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CONTENTS

A CONVERSE OF KY FAN'S INEQUALITY

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Abstract. In this note we prove the inequality

$$\sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1-a_i) \le \prod_{i=1}^{n} (a_i/(1-a_i))^{a_i/\sum_{j=1}^{n} a_j},$$

where $a_i \in (0,1), i = 1, \ldots, n$, which is a converse of the celebrated Ky Fan inequality

$$\prod_{i=1}^{n} (a_i/(1-a_i))^{1/n} \le \sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1-a_i),$$

where $a_i \in (0, 1/2], i = 1, \ldots, n$. -

In 1961 E.F. Beckenbach and R. Bellman [2, p.5] published in their well-known book "Inequalities" the following remarkable counterpart of the famous arithmetic-geometric inequality:

If $a_i \in (0, 1/2]$, $i = 1, \ldots, n$, are real numbers, then

$$\prod_{i=1}^{n} (a_i/(1-a_i))^{1/n} \le \sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1-a_i), \tag{1}$$

with equality holding if and only if $a_1 = \ldots = a_n$.

This inequality has been discovered by Ky Fan who used Cauchy's method of forward and backward induction for a proof. During the last years Fan's inequality has been studied intensively by different authors such that a variety of new proofs, extensions and sharpenings of inequality (1) can be found in literature; see [1], [3] and the references therein.

In 1979 E. El-Neweihi and F. Proschan [4] established inequality (1) by applying the "powerful tools of majorization and Schur-functions" [4, p.207]. We will use the same technique in order to prove a new converse inequality to (1).

A full discussion of the Theory of Majorization was given by A.W. Marshall and I. Olkin [5].

Theorem. If $a_i \in (0,1), i = 1,\ldots,n,$ are real numbers, then

$$\sum_{i=1}^{n} a_i / \sum_{i=1}^{n} (1-a_i) \le \prod_{i=1}^{n} (a_i/(1-a_i))^{a_i / \sum_{j=1}^{n} a_j}, \tag{2}$$

where equality holds if and only if $a_1 = \ldots = a_n$.

Proof. We denote by f the function

$$f : (0,1) \to \mathbb{R},$$

$$f(a) = (a/(1-a))^a.$$

Simple calculations yield

$$\frac{d^2}{da^2} \ln f(a) = \frac{1}{a(1-a)} + \frac{1}{(1-a)^2} > 0.$$

Hence $\ln f$ is strictly convex on the interval $(0,1)$. From [5, Chapter 3, E.1.Proposition, p.73] we conclude that the function

$$g : (0,1)^n \to \mathbb{R},$$

$$g(a_1,\ldots,a_n) = \prod_{i=1}^{n} f(a_i)$$

is strictly Schur-convex. Next we set $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$, then

$$(a_1,\ldots,a_n) \succ (A_n,\ldots,A_n)$$

and we obtain an inequality which is equivalent to (2) :

$$g(a_1,\ldots,a_n) \ \geq \ g(A_n,\ldots,A_n) = (\sum_{i=1}^{n} a_i / \sum_{i=1}^{n}(1-a_i))^{\sum_{j=1}^{n} a_j},$$

where the sign of equality holds if and only if $(a_1,\ldots,a_n)$ is a permutation of $(A_n,\ldots,A_n)$, i.e., if and only if $a_1 = \ldots = a_n$.

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Fermat's Last Theorem: A Note about Abel's Conjecture

by

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Abstract. In this paper we improve on a bound of Inkeri [5] for Abel's Conjecture.

Introduction. In 1823 Abel [1] made a conjecture in a particular case of Fermat's Last Theorem: if $n>2$ and x,y and z are nonzero integers such that

$$x^n + y^n = z^n \tag*{(1)n}$$

then none of x,y or z can be a prime powers.

Many authors have contributed ideas to this question (see, for example, [4], [6], [7]) culminating in two important papers by Inkeri [5] and Möller [8]. It is clear that we may assume, without loss of generality, that x,y and z are pairwise coprime and $0<x<y<z$.

In 1955 Möller [8] showed that if (1)n has solutions then neither y nor z are prime powers and if x is a prime power then $z=y+1$ and n is an odd prime.

In 1946, however, Inkeri [5] had shown that if (1)n has solutions where n is an odd prime, and $z=y+1$ then n divides x, y or z. Furthermore it is easy to show that, in this case, if @x is a prime power then it can't be a power of n.

So we may restate Abel's conjecture to read: For all odd primes p and q there does not exist positive integers r and y for which

$$p \text{ divides } y(y+1) \quad \text{and} \quad (y+1)^p - y^p = q^{rp} \tag*{(2)p,q}$$

Inkeri [5] used Baker's method [2] to show that if (2)p,q does have a solution then

$$q^r < y < \exp\exp\{(2^p(p-1)^{10(p-1)})^{(p-1)^2}\}$$

In this paper we will considerably improve Inkeri's bound by applying Baker's method to a superelliptic equation of half the degree of that considered by Inkeri. We prove

Theorem. If n,x,y,z are integers for which $0<x<y<z$, $n>2$, $x^n+y^n=z^n$ and one of x,y,z is a prime power then x is the prime power $n=p$ is an odd prime, $z=y+1$, p divides $y(y+1)$ and

$$y<\exp\{\frac{1}{2}\exp((8(\frac{p-1}{2})^{10})^{(\frac{p-1}{2})^3})\}$$

In order to prove the theorem we make use of two key ingredients: Firstly the famous application of Baker's method to superelliptic equations given by Schinzel and Tijdeman [10]:

Lemma 1. Suppose that $f(x)\in\mathbf{Z}[x]$ has degree $n\geq 3$, at least two distinct roots and let A be the maximum of the absolute values of the coefficients of $f(x)$. If x,y and m are integers with $m\geq 3$ and $f(x)=y^m$ then

$$|x|,\ |y|<\exp\exp\{(5m)^{10}n^{10n^3}A^{n^2}\}$$

Now define $f_n(x)=\sum_{j=0}^{\lfloor n/2\rfloor}\frac{n}{n-j}\binom{n-j}{j}x^j$ for each positive integer n. We note the following

Lemma 2. For each positive integer n, $f_n(x)\in\mathbf{Z}[x]$ has $\left\lfloor\frac{n}{2}\right\rfloor$ distinct roots, namely $-(1+\cos(j\pi/n))^{-1}/2$ for each odd integer j, $1\leq j\leq n-1$. Furthermore if $\alpha+\beta=1$ then $f(-\alpha\beta)=\alpha^n+\beta^n$.

Proof: $\frac{n}{n-j}\begin{bmatrix}n-j\\ j\end{bmatrix}=\begin{bmatrix}n-j\\ j\end{bmatrix}+\begin{bmatrix}n-j-1\\ j-1\end{bmatrix}$ so that $f_n(x)\in\mathbf{Z}[x]$. Now it is easy to prove that $f_n(-\alpha\beta)=\alpha^n+\beta^n$ by induction on n, using the fact that $(\alpha^n+\beta^n)=(\alpha^{n-1}+\beta^{n-1})+x(\alpha^{n-2}+\beta^{n-2})$ where $x=-\alpha\beta$.

Furthermore if $w=e^{2i\pi/2n}$ for j is odd then $w^n=-1$ and $r=(1+\cos(j^\pi/n))^{-1}/2=-w/(1+w)^2$. So taking $\alpha=w/(1+w)$ and $\beta=1/(1+w)$ gives

$$f_n(r)=\alpha^n+\beta^n=(w^n+1)/(w+1)^n=0\,.$$

Proof of the theorem: By Möller's theorem we know that if (1)n has solutions then $z=y+1$

and n is an odd prime. Let $\alpha = y+1$ and $\beta = -y$ so that if $r = y(y+1)$, then by Lemma 2, $f_n(r) = z^n - y^n = x^n$.

Now by lemma 2, for $n \geq 6$, $f_n(x)$ has at least three distinct roots and it is easy to show that $A < 2^{n-2}$. Applying Lemma 1 gives, for $n = p$,

$$y < r^{\frac{1}{2}} < \exp\{\frac{1}{2}\exp\{(5p)^{10}(\frac{p-1}{2})^{10(\frac{p-1}{2})^3} 2^{(p-2)(\frac{p-1}{2})^2}\}\}$$

$$< \exp\{\frac{1}{2}\exp\{(8(\frac{p-1}{2})^{10})^{(\frac{p-1}{2})^3}\}\}$$

for each $p \geq 7$.

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A remark on the locus of abelian surfaces of p-rank ≤ 1

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§1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and let $\mathcal{A}_{2,1}$ (resp. $\mathcal{A}_{2,1,2}$) be the coarse moduli variety of principally polarized abelian surfaces (resp. principally polarized abelian surfaces with level 2-structure). In the previous paper [6], we investigated the locus V (resp. V_2) of supersingular abelian surfaces in $\mathcal{A}_{2,1}$ (resp. $\mathcal{A}_{2,1,2}$), and showed that the irreducible components of the locus consist of rational curves (see also Oort [8]). Moreover, we showed that the number of irreducible components of V is equal to a class number of the non-principal genus of a quaternion hermitian form, which was concretely calculated in Hashimoto and Ibukiyama [2]. In this note, we give a remark on the locus $V_{1,1}$ (resp. $V_{1,2}$) of abelian surfaces of p-rank ≤ 1 in $\mathcal{A}_{2,1}$ (resp. $\mathcal{A}_{2,1,2}$) in the case of $p = 3$. The main theorem is as follows.

