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CONTENTS

A CHARACTERIZATION OF QR-DOMAINS

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Abstract. It is shown that an integral domain R is a QR-domain if and only if $U(T_1) \cap R \neq U(T_2) \cap R$ for distinct overrings $T_1 \subset T_2$ of R; here, $U(A)$ denotes the units of a ring A.

Let R be a (commutative integral) domain. Following [3], we say that R is a QR-domain in case each overring of R is a ring of fractions of R. Each Bézout domain is a QR-domain; and each QR-domain is a Prüfer domain ([2, Proposition 1, Theorem 1], [3, Theorem 2.5(a)]). Both converses are false. Pendleton [6, Theorem 5] showed that a Prüfer domain is a QR-domain if and only if each radical of a finitely generated ideal is the radical of a principal ideal. Characterizations of QR-domains are known in other classes of domains. For instance, a Noetherian domain is a QR-domain if and only if it is a Dedekind domain with torsion class group ([2, Theorem 2], [3, Corollary 2.6], [4, Corollary (1), page 114]). Recently, Daniel and David Anderson [1, page 9] have asked whether a QR-domain R is characterized by the property that $U(R) \neq U(T) \cap R$ for each proper overring T of R. (As usual, $U(A)$ denotes the group of units of a ring A.) In [5, Examples 3.1 and 3.2], Heinzer-Lantz have recently shown that the answer to this question is "no" (and the authors of [5] attribute the question to Gilmer). The purpose of this note is to prove that QR-domains *are* characterized by properties similar to the one proposed by the Andersons. More precisely, we have the following result.

THEOREM. *For a domain R, the following conditions are equivalent:*

(1) $U(T_1) \cap R \neq U(T_2) \cap R$ if $T_1 \subset T_2$ are distinct overrings of R;

(2) $U(T_1) \cap R \neq U(T_2) \cap R$ if T_1 and T_2 are distinct overrings of R;

(3) R is a QR-domain.

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PROOF. (3) ⇒ (2): Let $T_i (i = 1, 2)$ be distinct overrings of a QR-domain R. Then $T_i = S_i^{-1}R$ for multiplicatively closed $S_i \subset R$. Without loss of generality, S_i is saturated. Hence, $U(T_i) \cap R = S_i$ (cf. [3, Proposition 1.1]), yielding (2).

(2) ⇒ (1): Trivial.

(1) ⇒ (3): Assume (1). We shall show that each overring T of R is a ring of fractions of R. Put $S = \{r \in R\backslash\{0\} : r^{-1} \in T\}$. Evidently, S is a saturated multiplicatively closed subset of R. Put $A = S^{-1}R$. Hence, $U(A) \cap R = S$ (cf. [3, Proposition 1.1]). Observe, by the definition of S, that $A \subset T$ and $U(T) \cap R = S$. Thus, by (1), $A = T$; that is, $T = S^{-1}R$. ∎

REFERENCES

1. D.D. Anderson and D.F. Anderson, *Multiplicatively closed subsets of fields*, Houston J. Math. 13 (1987), 1-11.
2. E.D. Davis, *Overrings of commutative rings. II. Integrally closed overrings*, Trans. Amer. Math. Soc. 110 (1964), 196-212.
3. R. Gilmer and J. Ohm, *Integral domains with quotient overrings*, Math. Annalen 153 (1964), 97-103.
4. O. Goldman, *On a special class of Dedekind domains*, Topology 3 (1964), 113-118.
5. W. Heinzer and D. Lantz, *Integral domains that lose ideals in overrings*, submitted for publication.
6. R.L. Pendleton, *A characterization of Q-domains*, Bull. Amer. Math. Soc. 72 (1966), 499-500.

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Real Quadratic Fields of Class Number One and Continued Fraction Period Less Than Six

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The purpose herein is to provide criteria for the class number h(d), of the real quadratic field $Q(\sqrt{d})$, to be one when the period k, of the continued fraction expansion of w is less than 6, where $w = (1 + \sqrt{d})/2$ if $d \equiv 1 (\text{mod } 4)$ and $w = (\sqrt{d})$ if $d \equiv 2,3 (\text{mod } 4)$. This includes all Richaud-Degert (R-D)-types. Moreover we pose conjectures as to exactly those d with $h(d) = 1$ and $k \leq 5$ when $d \equiv 5 (\text{mod } 8)$. We actually determine all such d when $d \equiv 1 (\text{mod } 8)$, and all such d (with possibly only one more value remaining) when $d \equiv 2,3 (\text{mod } 4)$.

Introduction.

Throughout d will denote a positive square-free integer. Several results in the literature give general criteria for the class number h(d) of $Q(\sqrt{d})$ to be one, (eg. see [2], [4] and [5]). However, none of these results gives a general class number one criterion in terms of a specific prescribed factorization of a certain quadratic polynomial (over the integers), as does the well-known Rabinowitsch result for complex quadratic fields which says: If $d \equiv 3 (\text{mod } 4)$ then; $x^2 - x + (1 + d)/4$ is prime for all integers x with $1 \leq x \leq (d - 3)/4$ if and only if $h(-d) = 1$. This result embodies, for example, the celebrated Euler polynomial $x^2 - x + 41$ being prime for all integers x with $1 \leq x \leq 40$. We found a similar result (as did Louboutin [3] and Yokoi [12] independently) for d of narrow R-D type (see Mollin [6] - [7] and Mollin - Williams [8]). Narrow R-D-types are of the form $d = \ell^2 + r$

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where $|r| \in \{1,4\}$. General R-D types are of the form $\ell^2 + r$ where $4\ell \equiv 0 \pmod r$ and $-\ell < r \leq \ell$. If we remove the condition $-\ell < r \leq \ell$ we call such d of extended R-D type. This term was used in the Master's thesis (1988) of the first author's student P.G. Walsh (as well as in the authors' [11]). In [9], the authors found all extended R-D types of class number one under the generalized Riemann hypothesis (GRH). Later we were able to remove the GRH assumption and found in [11], all extended R-D types with class number one, (with the possibility of only one more value remaining.) This proves that 5 of the 6 outstanding conjectures, in the literature, concerning extended R-D types, are true with the sixth failing for at most one value. Three of the 6 conjectures were posed by: S. Chowla [1], R. Mollin [7], and H. Yokoi [12] for narrow R-D types. The remaining 3 conjectures for extended R-D types were posed by the authors in [9].

