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CONTENTS

ISOMETRIES ON W-ALGEBRAS

by

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ABSTRACT: In this paper we introduce the concept of metric space over a W-algebra. We prove that each W-algebra is a metric space over itself. Then we study the isometries of W-algebras as metric spaces.

INTRODUCTION: The W-algebras are the most natural algebraic models of $\aleph_0$-valued propositional Lukasiewicz's calculus. Formally, a W-algebra is an algebra $(A, \neg, \rightarrow, u)$ of type (1,2,0) satisfying the following equations:

(W.1) $u \rightarrow x = x$

(W.2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = u$

(W.3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(W.4) $(\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) = u$

For further information on this subject, see [1].

Definition-1: Let E be a non-empty set and A a W-algebra. $d: E \times E \longrightarrow A$ is an A-metric on E if and only if the following conditions hold:

(i) $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq \neg d(x,z) \rightarrow d(y,z)$

Definition-2: Let A be a W-algebra and E a non-empty set. An A-metric space is a couple (E,d) where d is an A-metric on E.

An exemple of A-metric space is the own W-algebra A, endowed this one with the metric: $d(x,y) = (x \rightarrow y) \rightarrow \neg(y \rightarrow x)$. We are going to show that effectively d is an A-metric on A.

Proposition-1: $d(x,y) = 0$ iff $x = y$

Proof: $d(x,y) = 0$ iff $(x \to y) \to \neg(y \to x) = 0$ iff $\{x \to y = u$ and $\neg(y \to x) = 0\}$ iff $x \to y = y \to x = u$ iff $x = y$.

Proposition-2: $d(x,y) = d(y,x)$

(Trivial)

To prove that the condition (iii) of Def-1 holds, we need the following

Lemma: For all $x,y,z \in A$ $\neg[(x \to z) \to \neg(z \to y)] \leqq x \to y$

Proof: By (W.2) we have $(x \to z) \to ((z \to y) \to (x \to y)) = u$. This implies $x \to z \leqq (z \to y) \to (x \to y) = \neg(x \to y) \to \neg(z \to y)$. Therefore, $\neg(x \to y) \leqq (x \to z) \to \neg(z \to y)$ and $\neg[(x \to z) \to \neg(z \to y)] \leqq x \to y$ as we wanted to prove.

Proposition-3: $d(x,y) \leqq \neg d(x,z) \to d(y,z)$

Indeed, by the previous lemma, we have $\neg[(x \to z) \to \neg(z \to y)] \leqq x \to y$ and $\neg[(y \to z) \to \neg(z \to x)] \leqq y \to x$. Thus, $d(x,y) = (x \to y) \to \neg(y \to x) \leqq (x \to y) \to [(y \to z) \to \neg(z \to x)] \leqq \neg[(x \to z) \to \neg(z \to y)] \to [(y \to z) \to \neg(z \to x)] =$

$$(y \to z) \to \{\neg[(x \to z) \to \neg(z \to y)] \to \neg(z \to x)\} =$$

$$(y \to z) \to \{(z \to x) \to [(x \to z) \to \neg(z \to y)]\} =$$

$$(y \to z) \to \{(x \to z) \to [(z \to x) \to \neg(z \to y)]\} =$$

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$$\neg[(x \to z) \to \neg(z \to x)] \to [(y \to z) \to \neg(z \to y)] = \neg d(x,z) \to d(y,z)$$

Definition-3: $\sigma : A \longrightarrow A$ is an isometry on A ($\sigma \in \mathrm{Isom}(A)$) if and only if for all $f,g \in A$ $d(f,g) = d(\sigma f, \sigma g)$

We study now the isometries on A. Begin with the simplest case: when A is a chain.

Theorem-1: If A is a chain then the single isometries on A are the identity and the negation.

Proof: Let $\sigma \in \mathrm{Isom}(A)$. We have that for all $f \in A$, $f = d(f,0) = d(\sigma f, \sigma 0) = (\sigma f \to \sigma 0) \to \neg(\sigma 0 \to \sigma f)$. Since A is a chain, then $f \in \{\neg(\sigma f \to \sigma 0), \neg(\sigma 0 \to \sigma f)\}$. If we take $f = u$ then we have $\sigma u \to \sigma 0 = 0$ or $\sigma 0 \to \sigma u = 0$. In the first case, we have $\sigma u = u$ and $\sigma 0 = 0$ and, in the second, $\sigma 0 = u$ and $\sigma u = 0$.

(i) If $\sigma 0 = 0$ then for all $f \in A$, $f = d(f,0) = d(\sigma f, 0) = \sigma f$ and therefore $\sigma = \mathrm{Id}_A$.

