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M.D.S. codes and arcs in projective space II

by

A.A. Bruen, J.A. Thas and A. Blokhuis

Presented by G. de B. Robinson, F.R.S.C.

Section 3. A general result.

Recall that in $\Sigma = PG(r,F)$, F a field, any finite number of homogeneous polynomials $H_1,\dots,H_m$ in $F[X_0,\dots,X_r]$ define a variety $V(H_1,\dots,H_m)$ whose point set is the set of all simultaneous zeros (over F) of the polynomials $H_1,\dots,H_m$. If $A = V(H_1,\dots,H_m)$ and $B = V(G_1,\dots,G_v)$ are varieties in Σ we say that A is algebraically contained in B and write $A \subset B$ provided that over every algebraic extension of F the point set of B contains the point set of A, i.e. provided that over every algebraic extension of F all simultaneous zeros of $H_1,\dots,H_m$ are simultaneous zeros of $G_1,\dots,G_v$.

We say that the varieties A and B are equal or coincide if $A \subset B$ and $B \subset A$. Further, $A \cap B$ denotes the intersection of the point sets of A and B, and $V(A \cap B)$ is the variety of Σ defined by the polynomials $H_1,\dots,H_m$, $G_1,\dots,G_v$.

In this note an algebraic curve C (respectively, surface Ψ) in $PG(2,F)$ (respectively, $PG(3,F)$) is the variety defined by a single homogeneous polynomial over F. By abuse of notation the point set of C (respectively, Ψ) in Σ is denoted also by C (respectively, Ψ).

3.1 Theorem. In $\Sigma = PG(3,q)$ let K denote a set of k distinct planes $\pi_1,\pi_2,\dots,\pi_k$. In each plane π_i let there be given an algebraic curve C_i of degree t, with $t < q + 1$. Assume that

(1) $\pi_j \cap C_i = \pi_i \cap C_j$ for all $i \neq j$,

(2) $|\pi_j \cap C_i| = |\pi_i \cap C_j| = t$ for all $i \neq j$,

(3) $\pi_j \cap \pi_u \cap C_i = \emptyset$ for all distinct i,j,u.

Then there exists an algebraic surface Φ (over $GF(q)$) in Σ of degree t such that

$$V(\Phi \cap \pi_i) = C_i,\quad 1 \le i \le k.$$

In particular, Φ algebraically contains C_i, $1 \le i \le k$.

Section 4. Arcs in $\Sigma = PG(3,q)$, $q = 2^h$, $h \geq 2$.

We first want to generalize 2.1 by showing that those planes of Σ containing exactly two points of an arc B belong to a dual algebraic surface $\Phi(B)$. To see this it is easier to work in the dual formulation. So let B be an arc of planes in Σ, with $B = \{\pi_1,\pi_2,\ldots,\pi_\beta\}$. The planes of $B - \{\pi_i\}$ cut out in π_i an arc Γ_i of lines. Using 3.1, the dual of 2.1 and some further properties of the algebraic curve B_i corresponding to Γ_i by 2.1, the existence of the algebraic surface $\Phi = \Phi(B)$ can be shown. The degree of Φ is $t = q + 3 - \beta$. Using properties of Φ one can then show an analogue of 2.2 as follows.

4.1 **Theorem**. *Assume that $\beta \geq q + 3 - \sqrt[3]{q}$. Then B can be embedded in a $(q + 1)$-arc L. Moreover L is completely determined by B.*

Sketch of proof (of the first part). Take B to be an arc of planes, $B = \{\pi_1,\pi_2,\ldots,\pi_\beta\}$. The planes of $B - \{\pi_i\}$ cut out an arc Γ_i of lines in π_i, $1 \leq i \leq \beta$. By our restriction on β the arc Γ_i can be extended to a $(q + 2)$-arc by adjoining a set L_i of t S-lines (Segre lines) in π_i. The lines of L_i lie on a surface $\Phi(B)$ of degree t. By way of contradiction assume B complete with $\beta < q + 1$. Analogous to the situation in 2.2 there is a one-to-one correspondence between planes of Σ extending B and linear factors of $\Phi(B)$. Since B is complete no plane of Σ contains more than t lines of $\Phi(B)$ and so no plane of Σ contains more than t S-lines. One can then argue that t is even and that $\Phi(B)$ factors into $\frac{t}{2}$ hyperbolic quadrics $\Delta_1,\Delta_2,\ldots,\Delta_{t/2}$ such that each plane of B is a tangent plane of each of these quadrics. Using classical properties of the intersection of quadrics gives that the planes of B lie on (the space-dual of) a normal rational curve. Therefore B is not complete, giving the required contradiction. Consequently either B is a $(q + 1)$-arc or B can be extended to a $(\beta + 1)$-arc. It then follows that B is contained in a $(q + 1)$-arc.

Section 5. Arcs in $PG(r,q)$, $r \geq 4$, $q = 2^h$, $h \geq 2$.

Using 4.1, a generalization to four dimensions of 3.1, 2.6 and the method of proof in 2.4 one then obtains one of the main results in [3].

5.1 **Theorem**. *Let B be an arc of points in $\Sigma = PG(r,q)$, $r \geq 4$, $q = 2^h$, $h \geq 2$ and $|B| \geq q + r - \sqrt[3]{q}$. Then B lies in a normal rational curve L of Σ. Moreover L is completely determined by B.*

The following is a kind of analogue of 2.3 for even q and higher dimensions.

5.2 *Corollary*. In $PG(r,q)$, $r \geq 4$, $q = 2^h$ *and* $q \geq (r-1)^3$, *every* $(q+1)$*-arc is the point set of a normal rational curve*.

The following gives a partial answer to Conjecture 1.

5.3 *Corollary*. *In* $PG(r,q)$, $r \geq 4$, $q = 2^h$ *and* $q \geq (r-2)^3$ *there holds* $|B| \leq q+1$ *for every arc* B.

By [20] from 5.1, 5.2, 5.3 we obtain

5.4 *Corollary*. *Let* B *be an arc of points in* $\Sigma = PG(r,q)$, $r \geq q - 2 - \sqrt[3]{q}$, $q = 2^h$, $h \geq 2$ *and* $|B| \geq 6 + r$. *Then* B *lies in a normal rational curve* L *of* Σ. *Moreover* L *is completely determined by* B.

5.5 *Corollary*. *In* $PG(r,q)$, $q - 5 \geq r \geq q - \sqrt[3]{q} - 2$ *and* $q = 2^h$, *every* $(q+1)$*-arc is the point set of a normal rational curve*.

5.6 *Corollary*. *In* $PG(r,q)$, $q - 4 \geq r \geq q - \sqrt[3]{q} - 2$ *and* $q = 2^h$ *there holds* $|B| \leq q+1$ *for every arc* B.

Corollary 5.6 again is a partial answer to Conjecture 1.

Remarks. Various generalizations of the surface $\Phi(B)$ in Section 4 are available for higher dimensions and also for q odd. These and other matters are pursued elsewhere.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, N6A 5B7
CANADA

SEMINAR OF GEOMETRY AND COMBINATORICS
STATE UNIVERSITY OF GHENT,
KRIJGSLAAN 281, B-9000
GHENT, BELGIUM

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF EINDHOVEN
EINDHOVEN, THE NETHERLANDS

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ON THE MACKEY CONVERGENCE CONDITIONS

BY
M. A. Canela and M. Ralló

Presented by H.S.M. Coxeter, F.R.S.C.

ABSTRACT: In this paper we study the Mackey convergence conditions introduced by Grothendieck, from a nonstandard point of view. This leads us to a property for a locally convex space (E,τ) between the strict Mackey convergence condition and the density condition for its strong dual. We study the stability of this property under the usual operations with locally convex spaces and provide examples that show the difficulties involved in improving the obtained stability results.

Dealing with locally convex spaces, two different kinds of convergence can be considered: topological convergence and convergence in the sense of Mackey or Mackey convergence. Recall that a sequence $(x_n)_n$ in a locally convex space E is said to *Mackey converge* to a point x in E if and only if $(x_n - x)_n$ converges to zero in some normed space E_B (the space spanned by a closed bounded disk B and seminormed by the Minkowski functional of B).

It is clear that convergence in the sense of Mackey implies topological convergence, but the converse is not true generally. In order to study the class of locally convex spaces for which these convergences coincide, Grothendieck introduced the condition of Mackey convergence and the strict condition of Mackey convergence in 4.4 of [2]. A locally convex space E is said to satisfy the *condition of Mackey convergence* if and only if every sequence tending to zero in E tends to zero in the sense of Mackey, and E satisfies the *strict condition of Mackey convergence* if and only if for every bounded disk A in E, there exists a bounded disk $B \supset A$ such that on A, the topology induced by E or by E_B is the same. Obviously the strict Mackey convergence condition implies the Mackey convergence condition. Grothendieck related these conditions for spaces admitting a fundamental sequence of bounded sets. For such spaces he noticed that both conditions coincide when bounded sets are metrizable.

In [5] Neus has proved that the strict condition of Mackey convergence is equivalent to the condition of Mackey convergence for *LN-spaces.*

In this paper we consider Mackey convergence and the conditions of Mackey convergence from a nonstandard point of view. We shall use nonstandard analysis as developed in [4,6]. So let E be a locally convex space an let $\mathfrak{M}$ be a superstructure containing it and containing R the field of real numbers. $^*\mathfrak{M}$ will be a *polysaturated nonstandard enlargement* of $\mathfrak{M}$. $^*R_I$ will stand for the set of *infinitesimal real numbers* and $^*R_0$ for the set of *finite real numbers.* We recall that the *union monad* of the family of τ-bounded sets of a locally convex space (E,τ) is $bd_\tau = \cup \{ {}^*B \mid B \text{ } \tau\text{-bounded set in } E \}$, and that the set $\mu_\tau = \cap \{ {}^*U \mid U \text{ } \tau\text{-neighborhood of } 0 \text{ in } E \}$ is the *intersection monad* of the filter of neighborhoods of zero. The nonstandard version of the strict Mackey convergence condition leads us to the following nonstandard condition for a locally convex space (E,τ):

(1) $$\mu_\tau \cap bd_\tau = {}^*R_l.bd_\tau$$

for which we obtain a standard equivalent version. We study the stability of this condition under usual operations with locally convex spaces and we give examples that show the difficulties involved in improving these results. In Prop. 6 we prove that the class $\mathcal{C}$ of spaces satifiying (1) is stable under subspaces, countable products, direct sums, and strict inductive limits (in the sense of [2]). It is clear that the class $\mathcal{C}$ contains all normed spaces. In particular, as every metrizable space is a subspace of a countable product of normed spaces, it follows that metrizable spaces are in $\mathcal{C}$. More generally, in Corollary 3 below, we prove that all spaces satisfiying the strict Mackey convergence condition are in $\mathcal{C}$.

We also show that this condition is closely related to the density condition introduced by S. Heinrich in [3]. Recall that a locally convex space (E,τ) is said to satisfy the *density condition* if and only if bd_τ is a dense subset of fin_τ, endowed with the vector topology having $\{{}^*U \cap fin_\tau \mid U \in \mathcal{U}_\tau\}$ as a base of the filter of neighborhoods of zero, where $\mathcal{U}_\tau$ is the filter of τ-neighborhoods of zero in E. The standard equivalent version of the density condition is as follows: For every mapping λ from the set $\mathcal{U}_\tau$ into R^+ and every $V \in \mathcal{U}_\tau$, there exists a finite subset $U_1,\dots, U_n$ of $\mathcal{U}_\tau$ and a closed bounded set B such that

$$\lambda(U_1).U_1 \cap \dots \cap \lambda(U_n).U_n \subseteq B + V. \qquad \text{(Th. 1.4 of [3]).}$$

Let (E,τ) be a locally convex space and let $(x_n)_n$ be a sequence of elements of E. We are going to give nonstandard characterizations of the convergence in the sense of Mackey and the strict Mackey convergence condition.

PROPOSITION 1. *The following are equivalent:*

(i) *$(x_n)_n$ Mackey converges to x.*

(ii) *There exists a closed bounded disk B such that $x_n \in x + {}^*R_l.{}^*B$ for every infinite integer n.*

PROPOSITION 2. *The following are equivalent:*

(i) *(E,τ) satisfies the strict Mackey convergence condition.*

(ii) *For every bounded disk A in E, there exists a bounded disk $B \supseteq A$ such that*

$$^*A \cap \mu_\tau \subseteq {}^*R_l.{}^*B.$$

The proofs of these propositions are straightforward from the definitions of Mackey convergence, strict Mackey convergence condition and the nonstandard characterization of the convergence in the Banach space E_B, B being a closed bounded disk of E..

Proposition 1 may suggest a simpler characterization of the Mackey convergence. We will give an example of a sequence with all its infinite terms in ${}^*R_l.bd_\tau$ and not Mackey converging to zero. Notice that from proposition 2 we deduce the

COROLLARY 3. *If a locally convex space* (E,τ) *satisfies the strict Mackey convergence condition, then* $\mu_\tau \cap bd_\tau = {}^*R_1.bd_\tau$.

The standard version of condition (1) is similar to the standard version of the density condition.

PROPOSITION 4. *Let* (E,τ) *be a locally convex space. The following are equivalent:*

(i) $\mu_\tau \cap bd_\tau = {}^*R_1.bd_\tau$.

(ii) *For every closed bounded disk* B *of* E *and for every mapping* φ *from the family of closed bounded disks of* E *into* $R^+\backslash\{0\}$, *there exist a neighborhood* U *of zero and closed bounded disks* $B_1,\dots, B_n$ *such that:*

$$B \cap U \subseteq \varphi(B_1).B_1 \cup \dots \cup \varphi(B_n).B_n.$$

Proof:

(i)$\Rightarrow$(ii). Assume (ii) does not hold. Then there exists a closed bounded disk B_0 such that the family of sets $(B_0 \cap U)\backslash\varphi(B).B$, B ranging over the closed bounded disks and U ranging over the τ-neighborhoods of zero, satisfies the finite intersection property. It follows that $\mu_\tau \cap bd_\tau \subseteq {}^*R_1.bd_\tau$ does not hold.

(ii)$\Rightarrow$(i). Notice that $\mu_\tau \cap bd_\tau \supseteq {}^*R_1.bd_\tau$, so if (i) does not hold, then there exists a point x_0 in ${}^*E$ such that $x_0 \in (\mu_\tau \cap bd_\tau)\backslash{}^*R_1.bd_\tau$. Let B be a closed bounded disk in E such that $x_0 \in {}^*B$, then ${}^*B$ does not satisfy (ii).

We actually don't know if condition (1) is equivalent to the strict Mackey convergence condition. The next proposition states that it is stronger than the density condition for the strong dual of E.

PROPOSITION 5. *If* (E,τ) *satifies condition (1), then its strong dual satifies the density condition.*

Proof:

Taking polars in (ii) of prop. 4, we obtain that for every absolutely convex closed $\beta(F,E)$-neighborhood of zero V of E' and for every mapping λ from the family of absolutely convex closed $\beta(F,E)$-neighborhoods of zero into $R^+\backslash\{0\}$, $(\lambda(U) = \varphi(U^\circ)^{-1})$, there exist an equicontinuous set A and absolutely convex closed $\beta(F,E)$-neighborhoods of zero $U_1,\dots,U_n$, such that

$$\lambda(U_1).U_1 \cap \dots \cap \lambda(U_n).U_n \subseteq (A \cup V)^{\circ\circ} \subseteq A + 2V.$$

and by theorem 1.4 of [3], we obtain that E'_β satifies the density condition.

A necessary condition for the converse of Prop. 5 can be *quasi-barrelledness*, i.e. every $\beta(F,E)$-bounded subset of E' is equicontinuous. Equivalence is not clear, even in this case.

It is known that a subspace of a space which satisfies one of the two conditions of Mackey convergence, the topological vector product and the strict inductive limit (in the sense of [2]) of a sequence of spaces or the topological direct sum of a family of spaces which satisfy one of the two properties, satisfies also the same property. We will prove that condition (1) is stable under the same operations.

PROPOSITION 6. *Condition (1) is stable under:*

(i) *subspaces,*

(ii) *countable products,*

(iii) *direct sums,*

(iv) *strict inductive limits of sequences $(E_n)_n$ of spaces such that E_n is closed in E_{n+1}.*

Proof:

(i) Let F be a subspace of E and let $\tau_{/F}$ be the topology induced by τ in F. Then

$$\mu_{\tau_{/F}} = \mu_\tau \cap {}^*F \quad \text{and} \quad bd_{\tau_{/F}} = bd_\tau \cap {}^*F.$$

It is clear that F satisfies (1) if E does.

(ii) Let $(E_n,\tau_n)_n$ be a sequence of locally convex spaces satisfying condition (1). Let E be its topological product and let τ be the product topology. By Theorem 3-2.4 of [1], an element $x = (x_n)_n$ of ${}^*E$ is in μ_τ if and only if $x_n \in \mu_{\tau_n}$ for every non negative standard integer n. Assume that x belongs to $\mu_\tau \cap bd_\tau$. Then there exists a τ-bounded balanced set $B = \Pi_{n\geq 1} B_n$ such that x belongs to ${}^*B$. Every standard component of x satisfies:

$$x_n \in \mu_{\tau_n} \cap {}^*B_n \subseteq {}^*R_1.bd_{\tau_n},$$

so we can find a bounded balanced subset B'_n of E_n with $x_n \in {}^*R_1.{}^*B'_n$. Now let us consider the bounded set $B' = \Pi_{n\geq 1}(B'_n \cup n.B_n)$. It is clear that $x \in {}^*R_1.\ {}^*B'$. So we have proved that (E,τ) satisfies (1).