Theorem. *Assume* $p = 3$. *Then, both* $V_{1,1}$ *and* $V_{1,2}$ *are irreducible.* $V_{1,1}$ *is birationally equivalent to a rational surface, and* $V_{1,2}$ *is birationally equivalent to a unirational* K3 *surface. Moreover,* $V_{1,2}$ *is a Zariski surface.*

Here, we call an algebraic surface X a Zariski surface if there exists a purely inseparable rational mapping of degree p from the projective space $\mathbb{P}^2$ of dimension two to X. We can find other examples of unirational K3 surfaces in Artin [1], Rudakov-Shafarevich

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[10], Shioda [11],[12], Katsura [4] and [5].

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§2. Proof of Theorem.

Let k be an algebraically closed field of characteristic $p = 3$, and let $M_{2,1,2}$ be the coarse moduli variety of non-singular curves of genus two with level 2-structure. We have an immersion at almost all points (cf. Oort and Steenbrink [9]):

$$\begin{array}{ccc} \mathcal{M}_{2,1,2} & \longrightarrow & \mathcal{A}_{2,1,2} \\ \Psi & & \Psi \\ (C,\sigma) & \longmapsto & (J(C),C,\sigma) \end{array} ,$$

where $J(C)$ is the Jacobian variety of a non-singular complete algebraic curve C of genus two, and where σ is a level 2-structure of $J(C)$. We denote by $MV_{1,2}$ the locus of non-singular curves C of genus two in $\mathcal{M}_{2,1,2}$ such that $J(C)$ has p-rank ≤ 1. It is well-known that the closure of $MV_{1,2}$ in $\mathcal{A}_{2,1,2}$ coincides with $V_{1,2}$. As for $\mathcal{A}_{2,1}$, there exists a Galois covering

$$\Pi : \mathcal{A}_{2,1,2} \longrightarrow \mathcal{A}_{2,1} \tag{1}$$

with Galois group S_3 (the symmetric group of degree 3), and we have $V_{1,1} = \Pi(V_{1,2})$.

Now, we examine $MV_{1,2}$. For this purpose, we consider the non-singular complete model C of the curve defined by the equation

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

with $\lambda_1,\lambda_2,\lambda_3 \in k$; $\lambda_1,\lambda_2,\lambda_3 \neq 0,1$; $\lambda_i \neq \lambda_j$ if $i \neq j$.

We note that $(\lambda_1, \lambda_2, \lambda_3)$ gives a parameter of $\mathcal{M}_{2,1,2}$. We set

$$\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 , \sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 , \sigma_3 = \lambda_1\lambda_2\lambda_3 .$$

We denote by $H^0(C, \Omega^1_C)$ the vector space over k of regular 1-forms on C. Then, with respect to the basis $\{dx/y , xdx/y\}$ of $H^0(C, \Omega^1_C)$, the Cartier operator of C is given by the matrix $A^{(1/3)}$:

$$A^{(1/3)} = \begin{pmatrix} -(\sigma_2+\sigma_3)^{1/3} & (\sigma_3)^{1/3} \\ 1 & -(1+\sigma_1)^{1/3} \end{pmatrix} .$$

We set

$$A = \begin{pmatrix} -(\sigma_2+\sigma_3) & \sigma_3 \\ 1 & -(1+\sigma_1) \end{pmatrix} . \tag{2}$$

then, $MV_{1,2}$ is given by the locally closed set in

$$\mathbb{A}^3 = \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i \in k \ (i=1,2,3)\}$$

defined by

$$\begin{aligned} \text{(3)}\quad \det A &= \sigma_2 + \sigma_1\sigma_2 + \sigma_1\sigma_3 \\ &= (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3) + (\lambda_1+\lambda_2+\lambda_3)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3) + \\ &\quad (\lambda_1+\lambda_2+\lambda_3)\lambda_1\lambda_2\lambda_3 = 0 \end{aligned}$$

with $\lambda_1, \lambda_2, \lambda_3 \neq 0, 1$; $\lambda_i \neq \lambda_j$ if $i \neq j$.

Now, we consider the hypersurface S in $\mathbb{P}^3$ defined by the equation

$$\sum_{(i,j,\ell)} \lambda_i\lambda_j\lambda_\ell^2 = 0 , \tag{4}$$

where (i,j,ℓ) runs through elements of the following set:

$$\{(i,j,\ell) \mid i,j,\ell : \text{integers} ; 1 \leq i,j,\ell \leq 4 ; i<j ; i\neq j, j\neq \ell, \ell\neq i\} .$$

Since char.k = p = 3 , the affine algebraic set defined by $\det A = 0$ in $\mathbb{A}^3$ is given by the equation (4) with $\lambda_4 \neq 0$. Using the symmetry of the equation (4), we can easily check that S is irreducible. Therefore, $MV_{1,2}$ is irreducible, hence, $V_{1,2}$ is irreducible. Since

$V_{1,1} = \Pi(V_{1,2})$, we conclude that $V_{1,1}$ is also irreducible.

We return to the equation in (3). Set

$$\mu_1 = \lambda_1 + 1\ ,\quad \mu_2 = \lambda_2 + 1\ ,\ \mu_3 = \lambda_3 + 1\ .$$

Since $p = 3$, we have

$$(\mu_1+\mu_2+\mu_3)^2 - (\mu_1\mu_2+\mu_2\mu_3+\mu_1\mu_3) - (\mu_1+\mu_2+\mu_3)\mu_1\mu_2\mu_3 = 0\ .$$

We set $u = (1-\mu_1\mu_2)(\mu_1+\mu_2+\mu_3)$. Then, we have

$$u^2 - (1-\mu_1\mu_2)(\mu_1+\mu_2)u + (1-\mu_1\mu_2)(\mu_1{}^2+\mu_1\mu_2+\mu_2{}^2) = 0$$

Set

$$v = \{u + (1-\mu_1\mu_2)(\mu_1+\mu_2)\}/\mu_1{}^2\ ,\ \mu = 1/\mu_1 \text{ and } t = \lambda_2\ .$$

Then, we have

$$v^2 = t(1-t^2)\mu^3 + t^4\mu^2 - t(1+t^2)\mu + t^2\ .$$

Setting $Y = t(1-t^2)v$ and $X = t(1-t^2)\mu$, we have

(5) $$Y^2 = X^3 + t^4X^2 + 2t^2(1-t^4)X + t^4(1-t^2)^2\ .$$

Therefore, considering t as a global coordinate of the affine line $\mathbb{A}^1$ in $\mathbb{P}^1$, we have an elliptic surface over $\mathbb{A}^1$. The discriminant $\Delta(t)$ of the Weierstrass normal form (5) in the sense of Ogg [7] is given by $\Delta(t) = t^6(t-1)^6(t+1)^6$. Set

$$s = 1/t\ ,\ y = Y/t^6 \text{ and } x = X/t^4\ .$$

Then, we have

(6) $$y^2 = (x-s^2+1)(x-s^3-s^2)(x+s^3-s^2)\ .$$

This gives the structure of an elliptic surface around the point at infinity of $\mathbb{P}^1$. The discriminant $\Delta(s)$ of the Weierstrass normal form (6) is given by $\Delta(s) = s^6(s-1)^6(s+1)^6$. We denote by $f: X \longrightarrow \mathbb{P}^1$ the relatively minimal non-singular complete model of the elliptic surface defined by (5) and (6). We denote by Δ the discriminant of this elliptic surface. Since the equations (5) and (6) are minimal Weierstrass forms in the sense of Ogg [7], the degree of the second Chern class $c_2(X)$ is equal to $\sum_{P\in\mathbb{P}^1} \mathrm{ord}_P \Delta = 24$. Hence, X is a K3 surface. Incidentally, $f: X \longrightarrow \mathbb{P}^1$ has four singular fibres. One of them is of type c_4 and other singular fibres are of type b_6 (for notation, see Ogg [7]) .

We consider the extension $k(x,y,s,z)/k(x,y,s)$ defined by $x - s^2 + 1 = z^3$. This is a purely inseparable extension of degree 3 . Since $x - s^2 + 1 = z^3$, $x - s^3 - s^2 = (z-1-s)^3$ and $x + s^3 - s^2 = (z-1+s)^3$, we have

$$y^2 = \{z(z-1-s)(z-1+s)\}^3 .$$

Setting $y/z(z-1-s)(z-1+s) = w$, we have

$$w^2 = z(z-1-s)(z-1+s) .$$

Considering s as a global coordinate of $\mathbb{A}^1$ in $\mathbb{P}^1$, we have an elliptic surface over $\mathbb{P}^1$. We denote by $\tilde{f}: \tilde{X} \longrightarrow \mathbb{P}^1$ the relatively minimal non-singular complete model of this elliptic surface. By the same method as above, we have $\deg c_2(\tilde{X}) = 12$. Therefore, $\tilde{X}$ is a rational surface. Therefore, X is a Zariski surface. Since X is birationally equivalent to $V_{1,2}$, we conclude that $V_{1,2}$ is a Zariski K3 surface.

