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CONTENTS

Series Permutation in Infinite-Dimensional Spaces

(Main Results and Open Problems)

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Presented by Israel Halperin, F.R.S.C.,
a sequel to his memoir of April 1986, these Comptes Rendus.

In each standard course of analysis one can find the famous Riemann theorem: if $\sum t_n$ is a convergent series of reals, then exactly one of the following two conditions holds:

(a) $\sum t_n$ converges absolutely (i.e. $\sum |t_n| < \infty$)

(b) for each $t \in \mathbf{R}$ there exist a permutation $\pi : \mathbf{N} \to \mathbf{N}$ of the set of natural numbers such that $\sum\limits_{n=1}^{\infty} t_{\pi(n)} = t$. A natural generalization of the Riemann theorem is the Steinitz theorem (1913) formulated earlier by P. Levi (1905): if $\sum x_n$ is a series with terms in a topological linear space X, we call

$$SS(\sum_{n=1}^{\infty} x_n) \equiv \{x : \exists \pi : \mathbf{N} \to \mathbf{N}, x = \sum_{n=1}^{\infty} x_{\pi(n)}\}$$

the *sum-set* of the series; then the sum-set is empty, or $\sum \|x_n\| < \infty$ and consequently $\sum\limits_n x_{\pi(n)} = \sum\limits_n x_n$ for each permutation and the sum-set consists of a single value, or the sum-set is infinite and for each two distinct points x and y it contains the whole line $tx + (1-t)y : t \in \mathbf{R}$. The history of the problem, different proofs of Steinitz theorem and related questions, especially in finite dimensional spaces, are considered by I. Halperin [1]. In our paper we shall deal with infinite-dimensional spaces.

Note: For definitions of (i) moduli of convexity, (ii) smoothness, consult *Geometry of Banach Spaces* (Springer Graduate Texts in Mathematics), 1984 by Joseph Diestel.

1. Absolute and unconditional convergence

Definition 1. Series $\sum x_n$ in a locally convex linear topological space X is said to be absolutely convergent if $\sum\limits_n p(x_n) < \infty$ for each continuous semi-norm p on X.

Evidently, in sequentially complete spaces absolute convergent series must be convergent. In particular, if $\sum x_n$ is a series in Banach space, then condition $\sum \|x_n\| < \infty$ implies convergence of the series.

Definition 2. Series $\sum x_n$ is said to be unconditionally convergent if for every permutation $\pi : \mathbb{N} \to \mathbb{N}$ series $\sum x_{\pi(n)}$ converges.

Since the condition $\sum p(x_n) < \infty$ does not depend on the order of the terms, every absolutely convergent series in sequentially complete space converges unconditionally. It is easy to prove that the sum of unconditionally convergent series does not depend on the order ot terms and that unconditional convergence of the series $\sum x_n$ in sequentially complete locally convex space is equivalent to convergence of all series $\sum\limits_n \pm x_n$. These useful facts were firstly obtained by W. Orlicz for the normed case.

Absolute and unconditional convergence coincide in finite dimensional spaces. Let us consider an orthonormal sequence $\{e_n\}$ in Hilbert space. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n} e_n$ is unconditionally convergent but does not converge absolutely. This simple example shows that in infinite dimensional spaces unconditional convergence differs from an absolute one. Significantly more says the Dvoretsky-Rogers theorem (1950).

Theorem 1 [2]. If in a normed space X absolute convergence is the same as an unconditional one, then X is finite dimensional. Moreover, in each infinite dimensional normed space for every sequence $\{a_n\}$, $a_n \in \mathbb{R}_+$, $\sum a_n^2 < \infty$ there exists an unconditionally convergent series $\sum x_n$ in X such that $\|x_n\| = a_n$, $n \in \mathbb{N}$.

An important generalization of Dvoretsky-Rogers theorem is the Grothendieck criterion of nuclearity (1956):

Theorem 1 [3]. A complete metrizable locally convex (i.e. Frechet) space X is nuclear, iff every unconditionally convergent series in X converges absolutely.

As a manual where one can find the Dvoretsky-Rogers and Grothendieck theorems we can recommend the monograph by Pietsch [4].

Interesting results were obtained while solving the following problem: what one can say about the norms of unconditionally convergent series terms in concrete Banach spaces. This subject was developed first by W. Orlicz (1930):

Theorem 3 [5]. Let $\sum x_n$ be an unconditionally convergent series in $L_p[0,1]$, $1 \leq p < \infty$, $r = \max\{2, p\}$. Then

$$\sum_n \|x_n\|^r < \infty$$

Investigations were continued by M.I. Kadets (1956):

Theorem 4 [6]. Let X be a uniformly convex normed space and $\delta(t)$ its modulus of convexity. The unconditional convergence of series $\sum x_n$ implies the convergence of series $\sum\limits_n \delta(\|x_n\|)$.

One can observe that in Theorem 4 the modulus of convexity can be replaced by $\delta^c(t)$ - the modulus of complex convexity [7, 8, 9].

Definition 3. A normed linear space X is said to have Orlicz property if there exists a sequence $\{a_k\}$ of positive reals, $\lim\limits_{n\to\infty} a_n = 0$, such that there is no unconditionally convergent series $\sum x_n$ in X with $\|x_n\| = a_n$, $n \in \mathbb{N}$. A normed space X is said to have p-Orlicz property if unconditional convergence of series $\sum x_n$ in X implies convergence of the series $\sum \|x_n\|^p$ (here the terminology is of our own).

Definition 4. A normed space X is said to be uniformly non-$\ell_\infty^{(n)}$ for some integer $n \in \mathbb{N}$ if there exists such $\varepsilon > 0$ that for every sequence $\{x_k\}_{k=1}^n \subset X$, $\|x_k\| \geq 1$, $k = 1, 2, \ldots, n$ the inequality holds:

$$\max_{\pm} \|x_1 \pm x_2 \pm x_3 \pm \ldots \pm x_n\| \geq C(\sum \|x_k\|^p)^{1/p}$$

for each finite set $\{x_k\}_1^n \subset X$.

The following results were obtained by B. Maurey and G. Pisier [10] and independently by S. Rakov [11, 12]:

Theorem 5. The following properties of normed space X are equivalent:

(1) X has the Orlicz property

(2) X is uniformly non-$\ell_\infty^{(n)}$

(3) X has M-cotype p for some p

(4) X has p-Orlicz property for some p

(p is the same in (3) and (4)).

As a continuation of former research one can consider the theory of (p, q)-summing operators and on the other hand the theory of random series in Banach spaces. The most essential part of these theories and a good bibliography can be seen in the book by Pisier [13].

At the end of this chapter we shall formulate some results on weak unconditional convergence in Banach spaces. The first result was obtained in 1929 by W. Orlicz [14], but after it had been discovered anew by B.J. Pettis in 1938 [15], it was often called the Pettis theorem:

Theorem 6. Let X be a Banach space and $\sum x_n$ such a series in X that $\sum \pm x_n$ converges in weak topology for any choice of signs. Then the series is norm unconditionally convergent.

The following result by C. Bessaga and A. Pelczynsky (1958) shows an interesting correspondence between the theory of weak convergence and the structure of a Banach space subspaces:

Theorem 7 [16]. A Banach space X does not contain isomorphic copies of C_0 iff every series absolutely convergent in weak topology of X converges.

We note that a space without c_0-subspaces need not be weakly sequentially complete.

2. The structure of the sum-set

Definition 5. A point x is said to be a sum of series $\sum x_n$ (permitting permutations) if there exists such a permutation $\pi : \mathbb{N} \to \mathbb{N}$ that $\sum\limits_{n=1}^{\infty} x_{\pi(n)} = x$. The set of all such sums of series $\sum x_n$ permutations is called the *sum-set* of the series $\sum x_n$ and denoted by $SS(\sum x_n)$ (in Russian: $OC(\sum x_n)$).

Definition 6. A set A in a linear space is called linear if with each two distinct points x and y it contains the whole line $tx + (1-t)y$, $t \in \mathbb{R}$.

The Steinitz theorem says that for every series $\sum x_n$ in finite dimensional linear space $SS(\sum x_n)$ is a linear set. In the problem 106 of the Scottish Book, S. Banach asked: does the Steinitz theorem hold in infinite dimensional normed spaces? A simple and elegant counter-example in $L_2[0,1]$ to this conjecture was given by I. Marcinkiewicz. The main idea of his construction (see [1]) is that a series of $\mathbb{Z}$-valued functions can not converge in strong L_2 topology to 1/2.

The following theorem of E.M. Nikishin [17] answers another question of Banach concerning a series of measurable functions: is the set-sum of such a series (in the sense of almost everywhere convergence) a linear space?

Theorem 8. There exists a series $\sum f_n(t)$ of measurable functions on the interval $[-1,1]$ such that:

(1) the series converges uniformly to some $V(t)$,

(2) there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ for which the series $\sum f_{\pi(n)}(t)$ converges uniformly to $U(t) \neq V(t)$, and

(3) there is no permutation σ for which the series $\sum f_{\sigma(n)}(t)$ converges almost everywhere to $\frac{U(t)+V(t)}{2}$.

Another interesting infinite-dimensional effect is demonstrated by

Theorem 9. In every infinite-dimensional Banach space there exists a series $\sum x_n$ which does not converge unconditionally, but $SS(\sum x_n)$ consists of exactly one point.

The existence of series with non-closed sum-set in Hilbert space was shown by M.I. Ostrovsky [18]. The series in his example has a non-linear sum-set. So the following problem seems to be open:

Problem 1. Does there exist in every Banach space a series with linear but non-closed sum-set?

In 1987 V.M. Kadets constructed an example [22] of series $\sum x_n$ in Hilbert space equipped by weak topology, for which $SS(\sum x_n)$ is non-linear.

An even more pathological example: a series $\sum x_n$ in Hilbert space with $SS(\sum x_n)$ consisting of exactly two points, was constructed (without proof) some years ago by M.I. Kadets, but the proof of its correctness were given independently by K. Wozniakowski and P.A. Kornilov only in fall, 1987 (to appear). I. Halperin has kindly informed us that since then an example with two-point sum-set was obtained by P. Enflo. The two-point example can be easily generalized [19] to an n-point example, $n \in \mathbb{N}$.

Problem 2. Does there exist for any subset A of a separable Hilbert space H a series $\sum x_n$ in H such that $SS(\sum x_n) = A$? The two most interesting cases are:

(1) A is finite;

(2) $A = H \setminus \{0\}$.

The technique originated by V.M. Kadets [20, 21, 22] allows the transfer of examples of pathological sum-sets in Hilbert space to any infinite dimensional Banach space. In particular,

Theorem 10. In every infinite dimensional Banach space there exists

(1) two-point [24] and n-points [19] sum-sets;

(2) non-closed sum-sets [22];

(3) non-linear weak sum-sets [22].

The following is a hybrid of examples by E.M. Nikishin and M.I. Kadets-K. Wozniakowski-P.A. Kornilov:

Theorem 11 [23]. There exist a series of continuous functions $\sum f_n(t)$ on the interval $[0, 1]$ which

(1) converges uniformly to some $U(t)$,

(2) after some permutation of terms it converges to $V(t) \neq U(t)$, and

(3) if $W(t)$ is distinct from $U(t)$ and from $V(t)$ then the series $\sum\limits_n f_{\pi(n)}(t)$ cannot converge to $W(t)$ even in the sense of almost everywhere convergence.

The only known non-trivial example of a linear-topological space in which the Steinitz theorem holds is the space s: all real sequences with point-wise convergence [25], [38], [1].

In this connection a most interesting unsolved question in the series permutation theory, it seems to us, is the following:

Problem 3. Does the Steinitz theorem hold in metrizable locally convex nuclear spaces?

An important direction of research is to obtain sufficient conditions (weaker than absolute convergence) for linearity of the sum-set. The first result in this direction is the M.I. Kadets theorem (1954):

Theorem 12 [26]. Suppose $1 < p < \infty$, $r = \min\{2, p\}$, $\sum x_n$ a series in L_p and $\sum \|x_n\|^r < \infty$. Then $SS(\sum x_n)$ is a linear set.

S. Troyanski (1967) has generalized this result to the unformly smooth spaces:

Theorem 13 [25]. Let X be a uniformly smooth Banach space and $\rho(t)$ its modulus of smoothness. Then the condition $\sum_1^\infty \rho(\|x_k\|) < \infty$ implies that the series $\sum x_k$ in X has a linear sum-set.

