}^{2n+1} \frac{x^{\nu+1}}{(\nu+1)!} \right] > 0 \quad \text{for } x > 0.$$

Differentiation yields

$$\begin{aligned} g'(x) &= \frac{e^x}{2} \left[1 + \frac{x^{2n+2}}{(2n+2)!} \right] - \frac{1}{2} \sum_{\nu=0}^{2n+1} \frac{x^\nu}{\nu!} \\ &= \frac{1}{2} \sum_{\nu=2n+2}^{\infty} \frac{x^\nu}{\nu!} + \frac{e^x}{2} \frac{x^{2n+1}}{(2n+2)!} > 0 \end{aligned}$$

which implies

$$g(x) > g(0) = 0 \quad \text{for } x > 0. \quad \blacksquare$$

Remark. Since the left-hand side and the right-hand side of (7) converge to e^{-x} if $n \rightarrow \infty$ we are able to approximate e^{-x} to any desired accuracy by rational functions. In particular we can approximate Euler's number e by rational numbers. For instance, setting $n = 2$ and $x = 1$ in (7) we obtain

$$2.717\dots = 2677/985 < e < 223/82 = 2.719\dots$$

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A SHORT NOTE ON FERMAT'S LAST THEOREM

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Throughout this note, by a segment of integers we shall mean a finite set of consecutive integers. We shall also use the abbreviation (FLT n) to denote the assertion of Fermat's Last Theorem for the exponent n .

The result we present here is that for any positive integer s , no matter how large, we can always find an infinity of segments of integers, all of length s , such that Fermat's Last Theorem holds for any exponent which lies in the union of these segments. In more detail, we have:

Proposition

For any positive integer s there exists an increasing sequence $\{a_n\}$ of positive integers such that the sets $\{a_n, a_n+1, \dots, a_n+s-1\}$ are mutually disjoint and (FLT k) is true for every k in the union of these sets.

Proof:

Let s be a positive integer.

For the positive integer 4 there exists, by Filaseta's theorem, a positive integer M_4 such that for $m \geq M_4$, (FLT $4m$) is true.

Also for the positive integer s there exists, by Filaseta's theorem a positive integer M_s such that for $m \geq M_s$, (FLT $5m$) is true. By a similar argument we may define a finite sequence M_4, M_5, \dots, M_{s+3} corresponding to the sequence $4, 5, \dots, s+3$ via Filaseta's theorem.

Choosing $M = \max\{M_4, M_5, \dots, M_{s+3}\}$ we have for $m \geq M$, (FLT mi) is true for each $i \in \{4, 5, \dots, s+3\}$.

Now choose a positive integer a such that $a \geq s+2$ and $a! \geq M$. Let $a_n = \left((a+n)! \right) + 4$, for every positive integer n . We assert that this increasing sequence satisfies the conclusion of the proposition. Since $a_{n+1} - a_n = \left((a+n+1)! \right) - \left((a+n)! \right) = (a+n) \left[(a+n)! \right] > a > s$, the segments of integers

$\{a_n, a_{n+1}, \dots, a_{n+s-1}\}$ are mutually disjoint. Furthermore, let $k \in \{0, 1, \dots, s-1\}$ and $n > 0$. Then,

$$a_n + k = \left((a+n)! \right) + (k+4) = 1 \cdot 2 \dots (a+n) + (k+4) = (k+4) \left(\frac{(a+n)!}{k+4} + 1 \right) = 1(k+4), \text{ where } 1 = \frac{(a+n)!}{k+4} + 1 \geq \frac{(a+n)!}{a+n} + 1 > (a+n-1)! \geq M.$$

Since $1 \geq M$ and $(k+4) \in \{4, 5, \dots, s+3\}$, the choice of M implies that (FLT $1(k+4)$) is true, and the proof is complete.

We note that in the definition of a_n one could also choose $a_n = \left(\ell_{cm}(1, 2, \dots, a+n) \right) + 4$, ℓ_{cm} meaning the least common multiple. The definition of a would, of course, have to be different in this case.

Reference: M. Filaseta, An application of Faltings' result to Fermat's Last Theorem, C.R. Math. Rep. Acad. Sci. Canada, 6, 1984, 31-32.

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ADDENDUM TO
CONGRUENCE LATTICES, AUTOMORPHISM GROUPS
OF FINITE LATTICES AND PLANARITY

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In our research announcement (these Mathematical Reports, Volume XI, No. 4, August 1989), the first major result (Theorem 1) has two applications: Theorem 2 and Theorem 3. We had overlooked the references given in this Addendum. Theorem 2 was proved independently by V. A. Baranskii and A. Urquhart, see [1], [2], and [3]. Theorem 3 was proved by V. A. Baranskii, see [1] and [2].

These new references have no bearing on the other two major results: Theorem 4 and Theorem 5.

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