Finally, by (1) and (3) the function field of $V_{1,1}$ is isomorphic to $k(\sigma_1, \sigma_2, \sigma_3)$ with relation (3) . Since $k(\sigma_1, \sigma_2, \sigma_3) = k(\sigma_2, \sigma_3)$, $V_{1,1}$ is birationally equivalent to a rational surface.

Remark 1. Assume char. $k = p = 2$. In this case, by the result of Igusa [3], we see that the locus of abelian surfaces of p-rank ≤ 1 is an irreducible rational surface.

Remark 2. Assume char. $k = p > 0$. Let $\mathcal{A}_{2,1,n}$ be the moduli variety of principally polarized abelian surfaces with level n-structure, $(n,p) = 1$. It seems that T. Ekedahl and F. Oort proved the irreducibility of the locus $V_{1,n}$ of abelian surfaces of p-rank ≤ 1 in

$\mathcal{A}_{2,1,n}$. Considering this fact, we propose the following problem.

Problem. Is $V_{1,n}$ unirational?

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ROOT MULTIPLICITIES FOR GENERAL KAC-MOODY ALGEBRAS

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In this note we announce a new algorithm for obtaining the multiplicity of the roots of any Kac-Moody algebra. This problem has gained heightened interest since Kumar [1987] extended the scope of the Weyl-Kac denominator identity and Mathieu [1988] proved that the radical of a Kac-Moody algebra is zero.

To our knowledge, there are currently two available procedures for finding the root multiplicities due to Peterson [1982] and to Berman and Moody [1979] . The former is limited in application to symmetrisable algebras while the latter requires the solution of the multi-partite partition problem for vectors of length equal to the rank of the algebra . It is well-known that this demands large memory storage and is time-consuming.

Our method which is based on the paper of Berman and Moody reduces the problem of finding partitions to that for a *single* number, namely the height of the root. It proceeds by recursion on the height and in one step for each value obtains the multiplicities of all roots of given height.

For a Kac-Moody algebra, L , of rank n let α_i be n simple roots. We assume that the Cartan matrix $A = (a_{ij})$ corresponds to a connected Coxeter-Dynkin diagram, and denote the *fundamental* weights by α^i so $\alpha^i(h_j) = \delta^i_j$ where h_j are, as usual, generators of the Cartan subalgebra H. Then $\rho := \sum \alpha^i$, $1 \le i \le n$. For any $w \in W$ where W is the Weyl group we define $\rho(w) := \rho - w(\rho)$. It is trivial to show that $\rho(w_1 w_2) = \rho(w_1) + w_1\rho(w_2)$. Since $w(\rho)$ is a weight of a representation of L with top weight ρ, it is clear that $\rho(w)$ is a positive sum of the simple roots, that $\rho(w_1) = \rho(w_2)$ is equivalent to $w_1 = w_2$, and that for $S_i w$ in reduced form $ht\rho(S_i w) > ht(\rho(w))$. Here, of course, S_i denotes the reflection in the simple root α_1. Denote the neutral element of W by w_o and order W in a sequence $w_o, w_1, w_2 \ldots$ such that the lengths $\ell(w)$ are monotonely non-decreasing. We calculate $\rho(w_j)$ recursively and use the simple check that $\rho(w') = \rho(w'')$ is equivalent to $w' = w''$ to eliminate redundant group elements.

The list of $\rho(w)$ is crucial for finding both root and weight multiplicities. We were surprised that our programme, implemented in Pascal, showed that for the matrix

$$\begin{bmatrix} 2 & -2 \\ -3 & 2 \end{bmatrix} \tag{1}$$

there are only *26* $\rho(w)$ with height less than 10,000.

Our starting point is the Weyl denominator identity:

$$\Pi\ (1 - \exp(-\alpha))^{m_\alpha} = \sum \varepsilon(w) \exp(-\rho(w)) \tag{2}$$

where the product is over all positive grades α ; the sum is over all $w \in W$; m_α is the multiplicity of the grade α which is 0 if α is not a root; and $\varepsilon(w) = (-1)^{\ell(w)}$.

We define $x_i := \exp(-\alpha_i)$ for $1 \le i \le n$ and set $x^\lambda := \exp(-\lambda)$, so if ,for example $\lambda = 2\alpha_1 + 3\alpha_2$, $x^\lambda = x_1^2 x_2^3$. Further for $j \in Z^+$ $\varepsilon(s_j) := -\varepsilon(w_j)$ and $\Delta := \sum \varepsilon(s_j)\, x^{s_j}$ with $s_j := -\rho(w_j)$,for $j > 0$.

Taking the logarithm of (2) we obtain

$$\sum m_\alpha \ln(\, 1 - x^\alpha\,) = \ln(\, 1 - \Delta\,)\ . \tag{3}$$

Define $B(\lambda)$ such that

$$ht(\lambda)\ B(\lambda) = \sum m_\alpha\ ht(\alpha) \tag{4}$$

where the sum is on those α for which there is a $k \in Z^+$ such that $\lambda = k\alpha$. Then using the argument of Berman and Moody it is easy to show that

$$\sum B(\lambda)\ x^\lambda = \ln(\, 1 - \Delta\,)\ . \tag{5}$$

We use (5) to obtain $B(\lambda)$ for all λ of a given height which enables us by (4) to obtain the m_α. We shall indicate the procedure for L with Cartan matrix (1). The $\rho(w_j)$ for this algebra are such that setting $x_1 = x$ and $x_2 = y$, gives

$$\Delta = x + y - x^3y - xy^4 + x^8y^4 + x^3y^9 - \dots \tag{6}$$

The right hand side of (5) is

$$\Delta/1 + \Delta^2/2 + \Delta^3/3 + \dots \tag{7}$$

Suppose that the *height* of s_j is d_j and that $ht(\lambda) = \sum k^j d_j$ summed on $k^j \in Z^+$, and that $\sum k^j = k$. Then there will be a term of degree equal to $ht(\lambda)$ in Δ^k of the form

$$(x+y)^{k^1}(x^3y)^{k^3}(xy^4)^{k^4}(x^8y^4 + x^3y^9)^{k^5}\dots \tag{8}$$

with an appropriate sign and a coefficient obtained from the multinomial theorem involving only the k and k_j. The sum of all such terms for the various partitions of $ht(\lambda)$ in terms of the parts $d_j = ht(\rho(w_j))$ gives the $B(\lambda)$ for all λ of a given height.

The reader will quickly verify that this algorithm gives 1 for the multiplicity of a simple root, and 0 for any multiple of a simple root. Verifying that $m(4\alpha_1 + 5\alpha_2) = 6$, will demonstrate the reader's mastery of the algorithm!

As a check on our procedure we obtained the tables H_2 and H_3 on pages 164/5 of Kac[1985] and some roots and multiplicities for the Kac-Moody algebra with matrix (1) which were kindly supplied to us by Prof. J. Patera. These algebras are *symmetrisable*, as follows from the known symmetrisability conditions which were first exhibited by Killing in 1889 [cf. Coleman, 1989]. Of course we have calculated the multiplicities for several non-symmetrisable algebras. Our procedure can be easily adapted to finding the multiplicities of weights of highest weight representations for general Kac-Moody algebras. This together with numerical results and some anticipated modification of our algorithm will be reported elsewhere in due course. As a foretaste of these results we note that for the Kac-Moody algebra of rank *five* with the following Cartan matrix

$$\begin{bmatrix} 2 & -73 & -4 & -1 & -7 \\ -8 & 2 & -9 & -1 & -1 \\ -17 & -5 & 2 & -9 & -4 \\ -71 & -3 & -2 & 2 & -17 \\ -43 & -8 & -57 & -34 & 2 \end{bmatrix}$$

the multiplicity of the root (0,1,1,1,1) is *6*. Compiling and running our programmeme on the Queen's IBM 3081G in VM/CMS mode required 16 seconds to obtain the multiplicities of *98 roots of* lowest height in this case. We are confident that our algorithm can be significantly improved.

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ON FUNCTIONS WITH GRAPHS INVARIANT UNDER SOME ROTATIONS

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Abstract. If $\mathcal{R}$ is an appropriate set of rotations of R^2 around the origin $(0,0)$, then there exists an additive function $f:R \longrightarrow R$ such that the graph Γ_f of f is mapped onto itself by every rotation T belonging to $\mathcal{R}$.

1. In [1], p.114, Problem 4, C. Alsina and J.L. García-Roig posed a problem the open part of which may be stated as follows: Given a rotation $T:R^2 \longrightarrow R^2$ around the origin $(0,0)$, does there exist a function $f:R \longrightarrow R$ with the property $T(\Gamma_f) = \Gamma_f$? A. Járai gave an affirmative answer in [1], p.120, Remark 11. In an informal discussion on the same meeting, B. Schweizer raised the question whether there are also additive functions $f:R \longrightarrow R$ such that $T(\Gamma_f) = \Gamma_f$. It is the purpose of this note to establish also an affirmative answer to the second question, even in a generalized version (Theorem 3 below), the direction of generalization being stimulated by A. Járai [2].

As usual, Q and R denote the fields of rational and real numbers, respectively, and for a subset S of R, $Q(S)$ stands for the extension field of Q generated by S. We find it convenient to identify the rotation $T:R^2 \longrightarrow R^2$ with the real matrix $\begin{pmatrix} a_T & b_T \\ -b_T & a_T \end{pmatrix}$, $a_T^2 + b_T^2 = 1$, representing T with respect to the standard basis $\{(1,0),(0,1)\}$ of R^2. Therefore

(1) $T(x,f(x)) = (a_T x + b_T f(x), -b_T x + a_T f(x))$ for all $x \in R$.