(ii) If $\sigma 0 = u$ then for all $f \in A \backslash \{0,u\}$ we have $f \in \{\neg(\sigma f \to u), \neg(u \to \sigma f)\} = \{0, \neg \sigma f\}$ and therefore $\sigma f = \neg f$. For $f = u$, we have also $\sigma u = 0 = \neg u$.

If A is not a chain, then we can consider a concordant subdirect representation of A by chains: $A \subseteq {}_{SP}\otimes(A_x : x \in X)$. Let $\sigma \in \mathrm{Isom}(A)$; we have $u = d(u,0) = d(\sigma u, \sigma 0) = (\sigma u \to \sigma 0) \to \neg(\sigma 0 \to \sigma u)$ and therefore, for all $x \in X$, $u_x = ((\sigma u)_x \to (\sigma 0)_x) \to \neg((\sigma 0)_x \to (\sigma u)_x)$. Then, for each linear fiber A_x, we have (see the proof of the previous theorem) $(\sigma 0)_x \in \{0_x, u_x\}$.

Proposition-4: If $(\sigma 0)_x = 0_x$ then for all $f \in A$ $(\sigma f)_x = f_x$.

Indeed: $f_x = ((\sigma f)_x \to 0_x) \to \neg(0_x \to (\sigma f)_x) = (\sigma f)_x$

Proposition-5: If $(\sigma 0)_x = u_x$ then for all $f \in A$ $(\sigma f)_x = (\neg f)_x$

Proof: $f_x = ((\sigma f)_x \to u_x) \to \neg(u_x \to (\sigma f)_x) = \neg(\sigma f)_x$

Thus the only isometries on A are the $\xi_T : A \longrightarrow A$ for $T \subseteq X$, where $(\xi_T(f))_x = \neg f_x$ if $x \in T$ and $(\xi_T(f))_x = f_x$ if $x \notin T$.

The nest step is to eliminate any reference to the representation in the study of isometries on A.

Observe that if $b \in A$ si a boolean element of A ($b \in Bool(A)$) then

$$\sigma_b : A \longrightarrow A \text{ where}$$

$$\sigma_b(f) = \{[(\neg f \to b) \to b] \to [(\neg b \to f) \to f]\} \to \neg[(\neg f \to b) \to b]$$

$$(\text{shortly, } \sigma_b(f) = b \Delta f)$$

is an isometry on A; the proof of this fact is trivial by taking coordinates and using that for all $x \in X$, $b_x \in \{0_x, u_x\}$

Let's show that the only isometries on A are the mentioned ξ_T ($T \subseteq X$)

Proposition-6: The application ϕ: Isom(A) ——> Bool(A) where $\phi(\sigma) = \sigma(0)$ is a bijection.

Proof: (i) ϕ is one to one because if $\phi(\sigma) = \phi(\tau)$ then $\sigma(0) = \tau(0)$ and thus $\sigma = \tau$.

(ii) ϕ is exhaustive because if $b \in Bool(A)$ then $\phi(\sigma_b) = \sigma_b(0) = b \Delta 0 = b$

We finish the present paper with some corollaries

Theorem-2: The isometries on A are bijective functions.

Indeed, it is sufficient to prove that the σ_b ($b \in Bool(A)$) are exhaustive, which is obvious because for all $g \in A$, $\sigma_b(b \Delta g) = b \Delta b \Delta g = g$.

Theorem-3: (Isom(A), $\circ$) is a group such that for all $\sigma \in$ Isom(A), $\sigma^2 = Id$.

REFERENCES

[1] Font, Rodríguez, Torrens: 'Wajsberg Algebras'
Stochastica VII (1984) n° 1 pp 5-31

[2] Tiñena, F.: 'The Representation of W-algebras by Global Subdirect Products'
(pre-print)

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The Top Root of a Simple Lie Algebra

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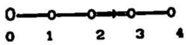
Presented by P. Ribenboim, F.R.S.C.

It is well-known that the integers n^i , $(1 \le i \le n)$, in the top root $\tau = \sum n^i \alpha_i$ of a simple complex Lie algebra (SLA), X_n , occur in many intriguing formulas. Here, $X \in \{A, B, \ldots, G\}$. For example, if $f-1$ is the number of i for which $n^i = 1$, the order of the Weyl Group [1] is $f.n!\Pi n^i$, and the order of the Coxeter transformation is $h = 1 + \sum n^i$ [2, Ch. VI, 1, 11, Prop. 31].

In the present paper I shall exhibit a new property of the integers n^i and at the same time add a foot-note to an important result of Borel and Siebenthal [3].