Now we will introduce two notions analogous to M-cotype and uniform non-$\ell_\infty^{(n)}$-ness:

Definition 7. A normed linear space X is said to be uniformly non-$\ell_1^{(n)}$ or B-convex, if there exists an integer $n \in \mathbb{N}$ and a real $\varepsilon > 0$ such that inequality

$$\min_{\pm} \|\sum_{i=1}^{n} \pm x_i\| \leq n - \varepsilon$$

holds for each $\{x_i\}_1^n \subset X$, $\max \|x_i\| \leq 1$.

A normed linear space X is said to have infratype $p > 1$ if there exists a constant $C > 0$ such that the inequality

$$\min_{\pm} \|\sum_{i=1}^{n} \pm x_i\| \leq C(\sum_i \|x_i\|^p)^{1/p}$$

holds for each finite set $\{x_i\}_1^n \subset X$.

Pisier Theorem [28]. A normed space X is B-convex iff it has an infratype $p > 1$.

The widest class of spaces for which Theorem 12 and the like hold is the class of uniformly non-$\ell_1^{(n)}$ (B-convex) spaces:

Theorem 14. (V.M. Kadets, 1984, [29, p. 136-158]). If a Banach space X has an infratype $p > 1$ and a series $\sum x_n$ in X is p-absolutely summable (i.e., $\sum \|x_n\|^p < \infty$), then $SS(\sum x_n) = \infty$ is a linear set. Conversely, if a space X is not uniformly non-$\ell_1^{(n)}$, then for each sequence $a_n \downarrow 0$ of positive numbers, $\sum a_n = \infty$, there exists a series $\sum x_n$ in X with non-linear $SS(\sum x_n)$ and $\|x_n\| \le a_n, n \in \mathbb{N}$.

In all Theorems 12-14 not only linearity but closedness of a sum-set holds too.

Another type of sufficient conditions was investigated by E.M. Nikishin (1970) in the case of functional series.

Theorem 15 [30]. Let $\sum f_n(t)$, $t \in [0,1]$ be a series of measurable functions with $\sum f_n^2(t) < \infty$ almost everywhere. Then $SS(\sum f_n)$, considered in the sense of almost everywhere convergence, is a linear set.

More precise characterization of the sum-set structure in terms of the "functionals of absolute convergence" can be found in a number of publications beginning with the original Steinitz work; and including a contemporary M.I. Ostrovsky paper: [31, 27, 18]. The exactness of the Theorems 12-15 was investigated in papers [17, 21, 32].

The next results are connected with the notion of unconditionally divergent series.

Definition 8. A series $\sum x_n$ is said to be unconditionally divergent if for each rearrangement of signs the series $\sum \pm x_n$ diverges.

The characterization of unconditionally divergent series in finite-dimensional spaces was given by A. Dvoretzky and C. Hanani in 1947:

Theorem 16 [33]. A series $\sum x_i$ in finite-dimensional space is unconditionally divergent iff $\lim\limits_{i\to\infty} \|x_i\| \neq 0$.

We recall an important J. Lindenstrauss result (1963):

Theorem 17 [34]. Let X be a uniformly smooth Banach space, $\rho(t)$ its modulus of smoothness and $\sum x_n$ an unconditionally divergent series in X. Then $\sum \rho(\|x_n\|) = \infty$.

It is easy to prove that

Theorem 18. Suppose X is a Banach space of type p and $\sum x_n$ is an unconditionally divergent series in X. Then $\sum \|x_n\|^p = \infty$. Conversely, if there exists such a sequence $\varepsilon_n \geq 0$, $\sum \varepsilon_n = \infty$, that there is no unconditionally divergent series $\sum x_n$ in X with $\|x_n\| \leq \varepsilon_n$, $n \in \mathbb{N}$, then X is B-convex and hence X has a type $p > 1$.

(Definition of type and its main properties one can find in [35].)

Problem 4. Suppose X is a Banach space of infratype $p > 1$ and $\sum x_n$ is an unconditionally divergent series in X. Is it true that under above assumptions $\sum \|x_n\|^p = \infty$?

Problem 5. Does the Dvoretzky-Hanani theorem (Theorem 16) hold in metrizable locally convex complete nuclear spaces?

A very important correspondence between unconditional divergence and the sum-set structure was obtained by D.V. Pechersky in 1988:

Theorem 19 [36]. Let $\sum x_n$ be a series in a Banach space such that for every permutation $\pi : \mathbb{N} \to \mathbb{N}$ the series $\sum x_{\pi(n)}$ is not unconditionally divergent. Then $SS(\sum x_n)$ is a closed linear set.

As a consequence the famous Chobanyan theorem (1984) follows:

Theorem 20 [37]. Let $\sum x_n$ be a series in a Banach space such that $\sum \pm x_n$ converges for almost all choices of signs. Then $SS(\sum x_n)$ is a closed linear set.

The last results in this review concern the notion of limit points of a series:

Definition 9. A point x is said to be a limit point of a series $\sum x_n$ if there exists a sequence of integers $n_1 < n_2 < n_3 \dots$ for which $x = \lim\limits_{k\to\infty} \sum\limits_{i=1}^{n_k} x_i$. For a fixed series $\sum x_n$ the set of all limit points of all permuted series $\sum x_{\pi(n)}$ is called the limit point-set of the series $\sum x_n$ and is denoted $LPS(\sum x_n)$ (in Russian: ОПТ ($\sum x_n$)).

The classic result (known in fact to Steinitz) is the following:

Theorem 21. Suppose that $\sum x_n$ is a series in a linear-topological space $X, x \in LPS(\sum x_n)$. Then the set $\{y - x : y \in LPS(\sum x_n)\}$ forms an additive subgroup in X.

The last result concerning LPS was obtained in 1987 by W. Banaszczyk (to appear):

Theorem 22. Let $\sum x_n$ be a series in a metrizable locally convex nuclear space, $\lim\limits_{n\to\infty} x_n = 0$. Then $LPS(\sum x_n)$ is a linear set.

It is easy to see (and probably it was obtained by somebody prior to me) that

Theorem 23. If $LPS(\sum x_n)$ for some series $\sum x_n$ in a Banach space X contains a subset of a form $X_1 + x$ where $x \in X$ and X_1 is a subspace of finite codimension, then $LPS(\sum x_n) = X_2 + x$, where $X_2 \supset X_1$ is a subspace.

Problem 6. Can every closed subset of a Banach space, which satisfies the conditions of Theorems 21 and 23 be the limit point-set for some series $\sum x_n$?

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FORD HYPERSPHERES : A GENERAL APPROACH

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ABSTRACT

An old theorem by Descartes is basic in the geometrical generalizations of Ford circles and of $PSL_2(\mathbb{Z})$ that we give here.

Particular cases will be detailed in *"quaternionic homographies"* and proofs will appear somewhere else.

1) PREREQUISITES

In this paper we are concerned with applications of inversive geometry to an arithmetical situation arising from Diophantine Approximation (Farey series, Ford circles) and concerning the model $\Pi = \mathbb{R}^n \cup \{\infty\}$ (see [1] and [2]).

This situation is best described in the setting of Wilker ([2]).

In this setting a point $u \in \Pi$ is represented by a ray :

$$[x] = \{\lambda x \; ; \; \lambda > 0\} \subset \mathbb{R}^{n+2}$$

in which

$$x = (2u, \|u\|^2 - 1, \|u\|^2 + 1)$$

Following Wilker we introduce a Lorentz bilinear form on $\mathbb{R}^{n+2}$:

$$x \bullet y = x_1 y_1 + \ldots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}.$$

Then a ray $[x] \subset \mathbb{R}^{n+2}$ names a point of Π iff :

$$x \bullet x = 0, \qquad p(x) > 0$$

where $p(x) := x_{n+2}$.

The image of an inversive ball of Π is the set of rays $[x]$ such that $x \bullet y \geq 0$ for a certain $y \in \mathbb{R}^{n+2}$ such that $y \bullet y = 1$.

The image of the sphere limiting this ball is :

$$\{[x] \; ; \; x \bullet y = 0\}.$$

In this context, Möbius transformations are just the linear transformations of $\mathbb{R}^{n+2}$ which preserve $\bullet$ and the positive cone :

$$\{x \in \mathbb{R}^{n+2} \; ; \; x \bullet x = 0, \; p(x) > 0\}.$$

If $\varphi \in \mathfrak{L}^{\uparrow}_{n+2}$ (the group of Möbius transformations of Π) the norm of φ [7] is defined by :

$$\|\varphi\| = \left[\mathrm{trace}\,(M.{}^tM)\right]^{1/2}$$

where M is the matrix of φ in the canonical basis $e_1, \ldots, e_{n+2}$ of $\mathbb{R}^{n+2}$.

$\Sigma^{\uparrow}_{n+2}$ is sharply transitive on (cartesian) clusters in $\mathbb{R}^{n+2}$, where "cluster" means an ordered set of n+2 points C_i of $\mathbb{R}^{n+2}$ such that :

$$C_i * C_j = (-1)^{\delta_{ij}} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

The vector $C \in \mathbb{R}^{n+2}$ attached to the Euclidean ball $\| u - \vec{a} \| \leq r = \varepsilon^{-1}$; is :

$$C = \left(\varepsilon\vec{a}, \frac{\varepsilon\|a\|^2 - \varepsilon - \varepsilon^{-1}}{2}, \frac{\varepsilon\|a\|^2 + \varepsilon - \varepsilon^{-1}}{2}\right) \tag{1}$$

and its bend $\varepsilon = r^{-1}$ is given by :

$$\varepsilon = E * C$$

with $E = (0,\ldots,0,-1,-1) \in \mathbb{R}^{n+2}$.

A cluster can be interpreted as n+2 externally tangent balls in Π. Given n+1 externally tangent balls $C_1,\ldots,C_{n+1}$ a necessary and sufficient condition for C_{n+2} to be externally tangent to the previous ones is "Descartes' condition" :

$$(\mathfrak{D}) : \quad 2nE + \left[E * (C_1 + \ldots + C_{n+2})\right](C_1 + \ldots + C_{n+2}) = n\left[(E * C_1)C_1 + \ldots + (E * C_{n+2})C_{n+2}\right]$$

Remark : equation $(\mathfrak{D})$ is impossible in C_{n+2} when :

$$(\varepsilon_1 + \ldots + \varepsilon_{n+1})^2 \leq (n-1)(\varepsilon_1^2 + \ldots + \varepsilon_{n+1}^2).$$

2. HOROSPHERIC CLUSTERS IN POINCARE'S HALFSPACE

We now change n to n+1 and consider Poincaré's half space ([1]) :

$$H^{n+1} = \{u \in \mathbb{R}^{n+1},\ u_{n+1} > 0\}.$$

In this context Π will be the hyperplane limiting H^{n+1} completed by ∞.

We are interested in the clusters in $\mathbb{R}^{n+1}$ which are contained in $H^{n+1} \cup \Pi$ and in which every ball is horospheric (i.e. tangent to Π). Representing the cluster $(\Pi, C_1,\ldots,C_{n+2})$ in inversive coordinates, Descartes' relation $(\mathfrak{D})$ gives :

$$(\mathfrak{D}') \quad 2(n+1)E + \left[E * (C_1 + \ldots + C_{n+2})\right](\Pi + C_1 + \ldots + C_{n+2}) = (n+1)\left[(E * C_1)C_1 + \ldots + (E * C_{n+2})C_{n+2}\right]$$

with $\Pi = (0,\ldots,-1,0,0)$.

Π being always present in all our clusters we will omit it systematically and write $(C_1,\ldots,C_{n+2})$ for a generic horospheric cluster.

Basic horospheric clusters

They are the clusters in which the first horosphere is the Euclidean hyperplane $P = \{x \in \mathbb{R}^{n+1} \,;\, x_{n+1} = 1\}$ which is represented by $P = (\vec{e}_{n+1}, 1, 1)$ in Wilker's coordinates.

For such a cluster we have $(e_2, \dots, e_{n+2}) = (2, \dots, 2)$ and the (Euclidean) centres of the horospheres $C_2, \dots, C_{n+2}$ form a regular simplex S.

Horospheric clusters posess a generating property which is already used in [5] and appears as a particular case in Wilker's relation p. 162 in [3].

THEOREM 1.- Each horospheric cluster $(C_1, \dots, C_{n+2})$ gives birth to $n+2$ new horospheric clusters $(C_1, \dots, C_i', \dots C_{n+2})$, with $i = 1, \dots, n+2$, where C_i' is determined by the equation :

$$C_i + C_i' = \frac{2}{n}(\Pi + C_1 + \dots + \overset{i}{\vee} + \dots + C_{n+2}) \tag{2}$$

Remark : When we apply (2) to a basic horospheric cluster $(P, C_2, \dots, C_{n+2})$ with $i \neq 1$, we have :

$$\frac{1}{2}(\vec{a}_i + \vec{a}_i') = \frac{1}{n}(\vec{a}_2 + \dots + \overset{i}{\vee} + \dots \vec{a}_{n+2}) \tag{3}$$

so $\vec{a}_i \longmapsto \vec{a}_i'$ is a reflexion in the i^{th}- face of the simplex S.