For the inverse rotation T^{-1} of T, we have $a_{T^{-1}} = a_T$, $b_{T^{-1}} = -b_T$,

(2) $T^{-1}(x,f(x)) = (a_T x - b_T f(x), b_T x + a_T f(x))$ for all $x \in R$.

Finally, if $f:R \longrightarrow R$ is an additive function, then

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$H_f := \{\lambda \in R;\ f(\lambda x) = \lambda f(x) \quad (\forall x \in R)\}$ is a subfield of R, the so-called <u>homogeneity field</u> of f.

<u>2</u>. <u>Lemma 1</u>. <u>For an additive function</u> $f: R \longrightarrow R$ <u>and a rotation</u> $T: R^2 \longrightarrow R^2$ <u>around</u> $(0,0)$, <u>the following statements are equivalent</u>:

(i) $T(\Gamma_f) = \Gamma_f$,

(ii) $a_T \in H_f$ <u>and</u> $f[b_T f(x)] = -b_T x \quad (\forall x \in R)$.

<u>Proof</u>. Let be $x \in R$ arbitrary. Then $(x, f(x)) \in \Gamma_f$, so $T(x,f(x)) \in \Gamma_f$, $T^{-1}(x,f(x)) \in \Gamma_f$, i.e., by (1), (2), and additivity of f,

(3) $\quad f(a_T x) + f(b_T f(x)) = f(a_T x + b_T f(x)) = -b_T x + a_T f(x)$,

(4) $\quad f(a_T x) - f(b_T f(x)) = f(a_T x - b_T f(x)) = b_T x + a_T f(x)$.

Adding and subtracting (3) and (4) and division by 2 lead to

(5) $\quad f(a_T x) = a_T f(x) \quad , \quad f(b_T f(x)) = -b_T x$.

Since $x \in R$ was arbitrary, the implication (i) $\Longrightarrow$ (ii) holds. For the converse, we remark that for an arbitrary $x \in R$, (ii) implies (5). By (5) and additivity of f we obtain (3) and (4), and then (1), (2) imply $T(x,f(x)) \in \Gamma_f$ and $T^{-1}(x,f(x)) \in \Gamma_f$. Since $x \in R$ was arbitrary, we have $T(\Gamma_f) \subset \Gamma_f$ and $T^{-1}(\Gamma_f) \subset \Gamma_f$, thus (i) holds.

<u>Lemma 2</u>. <u>Let</u> F <u>be a subfield of</u> R <u>such that the degree of</u> R <u>over</u> F <u>is infinite, and let be</u> $d \in R \setminus \{0\}$ <u>such that</u> $d^2 \in F$. <u>Then</u>:

a) <u>There exists an additive function</u> $f: R \longrightarrow R$ <u>such that</u> $F \subset H_f$ <u>and</u>

(6) $\quad f[df(x)] = -dx \quad (\forall x \in R)$.

b) <u>If</u> $d \in F$, <u>the function</u> f <u>of part</u> a) <u>moreover satisfies</u>

(7) $\quad f[f(x)] = -x \quad (\forall x \in R)$ ([3], p.117/118, Example 9).

c) <u>If</u> $d \in R \setminus F$, <u>then there exists an additive function</u> $f: R \longrightarrow R$ <u>satisfying</u> $F \subset H_f$ <u>and</u> (6), <u>but violating</u> $d \in H_f$ <u>and</u> (7).

<u>Proof</u>. a) Let B be a Hamel base of the F-vector space R. Since B is infinite, there exist subsets B_1, B_2 of B such that $B =$

$B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, card B_1 = card B_2. Let $\varphi: B_1 \longrightarrow B_2$ be bijective. We now define the function $f: R \longrightarrow R$ by the requirements

$$f(c) = d^{-1}\varphi(c) \quad (c \in B_1); \quad f(c) = -d\varphi^{-1}(c) \quad (c \in B_2); \quad f \text{ F-linear.}$$

Clearly $F \subset H_f$. For $c \in B_1$ we have $f[df(c)] = f[dd^{-1}\varphi(c)] = f[\varphi(c)]$ $= -d\varphi^{-1}[\varphi(c)] = -dc$, while for $c \in B_2$, by using $d^2 \in F \subset H_f$, we get $f[df(c)] = f[-d^2\varphi^{-1}(c)] = -d^2 f[\varphi^{-1}(c)] = -d^2 d^{-1}\varphi[\varphi^{-1}(c)] = -dc$, i.e., $f[df(c)] = -dc$ $(\forall c \in B)$, hence (6).

b) immediately follows from a) and $f[df(x)] = df[f(x)]$ $(\forall x \in R)$.

c) As $d \in R \setminus F$, the elements 1 and d are F-linearly independent. Now we proceed as in the proof of a) with the following specifications: $1, d \in B$, $1 \in B_1$, $d \in B_2$, $\varphi(1) = d$. The properties $F \subset H_f$ and (6) are ensured by a), and $f(1) = d^{-1}\varphi(1) = d^{-1}d = 1$, $f(d) =$ $-d\varphi^{-1}(d) = -d \neq d = df(1)$, $f[f(d)] = f(-d) = -f(d) = d \neq -d$, i.e., $d \notin H_f$, and (7) does not hold.

<u>Remark</u>. Lemma 2c) is to show that for $d \in R \setminus F$, the set of additive functions satisfying (6) is distinct from the set of those satisfying (7). Accordingly, we shall treat in the proof of Theorem 3 the case of a single rotation separately.

<u>Theorem 3</u>. <u>If $\mathcal{R}$ is a given set of rotations of</u> R^2 <u>around the origin</u> (0,0) , <u>and if</u> card $\mathcal{R}$ < card R , <u>then there exists an additive function</u> $f: R \longrightarrow R$ <u>such that</u> $T(\Gamma_f) = \Gamma_f$ <u>for all</u> $T \in \mathcal{R}$.

<u>Proof</u>. We put $S := \{a_T, b_T ; T \in \mathcal{R}\}$ (S may be empty) and $F :=$ $Q(S)$. Card $\mathcal{R}$ < card R implies card S < card R, so card F = max$\{$card Q, card S$\}$ < card R , therefore R has infinite degree over F . Let be $T \in \mathcal{R}$ arbitrary. Putting $d = 1$, we obtain from Lemma 2b) an additive function $f: R \longrightarrow R$ such that $F \subset H_f$ and (7) hold. Hence $a_T \in H_f$, $b_T \in H_f$, $f[b_T f(x)] = -b_T x$ $(\forall x \in R)$, and by Lemma 1 $T(\Gamma_f) = \Gamma_f$. Since $T \in \mathcal{R}$ was arbitrary, the assertion is proved.

Lemma 1 suggests a more adequate argument for the case $\operatorname{card} \mathcal{R} = 1$: For $b_T = 0$ we have $T = \pm \mathrm{id}_{R^2}$, and then every additive function $f: R \longrightarrow R$ satisfies $T(\Gamma_f) = \Gamma_f$. Let in the following be $b_T \neq 0$ and $F := Q(a_T)$. Then again R has infinite degree over F, and $b_T^2 = 1 - a_T^2 \in F$. Putting $d = b_T$, Lemma 2a) ensures the existence of an additive function $f: R \longrightarrow R$ such that $a_T \in F \subset H_f$ and $f[b_T f(x)] = -b_T x \quad (\forall x \in R)$. The assertion now follows from Lemma 1.

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ON THE LAGUERRE EXPANSIONS OF GENERALIZED FUNCTIONS

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Abstract. We give the intrinsic description of the basic spaces LG_α, $\alpha > -1$, which duals are generalized function spaces with elements having unique orthonormal expansions into Laguerre series.

Introduction. The spaces of generalized functions which elements have expansions with respect to Laguerre orthonormal bases depending on $\alpha > -1$, LG'_α, were introduced by Zemanian [4], (A' - type spaces). He gave structural properties of such spaces and Zayed [3] investigated them in connection with the boundary value problems. Zayed proved that

$$(1) \qquad LG_o = S_I = \{\phi \in C^\infty(0,\infty);\ \sup_{\substack{x\in(0,\infty)\\ k,r\le p}} |x^k\phi^{(r)}(x)| < \infty\ ,$$

$$p = 0,1,\dots\}\ .$$

We shall prove that the dual of LG_o is in fact S'_+, the space of tempered distributions on $(-\infty,\infty)$ supported by $[0,\infty)$. As well we shall prove that for $\alpha > -1$

$$LG_\alpha = x^{\alpha/2} LG_o = \{\psi \in C^\infty(0,\infty);\ \psi = x^{\alpha/2}\phi \quad \text{for some} \quad \phi \in LG_o\},$$

and quote some consequences of that.