We adopt the usual notation , so our Lie algebra , $\mathcal{L}$, is generated by e_i, f_i, h_i, $(1 \le i \le n)$, and has a set α_i of simple roots. The Cartan subalgebra, $\mathcal{H}$, has a dual basis h^i whose elements satisfy $\alpha_j(h^i) = \delta^i_j$. In the adjoint representation for which $\mathcal{L}$ is regarded as an $\mathcal{L}$ module , $\exp(sh^j)$ is an automorphism of $\mathcal{L}$ for any complex s. In particular if s is such that $2\pi i = sn^j$, the resulting automorphism , which, for fixed j, we denote by ω , satisfies $\omega\, e_i = e_i$ if $i \ne j$ and $\omega\, e_j = \zeta e_j$ where $\zeta = \exp(s)$ is a primitive n^j-th root of unity. If $\alpha = \Sigma\, k^i \alpha_i$ is any root then $\omega\, e_\alpha = \zeta^k e_\alpha$ where $k = k^j$ and hence $\omega\, e_\tau = e_\tau$ where τ is the top root. Thus if the j-th node is removed from the <u>extended</u> CDD, the result is the diagram of a semisimple subalgebra of $\mathcal{L}$ whose elements are fixed by ω . The extended Coxeter-Dynkin diagram of F_4, for example, is



where the nodes correspond respectively to $\alpha_0 = -\tau$, and to the simple roots $\alpha_1, \alpha_2, \alpha_3$, and α_4 . Recall that $\tau = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ in this case. The analogous diagrams for the other SLA are well-known, see for example [4, pp. 47-48]. Other roots can be inferred from the Appendices of [2, VI] but are in more convenient form in [5] which on p. 531 has the roots of F_4 which we shall need.

Thus for F_4, removing nodes 0 to 4 in succession gives F_4, $A_1 \oplus C_3$, $A_2 \oplus A_2$, $A_3 \oplus A_1$, B_4. For fixed j, we denote this subalgebra by $\mathcal{L}^0$. By looking at a table of extended CDD we see that $\mathcal{L}^0$ can have at most 4 factors and this occurs only for the algebra D_4.

According to an idea which seems to have originated with Gantmacher [6], we may use the automorphism ω which has period n^j to define a gradation on $\mathcal{L}$ such that

$$\mathcal{L} = \oplus \mathcal{L}^k \qquad (1)$$

where $\mathcal{L}^k = \{ x \in \mathcal{L} : \omega x = \zeta^k x \}$. Here k is to be understood modulo n^j.

It is easily seen that $\mathcal{L}^k$ is spanned by e_α where $\alpha = \Sigma k^i \alpha_i$ and k^j is congruent to k modulo n^j, except that $\mathcal{L}^0$ also includes $\mathcal{H}$. In particular for F_4 with j = 3 and $n^3 = 4$, if we denote the top root τ by 2342 and similarly for the other positive roots, we are led to define

R^0 = {1000, 0100, 0001, 1100, 1242, 1342, 2342 }

R^1 = {0010, 0110, 0011, 1110, 0111, 1111 }

R^2 = {0120, 1120, 0121, 1220, 0122, 1121, 1122, 1221, 1222 }

R^3 = {1231, 1232 }

and let $-R^k$ denote the negative of the roots in R^k.

Then $\mathcal{L}^0$ is spanned by $\mathcal{H}$ and the root vectors corresponding to R^0 and $-R^0$; whereas $\mathcal{L}^1$ corresponds to R^1 and $-R^3$; $\mathcal{L}^2$ to R^2 and $-R^2$; and $\mathcal{L}^3$ to R^3 and $-R^1$.

Since $\mathcal{L}^u \circ \mathcal{L}^v \subseteq \mathcal{L}^{u+v}$, it follows that for $k > 0$, $\mathcal{L}^k$ is an $\mathcal{L}^0$ module and that $\mathcal{L}^0$ is a subalgebra generated by e_τ and f_τ together with the e_i and f_i for which $i \neq j$. We can now formulate our main result:

Theorem. With the notation established above, the simple LA, $\mathcal{L}$, is a direct sum of the subalgebra $\mathcal{L}^0$ and n^j-1 *irreducible* $\mathcal{L}^0$-modules.

We still merely need to show that $\mathcal{L}^k$ is irreducible if $k > 0$. The proper approach can be seen by looking at the example and using the known property of simple LA's that each root is connected [4, §1.6, 5.3] and any two are the ends of a sequence obtained by successively adjoing simple roots. For F_4, it is easy to check that between any two roots of $\mathcal{L}^k$ one may insert a sequence of roots of $\mathcal{L}^k$ by adjoining successively roots of $\mathcal{L}^0$. We can formalize this argument for arbitrary j and X_n as follows.