Extended R-D types d have period k, of w, less than 5. It is the purpose of the next section to provide a class number one ("Rabinowitsch-like") criterion in terms of a specific factorization of a certain quadratic polynomial (over the integers) for all d with $k \leq 5$.

2. Results.

In what follows $\lfloor w \rfloor = a$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Also:

$$f_d(x) = \begin{cases} -x^2 - x + (d-1)/4 & \text{if } d \equiv 1 \pmod 4 \\ d - x^2 & \text{if } d \equiv 2,3 \pmod 4 \end{cases}$$

and;

Theorem.

If $k \leq 5$ for $Q(\sqrt{d})$ then:

(I) If $d \equiv 1 \pmod 8$ then $h(d) = 1$ if and only if $(d,k) \in \{(17,3), (33,4), (41,5)\}$.

(II) *If* $d \equiv 2,3 \pmod 4$ *then* $h(d) = 1$ *if and only if* $d = 2$ *or* $d = \ell^2 \pm 2$ *where* $\ell = a$ or $a + 1$ *and* $f_d(x)$ *is prime or twice a prime for all integers* x *with* $0 \le x \le a$.

(III) *If* $d \equiv 5 \pmod 8$ *then* $h(d) = 1$ *if and only if all of the following conditions hold*; (*where* $d = (2a - 1)^2 + 4r$ *and* $2a - 1 = br + s$ *with* $0 \le s < r$):

(a) r *is prime or* $r = (bs + 1)^2$

(b) $bs + 1$ *is prime or* 1

(c) *If* $r = (bs + 1)^2 \not\equiv 1$ *then* $f_d(x)/(bs + 1)$ *is prime whenever* $0 \le x \le a - 1$; $x \equiv 2^{-1}(\pm s - 1) \pmod{bs + 1}$, *and* $x \not\equiv 2^{-1}(\pm s - 1) \pmod{(bs + 1)^2}$. *Also* $f_d(x)/(bs + 1)^2$ *is* 1 *or prime whenever* $0 \le x \le a - 1$ *and* $x \equiv 2^{-1}(\pm s-1) \pmod{(bs + 1)^2}$

(d) *If* $r \not\equiv (bs + 1)^2$ *then* $f_d(x)/(bs + 1)$ *is prime or* r^2 *or* $(bs + 1)^2$ *whenever* $0 \le x \le a - 1$ *and* $x \equiv 2^{-1}(\pm(br - s)-1) \pmod{bs + 1}$

(e) *If* $r \not\equiv (bs + 1)^2$ *then* $f_d(x)/r$ *is* 1 *or prime or* r^2 *or* $r(bs + 1)$ *whenever* $0 \le x \le a - 1$ *and* $x \equiv 2^{-1}(\pm s - 1) \pmod r$.

(f) $f_d(x)$ *is prime whenever* $0 \le x \le a - 1$ *and* x *does not satisfy any of the congruences in* (c) - (e).

The proof for $k = 3$ appears in §4 of [10]. The proof for $k = 4$ will appear in Proceedings of the Japan academy, and the proof for $k = 5$ will appear elsewhere, since it is too long and intricate to reproduce here. The $k = 1,2$ cases do not appear as such in the literature. However, they are special cases of the results in [6] - [8]. Theorem (I) tells us exactly when $h(d) = 1$ for $d \equiv 1 \pmod 8$ and $k \le 5$. Also we have:

Corollary. *If* $d \equiv 2,3 \pmod 4$, $k \le 5$ *and* $h(d) = 1$ *then* $k \in \{2,4\}$ *and* $d \in \{2,3,6,7,11,14,23,38,47,62,83,167,227,398\}$ *with possibly only one more value remaining*.

Proof. We first note that it is a basic fact in the theory of continued fractions that if k is odd then d is a sum of two squares. Thus if $h(d) = 1$ either $d = 2$ or $k \in \{2,4\}$, because if $d \not\equiv 1 \pmod 4$ only 2 is a sum of two squares. Since $d = \ell^2 \pm 2$ or $d = 2$ for $\ell = a$ or $a + 1$ then by Theorem (II); either $d \in \{2,3\}$ or d is of non-narrow R-D type. In [11] the authors determined that, with one possible exception, the following set contains all non-narrow R-D types of class number one with $k \in \{2,4\}$: $\{6,7,11,14,23,33,38,47,62,69,83,93,141,167,213,227,237,398,413,453,573,717,1077,1133,1253,1293\}$. An examination of this set shows that the ones of the form $\ell^2 \pm 2$ are precisely those listed in the Corollary. □

Remark 1. It follows from the results in [11] that, with one possible exception, the following are all the $h(d) = 1$ for $k = 1,2$.

$k = 1$: $\{2,5,13,29,53,173,293\}$

$k = 2$: $\{3,6,11,21,38,77,83,93,227,237,437,453,1133,1253\}$.

We have done a computer check up to $d < 50,000$ and based on previously developed techniques we feel confident to pose the following:

Conjecture 1. If $k = 3$ and $h(d) = 1$ then $d \in \{17,37,61,101,197,317,461,557,677,773,1877\}$.

Conjecture 2. If $k = 4$ and $h(d) = 1$ then $d \in \{7,14,23,33,47,62,69,133,141,167,213,398,413,573,717,1077,1293,1397,1757,3053\}$.

Conjecture 3. If $k = 5$ and $h(d) = 1$ then $d \in \{41,149,157,181,269,397,941,1013,2477,2693,3533,4253\}$.

Conjecture 4. If $k = 6$ and $h(d) = 1$ then $d \in \{19,22,57,59,107,131,253,278,309,341,381,749,813,893,1893,2453,2757,3317\}$.

Conjecture 5. If $k = 7$ and $h(d) = 1$ then $d \in \{89,109,113,137,373,389,509,653,797,853,997,1493,1997,2309,2621,3797,4973\}$.

Conjecture 6. If $k = 8$ and $h(d) = 1$ then $d \in \{31,71,158,206,383,501,503,581,743,789,869,917,983,989,1333,1349,1437,2573,3093,6677,14693\}$.

We have several conjectures for higher periods, too numerous to mention here. We expect to be able to prove the above (and other) conjectures in the near future with the techniques which we are now developing, using results of Tatuzawa in connection with classical class number formulae.