(iii) Let $(E_\alpha,\tau_\alpha)_{\alpha\in A}$ be a family of locally convex spaces satisfying (1) and let (E,τ) be the direct sum of this family. We will denote by g_α the inclusion mapping from E_α into E and by p_α the projection from E onto E_α. It is not difficult to prove

$${}^*p_\alpha(\mu_\tau) = \mu_{\tau_\alpha} \quad \text{for every } \alpha\in A \text{ and}$$
$$bd_\tau = \oplus_{\alpha\in A}\, {}^*g_\alpha(bd_{\tau_\alpha}).$$

Now let $x \in \mu_\tau \cap bd_\tau$, then $x = x_1 + \ldots + x_n$ with

$$x_i \in {}^*g_{\alpha_i}(bd_{\tau_{\alpha_i}} \cap \mu_{\tau_{\alpha_i}}) = {}^*R_1.{}^*g_{\alpha_i}(bd_{\tau_{\alpha_i}})$$

for every $i = 1,\ldots, n$. It follows that $x \in {}^*R_1.bd_\tau$

(iv) Let (E,τ) be the strict inductive limit of an increasing sequence of locally convex spaces $(E_n,\tau_n)_n$ such that E_n is closed in E_{n+1}, satisfying (1). It is known that in this case the bounded subsets of E are the bounded subsets of some E_n, $n \in N$, and that τ induces in every E_n the topology τ_n. So we obtain $bd_\tau = \cup\{ bd_{\tau_n} \mid n \in N \}$. Now let $x \in {}^*E$ be an element of $\mu_\tau \cap bd_\tau$ Then there exists an $n \in N$ such that

$$x \in \mu_\tau \cap bd_{\tau_n} = \mu_{\tau_n} \cap bd_{\tau_n} = {}^*R_1 . bd_{\tau_n} \subseteq {}^*R_1 . bd_\tau$$

So (E,τ) satisfies condition (1).

The next examples show the difficulties involved in obtaining better stability results.

Example of an uncountable product of Banach spaces for which condition (1) fails.

Let X be an infinite dimensional Banach space, and let X^* be its dual endowed with the weak-star topology w^*. (X^*,w^*) embeds canonically in R^X when we endow this space with the product topology.

Now let B be a bounded subset of X^* and let B_{X^*} be the unit ball in X^*. Then there exists an strictly positive real number $\phi(B)$ such that:

$$\phi(B).B \subseteq {}^1/_2 . B_{X^*}$$

On the other hand B_{X^*} is a bounded subset of X^* and there is no w^*-neighborhood of zero U such that $B_{X^*} \cap U \subseteq ({}^1/_2 .) B_{X^*}$. So $B_{X^*} \cap U$ is not a subset of any finite union

$$\phi(B_1).B_1 \cup \phi(B_2).B_2 \cup \ldots \cup \phi(B_n).B_n$$

where the $B_1, B_2, \ldots, B_n$ are all bounded subsets. So we obtain that (X^*,w^*) does not fullfil condition (1). Condition (1) is stable for subspaces, as we know, so R^X can not satisfy it.

Example of a regular inductive limit of a sequence of Banach spaces for which condition (1) fails.

Let E_n be the Banach space of double sequences of scalars $x = (x_{ij})$ such that

$$\|x\|_n = \max\{ \sup_{i\geq 1} |x_{i1}|, \ldots, \sup_{i \geq 1} |x_{i\,n-1}|, (\textstyle\sum_{i\geq 1\, j\geq n} |x_{ij}|^2)^{1/2} \} < +\infty$$

Notice that $E_n \subseteq E_{n+1}$ and that the inclusion mapping is continuous. Let (E,τ) be the inductive limit of that increasing sequence of Banach spaces. We can consider in E the norm

$$\|x\|_\infty = \sup\{ |x_{ij}| : i,j \in N \},\ x \in E.$$

The inclusion mapping from (E,τ) into $(E,\|\,\|)$ is continuous, so (E,τ) is separated. We are going to prove that the unit ball in E_n is closed in (E,τ), for every $n \in N$. Then by 4.3.3, prop. 5 of [2] we shall deduce that E is *regular*, i. e. a subset of E is bounded if and only if it is a bounded subset of some E_n.

The unit ball of E_n can expressed as $A_n + C_n$, where:

$$A_n = \{ x \mid \sup_{i\geq 1} |x_{i1}| \leq 1, \ldots, \sup_{i\geq 1} |x_{i\,n-1}| \leq 1; x_{ij} = 0 \text{ if } j \geq n \}$$

$$C_n = \{ x \mid x_{ij} = 0 \text{ if } j \leq n; (\textstyle\sum_{i\geq 1\, j\geq n} |x_{ij}|^2) \leq 1 \}$$

A_n is $\|.\|_\infty$-closed in E, so it is also τ-closed. On the other hand C_n is weakly compact in E_n. We obtain that $A_n + C_n$ is weakly closed in E and from this we deduce that B_n, the unit ball in E_n, is τ-closed in E. Now let

$$x^{(k,n)}_{p,m} = \begin{cases} 1/\sqrt{n} & \text{if } m = k \text{ and } p \leq n \\ 0 & \text{otherwise} \end{cases}$$

then $\|x^{(k,n)}\|_j = 1$ if $j \leq k$ and $\|x^{(k,n)}\|_j = 1/\sqrt{n}$ if $j > k$, so $x^{(k,n)} \in B_1$ for every pair of natural numbers k,n.

Let U be a τ-neighborhood of zero and let $(\varepsilon_n)_n$ be a sequence of strictly positive real numbers with $\varepsilon_n < 1$, for every $n \in N$. If $n' \in N$ there exists $\rho > 0$ such that $\rho.B_{n'+1}$ is a subset of U. Given ρ, there exists a positive integer n such that $1/\sqrt{n} < \rho$. It follows that $\|x^{(n,n')}\|_j = 1$ if $j \leq n$ and $\|x^{(n,n')}\|_{n'+1} = 1/\sqrt{n} < \rho$, so $x^{(n,n')} \in B_1 \cap U$ and it is clear that $x^{(n,n')} \notin \varepsilon_1.B_1 \cup \ldots \cup \varepsilon_n.B_n$. Consequently E does not fulfil condition (1).

Example of a sequence with all its infinite terms in ${}^*R_1 bd_\tau$ and not Mackey converging to zero.

Let (E,τ) be the space of double sequences just considered in the former example, and let

$$x^{(r,n)}_{i,j} = \begin{cases} 1/r.n & \text{if } j = n \text{ and } i \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

then $\|x^{(r,n)}\|_j = 1/n$ if $j \leq n$ and $\|x^{(r,n)}\|_j = 1/r.n$ if $j > n$. So $(x^{(r,n)})_r$ does not converge to zero in E_n and it is a subsequence of $x^{(1,1)}, x^{(2,1)}, x^{(2,2)}, x^{(3,1)}, x^{(3,2)}, x^{(3,3)}, x^{(4,1)}, \ldots$. It follows that this sequence can not Mackey converge to zero in E. Notice that the infinite terms of this sequence are $x^{(\mu,\nu)}$ for infinite positive integers μ and $0 \leq \nu \leq \mu$. Now if ν is infinite $\|x^{(\mu,\nu)}\|_1 = 1/\nu$ and $x^{(\mu,\nu)} \in {}^*R_1.{}^*B_1$. When ν is finite $\|x^{(\mu,\nu)}\|_\nu = 1/\mu.\nu$ and we obtain that $x^{(\mu,\nu)} \in {}^*R_1.{}^*B_\nu$. We have proved that every infinite term of the considered sequence is in ${}^*R_1.{}^*bd_\tau$ and that this sequence does not Mackey converge to zero.

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Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
08071 - Barcelona
SPAIN

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LINEAR METHODS IN THE ANDRE-QUILLEN COHOMOLOGY

A. R-Grandjean and M. J. Vale

Presented by P. Ribenboim, F.R.S.C.

ABSTRACT. The main object of this note is to introduce in commutative algebra the 2-fold crossed modules of the group theory [4]. This allows us to define groups of cohomology which coincide with the André [2] and Quillen [9] groups in dimensions 0,1,2,3.

INTRODUCTION. The idea of this note is the translation of the usual homological algebra methods to non abelian contexts. We introduce the 2-fold crossed modules and the homotopy operators to obtain a cohomology theory which coincides with André-Quillen cohomology in dimensions 0,1,2,3 and to give an internal interpretation "by extensions" of these groups.

We want to point out that our result avoids the torsors theory used by Duskin [6]. An interpretation of torsors in terms of special extensions in a category of interest [8] is given in [10].

Moreover, we obtain in [5] the Jacobi-Zariski sequence (without using simplicial theory or spectral sequences) and we apply our theory to problems of commutative algebra.

This note is a summary of a paper which has won the prize of the "Real Academia de Ciencias Exactas, Fisicas y Naturales" of Spain, in 1986.

1. 2-FOLD CROSSED MODULES

Let K be a commutative ring with unit.

DEFINITION 1.1. A 2-fold crossed module of commutative algebras is a complex of R-modules

$$H \xrightarrow{\partial_2} Q \xrightarrow{\partial_1} R$$

where R is a commutative K-algebra, Q is a commutative R-algebra (not necessary with unit), ∂_1 is a homomorphism of R-algebras and there is an R-bilinear map $< , > : Q \times Q \longrightarrow H$ satisfying

(1) $\partial_2 < q_0, q_1 > = q_0 q_1 - \partial_1 q_1 \cdot q_0$

(2) $< q_0, q_1 q_2 > = < q_0 q_1, q_2 > + \partial_1 q_2 \cdot < q_0, q_1 >$

(3) $< \partial_2 h, q > = < q, \partial_2 h > - \partial_1 q . h$

with $q_0, q_1, q_2, q \in Q$ and $h \in H$.

Hence, H is an R-algebra with multiplication

$$h_0 h_1 = < \partial_2 h_0, \partial_2 h_1 >$$

and ∂_2 is a homomorphism of R-algebras. Moreover, H is a Q-algebra with action

$$q * h = < q, \partial_2 h > , \quad q \in Q, \ h \in H$$

A morphism of 2-fold crossed modules is a triplet (f_0, f_1, f_2): $(H \xrightarrow{\partial_2} Q \xrightarrow{\partial_1} R) \longrightarrow (H' \xrightarrow{\partial_2'} Q' \xrightarrow{\partial_1'} R')$, where $f_0 : R \longrightarrow R'$ is a homomorphism of K-algebras, $f_1 : Q \longrightarrow Q'$ is a homomorphism of R-algebras, with R acting on Q' vía f_0, and $f_2 : H \longrightarrow H'$ is a homomorphism of R-modules, such that $f_1 \partial_2 = \partial_2' f_2$, $f_0 \partial_1 = \partial_1' f_1$ and $f_2 < , > = < , > (f_1 \times f_1)$.

PROPOSITION 1.2. The category of 2-fold crossed modules of commutative algebras is equivalent to the category of 2-truncated simplicial algebras E. such that E_2 verifies the following conditions:

$$(\mathrm{Ker}\ d_0)(\mathrm{Ker}\ d_1 \cap \mathrm{Ker}\ d_2) = 0, \quad (\mathrm{Ker}\ d_1)(\mathrm{Ker}\ d_0 \cap \mathrm{Ker}\ d_2) = 0$$

$$(\mathrm{Ker}\ d_2)(\mathrm{Ker}\ d_0 \cap \mathrm{Ker}\ d_1) = 0$$

2. COHOMOLOGY THEORY

DEFINITION 2.1. A 2-crossed complex is a complex of K-modules

$$C : \dots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \longrightarrow \dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

where

(1) $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a 2-long crossed module.

(2) For $k \geqslant 3$, C_k is an A-module, being $A = C_0/\partial_1(C_1)$.

(3) ∂_3 is a homomorphism of C_0-modules, where C_0 acts on C_3 via the projection $p:C_0 \longrightarrow A$.

(4) For $k \geqslant 4$, ∂_k is a homomorphism of A-modules.

If the 2-crossed complex C is exact, then C is called a 2-crossed resolution of A.

A 2-crossed complex is called projective if it verifies the following conditions:

(5) C_0 is a K-algebra which is projective relative to the class of all surjective epimorphisms of K-algebras.

(6) C_1 is a C_0-algebra (not necessary with unit) which is projective relative to the class of all surjective epimorphisms of C_0-algebras without unit.

(7) $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a 2-fold crossed module which is projective relative to the class of epimorphisms of 2-crossed modules

$$E = \{(\alpha_2,1,1) \ / \ \alpha_2 \text{ is an epimorphism}\}$$

(8) For $k \geqslant 3$, C_k is a projective A-module.

If the 2-crossed complex C is projective and exact, then we call C a projective 2-crossed resolution of A.

A morphism of 2-crossed complexes $f:C \longrightarrow C'$ is a family of homomorphisms of K-modules $f = \{f_k\}_{k\geqslant 0}$, $f_k:C_k \longrightarrow C'_k$, such that

(1) $f_i\partial_{i+1} = \partial'_{i+1}f_{i+1}$, for $i \geqslant 0$.

(2) (f_0,f_1,f_2) is a morphism of 2-long crossed modules.

(3) For $k \geqslant 3$, f_k is a homomorphism of $C_0/\partial_1(C_1)$-modules.

PROPOSITION 2.2. *Any commutative K-algebra with unit has a projective 2-crossed resolution.*

PROPOSITION 2.3. *Let* C *be a projective 2-crossed complex and* C' *be a 2-crossed resolution of* A'. *Then there exists, to every homomorphism of K-algebras* $\alpha: C_0/\partial_1(C_1) \longrightarrow A'$, *a morphism* $f: C \longrightarrow C'$ *of 2-crossed complexes inducing* α.

DEFINITION 2.4. Let C and C' be 2-crossed complexes and $f,g: C \longrightarrow C'$ morphisms of 2-crossed complexes. A *homotopy between f and g*. is a family of K-module homomorphisms $\Sigma = \{\Sigma_k\}_{k\geq 0}$, $\Sigma_k: C_k \longrightarrow C'_{k-1}$, such that:

(1) Σ_0 is a g_0-derivation (i.e. Σ_0 is a K-lineal map and $\Sigma_0(x_0y_0) = \Sigma_0x_0 \cdot \Sigma_0y_0 + g_0x_0 \cdot \Sigma_0y_0 + g_0y_0 \cdot \Sigma_0x_0$, $x_0, y_0 \in C_0$) and $\partial'_1\Sigma_0 = f_0 - g_0$.

(2) $\Sigma_1(x_0 \cdot x_1) = g_0x_0 \cdot \Sigma_1x_1 + \Sigma_0x_0 * \Sigma_1x_1 - \langle f_1x_1, \Sigma_0x_0\rangle + \langle \Sigma_0x_0, g_1x_1\rangle$, $\Sigma_1(x_1y_1) = -\Sigma_1x_1 \cdot \Sigma_1y_1 + f_1x_1 * \Sigma_1y_1 + f_1y_1 * \Sigma_1x_1 - \langle \partial'_2\Sigma_1y_1, g_1x_1\rangle - \langle \partial'_2\Sigma_1x_1, g_1y_1\rangle + \langle f_1y_1 - g_1y_1, g_1x_1\rangle + \langle f_1x_1 - g_1x_1, g_1y_1\rangle$, for all $x_1, y_1 \in C_1$, $x_0 \in C_0$, and $\partial'_2\Sigma_1 + \Sigma_0\partial_1 = f_1 - g_1$.

(3) Σ_2 is a homomorphism of C_0-modules (C'_3 being a C_0-module via $p'g_0: C_0 \longrightarrow C'_0/\partial'_1(C'_1)$), $\Sigma_2\langle\ ,\ \rangle = 0$, and $\partial'_3\Sigma_2 + \Sigma_1\partial_2 = f_2 - g_2$.

(4) For $k \geq 3$, $\Sigma_k: C_k \longrightarrow C'_{k+1}$ is a homomorphism of $C_0/\partial_1(C_1)$-modules, C'_{k+1} being a $C_0/\partial_1(C_1)$-module via the induced homomorphism of K-algebras $f_{-1}: C_0/\partial_1(C_1) \longrightarrow C'_0/\partial'_1(C'_1)$ (note that f and g induce the same homomorphism f_{-1} from (1)), and $\partial'_{k+1}\Sigma_k + \Sigma_{k-1}\partial_k = f_k - g_k$.

PROPOSITION 2.5. *Let* C *be a projective 2-crossed complex and* C' *a 2-crossed resolution of* A'. *If* $f,g: C \longrightarrow C'$ *are morphisms of 2-crossed complexes inducing the same homomorphism* $\alpha: C_0/\partial_1(C_1) \longrightarrow A'$, *there is a homotopy betweeen* f *and* g.