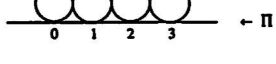
3) FORD'S CONFIGURATION

Starting with a basic cluster, putting P in the first place and taking $i = 2, \dots, n+2$, the generating process of theorem 1 gives us $n+1$ new basic cartesian sets, going on indefinitly we obtain Ford's initial configuration $\mathcal{F}_1$.

Apart from P, $\mathcal{F}_1$ contains only spheres of bend 2. The points where the latter are touching P form a special set L in $\mathbb{R}^n$ which can be constructed by repeated applications of (2). L is not always a lattice but we will suppose that $O \in L$ and we will denote by Λ the group generated by L : if $n > 2$, L is not discrete and Λ is not finitely generated, but if $n \leq 2$, L is a lattice.

Example for n=1 :

$\mathcal{F}_1$



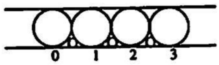
Remark : When $n \geq 3$, two spheres of $\mathcal{F}_1$ can intersect each other.

Taking $i = 1$ and applying theorem 1 to the clusters of $\mathcal{F}_1$ we get, by the same generating process, the first derived configuration $\mathcal{F}_2$.

$\mathcal{F}_2$ contains $\mathcal{F}_1$ and new spheres of bend $\dfrac{2(n+1)}{n}$.

Example for n=1 :

$\mathcal{F}_2$



Taking $i = 1 \dots n+1$ and applying theorem 1 to the clusters of $\mathcal{F}_2$ in hereditary order we get the second derived configuration $\mathcal{F}_3$, which, by definition, certains $\mathcal{F}_2$ and the new spheres.

DEFINITIONS and NOTATIONS

$\mathcal{F}_v$ will denote the v^{th} derived configuration and $\mathcal{F} = \bigcup_v \mathcal{F}_v$ will be called the Ford configuration of $\mathbb{R}^n$ (see [4] for $n = 1$).

$\mathcal{E}_v$ will denote the bends of the spheres of $\mathcal{F}_v$, and $\mathcal{E}$ will be $\bigcup_v \mathcal{E}_v$.

Θ_v will denote the set of points where the spheres of $\mathcal{F}_v$ touch Π and Θ will be $\bigcup_v \Theta_v$.

Remark : This construction seems to lie midway between Coxeter's sequences of successively tangent spheres [5] and Boyd's osculatory packings [6].
But it is not a "packing" at all if $n \geq 3$.

THEOREM 2.-

1) In a basis of $\mathbb{R}^{n+3}$ constituted by the images of a basic cluster (without omitting Π), the coordinates of any Ford sphere lie in $\mathbb{Z}[\frac{2}{n}]$.

2) $\mathcal{E} \subset 2\mathbb{Z}[\frac{2}{n}]$

3) $\Theta \subset \{\infty\} \cup (\mathbb{Q} \otimes \Lambda)$

4) If C_1 and $C_2 \in \mathcal{F}$, then $C_1 \bullet C_2 + 1 \in \mathcal{E}$.

5) If $C_1 \bullet C_2 + 1 \notin \mathcal{E}_v$, then C_1 and C_2 cannot be linked by a Coxeter sequence [5] of length $\leq v$.

6) If C_1 and $C_2 \in \mathcal{F}$ are touching each other at $\vec{a} + \lambda \vec{c}_{n+1} \in H^{n+1}$, then $\vec{a} \in \mathbb{Q} \otimes \Lambda$ and $\lambda \in \mathbb{Q}$.

4) THE GROUP OF THE FORD CONFIGURATION

DEFINITION 3.- $GM(\mathcal{F})$ (resp. $M(\mathcal{F})$) will denote the group of Möbius transformations (resp. orientation preserving Möbius transformations) which preserve $\mathcal{F}$ ([1] p. 22).

Example :

If $(C_1,\ldots,C_{n+2})$ is a cluster, the reflexion σ_D in the sphere D named by the ray $[\Pi + C_1 + \ldots + \overset{i}{\vee} + \ldots + C_{n+2} - n\,C_i]$ is in $GM(\mathcal{F})$. It sends C_i to C_i' and preserves C_j for $j \neq i$.

THEOREM 3.-

1) $GM(\mathcal{F})$ is made up of the isometries $\mathcal{I}s(L)$ of L and of the products $\sigma_D\,\varphi$ where D is as above and $\varphi \in \mathcal{I}s(L)$.

2) $M(\mathcal{F})$ is made up of $\mathcal{I}s^+(L)$ and of the products $\sigma_D\psi$, where D is as above and $\psi \in \mathcal{I}s^-(L)$.

3) If e_{n+1} denotes the unit vector of H^{n+1} orthogonal to P, then for all φ, $\varphi' \in GM(\mathcal{F})$, $\varphi(e_{n+1}) = \varphi'(e_{n+1})$ means that $\varphi' = \varphi\tau$ with an orthogonal τ preserving L and e_{n+1}.

COROLLARY .-

Θ is the orbit of ∞ under the action of $M(\mathcal{F})$ and any $\varphi \in GM(\mathcal{F})$ is determined by its action on Θ.

Remark :

This corollary makes it clear that $GM(\mathcal{F})$ is entirely determined by its action on Π. In fact it is composed of Poincaré's extensions of elements of $GM(\Pi)$.

Now denote by x_0 the vector $(0,\ldots,0,1) \in H^{n+1}$ and consider :

$$\rho(\varphi) := \text{hyperbolic distance } (e_{n+1}, \rho(e_{n+1}))$$

It is clear that we have :

$$\rho(\varphi\varphi') \leq \rho(\varphi) + \rho(\varphi')$$

$$\rho(\varphi) = 0 \Leftrightarrow \varphi \in \mathcal{I}s(L)$$

but we also have ([7] theorem 2 p. 306) :

$$\|\varphi\|^2 = \|I\|^2 + 4\sinh^2\rho(\varphi). \tag{4}$$

The vector z_0 of [7] p. 306 is here e_{n+3}, and if we choose now, as in theorem 2, the images of the basic cluster $(\Pi, P, C_2, \ldots, C_{n+2})$ as a basis for R^{n+3} we have :

$$2e_{n+3} = C_2 + 2\Pi + P = C_2 + \Pi - E \tag{5}$$

where C_2 denotes the Ford sphere touching Π at the origin 0.

Then the relations (4) and (5), theorem 2 and lemma 1 of [7] give the following result :

Theorem 4.- For any $\varphi \in GM(\mathcal{F})$ we have :

1) $$\|\varphi\|^2 \in \mathbb{Z}\left[\frac{2}{n}\right].$$

2) $$\cosh \rho(\varphi) \in \mathbb{Z}\left[\frac{2}{n}\right].$$

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QUATERNIONIC HOMOGRAPHIES : APPLICATION TO FORD HYPERSPHERES

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Algebraic tools (complex numbers and Hamilton quaternions) are used here to describe the geometrical objects of [1].

Ford's classical results in dimension 1 as well as Rieger's results in dimension 2 are recovered. In dimension 4 new results are obtained : they illustrate the usefulness of $GL_2(\mathbb{H})$ and $SL_2(\mathbb{H})$ in the study of Möbius transformations of the Poincaré space H^5.

NOTATIONS.- In general we will use the notations of [1], in particular n will be the dimension of the hyperspheres considered.

When we consider $\mathbb{C}$ or $\mathbb{H}$, $z \to \bar{z}$ denotes the basic involution and $|z|$ is $\sqrt{z\bar{z}}$.

1) QUATERNIONIC HOMOGRAPHIES

We will denote by $GL_2(\mathbb{H})$ the group of regular matrices with entries in $\mathbb{H}$ and by $SL_2(\mathbb{H})$ the subgroup of those matrices with unitary Dieudonné determinant. If $M \in GL_2(\mathbb{H})$ its Dieudonné determinant has an absolute value which is well defined and will be denoted by $|\det M|$

DEFINITION 1.-

For any $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{H})$, we define φ_M et $\psi_M : \hat{\mathbb{H}} \longrightarrow \hat{\mathbb{H}}$, where $\hat{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$, by :

$$\varphi_M(z) = (\alpha z + \beta)(\gamma z + \beta)^{-1}, \qquad \psi_M(z) = (\alpha \bar{z} + \beta)(\gamma \bar{z} + \delta)^{-1}$$

THEOREM 1.-

1) φ is an epimorphism of $GL_2(\mathbb{H})$ on the group $M(\hat{\mathbb{H}})$ of direct Möbius transformations of $\hat{\mathbb{H}}$

2) The Poincaré extension $\tilde{\varphi}_M$ of φ_M can be expressed by :

$$\tilde{\varphi}_M(z+te_5) = \frac{(\alpha z+\beta)\overline{(\gamma z+\delta)} + \alpha\bar{\gamma}t^2 + |\det M| te_5}{|\gamma z+\delta|^2 + |\gamma|^2 t^2}$$

COROLLARY 1.- For all $M \in GL_2(\mathbb{H})$, φ_M preserves cross-ratios in the sense of [4].

Remark : If we define $[z_1,z_2,z_3,z_4]$ by :

$$[z_1,z_2,z_3,z_4] = [(z_1-z_3)(z_2-z_3)^{-1}]\,[(z_1-z_4)(z_2-z_4)^{-1}]^{-1}$$

then one can see that the conjugacy class of $[z_1,z_2,z_3,z_4]$ is invariant by φ_M.

COROLLARY 2.- If $M \in SL_2(\mathbb{H})$, the image of $P := e_5 + \mathbb{H}$ by φ_M is the Euclidean sphere :

$$\Sigma\left(\alpha\gamma^{-1} + \frac{e_5}{2|\gamma|^2}, \frac{1}{2|\gamma|^2}\right).$$

COROLLARY 3.- Let :

$$\begin{cases} M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{H}), & \Sigma := \varphi_M(P) \\ M' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in SL_2(\mathbb{H}), & \Sigma' := \varphi_{M'}(P) \end{cases}$$

then the inversive product of Σ and Σ' is :

$$\Sigma * \Sigma' = 1 - 2\,\frac{\left|\,|\gamma'|^2 \alpha\bar{\gamma} - |\gamma|^2 \alpha'\bar{\gamma}'\right|}{|\gamma|^2\,|\gamma'|^2}$$

LEMMA 1.-

Let $M \in SL_2(\mathbb{H})$ and let $\tilde{\varphi}_M$ denote the Poincaré extension of φ_M, then we have :

$$2\cosh \rho(e_5, \tilde{\varphi}_M(e_5)) = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = \|M\|^2.$$

Taking in account [5] we can generalize theorem 10 of [6].

THEOREM 2.- The mapping $M \longrightarrow \varphi_M$ induces a homomorphism of $SL_2(\mathbb{H})$ into the proper orthochroneous Lorentz group $\mathfrak{L}_6^{\uparrow +}$. In the standard inversive coordinates the matrix of φ is the transpose of :

$$\begin{bmatrix} R(\alpha\bar\delta+\beta\bar\gamma) & I(\alpha\bar\delta+\beta\bar\gamma) & J(\alpha\bar\delta+\beta\bar\gamma) & K(\alpha\bar\delta+\beta\bar\gamma) & R(\alpha\bar\beta-\gamma\bar\delta) & R(\alpha\bar\beta+\gamma\bar\delta) \\ R(\alpha i\bar\delta-\beta i\bar\gamma) & I(\alpha i\bar\delta-\beta i\bar\gamma) & J(\alpha i\bar\delta-\beta i\bar\gamma) & K(\alpha i\bar\delta-\beta i\bar\gamma) & I(\bar\alpha\beta-\bar\gamma\delta) & I(\bar\alpha\beta+\bar\gamma\delta) \\ R(\alpha j\bar\delta-\beta j\bar\gamma) & I(\alpha j\bar\delta-\beta j\bar\gamma) & J(\alpha j\bar\delta-\beta j\bar\gamma) & K(\alpha j\bar\delta-\beta j\bar\gamma) & I(\bar\alpha\beta-\bar\gamma\delta) & J(\bar\alpha\beta+\bar\gamma\delta) \\ R(\alpha k\bar\delta-\beta k\bar\gamma) & I(\alpha k\bar\delta-\beta k\bar\gamma) & J(\alpha k\bar\delta-\beta k\bar\gamma) & K(\alpha k\bar\delta-\beta k\bar\gamma) & K(\bar\alpha\beta-\bar\gamma\delta) & K(\bar\alpha\beta+\bar\gamma\delta) \\ R(\alpha\bar\gamma-\beta\bar\delta) & I(\alpha\bar\gamma-\beta\bar\delta) & J(\alpha\bar\gamma-\beta\bar\delta) & K(\alpha\bar\gamma-\beta\bar\delta) & \frac{1}{2}(|\alpha|^2-|\beta|^2-|\gamma|^2+|\delta|^2) & \frac{1}{2}(|\alpha|^2-|\beta|^2+|\gamma|^2-|\delta|^2) \\ R(\alpha\bar\gamma+\beta\bar\delta) & I(\alpha\bar\gamma+\beta\bar\delta) & J(\alpha\bar\gamma+\beta\bar\delta) & K(\alpha\bar\gamma+\beta\bar\delta) & \frac{1}{2}(|\alpha|^2+|\beta|^2-|\gamma|^2-|\delta|^2) & \frac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2) \end{bmatrix}$$

where R, I, J, K denote the standard coordinates in $\mathbb{H}$.