We conjecture that for a given period k there are at most finitely many d with $h(d) = 1$. This conjecture follows from the generalized Reimann hypothesis using the techniques of [11]. Although the number of d with $h(d) = 1$ seems to increase (in some general way but not monotonically however) as k increases, we have not been able to prove that this is so. If we could then of course we would have proved Gauss' conjecture as to the infinitude of real quadratic fields with $h(d) = 1$. We do feel that we are making progress, however tentative, in that direction, if for no other reason than providing more affirmative computational and theoretical data in favour of the Gauss conjecture. There is much more work left to be done.

References

[1] S. Chowla and J. Freidlander, Class numbers and quadratic residues, Glasgow Math. J. 17 (1976), 47-52.

[2] M. Kutsuna, On a criterion for the class number of a quadratic number field to be one, Nagoya Math. J. 79 (1980), 123-129.

[3] S. Louboutin, Critères des principalité et minoration des nombres de classes l'ideaux des corps quadratiques réells à l'aide de la théorie des fractions continues, Université Paris VII, These de Doctorat, 22-06-87.

[4] S. Louboutin, Continued fractions and real quadratic fields J. Number Theory, 30 (1988), 167-176.

[5] H.-W. Lu, On the class-number of real quadratic fields, Sci. Sinica, (Special Issue) 2 (1979), 118-130.

[6] R.A. Mollin, Necessary and Sufficient Conditions for the Class Number of a Real Quadratic field to be one and conjecture of S. Chowla, Proc. Amer. Math. Soc. 102 (1988), 17-21.

[7] R.A. Mollin, Class number one criteria for real quadratic fields I, Proc. Japan Acad., Ser. A. 63, (1987), 121-125.

[8] R.A. Mollin and H.C. Williams, On prime-valued polynomials and class numbers of real quadratic fields, Nagoya Math. J. 112 (1988), 143-151.

[9] R.A. Mollin and H.C. Williams, Prime-producing polynomials and real quadratic fields of class number one (to appear: Proc. Internat. Conf. on Number Theory @ Quebec City 1987; Editors: J.-M. de Koninck and C. Levesque).

[10] R.A. Mollin and H.C. Williams, Class Number One for real quadratic fields, continued fractions, and reduced ideals (to appear: Proc. NATO ASI on Number Theory and Applications @ Banff, Canada 1988; Editor: R.A. Mollin).

[11] R.A. Mollin and H.C. Williams, Solution of the class number one problem for real quadratic fields of extended Richaud-Degert Type (with one possible exception), (to appear: Proc. of first conference of the Canadian Number Theory Assoc. @ Banff, Canada 1988; Editor: R.A. Mollin).

[12] H. Yokoi, Class number one problem for certain kind of real quadratic fields, Proceed. Internat. Conf. on class numbers and fundamental units, Katata, Japan 1986.

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THE CLASSIFICATION OF CERTAIN LOW-ORDER 2-EXTENSIONS OF A FIELD OF CHARACTERISTIC DIFFERENT FROM 2 .

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A reductive method for the resolution of central, simple field-imbedding problems is obtained and this is used to classify certain 2-extensions of a field of characteristic different from 2 .

1. Let p be a prime number. Let k be a field not of characteristic p and containing the p'th roots of unity. Let $\mathcal{G}$ be a finite group. A normal extension of k with Galois group $\mathcal{G}$ will be called a $\mathcal{G}$-extension of k .

Suppose that L/k is a $\mathcal{G}$-extension and that a simple, central imbedding problem with kernel $\mathbb{Z}/\mathbb{Z}p$ for this extension is given; thus a 2-cocycle, c , on $\mathcal{G}$ with coefficients in $\mathbb{Z}/\mathbb{Z}p$ is given and a normal extension of k containing L/k and with Galois group the extension of $\mathbb{Z}/\mathbb{Z}p$ by $\mathcal{G}$ given by c is sought. With c perceived as belonging to $Z^2(\mathcal{G}, L^x)$, the imbedding problem is solvable if and only if $c \in B^2(\mathcal{G}, L^x)$.

Hence we are confronted with the problem of expressing explicitly in terms of the structure of L/k the condition for c to split. This problem has been solved in [2] in the case where $\mathcal{G}$ is an elementary abelian p-group.

Assuming that a normal subgroup, $\mathcal{N}$, of $\mathcal{G}$ is given, that $\mathrm{res}(c) \in B^2(\mathcal{N}, L^x)$ where res denotes restriction to $\mathcal{N}$, and letting K denote the subfield of L fixed by $\mathcal{N}$, we reduce the problem of the splitting of c to the problem of the splitting of an explicitly constructed 2-cocycle $s \in Z^2(\mathcal{G}/\mathcal{N}, K^x)$. Here, "explicitly" is to be understood as "explicitly 'modulo' possible non-constructivity of the proof of the Hilbert theorem 90".

In this way a 'reductive' method for the construction of p-extensions of a field not of characteristic p and containing the p'th roots of unity is obtained.

We use this method to solve constructively the problems of imbedding a quadratic extension of k into a ($\mathbb{Z}/\mathbb{Z}8$)-extension and of imbedding a biquadratic extension of k into a dihedral-, quasidihedral- or quaternion-extension of degree 16. In [1] the problem of the existence of certain imbeddings of quadratic extensions into dihedral- or quaternion-extensions is solved in the case where k is an algebraic number field (not necessarily containing any other roots of unity than 1 and −1). However, the methods used in [1] do not work in the general case. Let us furthermore remark that the general central imbedding problem has been solved in [4] in the case where $\mathcal{G}$ is a p-group and k has characteristic p.

For elements G and H in a group we use the notation $G^H = H^{-1}GH$. In Theorem 1, δ denotes the relevant coboundary operator.