First, if j corresponds to a node for which n^j is 1, there is only one component which necessarily is the original LA, $\mathcal{L}$. This is certainly the case when $j = 0$. Otherwise, there are two or more components in the direct sum (1). The algebra $\mathcal{L}^0$ contains the root vectors for the simple roots of $\mathcal{L}$ other than α_j as well as α_0, together with those corresponding to α for which $k^j = 0$ or n^j. That $\mathcal{L}$ is simple then implies, as for F_4, that all root vectors in $\mathcal{L}^k$, for $k > 0$, can be obtained from one of them by the action of $\mathcal{L}^0$. Hence $\mathcal{L}^k$ is an irreducible $\mathcal{L}^0$- module. #

A result of Borel and Siebenthal [3] implies that $\mathcal{L}^0$ is a maximal subalgebra of $\mathcal{L}$ if and only if n^j is a prime. Our theorem leads to the following minor gloss on their paper. The only non-primes are 4 and 6. For $n^j = 4$, which occurs for F_4, E_7 and E_8, $\mathcal{L}^0 \oplus \mathcal{L}^2$ is a proper subalgebra of $\mathcal{L}$, and for $n^j = 6$, which occurs for E_8, $\mathcal{L}^0 \oplus \mathcal{L}^2 \oplus \mathcal{L}^4$ is also a proper subalgebra. This proves and makes more explicit the theorem of Borel and Siebenthal when n^j is not a prime.

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References.

1. H.S.M. Coxeter, *Regular Polytopes,* Third Edn. ,1973, Dover, NYC. Section 11.9.
2. N. Bourbaki, *Groupes et algèbres de Lie,* Hermann, Paris, 1975.
3. A. Borel & J.de Siebenthal , Comment. Math. Helv. ,23 (1949),200 - 221.
4. V. Kac, *Infinite Dimensional Lie Algebras,* Second Edn., Cambridge University Press, 1985.
5. H. Freudenthal & H. de Vries, *Linear Lie Algebras,* Academic Press,NYC, 1969.
6. F. Gantmacher, Math. Sb., 5 (1939) 101 - 144.

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Problèmes de Paul Lévy sur les polygones articulés

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Résumé :

Le but de ce travail est de donner quelques réponses à des questions soulevées par Paul Lévy relatives à des polygones articulés . Nous montrons en particulier, que la fonction $\varphi(\alpha)$ étudiée par Paul Lévy est strictement décroissante sur l'intervalle $[0,\pi]$, elle ne peut donc admettre d'extrémums . Nous déduisons alors des résultats pour des n-polygones inscriptibles .

Abstract :

The objective of this paper is to solve some of Paul Lévy's questions about articuled polygons . We show in particular that the function $\varphi(\alpha)$ calculated by Paul Lévy does not have extremas since it decreases strictly . As a corollary we deduce some results about inscriptible n-gons .

§1-Introduction :

Nous savons qu'un quadrilatère articulé plan entoure l'aire maximale lorsqu'il est convexe et inscriptible . Cette propriété s'étend aux polygones quelconques; l'aire entourée par un n-polygone dont les côtés sont de longueur $a_1, a_2, \ldots, a_n$ admet un maximum $f_n(a_1, a_2, \ldots, a_n)$ atteint quand il est convexe et inscriptible . Paul Lévy [L] a étudié certaines propriétés algébriques de la fonction f_n , mais la valeur exacte n'est toujours pas connue, sauf pour les polygones réguliers . La question intéressante est de savoir si f_n n'a pas toujours la valeur :

$$P_n = p^2\sqrt{(1-a_1/p)(1-a_2/p)\ldots.(1-a_n/p)} \quad \text{où} \quad p = \frac{1}{2}\sum_{i=1}^{i=n} a_i$$

comme dans le cas n=2,3 ou 4 .

Paul Lévy a conjecturé que la quantité $\varphi_n = f_n/P_n$ est comprise entre les valeurs e/π et 1 .

Pour un polygone régulier, nous avons l'égalité $a_i/p = 2/n$, d'où l'expression du rapport φ_n :

$$\varphi_n = \frac{(1-2/n)^{-n/2}}{n \operatorname{tg}\pi/n}$$

qui est effectivement comprise entre e/π et 1 pour $n \geq 3$.

§2- Considèrons un cercle de rayon 1, p_n désignant le demi-périmètre d'un polygone inscrit dans ce cercle, et S_n la surface qu'il entoure. Soit AB un arc de cercle de longueur 2α, on considère son contour comme limite d'un polygone de (n+1) côtés dont l'un d'eux est la corde AB de longueur $a = 2\sin\alpha$ et les autres sous-tendant des arcs de même longueur $2\alpha/n$. On a d'après [L], l'expression des limites de p_n, S_n et P_n respectivement, à savoir :

$$\lim p_n = (\alpha + \sin\alpha) = p(\alpha) \quad ; \quad \lim S_n = (\alpha - \tfrac{1}{2}\sin 2\alpha) = S(\alpha)$$

et $\lim P_n = (\alpha + \sin\alpha)^{3/2}(\alpha - \sin\alpha)^{1/2} \exp(-\alpha/p(\alpha)) = P(\alpha)$

d'où l'expression du rapport $\varphi(\alpha) = \dfrac{(\alpha - \frac{1}{2}\sin 2\alpha)(\alpha - \sin\alpha)^{-1/2}}{(\alpha + \sin\alpha)^{3/2} \exp(-\alpha/p(\alpha))}$

Proposition (2-1) : La fonction φ est strictement décroissante lorsque α varie sur l'intervalle $[0,\pi]$, avec $\varphi(0) = \sqrt{e/3}$ et $\varphi(\pi) = e/\pi$.