DEFINITION 2.6. Let A be a commutative K-algebra with unit and M an A-module (M is considered to be a trivial A-algebra). We define the *cohomology groups of A with coefficients in M*, *$H^*(A,M)$*, to be the cohomology groups of the complex:

$$H(C,M)\ :\ 0 \longrightarrow \mathrm{Der}_K(C_0,M) \xrightarrow{\partial_1^*} \mathrm{Hom}_{C_0\text{-}Alg}(C_1,M) \xrightarrow{\partial_2^*} H_{C_0}(C_2,M) \xrightarrow{\partial_3^*}$$

$$\longrightarrow \mathrm{Hom}_A(C_3,M) \longrightarrow \dots \longrightarrow \mathrm{Hom}_A(C_{k-1},M) \xrightarrow{\partial_k^*} \mathrm{Hom}_A(C_k,M) \longrightarrow \dots$$

where $C:\dots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \longrightarrow \dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$ is a projective 2-crossed resolution of A, $H_{C_0}(C_2,M)$ is the group of all homomorphisms $f:C_2 \longrightarrow M$ of C_0-modules such that $f\langle\ ,\ \rangle = 0$, and $\partial_k^*(h) = h\partial_k$ for $k\geqslant 1$.

Notice that $H^*(A,M)$ does not depend on the chosen projective 2-crossed resolution of A.

It is easy to prove that $H^0(A,M) = \mathrm{Der}_K(A,M)$.

3. INTERPRETATION OF THE COHOMOLOGY GROUPS

DEFINITION 3.1. Let M be an A-module and C a 2-crossed resolution of A with $C_k = 0$ for $k>n$ ($n\geqslant 1$) and $C_n = M$. For $n\geqslant 3$, we call C a *non abelian n-fold extension of A by M*. For $n = 2$, if $\langle\ ,\ \rangle = 0$ and the C_0-action of M coincides with the induced by p, we call C a *non abelian 2-fold extension* or a *crossed 2-fold extension* of A by M. For $n = 1$, if the C_0-action on M coincides with the induced by p, we call C a *non abelian 1-fold extension* or a *singular extension* of A by M.

Two non abelian n-fold extensions of A by M, E^n and E'^n, *are related* if there is a morphism $f:E^n \longrightarrow E'^n$ or $f:E'^n \longrightarrow E^n$ of 2-crossed complexes with $f_{-1} = 1$ and $f_n = 1$; this relation generates an equivalence relation and we write $[E^n]$ for the equivalence class of E^n and $S^n(A,M)$ for the quotient set.

THEOREM 3.2. *There exist bijections*

$$H^n(A,M) \simeq S^n(A,M),\quad n\geqslant 1$$

In [7] Iwai shows that there is a bijection between the André cohomology groups [2], $\mathrm{And}^n(A,M)$, and $S^n(A,M)$ for $n = 1,2,3$. Then, we have

THEOREM 3.3. There exist bijections

$$H^n(A,M) \simeq \mathrm{And}^n(A,M), \quad n = 0,1,2,3.$$

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Departamento de Algebra
Facultad de Matemáticas
Universidad de Santiago de Compostela
SPAIN

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TAME AND EXHAUSTIVE REPETITIVE ALGEBRAS

Ibrahim Assem and Andrzej Skowroński

Presented by V. Dlab, F.R.S.C.

ABSTRACT. Let A be a finite dimensional algebra over an algebraically closed field. We show that the repetitive algebra $\hat{A}$ of A is tame and exhaustive if and only if A is either iterated tilted of Dynkin or Euclidean type or tilting-cotilting equivalent to a tubular canonical algebra.

1. Let k denote a fixed algebraically closed field, and A be an associative finite dimensional k-algebra with an identity. By module is meant throughout a right module, finite dimensional over k. We shall denote by mod A the category of A-modules. The repetitive algebra $\hat{A}$ of A is the self-injective, locally finite dimensional matrix algebra without identity:

$$\hat{A} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & A_{-1} & & & \\ & E_{-1} & A_0 & & \\ & & E_0 & A_1 & \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero coefficients and, for all $i \in \mathbb{Z}$, A_i equals A, while E_i is the minimal injective cogenerator $E = \mathrm{Hom}_k(A,k)$. Addition is the usual addition of matrices, and multiplication is induced from the canonical A-A-bimodule structure of E and the zero map $E \otimes_A E \longrightarrow 0$ [HW].

Repetitive algebras are closely connected with derived categories. Indeed, it was shown by Happel in [H] that, if the global dimension of A is finite, then the derived category $D^b(A)$ of bounded complexes over mod A (see [V]) is equivalent to the stable category $\underline{\mathrm{mod}}\,\hat{A}$. Originally, however, repetitive algebras were introduced to study the finite dimensional self-injective algebras. Indeed, the identity maps $A_i \longrightarrow A_{i-1}$, $E_i \longrightarrow E_{i-1}$ induce an automorphism ν_A of $\hat{A}$ (the Nakayama automorphism) and the orbit space $\hat{A}/(\nu_A)$ inherits the structure of a finite dimensional algebra, isomorphic to the trivial extension T(A) of A by E: this is the algebra whose additive structure is that of the group $A \oplus E$ and whose multiplication is defined by:

$$(a,f)\ (b,g) = (ab, ag+fb)$$

for $a,b \in A$ and $f,g \in E$. It is well-known that $T(A)$ is a self-injective and, in fact, a symmetric algebra. Thus $\hat{A}$ is a Galois covering (in the sense of Gabriel, see [G]) of $T(A)$ with the infinite cyclic group (ν_A) generated by ν_A, and mod $\hat{A}$ is equivalent to the category of $\mathbb{Z}$-graded $T(A)$-modules. The connection between $\hat{A}$ and $T(A)$ is particularly close whenever the push-down functor F_λ : mod $\hat{A} \longrightarrow$ mod $T(A)$ associated with the covering $\hat{A} \longrightarrow T(A)$ is dense (that is, every $T(A)$-module is $\mathbb{Z}$-gradable). We then say that $\hat{A}$ is exhaustive. The aim of this note is to present a classification of the algebras A such that $\hat{A}$ is tame and exhaustive.

2. Following Drozd, [D], we shall say that a locally finite dimensional algebra B is tame if, for any dimension d, there exists a finite (parametrising) family of functors F_i : mod $k[X] \longrightarrow$ mod B, $i = 1, 2, \ldots, n_d$, where $k[X]$ is the polynomial algebra in one variable X, such that:

(a) For each i, $F_i = - \otimes_{k[X]} Q_i$, where Q_i is a $k[X]$-B-bimodule, which is finitely generated and free as a $k[X]$-module.

(b) All but finitely many isomorphism classes of indecomposable B-modules of dimension d are of the form $F_i(S)$ for some i, and some simple $k[X]$-module S.

That is, B is tame if and only if the classification of the indecomposable B-modules does not include the classical unsolved problem of the reduction of a pair of matrices under simultaneous conjugation.

3. Let A be a finite dimensional algebra. Following [HR], we shall call a module T_A a tilting (respectively, a cotilting) module if $\mathrm{Ext}_A^2(T,-) = 0$ (respectively, $\mathrm{Ext}_A^2(-,T) = 0$), $\mathrm{Ext}_A^1(T,T) = 0$, and the number of non-isomorphic indecomposable direct summands of T_A equals the rank of the Grothendieck group $K_0(A)$ of A. Two algebras A and B are called tilting-cotilting equivalent if there exists a sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ and a sequence of tilting or cotilting modules $T^i_{A_i}$ $(i = 0, 1, \ldots, m-1)$ such that $A_{i+1} = \mathrm{End}\, T^i_{A_i}$. If moreover B is the path algebra kQ of a quiver Q, and each T^i is a tilting module such that, for any indecomposable A_i-module M, we have either $\mathrm{Hom}_{A_i}(T^i,M) = 0$ or $\mathrm{Ext}^1_{A_i}(T^i,M) = 0$, we say that A is iterated tilted of type Q, see [AH]. If m is at most 1, we say that A is a tilted algebra, see [HR]. It is shown in [HRS] that A is iterated tilted of type Q if and only if A is tilting-cotilting equivalent to kQ, or if and only if $D^b(A) \simeq D^b(kQ)$. A complete

description of mod $\hat{A}$ for A iterated tilted of Dynkin (respectively, Euclidean) type is presented in [AHR] (respectively, [ANS]). Similarly, a complete description of mod $\hat{A}$ for A tilting-cotilting equivalent to a tubular canonical algebra (in the sense of Ringel [R]) is presented in [NS].

In all these cases, the repetitive algebra $\hat{A}$ is tame. It is also *locally support-finite* [DS], that is, for each indecomposable projective $\hat{A}$-module P, the set of isomorphism classes of indecomposable projective $\hat{A}$-modules P' such that there exists an indecomposable $\hat{A}$-module M with both $\mathrm{Hom}_{\hat{A}}(P,M) \neq 0$ and $\mathrm{Hom}_{\hat{A}}(P',M) \neq 0$ is finite. It is shown in [DS] that, if $\hat{A}$ is locally support-finite, then $\hat{A}$ is exhaustive.

4. Let A be a triangular algebra, that is, having no oriented cycles in its ordinary quiver (this is the case, for instance, if $\hat{A}$ is exhaustive, see [S]). Then A is called *simply connected*, see [AS1] or [AS3], if, for any presentation $A \cong kQ/I$ of A as a bound quiver algebra, then the fundamental group of the bound quiver (Q,I) (in the sense of [MP]) is trivial or, equivalently, if A has no proper Galois covering. Our first theorem is a criterion for the simple connectedness of the algebras with a tame and exhaustive repetitive algebra.

THEOREM (A). *Let* A *be a finite dimensional, basic and connected k-algebra. Assume that* $\hat{A}$ *is tame and exhaustive. Then* A *is simply connected if and only if* A *is not an iterated tilted algebra of type* $\tilde{A}_m$.

The proof uses essentially the classification of the iterated tilted algebras of type $\tilde{A}_m$ presented in [AS1] and [AS2]. Our second theorem completely identifies these algebras.

THEOREM (B). *Let* A *be a finite dimensional, basic and connected k-algebra. Assume that* $\hat{A}$ *is tame and exhaustive. Then there exists an algebra* C *which is either hereditary of Dynkin or Euclidean type, or tubular canonical, such that* A *and* C *are tilting-cotilting equivalent.*

It was shown in [AS4] (see also [AS1]) that these algebras coincide with the algebras A such that $D^b(A)$ (or, equivalently, *mod* $\hat{A}$) is cycle-finite, that is, every (oriented) cycle in this category lies in a tube (in the sense of [R]). Actually, we have:

COROLLARY. *Let* A *be a finite dimensional, basic and connected k-algebra. The following assertions are equivalent:*

(i) $\hat{A}$ *is tame and exhaustive.*

(ii) $\hat{A}$ *is tame and locally support-finite.*

(iii) T(A) *is tame and every indecomposable* T(A)*-module is* $\mathbb{Z}$*-gradable.*

(iv) *There exists an algebra* B *which is either tilted of Dynkin type or representation-infinite tilted of Euclidean type, or tubular, such that* $\hat{A} \cong \hat{B}$.

(v) *There exists an algebra* C *which is either hereditary of Dynkin type, or of Euclidean type, or tubular canonical, such that* A *and* C *are tilting-cotilting equivalent.*

(vi) *mod* $\hat{A}$ *is cycle-finite.*

(vii) *There exists an algebra* C *which is either hereditary of Dynkin type, or of Euclidean type, or tubular canonical, such that mod* $\hat{A} \cong$ *mod* $\hat{C}$.

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Ibrahim Assem
Département de Mathématiques et d'Informatique
Université de Sherbrooke
Sherbrooke, Québec
Canada, J1K 2R1

Andrzej Skowroński
Institute of Mathematics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń
Poland

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CONTACT THREE-MANIFOLDS WITH POSITIVE GENERALIZED TANAKA-WEBSTER SCALAR CURVATURE

S. I. Goldberg[1], D. Perrone[2] and G. Toth

Presented by P. Ribenboim, F.R.S.C.

ABSTRACT. The metric of a compact contact Riemannian three-manifold whose characteristic vector field generates a one-parameter group of isometries may be deformed to a contact metric of positive curvature if the generalized Tanaka-Webster scalar curvature r^* is positive. If, in addition, r^* is a constant then the metric may be deformed to a contact metric of constant curvature.

1. Introduction. Y. Carrière [2] has classified Riemannian flows on compact three-manifolds. The difficulty encountered in the study of Riemannian flows is that they are not automatically Killing flows. A compact three-manifold M admitting a nonsingular Killing vector field is a Seifert manifold, so if M is simply connected it is diffeomorphic to the standard three-sphere S^3. Chern and Hamilton [3] introduced the torsion $|\tau|$ (the length of τ) in their study of compact contact three-manifolds (M,g), where $\tau(= L_{X_0} g)$ is the Lie derivative of the contact metric g with respect to the characteristic vector field X_0 of the contact structure, and they conjectured that for fixed contact form $\omega = g(X_0,\cdot)$, with X_0 inducing a Seifert foliation, there exists a complex structure $\phi|B$ on $B = \ker \omega$ such that the Dirichlet energy

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$$(1.1) \qquad \mathcal{E}(g) = \frac{1}{2}\int_M |\tau|^2 d\ \mathrm{vol}(M,g)$$

is critical over all CR structures.

Let M be a contact manifold with a fixed contact form ω. Denote the space of all associated Riemannian metrics to the contact form ω by $\mathcal{M}(\omega)$. Let g be a point of $\mathcal{M}(\omega)$, and denote by $\{g(t)\}$ a curve in $\mathcal{M}(\omega)$ with $g(0) = g$. Tanno [8] showed that g is a critical point of $\mathcal{E}$, if and only if

$$(1.2) \qquad \nabla_{X_0}\tau = 2\tau \cdot \phi.$$

Thus, $\mathcal{E}(g)$ is critical over all CR structures if and only if (1.2) is satisfied. (This differs from the condition $\nabla_{X_0}\tau = 0$ incorrectly obtained in [3], Theorem 5.4.) In the sequel, a critical point of $\mathcal{E}$ will be called a *critical metric*. Note that g is a critical point of $\mathcal{E}$ if X_0 is a Killing vector field with respect to g.

In [5] it is shown that if the scalar curvature $r > -2$ on a compact contact three-manifold (M,g) whose characteristic vector field is a Killing field, then g may be deformed to a contact metric of positive Ricci curvature. It is the main purpose of this paper to show that g may in fact be deformed to a contact metric of positive sectional curvature. This is a consequence of Theorem 1 which also yields the statement that if r is a constant greater than -2, then g may be deformed to a contact metric of (positive) constant curvature.

2. *Compact three-manifolds*. To facilitate the study of compact three-manifolds, one may apply the following important result due to Lutz and Martinet [7], namely, 'every compact and orientable three-manifold has a contact structure.' The reader is referred to [1] for details and other properties of contact manifolds. In the sequel, we denote the Ricci tensor by S, and set $\sigma = S(X_0, \cdot)|B$.

THEOREM 1. *Let* M *be a compact and orientable three-manifold with contact metric structure* (ω, X_0, g), *where* g *is critical. Then, if the scalar curvature* r *satisfies the inequality*

$$r > 2\left(1 - \frac{c^2}{4}\right) + \frac{|\sigma|^2}{1 - \frac{c^2}{4}} + 2c, \quad c = |\tau| < 2, \tag{2.1}$$

g *has positive Ricci curvature.*

PROOF. As in the proof of the Theorem in [5], to show that the Ricci tensor S is positive definite, we determine at each point $x \in M$, a suitable ϕ-basis $\{E, \phi E, X_0\}$ of T_xM, and verify that the subdeterminants along the main diagonal are positive.

COROLLARY 1. *Let* M *be a compact and orientable three-manifold with contact metric structure* (ω, X_0, g), *where* X_0 *is a Killing vector field. Then, if* $r > -2$, M *admits a contact metric structure* $(a\omega, a^{-1}X_0, ag + a(a-1)\omega \otimes \omega)$ *of positive sectional curvature for some constant* a, $0 < a \leq 1$.

PROOF. Since X_0 is a Killing field, g is a critical metric. In addition, σ and τ both vanish. The matrix for S with respect to the ϕ-basis $\{E, \phi E, X_0\}$ is given by

$$S = \begin{bmatrix} \frac{r}{2} - 1 & 0 & 0 \\ 0 & \frac{r}{2} - 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The components of the Riemann curvature tensor R with respect to the orthonormal basis $\{e_1, e_2, e_3\} = \{E, \phi E, X_0\}$ are

$$(2.2)\quad R_{ijk\ell} = \delta_{ik}S_{j\ell} + \delta_{j\ell}S_{ik} - \delta_{i\ell}S_{jk} - \delta_{jk}S_{i\ell} - \frac{r}{2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}).$$

Consequently, since $S_{ij} = 0$ for $i \neq j$, the only nonvanishing components of R are those with two indices different.