THEOREM 1 : In the situation above we assume $\mathrm{res}(c) = \delta f$. *Let*

$$\mathcal{G} = \bigcup_{X \in \mathcal{G}/\mathcal{N}} G(X)\mathcal{N}$$

be a coset decomposition and let $N(\ ,\ )$ *denote the corresponding factor system, i.e.* $G(X)G(Y) = G(XY)N(X,Y)$ *for* $X,Y \in \mathcal{G}/\mathcal{N}$. *Then for* $X \in \mathcal{G}/\mathcal{N}$ *the function* $\Omega_X : \mathcal{N} \to L^x$ given by

$$\Omega_X(N) = \frac{c(N,G(X))}{c(G(X),N^{G(X)})} \cdot \frac{G(X)f(N^{G(X)})}{f(N)}$$

is a 1-cocycle on $\mathcal{N}$. *We choose* $a_X \in L^x$ *such that* $Na_X = a_X \Omega_X(N)$ *for all* $N \in \mathcal{N}$. *Then for* $X,Y \in \mathcal{G}/\mathcal{N}$,

$$s(X,Y) = \frac{c(G(X),G(Y))}{c(G(XY),N(X,Y))}(G(XY)f(N(X,Y)))\frac{a_X}{a_X(G(X)a_Y)}$$

is a 2-cocycle in $Z^2(\mathcal{G}/\mathcal{N}, K^x)$, *and defining* $h : \mathcal{G} \to L^x$ by

$$h(G(X)N) = a_X \cdot \frac{G(X)f(N)}{c(G(X),N)}$$

for $X \in \mathcal{G}/\mathcal{N}$, $N \in \mathcal{N}$, *we have for* $G_1, G_2 \in \mathcal{G}$ *with classes* X, *resp.* Y *in* $\mathcal{G}/\mathcal{N}$ *that*

$$c(G_1,G_2) = s(X,Y)(\delta h)(G_1,G_2) .$$

2. From now on k will denote a field of characteristic different from 2. As is well known or easily seen, the $(\mathbb{Z}/\mathbb{Z}4)$-extensions of k are $k((q\theta)^{\frac{1}{2}})/k$ where $\theta = a + u\sqrt{a}$ with $a = u^2 + v^2$ where $u,v \in k$ are such that a is not a square in k , and $q \in k^x$. Also, the D_4-extensions of k , D_4 being the dihedral group of order 8, are $k(\sqrt{a}, \sqrt{b}, \sqrt{(q\theta)})/k$ where $a,b \in k^x$ are such that

none of a,b,ab is a square in k, $b=a-1$ or $b=-1$, and $\theta = a+\sqrt{a}$ if $b=a-1$, $\theta = \sqrt{a}$ if $b=-1$.

Let us recall that the dihedral group of order 16, D_8, the quasi-dihedral group of order 16, QD_8, and the quaternion group of order 16, Q_{16}, are presented as follows:

$$D_8 = \langle N,S \mid N^8 = E,\ S^2 = E,\ N^S = N^{-1} \rangle \quad ,$$
$$QD_8 = \langle N,S \mid N^8 = E,\ S^2 = E,\ N^S = N^3 \rangle \quad ,$$
$$Q_{16} = \langle N,S \mid N^8 = E,\ S^2 = N^4,\ N^S = N^{-1} \rangle \quad .$$

Let us also recall that (a,b) for $a,b \in k^\times$ denotes the usual symbol in the part of the Brauer group of k annihilated by 2.

THEOREM 2 : *Let* $a \in k^\times \setminus (k^\times)^2$ *and assume that* a *is a sum of two squares in* k, *say* $a = u^2 + v^2$. *Put* $\theta = a + u\sqrt{a}$.

Then $k(\sqrt{a})/k$ *can be imbedded in a* $(\mathbb{Z}/\mathbb{Z}8)$-*extension of* k *if and only if the equation*

$$X^2 - aY^2 - \frac{v}{u}Z^2 - a\frac{v}{u}V^2 = 0$$

has a non-trivial solution over k.

For $q \in k^\times$ *the* $(\mathbb{Z}/\mathbb{Z}4)$-*extension* $k((q\theta)^{\frac{1}{2}})/k$ *can be imbedded in a* $(\mathbb{Z}/\mathbb{Z}8)$-*extension of* k *if and only if*

$$(a,\tfrac{v}{u})(q,-1) = 1 \ .$$

THEOREM 3 : *Let* $a,b \in k^\times$ *be such that none of* a,b,ab *is a square in* k. *Then* $k(\sqrt{a},\sqrt{b})/k$ *can be imbedded in a* Q_8-*extension of* k *if and only if* $(a,b)(ab,-1) = 1$ *and this is the case if and only if there exist* $\alpha,\beta,\gamma,\lambda,\mu,\nu \in k$ *such that*

$$a = \alpha^2 + \beta^2 + \gamma^2 \ ,$$
$$b = \lambda^2 + \mu^2 + \nu^2 \quad \text{and}$$
$$0 = \alpha\lambda + \beta\mu + \gamma\nu \ .$$

REMARK: In [4] Witt obtained a parametrisation of the Q_8-extensions of k but the parametrisation given in Theorem 3 is somewhat different.

THEOREM 4 : Let $a,b \in k^x$ *be such that none of* a,b,ab *is a square in* k , *and either* $b=a-1$ *or* $b=-1$. *Put* $\theta = a + \sqrt{a}$ *if* $b=a-1$ *and* $\theta = \sqrt{a}$ *if* $b=-1$. *For* $q \in k^x$ *we consider the* D_4*-extension* $K_q/k = k(\sqrt{a}, \sqrt{b}, (2q\theta)^{\frac{1}{2}})/k$.

(1) K_q/k *can be imbedded in a* D_8*-extension of* k *cyclic over* $k(\sqrt{b})$ *if and only if* $(a,2)(q,-b) = 1$.

There exists $q \in k^x$ *such that* $(a,2)(q,-b) = 1$ *if and only if the equation*

$$X^2 - aY^2 - 2Z^2 - 2abV^2 = 0$$

has a non-trivial solution over k .

(2) K_q/k *can be imbedded in a* QD_8*-extension of* k *cyclic over* $k(\sqrt{b})$ *if and only if* $(a,-2)(q,-b) = 1$.

There exists $q \in k^x$ *such that* $(a,-2)(q,-b) = 1$ *if and only if the equation*

$$X^2 - aY^2 + 2Z^2 + 2abV^2 = 0$$

has a non-trivial solution over k .

(3) K_q/k *can be imbedded in a* Q_{16}*-extension of* k *cyclic over* $k(\sqrt{b})$ *if and only if* $(a,2)(b,-1)(q,-b) = 1$.