En effet, le calcul de la dérivée de $P(\alpha)$ donne :

$P'(\alpha) = P(\alpha)(\alpha + \sin\alpha)^{-2}(\alpha - \sin\alpha)^{-1}\left[2\alpha^2 - 2\alpha\sin\alpha\cos\alpha - 2\cos\alpha\sin^2\alpha + 2\alpha^2\cos\alpha\right.$

et celui de $S(\alpha)$: $S'(\alpha) = 1 - \cos 2\alpha$. La dérivée de la fonction $\varphi(\alpha)$ sera de même signe que l'expression suivante :

$$\left[\alpha^2(\cos\alpha + 1) - \alpha\sin\alpha - \sin^2\alpha\right](\sin\alpha - \alpha\cos\alpha)$$

Lemme (2-2) : L'expression $-\alpha^2(\cos\alpha + 1) + \sin^2\alpha + \alpha\sin\alpha = F$ est strictement positive pour $\alpha \in]0,\pi]$.

On a $-\alpha^2(\cos\alpha + 1) + \sin^2\alpha + \alpha\sin\alpha = 2\cos^2\frac{\alpha}{2}\left[\alpha\,\mathrm{tg}\,\frac{\alpha}{2} + 2\sin^2\frac{\alpha}{2} - \alpha^2\right]$

Or $\mathrm{tg}\,\frac{\alpha}{2} \geq \frac{\alpha}{2} + \frac{\alpha^3}{24} + \frac{\alpha^5}{240}$ et $\sin^2\frac{\alpha}{2} \geq \frac{\alpha^2}{4} - \frac{\alpha^4}{48} + \frac{\alpha^6}{2304}$

d'où $F > \dfrac{197\,\alpha^6}{5760}$.

Remarque (2-3) : 1) Ce résultat contredit évidemment celui annoncé par [L] page 111, où il y a dû avoir confusion entre les valeurs de $\varphi(\pi/4)$ et $\varphi(\pi/2)$.

2) Il n'y a pas lieu de distinguer deux cas, selon que le centre O du cercle est intérieur au "polygone" ou ne l'est pas; le premier correspond à $0 \leq \alpha \leq \pi/2$ et le second à $\pi/2 \leq \alpha \leq \pi$. En effet, cela correspond au changement de α par $\pi - \alpha$ et, l'expressio de φ est identique dans les deux cas.

§3- On considère sur l'arc AB opposé à celui qui entoure l'aire $S(\alpha)$, un point C variable . Si 2θ désigne l'angle AOC, $S(\alpha,\theta)$ la somme de S et de l'aire du triangle ABC , $\varphi(\alpha,\theta)$ est la valeur du rapport φ_n qui correspond à ce nouveau "polygone" . Nous trouvons :

$$S(\alpha,\theta) = \alpha - \sin\alpha\cos(\alpha+2\theta) \quad ; \quad p(\alpha,\theta) = \alpha + \sin\theta + \sin(\alpha+\theta)$$

et $P(\alpha,\theta) = p(\alpha,\theta)\left[\alpha^2 - 4\sin^2\frac{\alpha}{2}\cos^2(\frac{\alpha}{2}+\theta)\right]^{1/2}\exp(-\alpha/p)$.

Notons que la variable θ est telle que : $0 \leq \theta \leq \pi-\alpha$ et, pour $\theta = 0$ on retrouve l'expression du §2 $\varphi(\alpha,0) = \varphi(\alpha)$.

<u>Proposition (3-1)</u> : <u>Pour α fixé entre 0 et π, la fonction $\varphi(\alpha,\theta)$ admet un maximum en la valeur $\theta_1 = (\pi-\alpha)/2$ et deux minimums symétriques par rapport à θ_1</u> .

<u>Démonstration</u> : Un calcul explicite des dérivées donne

$$S'(\alpha,\theta) = 2\sin\alpha\sin(2\theta+\alpha) \quad ; \quad p'(\alpha,\theta) = 2\cos\frac{\alpha}{2}\cos(\frac{\alpha}{2}+\theta)$$

et $P'(\alpha,\theta) = \left[(1+\frac{\alpha}{p})p'A(\alpha,\theta) + 4\sin^2\frac{\alpha}{2}\cos(\frac{\alpha}{2}+\theta)\sin(\frac{\alpha}{2}+\theta)\frac{p}{A(\alpha,\theta)}\right]\exp(-\alpha/p)$

où $A(\alpha,\theta) = \left[\alpha^2 - 4\sin^2\frac{\alpha}{2}\cos^2(\frac{\alpha}{2}+\theta)\right]^{1/2}$. Ainsi $S'(\alpha,\frac{\pi-\alpha}{2}) = P'(\alpha,\frac{\pi-\alpha}{2}) = 0$.