Notation and notions. We shall alwaus denote by α a number greater than -1. The Laguerre polynomials L_n^α, $n=0,1,\dots$, are defined by

$$L_n^\alpha(x) = \sum_{m=o}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}\ ,\ x \in [0,\infty),$$

and the Laguerre orthonormal base $\ell_{n,\alpha}$ of the space $L^2(0,\infty)$ is defined by

$$\ell_{n,\alpha}(x) = x^{\alpha/2}e^{-x/2}L_n(x),\ x \in (0,\infty),\ n=0,1,\dots .$$

The Zemanian space LG_α is the space of all $\phi \in C^\infty(0,\infty)$ for which we have

(2) $\|\phi\|_k = \|R_\alpha^k\phi\|_2 = (\int_0^\infty |R_\alpha^k\phi(t)|^2 dt)^{1/2} < \infty,\ k=0,1,\dots ,$

(3) $(R_\alpha^k\phi, \ell_{n,\alpha}) = (\phi, R_\alpha^k\ell_{n,\alpha}),\ k,n = 0,1,\dots ,$

where

$$R_\alpha = x^{-\alpha/2}e^{x/2}Dx^{\alpha+1}e^{-x}Dx^{-\alpha/2}e^{x/2},$$

$R_\alpha^k = R_\alpha^{k-1}R_\alpha$, R_α^o is the identity operator, $k=0,1,\dots ,$

$(\phi,\psi) = \langle\phi,\bar{\psi}\rangle = \int_0^\infty \phi(t)\bar{\psi}(t)dt,\ \phi,\psi \in L^2(0,\infty).$

It is well-known that $R_\alpha\ell_{n,\alpha} = n\ell_{n,\alpha}$, $n=0,1,\dots$. For $\alpha = 0$ we shall use the notation $\ell_{n,o} = \ell_n$, $LG_o = LG$, ...

From the general theory of Zemanian we have:

(a) With seminorms (2) LG_α is a Fréchet space.

Let $\phi \in LG_\alpha$, $f \in LG'_\alpha$. Then

(b) $\phi = \sum_{n=o}^\infty (\phi,\ell_{n,\alpha})\ell_{n,\alpha}$,where the series converges in LG_α.

(c) $\sum_{n=o}^\infty a_n\ell_{n,\alpha}$ converges in LG_α iff $\sum_{n=1}^\infty n^{2k}|a_n|^2 < \infty$, $k=0,1,\dots$.

(d) $f = \sum_{n=o}^\infty (f,\ell_{n,\alpha})\ell_{n,\alpha}$ where the series converges in LG'_α.

(e) $\sum_{n=o}^\infty b_n\ell_{n,\alpha}$ converges in LG'_α if for some non-negative integer r, $\sum_{n=1}^\infty n^{-2r}|b_n|^2 < \infty$. Moreover, if f is the limit of this series in LG'_α, then $b_n = (f,\ell_{n,\alpha})$, $n=0,1,\dots$.

As we mentioned, with the corresponding sequence of seminorms in S_I:

(f) $S_I = LG.$

Results

Proposition 1. *S_I is the space of all $\phi \in C^\infty[0,\infty)$ for which the norms*

$$\sup_{\substack{x\in[0,\infty)\\ k,r\leq p}} |x^k \phi^{(r)}(x)|, \quad p=0,1,\dots ,$$

are all finite.

Proof. For the properties of Leguerre polynomials we refer to [1]. Let $\phi = \sum_{n=0}^{\infty} a_n \ell_n \in S_I$. It is enough to prove that for any non-negative integer r, $\phi^{(r)}$ has a continuous extension from $(0,\infty)$ on $[0,\infty)$. Since $\ell_n(0)=1$, $n=0,1,\dots$, we have $\phi_0 = \sum_{n=0}^{\infty} a_n < \infty$. Since

(4) $$|\ell_n(x)| \leq 1, \; x\in[0,\infty), \quad n=0,1,\dots ,$$

we have

$$\lim_{x\to+0} \sum_{n=0}^{\infty} a_n \ell_n(x) = \phi_0 = \phi(0).$$

From [1, p.189 (15) and p.192 (38)] follows

(5) $$\ell_n' = \sum_{m=0}^{n-1} \ell_m - \frac{1}{2}\ell_n, \quad n=0,1,\dots \quad \left(\sum_{0}^{-1} = 0\right).$$

We need the following assertion as well:

(6) $$\begin{cases} \sum_{n=1}^{\infty} |a_n| n^{2k} < \infty \text{ for every non-negative integer } k \\ \sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} |a_n|\right)^2 n^{2k} < \infty \text{ for every non-negative integer } k. \end{cases}$$

On $[0,\infty)$ we have

$$\sum_{n=0}^{\infty} a_n \ell_n' = \sum_{n=0}^{\infty} a_n \left(\sum_{m=0}^{n-1} \ell_m - \frac{1}{2}\ell_n\right) =$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2} a_n + \sum_{j=n+1}^{\infty} a_j\right)\ell_n ,$$

where the second equality follows from (4) and the fact that $\sum_{n=0}^{\infty} a_n \ell_n'$ uniformly converges on $[0,\infty)$. So we get

$$\phi'(x) = \left(\sum_{n=0}^{\infty} a_n \ell_n(x)\right)' = \sum_{n=0}^{\infty} a_n \ell_n'(x), \; x\in[0,\infty)$$

and that ϕ' is continuous in 0 from the right.

Now, by induction, we prove the same for $\phi^{(r)}$, $r = 2,3,\ldots$.

Corollary 2. *LG' is the space S'_+ consisted of tempered distributions on $(-\infty,\infty)$ supported by $[0,\infty)$.*

Proof. This follows from [2, p.24] where it is proved (in Hörmander's notation) that

$$LG = S(0,\infty) \quad \text{and} \quad LG' = \overset{o}{S}'([0,\infty)) = S'_+ .$$

Proposition 3. $LG_\alpha = \{\psi \in C^\infty(0,\infty);\ \psi(x) = x^{\alpha/2}\phi(x)$, $x \in (0,\infty)$ *for some* $\phi \in S_I\}$.

Proof. Let $\psi = \sum_{n=0}^{\infty} a_n \ell_{n,\alpha} \in LG_\alpha$. From

$$L_n^\alpha(x) = \sum_{m=0}^{n} \frac{\Gamma(m+\alpha)}{\Gamma(m)} L_{n-m}(x), \quad x \in [0,\infty), ([1, p.192(39)])$$

(4) and (6) and the fact that $\{\frac{\Gamma(\alpha+n)}{\Gamma(n)}, n = 0,1,\ldots\}$ is bounded, we have

$$\psi(x) = x^{\alpha/2} \sum_{n=0}^{\infty} a_n e^{-x/2} \sum_{m=0}^{n} \frac{1}{m!} \frac{\Gamma(\alpha+m)}{\Gamma(m)} L_{n-m}(x) =$$

$$= x^{\alpha/2} \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(\alpha+n)}{\Gamma(n)} a_{n+m} \Big) \ell_m(x) =$$

$$= x^{\alpha/2}\phi(x), \quad x \in (0,\infty), \text{ where } \phi \in S_I .$$

Conversely, if $\phi \in S_I$ and $x \in (0,\infty)$ we have

$$R_\alpha(x^{\alpha/2}\phi(x)) = \frac{\alpha+1}{2} x^{\alpha/2}\phi(x) - \frac{1}{4} x^{\alpha/2}(x\phi(x)) + (\alpha+1)x^{\alpha/2}\phi'(x) +$$

$$+ x^{\alpha/2}(x\phi''(x)) = x^{\alpha/2}\phi_1(x),$$

where $\phi_1 \in S_I$. The proof is complete.

Corollary 4. *A sequence ψ_n from LG_α converges to 0 in LG_α iff $\psi_n(x) = x^{\alpha/2}\phi_n(x)$, $x \in (0,\infty)$ where ϕ_n is a sequence from S_I which converges to 0 in S_I.*

Note that the derivation in LG_α , $\alpha \neq 0$, is not an inner operation.

Corollary 5. S_I *is an ideal in* LG_α.

Let $\lambda_o \in C^\infty(0,\infty)$, $\lambda_o = 1$ on $(0,\varepsilon)$, $\lambda_o = 0$ on $(2\varepsilon,\infty)$. This function is from S_I. By a partition of unity ψ_n, $n = 0,1,\ldots$ such that $\psi_o = \lambda_o$ and $\psi_i \in \mathcal{D}(0,\infty)$, $i = 1,2,\ldots$, one can localize the properties of elements in LG'_α.

Corollary 6. *An* f *is from* LG'_α *iff it is of the form*

$$f(x) = x^{-\alpha/2}F^{(m)}(x), \quad x \in (0,\infty)$$

where F *is a continuous function on* $(-\infty,\infty)$ *supported by* $[0,\infty)$, *and the dual pairing between* f *and* $\psi = x^{\alpha/2}\phi \in LG_\alpha$ *is given by*

$$< f,\psi > = < F^{(m)},\phi > .$$

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On entire functions additive on graphs

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Presented by J. Aczél, F.R.S.C.

1. Studying Cauchy functional equations on restricted domains interesting situations appear when the set where the equation holds is extremally thin, for instance when it is the graph of a given or of the unknown function. This leads to the following two functional equations

$$f(x + f(x)) = f(x) + f(f(x)) \tag{1}$$

$$f(x + h(x)) = f(x) + f(h(x)) \tag{2}$$

(h given function).

About equations (1) and (2) many results have been obtained when $f : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ and $h(0) = 0$ (see [1]-[6]); since all proofs use essentially the ordering of the real line they cannot be generalized to higher dimensions.