Furthermore, we have obtained a classification of $(\mathbb{Z}/\mathbb{Z}8)$-, D_8- and QD_8-extensions of k by the non-trivial solutions over k to the equations mentioned (considered for such values of u,v,a,b respectively for which solutions exist). If a and b are as in Theorem 3 and $(a,b)(ab,-1) = 1$ then all Q_8-extensions of k containing $k(\sqrt{a}, \sqrt{b})$ are constructed from given field elements $\alpha, \beta, \gamma, \lambda, \mu, \nu$ with the properties given in the theorem. If the extension K_q/k can be imbedded in a Q_{16}-extension of k cyclic over $k(\sqrt{b})$ then it is shown how to construct all such imbeddings from a fixed Q_8-extension of $k(\sqrt{a})$ containing K_q . In particular, with $k = \mathbb{Q}$, one obtains that the extension

$$\mathbb{Q}(\sqrt{6}, \sqrt{7}, (ab)^{\frac{1}{2}})/\mathbb{Q}$$

where

$$a = \sqrt{6}\sqrt{7}(41 + 38\sqrt{7}) ,$$

$$b = -1 + \sqrt{6} + (-4 + \sqrt{6} - \sqrt{7})(7 + \sqrt{7})^{-\frac{1}{2}}$$

is a Q_{16}-extension of $\mathbb{Q}$.

Also, the existence of 'regular' $\mathcal{G}$ -extensions of the rational function field $\mathbb{Q}(t)$ when $\mathcal{G}$ is one of the 2-groups considered is verified in a direct way from the theorems above, a 'regular' extension of $\mathbb{Q}(t)$ being an extension $K/\mathbb{Q}(t)$ with the property that any element of K which is algebraic over $\mathbb{Q}$ belongs to $\mathbb{Q}$. The existence of a regular $\mathcal{G}$ -extension of $\mathbb{Q}(t)$ is, by use of the Hilbert irreducibility theorem and elementary Galois theory, seen to imply the existence of infinitely many $\mathcal{G}$ -extensions of $\mathbb{Q}$. Using different methods, Lamprecht has given some constructions of cyclic, regular extensions of $\mathbb{Q}(t)$ in [3] .

REFERENCES.

[1] P. Damey, J. Martinet: Plongement d'une extension quadratique dans une extension quaternionienne.
J. reine angew. Math. 262/263 (1973) pp.323-338 .

[2] R. Massy: Construction de p-extensions galoisiennes d'une corps de caractéristique différente de p .
J. Algebra 109 (1987) pp. 508-535 .

[3] E. Lamprecht: Zur Charakterisierung zyklischer Erweiterungen rationaler Funktionenkörper. II.
Arch.Math. 13 (1962) pp.488-497 .

[4] E. Witt: Konstruktion von galoisschen Körpern der Charakteristik p zu vorgegebener Gruppe der Ordnung p^f .
J. reine angew. Math. 174 (1936) pp.237-245.

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A SUFFICIENT CONDITION FOR A POLYNOMIAL TO BE SUM OF 2M-TH POWERS OF RATIONAL FUNCTIONS

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Presented by P. Ribenboim, F.R.S.C.

Abstract.- Using some model theoretic arguments applied to the theory of real valuations, we obtain a sufficient condition based on elementary properties of $f(\underline{x})$, $f(\underline{x}) \in K[x_1,\ldots,x_n]$, K any real closed field, for f to be sum of $2m^{th}$ powers of rational functions in $K(\underline{x})$.

Introduction.- The property of being sum of 2m-th powers of rational functions is not an elementary property as was proved by A. Prestel ([P], Thm. 2). Hence, it seems a reasonable question to ask, whether once we have fixed a number n of indeterminates, we are able to decompose the space of coefficients of polynomials in n indeterminates (belonging to an arbitrary real closed field), into semialgebraic sets, described by elementary properties, such that any polynomial with coefficients in one of those sets is a sum of $2m^{th}$ powers of rational functions, and there is a bound of the degree and the number of rational functions which appear in the actual decomposition as sum of $2m^{th}$ powers. To establish such a decomposition in "stable sets" we give a sufficient criterium to be sum of 2m-th powers, based on elementary geometric properties of the polynomials. Note that the criterium due to Becker ([B]) is not adequate for such purposes.

§1. We introduce the following definitions:

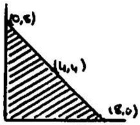
Definition.- Given $f \in K[x_1,\dots,x_n]$, K any real closed field and $f=\sum_{\nu} a_{\nu} x_1^{\nu_1}\dots x_n^{\nu_n}$ we define the convex hull of f as the convex hull in $\mathbb{R}^n$ determined by the exponents of the terms of f, i.e. by the points $(\nu_1,\dots,\nu_n) \in \mathbb{N}^n$. We denote it by CH(f),

$CH(f) = \{z \in \mathbb{R}^n:\ z=\Sigma\lambda_i\nu_i \quad \nu_i=(\nu_{1i},\dots,\nu_{ni})\ \Sigma\lambda_i=1,\ \lambda_i \geq 0 \quad \forall i\}$

Example.- Given f as above and a supporting hyperplane Π of CH(f), we say g is a supporting polynomial of f, if g is formed by the terms of f corresponding to the points of $CH(f) \cap \Pi$.

Example.- Let $f(\underline{x})=x^8+y^8+3x^4y^4+1$

CH(f)



Convex Hull of f.

The supporting polynomials are: x^8, y^8, 1, x^8+1, y^8+1, $x^8+y^8+3x^4y^4$.

Definition.- We say a polynomial $f \in K[x_1,\dots,x_n]$, K any real closed field, verifies (*) if $f(a_1,\dots,a_n)=0$ $(a_1,\dots,a_n)\in K^n$ implies $a_1=0 \vee\dots\vee a_n=0$.

In what follows we shall be working with real valuations $v:K(\underline{x})\longrightarrow\Gamma$ where Γ is an ordered abelian group, with an archimedean, formally real residue field, $(K(\underline{x}))_v$. Av denotes the valuation ring and v/k the restriction of v to K.