Remark : The entries in this matrix have been corrected by J.B. Wilker who calculated them independently.

COROLLARY 4.- If $M \in SL_2(\mathbb{H})$, the norm of the application φ_M in the sense of [5] § 4 is given by :

$$\|\varphi_M\|^2 = 2 + \|M\|^4.$$

2) CONTINUED FRACTION ALGORITHM

In this section we will use the following notations :

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

E being any set in $\mathbb{H}$ we consider the matrices :

$$M = S_{\alpha_0} T S_{\alpha_1} \dots S_{\alpha_\nu} T = \begin{pmatrix} p_\nu & p_{\nu-1} \\ q_\nu & q_{\nu-1} \end{pmatrix}$$

with $\alpha_0, \dots, \alpha_\nu \in E$.

Clearly $M \in SL_2(\mathbb{H})$ and those matrices make up a multiplicative monoid. We will write :

$$[\alpha_0, \dots, \alpha_\nu, z] = \varphi_M(z)$$

so that :

p_ν and q_ν being defined by the usual formulas :

$$(1)\qquad \begin{aligned} &(p_{-1},q_{-1}) = (1,0), \qquad (p_o,q_o) = (\alpha_o,1) \\ &(p_\nu,q_\nu) = (p_{\nu-1}\alpha_\nu+p_{\nu-2},\ q_{\nu-1}\alpha_\nu+q_{\nu-2}) \\ &\alpha_o,\dots,\alpha_\nu \in E. \end{aligned}$$

We will suppose that $O \in L$ and we will denote by $\Omega(L)$ the group of all orthogonal transformations θ such that $\theta(L)$ is in the smallest subfield of $\mathbb{H}$ containing L

THEOREM 3.- Each Ford hypersphere Σ can be written :

$$\Sigma = [\alpha_o,\alpha_1,\dots,\alpha_\nu](P)$$

with $\alpha_o \in L$ and $\alpha_i = \theta_i(\lambda_i)-\theta_{i-1}(\lambda_{i-1})$ with $\lambda_i \in L$ and $\theta_i \in \Omega(L)$ for $1 \le i \le \nu$.

COROLLARY 5.- The Ford hyperspheres can be written :

$$\Sigma\Big(p_\nu q_\nu^{-1} + \frac{e_5}{2\,|q_\nu|^2},\ \frac{1}{2\,|q_\nu|^2}\Big)$$

with p_ν and q_ν as in (1) with $\alpha_o,\dots,\alpha_\nu$ as in th. 3.

COROLLARY 6.- $\mathcal{E} \subset \{2\,|q_\nu|^2\,;\ q_\nu$ as in Corollary 1$\}$.

3) DESCRIPTION OF THE DIFFERENT CASES

3,1) If $n=4$, we can choose the axes in such a way that the vertices of the regular simplex where the spheres of a basic molecule touch Π are :

$$0,\ \rho = -\frac{1}{2}+i\frac{\sqrt{3}}{2},\ -\rho^2,\ \sigma = \frac{i}{\sqrt{3}}+\sqrt{\frac{2}{3}}\,j,\ \tau = \frac{i}{\sqrt{3}}+\frac{j}{2\sqrt{6}}+\sqrt{\frac{5}{8}}\,k.$$

The ring A generated by ρ, σ, τ is :

$$\mathbb{Z}[\rho,\sigma,\tau] = \mathbb{Z}+\mathbb{Z}\rho+\mathbb{Z}\sigma+\mathbb{Z}\tau+\mathbb{Z}\rho\sigma+\mathbb{Z}\sigma\tau+\mathbb{Z}\tau\rho+\mathbb{Z}\rho\sigma\tau$$

and all the elements $z \in A$ are integral in the sense that $z+\bar{z} \in \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ and $z\bar{z} \in \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

All the numbers p_v, q_v are in the field of fractions of A.

3,2) If $n=2$, we can choose the axes in such a way that the fundamental simplex is :

$$0, \rho, -\rho^2$$

Then $L = \mathbb{Z}[\rho]$ and, sinc $\mathbb{Z}[\rho]$ is a Euclidean ring, we have :

THEOREM 4.-

1) $M(\mathcal{F}) = PSL_2(\mathbb{Z}[\rho])$.

2) Our Ford spheres are the Rieger spheres :

$$\Sigma(\alpha\gamma^{-1} + \frac{e_3}{2|\gamma|^2}, \frac{1}{2|\gamma|^2})$$

with α and $\gamma \in \mathbb{Z}[\rho]$ relatively prime.

Corollary 3 gives Rieger's condition of tangency for two Ford spheres :

$$|\alpha\gamma'-\alpha'\gamma| = 1.$$

3,3) If $n=1$, we can choose the axes in such a way that the fundamental simplex is : $\{0, 1\}$. Then $L=\mathbb{Z}$ and, since $\mathbb{Z}$ is a Euclidean ring, we have :

THEOREM 5.-

1) $M(\mathcal{F}) = PSL_2(\mathbb{Z})$.

2) Our Ford spheres are the usual Ford circles [2] :

$$\Sigma(\alpha\gamma^{-1} + \frac{e_2}{2\gamma^2}, \frac{1}{2\gamma^2})$$

with α and $\gamma \in \mathbb{Z}$ relatively prime.

Corollary 3 gives a well known condition of tangency for two Ford circles :

$$|\alpha\gamma'-\alpha'\gamma| = 1.$$

Special features in dimension 1.

Theorem 5 suggests to put $\varepsilon_v = 2\,\theta_v{}^2$ in relation (2) of [1], then multiplying by E one gets :

$$(\theta_1+\theta_2+\theta_3)\,(-\theta_1+\theta_2+\theta_3)\,(\theta_1-\theta_2-\theta_3)(\theta_1+\theta_2-\theta_3) = 0$$

so we déduce :

PROPOSITION.- If the bends $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ of the cluster $(\Sigma_1,\Sigma_2,\Sigma_3)$ are such that $0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_3$ we have :

$$\varepsilon_3 = (\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2})^2$$

Example : Applying relation 2 of [1] we get the following sequence of successively tangent circles [7]

$$\begin{tikzcd}
(P,\Sigma_2,\Sigma_3) \arrow[dr, "\text{TH 2 (i=1)}"] & & (\Sigma_2,\Sigma_3,\Sigma_4) \arrow[dr] & & (\Sigma_3,\Sigma_4,\Sigma_5) \\
& (\Sigma_4,\Sigma_2,\Sigma_3) \arrow[ur, "\text{cyclic permutation}"] & & (\Sigma_5,\Sigma_3,\Sigma_4) \arrow[ur] &
\end{tikzcd}$$

Then the θ_V in the first line make up the Fibonacci sequence, applying again relation 2, with $i = 2$, to clusters of this line we get the Lucas sequence.

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Remarks on group actions and induction theorems

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§1: In [Sn] I used group actions on spaces related to representations of a finite group to give a natural canonical form for Brauer's induction theorem [S, p. 78, Theorem 20]. However, the canonical form does not establish the complete form of Brauer's theorem since it does not specify the type of subgroups from which one must induce the linear characters.

In this note I make the remark that one may, indeed, recover the original result of Brauer using only the group actions of [Sn].

§2: In this section let C_n denote the cyclic group of order n, and let $R(C_n)$ denote the complex representation ring C_n. Let V be a virtual representation which is a $\mathbb{Z}$-linear combination of permutation representations

$$(2.1) \qquad V = \sum_{d|n} a_d \mathrm{Ind}_{C_d}^{C_n}(1) \in R(C_n).$$

Here V in (2.1) is the image of the element $\Sigma\, a_d[C_n/C_d]$ under the canonical map from the Burnside ring

$$(2.2) \qquad \begin{cases} b\colon A(C_n) \longrightarrow R(C_n) \\ b[(C_n/H]) = \mathrm{Ind}_H^{C_n}(1) \end{cases}.$$

The map, b, in (2.2) is injective. This is easy to see by use of the Mobius inversion formula. Let ϕ_e be any one-dimensional representation of the form

$$(2.3) \qquad \phi_e\colon C_n \twoheadrightarrow C_n/C_e \rightarrowtail \mathbb{C}^*.$$

If $\dim_{\mathbb{C}}\mathrm{Hom}_{C_n}(\phi_e, V) = F(e)$ then the Schur inner product satisfies

$$<\phi_e, \mathrm{Ind}_{C_d}^{C_n}(1)>_{C_n} = <\phi_e, 1>_{C_d} = \begin{cases} 1 & \text{if d divides e,} \\ 0 & \text{if not.} \end{cases}$$

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Hence

(2.4)
$$\begin{cases} F(e) = \sum_{d|e} a_d & \text{and} \\ a_d = \sum_{m|d} \mu(m)F(d/m) \end{cases}$$

where μ is the Mobius function [H, pp. 37/38].

2.5: Example

Let W be a compact manifold upon which C_n acts. By virtue of classical triangulation theorem we may assume that W is a finite simplicial complex and that C_n acts cellularly by permuting the simplices. Let $(C_*(W;\mathbb{C}),d)$ denote the resulting simplicial chain complex [Sp]. Set V equal to the Euler characteristic

(2.6)
$$V = \sum_i (-1)^i C_i(W;\mathbb{C}) = \Sigma(-1)^i H^i(W;\mathbb{C}) \in R(C_n)$$

where $H^i(W;\mathbb{C})$ denotes the i–th simplicial homology group, $H_i(W;\mathbb{C}) = (\ker (C_i(W) \xrightarrow{d} C_{i-1}(W)))/(\mathrm{im}(d_i C_{i+1}(W) \to C_i(W)))$.

We will require the following simple consequence of (2.4) for the injectivity of (2.2)) for this example.

2.7: Proposition

Let W be a compact manifold with a C_n–action. If $\chi(H_*(W;\mathbb{C})) = 1 \in R(C_n)$ then the Euler characteristic of the fixed–point set, W^{C_n}, satisfies $\chi(H_*(W^{C_n}); \mathbb{C}) = 1 \in \mathbb{Z}$.

2.8: Corollary

Let p be a prime and let G be an extension of the form $C_n \rightarrowtail G \twoheadrightarrow P$ where P is finite p–group and $(n,p) = 1$. Then, if G acts on W of §2.7 then W^G is non–empty.

Proof: P acts on $W^{C_n} = Y$ in a cellular manner so that (see [Sn])

$$\Sigma(-1)^i C_i(Y;\mathbb{C}) = \Sigma\alpha_H \mathrm{Ind}_H^P(1) \in R(P).$$

Therefore

$$\begin{aligned} 1 &= \Sigma(-1)^i \dim_{\mathbb{C}} H_i(Y;\mathbb{C}), \text{ by §2.7,} \\ &= \Sigma(-1)^i \dim_{\mathbb{C}} C_i(Y;\mathbb{C}) \equiv \alpha_G \pmod p. \end{aligned}$$

Hence α_G is non–zero which implies that Y^P, and hence $X^G = Y^P$, is non–empty, as required.

§3: Applications to induction theorems

In this section we are going to give, proofs of three well-known induction theorems. The only novelty of our proofs consists in the group action which we use repeatedly to avoid steps which are usually accomplished algebraically.

We will study representations over the classical fields, $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If ν is a representation on a K–vector space of dimension n then we may assume that ν lands in the compact subgroup of GL_nK given by $O(n)$, $U(n)$ or $Sp(n)$ – the orthogonal, unitary or symplectic group, respectively. When $K = \mathbb{R}$ we will assume that n is even and we will set $X_n = O(2n)$, $U(n)$ or $Sp(n)$ when $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively. The group X_1^n sits inside X_n as the subgroup of "diagonal matrices". The normaliser of X_1^n in X_n is $Y_n = \Sigma_n \int X_1$, the wreath product generated by the permutation matrices, Σ_n, permuting the factors of X_1^n by conjugation.