Let P be a 2-dimensional subspace of the tangent space T_xM at $x \in M$, and let $\{X = \Sigma a_i e_i,\ Y = \Sigma b_j e_j\}$ be an orthonormal basis of P. Then, the sectional curvature $K(X,Y)$ of P is given by

$$\begin{aligned} g(R(X,Y)X,Y) = {} & (a_1b_2 - a_2b_1)^2R_{1212} + (a_1b_3 - a_3b_1)^2R_{1313} \\ & + (a_2b_3 - a_3b_2)^2R_{2323}\ . \end{aligned}$$

Thus, if $r > 4$, $K(X,Y) > 0$.

Assume that there is a point $x_0 \in M$ such that $0 < (r(x_0) + 2)/6 \leq 1$. The minimum k of the function $(r(x) + 2)/6$ then lies in the interval $(0,1]$. Consider the metric $\tilde{g}$ on M defined by

$$\tilde{g} = ag + a(a-1)\omega \otimes \omega$$

for some constant a, $0 < a < k \leq (r(x) + 2)/6$. If we put $\tilde{\omega} = a\omega$ and $X_0 = a^{-1}X_0$, then $(\tilde{\omega}, \tilde{X}_0, \tilde{g})$ is a contact metric structure whose characteristic vector field is a Killing field. By a direct computation (see [4]), the Ricci tensors S and $\tilde{S}$ of the metrics g and $\tilde{g}$, respectively, are related by

$$\tilde{S} = S + 2(1-a)g - 2(1-a)(2+a)\omega \otimes \omega,$$

so since $\tilde{g}^{ij} = a^{-1}g^{ij} + (1-a)a^{-2}X_0^iX_0^j$, $\tilde{r} - 4 = \frac{6}{a}(\frac{r+2}{6} - a)$. Thus, $\tilde{r} > 4$, from which $\tilde{K}(X,Y) > 0$.

3. *Constant curvature*. Hamilton [6] showed that a metric g of positive Ricci curvature on a compact three-manifold can be deformed to a metric of (positive) constant curvature. If g is a contact metric, we obtain the following

COROLLARY 2. *Let* M *be a compact and orientable three-manifold with contact metric structure* (ω, X_0, g) *where* X_0 *is a Killing vector field. Then, if* r *is a constant greater than* -2, *the metric* g *may be deformed to a contact metric of constant curvature* 1.

PROOF. It is well-known that a Riemannian three-manifold (M, g) is an Einstein manifold if and only if it has constant curvature (cf. (2.2)). The matrix S in the proof of Corollary 1 says that g is an Einstein metric $\Leftrightarrow S = (r/3)g \Leftrightarrow |S|^2 = r^2/3 \Leftrightarrow r = 6$. If r is a constant greater than -2, then the contact metric defined by $\tilde{g} = ag + a(a - 1)\omega \otimes \omega$, $a = (r + 2)/8$, has constant scalar curvature $\tilde{r} = ((r + 2)/a) - 2 = 6$.

More generally, if the contact metric is critical, we obtain

THEOREM 2. *Let* M *be a compact and orientable three-manifold with contact metric structure* (ω, X_0, g), *where* g *is critical. Then,* g *is of constant curvature* k, *if and only if* X_0 *is a Killing vector field and* $k = 1$.

COROLLARY. *A critical metric on the three-sphere with constant curvature is the standard normal contact metric*.

REMARK: The quantity $r + c^2/2$ appearing in (2.1) is equal to $r^* - 2$, where r^* is the generalized Tanaka-Webster scalar curvature defined in [8].

The condition on the scalar curvature r in Corollaries 1 and 2 may therefore be replaced by the assumption that r^* be positive. The Webster curvature studied by Chern and Hamilton in [3] is equal to $r^*/8$. Their main result says that every contact structure on a compact and orientable three-manifold has a contact Riemannian metric whose Webster curvature is either a constant ≤ 0 or it is strictly positive everywhere.

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Department of Mathematics,
University of Illinois
1409 West Green Street
Urbana, Illinois 61801

Dipartimento di Matematica
Facolta' di Scienze
Universita' degli studi di Lecce
Via Arnesano
73100 Lecce, Italy

and

Department of Mathematics
and Statistics
Queen's University
Kingston, Canada K7L 3N6

Department of Mathematics
Rutgers University
Camden, NJ 08102

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A NOTE ON QUASI-CONVEX FUNCTIONS

Khalida Inayat Noor

Presented by J.G. Arthur, F.R.S.C.

ABSTRACT: The class C^* of quasi-convex functions is a subclass of K, the class of close-to-convex univalent functions. In this paper the concept of order and type is used to prove that $f \in C^*(\beta, \gamma)$ implies $f \in K(\alpha(\beta), \lambda(\gamma))$.

MATHEMATICAL SUBJECT CLASSIFICATIONS (1980): 30A35; 30A32; 30C45.

KEYWORDS AND PHRASES: Starlike, convex, order, quasi-convex, close-to-convex, univalent.

1. INTRODUCTION:

Let A denote the class of functions $f: f(z) = z + \sum_{n=2}^{+\infty} a_n z^n$, which are analytic in the unit disc $E = \{z: |z|<1\}$. By S, K, S^* and C, denote the subclasses of A which are respectively univalent, close-to-convex, starlike and convex in E. In [7], Robertson defined the subclasses of C and S^* by using the order of the class as follows. A function $f \in S$ is called convex function of order γ, $0 < \gamma < 1$, if and only if

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} > \gamma, \quad z \in E.$$

We denote this class of functions by $C(\gamma)$. Also, a function $f \in S$ is called starlike function of order γ, $0 < \gamma < 1$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \gamma, \quad z \in E.$$

We call this class $S^*(\gamma)$. Obviously $f \varepsilon C(\gamma)$ if and only if $zf' \varepsilon S^*(\gamma)$, and for $\gamma = 0$ we have $C(0) = C$, $S^*(0) = S^*$.

Libera [3] introduced the terminology of order and type together in the class of close-to-convex functions as follows:

A function $f \varepsilon A$ is said to be close-to-convex of order β and of type γ where $0 \le \beta < 1$, and $0 \le \gamma \le 1$ if and only if there exists a function $g \varepsilon S^*(\gamma)$ such that

$$\mathrm{Re}\ \frac{zf'(z)}{g(z)} > \beta, \quad z \varepsilon E.$$

We denote this class as $K(\beta, \gamma)$. It is clear that $K(0,0) = K$ and $K(1, 0) = C$.

In 1980, Noor [5] introduced the class C^* of quasi-convex univalent functions. We define the class $C^*(\beta, \gamma)$ as follows: A function $f \varepsilon A$ is said to be quasi-convex of order β and of type γ if and only if there exists a function $g \varepsilon C(\gamma)$ such that

$$\mathrm{Re}\ \frac{(zf'(z))'}{g'(z)} > \beta, \quad z \varepsilon E,$$

where $0 \le \beta < 1$ and $0 \le \gamma \le 1$. Clearly $C^*(0, 0) = C^*$. Also $f \varepsilon C^*(\beta, \gamma)$ if and only if $zf' \varepsilon K(\beta, \gamma)$. For more details of the class $C^*(\beta, \gamma)$, see [6].

2. MAIN RESULTS:

The following result is proved in [2].

THEOREM 1: *If $f \varepsilon C(\gamma)$, then $f(z)$ is analytic, univalent and starlike of order $\lambda(\gamma)$, where*

$$\lambda(\gamma) = \begin{cases} 4^{\gamma}\, \dfrac{(1-2\gamma)}{4 - 2^{2\gamma+1}}, & \gamma \neq 1/2 \\[2ex] (\log 4)^{-1}, & \gamma = 1/2 \end{cases} \tag{1}$$

The result is sharp.

We shall use the same technique to prove the main result of this paper.

THEOREM 2: Let $f \varepsilon C^*(\beta, \gamma)$. Then $f \varepsilon K(\alpha(\beta), \lambda(\gamma))$, where $\lambda(\gamma)$ is given by (1) and

$$\alpha(\beta) = \begin{cases} 4^{\beta} \dfrac{(1-2\beta)}{4-2^{2\beta+1}}, & \beta \neq 1/2 \\ (\log 4)^{-1}, & \beta = 1/2 \end{cases} \tag{2}$$

This result is sharp for $\beta = \gamma$.

PROOF: Since $f \varepsilon C^*(\beta, \gamma)$, there exists a $g \varepsilon C(\gamma)$ such that

$$\frac{(zf'(z))'}{g'(z)} = (1-\beta)\, h(z) + \beta, \quad \text{Re } h(z) > 0, \ z \varepsilon E \tag{3}$$

Now, using Theorem 1, $g \varepsilon C(\gamma)$ implies that $g \varepsilon S^*(\lambda(\gamma))$ where $\lambda(\gamma)$ is given by (1).

Furthermore, we can write (3) as

$$\frac{(zf'(z))'}{g'(z)} = (1-\beta)\frac{z\phi'(z)}{\phi(z)} + \beta = \frac{N'(z)}{D'(z)}$$

for some $\phi \varepsilon S^*$.

So

$$\frac{N(z)}{D(z)} = \frac{zf'(z)}{g(z)} = \frac{z\left(\dfrac{\phi(z)}{z}\right)^{1-\beta}}{\displaystyle\int_0^z \left(\frac{\phi(t)}{t}\right)^{1-\beta} dt}$$

$$= \frac{1}{\displaystyle\int_0^z \left(\frac{z}{t}\right)^{1-\beta} \left[\frac{\phi(t)}{\phi(z)}\right]^{1-\beta} \frac{dt}{z}}, \tag{4}$$

where we integrate along the straight line segment $[0, z]$, $z \varepsilon E$.

Using a well known result [1], we conclude that $\text{Re } \dfrac{N(z)}{D(z)} = \text{Re } \dfrac{zf'(z)}{g(z)} > \beta > 0$, and since $\dfrac{zf'(z)}{g(z)} = 1$ at $z = 0$, we have

$$\left| \frac{zf'(z)}{g(z)} - \frac{1 + r^2}{1 - r^2} \right| < \frac{2r}{1 - r^2}, \tag{5}$$

$|z| = r$, $z \in E$; see [4, p 173].

From (5), it is clear that

$$\min_{f \in C^*(\beta,\gamma)} \min_{|z|=r} \mathrm{Re}\, \frac{zf'(z)}{g(z)} = \min_{f \in C^*(\beta,\gamma)} \min_{|z|=r} \left| \frac{zf'(z)}{g(z)} \right|,$$

and hence it is sufficient to find the minimum of the right-hand side of (4). Then, from [2] we have

$$\alpha(\beta) = \mathrm{Min} \left[\left| \int_0^z \left(\frac{z}{t} \right)^{1-\beta} \left(\frac{\phi(t)}{\phi(z)} \right)^{1-\beta} \frac{dt}{z} \right| \right]^{-1},$$

for $\phi \in S^*$, $z \in E$ and $\alpha(\beta)$ is as given in (2). This proves our result. Sharpness for $\beta = \gamma$ follows by taking

$$f_\gamma(z) = g_\gamma(z) = \begin{cases} \dfrac{1 - (1-z)^{2\gamma-1}}{2\gamma-1}, & \gamma \neq 1/2 \\ \log \dfrac{1}{1-z}, & \gamma = 1/2 \end{cases}$$

Remark: If we take $\beta = \gamma = 0$, then $\alpha(0) = 1/2$ and $\lambda(0) = 1/2$. This gives us that $f \in C^*$ implies $f \in K(1/2, 1/2)$.

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K.I. Noor
Mathematics Department
Punjab University
New Campus, Lahore, Pakistan

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ON THE POWER SERIES RING OVER A MORI DOMAIN

Valentina Barucci

Presented by P. Ribenboim, F.R.S.C.

Introduction. In this paper we consider the following problem about Mori domains: if A is a Mori domain (i.e. a domain with the ascending chain condition on divisorial ideals), when are A[X] and A[[X]] also Mori domains ?

For the polynomial ring case there are positive answers, e.g. if A is integrally closed (J.Querré, cf.[7, Section 3, Théorème 5]) or if A contains an uncountable field (M. Roitman, cf.[9]). There are no example where A is Mori and A[X] is not. For the power series ring case there is such an example (cf. [10]).

In this paper we shall show that if A is a Mori domain of a particular type, then both A[[X]] and A[X] are Mori domains.

As a consequence, we get that if A is an integrally closed Mori domain, with the conductor in its complete integral closure different from (0), then A[[X]] and A[X] are also Mori domains.

The theorem. We begin by explaining the particular type of Mori domains that we shall consider. Suppose A is a Mori domain and A^* is the complete integral closure of A. Suppose further that $(A:A^*) \neq (0)$. Let

$$A = A_0 \subset A_1 \subset \ldots \subset A_m = A^* \qquad (*)$$

be the sequence of Mori overrings of A constructed as in [2, Section 1]. We have that for each i, i = 0,...,m-1, $\mathfrak{R}_i = (A_i:A_{i+1})$ is the intersection of all the strong maximal divisorial ideals of A_i. We are interested in the particular case in which $\mathfrak{R}_i$ is a radical ideal of A_{i+1} for each i, $i = 0,\ldots,m-1$, and we shall call in this case A "*seminormal in* A^*". It turns out that if A is a Noetherian domain then A^* is the integral closure of A and our particular condition holds if and only if A is seminormal in the usual sense (cf. [2, Theorem 3.8]).

For examples of Mori domains of this type, see [2, Examples 3.12].

THEOREM 1. *Let* A *be a Mori domain such that* $(A:A^*) \neq (0)$. *If* A *is "seminormal in* A^*", *then* $A[[X]]$ *is also a Mori domain*.

Proof. Let $A_i = B \subset A_{i+1} = C$ be the generic step of the sequence (*). Thus $C = (B:\mathfrak{R})$, where, if $P_1,\ldots,P_n$ are the strong maximal divisorial ideals of B (cf. [2, Proposition 1.5]), we have $\mathfrak{R} = P_1 \cap \ldots \cap P_n$.

By [2, Proposition 2.7 and Corollary 2.8], we know that $B = C \cap B_1 \cap \ldots \cap B_n$, where for each j, $j = 1,\ldots,n$, B_j is the pullback of the diagram

$$\begin{tikzcd}
 & k(P_j) = B_{P_j}/P_jB_{P_j} \arrow[d, hook] \\
S_j^{-1}C \arrow[r] & S_j^{-1}C/S_j^{-1}P_j
\end{tikzcd}$$

(where $S_j = B - P_j$).

Since A^* is a Krull domain (cf.[1, Corollary 18]), it is well known that $A[[x]]$ also is a Mori (in fact Krull) domain. Thus to prove the proposition it is enough to show (in the generic step of the sequence (*), $B \subset C$) that if $C[[X]]$ is a Mori domain, then $B[[X]]$ is also a Mori domain.

Indeed $B[[X]] = C[[X]] \cap B_1[[X]] \cap \ldots \cap B_n[[X]]$. So, by [8, Théorème 2], it is enough to show that $B_j[[X]]$ is a Mori domain (for $j = 1,\ldots,n$).

Let's fix an index j and let's denote, for simplicity, B_j by $\mathcal{B}$, $S_j^{-1}C$ by $\mathcal{C}$, $S_j^{-1}P_j$ by $\mathcal{M}$ and $k(P_j)$ by k. From the previous pullback diagram, we get the following pullback diagram (cf.[4, Lemma 2]):

$$\begin{tikzcd}
\mathcal{B}[[X]] \arrow[r] & k[[X]] \arrow[d, hook] \\
\mathcal{C}[[X]] \arrow[r] & \mathcal{C}/\mathcal{M}\,[[X]] = \mathcal{C}[[X]]/\mathcal{M}[[X]]
\end{tikzcd}$$

If L is the quotient field of $k[[X]]$, we have $k[[X]] = \mathcal{C}/\mathcal{M}[[X]] \cap L$, where the intersection is made in the total quotient ring of $\mathcal{C}/\mathcal{M}[[X]]$. Thus, by a result of Roitman (cf.[10, Theorem 4.16]), $\mathcal{B}[[X]]$ is a Mori domain if: i) $\mathcal{C}[[X]]$ is a Mori domain, ii) $\mathcal{M}[[X]]$ is a Mori ideal, iii) $\mathcal{M}[[X]]$ is a prime ideal of $\mathcal{B}[[X]]$. Actually $\mathcal{C}[[X]] = S_j^{-1}C[[X]] = S_j^{-1}(C[[X]])$ is a Mori domain, because $C[[X]]$ is Mori (cf.[6,Corollaire 3]). Moreover $\mathcal{M}$ is a radical ideal of $\mathcal{C}$ (cf.[2, Proposition 3.3,2)]) and is also a Mori ideal, because it is a prime (in fact maximal) ideal of the Mori domain $\hat{B}$ (cf.[9, Theorem 6.2]). Thus, by [10, Proposition 4.9, (b)], $\mathfrak{A}$ is a finite intersection of prime ideals of C. We easily deduce that $\mathfrak{A}[[X]]$ is also a finite intersection of prime ideals of the Mori domain $C[[X]]$. Thus, by [9, Theorem 6.2], $\mathfrak{A}[[X]]$ is a Mori ideal. Moreover $\mathfrak{A}[[X]]$ is prime in $\hat{B}[[X]]$ and so we conclude that $\hat{B}[[X]]$ is a Mori domain.