<u>Definition</u>.- Given $f\in Av/k[x_1,\dots,x_n]$ $(Av/k=Av\cap K)$

$f=\sum_{\nu} a_\nu x_1^{\nu_1}\dots x_n^{\nu_n}$, with $a_\nu\in A_v$, we define the <u>residue polynomial of f with respect to V</u> as $f_{V/K}(x_1,\dots,x_n)=\sum_{\nu}\underline{a}_\nu x_1^{\nu_1}\dots x_n^{\nu_n}$ where $\underline{a}_\nu$ is the image of a_ν in the residue field $K_{v/k}$.

The main result we prove is: (for an archimedean version of the theorem see [Br])

<u>Theorem</u>.- <u>Let</u> K <u>be a real closed field and let</u> $f(\underline{x})\in K[x_1,\dots,x_n]$. <u>If we have</u>

a) f(x) <u>is positive semidefinite</u> (p.s.d.)

b) 2m <u>divides the degree of</u> $f(\underline{x})$

c) 2m <u>divides the extreme points (everyone of the coordinates) of</u> CH(f)

<u>and for every real valuation</u> v <u>of</u> $K(\underline{x})$

d) $v/k(c)=0$ <u>for every coefficient</u> $c\neq 0$ <u>of</u> f.

e) <u>every supporting polynomial</u> g <u>of</u> $f(\underline{x})$ <u>and</u> $f(\underline{x})$ <u>itself verify that their residue polynomials</u> $g_{v/k}(\underline{x})$, $f_{v/k}(\underline{x})$ <u>satisfy</u> (*).

<u>then</u> $f(\underline{x})\in\Sigma K(\underline{x})^{2m}$.

The first step of the proof is the following lemma.

Definition.- We say a polynomial $f\in K[x_1,\dots,x_n]$ *is homogeneous with respect to* $\underline{v}$, real valuation of $K(\underline{x})$, if every monomial of f has the same value, which shall be called *homogeneous value of f*.

Lemma.- *Let* $f(\underline{x})\in K[x_1,\dots,x_n]$. *If* v *is a real valuation of* $K(\underline{x})$ *such that* $v(c_\nu)=0$ *for every coefficient* $c_\nu (c_\nu\neq 0)$ *of* $f(\underline{x})$, *then there exists a supporting polynomial* g *of* f *homogeneous with respect to* v *and of homogeneous value minimal in* f, *or* f *is homogeneous with respect to* v.

Sketch of the proof.- We consider a universal formula of first order language, which expresses the stating of the lemma and which we know ([Br]) is true in $(\mathbb{R},+,0,<)$. Hence it is true in $(\mathbb{Q},+,0,<)$ and we may transfer it to $(\tilde{\Gamma},+,0,<)$ where $\tilde{\Gamma}$ is the divisible hull of the ordered abelian group Γ.

Sketch of the proof of the Theorem.- By a theorem of Becker [B] it suffices to show that every formally real valuation v of $K(x_1,\dots,x_n)$, $v(f)$ is divisible by $2m$. Now consider a valuation v. Let g be the supporting polynomial or $g=f$ which exists by the previous lemma. The essential point is to prove $v(f)=v(g)=v(x_1^{\nu_1}\dots x_n^{\nu_n})$ with $(\nu_1,\dots,\nu_n)$ extreme point of $CH(g)$, for every real valuation v of $K(\underline{x})$ verifying condition d). Hence $2m$ divides $v(f)$ and $f\in\Sigma K(\underline{x})^{2m}$.

To prove $v(g)=v(x_1^{\nu_1},\dots,x_n^{\nu_n})$ we first show it is true for $g(x_i)\in \cup_r\mathbb{R}((t^{1/r}))$-where $\cup_r\mathbb{R}((t^{1/r}))$ denotes the field of Puiseux series and $x_i \in U_r\mathbb{R}((t^{1/r}))$- with the canonical valuation defined by the order, by considering the residue polynomial $g_{v/k}$ and using the fact that it verifies (*) (cf. [Br]). We now express $v(g)=v(x_1^{\nu_1}\dots x_n^{\nu_n})$ as a first order formula which is true in $(U_r\mathbb{R}((t^{1/r})), U_r\mathbb{R}[[t^{1/r}]],<)$. Hence, we transfer it to $(\underline{K(\underline{x})},\underline{\theta},<)$ where $\underline{K(\underline{x})}$ is the real closure of $K(\underline{x})$ with respect to an order compatible with the considered valuation, and $\underline{\theta}$ is the unique extension of θ, the convex valuation ring considered. Hence, since $K(\underline{x}) \longrightarrow \underline{K(\underline{x})}$ we obtain the result for $(K(\underline{x}),\theta,<)$ itself.■

The importance of this sufficient criterium is that it enables us to show that the following sets $S^{(N)}$ are stable, by proving $S^{(N)} \cap \Sigma k(x)^{2m} = S^{(N)}$ for every K, real closed field ([Br])

$S^{(N)} = \{\underline{c}/\ 1/N \leq |c_i| \leq N$ if $c_i \neq 0$ and $P(\underline{c}\ \underline{x})$ is such that
a) $2m/\deg P(\underline{c}\ \underline{x})$
b) 2m/extreme points of CH(P)
c) $P(\underline{c}\ \underline{x})$ is p.s.d.
d) for all $\underline{d} \in U^{1/N}(\underline{c})$ with $P(\underline{d}\ \underline{x})$ p.s.d.,$p(\underline{d}\ \underline{x})$ verifies (*) and so do its supporting polynomials.
(i.e. $\underline{d} \in U^{1/N}(\underline{c})$ if $|d_i - c_i| < 1/N$ for all i with $c_i \neq 0$, otherwise $d_i = 0$ and $P(\underline{d}\ \underline{x})$ is the polynomial obtained by substituting the coefficients $\underline{c}$ of $P(\underline{x})$ by $\underline{d}$) $\}$

Condition (d) implies the fact that every residue polynomial $g_{v/k}$ of g, supporting polynomial of $P(\underline{c}\ \underline{x})$ and $P_{v/k}$ itself verify (*). Observe that the crucial point is that this condition is first order, hence $S^{(N)}$ is semialgebraic, and we may apply the compactness theorem which gives us the required bound.

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References

[B] E. BECKER: Summen n-ter Potenzen in Körpern. J. reine angew. Math. 307/308 (1979), 8-30.

[Br] M. BRADLEY: PhD Dissertation. Univ. Cantabria. Spain (1987).