D'une manière plus précise, pour θ voisin de la valeur θ_1 , on trouve :

$$P'(\alpha,\theta) \simeq \frac{4}{p}(\theta-\theta_1)(\alpha^2\cos\frac{\alpha}{2} + \alpha\cos\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \sin\alpha\sin\frac{\alpha}{2})\exp(-\alpha/p) \quad .$$

Par conséquent, la fonction $\varphi'(\alpha,\theta)$ est de même signe que la fonction $4(\theta-\theta_1)\Psi(\alpha)$ pour θ voisin de θ_1 , où :

$$\Psi(\alpha) = \alpha^3\cos\frac{\alpha}{2} - \alpha^3\sin\alpha + \alpha^2\cos\frac{\alpha}{2} - 4\alpha^2\sin\alpha\cos\frac{\alpha}{2} + \alpha\sin^2\frac{\alpha}{2} - 4\alpha\sin\alpha\cos^2\frac{\alpha}{2} + \sin\alpha\sin\frac{\alpha}{2}$$

Par ailleurs, sachant que pour $0 \leq x \leq \pi/2$ on a :

$$x - \frac{x^3}{6} \leq \sin x \leq x \qquad \text{et} \qquad 1 - \frac{x^2}{2} \leq \cos x \leq 1$$

on en déduit l'inégalité suivante :

$$\Psi(\alpha) \leq \alpha^3 - \alpha^3(\alpha - \frac{\alpha^3}{6}) + \alpha^2 - 4\alpha^2(\alpha-\frac{\alpha^3}{6})(1-\frac{\alpha^2}{8}) + \frac{\alpha^3}{4} - 2\alpha(\alpha-\frac{\alpha^3}{6})(1-\frac{\alpha^2}{2}) - 2\alpha(\alpha-\frac{\alpha^3}{6}) + \frac{\alpha^2}{2}$$

d'où $\Psi(\alpha) = -\frac{\alpha^2}{12}(30 + 33\alpha - 8\alpha^2 - 14\alpha^3 + \alpha^5) \leq 0$.

D'autre part, pour $\pi/2 \leq \alpha \leq \pi$ posons $\gamma = \alpha - \pi/2$ d'où :

$$\Psi(\alpha) \leq \alpha^3 - \alpha^3(1-\frac{\gamma^2}{2}) + \alpha^2 - 4\alpha^2(1-\frac{\gamma^2}{2})(1-\frac{\alpha^2}{8}) + \frac{\alpha^3}{4} + 2\alpha\gamma - 2\alpha(1-\frac{\gamma^2}{2}) + \frac{\alpha}{2}$$

et $\Psi(\alpha) \leq \frac{\alpha}{16}\left[4\pi^2 - 16\pi - 24 + (8\pi^2 - 16 - 16\pi)\alpha + (4\pi^2 - 32\pi + 20)\alpha^2 + (40 - 40\pi - \pi^2)\alpha^3 + (4\pi+8)\alpha^4\right] = \frac{\alpha}{16}g(\alpha)$.

Or la dérivée $g'(\alpha)$ admet un minimum négatif pour une valeur de voisine de 2,4579 , et du fait que $g'(\pi/2)$ et $g'(\pi)$ sont négatifs, par conséquent la fonction g est décroissante sur l'intervalle $[\pi/2,\pi]$. Ainsi, $g(\alpha) \leq g(\pi/2) < 0$ et, la fonction $\Psi(\alpha)$ est négative pour $0 \leq \alpha \leq \pi$. θ_1 est donc un maximum pour $\varphi(\alpha,\theta)$.

Pour la deuxième partie de la proposition, lorsque θ est voisin de 0, ce qui revient en fait à donner un accroissement de θ à l'angle α , l'accroissement de l'aire et du périmètre correspondant serait le même, autrement dit :

$$\frac{\partial S}{\partial \theta}(\alpha,0) = \frac{\partial S}{\partial \alpha}(\alpha,0) \quad ; \quad \frac{\partial p}{\partial \theta}(\alpha,0) = \frac{\partial p}{\partial \alpha}(\alpha,0)$$

de même $\frac{\partial P}{\partial \theta}(\alpha,0) = \frac{\partial P}{\partial \alpha}(\alpha,0)$ et par conséquent $\frac{\partial \varphi}{\partial \theta}(\alpha,0) = \frac{\partial \varphi}{\partial \alpha}(\alpha,0)$.