COROLLARY 2. *Let* A *be a Mori domain such that* $(A:A^*) \neq (0)$. *If* A *is "seminormal in* A^*", *then* $A[X]$ *is a Mori domain*.

Proof. The Corollary follows from Theorem 1 and [8, Théorème 2], observing that $A[X] = A[[X]] \cap K[X]$, where K is the quotient field of A.

COROLLARY 3. *Let* A *be a Mori domain such that* $(A:A^*) \neq (0)$. *If* A *is integrally closed*, *then* $A[[X]]$ *and* $A[X]$ *are Mori domains*.

Proof. By Theorem 1 and Corollary 2, it is enough to show that, if A is integrally closed, then A is "seminormal in A^*". Consider the sequence (*) associated to A. We want to show first that for each i, $i = 0,\dots,m$, A_i is integrally closed. By induction: $A_0 = A$ is integrally closed. Suppose now that A_i is integrally closed. Let P be a maximal divisorial ideal of A_{i+1} and let $\mathfrak{R}_i = (A_i:A_{i+1})$. If $P \supset \mathfrak{R}_i$, then P is not strong (cf.[2, Lemma 3.1]) and $(A_{i+1})_P$ is a DVR (cf.[3, Theorem (2.5)]). If $P \not\supset \mathfrak{R}_i$, then $(A_{i+1})_P = (A_i)_{\underline{p}}$ (where $\underline{p} = P \cap A_i$) (cf.[5, Theorem 1.4,c]). Thus, for each maximal divisorial ideal P of A_{i+1}, $(A_{i+1})_P$ is integrally closed and

$A_{i+1} = \cap\{ (A_{i+1})_P ;P$ maximal divisorial $\}$ is integrally closed. We have to show now that, for each i, $i = 0,\ldots,m-1$, $\mathfrak{R}_i$ is a radical ideal of A_{i+1}. In fact if $x \in A_{i+1}$ and $x^n \in \mathfrak{R}_i$, for some positive integer n, then $x \in A_i$, because $\mathfrak{R}_i \subset A_i$ and A_i is integrally closed. Since $\mathfrak{R}_i$ is a radical ideal of A_i by construction, we have that $x \in \mathfrak{R}_i$.

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Dipartimento di Matematica
Istituto 'G.Castelnuovo'
Università di Roma 'La Sapienza'
Piazzale Aldo Moro 2
00185 Roma, Italia

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DETERMINATION OF A FROM $M(A)$ AND RELATED MATTERS

LAWRENCE G. BROWN

Presented by G.A. *Elliott*, F.R.S.C.

Abstract: We show that for A_1 and A_2 separable C^*–algebras, the multiplier algebras $M(A_1)$ and $M(A_2)$ are isomorphic if and only if A_1 and A_2 are isomorphic. This is done by characterizing A as the largest separable ideal of $M(A)$. In fact, A is the largest separable hereditary C^*–subalgebra of $M(A)$. The results we care more about are characterizations of those hereditary C^*–subalgebras, B, of A for which $M(A,B) \neq B$ (A σ–unital) and a proof that for such B $M(A,B)/B$ is non–separable.

If B is a hereditary C^*–subalgebra of A, $M(A,B)$ denotes $M(A) \cap B^{**} \subset A^{**}$, where A^{**} is the enveloping W^*–algebra of A. In other words $M(A,B)$ is the set of multipliers of A which are supported by the open projection of B. Clearly, $M(A,B) \cap A = B$. For A σ–unital we give two necessary and sufficient conditions for $M(A,B) \neq B$. The first is an easy consequence of the "Urysohn lemma" for multipliers of [4]. In the commutative case the question is: If U is an open subset of a locally compact Hausdorff space X, does there exist a bounded continuous function f on X such that f is supported by U and f does not vanish at ∞? If X is σ–compact, the answer is "yes if and only if $\overline{U}$ is not compact". (If X is not σ–compact, the answer is definitely more complicated.) Our answer in the non–commutative case is a little different because the non–commutative analogue of compactness is not always easy to work with.

Theorem 1. Let A be a C^*–algebra, q an open projection in A^{**}, and $B = \mathrm{her}(q)$. Then either of the following conditions implies $M(A,B) = B$.

1. If p is a closed projection in A^{**} such that $p \leq q$, then p is compact.
2. There is a in A_{sa} such that $qaq \geq q$.

Even more, condition 2 implies $QM(A,B) = B$.

Proof: 1. Let $h \in M(A,B)_{sa}$. If F is a closed subset of $\mathbb{R}$ such that $0 \notin F$, then the spectral projection $p = E_F(h)$ is closed and $p \leq q$. By hypothesis p is compact. Thus h is strongly q–continuous, and a result of

Akemann, III.3 of [2], implies $h \in A$.

2. Let $h \in M(A, B)$. Then $h^*h = h^*qh \le h^*qaqh = h^*ah \in A$. Thus h^*h, and hence h, are in A (since A is an ideal of $M(A)$).

Recall that $QM(A, B)$ is defined as $QM(A) \cap B^{**}$, where $QM(A)$ is the set of quasi-multipliers of A. We may assume $a \ge 0$. Then $q \le qaq$ implies the existence of s in $A^{**}(\|s\| \le 1)$ such that $q = sa^{\frac{1}{2}}q$. Then if $b \in B$, $b = qbq^* = sa^{\frac{1}{2}}ba^{\frac{1}{2}}s^*$. Let $L = \left(a^{\frac{1}{2}}Ba^{\frac{1}{2}}\right)^-$. ($L$ is only a norm closed linear subspace of A.) Then $sLs^* \subset B$. Let $h \in QM(A, B)$. Then $h \in B^{**}$ and $a^{\frac{1}{2}}Ba^{\frac{1}{2}} \subset L$ imply $a^{\frac{1}{2}}ha^{\frac{1}{2}} \in L^{**}$. Thus $a^{\frac{1}{2}}ha^{\frac{1}{2}} \in L^{**} \cap A = L$. Thus $h = qhq^* = sa^{\frac{1}{2}}qhq^*a^{\frac{1}{2}}s^* = s\left(a^{\frac{1}{2}}ha^{\frac{1}{2}}\right)s^* \in sLs^* \subset B$.

Theorem 2. If A is a σ–unital C^*–algebra, q is an open projection in A^{**}, and $B = \text{her}\ (q)$, then $M(A, B) = B$ if and only if every closed projection p such that $p \le q$ is compact.

Proof: One direction has already been proved. For the other, if p is closed and $p \le q$, then Urysohn's lemma (3.31 of [4]) implies the existence of h in $M(A)_{sa}$ such that $p \le h \le q$. If $M(A, B) = B$, then $h \in B \subset A$, and this implies that p is compact.

The following lemma allows condition 1 in Theorem 2.7 of [3] to be weakened. The idea of using it to attack the question "$M(A, B) = B$?" was probably inspired by our hearing a lecture on the subject of [3].

Lemma 3. Let A be a C^*–algebra, (p_n) a sequence of mutually orthogonal rank one projections in A^{**}, and let φ_n be the pure state of A associated to p_n. If there is a strictly positive element e of A such that $\Sigma_1^\infty \varphi_n(e) < \infty$, then $\Sigma_1^\infty p_n$ is a closed projection.

Proof. Let $\{\pi_i : i \in I\}$ be the set of *inequivalent* irreducible representations associated to the φ_n's and $\pi = \oplus_i \pi_i$. All of the p_n's can be identified with elements of $\pi(A)'' \subset B(H_\pi)$. Let $p = \Sigma_1^\infty p_n \in B(H_\pi)$. The hypothesis implies that $p\pi(e)p$ is trace class and hence compact. Thus $\pi\left(e^{\frac{1}{2}}\right)p$ is compact, and hence $\pi\left(Ae^{\frac{1}{2}}\right)p \subset \mathcal{K}(H_\pi)$. Since $Ae^{\frac{1}{2}}$ is dense in A, we conclude that $p\pi(A)p \subset \mathcal{K}(H_\pi)$.

Now let $\mathcal{P} = \{T \in B(H_\pi) : T \ge 0, tr(T) \le 1, T = pTp,$ and $T = \oplus_i T_i$ where $T_i \in B(H_{\pi_i})\}$. Then $\mathcal{P}$ can be identified with F, the norm closed face of $\Delta(A)$ associated to p, where $\Delta(A)$ is the quasi–state space of A (the set of positive functionals of norm at most 1). Our task is to prove that F is weak* closed in $\Delta(A)$. But obviously $\mathcal{P}$ is compact in the weak operator topology. Since $tr(p\pi(a)pT) = tr(\pi(a)T)$ for $T \in \mathcal{P}$, and since $p\pi(A)p \subset \mathcal{K}(H_\pi)$, the weak operator topology on $\mathcal{P}$ dominates the weak* topology. $\square$

Theorem 4. Let A be a σ-unital C^*-algebra, q an open projection in A^{**}, and $B = \text{her }(q)$. Then $M(A,B) = B$ if and only if there is a in A_{sa} such that $qaq \geq q$. Consequently, $M(A,B) = B$ implies $QM(A,B) = B$.

Proof. One direction has already been proved. Thus we assume that $\nexists a$ with $qaq \geq q$ and seek to prove $M(A,B) \neq B$.

Let e be a strictly positive element of A. Then $\nexists \varepsilon > 0$ such that $qeq \geq \varepsilon q$. We can regard $S(B)$, the state space of B, and $P(B)$, the set of pure states of B, as subsets of A^*. The weak* topologies induced by A and B do, in fact, agree on $S(B)$ (though not on $\Delta(B)$). If we regard elements of A as affine functionals on $\Delta(A)$, then the restriction of e to $\Delta(B)$ is most naturally identified with $qeq \in B^{**}$. By what has already been said, qeq is weak* continu ous on $S(B)$ (another way to see this continuity is to note that qeq is a quasi-multiplier of B), and $\inf\{\varphi(qeq) : \varphi \in S(B)\} = 0$. It follows that $\inf\{\varphi(qeq) : \varphi \in P(B)\} = 0$. Thus we can find a sequence (φ_n) such that $\varphi_n \in P(B) \subset P(A)$ and $\varphi_n(e) \to 0$.

Let p_n be the rank one projection in A^{**} which is the support projection of φ_n. Then p_n is compact (II.4 or II.8 of Akemann [1] imply this for A unital, and this combined with results of [2] (the equivalence of compactness with 2.47(iii) of [4]) extend it to the non-unital case). Thus there is a_n in A such that $p_n \leq a_n$. Now $\varphi_n(e) \to 0$ implies $\varphi_n \to 0$ weak*. Therefore $\varphi_n(a_m) \to 0$ for each fixed m, and hence $\varphi_n(p_m) \to 0$. If we consider the reduced atomic representation of A and unit vectors v_n in range (p_n), we see that there are a subsequence (φ_{n_k}) and states ψ_{n_k} in $P(B)$ such that $\|\varphi_{n_k} - \psi_{n_k}\| \to 0$ and the support projections of the ψ_{n_k}'s are mutually orthogonal. (The n_k's are chosen recursively and ψ_{n_k} is obtained by applying the Gram–Schmidt process to $v_{n_1}, \ldots, v_{n_k}$. It is easy to see that the ψ_{n_k}'s are still supported by q.) Changing notation, we assume that the p_n's are mutually orthogonal and also (passing to a further subsequence) that $\Sigma_1^\infty \varphi_n(e) < \infty$.

Now let $p = \Sigma_1^\infty p_n$. Then p is closed by Lemma 3, and $p \leq q$ by construction. We claim that p is not compact. If p were compact, there would be a in A such that $p \leq a$. Then $\varphi_n \to 0$ weak* would imply

$\varphi_n(a) \to 0$ and $\varphi_n(p) \to 0$. But $\varphi_n(p) \geq \varphi_n(p_n) = 1$. Now Theorem 2 implies $M(A,B) \neq B$.

Remarks. 1. It is easy to see *a priori* that if q is any projection in A^{**}, and if there is a in A such that $qaq \geq q$, then there is $\varepsilon > 0$ such that $qeq \geq \varepsilon q$ (e strictly positive, given in advance).

2. $M(A,B) = B$ does not imply $\overline{q}$ is compact. In fact, with $p = \overline{q}$ it is possible to achieve that $\exists a \in A$ such that $pap \geq p$ (this is stronger than $qaq \geq q$), and $\nexists a' \in A$ such that $a' \geq q$. (Thus $\nexists a' \in A$ such that $a' \geq p$. Note that for a closed projection p, p is compact if and only if $\exists a' \in A$ such that $a' \geq p$; but for a non–closed projection q, $a' \geq q$ does not imply $\overline{q}$ compact (IV.5 of [2]).) The example is very simple:

Let $A = c \otimes \mathcal{K}$, where $\mathcal{K}$ is the set of compact operators on the separable infinite dimensional Hilbert space H. Thus A^{**} can be identified with the set of bounded collections $\{T_n : 1 \leq n \leq \infty, T_n \in B(H)\}$. Define q in A^{**} by $q_\infty = 0$ and $q_n = v_n \times v_n$, $v_n = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_{n+1}$, for $n < \infty$, where $\{e_n\}$ is an orthonormal basis of H. Then q is open. Since $q_n \to \frac{1}{2}(e_1 \times e_1)$ weakly, any closed projection $p \geq q$ must satisfy $p_\infty \geq \frac{1}{2}(e_1 \times e_1)$. Thus the closure, p, of q is given by $p_\infty = e_1 \times e_1$, $p_n = q_n$ for $n < \infty$. If $a \in A$ is given by $a_n = 2(e_1 \times e_1)$, $1 \leq n \leq \infty$, then $pap \geq p$. Suppose $a' \in A$ and $a' \geq q$. Then $(a'_n e_{n+1}, e_{n+1}) \geq (q_n e_{n+1}, e_{n+1}) = \frac{1}{2}$, $n < \infty$. Since $a'_n \to a'_\infty$ in norm, $(a'_\infty e_{n+1}, e_{n+1}) > \frac{1}{4}$ for n sufficiently large. This contradicts $a'_\infty \in \mathcal{K}$.

3. If q is central, in other words if B is an ideal, then $M(A,B) = B$ if and only if $\overline{q}$ is compact. It is easy by spectral theory to deduce from $qaq \geq q$ that there is a b in A such that $q \leq b \leq 1$. This implies that $\overline{q}$ is compact.

4. A modification of the example in Remark 2 gives an open projection q such that $\exists a \in A_+$ with $qaq \geq q$, but $\nexists b \in A_+$ with $\overline{q}b\overline{q} \geq \overline{q}$. Let (m_n) be a sequence which includes each positive integer infinitely often, and define q by $q_\infty = 0, q_n = v_{m_n} \times v_{m_n}$. Then q is open and $qaq \geq q$, where $a_\infty = a_n = 2(e_1 \times e_1)$. Then $p = \overline{q}$ is given by $p_n = q_n, p_\infty = 1$. Since p_∞ has infinite rank, $\nexists b \in A$ with $pbp \geq p$. Moreover, p could not satisfy any reasonable compactness condition.

Theorem 5. If A is a σ–unital C^*–algebra and B is a hereditary C^*–subalgebra such that $M(A,B) \neq B$, then $M(A,B)$ and $M(A,B)/B$

are non-separable.

Proof. Let e be a strictly positive element of A, and let (e_n) be a sequential approximate identity of A such that $e_{n+1}e_n = e_n$. Choose h in $M(A,B)\backslash B$ such that $0 \le h \le 1$. Then $\left\| h^{\frac{1}{2}}(1-e_n)h^{\frac{1}{2}} \right\|$ decreases to a limit c. Since $h \notin A$, $c > 0$. For each k, $\liminf_{n\to\infty} \left\| h^{\frac{1}{2}}(e_n - e_k)h^{\frac{1}{2}} \right\| \ge \left\| h^{\frac{1}{2}}(1-e_k)h^{\frac{1}{2}} \right\| \ge c$. Thus we can recursively choose $b_k = h^{\frac{1}{2}}\left(e_{n'_k} - e_{n''_k}\right)h^{\frac{1}{2}}$, where $n'_k > n''_k > n'_{k-1}$, such that $\left\| b_k \left(\Sigma_1^{k-1} t_\ell b_\ell\right) \right\| < 2^{-k}$ if $|t_\ell| \le 1$ for $\ell = 1, \dots, k-1$, $\|b_k e\| < 2^{-k}$, and $\|b_k\| > \frac{c}{2}$. (Note that $h^{\frac{1}{2}}(1-e_n)h^{\frac{1}{2}} \to 0$ strictly.) For $k \ne \ell$, $f_k = e_{n'_k} - e_{n''_k}$ and $f_\ell = e_{n'_\ell} - e_{n''_\ell}$ are orthogonal positive elements of A. Hence for any subset S of $\mathbb{N}$, $b_S = \sum_{k\in S} b_k = h^{\frac{1}{2}} \sum_{k\in S} f_k \, h^{\frac{1}{2}}$ is bounded (the sum converges strongly in A^{**}). The condition $\|b_k e\| < 2^{-k}$ implies that $b_S e \in A$, and hence $b_S \in M(A,B)$. We claim that if $S_1 \Delta S_2$ is infinite, then $\|b_{S_1} - b_{S_2}\| \ge \frac{c^2}{4}$. Assume for example that $\{k_i : i = 1,2,\dots\} \subset S_1 \backslash S_2$, where $k_1 < k_2 < \dots$. Then

$$\|b_{k_i} b_{S_2}\| \le \left\| b_{k_i} \sum_{\substack{\ell \in S_2 \\ \ell < k_i}} b_\ell \right\| + \sum_{\ell > k_i} \|b_{k_i} b_\ell\| \le 2^{-k_i} + \Sigma_{k_i+1}^{\infty} 2^{-\ell} = 2^{1-k_i}.$$

$$\|b_{k_i} b_{S_1}\| \ge \|b_{k_i}\|^2 - \left\|b_{k_i} b_{S_1 \backslash \{k_i\}}\right\| \ge \frac{c^2}{4} - 2^{1-k_i}.$$

Since $\|b_{k_i}\| \le 1$ and $\|b_{S_1} - b_{S_2}\| \ge \frac{\|b_{k_i}(b_{S_1} - b_{S_2})\|}{\|b_{k_i}\|}$, we find that $\|b_{S_1} - b_{S_2}\| \ge \frac{c^2}{4} - 2^{2-k_i}, \forall i$. Thus the claim is proved, and it follows easily that $M(A,B)$ is non-separable.