Suppose now that

$$\nu: G \to X_n \tag{3.1}$$

is a K–representation (for $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). Set $W = X_n/Y_n$ with G acting, via ν, by left translation. Assume, henceforth, that this action is triangulated, as in §2.5.

3.2: Lemma

If H fixes gY_n then

(i) $\mathrm{Res}_H^G(g^{-1}\nu g)$ (and hence ν) is a sum of induced representations of the form $\mathrm{Ind}_J^H(\phi: J \to X_1)$.

(ii) If ν is not the sum of induced representations of the form $\mathrm{Ind}_L^G(\phi: L \to X_1)$ then

$$1 = \sum_{H \underset{\neq}{<} G} a_H \mathrm{Ind}_H^G(1) \in R(G),$$

for suitable integers, a_H.

Proof: To prove part(i) observe that $(g^{-1}\nu g)(H) \le Y_n$. However, for $\psi: H \to Y_n$, consider the composition $\lambda: H \to Y_n \to Y_n/(X_1)^n \cong \Sigma_n$. Now suppose that $i_1, \ldots, i_s \in \{1, \ldots, n\}$ are the H–orbit representatives of the H–action on $\{1, \ldots, n\}$ via λ. For each i_j set $J_j = \mathrm{stab}_H(i_j) \le H$. Let $\phi_j: J_j \to X_1$ denote the homomorphism given by the (j,j)–entry (where, in the case $K = \mathbb{R}$, this means the (j,j) diagonal 2×2 block). An easy character calculation shows that $\psi = \sum_j \mathrm{Ind}_{J_j}^H(\phi_j) \in R_K(H) \subset R(G)$.

To prove part (ii) we use the fact that $H_*(W;\mathbb{C}) \cong H_*(\text{point};\mathbb{C})$ so that, in $R(G)$

$$(3.3)\qquad \begin{cases} 1 = \Sigma(-1)^i H_i(W;\mathbb{C}) \\ \quad = \Sigma(-1)^i C_i(W;\mathbb{C}) \end{cases}.$$

However, the representation given by $\mathbb{C}[G](\sigma) \subset C_i(W;\mathbb{C})$ is isomorphic to $\mathrm{Ind}^G_{\mathrm{stab}_G(\sigma)}(1)$ where $\mathrm{stab}_G(\sigma)$ is the stabiliser of the i-simplex, σ. Since $\mathrm{stab}_G(\sigma) \lneq G$, by part (i), we see that part (ii) follows at once from (3.3).

3.4: Definition

We recall some well-known definitions. G is an M-group if every irreducible complex representation of G is of the form $\mathrm{Ind}^G_H(\phi\colon H \to \mathbb{C}^*)$.

An elementary group is a finite of the form $C \times P$ where P is a ρ-group, for some prime ρ, and C is cyclic of order prime to ρ.

An $\mathbb{R}$-elementary group is semi-direct product of the form $P \propto C$ where P is a finite p-group, for some prime p, and C is a cyclic group of order prime to P so that $y \in P$ acts on C by the formula $yxy^{-1} = x^{\epsilon}$ for $\epsilon = \mp 1$ for all $x \in C$. Notice that an elementary group is $\mathbb{R}$-elementary.

We will need two corollaries of §3.2.

3.5: Corollary

There exist M-groups, H_α, such that $1 = \sum_\alpha a_\alpha \mathrm{Ind}^G_{H_\alpha}(1) \in R(G)$.

Proof: If G is not an M-group there exists a $\mathbb{C}$-representation, ν, of G to which to apply §3.2(ii). The result follows by induction on the order of G, for if $1 = \sum_{H \lneq G} a_H \mathrm{Ind}^G_H(1)$ and H is not an M-group then there exist M-groups, $J_\beta \lneq H$ such that

$$1 = \Sigma\, b_\beta \mathrm{Ind}^H_{J_\beta}(1) \in R(H) \text{ and } \mathrm{Ind}^G_H(1) = \sum_\beta b_\beta \mathrm{Ind}^G_{J_\beta}(1) \in R(G).$$

3.6: Corollary

Suppose that G is as in §2.8 then every irreducible K-representation (even dimensional if $K = \mathbb{R}$) of G has the form, $\mathrm{Ind}^G_H(\phi\colon H \to X_1)$, for some (H,ϕ).

Proof: By §2.8 $W^G \neq \phi$ so the result follows from §3.2(i).

3.7: Theorem (Brauer)

If $x \in R(G)$ then there exist elementary subgroups, $J_\delta \leq G$, and integers c_δ such that

$$x = \sum_\delta c_\delta \operatorname{Ind}_{G_\delta}^G(\phi_\delta : J_\delta \to \mathbb{C}^*) \in R(G).$$

Proof: We proceed by induction on the order of G. Let $1 = \sum_\alpha a_\alpha \operatorname{Ind}_{H_\alpha}^G(1)$ be as in §3.5. By Frobenius reciprocity

$$x = x.1 = \sum_\alpha a_\alpha \operatorname{Ind}_{H_\alpha}^G(1)x = \sum_\alpha a_\alpha \operatorname{Ind}_{H_\alpha}^G(\operatorname{Res}_{H_\alpha}^G(x)).$$

Therefore we may assume G is an M–group. Let A denote the subgroup of R(G) consisting of the $\mathbb{Z}$–linear span of elements of the form $\operatorname{Ind}_J^G(\phi: J \to \mathbb{C}^*)$ with J an elementary group. By Frobenius reciprocity together with §3.6 A is an ideal of R(G). Furthermore, if ν is an irreducible representation and $\dim(\nu) \geq 2$ then $\nu \in A$, since $\nu = \operatorname{Ind}_H^G(\phi: H \to \mathbb{C}^*)$ with $H \underset{\neq}{<} G$. If G is not abelian there exists an irreducible, ν, with $\dim(\nu) \geq 2$. Hence, if $\bar{\nu}$ is the dual of ν then $\nu\bar{\nu} \in A$. However, if ψ is a one–dimensional representation of G then

$$\langle \psi, \nu\bar{\nu} \rangle_G = \langle \psi\nu, \nu \rangle_G = \begin{cases} 1 & \text{if } \psi = 1, \\ 0 & \text{if not}. \end{cases}$$

Hence $\nu\bar{\nu} = 1 + \sum_i \lambda_i$ with λ_i irreducible and $\dim(\lambda_i) \geq 2$. Therefore $1 \in A$ and $A = R(G)$.

The remaining case is when G is abelian, for which the result is easy since all the irreducible representations of G are one–dimensional. We leave this case to the reader.

We will conclude with the results which correspond to §3.7 in the case when $K = \mathbb{R}$ or $\mathbb{H}$. Here my results are less satisfactory since I have not been able to find a self–contained proof which uses only the results of §2.. Instead I have had to content myself with a proof which follows that of [M] and merely uses §2.8 to replace Martinet's use of the Borel–Serre result that supersolvable groups normalise a maximal torus.

3.8: Theorem

(i) $R_{\mathbb{R}}(G)$ is generated by $\mathbb{R}$–representations of the form $\operatorname{Ind}_K^G(\phi: K \to O(n_K))$ where K is $\mathbb{R}$–elementary and $n_K = 1$ or 2.

(ii) $R_{\mathbb{H}}(G)$ is generated by $\mathbb{H}$–representations of the form $\operatorname{Ind}_K^G(\psi: K \to Sp(1))$ where K is $\mathbb{R}$–elementary.

<u>Proof:</u> By [S, p. 98, Theorem 27] $R_{\mathbb{R}}(G)$ is generated by the images of $\mathrm{Ind}_K^G\colon R_{\mathbb{R}}(K) \to R_{\mathbb{R}}(G)$ as K runs through $\mathbb{R}$–elementary groups. Let ν be an irreducible orthogonal representation. If $\dim(\nu)$ is even set $w = \nu$ and if $\dim(\nu)$ is odd set $w = \nu \oplus 1$. Let K act on $W = O(2n)/\Sigma_n \int O(2) = X_n/Y_n$ $(2n = \dim(w))$ as in §3.1. by §2.7, $W^K \neq \phi$, so that the argument used to prove §3.2 shows that $w = \sum_i \mathrm{Ind}_{J_i}^K(\phi_i\colon J_i \to O(2)) \in R_{\mathbb{R}}(K)$ and part (i) follows, since J_i is $\mathbb{R}$–elementary.

To prove part (ii) we remark that $R_{\mathbb{H}}(G)$ is an $R_{\mathbb{R}}(G)$–module. Hence $R_{\mathbb{H}}(G)$ is generated by the images of $\mathrm{Ind}_K^G\colon R_{\mathbb{H}}(K) \to R_{\mathbb{H}}(G)$ by part (i) and Frobenius reciprocity. The proof of part (ii) is completed by the symplectic analogue of the argument used in part (i). That is, if $\nu\colon K \to Sp(n)$ is an irreducible symplectic representation of an $\mathbb{R}$–elementary group then ν is a sum of representations of the form $\mathrm{Ind}_J^K(\psi\colon J \to Sp(1))$, as is seen by applying §2.7 and §3.2 to the action of K on $W = Sp(n)/\Sigma_n \int Sp(1) = X_n/Y_n$.

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Jacobi sums, Fermat motives and the Artin-Tate formula

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Let $m, n \in \mathbb{N}$, $m \geq 3$ and $n \geq 1$. Let μ_m $(= \mu_m(\mathbb{C}))$ denote the group of m^{th} roots of unity. Let $k = \mathbb{F}_q$ be a finite field of characteristic $p > 0$ such that $q \equiv 1 \pmod m$ (i.e., $k \supset \mu_m(\mathbb{C})$), and let $k^\times = \mathbb{F}_q^\times = \langle z \rangle$. Choose, once and for all, a multiplicative character $\chi : k^\times \longrightarrow \mu_m$, $\chi(z) = e^{2\pi i/m} =: \zeta$. Put $L = \mathbb{Q}(\zeta)$. Let $G = \mu_m^{n+2} \big/ \{\text{diagonal}\} = \{g = (\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \in \mu_m^{n+2}\} \big/ \{\text{diagonal}\}$, and let $\hat{G}$ be its character group. Then $\hat{G}$ is identified with the set

$$\left\{ a = (a_0, a_1, \ldots, a_{n+1}) \in (\mathbb{Z}/m\mathbb{Z})^{n+2} \;\middle|\; \sum_{i=0}^{n+1} a_i \equiv 0 \pmod m \right\}$$

by $\hat{G} \times G \longrightarrow L : a(g) = \prod_{i=0}^{n+1} \zeta_i^{a_i}$.

Let $\mathfrak{A}(= \mathfrak{A}_m^n) = \{a = (a_0, a_1, \ldots, a_{n+1}) \in \hat{G} \mid a_i \not\equiv 0 \pmod m \text{ for all } i\}$.

1. The Jacobi sum, $j(a)(= j(a)_{q,\chi})$, $a \in \mathfrak{A}$, of dimension n and degree m is defined by

$$j(a) = (-1)^n \sum \chi(v_1)^{a_1} \chi(v_2)^{a_2} \cdots \chi(v_{n+1})^{a_{n+1}}$$

where the sum is taken over all $(n+1)$-tuples $(v_1, v_2, \ldots, v_{n+1}) \in (k^\times)^{n+1}$ with a linear relation $v_1 + v_2 + \ldots + v_{n+1} = -1$. Properties of Jacobi sums which are relevant to our discussions are summarized as follows: Put $\Gamma = \mathrm{Gal}(L/\mathbb{Q})$.

(a) $j(a) \in \mathbb{Z}[\zeta] \subset L$ with the complex absolute value $q^{n/2}$.

(b) $\Gamma = \{ \sigma_t \mid \sigma_t(\zeta) = \zeta^t, (t,m) = 1 \} \cong (\mathbb{Z}/m\mathbb{Z})^\times$ acts on $j(a)$ by $j(a)^{\sigma_t} = j(ta)$.

(c) If m is prime, $j(a) \equiv 1 \pmod{(1-\zeta)^3}$. (The Iwasawa congruence). ([1].)