Du fait que $\frac{\partial \varphi}{\partial \alpha}(\alpha,0)$ est négative, φ admet donc un minimum inférieur à $\varphi(\alpha,0)$. Compte tenu de l'égalité $\varphi(\alpha,\theta) = \varphi(\alpha,\pi-\alpha-\theta)$, la fonction $\varphi(\alpha,\theta)$ admet donc deux minimums symétriques par rapport à θ_1 .

Remarque (3-2) : 1) Notons que ce résultat est plus général que celui pressenti par Paul Lévy, où il supposait α voisin d'une valeur β solution de l'équation $\varphi(\alpha,(\pi-\alpha)/2) = \varphi(\alpha,0)$.

2) Pour un polygone de n côtés, si l'on suppose que tous ses sommets sont fixes sauf un, qui lui se déplace sur l'arc formé par deux sommets consécutifs A_1A_3 , la fonction φ_n réalise un maximum lorsque $a_1 = a_2$. Contrairement à ce qu'on attendait, φ_n n'a pas d'extrémums lorsque $a_1 = 0$ ou $a_2 = 0$. Pourtant, lorsque tous les sommets se rapprochent indéfiniment de l'un d'eux ou de son diamétra-lement opposé, la fonction φ_n correspondante tend vers la valeur 1 .

§4- Soit un n-polygone de côtés $a_1,a_2,\ldots,a_n$ et de demi-périmètre p , considèrons la condition suivante :

$$(4\text{-}1) \qquad a_i/p \leq 2k/n \qquad \text{pour } 1 \leq i \leq n$$

où k est une valeur (nécessairement supérieure ou égale à 1) ne dépendant que de n , $k = 1$ correspondant au n-polygone régulier .

Proposition (4-2) : Le rapport φ_n correspondant à ce polygone est tel que :

$$(4\text{-}3) \qquad \varphi_n \leq \frac{e^{kn/n-2k}}{n\,\mathrm{tg}\frac{\pi}{n}}$$

De plus, une condition suffisante pour avoir $\varphi_n \leq 1$ est que :

$$k \leq \frac{n\log(n\,\mathrm{tg}\frac{\pi}{n})}{n+2\log(n\,\mathrm{tg}\frac{\pi}{n})}$$

inégalité qui n'est possible que pour $n \geq 15$.

Lemme (4-4) : Etant donné l'hypothèse (4-1) , alors on a l'inégalité :

$$(1 - a_1/p)(1 - a_2/p)\ldots.(1 - a_n/p) \geq e^{-2kn/n-2k}$$

De ce lemme, on en déduit que : $P_n \geq p^2 e^{-kn/n-2k}$

et, la proposition (4-2) résulte du fait que l'aire du polygone est telle que : $S_n \leq \frac{p^2}{n\,tg\frac{\pi}{n}}$

Démonstration du lemme (4-4) :

Il suffit de remarquer que l'inégalité $\frac{a_i}{p-a_i} \leq \frac{2k}{n-2k}$ entraine le résultat car $\log(1-a_1/p)(1-a_2/p)\ldots.(1-a_n/p) \geq -(\frac{a_1}{p-a_1}+\ldots+\frac{a_n}{p-a_n})$

Le résultat suivant précise une minoration du rapport φ_n .

Proposition (4-5) : Pour tout n-polygone inscriptible, la valeur de φ_n est telle que :

$$\varphi_n \geq \frac{\sqrt{1-k^2\sin^2\pi/n}}{n\sin\frac{\pi}{n}\,(1-\frac{2}{n})^{n/2}}$$

avec égalité si et seulement si le polygone est régulier .

Lemme (4-6) : Pour tout n-polygone de côté $a_1, a_2, \ldots, a_n$, on a l'inégalité suivante (qui généralise celle des triangles):

$$(1 - a_1/p)(1 - a_2/p)\ldots.(1 - a_n/p) \leq (1 - 2/n)^n$$

avec égalité si et seulement si le polygone est régulier .

En effet, soit l'expression suivante $A = (1-a_1/p)\ldots.(1-a_n/p)(\frac{n}{n-2})^n$

du fait que $(1-a_i/p)(1+\frac{2}{n-2}) = 1 + (2p-na_i)/(n-2)p$

alors $\log A \leq \frac{2p-na_1}{(n-2)p} + \ldots. + \frac{2p-na_n}{(n-2)p} = 0$

Par conséquent, $A \leq 1$ et, $A = 1$ correspond au polygone régulier .

Pour démontrer la proposition (4-5), remarquons que le demi-périmètre vérifie l'inégalité $p \leq n\sin\pi/n$ et, l'aire S_n est telle que

$$S_n = \frac{1}{2}\sum_{i=1}^{i=n} h_i a_i \quad \text{où} \quad h_i^2 = 1 - a_i^2/4 \geq 1 - (kp/n)^2$$

Ainsi $s_n \geq \frac{p^2}{n\sin\frac{\pi}{n}}\sqrt{1-k^2\sin^2\frac{\pi}{n}}$ d'où le résultat .