To see that $M(A,B)/B$ is non-separable, first replace B by $B_1 =$ her (q_1), where q_1 is the range projection of h. Since $h \in M(A, B_1)$, we still have $M(A, B_1) \ne B_1$, and B_1 is σ-unital. Then choose a separable C^*-subalgebra, A_0, of A such that $A =$ her (A_0), $B_1 =$ her $(B_1 \cap A_0)$, and $h \in A_0^{**}$. Thus with $B_0 = B_1 \cap A_0$, B_0 is separable and $h \in M(A_0, B_0) \backslash B_0$ (cf 2.15 and 2.16 of [4]). It follows that $M(A_0, B_0)$ and hence $M(A_0, B_0)/B_0$ are non-separable. Since $M(A_0, B_0) \cap B = B_0$, it follows that $M(A,B)/B$ is non-separable.

Remark. We have not used Theorem 4 in the proof of Theorem 5, but a formally shorter, and in reality much more technical, proof could be given by combining the facts about the p_n's established in the proof of Theorem 4 with 3.43(b) of [4]. (The idea of looking at it in this way was suggested to us by Theorem 4.7 of [3].)

Corollary 6. If A is a separable C^*-algebra, then A is the largest separable hereditary C^*-subalgebra of $M(A)$. In particular A is the largest separable,closed, two-sided ideal of $M(A)$.

Proof. Let C be a separable hereditary C^*-subalgebra of $M(A)$ such that $C \not\subset A$. Choose h in $C_+ \backslash A$. Since h is not strongly q-continuous, there is $\varepsilon > 0$ such that $E_{[\varepsilon,\infty)}(h)$ is not compact. Let $q = E_{(\frac{\varepsilon}{2},\infty)}(h)$ and $B =$ her(q).

Then it is easy to see (either from Theorem 2 or by a direct argument) that $M(A,B) \neq B$. Also $h \geq \frac{\varepsilon}{2} q$ implies $M(A,B) \subset C$. This contradicts Theorem 5.

Corollary 7. If A is a σ-unital C^*-algebra, then any non-zero hereditary C^*-subalgebra of $M(A)/A$ is non-separable.

Proof. If C_1 is hereditary in $M(A)/A$, let $C = \pi^{-1}(C_1)$, where $\pi : M(A) \to M(A)/A$ is the natural map. Apply the arguments of the previous proof to C.

Corollary 8. If A_1 and A_2 are separable C^*-algebras, then any isomorphism of $M(A_1)$ with $M(A_2)$ arises from an isomorphism of A_1 with A_2.

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Department of Mathematics, Purdue University
West Lafayette, IN 47907

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REMARKS ON MATERIAL IMPLICATION IN ORTHOMODULAR LATTICES

L. Román and B. Rumbos

Presented by J. Lambek, F.R.S.C.

Abstract

The problem of defining an adequate implication in orthomodular lattices is important if one is to take these lattices as models for the logic of quantum mechanics. The best choice for such an implication is, so far, the Sasaki hook; it's main drawback being that it does not combine smoothly, in a sense that will be made precise later, with the usual logical conjunction. Here, we attempt to solve this problem by changing the conjunction as suggested by Finch in [F.1].

A complete lattice $(Q, \leq)$ is called orthomodular iff the following are satisfied:

I) There exists an orthocomplementation

$.\ ^{\perp} : Q \to Q$ satisfying

.i) $a \leq b \iff b^{\perp} \leq a^{\perp}$

.ii) $(a^{\perp})^{\perp} = a$

.iii) $a^{\perp} \vee a = 1$

.iv) $a^{\perp} \wedge a = 0$. for all a, b in Q.

II) Weak modularity is satisfied i.e., given $a \leq b$ in Q then $b = a \vee (b \wedge a^{\perp})$ holds.

Perhaps the best known example of an orthomodular lattice is the lattice of closed subspaces of a Hilbert space H. It is well known (see [K.] for example), that if H is finite dimensional then this lattice is actually modular, but for an arbitrary H only weak modularity is satisfied.

For reasons that do not concern us here, orthomodular lattices have been linked to the creation of a logic for quantum mechanics, on the assumption that the usual boolean and intuitionistic logics are not applicable to quantic phenomenae. In order to produce a logic out of an orthomodular lattice Q, it became necessary to construct a "well-behaved" implication "$\rightarrow$":$Q \times Q \rightarrow Q$. It is now generally agreed that the best choice for an implication is the so called Sasaki hook, defined as follows:

$$a \rightarrow b = (a \wedge b) \vee a^{\perp} \text{ for all } a, b \in Q$$

Hardegree [H.] proves that "$\rightarrow$" satisfies certain basic implicative criteria but the problem, as we see it, is that "$\wedge$" does not behave exactly as the logical conjunction should with respect to "$\rightarrow$".

Note that in both Boolean and Heyting algebras, the following condition is satisfied:

$a \wedge b \leq c$ iff $a \leq b \rightarrow c$ for all a, b and c in the lattice.

In categorical terms, this is just saying that $- \wedge b$ is left adjoint to $b \rightarrow -$ for all b in the given lattice. We are thus tempted to define the logical conjunction in the orthomodular lattice Q, as the left adjoint (if it exists) to $b \rightarrow -$, where, "$\rightarrow$" is the

Sasaki hook. In [F.2], Finch proves that such an adjoint exists; we shall denote it by $-\&b$. The relation $a\&b \leq c$ iff $a \leq b \to c$ holds for all a, b and c in Q. An explicit description of $a\&b$ is given by $a\&b = (a \vee b^{\perp}) \wedge b$ for all a and b in Q. In [F.1] it is also suggested that logical disjunction be given by "$*$" where $a*b = (a^{\perp}\&b^{\perp})^{\perp} = (a \wedge b^{\perp}) \vee b$ for all a and b in Q. We shall make use of this later on.

We now prove that "&" is "well-behaved" with respect to "→" in the sense that they satisfy the following basic implicative criteria.

(Q.1) $a\&b \leq c$ iff $a \leq b \to c$ This is satisfied by definition.

(Q.2) $a \leq b$ iff $a \to b = 1$

Proof $1\&a = a \leq b$ iff $1 \leq a \to b$ iff $1 = a \to b$

(Q.3) $(a \to b)\&a \leq b$

Proof again by the adjunction since $a \to b \leq a \to b$

(Q.4) $a\&(a \to b) \leq a \wedge b$

Proof

$a\&(a \to b) \leq a \wedge b$ iff $a \leq (a \to b) \to (a \wedge b)$

$= [(a \to b) \wedge (a \wedge b)] \vee (a \to b)^{\perp}$

$= (a \wedge b) \vee (a \to b)^{\perp}$

$= (a \wedge b) \vee [(a \wedge b)^{\perp} \wedge a] = a$

(Q.5) $b^{\perp}\&(a \to b) \leq a^{\perp}$

Proof

$b^{\perp}\&(a \to b) \leq a^{\perp}$ iff $b^{\perp} \leq (a \to b) \to a^{\perp}$ but

$(a \to b) \to a^{\perp} = [(a \to b) \wedge a^{\perp}] \vee (a \to b)^{\perp} =$

$\{[(a \wedge b) \vee a^{\perp}] \wedge a^{\perp}\} \vee [(a \wedge b)^{\perp} \wedge a] =$

$a^{\perp} \vee [(a^{\perp} \vee b^{\perp}) \wedge a] = a^{\perp} \vee b^{\perp}$

that last equality holds by $a^{\perp} \leq a^{\perp} \vee b^{\perp}$ and weak modularity. So we have $b^{\perp} \leq a^{\perp} \vee b^{\perp}$ and the result holds.

(Q.6) $b^{\perp} \& a = (a \to b)^{\perp}$

Proof just express in terms of $\wedge$ and $\vee$.

The following assertions are inmediate consequences of the definitions of "$\to$" and "&"

i) $a \to 0 = a^{\perp}$

ii) $0 \to a = 1 = a \to 1$

iii) $1 \& a = a = a \& 1$

iv) $a \& 0 = 0 = 0 \& a$

v) $a \& b \leq b$

vi) $a \& a = a$

vii) $a \wedge b \leq a \& b$

Adjunction gives

viii) $(\vee_i a_i) \& b = \vee_i (a_i \& b)$

ix) $b \to \wedge_i a_i = \wedge_i (b \to a_i)$

In boolean algebras we have $a \to b = a^{\perp} \vee b$, here, the analogous result holds for the disjunction * since it is easily seen that for Q orthomodular, $\to$ the Sasaki hook and $*$ as given before then $a \to b = b * a^{\perp}$

The new conjunction & differs from the usual meet in that it is not necessarily commutative or associative. When $a\&b = b\&a$ for all a, b in Q, then $\& = \wedge$ and Q turns out to be a boolean algebra. This can be easily checked by using properties v) and vii) above. One also notices that this logic is quite different from Girard's linear logic [G.L.], where, he defines a second conjunction which is both associative and commutative, unlike &.

Another two conjunction logic that has been around lately is the logic derived-from Borceux's "quantales" [B.V.]. His & operation is not necessarily commutative but it is nontheless associative. A comparison of quantales and orthomodular lattices is given in [Ro.Ru.] where a detailed exposition of lattices of this type is given.

Given Q orthomodular, a probability measure on Q is a function $\alpha : Q \to [0,1]$ such that $\alpha(0) = 0$, $\alpha(1) = 1$ and for every countable orthogonal sequence $\{a_i\}$ in Q we have $\sum_i \alpha(a_i)$ converges and $\alpha(\vee_i a_i) = \sum_i \alpha(a_i)$. This last property is usually referred to as σ-additivity. If Q is the lattice of closed subspaces of a Hilbert space H, Gleason's theorem asserts that the set of probability measures is complete in the sense that every probability measure α determines a density operator in H and viceversa.

Noting that for a and b in Q such that $a \perp b$ i.e., $a \leq b^{\perp}$, we have $a \vee b = a * b$, we can take the above set of measures and express σ-additivity in terms of $*$, that is, $\alpha(*_i a_i) = \sum_i \alpha(a_i)$. From [F.1] we know that $\alpha(a * b) = \alpha(a) + \alpha(b) - \alpha(a\&b)$ holds. This suggests that the usual probability measures on an orthomodular lattice behave "better" with respect to $*$ and & than they do with respect to $\vee$ and $\wedge$.

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Address:

Instituto de Matemáticas U.N.A.M.

Area de la Investigación Científica

Circuito Exterior, Cd. Universitaria

04510 México, D.F. MEXICO

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MODELES PREMIERS
ET CORPS REGULIEREMENT CLOS

Luc Bélair(1)

Presented by A.H. Lachlan, F.R.S.C.

RESUME. Dans le langage d'élimination des quantificateurs, et hormis le cas naturel, il n'y a pas de modèle premier au-dessus d'un ensemble de paramètres (dénombrable) dans la théorie des corps régulièrement clos de caractéristique 0 et dont le groupe de Galois absolu est (top.) libre sur e générateurs, $e \in \omega$.

§ 0. INTRODUCTION

Un corps K est dit régulièrement clos si toute variété affine sur K, absolument irréductible, possède un point rationnel sur K. Cette propriété est apparue dans l'étude de la théorie du premier ordre des corps finis [Ax] (voir aussi [FJ]). Soit $\mathcal{L}$ le langage des anneaux, RGC la théorie des corps régulièrement clos, $\mathrm{Sol}_n(x_1, \ldots, x_n)$, $n \in \omega$, des prédicats dont l'interprétation est RGC $\models \mathrm{Sol}_n(\vec{x}) \leftrightarrow \exists y\,(y^n + x_1 y^{n-1} + \ldots + x_n = 0)$, et $\mathcal{L}(\mathrm{Sol}_\omega) = \mathcal{L} + \{\mathrm{Sol}_n : n \in \omega\}$. Cherlin, Macintyre et van den Dries [CMD] ont montré que la théorie des corps d'Iwasawa parfaits régulièrement clos élimine les quantificateurs dans le langage $\mathcal{L}(\mathrm{Sol}_\omega)$ muni de prédicats témoignant de chaque type d'isomorphisme des images finies du groupe de Galois absolu. On obtient en particulier, pour tout $e \geq 1$, l'élimination des quantificateurs (E.Q.) de la théorie RGC_e des corps parfaits régulièrement clos dont le groupe de Galois absolu est (topologiquement) libre sur e générateurs (voir aussi [JK]). Dans cette note nous démontrons un résultat sur les modèles premiers au-dessus d'ensembles de paramètres pour RGC_e dans le langage d'élimination $\mathcal{L}(\mathrm{Sol}_\omega)$, résultat déjà connu de Macintyre et van den Dries (ibid.), mais

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(1) Les résultats de cette note ont été obtenus alors que l'auteur était boursier postdoctoral du C.R.S.N.G. dans l'Equipe de logique mathématique C.N.R.S., Université de Paris-VII. L'auteur remercie également A.Macintyre pour plusieurs discussions utiles.

dont la preuve semble avoir été oubliée. Si F est un corps, alors $F^{\bullet}$ désigne son groupe multiplicatif, G(F) son groupe de Galois absolu, et pour $\vec{\sigma} \in G(F)^e$, $\mathrm{Inv}(\vec{\sigma})$ est le corps des invariants de $\vec{\sigma}$. Notre démonstration utilise des techniques telles qu'indiquées dans [CMD]: en particulier le lemme de Gaschutz ([Ga], théorème I), ainsi que le théorème de compacité en conjugaison avec les méthodes de Jarden pour construire des modèles de RGC_e [Ja]. Notons que dans la théorie RGC, les prédicats Sol_n se prolongent de façon unique d'un anneau intègre à son corps des fractions.

§ 1. LE RESULTAT

Soit $M \models RGC_e$ et $A \subseteq M$ un sous-corps de M. Si M/A est algébrique, alors, dans le langage $\mathcal{L}(Sol_\omega)$, M est premier au-dessus de A. En effet, si $\iota: A \hookrightarrow N$ est un $\mathcal{L}(Sol_\omega)$-plongement de A dans un autre modèle N, alors tout polynôme sur A a un zéro dans M si et seulement si il en a un dans N. Par un lemme de Ax (voir [Po]), ι se prolonge en un $\mathcal{L}(Sol_\omega)$-plongement $M \hookrightarrow N$ au-dessus de A. Le théorème ci-dessous montre que, au moins en caractéristique zéro et si A est dénombrable, c'est le seul cas possible. Notons, par exemple, qu'on peut plonger un corps algébriquement clos dénombrable dans un modèle de RGC_e (voir [Ja]).

Théorème. *Soit* $M \models RGC_e$, *de caractéristique zéro, et* $A \subseteq M$ *un sous-corps dénombrable de* M. *Alors, dans* $\mathcal{L}(Sol_\omega)$, M *est premier au-dessus de* A *si et seulement si l'extension* M/A *est algébrique*.