2. Let X $(= X_m^n)$ be the Fermat variety over k of dimension n and degree m defined by the equation

$$X_0^m + X_1^m + \ldots + X_{n+1}^m = 0 \subset \mathbb{P}_k^{n+1} .$$

The zeta-function of X, $Z(X,T)$, has the form

$$Z(X,T) = \frac{P(T)^{(-1)^{n+1}}}{(1-T)(1-qT)\ldots(1-q^nT)}$$

where $P(T) = \det(1-\Phi^{*}T)$ is the characteristic polynomial of the Frobenius endomorphism Φ^{*} of the ℓ-adic étale cohomology group $H^n(\bar{X},\mathbb{Q}_\ell)$ (ℓ prime $\neq p$), or of the cyrstalline cohomology group $H^n(X/W)_K$, induced by the Frobenius endomorphism Φ of X relative to k. Then $P(T) \in 1 + T\mathbb{Z}[T]$ with $\deg P = B_n(X)$ (the n^{th} Betti number of X), and furthermore, over $\mathbb{C}$,

$$P(T) = \begin{cases} \prod_{a\in\mathfrak{A}} (1 - j(a)T) & \text{if } 2\nmid n \\ (1-q^{n/2}T) \prod_{a\in\mathfrak{A}} (1 - j(a)T) & \text{if } 2\mid n \end{cases} .$$

3. <u>Fermat submotives</u> of X are defined as follows. For $a \in \hat{G}$, put $p_a = \frac{1}{\#G} \sum_{g\in G} a(g)^{-1} \cdot g \in L[G]$ and for $A = [a]$ (the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of a), let $P_A = \sum_{a\in A} p_a \in \mathbb{Z}[\frac{1}{m}][G]$. Then p_a and p_A are idempotents with $\sum_{a\in\hat{G}} p_a = 1$ and $\sum_{A\in O(\hat{G})} p_A = 1$ where the latter sum runs over the set $O(\hat{G})$ of $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\hat{G}$. Identifying $g \in G \subset \mathrm{Aut}(X_L)$ with its graph, p_A may be regarded as an algebraic cycle on $(X \times X)_L$ with coefficients in $\mathbb{Z}[\frac{1}{m}]$. The pair $M_A := (X,p_A)$ is called a Fermat submotive of X corresponding to the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit A in $\hat{G}$. ([2].) The field of definition of M_A is the prime field $\mathbb{F}_p$ with $(p,m) = 1$ or $\mathbb{Q}$.

4. The motivic decomposition of X , $\tilde{X} = (X, \Delta_X) = \oplus M_A$, commutes with cohomology functors with various coefficients, and this enables us to define some numerical invariants of M_A . For $a \in \mathfrak{A}$ and $t \in (\mathbb{Z}/m\mathbb{Z})^\times$, let $\|ta\| = \sum_{i=0}^{n+1} \left\langle \frac{ta_i}{m} \right\rangle - 1$ where $\langle x \rangle$ stands for the fractional part of $x \in \mathbb{Q}$.

(a) The n^{th} Betti number of M_A is $B_n(M_A) := \dim_{\mathbb{Q}_\ell} H^n(M_{A,\bar{k}}, \mathbb{Q}_\ell)$ $= \# A$ if $A \subset \mathfrak{A}$.

(b) The (i,j)-th Hodge number of M_A is $h^{i,j}(M_A) := \dim_k H^j(M_A, \Omega^i)$ $= \#\{ a \in A \mid \|a\| = i \}$ if $i + j = n$ and $A \subset \mathfrak{A}$. In particular, $h^{0,n}(M_A) = p_g(M_A)$ (the geometric genus of M_A).

(c) The slopes of M_A are $\{A_H(a)/f\}_{a\in A}$ where $H = \{p^i \bmod m \mid 0 \le i < f\}$, f the order of p mod m and $A_H(a) = \sum_{t\in H} \|ta\|$.

We have $P(T) = \prod_{A\in O(\hat{G})} P_A(T)$ if $2 \nmid n$, and $(1-q^{n/2}T) \prod_{a\in O(\hat{G})} P_A(T)$ if $2 \mid n$, where $P_A(T) = \prod_{a\in A} (1 - j(a)T) \in \mathbb{Z}[T]$.

5. A theorem of Mazur that the Newton polygon lies over or on the Hodge polygon is also valid for Fermat submotives M_A . Now we can make the following definition.

(a) M_A is said to be **ordinary** if the Newton polygon of M_A coincides with the Hodge polygon of M_A .

(b) M_A is said to be **supersingular** if the Newton polygon of M_A has the pure slope $n/2$.

(c) M_A is said to be of **Hodge-Witt type** if $H^j(M_A, W\Omega^i)$ is of finite type over W for all pairs (i,j) with $i + j = n$.

Combinatorial characterizations are given in the following theorem.

Theorem. *Let* M_A *be a Fermat submotive.*

(a) M_A *is ordinary if and only if* $\|pa\| = \|a\|$ *for every* $a \in A$.

(b) M_A *is of Hodge-Witt type if and only if for each* j , $0 < j < f$, $\|p^j a\| - \|a\| \in \{0\}$, $\{0,1\}$ *or* $\{-1,0\}$ *for every* $a \in A$.

(c) M_A *is supersingular if and only if* $A_H(a) = nf/2$ *for every* $a \in A$.

Examples. (a) Let $m = 7$, $n = 2$ and $A = [1,1,2,3]$. Then M_A is not ordinary but of Hodge-Witt type.

(b) Let $m = 7$, $n = 3$ and $A = [1,1,2,4,6]$. Then M_A is ordinary.

(c) Let $m = 7$, $n = 4$ and $A = [1,1,1,6,6,6]$. Then M_A is ordinary and supersingular.

6. Let $n = 2d$ and let M_A be a Fermat submotive over $k = \mathbb{F}_q$.

Theorem. *The following statements are all equivalent.*

(i) M_A *is supersingular.*

(ii) $j(a)/q^d$ *is a root of unity for some* $a \in A$.

(iii) $j(a)/q^d$ *is a root of unity for every* $a \in A$.

Theorem $(n = 2)$. *The following assertions hold.*

(a) The Picard number $\rho(X_{\bar{k}})$ *of the Fermat surface* $X_{\bar{k}}$ *is equal to* $1 + \sum B_2(M_{A,\bar{k}})$ *where the sum runs over all supersingular Fermat submotives* $M_{A,\bar{k}}$

(b) (The Artin-Tate formula). *Let* M_A *be supersingular. Then*

$$|Br(M_A)_{p\text{-tors}}|\,|\det NS(M_A) \otimes_{\mathbb{Z}} \mathbb{Z}_p| = q^{p_g(M_A)} \quad ;$$

$$Br(M_A)_{\ell\text{-tors}} = \{0\} \ \textit{and} \ |\det NS(M_A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell| = 1 \ \textit{for any}$$

prime ℓ *with* $(\ell, mp) = 1$.

(c) (The Artin-Tate formula). *Suppose that* M_A *is not supersingular. Then*

$$|Br(M_A)_{p\text{-tors}}|/q^{p_g(M_A)} = |P_A(1/q)|_p^{-1} = \Big|\prod_{a\in A}(1-j(a)/q)\Big|_p^{-1} \ ;$$

$$|Br(M_A)_{\ell\text{-tors}}| = |P_A(1/q)|_\ell^{-1} = \Big|\prod_{a\in A}(1-j((a)/q)\Big|_\ell^{1} \quad \textit{for each}$$

prime ℓ with $(\ell, mp) = 1$.

(The ℓ-part is due to Shioda [2].)

Corollary. *If* X *is a Fermat surface of Hodge-Witt type over* k , *then* $\det NS(X)$ *divides a power of* m .

(This extends a result of Shioda [2] for ordinary Fermat surfaces to a larger class of Fermat surfaces of Hodge-Witt type.)

7. **Theorem** (n = 2). *Let $m > 3$ be prime. Let M_A be a Fermat submotive of Hodge-Witt type over $k = \mathbb{F}_q$. Suppose that M_A is not supersingular. Then*

$$\mathrm{Norm}_{L/\mathbb{Q}}(1 - j(a)/q) = \pm Bm^3/q^{w^0(M_A)}$$

where B is a positive integer which is a square or twice a square, possibly multiplied by a divisor of 2m, and $w^0(M_A)$ is the dimension of the p-divisible formal group in $H^2(M_A, \hat{\mathbb{G}}_m)$.

(This generalizes a result to Shioda [2] on ordinary Fermat submotives.)

8. Let $n = 2d$, $m > 3$ prime and $k = \mathbb{F}_p$ with $p \equiv 1 \pmod m$. Let $j(a)$ be a Jacobi sum of dimension n and degree m over k. Let $P_A(T) = \prod_{a \in A} (1 - j(a)T) \in \mathbb{Z}[T]$. Then for each r, $0 \le r \le n$, $\mathrm{Norm}_{L/\mathbb{Q}}(1 - j(a)/p^r) = P_A(1/p^r)$ is a rational number, and there exists a certain duality between $\mathrm{Norm}_{L/\mathbb{Q}}(1 - j(a)/q^r)$ and $\mathrm{Norm}_{L/\mathbb{Q}}(1 - j(a)/q^{n-r})$.

Examples. (a) Let $m = 5$, $n = 4$ and $a = (1,1,1,1,2,4)$.

p	11	31	41
$Nr(1-j(a))$	$5^3 31\cdot 41\cdot 1381$	$5^3 58771\cdot 116101$	$5^3 11\cdot 5808146251$
$Nr(1-j(a)/p)$	$5^3 151$	$5^3 7411$	$3^4 5^3 281$
$Nr(1-j(a)/p^2)$	$\frac{5^3}{11}$	$\frac{5^3}{31}$	$\frac{5^3}{41}$
$Nr(1-j(a)/p^3)$	$\frac{5^3 151}{11^4}$	$\frac{5^3 7411}{31^4}$	$\frac{3^4 5^3 281}{41^4}$
$Nr(1-j(a)/p^4)$	$\frac{5^3 31\cdot 41\cdot 1381}{11^8}$	$\frac{5^3 58771\cdot 116101}{31^8}$	$\frac{5^3 11\cdot 5808146251}{41^8}$

(b) Let $m = 7$, $n = 6$ and $a_1 = (1,1,1,1,1,2,2,5)$, and $a_2 = (1,1,1,1,3,3,5,6)$.

p	29	43
$Nr(1-j(a_1)/p^3$	$\frac{7^3}{29^3}$	$\frac{7^3 83^2}{43^3}$
$Nr(1-j(a)_2)/p^3)$	$\frac{7^2 7^3}{29^2}$	$\frac{7^2 7^3}{43^2}$

Theorem. *Let* $j(a)$ *be a Jacobi sum of dimension* $n = 2d$ *and degree* m prime > 3. *Then* m^3 *always divides* $Nr(1 - j(a)/q^r)$ *for any* r, $0 \le r \le n$. *Furthermore for* $r = d$, *under certain additional conditions, if* m^{t+3} $(t \ge 0)$ *divides* $Nr(1 - j(a)/q^d)$, *then* t *must be even.*

(The first assertion is due to Shioda [2]; cf. [1].)

Conjecture $(n = 2d)$. *Let* $j(a)$ *be a Jacobi sum of dimension* n *and degree* m *prime* > 3 *over* $k = \mathbb{F}_p$ *with* $p \equiv 1 \pmod m$. *Suppose that the Fermat submotive* M_A *corresponding to* $j(a)$ *is not supersingular. Then*

$$Nr(1 - j(a)/p^d) = \pm \frac{Bm^3}{q^{dh^{0,2d}+(d-1)h^{1,2d-1}+\dots+h^{d-1,d+1}}}$$

where B is a square.

The detailed account of this paper can be found in [3].

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Monstrous E_{10}'s and a generalization of a theorem of L. Solomon

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Abstract. We generalize a theorem of Solomon to give a formula for counting the number of conjugacy classes of elements of prime order of a compact simple Lie group with character values in a number field of a given degree.

Using the "monster Lie algebra" [BCQS] as motivation we consider the following situation for the Kac-Moody Lie algebra of type E_{10}

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & & & & & 0 & 8 & & & \\
 & & & & & & & & & & & & | & & & & \\
0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\
-1 & & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7
\end{array}$$

If $Q := \bigoplus_{i=-1}^{8} \mathbb{Z}\alpha_i$ is the root lattice of E_{10} then Q is isomorphic to the Lorentzian 10 dimensional lattice. Let $\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ be the null root of affine E_8. The element

$$\omega = \alpha_{-1} + \delta$$

is an isotropic element of Q. Consider the set of "Leech roots"

$$\Pi = \{\alpha \in Q \mid \omega \cdot \alpha = -1 \quad \text{and} \quad \|\alpha\| := \alpha \cdot \alpha = 2\}.$$

If $\Lambda_8 = \bigoplus_{i=1}^{8} \mathbb{Z}\alpha_i$ then

$$Q \cap \mathbb{Z}\omega^{\perp}/\mathbb{Z}\omega \simeq \Lambda_8$$

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and we have a bijection

$$\Lambda_8 \longleftrightarrow \Pi$$

given by

$$\alpha \mapsto \tilde{\alpha} := \alpha + \delta + \left(\frac{\|\alpha\| - 2}{2}\right)\omega.$$

Consider the matrix $A = A_{\alpha,\beta}$ indexed by a given sublattice Λ of Λ_8 defined by

$$A_{\alpha,\beta} := \tilde{\alpha} \cdot \tilde{\beta} = -\frac{1}{2}(\|\alpha - \beta\| - 4).$$

Then A is a Cartan matrix if and only if Λ does not have any elements of norm 2 (In the monstrous case one can use the full Leech lattice). The following construction shows how to produce such sublattices.