In this paper we study equations (1) and (2) when $f : \mathbb{C} \to \mathbb{C}$, $h : \mathbb{C} \to \mathbb{C}$, $f(0) = h(0) = 0$ and both functions are entire.

2. From now on the functions f and h will be expressed as power series

$$f(z) = \sum_{n=1}^{\infty} c_n z^n, \quad h(z) = \sum_{n=1}^{\infty} d_n z^n ,$$

thus equations (1) and (2) become

$$\sum_{n=2}^{\infty} c_n z^n \left[\sum_{k=1}^{n-1} \binom{n}{k} \left(\sum_{m=1}^{\infty} c_m z^{m-1} \right)^k \right] = 0 \tag{3}$$

and

$$\sum_{n=2}^{\infty} c_n z^n \left[\sum_{k=1}^{n-1} \binom{n}{k} \left(\sum_{m=1}^{\infty} d_m z^{m-1} \right)^k \right] = 0 , \tag{4}$$

respectively. The following sequence of polynomials

$$P_n(z) := (1 + z)^{n+1} - (1 + z^{n+1}) = \sum_{k=1}^{n} \binom{n+1}{k} z^k, \ n \geq 1$$

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is of interest. We denote by $\Re_n$ the set of the roots of P_n and we define $\Re := \bigcup_{n=1}^{\infty} \Re_n$. The first result is given by the following

Theorem 1. *If $c_1 \notin \Re$ then equation (1) has the only solution $f(z) = c_1 z$. If $d_1 \notin \Re$ then equation (2) has only affine (first degree polynomial) solutions.*

Proof. The coefficient of z^2 on the left hand side of (3) is $2c_1c_2 = c_2 P_1(c_1)$. Since $P_1(c_1) \neq 0$ we must have $c_2 = 0$. Assume now $c_2 = \ldots = c_n = 0$: the coefficient of z^{n+1} in (3) is $c_{n+1}P_n(c_1)$ then $c_{n+1} = 0$ and, by induction, we get the result. In the same way we prove the result for equation (2).

Q.E.D.

It is easy to prove that in $\Re$ there are only two real numbers: 0 and -1. More exactly, $0 \in \bigcap_{n=1}^{\infty} \Re_n$ and $-1 \in \bigcap_{n=1}^{\infty} \Re_{2n}$. We now examine equations (1) and (2) when $c_1 \in \{0, -1\}$ and $d_1 \in \{0, -1\}$ respectively.

Theorem 2. *If $c_1 \in \{0, -1\}$ then the only solution of equation (1) is $f(z) = c_1 z$ (i.e. $f(z) = 0$ or $f(z) = -z$).*

Proof. The case $c_1 = 0$: on the left hand side of (3) the coefficient of z^3 is $2c_2^2$, hence $c_2 = 0$. Assume $c_2 = \ldots = c_n = 0$; the coefficient of z^{2n+1} is $(n+1)c_{n+1}^2$, so $c_{n+1} = 0$.

The case $c_1 = -1$: from (3), we have $c_2 P_1(-1) = 0$, so $c_2 = 0$; from $c_3 P_2(-1) = 0$ we have no information, since $P_2(-1) = 0$; $c_4 P_3(-1) = 0$ implies $c_4 = 0$; the coefficient of z^5 is

$$c_5 P_4(-1) + c_3^2 P_2'(-1) = c_3^2 P_2'(-1)$$

and since $P_n'(-1) \neq 0$ for every n, we have $c_3 = 0$. Assume now $c_2 = c_3 = \ldots = c_n = 0$: if n is odd, then $P_n(-1) \neq 0$ and we get $c_{n+1} = 0$. If n is even, we compute the coefficient of z^{2n+1}, obtaining

$$c_{2n+1} P_{2n}(-1) + c_{n+1}^2 P_n'(-1) = c_{n+1}^2 P_n'(-1)\,,$$

thus $c_{n+1} = 0$. By induction we have $f(z) = -z$.

Q.E.D.

About equation (2) we notice that if h is identically zero (so $d_1 = 0$) then any function f is a solution while if $h(z) = -z$ (so $d_1 = -1$) then any odd function f is a solution. So we assume $d_1 \in \{0, -1\}$ and $d_n \neq 0$ for some $n > 1$.

Theorem 3. *If $d_1 = 0$ (and $d_n \neq 0$ for some $n > 1$) then equation (2) has only affine solutions.*

Proof. The coefficient of z^3 in (4) must vanish so we obtain $c_2 d_2 = 0$. There are two cases.

α) $d_2 = 0$:

Assume the existence of a non-affine solution of (2) and let c_k, $k \geq 2$, be the first coefficient different from zero. If $d_2 = d_3 = \ldots = d_n = 0$, then computing the coefficient of z^{n+k}, we must have $c_k d_{n+1} = 0$, so $d_{n+1} = 0$. This implies that h is identically zero contrary to our hypothesis.

β) $d_2 \neq 0$, $c_2 = 0$:

Assume $c_2 = c_3 = \ldots = c_n = 0$; setting the coefficient of z^{n+2} equal to zero we get $c_{n+1}d_2 = 0$, thus $c_{n+1} = 0$. Hence there are only linear solutions.

Q.E.D.

Theorem 4. *Let* $d_1 = -1$ *and* $d_n \neq 0$ *for some* $n > 1$.

(i) *If either* $d_{2k+1} = 0$, $k = 1, 2, \ldots$, *or* $d_{2k} = 0$, $k = 1, 2, \ldots$, *then equation (2) has only linear solutions.*

(ii) *The only even solutions of (2) are the linear functions.*

Proof. (i) In the first case assume d_{2r} to be the first non-zero coefficient in h (after d_1). We immediately see that $c_2 = c_4 \ldots = c_{2r} = 0$. Taking the coefficients of z^{2r+2} and z^{4r+1} in (4) we get

$$\begin{cases} c_{2r+2}P_{2r}(-1) + c_3 d_{2r} P_2'(-1) = 0 \\ c_{2r+2}P_{2r+1}'(-1) + \frac{1}{2} c_3 d_{2r} P_2''(-1) = 0 \end{cases}$$

so $c_3 = 0$ and $c_{2r} = 0$.

Assume now $c_2 = c_4 = \ldots = c_{2(r+s)} = 0$ and $c_3 = c_5 = \ldots = c_{2s+1} = 0$ and take the coefficients of $z^{2(r+s)+2}$ and $z^{4r+2s+1}$, we have

$$\begin{cases} c_{2r+2s+2}P_{2r+2s+1}(-1) + c_{2s+3} d_{2r} P_{2s+2}'(-1) = 0 \\ c_{2r+2s+2}P_{2r+2s+1}'(-1) + \frac{1}{2} c_{2s+3} d_{2r} P_{2s+2}''(-1) = 0 \end{cases}$$

and again $c_{2r+2s+2} = 0$ and $c_{2s+3} = 0$.

In the second case, if d_{2r+1} is the first non-zero coefficient in h (after d_1), we have $c_2 = c_4 = \ldots = c_{2r+2} = 0$. The coefficient of z^{2r+3} is $c_3 d_{2r+1} P_2'(-1)$, so $c_3 = 0$. The result follows by induction as before.

(ii) Let d_r be the first non-zero coefficient of h (after d_1) and compute in (4) the coefficient of z^{r+2}: we have $c_{r+2}P_{r+1}(-1) + c_3 d_r P_2'(-1) = 0$. If r is even $c_{r+2} = 0$, if r is odd $P_{r+1}(-1) = 0$, so we have $c_3 = 0$. By induction we have the result.

Q.E.D.

About even or odd solutions of equation (2) we can prove a more general result. First we need a lemma about polynomials P_n, which we state without giving the simple proof.

Lemma 1. *A polynomial P_n has multiple roots if and only if $n \equiv 0$ (mod 6) and the multiple roots are $e^{\pm i\frac{2}{3}\pi}$.*

Theorem 5. *Suppose that in h the first non-zero coefficient (after d_1) is d_{2s} (i.e. of even index). Then the only even solution of (2) is the zero function and, if $d_1 \neq e^{\pm i\frac{2}{3}\pi}$, the only odd solutions of (2) are the affine ones.*

Proof. Assume f even. If $d_1 \notin \Re$ we refer to Theorem 1. Assume $d_1 \in \Re$ but $d_1 \notin \bigcup_{n=0}^{\infty} \Re_{2n+1}$: from (4) it is $c_2 P_1(d_1) = 0$, so $c_2 = 0$; if $c_{2r} = 0$, $r = 1, ..., n$, then we must have $c_{2r+2}P_{2r+1}(d_1) = 0$, thus $c_{2r+2} = 0$ and, by induction, $f(z) \equiv 0$.

Suppose now $d_1 \in \bigcup_{n=0} \Re_{2n+1}$ and let P_{2r-1} be the polynomial with minimum odd degree having d_1 as a root. As before, we have $c_{2m} = 0$ for $m = 1, ..., r-1$ (also if d_1 is a root of some even polynomial). Equating to zero the coefficient of $z^{2r+2s-1}$ we have $c_{2r} d_{2s} P_{2r-1}'(d_1) = 0$ and, by Lemma 1, $c_{2r} = 0$.

By induction we have again $f(z) = 0$.

In the same way, interchanging the role of even and odd polynomials, we prove the case where f is odd.

Q.E.D.