Démonstration. Supposons que M/A n'est pas algébrique. Si M est premier au-dessus de A alors, par le lemme de Ax, M est aussi premier au-dessus de la clôture algébrique relative de A dans M. On peut donc supposer que A est relativement algébriquement clos dans M. Il suffit de montrer que M n'est pas atomique sur A. Soit donc $t \in M$, un élément transcendant sur A. On montre que le type tp(t/A) de t au-dessus de A n'est pas isolé. Soit $\alpha(v, \vec{a}) \in$ tp(t/A), nous allons montrer que α n'isole pas tp(t/A). Par E.Q. on a:

$$RGC_e \models \alpha(v, \vec{a}) \leftrightarrow \bigvee_j \bigwedge_i \exists w_i(F_{ij}(w_i, v, \vec{a}) = 0) \wedge \neg \exists w_j(H_j(w_j, v, \vec{a}) = 0)$$

où $F_{ij}, H_j \in \mathbb{Z}[X, Y, \vec{Z}]$. Il suffit de considérer l'une de ces conjonctions satisfaite par t. On peut donc supposer:

$$RGC_e \models \alpha(v, \vec{a}) \leftrightarrow \bigwedge_i \exists w_i(F_i(w_i, v, \vec{a}) = 0) \wedge \neg\exists w(H(w, v, \vec{a}) = 0) .$$

On travaille dans une clôture algébrique fixée de M. Soit L le corps de décomposition des F_i, H au-dessus de A(t), M' la clôture algébrique relative de A(t) dans M, et $B = L \cap M'$, de sorte que M' et L sont linéairement disjoints au-dessus de B. Notons que G(M'), G(A) sont tous deux (topologiquement) engendrés par e générateurs. Soit A" la clôture algébrique relative de A dans M'L, alors pour tout corps intermédiaire $B \subseteq E \subseteq M'$ on a $[EA'': BA''] = [A'': A] = [A''M': M']$. Notons que Gal(L/B) est aussi engendré par e générateurs. Le lemme suivant est un corollaire immédiat de la Proposition 7 de [Du].

Lemme 1. Soit $\{k_1, k_2, \ldots\}$ un ensemble infini d'entiers distincts, et $f_n(X) = X^2 - (t + k_n)(t + k_{n+1})$, pour n = 1, 2, ... Alors les polynômes $f_n(X, t)$ sont absolument irréductibles et engendrent des extensions linéairement disjointes au-dessus de A(t).

Par hypothèse M' possède au plus e extensions de chaque degré dans une clôture algébrique fixée, et donc l'indice des carrés $(M')^{\cdot 2}$ dans $(M')^{\cdot}$ est au plus e + 1. On peut donc trouver un ensemble infini d'entiers distincts $\{k_1, k_2, \ldots\}$ tel que $X^2 - (t + k_n)(t + k_{n+1})$ ait une racine dans M' pour tout n = 1, 2, ... Le lemme 1 implique alors que pour toute extension finie de A(t) dans M', il existe des entiers k, k' tel que $X^2 - (t + k)(t + k')$ n'a pas de racine dans cette extension mais en possède une dans M'. Soit $f(X, t)$ un tel polynôme de degré 2 pour B, et B_1/B l'extension quadratique engendrée par f; ainsi $M \models \alpha(t) \wedge \exists x(f(x, t) = 0)$. Nous allons construire un modèle $N \models RGC_e$ tel que $A(t) \subseteq N$, $N \models \alpha(t) \wedge \neg\exists x(f(x, t) = 0)$, et A soit relativement algébriquement clos dans N, ce qui assurera que $\alpha(t)$ n'isole pas $tp(t/A)$. Il suffit de construire un modèle de la théorie suivante du langage $\mathcal{L}(B)$, où $\Delta(B)$ est le diagramme de B:

$$RGC_e + \Delta(B) + \alpha(t) + \neg\exists x(f(x, t) = 0) + \{\neg\exists x(g(x) = 0) \ : \ 0 \neq g \in A[X]\}\,.$$

Pour satisfaire les conditions sur A, il suffit, par compacité, de considérer une extension galoisienne finie A'/A à éviter. Notons que A'' est aussi la clôture algébrique relative de A dans L; on peut supposer que A' est une extension de A'' et posons $d = [A'\colon A'']$. Les extensions A''/A et BA''/B sont aussi galoisiennes. Par hypothèse, A'' coïncide avec la clôture algébrique relative de A dans B_1L. Il s'ensuit que A' et B_1L sont linéairement disjoints au-dessus de A'', d'où $[B_1LA'\colon B_1L] = [A'\colon A''] = d = [A'L\colon L]$. Comme $[B_1L\colon B_1] = [L\colon B]$, on en conclut que B_1 et $A'L$ sont linéairement disjoints au-dessus de B. Il est clair que L et $A'B$ sont linéairement disjoints au-dessus de $A''B$ ($[A'B\colon A''B] = d$), et donc $Gal(A'L/A''B)$ est produit direct de $Gal(A'B/A''B)$ et $Gal(L/A''B)$, et les applications de restriction $Gal(A'L/A'B) \longrightarrow Gal(L/A''B)$, $Gal(A'L/L) \longrightarrow Gal(A'B/A''B)$ sont des isomorphismes.

Lemme 3. Soit $x \in Gal(A'B/B)$ et $y \in Gal(L/B)$ tel que les restrictions $x|A''B$ et $y|A''B \in Gal(A''B/B)$ coïncident. Alors il existe $z \in Gal(A'L/B)$ tel que $z|A'B = x$ et $z|L = y$.

Lemme 4. Il existe $\sigma_1, \ldots, \sigma_e \in Gal(A'L/B)$ tel que $Inv(\vec{\sigma}) \cap L = B$ et $Inv(\vec{\sigma}) \cap A'B = B$.

Démonstration. Les groupes $Gal(A'B/B)$, $Gal(L/B)$, $Gal(A''B/B)$ sont des images homomorphes de $G(M')$ et par conséquent engendrés chacun par e éléments. Soit $\theta_1, \ldots, \theta_e$ des générateurs de $Gal(A''B/B)$, alors par le lemme de Gaschutz les θ_i se relèvent en des générateurs g_i et h_i de

$\mathrm{Gal}(A'B/B)$ et $\mathrm{Gal}(L/B)$ respectivement. Par le lemme précédent il existe $\sigma_1, \ldots, \sigma_e \in \mathrm{Gal}(A'L/B)$ tel que $\sigma_i|A'B = g_i$ et $\sigma_i|L = h_i$. □

Comme B_1 et $A'L$ sont linéairement disjoints au-dessus de B, il existe $\theta_1, \ldots, \theta_e \in \mathrm{Gal}(BA'L/B)$ tel que $\mathrm{Inv}(\vec{\theta}) \cap L = B$, $\mathrm{Inv}(\vec{\theta}) \cap A'B = B$ et $\mathrm{Inv}(\vec{\theta}) \cap B_1 = B$. Notons que B, une extension finie de $A(t)$, est dénombrable et hilbertien. Par les résultats de Jarden [Ja], il existe $\vec{\sigma} \in G(B)^e$ tel que $\mathrm{Inv}(\vec{\sigma}) \models RGC_e$ et $\vec{\sigma}|A'LB = \vec{\theta}$. Alors $\mathrm{Inv}(\vec{\sigma})$ est un modèle de $RGC_e + \Delta(B) + \alpha(t) + \neg\exists x(f(x, t) = 0)$ et ne contient aucun élément de A'. □

Notre démonstration montre aussi qu'en toute cardinalité d'ensemble de paramètres A, ou bien A se plonge dans un modèle M tel que M/A est algébrique, et alors M est premier au-dessus de A, ou bien il n'y a pas de modèle atomique sur A.

§ 2. APPLICATION AUX CORPS p-ADIQUES

Dans le cas particulier $e = 1$, la théorie RGC_1 a pour modèles les modèles infinis de la théorie des corps finis, et en particulier de la théorie des corps premiers finis $T = \mathrm{Th}(\{\mathbb{F}_p : p \text{ premier}\})$. Soit $\mathbb{Q}_p$ le corps des nombres p-adiques, et considérons la théorie de corps valué $\Sigma = \mathrm{Th}(\{\mathbb{Q}_p : p \text{ premier}\})$ (voir [Bé]). La théorie T apparaît alors comme théorie résiduelle des modèles de Σ d'égale caractéristique zéro. La théorie Σ élimine les quantificateurs dans le langage $\mathfrak{L}'$, obtenu de $\mathfrak{L}$ en ajoutant pour chaque $n \geq 2$: un nombre fini de constantes, un prédicat $\underset{\sim}{P}_n(x)$ interprété par $\underset{\sim}{P}_n(x) \longleftrightarrow \exists y(y^n = x)$, et un prédicat $S_n(x_1, \ldots, x_n)$ interprété comme Sol_n au niveau du corps des restes (ibid.). Des techniques standard de la théorie des modèles des corps valués permettent de démontrer que pour $M \models \Sigma$, d'égale caractéristique 0, et $A \subset M$ un sous-corps, si, dans $\mathfrak{L}'$, M est premier au-dessus de A, alors, dans $\mathfrak{L}(\mathrm{Sol}_\omega)$, le corps des restes de M est premier au-dessus du corps des restes de A. Ceci permet de "relever" le résultat du paragraphe § 1. au niveau de la théorie Σ: soit M, A comme ci-dessus et A dénombrable, alors, dans le langage $\mathfrak{L}'$, M est premier au-dessus de A si et seulement si l'extension M/A est algébrique. On peut trouver un tel A dont le corps des restes est algébriquement clos, de sorte qu'il n'y a pas de modèle de Σ qui, dans $\mathcal{B}'$, soit premier au-dessus de ce A. Il s'ensuit que, dans $\mathcal{B}'$, la théorie Σ ne possède pas de fonctions de Skolem définissables (voir [VdD]). Ceci a une incidence sur l'étude de l'uniformité par rapport au paramètre p dans les résultats de Denef sur la rationalité de séries de Poincaré p-adiques (voir [Ma]).

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Département de Mathématiques et Informatique,
Université du Québec à Montréal,
Québec, Canada H3C 3P8

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Approximately quasiconvex functions

Kazimierz Nikodem

Presented by J. Aczél, F.R.S.C.

Abstract. We prove a stability theorem of Hyers-Ulam type for quasiconvex functions.

The classical stability theorem of D.H. Hyers and S.M. Ulam (cf.[3]) states that, if a function $f: D \to \mathbb{R}$, where $\mathbb{R}$ denotes the real line and D is a convex subset of $\mathbb{R}^n$, satisfies the inequality

$$f(tx+(1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon$$

for all $x,y \in D$, $t \in [0,1]$ and some $\varepsilon > 0$, then there exists a convex function $g: D \to \mathbb{R}$ such that

$$|f(x)-g(x)| \leq k_n\varepsilon, \quad x \in D,$$

where the constant k_n depends only on the dimension of the domain (cf. also [1]).

In the present note we prove an analogous theorem for quasiconvex functions.

In what follows we assume that D is a non-empty convex subset of $\mathbb{R}^n$ and ε is a positive constant. Recall that a function $f: D \to \mathbb{R}$ is said to be quasiconvex if

$$f(tx+(1-t)y) \leq \max\{f(x),f(y)\}$$

for all $x,y \in D$, $t \in [0,1]$(cf. e.g. [4]). We say that a function $f: D \to \mathbb{R}$ is ε-quasiconvex if

$$f(tx+(1-t)y) \leq \max\{f(x),f(y)\} + \varepsilon \tag{1}$$

for all $x,y \in D$, $t \in [0,1]$.

Assume that $f\colon D \to \mathbb{R}$ is an ε-quasiconvex function and consider the level sets $L_a := \{x \in D\colon f(x) \leq a\}$, $a \in \mathbb{R}$. It is clear that $\bigcup_{a\in\mathbb{R}} L_a = D$ and $L_a \subset L_b$ whenever $a \leq b$. We have the following

Lemma. *Let* $m \in \mathbb{N}$ *(positive integers) and* $a \in \mathbb{R}$. *If* $x_1,\dots,x_{m+1} \in L_a$, $t_1,\dots,t_{m+1} \in [0,1]$ *and* $t_1 + \dots + t_{m+1} = 1$, *then* $t_1x_1 + \dots + t_{m+1}x_{m+1} \in L_{a+k(m)\varepsilon}$, *where* $k(m) := [\log_2 m] + 1$.

Proof. Notice first that, if $x_1,x_2 \in L_a$ and $t_1,t_2 \in [0,1]$, $t_1 + t_2 = 1$, then, because of (1), $t_1x_1 + t_2x_2 \in L_{a+\varepsilon}$. By induction we can show that

$$t_1x_1 + \dots + t_{2^p}x_{2^p} \in L_{a+p\varepsilon} \tag{2}$$

for all $p \in \mathbb{N}$, $x_1,\dots,x_{2^p} \in D$ and $t_1,\dots,t_{2^p} \in [0,1]$ with $t_1 + \dots + t_{2^p} = 1$. Now, fix an $m \in \mathbb{N}$ and assume that $x_1,\dots,x_m \in D$, $t_1,\dots,t_m \in [0,1]$ and $t_1 + \dots + t_m = 1$. Take the minimal $p \in \mathbb{N}$ such that $2^p \geq m+1$. One can easily check that $p = [\log_2 m] + 1 = k(m)$. In the case where $2^p > m+1$, let us put $t_{m+2} = \dots = t_{2^p} := 0$ and $x_{m+2} = \dots = {}_{2^p} := x_1$. Then, by (2), we obtain

$$t_1x_1 + \dots + t_{m+1}x_{m+1} = t_1x_1 + \dots + t_{2^p}x_{2^p} \in L_{a+k(m)\varepsilon},$$

which was to be proved.

The main result of this paper is the following.

Theorem. *Let D be a convex subset of $\mathbb{R}^n$. If a function $f: D \to \mathbb{R}$ is ε-quasiconvex, then there exists a quasiconvex function $g: D \to \mathbb{R}$ such that*

$$g(x) \le f(x) \le g(x) + k(n)\varepsilon, \quad x \in D, \tag{3}$$

where $k(n) := [\log_2 n] + 1$. ([illegible])

Proof. Fix a point $x \in D$ and take an $a \in \mathbb{R}$ such that $x \in \operatorname{conv} L_a$ (the convex hull of L_a). By the Caratheodory theorem (cf. [4, Th. E, Sec. 3]) $x = t_1x_1 + \ldots + t_{n+1}x_{n+1}$ for some $x_1, \ldots, x_{n+1} \in L_a$ and $t_1, \ldots, t_{n+1} \in [0,1]$ with $t_1 + \ldots + t_{n+1} = 1$. Using our lemma we get $x \in L_{a+k(n)\varepsilon}$, which means that $f(x) \le a + k(n)\varepsilon$. Since this inequality holds for every $a \in \mathbb{R}$ such that $x \in \operatorname{conv} L_a$, we have also

$$f(x) \le \inf\{a \in \mathbb{R}: x \in \operatorname{conv} L_a\} + k(n)\varepsilon. \tag{4}$$

Let us define a function $g: D \to \mathbb{R}$ putting

$$g(x) := \inf\{a \in \mathbb{R}: x \in \operatorname{conv} L_a\}, \quad x \in D.$$

By (4) we obtain $f(x) \le g(x) + k(n)\varepsilon$. Since $\{a \in \mathbb{R}: x \in \operatorname{conv} L_a\} \supset \{a \in \mathbb{R}: x \in L_a\}$, we have also

$$g(x) = \inf\{a \in \mathbb{R}: x \in \operatorname{conv} L_a\} \le \inf\{a \in \mathbb{R}: x \in L_a\} = f(x).$$

Now we shall show that g is quasiconvex (cf. [2]). For this purpose fix $x, y \in D$ and assume that $g(x) \le g(y)$. Take an arbitrary $c > g(y)$. By the definition of $g(y)$, there exists an $a < c$ such that $y \in \operatorname{conv} L_a$. Then also $y \in \operatorname{conv} L_c$, because $L_a \subset L_c$. Analogously we show that $x \in \operatorname{conv} L_c$. Hence $tx + (1-t)y \in \operatorname{conv} L_c$ for every $t \in [0,1]$.

Since this relation holds for all $c > g(y)$, we obtain

$$g(tx+(1-t)y) = \inf\{c \in \mathbb{R}: tx+(1-t)y \in \operatorname{conv} L_c\} \leq g(y) = \max\{g(x), g(y)\}.$$

This shows that g is quasiconvex and finishes our proof.

<u>Remark 1.</u> For $n = 1$ the constant $k(n) = 1$ and the example $f(x) = \sin x$, $x \in \mathbb{R}$, shows that (3) can not be reduced.

2. The condition (3) can be rewritten in the form $|f(x)-h(x)| \leq \frac{1}{2}k(n)\varepsilon$, where $h(x) := g(x) + \frac{1}{2}k(n)\varepsilon$ is also a quasiconvex function.

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Department of Mathematics
Technical University
43-300 Bielsko-Biała, POLAND

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Equations arising from the theory of orthogonally additive and quadratic functions

By Gy. Maksa, Gy. Szabó and L. Székelyhidi

Presented by J. Aczél, F.R.S.C.