Let Δ be a indecomposable irreducible root system and let $P, Q, P^\vee, Q^\vee$ be its weight (respectively root, coweight, coroot) lattice. Fix $N \in \mathbb{Z}_{>0}$, let

$$S = \{f \in \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z}/N\mathbb{Z}) \mid f \text{ is surjective}\}.$$

The Weyl group W of Δ acts on S via $(wf)(\mu) = f(w^{-1}\mu)$ and the orbits of W on S define an equivalence relation $\sim$ on S. We let $\tilde{S} := S/\sim$ and let $\mathbf{G}$ be the simply connected compact Lie group of Δ. Finally we let $\mathcal{E}$ denote the set of conjugacy classes of elements of $\mathbf{G}$ of order N.

If $x \in \mathbf{G}$ is of order N then $x \sim \exp i2\pi N^{-1}z$ for some $z \in Q^\vee$ and we can define

$$f_z : P \to \mathbb{Z}/N\mathbb{Z}$$

by

$$f_z(\mu) := \langle \mu, z \rangle + N\mathbb{Z}$$

for all $\mu \in P$.

LEMMA 1. *The procedure $x \mapsto z \mapsto f_z$ establishes a bijection of $\mathcal{E}$ with $\tilde{S}$.*

□

Let x be as above and let $A(x)$ be the smallest number field containing all the character values of x at every finite dimensional representation of $\mathbf{G}$. Clearly $\mathbb{Q} \subset A(x) \subset \mathbb{Q}(e^{i2\pi N^{-1}})$ and we define $a(x) := \dim_{\mathbb{Q}} A(x)$ and $b(x) := \dim_{A(x)} \mathbb{Q}(e^{i2\pi N^{-1}})$ (called the *height* and *depth* of x respectively).

Let $\mathcal{L}$ be the set of sublattices L of P satisfying $P/L \simeq \mathbb{Z}/N\mathbb{Z}$. The Weyl group acts on $\mathcal{L}$ inducing an equivalence relation $\sim$ and a quotient set $\tilde{\mathcal{L}} := \mathcal{L}/\sim$.

LEMMA 2. *There exists a well defined surjection*

$$\tilde{S} \to \tilde{\mathcal{L}}$$

given by

$$\tilde{f}_x \mapsto \widetilde{\ker f_x}.$$

Moreover if $x = \exp i2\pi N^{-1} z$ *then*

- *The preimage of* $\widetilde{\ker f_x}$ *has cardinality* $a(x)$.
- *(Δ of type E_8). For an element of* $\widetilde{\ker f_x}$ *to have no elements of norm 2 it is necessary and sufficient that* x *be a regular element of* $\mathbf{G}$.

□

Going back to E_8 we see that a suitable sublattice Λ of the E_8 lattice can be constructed for $N = 31$ using the unique conjugacy class of regular elements of order 31. Such an element is given by

$$x := \exp i2\pi z/31$$

where $z = \omega_1^\vee + \cdots + \omega_8^\vee$. (Here the ω_i's are fundamental coweights relative to a base $\alpha_1, \ldots, \alpha_8$ of Δ.) We have

$$\Lambda = \Big\{ \sum_{i=1}^{8} a_i \alpha_i \mid \sum_{i=1}^{8} a_i \equiv 0(31) \Big\}.$$

Let $W_D := \{w \in W \mid w\Lambda = \Lambda\}$. It is known that W_D is generated by a Coxeter transformation [Pzℓ1]. Since $a(x) = 1$ we can use Lemma 2 to conclude that

$$\text{Card}\ \tilde{\Lambda} = [W : W_D] = 23.224.320.$$

The numerology of these lattices and of the "monstrous like" Kac-Moody algebra generated by their associated Cartan matrices is suggesting. There is also an induced "root-data" geometry [MP] inside the E_{10} lattice obtained from $\tilde{\Lambda}$.

The last lemma indicates that counting W-orbits of lattices can be accomplished by counting conjugacy classes of a given height. This problem is of interest on its own and can be solved as follows:

Given $D,\ b \in \mathbb{Z}_{>0}$ consider the polynomial

$$B_{D,b}(t) := \sum_{\substack{m|D \\ b|m}} \mu(\frac{m}{b}) A_m(t)$$

where μ is the Möbius function and for each $m \in \mathbb{Z}_{>0}$

$$A_m(t) := \frac{1}{|W|} \sum_{w \in W} (t^{f_d(w)} - 1)$$

where $f_d(w)$ is the multiplicity of a primitive dth root of unity as an eigenvalue of w.

THEOREM 1. *Let p be a prime which is relatively prime to the order of W and let D be the greatest common divisor of $p-1$ and the exponent of W (i.e. the least common multiple of the orders of elements of W). For all $b \in \mathbb{Z}_{>0}$ the number of conjugacy classes of elements of G of order p and height b is given by evaluating $B_{D,b}(t)$ at p.*

□

Recall that if $x \in \mathrm{G}$ is of prime order p then [Pzℓ1]

$$p - 1 \leq a(x)h$$

where h is the Coxeter number of Δ. The limit case when $p-1=a(x)h$ forces x to be regular [Pzℓ2] and with the aid of Theorem 1 we can show that

THEOREM 2. *If p is a prime such that $p-1=ah$ then the number of conjugacy classes of elements of* G *of order p and height a is a*

□

Remark. There are 2 conjugacy classes of quadratic (i.e. height =2) elements of order 61 in G of type E_8. By Lemma 2 these produce a "distinguished" set of index 61 sublattices of Λ_8 with no elements of norm 2. This set has again 23.224.320 elements.

We finish by describing an easy way of computing the number $s_k(d)$ of elements of W that admit a primitive d^{th} root of unity as an eigenvalue of multiplicity k. This permits an immediate calculation of the A and hence B polynomials.

THEOREM 3. *Let $d \in \mathbb{Z}_{>0}$ and let $m_1, \ldots, m_\ell$ be the exponents of W. Define*

$$I(d) := \big\{ i \in \{1, \ldots, \ell\} \mid d \text{ divides } (m_i + 1) \big\}.$$

Then

$$\frac{1}{|W|} \sum_{k=0}^{\ell} s_k(d) t^k = \prod_{i \in I(d)} \left(\frac{m_i + t}{m_i + 1} \right) = A_d(t) + 1.$$

□

The case $d=1$ gives a well known result of Solomon [Slm]. This last theorem is of its own interest and we finish by listing three immediate consequences of it (all known and listed in order of increasing difficulties of the original proofs).

COROLLARY 1. *For -1 to belong to W it is necessary and sufficient that all exponents of W be odd.*

□

COROLLARY 2. *(Springer). If $w \in W$ admits a primitive d^{th} root of unity as an eigenvalue then d divides $(m_i + 1)$ for some exponent m_i of W.* □

COROLLARY 3. *(Kostant). If $w \in W$ admits a primitive h^{th} root of unity as an eigenvalue then w is a Coxeter transformation.* □

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Navier Stokes Derivative Estimates in Three Space Dimensions with Boundary Values and Body Forces.

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Abstract. Conditions are stated on boundary values and body forces that ensure the derivative estimates of [3] in the initial and boundary value problem for the Navier Stokes equations in three space dimensions. Interpretation of these results, and an application to fluid layers, are given.

1. Introduction.

Consider the initial and boundary value problem for the Navier Stokes equation

$$u_{i,t} + u_k u_{i,k} = -p_{,i} + \nu\Delta u_i + B_i(x,t)$$

where $i,k = 1,2,3$, subscript commas denote derivatives, and Δ is the Cartesian Laplace operator. The velocity vector $u_i = u_i(x,t)$ describes the flow of a viscous incompressible fluid; thus ν denotes the constant viscosity and

$$u_{i,i} = 0.$$

The impressed force or body force per unit mass is denoted by $B_i(x,t)$.

On a region $\Omega \subseteq R^3$ we impose initial values

$$u_i(x,0) = u_{i0}(x) \in L^2(\Omega)$$

and boundary values

$$u_i(x,t) = w_i(x,t) \quad \text{on} \quad \partial\Omega,\ t > 0,$$

where $w_i(x,t)$ is defined on $\overline{\Omega}$ and is subject to conditions stated below.

When boundary values and body forces are zero, it has been shown in [3] that the $L^2(\Omega)$ norms of typical derivatives satisfy an integrability condition over time:

$$\|D_t^r D_x^s u\|_2 \in L^{2(4r+2s-1)^{-1}}(0,T)$$

where $0 < T < \infty$, and that $\max_{x\in\Omega} |D_t^r D_x^s u| \in L^{(2r+s+1)^{-1}}(0,T)$. Herein are stated conditions on general boundary values and body forces that ensure the same integrability results for the solutions $u_i(x,t)$ in the nonhomogeneous case. Note however that the incompressibility condition still holds.

2. Statement of the Main Theorem.

Throughout this work the order of a partial derivative $D_t^r D_x^s u$ is effectively the weighted order $2r + s = 2r + s_1 + s_2 + s_3$ where s_i are the components of the 3-index s. The weighted order will often be subject to an integer or half-odd integer correction according to the index of the space norm in which it appears.

Theorem. *Let $u_0 \in L^2(\Omega)$ and let ρ be an odd positive integer, r, s be non-negative integers where s is a 3-index. Then*

a) if $\|\nabla w\|_2 \in L^4(0,T)$ *and* $\|w_t\|_{6/5}, \|B\|_{6/5} \in L^2(0,T)$, *then* $\|u\|_2 \in L^\infty(0,T)$ *and* $\|\nabla u\|_2 \in L^2(0,T)$.

b) if we have

$$\|D_t^{r+1} D_x^s w\|_{6/5} \in L^{\frac{2\rho}{4r+2s+1}}(0,T), \qquad 2r + s \le \frac{1}{2}(\rho - 1)$$

$$\|D_t^r D_x^s w\|_2 \in L^{\frac{2\rho}{4r+2s-1}}(0,T), \qquad 0 < 2r + s \le \frac{1}{2}(\rho + 1)$$

and

$$\|D_t^r D_x^s B\|_{6/5} \in L^{\frac{2\rho}{4r+2s+1}}(0,T), \qquad 2r + s \le \frac{1}{2}(\rho - 1)$$

$$\|D_t^r D_x^s B\|_2 \in L^{\frac{2\rho}{4r+2s+3}}(0,T), \qquad 2r + s \le \frac{1}{2}(\rho - 3)$$

then

$$\|D_t^r D_x^s u\|_2 \in L^{\frac{2}{4r+2s-1}}(0,T), \qquad 0 < 2r + s \le \frac{1}{2}(\rho + 1)$$

and

$$\max_{x\in\Omega} |D_t^r D_x^s u|_2 \in L^{\frac{1}{2r+s+1}}(0,T), \qquad 2r + s \le \frac{1}{2}(\rho - 3)$$

c) if the conditions of b) hold for all odd positive integers ρ then the conclusions hold for all orders of partial derivatives.

A complete proof of this result will be given elsewhere. Here we remark only that the overall course of the proof is similar to the proof for the initial value problem only [3], but the details of the estimates are considerably more complicated. The conditions sufficient for each weighted order are stated separately; it would be possible for these to fail from some order onward initially or after a certain time interval. Because of interpolation and embedding theorems, these conditions are not all independent in general, but we do not discuss any details here.

3. A nonhomogeneous integrability lemma.

The proof proceeds by reducing certain integral estimates to a standard form and then applying an integrability lemma for the time variable (Lemma 3 of [3]). The modified form of this lemma, suitable for the non-homogeneous problem, is the following:

Lemma *Let* $a > 1,\ p > 0,\ F(t) \geq F_0 > 0, G(t) \geq 0$ *and* $N(t) \geq 0$. *Let* $F(t) \in L^p(0,T)$ *and assume for* $0 < t < T$ *that*

$$F'(t) + G(t) \leq KF^{a+p}(t) + N(t)$$

where $N(t) \leq CF^a(t)Q(t)$ *with* $Q(t) \in L^1(0,T)$.