From the proof of Theorem 5 we get immediately the following.

Corollary 1. *If $d_1 \notin \bigcup_{n=0}^{\infty} \Re_{2n+1}$ (for instance $d_1 = -1$) and $h(z) \neq d_1 z$, then the only even solution of (2) is $f(z) \equiv 0$. If $d_1 \notin \bigcup_{n=1}^{\infty} \Re_{2n}$ and $h(z) \neq d_1 z$, then the only odd solutions of (2) are the linear polynomials. If $c_1 \notin \bigcup_{n=0}^{\infty} \Re_{2n}$ the only odd solution of (1) is $f(z) = c_1 z$.*

The last result we prove is about polynomial solutions of equation (2). It is only for this theorem that we need f and h be entire functions, all other results hold in the ring of complex formal power series.

Theorem 6. *If h is not a polynomial, then the only polynomial solutions of equation (2) are the linear ones.*

Proof. By the "Great Picard Theorem" if we fix an arbitrary (with at most one exception) $\alpha \in \mathbb{C}\setminus\{0\}$, then there exist a sequence $\{z_n\}$, $z_n \to \infty$, such that $h(z_n) = \alpha$ for each n. So we have

$$f(z_n + \alpha) = f(z_n) + f(\alpha), \quad n \in \mathbb{N}$$

and, since f is polynomial,

$$f(z + \alpha) = f(z) + f(\alpha), \quad z \in \mathbb{C}.$$

This means

$$\sum_{k=2}^{n} c_k \sum_{s=1}^{k-1} \binom{k}{s} z^s \alpha^{k-s} = 0, \quad z \in \mathbb{C},$$

so all coefficients of the powers of z must be zero; the maximum power is z^{n-1} and its coefficient is $c_n \binom{n}{n-1} \alpha$, so $c_n = 0$. In the same way we get $c_{n-1} = c_{n-2} = \ldots = c_2 = 0$.

Q.E.D.

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ON THE LIFTING PROBLEM OVER AN ALGEBRAICALLY CLOSED FIELD

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Presented by G. de B. Robinson, F.R.S.C.

Abstract. I show that in a polynomial ring over an algebraically closed field there exist homogeneous ideals that cannot be lifted, in the sense of Geramita, Gregory, and Roberts, Journal of Pure and Applied Algebra 40(1986), 33-62.

In [2, Definition 1.7] the following "lifting" problem is discussed:

Let J be a homogeneous ideal in $S = k[X_1,\ldots,X_n]$ (k a field). Does there exist a homogeneous radical ideal I in $R = k[X_0,\ldots,X_n]$ such that

(a) X_0 is a non-zero divisor mod I

(b) $(I,X_0)/(X_0) \cong J$ (under an isomorphism that sends X_i to X_i for $i \geq 1$)?

If such an I exists, we say that I lifts J, or that J can be lifted (to I).

In [2] we forgot to state that I is homogeneous, and that under the isomorphism in (b), X_i is mapped to X_i for $i \geq 1$, but we certainly intended that these hypotheses hold. In [2, Theorem 2.2] we prove, following Hartshorne [3, section 4], that J can be lifted if J is generated by monomials, and the cardinality $|k|$ of k is large enough (i.e. either infinite, or finite, but larger

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that a certain finite number that can be given explicitly if one knows generators of J). When [2] was written we did not know any homogeneous ideals J that could not be lifted. Later Moshe Roitman in [6, Theorem 9] gave an example (with k finite) of a homogeneous ideal J in S that cannot be lifted. In [7] it was proved in a different way that such examples exist. Both methods depend on the finiteness of the field k. The purpose of this note is to show that non-liftable homogeneous ideals exist in abundance when k is an algebraically closed field, as a consequence of well known properties of the Hilbert scheme. First we prove:

Lemma *Let k be an algebraically closed field. Let* $J \subset S = k[X_1,\ldots X_n]$ *be a saturated homogeneous ideal such that* S/J *is of Krull dimension one (i.e.* J *is the homogeneous ideal of the zero-dimensional closed subscheme* $Z = \mathrm{Proj}\,(S/J) \subset \mathbb{P}_k^{n-1} = \mathrm{Proj}\, S)$. *Let* d *be the degree of* Z. *Suppose* J *can be lifted to* $I \subset R=k[X_0,\ldots,X_n]$. *Then* Z *can be deformed (over an open subscheme of* $\mathbb{A}_k^1$) *into d distinct reduced closed points.*

Proof The ring R/I is of Krull dimension 2. We have $R/(I,X_0) \cong S/J$, the homogeneous co-ordinate ring of the zero dimensional scheme Z. Thus $R/(I,X_0)$ contains a non-zero-divisor, so R/I is of depth 2, (hence Cohen-Macaulay). Since R/I is reduced, the scheme Proj(R/I) is a reduced, but possibly reducible, curve C of degree d. Hence for sufficiently general f of degree 1, the hyperplane $X_0+f = 0$ intersects C in d distinct points. Then there exists a finite set $T \subset k$ such that for $\lambda \notin T$ the hyperplane $X_0 + \lambda f = 0$ intersects C in d distinct points. For $\lambda \notin T$, $X_0 + \lambda f$ is a not a zero divisor in R/I. (The only way $X_0+\lambda f$ could be a zero divisor in R/I is for the hyperplane $X_0+\lambda f = 0$ to contain an entire irreducible component of C.) For $\lambda \notin T$ we have that $R/(I,X_0 + \lambda f)$ is Cohen-Macaulay (one dimensional), and hence contains a non-zero divisor, which implies that $(I,X_0+\lambda f)$ is saturated in R for $\lambda \notin T$. Now let W be the closed subscheme of $\mathbb{P}^{n-1} \times \mathbb{A}^1$ defined by the ideal $(I,X_0+tf) \subset R[t]$. Consider the projection $\pi: W \longrightarrow \mathbb{A}^1_k$. Let U be the (non-empty) open subscheme of $\mathbb{A}^1_k$ consisting of the prime ideals $(t-\lambda)$, such that $\lambda \notin T$, together with (t). For each closed point $p = (t-\lambda) \in U$, the fibre $\pi^{-1}(p) \subset \mathbb{P}^{n-1}_k$ has (reduced) homogeneous co-ordinate ring $R/(I,X_0+\lambda f)$. Thus over U the fibres of π have constant Hilbert function. Hence by [4, Theorem 9.9, page 261] the morphism π: $\pi^{-1}(U) \longrightarrow U$ is flat, thus giving a deformation of $\pi^{-1}(t) = Z$ into d reduced closed points. ∎

Iarrobino has proved [5] that the Hilbert scheme $\mathcal{H}_{d,n}$ of 0-dimensional subschemes of degree d in $\mathbb{P}^n$ ($n\geq3$, d sufficiently large) is reducible. (see [1, page 5] for another exposition of this.) Let Z correspond to a point z not in the closure of U_d = the d-fold symmetric product of $\mathbb{P}^n_k$ minus the "diagonals" (the open subscheme of $\mathcal{H}_{d,n}$ corresponding to d distinct points). Let J be the saturated ideal of Z. If J could be lifted, then the fibres of the deformation π given by our lemma consist of d distinct points over all but a finite number of $p \in U$. Thus the characteristic map of the flat family π will have to map U into the closure of U_d, thus missing z, contradiction. This proves:

<u>Theorem</u> <u>Let k be an algebraically closed field and let</u> $n\geq4$. <u>Then</u>

for d sufficiently large (depending on n) there is a zero dimensional subscheme Z of degree d in $\mathbb{P}_k^{n-1}$ such that the saturated ideal J of S cannot be lifted.

Roitman [6] has shown that if k has characteristic 0, then every homogeneous ideal in $k[X_1,X_2]$ can be lifted. Thus there remains the case n=3 about which I have no information. Note however that $\mathcal{H}_{d,2}$ is irreducible, so the above argument cannot be used to prove that non-liftable J exist if n=3.

Furthermore, in [2] we were primarily interested in the case where S/J was a zero-dimensional ring, whereas here I have been able to handle only the case where S/J is one dimensional.

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ON IRREDUCIBLE COMPONENTS OF A RESTRICTION OF AN IRREDUCIBLE INDUCED CHARACTER TO A NORMAL SUBGROUP

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ABSTRACT. A finite group is called an M-group if all of its irreducible characters are monomial. L. Dornhoff proved that a normal Hall subgroup of an M-group is an M-group also. We prove a one character version of this result.

Theorem. Let H be a subgroup, and N a normal subgroup of a finite group. Suppose that $\lambda \in \mathrm{Irr}(H)$, $\lambda^G = \chi \in \mathrm{Irr}(G)$ and ψ is an irreducible component of χ_N, a restriction of χ to N. If g.c.d.$(|NH:N|, |NH:H|\lambda(1)) = 1$ then $\psi = \{(\lambda_{N\cap H})^N\}^t$ for certain $t \in G$.

Remark. Observe that all irreducible components of χ_N are G-conjugate by Clifford's theorem.

Proof. Let $\phi = \lambda^{NH}$. Then $\phi^G = (\lambda^{NH})^G = \lambda^G = \chi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{Irr}(NH)$. Next

$$\chi_N = |NH|^{-1} \sum_{t \in G} (\phi_N)^t$$

by the formula for induced characters. Here $\phi^t : (NH)^t \to \mathbb{C}$ is defined by

$$\phi^t(x) = \phi(txt^{-1}) \quad \text{for} \quad x \in (NH)^t = NH^t.$$

Obviously, $\phi^t \in \mathrm{Irr}(NH^t)$.