[P] A. PRESTEL: Model Theory of fields: An application to positive semidefinite polynomials. Soc. Math. de Frances 2º série, mémoires 16 (1984), 53-65.

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NATURALLY REDUCTIVE QUASI-KÄHLER MANIFOLDS

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Presented by G. de B. Robinson, F.R.S.C.

We derive a characterization of naturally reductive homogeneous Riemannian manifolds with an invariant quasi-Kähler structure on it.

1. INTRODUCTION

Naturally reductive homogeneous Riemannian spaces provide a large class of Riemannian manifolds which have a lot of remarkable properties. This class contains the symmetric spaces but there are also a lot of non-symmetric examples. We refer to [6],[11],[14],[18],[19],[20] for the basic definitions and for examples. It is worthwile to derive results which make it possible to conclude whether a naturally reductive structure is symmetric or not. This is the aim of this note.

For example, let (M, g, J) be a Kähler manifold and let γ be a unit speed geodesic. Then the plane $\{\gamma', J\gamma'\}$ is parallel along γ and the Kähler manifold is locally symmetric if and only if the restriction of the sectional curvature function to these holomorphic planes is a constant along γ, for all geodesics γ. In fact we have [10],[17]

Proposition 1. A Kähler manifold (M, g, J) is locally symmetric if and only if

$$\nabla_X R_{XJXXJX} = 0 \tag{1}$$

for all tangent vector fields X.

Here ∇ denotes the Levi Civita connection and R is the Riemannian curvature tensor. This criterion has been proved to be very useful in complex and symplectic geometry (see [8],[19]). In particular it led quickly to the following result [19] which was also proved, in a different way, by E.D.Deloff [7]:

Proposition 2. Let (M, g, J) be a naturally reductive homogeneous space with an invariant Kähler structure J. Then (M, g, J) is locally Hermitian symmetric.

Moreover, Proposition 1 provided an elegant proof for the following extension :

Proposition 3[15]. Let (M, g, J) be a homogeneous Kähler manifold all of whose geodesics are orbits of one-parameter subgroups of holomorphic isometries. Then (M, g, J) is locally Hermitian symmetric.

The main purpose of this note is to extend these results to a broader class of almost Hermitian manifolds.

2. PRELIMINARIES

We start by giving some of the basic definitions and results which will be needed to prove our results. In what follows we will always suppose that the manifold is connected and of class C^∞.

Let (M,g,J) be an almost Hermitian manifold, i.e., J is a (1,1)-tensor field such that

$$J^2=-1,\quad g(JX,JY)=g(X,Y)$$

for all tangent vector fields X,Y. (M,g,J) is said to be a Kähler manifold if J is parallel and it is an almost Kähler manifold if the Kähler two-form Ω is closed. Moreover, (M,g,J) is quasi-Kählerian if

$$(\nabla_X J)Y+(\nabla_{JX}J)JY=0 \tag{2}$$

for all X,Y. Finally, an almost Hermitian manifold is called a nearly Kähler manifold if

$$(\nabla_X J)X=0 \tag{3}$$

for all vector fields X. Note that a nearly Kähler manifold is automatically quasi-Kählerian. See [9] for examples.

The following proposition will be a key result. It is proved in [10].

Proposition 4. Let (M,g,J) be an analytic nearly Kähler manifold such that

$$\nabla_X R_{XJXXJX}=0. \tag{1'}$$

Then (M,g,J) is a locally 3-symmetric space with canonical almost complex structure J.

(1′) is again equivalent to the following condition: the sectional curvature of the parallel plane $\{\gamma',J\gamma'\}$ is constant along any geodesic γ on the nearly Kähler manifold.

We note that a locally 3-symmetric space is a C^∞ Riemannian manifold (M,g) together with a family of local cubic diffeomorphisms $p\mapsto s_p$, $p\in M$, such that each s_p is a holomorphic isometry in a neighborhood of p with respect to the canonical almost complex structure J of the family, determined by

$$S_p=s_{p*}(p)=-\frac{1}{2}I_p+\frac{\sqrt{3}}{2}J_p.$$

By a family of local cubic diffeomorphisms we mean a differentiable function $p \mapsto s_p$ which assigns to each $p \in M$ a diffeomorphism s_p on a neighborhood $\mathcal{U}(p)$ of p such that $s_p^3 =$ identity and p is the unique fixed point of s_p. See [10],[13] for more details.

Locally 3-symmetric spaces are locally homogeneous quasi-Kähler manifolds (see for example [18]). Moreover we have

Proposition 5[18]. A connected, complete and simply connected locally 3-symmetric space is nearly Kählerian if and only if it is a naturally reductive space.

This result has been proved in [18] by using the following infinitesimal characterization:

Proposition 6[3],[18]. Let (M,g) be a connected, complete and simply connected Riemannian manifold. Then (M,g) is a homogeneous Riemannian space if and only if there exists a (1,2)-tensor field T on M such that with

$$\overline{\nabla} = \nabla - T$$

we have

$$\overline{\nabla} g = \overline{\nabla} R = \overline{\nabla} T = 0.$$

Moreover, (M,g) is naturally reductive if and only if in addition

$$T_X X = 0 \tag{4}$$

for all tangent vector fields X.

We refer to [6],[11],[18] for more details about naturally reductive spaces.

3. THE MAIN RESULT

We are now ready to prove our

MAIN THEOREM. Let (M,g) be a homogeneous Riemannian manifold with an invariant quasi-Kähler structure J. If (M,g,J) is naturally reductive, then it is a locally 3-symmetric space with canonical nearly Kähler structure J.

Proof. Following Proposition 6 there exists a homogeneous structure T on the manifold (M,g,J) and $\overline{\nabla} = \nabla - T$ is the associated canonical connection. Since J is invariant we have $\overline{\nabla} J = 0$, i.e.,

$$(\nabla_X J)Y = T_X JY - JT_X Y. \tag{5}$$

Further, since J is quasi-Kählerian, (2) may be written, by using (5), as

$$T_X JY - JT_X Y - T_{JX} Y - JT_{JX} JY = 0. \tag{6}$$

Now, put $Y = X$ in (6). Then, using (4), we obtain

$$T_X JX - T_{JX} X = 0. \tag{7}$$

But (4) implies $T_X Y + T_Y X = 0$, and so (7) becomes

$$T_X JX = 0. \tag{8}$$

Then, from (5) and (8) we get (3) which means that J is nearly Kählerian.