Then $G(t) \leq CF^a(t)Q_1(t)$ *where* $Q_1(t) \in L^1(0,T)$; $G(t)$ *and* $N(t) \in L^{\frac{p}{a+p}}(0,T)$ *and*

$$\int_0^T \frac{G(t)}{F(t)^a}dt \ \leq\ K\int_0^T F^p(t)dt + \int_0^T Q(t)dt + \frac{F_0^{1-a}}{a-1}$$

$$\int_0^T G(t)^{\frac{p}{a+p}}dt \ \leq\ (K+1)\int_0^T F^p(t)dt + \frac{p}{a+p}\int_0^T Q(t)dt + \frac{p}{a+p}\cdot\frac{F_0^{1-a}}{a-1}.$$

The straightforward proof of this lemma is omitted; we shall comment instead on the nature of the hypothesis '$N(t)F(t)^{-a} \in L^1(0,T)$ made on the nonhomogeneous term $N(t)$. In practice singularities of $F(t)$ are determined by u and its derivatives and are not predictable or controllable in advance. Thus the assignable or controllable singularities of

$N(t)$ are limited to those of the integrable function $Q(t)$ and while stronger singularities of $N(t)$ may occur they are limited to those of $F(t)$, that is, to singular instants, regarded as unpredictable in advance, of the solution u.

The behaviour at any time of the boundary values and body forces can contribute to the formation of later singularities in the solution field u, even if these nonhomogeneous terms are themselves well-behaved. Thus later singularities of $F(t)$ may, in theory, be causally related to those of $N(t)$ through the overall behaviour of the nonhomogeneous terms and the solution.

4. Interpretation of results.

The well known energy relation

$$\|u(,t)\|_2^2 + 2\nu \int_0^t \|\nabla u(,T)\|_2^2 dT = \|u(,0)\|_2^2$$

and its counterpart in the nonhomogeneous case, show that the kinetic energy $\frac{\rho}{2}\|u\|_2^2$ is bounded, and the rate of viscous dissipation of energy $\nu\|\nabla u\|_2^2$ is integrable over time; that is, $\|\nabla u\|_2 \in L^2(0,T)$ for $T > 0$. At the next level, the space rate of change of shear (or space rate of change of velocity which generates viscous dissipation of energy) is majorized overall by $\|\Delta u\|_2$ which is time integrable to the $\frac{2}{3}$ power: $\|\Delta u\|_2 \in L^{\frac{2}{3}}(0,T)$. The acceleration $a = \frac{Du}{Dt} = u_t + u \cdot \nabla u$ also satisfies $\|a\|_2 \in L^{\frac{2}{3}}(0,T)$ while the time rate of change of shear satisfies $\|\frac{D}{Dt}\nabla u\|_2 = \|\nabla(u_t + u \cdot \nabla u)\|_2 \in L^{\frac{2}{5}}(0,T)$. The time-integrability of the maximum values over Ω and of the intermediate $L^p(0,T)$ norms for $2 \le p \le \infty$ are as shown.

Table 1 Time Integrability of $\| \ \|_p$.

Quantity	Symbol	$p=2$	$2<p<\infty$	$p=\infty$
velocity	u	∞	$\begin{cases} \frac{4p}{3p-6} & (2 \le p \le 6) \\ \frac{p}{p-3} & (6 \le p) \end{cases}$	1
shear	∇u	2	$\frac{p}{2p-3}$	$\frac{1}{2}$
acceleration	$a = \frac{Du}{Dt}$	$\frac{2}{3}$	$\frac{p}{3p-3}$	$\frac{1}{3}$
time rate of change of shear	$\frac{D}{Dt}(\nabla u)$	$\frac{2}{5}$	$\frac{p}{4p-3}$	$\frac{1}{4}$

5. Averages over Layers:

By means of the gradient inequalities in [1,2], the foregoing integrability results can also be applied to averages taken over layers of fluid defined by the motion. In the first instance, consider a fluid layer, such as a boundary layer, with velocity of magnitude u, where $u_1 \leq u \leq u_1 + du$. The layer thickness at any point is $du/|\nabla u|$. Thus we define the spherically symmetric equimeasurable decreasing rearrangement u^* of u in Ω or a suitable subdomain of $\Omega \subseteq R^3$. Then [1,2]

$$\frac{1}{|u^{*\prime}(x^*)|} = \int_S \frac{dS}{|\nabla u|}$$

where S is the surface area, dS the surface element, defined by $u^* \leq u \leq u^* + du$. Thus $S|u^{*\prime}(x^*)|$ is the harmonic average thinness (reciprocal thickness) of the layer, per unit of velocity increase, averaged over the layer surface. Here also $x^* = V^* = V = \frac{4\pi}{3}r^3$ where r is the corresponding spherical radius and dV the volume element of the layer. Then

$$\|Su^{*\prime}\|_p^p \equiv \int_{\Omega^*} (S|u^{*\prime}(x^*)|)^p dx^* \leq \int_{\Omega} |\nabla u|^p dx,$$
$$\leq \|\nabla u\|_p^p \in L^{\frac{1}{2p-3}}(0,T) \qquad p \geq 2$$

Thus the average thinness of the u-layers, integrated by volume over the region, has the same integrability over time as the corresponding gradient norm. Again a table may be constructed as follows:

Table 2

Layers defined by ranges of: (rearranged quantity)	Symbol	Gradient or majorizing quantity	Time Integrability, $\| \|_p$ of layer thinness $p = 2$	$2 < p < \infty$	$p = \infty$
velocity	u	∇u	2	$\frac{p}{2p-3}$	$\frac{1}{2}$
shear rate	∇u	Δu	$\frac{2}{3}$	$\frac{p}{3p-3}$	$\frac{1}{3}$
acceleration	$a = \frac{Du}{Dt}$	$\nabla(u_t + u \cdot \nabla u)$	$\frac{2}{5}$	$\frac{p}{4p-3}$	$\frac{1}{4}$
time rate of change of shear	$\frac{D}{Dt}(\nabla u)$	$\Delta(u_t + u \cdot \nabla u)$	$\frac{2}{7}$	$\frac{p}{5p-3}$	$\frac{1}{5}$

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A GENERAL INTEGRAL INEQUALITY FOR THE DERIVATIVE OF AN EQUIMEASURABLE REARRANGEMENT

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1. Introduction. For a real valued measurable function f on the domain $[0,b]$ the equimeasurable decreasing rearrangement f^* of f is defined as a function μ^{-1} inverse to μ, where $\mu(y)$ is the measure of the set $\{x \mid f(x) > y\}$. Since f^* is monotonic $f^{*\prime}$ is defined almost everywhere on $[0,b]$, and the following inequality is valid ([1,2]):

Let f be differentiable almost everywhere in $[0,b]$. If $p > 0$, then

$$(1) \qquad \int_0^b |f^{*\prime}(x)|^p \, dx \leq \int_0^b |f'(x)|^p \, dx,$$

and the reverse inequality is valid if $p < 0$.

A related inequality is also given in [2], and a generalization of this result for convex functions is given in [3].

In this paper we shall show that a similar generalization of (1) can be given, but for monotonic functions.

2. The general inequality

Theorem 1. *Let f be differentiable almost everywhere in $[0,b]$ and let $G(y)$ be a nondecreasing function for $y > 0$. Then*

$$(2) \qquad \int_0^b G(|f^{*\prime}(x)|) \, dx \leq \int_0^b G(|f'(x)|) \, dx.$$

Proof. As in [2,3] let the multiplicity $n(y)$ of f at the level y be the number of roots $x_k = x_k(y)$, $k = 1,\dots,n(y)$ of the equation $y = f(x)$, in $[0,b]$. The basic relation connecting the derivatives of f is obtained in [2]:

$$|f^{*\prime}(x)|^{-1} = \sum_{k=1}^{n} |f'(x_k)|^{-1}.$$

Using x^* as independent variable for the rearranged function f^*, we have as in [3]

$$|f^{*\prime}(x^*)| = |dy/dx^*|,$$

hence from the basic relation

$$dx^* = |f^{*\prime}(x^*)|^{-1}dy = \sum_{k=1}^{n}|f'(x_k)|^{-1}dy = \sum_{k=1}^{n}dx_k.$$

Therefore

$$G(|f^{*\prime}|)dx^* = G((\sum_{k=1}^{n}|f'(x_k)|^{-1})^{-1})\sum_{k=1}^{n}|f'(x_k)|^{-1}dy$$

$$= H(\sum_{k=1}^{n}|f'(x_k)|^{-1})dy$$

where $H(x) = xG(1/x)$. It is obvious that the function $H(x)/x$ is nonincreasing, so the following inequality is valid (see [4, p. 83])

$$H(\Sigma x) \leqq \Sigma H(x).$$

Thus we obtain

$$G(|f^{*\prime}(x^*)|)dx^* = H(\sum_{k=1}^{n}|f'(x_k)|^{-1})dy$$

$$\leqq \sum_{k=1}^{n}H(|f'(x_k)|^{-1})dy$$

$$= \sum_{k=1}^{n}|f'(x_k)|^{-1}\,G(|f'(x_k)|)dy$$

$$= \sum_{k=1}^{n}G(|f'(x_k)|)dx_k$$

using in the last step the relation $dy = |f'(x_k)|dx_k$. Integration over the domain $[0,b]$ now yields the stated result, as the integral elements based on the dx_k exactly cover the interval $[0,b]$ once when the summation over all integral elements based on dx^* is performed. This completes the proof of Theorem 1.

In fact, we can prove the following result (see [3]):

Let $\lambda(f)$ and $a(f)$ be arbitrary positive functions on the range of f. Then, assuming the indicated expressions are defined and that $G(x)$ is nondecreasing, we have

$$\int \lambda(f^*)\, G(a(f^*)|f^{*\prime}|)dx \leq \int \lambda(f)G(a(f)|f'|)dx$$

and the reverse inequality if G(x) is nonincreasing.

3. The m-dimensional case. Recall from [2] that a function $f(x) = f(x_1,\dots,x_m)$ has a spherically symmetric equimeasurable decreasing rearrangement which is essentially a function f^* of volume or of radial distance only. Let

$$\mu(z) = \text{meas.}\{(x_1,\dots,x_m) \mid f(x_1,\dots,x_m) > z\},$$

and let

$$f^*(x) = \mu^{-1}(x).$$

The basic relation for $f \in PC^1$ is derived by integration over the lever surface $f = f^*$ in the domain D involved. If dn denotes the inward normal differential, and ∇f the gradient, then $|\nabla f|dn = df$. Since

$$\mu(z) = \int_{f \geq z} dV = \int_{f \geq z} dndS$$

we find

$$d\mu = \Sigma \int dndS = \Sigma \int \frac{df}{|\nabla f|} dS.$$

The summation runs over all components of the level surface $f = f^*$. However,

$$d\mu = -\frac{d\, f^*}{|f^{*\prime}(x)|}$$

while $|df| = |df^*|$. Comparing, we obtain the m-dimensional basic relation (see [2,3]):

$$\frac{1}{|f^{*\prime}(x)|} = \Sigma \int_{f=f^*} \frac{dS}{|\nabla f|} .$$

The m-dimensional analogue of the multiplicity function is now the level surface area

$$S = \Sigma \int_{f=f^*} dS.$$

Theorem 2 . Let $G(x)$ be nondecreasing for $x > 0$. Then

$$\int G(|f^{*\prime}(x)|)\,dV \leq \int G(|\nabla f|)\,dV.$$

Proof. The integral on the left has the differential

$$\begin{aligned}
G(|f^{*\prime}(x)|)dV &= G((\Sigma \int (|\nabla f|)^{-1}\,dS)^{-1})\ \Sigma \int dS dn \\
&= G((\Sigma \int (|\nabla f|)^{-1}\,dS)^{-1})\ \Sigma \int |\nabla f|^{-1}\,dS df \\
&= H(\Sigma \int |\nabla f|^{-1}\,dS)df \\
&\leq \Sigma \int_{f=f^*} H(|\nabla f|^{-1})\,dS\,df \\
&= \Sigma \int_{f=f^*} |\nabla f|^{-1}\,G(|\nabla f|)\,dS\,df \\
&= \Sigma \int_{f=f^*} G(|\nabla f|)\,dS\,dn \\
&= \Sigma \int_{f=f^*} G(|\nabla f|)\,dV.
\end{aligned}$$

The result now follows, as in the onedimensional case, by integration over the domain D.

Again it is possible to include positive weight functions $\lambda(f) = \lambda(f^*)$ on the range in the integration, and an arbitrary positive function $a(f)$ in the argument of G, i.e. if $G(x)$ is a nondecreasing function we have

$$\int \lambda(f^*)G(a(f^*)|f^{*\prime}(x)|)dV \leq \int \lambda(f)G(a(f)|\nabla f(x)|)\,dV.$$

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