By assumption, $\langle \chi_N, \psi \rangle > 0$. Thus $\langle (\phi^t)_N, \psi \rangle > 0$ for certain $t \in G$. First we prove that $\psi = \phi_N^t = (\phi_N)^t$. For this it is sufficient to show that $\psi(1) = \phi(1)$.

Let $(\phi^t)_N = e(\psi_1 + \dots + \psi_k)$ be Clifford's decomposition with $\psi_1 = \psi$.

Let $I_{NH}(\psi)$ be the inertia group of ψ in NH^t. Then k is equal to the index of $I_{NH^t}(\psi)$ in NH^t. By Clifford's theory the ramification e equals the degree of irreducible projective representation of $I_{NH^t}(\psi)/N$, so e divides $|I_{NH^t}(\psi)/N|$ by Schur's theorem. Hence $ek = \frac{\phi(1)}{\psi(1)}$ divides

$$|I_{NH^t}(\psi):N|\cdot|NH^t:I_{NH^t}(\psi)| = |NH^t:N| = |NH:N|.$$

Since $\phi(1) = \lambda^{NH}(1) = |NH:H|\lambda(1)$, then $\frac{\phi(1)}{\psi(1)}$ divides $|NH:H|\lambda(1)$. Finally, $\frac{\phi(1)}{\psi(1)}$ divides

$$\text{g.c.d.}(|NH:N|, |NH:H|\lambda(1)) = 1.$$

Hence $\phi^t(1) = \phi(1) = \psi(1) \Rightarrow \psi = (\phi^t)_N = (\phi_N)^t$.

By the Mackey decomposition theorem

$$\phi_N = (\lambda^{NH})_N = (\lambda_{N\cap H})^N,$$

since $\phi_N \in \mathrm{Irr}(N)$, and $\psi = (\phi_N)^t = \{(\lambda_{N\cap H})^N\}^t$. □

Dornhoff's theorem, mentioned earlier, is an immediate corollary to our theorem.

It may be worthwhile to note the following:

Corollary. Suppose that $H < G$, $1 < N \triangleleft G$, and g.c.d.$(|N|,|H|) = 1$. If λ is a linear character of H, then λ^G is reducible.

Proof. Suppose that $\lambda^G = \chi \in \mathrm{Irr}(G)$. Let ψ be an irreducible component of χ_N. Then by Theorem $\psi = \{(\lambda_{N\cap H})^N\}^t$ for certain $t \in G$. Since $H \cap N = 1$, $(\lambda_{N\cap H})^N = \rho_N$ is the regular character of N. But $N > 1$, and so ρ_N is reducible. Then ψ is also reducible which is a contradiction. □

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GEOMETRIC PROPERTIES OF UNITARY-SYMMETRIC KAEHLER MANIFOLDS

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Presented by J. Arthur, F.R.S.C.

1. Introduction.

The notion of unitary-symmetry on Kaehler manifolds is a differential geometric version of Kaup's result on the isotropy subgroups of connected, complex manifolds(see Folgerung 1.10 in [2]). The second author [6] has characterized such a Kaehler manifold in terms of pulled back metric through the exponential mapping at a point where the manifold is unitary-symmetric. And the present authors [3] then have, using the notion of unitary-symmetry, constructed a one parameter family of complete Kaehler metrics on CP^n, the complex projective n-space, which are compatible with the usual complex structure on it, and any two of which are not homothetic to each other.

In this paper, we shall continue the study of differential geometric properties of unitary-symmetric Kaehler manifolds. The first result is as follows.

Theorem 1. Let (M,g,J) be a connected, complete, unitary-symmetric Kaehler manifold of complex dimension n. If the scalar curvature of (M,g,J) is equal to a constant $4n(n+1)c$, then it is biholomorphically isometric to a complex space form of constant holomomorphic sectional curvature $4c$.

Warner [5] showed as a corollary that if the first tangential conjugate locus of a point m in a connected, complete Riemannian manifold of dimension n is a sphere of radius a, and consists of points with constant order, then each geodesic starting from m returns to m at length $2a$. It seems to us without any further condition on the manifold that no one can show that such a geodesic is closed. The second result of this paper is as follows.

Theorem 2. *If a compact, connected Kaehler manifold M is unitary-symmetric at a point m of M, then the first tangential conjugate locus of m consists of points with constant order one and each geodesic in M starting from m is a simple closed one with common length.*

2. Proof of Theorem 1.

A connected Kaehler manifold (M,g,J) of complex dimension n is unitary-symmetric at a point m of M if the linear isotropy group at m of the automorphism group of (M,g,J) is $U(n)$. Then we have (see [6]) that

$$\exp_m^* g = dr^2 + f(r)^2 d\theta^2 + f(r)^2(f'(r)^2 - 1)\eta \otimes \eta \tag{2.1}$$

where (r,θ) is the usual polar coordinate system of $C^n \equiv R^{2n}$, $f(r)$ is a C^∞, odd function on $[0,\delta)$ (resp. $[0,\infty)$) satisfying $f'(0) = 1$ and $f(r) > 0$ for $0 < r < \delta$ (resp. for $0 < r < \infty$) when M is compact (resp. noncompact). Here $(d\theta^2, \phi, \xi, \eta)$ is the standard Sasakian structure on the $(2n-1)$-sphere S^{2n-1} in C^n (cf. [4]).

Now, from (2.1) and constancy of the scalar curvature, we have the following differential equation

(2.2) $$2n(n-1)(1-(f')^2)f' - (4n-1)ff'f'' - f^2f''' = 2n(n+1)cf^2f'.$$

Regarding $p := f'$ as a function of $y := f$, dividing by p on the both sides and changing variables $u = 1 - p^2$ we find that (2.2) is transformed into the following differential equation

(2.3) $$y^2(d^2u/dy^2) + (4n-1)ydu/dy + 4n(n-1)u = 4n(n+1)cy^2.$$

Note that the initial condition $f(0) = 0$, $f'(0) = 1$ of (2.2) implies the initial condition $y = 0$, $u = 0$ of (2.3). It is shown that a solution u of (2.3) with $u = 0$ at $y = 0$ is of the form $u = cy^2$, which implies that

(2.4) $$(f')^2 = 1 - cf^2.$$

We can easily shown that the solution $f(t)$ of (2.4) satisfying the initial condition $f(0) = 0$, $f'(0) = 1$ is given by

(2.5) $$f(t) = \begin{cases} \sin(\sqrt{c}t)/\sqrt{c}, & \text{for } c > 0, \\ t, & \text{for } c = 0, \\ \sinh(\sqrt{-c}t)/\sqrt{-c}, & \text{for } c < 0. \end{cases}$$

Finally, note that if $f(t)$ in the expression of (2.1) is given by the function in (2.5), then (M,g,J) is of constant holomorphic sectional curvature $4c$. This completes the proof, since M is simply-connected by the Kaup's result in [2].

3. Proof of Theorem 2.

First, it follows from theorem B in [6] that the first tangential conjugate locus $Q(m)$ of the point m consists of points with constant order one and is equal to the sphere of radius a

on the tangent space $T_m(M)$ to M to m, where a is a geodesic distance in M from m to $\widetilde{Q}(m) = \exp_m Q(m)$. Since M is simply-connected, by using this fact and one of Warner's results (see the proof of theorem 4.4 in [5]) we have that

(3.1) $Q(m)$ is equal to the cut locus of m in $T_m(M)$.

Next, for each unit vector X in $T_m(M)$ we define the functions $X(\theta) = \cos\theta X + \sin\theta JX$, $\omega(r,\theta) = \exp_m rX(\theta)$, θ, r in R. Then we have the following facts (see [6]).

(i) Each maximal connected integral manifold of the distribution, defined by the kernel of differential of $\exp_m$ restricted to the tangent bundle of $\widetilde{Q}(m)$, is a curve $aX(\theta)$.

(ii) $\omega(a,\theta)$ = one point, say q, for all θ, and the tangent vectors $\partial\omega(a,\theta)/\partial r$ lie in the 2-dimensional plane Π_q of the tangent space $T_q(M)$, where Π_q is the orthogonal complement of the tangent space $T_q(\widetilde{Q}(m))$ at q.

(iii) The velocity vectors of the curve $\partial\omega(a,\theta)/\partial r$ in Π_q (with respect to θ) are always of constant length $|f(a)f''(a)|$.

Combining (i) - (iii) and proposition 4.1 in [5], we find that for each fixed unit vector X in $T_m(M)$, the mapping : $X(\theta) \longrightarrow \partial\omega(a,\theta)/\partial r$ is an isometry between the unit circles in span $\{X, JX\}$ and Π_q. From this observation we have that

$$(3.2) \qquad f(a)f''(a) = -1,$$

by virtue of $f'(r) > 0$, $0 \leq r < a$ and $f'(a) = 0$. Using (3.1) and (3.2), we see that $\omega(r,\theta) = \omega(r+2a,\theta)$, r, θ in R, and that the mapping $r \longrightarrow \omega(r,\theta)$, $0 \leq r < 2a$, for fixed θ, is injective. This completes the proof.

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