Finally, $\overline{\nabla} R = 0$ yields

$$\nabla_X R_{XJXXJX} = -2R_{T_X XJXXJX} - 2R_{XT_X JXXJX},$$

which, together with (4) and (8), implies

$$\nabla_X R_{XJXXJX} = 0.$$

Then the desired result follows from Proposition 4.

From this proof and Proposition 1 we obtain easily Proposition 2. Moreover, we have

Corollary 7[1]. *Let* (M, g) *be a naturally reductive manifold with an invariant almost Kähler structure* J. *Then* (M, g, J) *is locally Hermitian symmetric.*

Proof. An almost Kähler manifold is necessarily quasi-Kählerian. Then the hypothesis implies that (M, g, J) is a nearly Kähler manifold. Further, as is well-known, a nearly Kähler almost Kähler manifold is Kählerian. So, the required result follows from Proposition 2.

4. SOME EXTENSIONS

To finish this note we give two extensions of the results cited above. We start with our Main Theorem and replace the invariance condition for J by the irreducibility of (M, g).

Theorem 8. *Let* (M, g) *be an irreducible naturally reductive space and suppose there exists a Kähler structure on it. Then* (M, g) *is locally symmetric.*

Proof. Let (M, g) be a naturally reductive space. Then the local geodesic symmetries are volume-preserving (see [4],[5],[14]). This implies that the Ricci tensor ϱ is cyclic parallel, i.e.,

$$\nabla_X \varrho_{XX} = 0$$

for any tangent vector field X [19]. Moreover, since (M, g) is a Kähler manifold, we have [17]

$$\nabla \varrho = 0$$

and so, because (M,g) is also irreducible, it is an Einstein space.

Next, suppose (M,g) is Ricci flat. A well-known result [2] then implies that (M,g) is flat.

Finally, when (M,g) has non-vanishing scalar curvature, then the Ricci tensor is non-degenerate and hence, following [12],[16], the connected component of the identity of the group of holomorphic isometries coincides with that of the full group of isometries. So, J is an invariant Kähler structure. This, together with Proposition 2, implies the desired result.

From this we get easily

Corollary 9. Let (M,g) be a non-symmetric irreducible naturally reductive space. Then there does not exist a Kähler structure on (M,g).

Finally, we prove an extension of Proposition 3.

Theorem 10. Let (M,g,J) be a simply connected homogeneous nearly Kähler manifold all of whose geodesics are orbits of one-parameter subgroups of holomorphic isometries. Then (M,g,J) is a naturally reductive 3-symmetric space with canonical almost complex structure J.

Proof. Let γ be a unit speed geodesic on (M,g). Then the hypothesis implies that the holomorphic sectional curvature $R_{\gamma' J\gamma'\gamma' J\gamma'}$ is constant along γ. This is equivalent to

$$\nabla_X R_{XJXXJX} = 0$$

for all vector fields X and hence Proposition 4 and Proposition 5 imply the result.

REFERENCES

[1] E.Abbena and S.Garbiero, Almost Hermitian homogeneous structures, Proc. Edinburgh Math. Soc., to appear.

[2] D.V.Alekseevskii and B.N.Kimel'fel'd, Structure of homogeneous Riemann spaces with zero Ricci curvature, Funktional Anal. i Priložen 9 (1975), 5–11, (English translation : Functional Anal. Appl. 9 (1975), 97–102).

[3] W.Ambrose and I.M.Singer, On homogeneous Riemannian manifolds, Duke Math. J. 25 (1958), 647–669.

[4] J.E.D'Atri and H.K.Nickerson, Geodesic symmetries in spaces with special curvature tensor, J. Differential Geometry 9 (1974), 251–262.

[5] J.E.D'Atri, Geodesic spheres and symmetries in naturally reductive homogeneous spaces, Michigan Math. J. 22 (1975), 71–76.

[6] J.E.D'Atri and W.Ziller, Naturally reductive metrics and Einstein metrics on compact Lie groups, Mem. Amer. Math. Soc. 215, 1979.

[7] E.D.Deloff, Naturally reductive metrics and metrics with volume-preserving geodesic symmetries on NC-algebras, Ph.D.thesis, Rutgers Univ. New Brunswick, New Jersey, 1979.

[8] M.Djorić and L.Vanhecke, Almost Hermitian geometry, geodesic spheres and symmetries, to appear.

[9] A.Gray, Some examples of almost Hermitian manifolds, Illinois J. Math. 10 (1966), 353–366.

[10] A.Gray, Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geometry 7 (1972), 343–369.

[11] S.Kobayashi and K.Nomizu, Foundations of differential geometry, I, II, Interscience, New York, 1963 and 1969.

[12] S.Kobayashi and K.Nomizu, On automorphisms of a Kählerian structure, Nagoya Math. J. 11 (1957), 115–124.

[13] O.Kowalski, Generalized symmetric spaces, Lecture Notes in Mathematics 805, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[14] O.Kowalski and L.Vanhecke, A generalization of a theorem on naturally reductive homogeneous spaces, Proc. Amer. Math. Soc. 91 (1984), 433–435.

[15] O.Kowalski and L.Vanhecke, The Gelfand theorem and its converse for Kähler manifolds, Proc. Amer. Math. Soc. 102 (1988), 150–152.

[16] A.Lichnerowicz, Sur les groupes d'automorphismes de certaines variétés kählériennes, C. R. Acad. Sci. Paris 239 (1954), 1344–1346.

[17] K.Sekigawa and L.Vanhecke, Symplectic geodesic symmetries on Kähler manifolds, Communications Dep. Math. Univ. Leuven 183 (1985) and Quart. J. Math. Oxford 37 (1986), 95–103.

[18] F.Tricerri and L.Vanhecke, Homogeneous structures on Riemannian manifolds, Lecture Notes Series London Math. Soc. 83, Cambridge Univ. Press, 1983.

[19] L.Vanhecke, Symmetries on homogeneous Kähler manifolds, Differential Geometry and its Applications, Reidel Publ. Co., Dordrecht, 1987, 339–357.

[20] L.Vanhecke, Geometry in normal and tubular neighborhoods, Proc. Workshop on Differential Geometry and Topology, Cala Gonone (Sardinia) 1988, to appear.

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