Throughout this paper[1] , $(\mathcal{F},+,\cdot)$ and $(\mathcal{G},+),(\mathcal{S},+)$ denote a commutative field and two abelian groups, respectively. The study of orthogonally additive resp. quadratic mappings on abstract orthogonality spaces (see [2] and some forthcoming papers of the second author) leads to the following unrestricted equations for the unknown functions $g,h:\mathcal{F}\to\mathcal{S}$, respectively:

$$(1)\quad g(ax+by)-g(ax)-g(by)=g(ay+bx)-g(ay)-g(bx),\qquad x,y\in\mathcal{F},$$

$$(2)\qquad h(ax+by)+h(ax-by)-2h(ax)-2h(by)=$$
$$=h(ay+bx)+h(ay-bx)-2h(ay)-2h(bx),\qquad x,y\in\mathcal{F},$$

where $a,b\in\mathcal{F}$ are fixed elements such that $a,b,a\pm b\neq 0$. In fact, (1) is a consequence of the more complex equation with two unknown functions $f,g:\mathcal{F}\to\mathcal{S}$, as follows

$$(3)\qquad f(a_1x+b_1y)+g(a_2x+b_2y)=$$
$$=f(a_1x)+f(b_1y)+g(a_2x)+g(b_2y),\qquad x,y\in\mathcal{F},$$

where $a_1,a_2,b_1,b_2\in\mathcal{F}\setminus\{0\}$ are fixed elements such that $a_1b_2\neq a_2b_1$. Our aim with this paper is to solve these equations under fairly weak conditions on $\mathcal{F}$ and $\mathcal{S}$. We shall make use of the following result, which somewhat generalizes that of Székelyhidi (see [3] and also [1]; here for $p\in\mathbb{N}$, a group is *p-divisible, p-torsion-free* or *uniquely p-divisible*, if the p-multiplication on the group is surjective, injective or bijective, respectively):

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THEOREM 1 *Let $n \in \mathbf{N}$ and suppose that $\mathcal{G}$ or $\mathcal{S}$ is $n!$-divisible and $\mathcal{S}$ is also $n!$-torsion-free. If the function $P : \mathcal{G} \to \mathcal{S}$ is of degree n, i.e.*

$$P(x) + \sum_{k=0}^{n} P_k(\phi_k(x) + \psi_k(y)) = 0, \qquad x, y \in \mathcal{G}, \tag{4}$$

with some functions $P_k : \mathcal{G} \to \mathcal{S}$ and homomorphism $\phi_k, \psi_k : \mathcal{G} \to \mathcal{G}$ such that $\phi_k(\mathcal{G}) \subset \psi_k(\mathcal{G})$ ($k = 0, 1, \ldots, n$), then it has the form

$$P(x) = P(0) + \sum_{k=1}^{n} A_k(x, x, \ldots, x), \qquad x \in \mathcal{G}, \tag{5}$$

where $A_k : \mathcal{G}^k \to \mathcal{S}$ are symmetric k-additive functions ($k = 1, \ldots, n$). The converse implication also holds whenever $\mathcal{G}$ is $(n+1)!$-divisible.

Proof. (4) $\Longrightarrow$ (5): Without loss of generality, we may assume that $P(0) = 0$. Also, it is clear from the proof of [3], Thm. 3.6 that P is a polynomial of degree n (no divisibility is required). Thus for $n!$-divisible $\mathcal{S}$, Thm. 3 in [1] completes the proof. On the other hand, when $\mathcal{G}$ is the $n!$-divisible group, the proof of the quoted Theorem 3 in [1] yields only that

$$rP(x) = \sum_{k=1}^{n} r^k A_k^*\left(\frac{x}{r}, \frac{x}{r}, \ldots, \frac{x}{r}\right), \qquad x \in \mathcal{G}, \tag{6}$$

with some symmetric k-additive functions $A_k^* : \mathcal{G}^k \to \mathcal{S}$ ($k = 1, 2, \ldots, n$) and $r = n!(n-1)!\ldots 2!$, where x/r is a symbol for any element in $\mathcal{G}$ with $r(x/r) = x$. Next observe that for any $k \in \mathbf{N}$ and arbitrary $u_i, u_i' \in \mathcal{G}$ such that $ru_i = ru_i'$ ($i = 1, 2, \ldots, k$),

$$r^k A_k^*(u_1, u_2, \ldots, u_k) = r^k A_k^*(u_1', u_2', \ldots, u_k')$$

holds. Thus since $\mathcal{S}$ is $n!$-torsion-free, the functions $A_k : \mathcal{G}^k \to \mathcal{S}$ ($k = 1, 2, \ldots, n$) are well defined by

$$A_k(x_1, x_2, \ldots, x_k) = r^{k-1} A_k^*\left(\frac{x_1}{r}, \frac{x_2}{r}, \ldots, \frac{x_k}{r}\right),$$

where $x_i \in \mathcal{G}$ and x_i/r are arbitrary elements of $\mathcal{G}$ such that $r(x_i/r) = x_i$ ($i = 1, 2, \ldots, k$). Clearly A_k is symmetric and also it is additive in each variable. This latter assertion immediately follows from the k-additivity of A_k^* and the fact $r([x_i + x_i']/r) = r(x_i/r + x_i'/r)$. Thus (6) turns into

$$rP(x) = r \sum_{k=1}^{n} A_k(x, x, \ldots, x), \qquad x \in \mathcal{G},$$

which since $\mathcal{S}$ is $n!$-torsion-free, proves the first implication.

(5) $\Longrightarrow$ (4) is immediate from the proof of [3], Thm. 3.6.

COROLLARY 2 *Suppose that either* char$\mathcal{F} \neq 2$ *and* $\mathcal{S}$ *is* 2*-torsion-free, or* $\mathcal{S}$ *is uniquely* 2*-divisible. Then* g *is a solution of equation* (1) *if, and only if, it is of the form*

$$g(x) = g(0) + A(x) + B(x,x), \qquad x \in \mathcal{F}, \tag{7}$$

with an additive $A : \mathcal{F} \to \mathcal{S}$ *and symmetric biadditive* $B : \mathcal{F}^2 \to \mathcal{S}$ *such that*

$$B(ax, by) = B(ay, bx), \qquad x, y \in \mathcal{F}. \tag{8}$$

Proof. Necessity. Introduce the new variables $u = ax + by$ and $v = ay + bx$. Then

$$x = \frac{au - bv}{a^2 - b^2}, \qquad y = \frac{av - bu}{a^2 - b^2}.$$

Thus (1) turns into the form

$$g(u) - g(v) + \left[g\left(\frac{b}{a^2 - b^2}[au - bv]\right) - g\left(\frac{a}{a^2 - b^2}[au - bv]\right)\right] +$$

$$+ \left[g\left(\frac{a}{a^2 - b^2}[-bu + av]\right) - g\left(\frac{b}{a^2 - b^2}[-bu + av]\right)\right] = 0, \qquad u, v \in \mathcal{F},$$

i.e. with suitably defined functions $g_k : \mathcal{F} \to \mathcal{S}$ $(k = 0, 1, 2)$,

$$g(u) + g_0(0u + 1v) + g_1(au - bv) + g_2(-bu + av) = 0, \qquad u, v \in \mathcal{F}, \tag{9}$$

and so the above Theorem implies (7). Finally, (8) comes from equation (1).

Sufficiency. Obvious.

COROLLARY 3 *Suppose that either* char$\mathcal{F} \neq 2, 3$ *and* $\mathcal{S}$ *is* 6*-torsion-free, or* $\mathcal{S}$ *is uniquely* 6*-divisible. Then* h *is a solution of equation* (2) *if, and only if, it has the form*

$$h(x) = h(0) + B(x,x) + D(x,x,x,x), \qquad x \in \mathcal{F}, \tag{10}$$

with some symmetric 2*-additive* $B : \mathcal{F}^2 \to \mathcal{S}$ *and a symmetric* 4*-additive function* $D : \mathcal{F}^4 \to \mathcal{S}$ *such that*

$$D(ax, ax, by, by) = D(ay, ay, bx, bx), \qquad x, y \in \mathcal{F}. \tag{11}$$

Proof. Necessity. Substitute in (2) $y = 0$. Then $h(bx) = h(-bx)$ for all $x \in \mathcal{F}$, i.e. h is even. Now introduce the new variables $u = ax + by$ and $v = ay + bx$, whence

$$x = \frac{au - bv}{a^2 - b^2}, \qquad y = \frac{av - bu}{a^2 - b^2}.$$

Then (2) turns into the form

$$\begin{aligned} h(u) - h(v) + 2\left[h\left(\frac{b}{a^2 - b^2}[au - bv]\right) - h\left(\frac{a}{a^2 - b^2}[au - bv]\right)\right] + \\ +2\left[h\left(\frac{a}{a^2 - b^2}[-bu + av]\right) - h\left(\frac{b}{a^2 - b^2}[-bu + av]\right)\right] + \\ +h\left(\frac{a^2 + b^2}{a^2 - b^2}u - \frac{2ab}{a^2 - b^2}v\right) - h\left(\frac{-2ab}{a^2 - b^2}u + \frac{a^2 + b^2}{a^2 - b^2}v\right) = 0, \qquad u, v \in \mathcal{F}, \end{aligned}$$

i.e. with suitably defined functions $h_k : \mathcal{F} \to \mathcal{S}$ $(k = 0, 1, \ldots, 4)$ and elements $c = (a^2 + b^2)/(a^2 - b^2)$, $d = 2ab/(a^2 - b^2)$, we have

$$\begin{aligned} (12) \qquad & h(u) + h_0(0u + 1v) + h_1(au - bv) + h_2(-bu + av) + \\ & +h_3(cu - dv) + h_4(-du + cv) = 0, \qquad u, v \in \mathcal{F}. \end{aligned}$$

Thus by Theorem 1, h is of form

$$h(x) = h(0) + \sum_{k=1}^{4} A_k(x, x, \ldots, x), \qquad x \in \mathcal{F},$$

with certain symmetric k-additive functions $A_k : \mathcal{F}^k \to \mathcal{S}$ $(k = 1, 2, 3, 4)$. Here, since h is even and $\mathcal{S}$ is 6-torsion-free, we have for all $x \in \mathcal{F}$ that $A_1(x) + A_3(x, x, x) = 0$, whence $8A_3(x, x, x) = -2A_1(x)$ and therefore $4A_3(x, x, x) = -A_1(x) = A_3(x, x, x)$, i.e.

$$A_3(x, x, x) = 0 = A_1(x), \qquad x \in \mathcal{F}.$$

Finally, the formula (11) for $D = A_4$ can be verified by simple computations.

Sufficiency. Obvious.

In the rest of the paper we discuss equation (3) showing that the restrictions on $\mathcal{S}$ can be dropped as soon as the even and odd solutions are treated separately.

THEOREM 4 *Assume that* $\mathrm{char}\mathcal{F} \neq 2$ *and* f, g *are solutions of equation* (3). *Then*

i) f *or* g *is odd if and only if, both of them are additive;*

ii) g *is even if and only if,* $g - g(0)$ *is quadratic while* f, *with an additive* $A : \mathcal{F} \to \mathcal{S}$ *and* $a = a_2/a_1$, $b = b_2/b_1$, *has the form*

$$f(x) = f(0) + A(x) - g\left(\frac{a+b}{2}x\right) + g\left(\frac{a-b}{2}x\right), \qquad x \in \mathcal{F}. \tag{13}$$

Proof. By (3), $f(0) + g(0) = 0$ and so $f - f(0)$, $g - g(0)$ satisfy (3), too. Thus we may and do assume that $f(0) = g(0) = 0$. Now define $F, G : \mathcal{F}^2 \to \mathcal{S}$ by

$$F(u,v) = f(u+v) - f(u) - f(v), \qquad u, v \in \mathcal{F},$$

$$G(u,v) = g(u+v) - g(u) - g(v), \qquad u, v \in \mathcal{F}.$$

Then (3) turns into

$$F(u,v) + G(au, bv) = 0, \qquad u, v \in \mathcal{F}. \tag{14}$$

Hence, by the definition of F and G, for all $u, v, w \in \mathcal{F}$, we have

$$G(au, bv) + G(au + bv, bw) = G(au, bv + bw) + G(bv, bw), \tag{15}$$

$$G(au, bv) + G(au + av, bw) = G(au, bv + bw) + G(av, bw). \tag{16}$$

Subtracting (15) from (16), it follows for all $u, v, w \in \mathcal{F}$ that

$$G(au + av, bw) - G(au + bv, bw) = G(av, bw) - G(bv, bw). \tag{17}$$

Since $G(0, bw) = 0$, therefore the substitution $u = -(b/a)v$ in (17) yields

$$G(av - bv, bw) = G(av, bw) - G(bv, bw), \qquad v, w \in \mathcal{F},$$

which implies by (17) that

$$G(au + av, bw) - G(au + bv, bw) = G(av - bv, bw), \qquad u, v, w \in \mathcal{F}. \tag{18}$$

Because of $\mathrm{char}\mathcal{F} \neq 2$, this means that G is additive in its first variable, and regarding the symmetry, G is actually biadditive. Thus

$$g(u + v + w) - g(u + v) - g(w) = G(u + v, w) = G(u, w) + G(v, w) =$$

$$= g(u+w) - g(u) - g(w) + g(v+w) - g(v) - g(w), \qquad u, v, w \in \mathcal{F},$$

whence letting $w = -v$,

$$g(u+v) + g(u-v) = 2g(u) + g(v) + g(-v), \qquad u, v \in \mathcal{F}. \tag{19}$$

In an analogous way, one can gain the same for f:

$$f(u+v) + f(u-v) = 2f(u) + f(v) + f(-v), \qquad u, v \in \mathcal{F}. \tag{20}$$

Now if, say, g is odd, then for $u = v$, (19) yields $g(2u) = 2g(u)$, whence

$$g(u+v) + g(u-v) = g(2u), \qquad u, v \in \mathcal{F},$$

i.e. regarding again the condition $char\mathcal{F} \neq 2$, g is additive. By the original equation (3), so is f, proving part i).

Next suppose that g is even. Then by (19), it is quadratic and due to the symmetry of F, for all $u, v \in \mathcal{F}$ we have $G(au, bv) = -F(u,v) = -F(v,u) = G(av, bu)$ and so

$$f(u+v) - f(u) - f(v) = -G(au, bv) = -2G\left(\frac{a}{2}u, \frac{b}{2}v\right) - 2G\left(\frac{a}{2}v, \frac{b}{2}u\right) =$$

$$= 2G\left(\frac{a}{2}u, \frac{b}{2}u\right) + 2G\left(\frac{a}{2}v, \frac{b}{2}v\right) - 2G\left(\frac{a}{2}[u+v], \frac{b}{2}[u+v]\right).$$

This means that the function $A : \mathcal{F} \to \mathcal{S}$, defined for each $x \in \mathcal{F}$ by

$$A(x) = f(x) + 2G\left(\frac{a}{2}x, \frac{b}{2}x\right) = f(x) + g\left(\frac{a+b}{2}x\right) - g\left(\frac{a-b}{2}x\right)$$

is additive, giving the formula (13).

The converse implications are obvious.

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H-4010 Debrecen, Hungary

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Mailing Addresses

1. I. Assem
Département de Mathématiques et d'Informatique
Université de Sherbrooke
Sherbrooke, Québec, Canada J1K 2R1

2. V. Barrucci
Dipartimento di Matematica
Istituto 'G. Castelnuovo'
Universita di Roma 'La Sapienza'
Piazzale Aldo Moro 2, 00185 Roma, Italia

3. L. Bélair
Département de Mathematiques et Informatique
Université de Québec à Montréal
Québec, Canada H3C 3P8

4. A. Blokhuis
Department of Mathematics
Technical University of Eindhoven
Eindhoven, The Netherlands

5. L.G. Brown
Department of Mathematics
Purdue University
West Lafayette, IN 47907 U.S.A.

6. A.A. Bruen
Department of Mathematics
University of Western Ontario
London, Ontario, Canada N6A 5B7

7. M.A. Canela
Departamento de Matematica Aplicada i Anàlisi
Universidad de Barcelona
08071 - Barcelona, Spain

8. S.I. Goldberg
Department of Mathematics
University of Ilinois
1409 West Green Street
Urbana, IL 61801 U.S.A.

9. A.R. Grandjean
Departamento de Algebra
Facultad de Matematicas
Universidad de Santiago de Compostela, Spain

10. Gy. Maksa
KLTE Matem. Int., Pf. 12
H-4010 Debrecen, Hungary

11. K. Nikodem
Department of Mathematics
Technical University
43-300 Bielsko-Biala, Poland

12. K.I. Noor
Mathematics Department
Punjab University
New Campus, Lahore, Pakistan

13. D. Perrone
Dipartimento di Matematica
Facolta di Scienze
Universita degli Studi di Lecce
Via Arnesano, 73100 Lecce, Italia

14. M. Ralló
Departamento di Matemàtica Aplicada i Analisi
Universidad de Barcelona
08071 - Barcelona, Spain

15. L. Román
Instituto de Matemáticas U.N.A.M.
Area de la Investigación Cientifica
Circuito Exterior, Cd. Universitaria
0 4510 México, D.F. Mexico

16. B. Rumbos
Instituto de Matemáticas U.N.A.M.
Area de la Investigación Cientifica
Circuito Exterior, Cd. Universitaria
0410 México, D.F. Mexico

17. A. Skowroński
Institute of Mathematics
Nicholas Copernicus University
Chopina 12/18, 87-100 Toruń, Poland

18. Gy. Szabó
KLTE Matem. Int., Pf. 12,
H-4010 Debrecen, Hungary

19. L. Székelyhidi
KLTE Matem. Int., Pf. 12,
H-4010 Debrecen, Hungary

20. J.A. Thas
Seminar of Geometry and Combinatorics
State University of Ghent
Krijgslaan 281, B-9000, Ghent, Belgium

21. G. Toth
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario, Canada K7L 3N6

22. M.J. Vale
Departamento de Algebra
Facultad de Matemáticas
Universidad de Santiago de Compostela, Spain

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