Remarque (4-7) : 1) Notons que cette borne inférieure du rapport φ_n est nécessairement supérieure à e/π , car on sait que pour un polygone régulier $\varphi_n \geq e/\pi$.

2) Il semble qu'on puisse améliorer l'inégalité (4-3) car pour le cas du polygone régulier (k = 1) on a :

$$\varphi_n < \frac{e^{n/n-2}}{n\,tg\frac{\pi}{n}} .$$

Référence :

[L] Paul Lévy : "Le problème des isopérimètres et des polygones articulés" Bull. Sc. Math. , 2ème série, 90 (1966) p. 103 à 112 .

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Modular Elliptic Curves and Values of L-Functions

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Abstract: Let A be a Q-isogeny class of modular elliptic curves and let f be the associated weight two newform. In [9] a conjecture was formulated which would identify two lattices, $\mathcal{L}(A_{min}) \stackrel{?}{=} \mathcal{L}(f)$, where $\mathcal{L}(A_{min})$ is the minimal Neron lattice of A (see [9]) and $\mathcal{L}(f)$ is defined purely in terms of the modular data. In §1 of the present note, this conjecture is reformulated (see (1.3)) in terms of the arithmetic of modular parametrisations. In §2, we describe two consequences of the conjecture for the arithmetic of values of L-series.

§1 Modular Parametrizations.

For a positive integer N we let $X_1(N)_{/\mathbf{Q}}$ (resp. $X_0(N)_{/\mathbf{Q}}$) denote Shimura's canonical model over Q of $X_1(N)$ (resp. $X_0(N)$) in which the 0-cusp is a rational point ([8], Chapter 6). In this model, a rational function on $X_1(N)$ (resp. $X_0(N)$) is defined over Q if and only if its q-expansion at the 0-cusp has coefficients in Q. The following theorem follows from the work of Shimura.

(1.1) Theorem. *An elliptic curve $A_{/\mathbf{Q}}$ of conductor N is modular if and only if there is a morphism $X_1(N) \xrightarrow{\pi} A$ of algebraic curves over Q such that*

(i) π sends the 0-cusp to the origin, and

(ii) $\pi^*\omega_A = c(\pi)\cdot\omega_f$

where ω_A is a Neron differential on A, $\omega_f = f(q)\frac{dq}{q}$ is the differential 1-form on $X_1(N)$ associated to a normalized newform f of level N, and $c(\pi) \in \mathbf{Q}^*$. ∎

The map π is called a *modular parametrization* of A and the constant $c(\pi)$ is called the *Manin constant* of the parametrization π. The following is a consequence of the theory of the q-expansion [1,4]

(1.2) Theorem. *For any modular parametrization $\pi: X_1(N) \to A$ we have $c(\pi) \in \mathbf{Z}$.* ∎

In fact, we suspect that more is true.

(1.3) Conjecture. *Let $A_{/\mathbf{Q}}$ be a modular elliptic curve of level N. Then there is a modular parametrization $\pi : X_1(N) \longrightarrow A$ for which $c(\pi) = \pm 1$.* ∎

This conjecture is related to a conjecture of Manin (see [5,6]), but is stronger than that conjecture. Manin's conjecture asserts that *some* curve in the isogeny class of A (namely the strong one) admits a parametrization π by $X_0(N)$ with $c(\pi) = \pm 1$. The above conjecture asserts that if we replace $X_0(N)$ by $X_1(N)$ then *every* curve isogenous to A admits such a parametrization. The analogous statement for $X_0(N)$ is false.

(1.4) Theorem. *Conjecture 1.3 is equivalent to Conjecture 2.3 of* [9]

The proof is based on the following fact about modular parametrizations.

(1.5) Proposition. (compare [6]) *Let $\mathcal{A}$ be an isogeny class (over $\mathbf{Q}$) of modular elliptic curves of conductor N. Then there is a curve $A_1 \in \mathcal{A}$ and a modular parametrization π_1 of A_1 which satisfies the following equivalent conditions.*

(1) *π_1 is optimal in the following sense. If $\pi : X_1(N) \to A$ is a parametrization of a curve $A \in \mathcal{A}$, then there is an isogeny $\beta : A_1 \to A$ which makes the following diagram commutative:*

$$\begin{array}{ccc} X_1(N) & \xrightarrow{\pi_1} & A_1 \\ & \searrow^{\pi} & \downarrow \beta \\ & & A. \end{array}$$

(2) *The induced map on singular homology, $\pi_{1*} : H_1(X_1(N);\mathbf{Z}) \longrightarrow H_1(A_1;\mathbf{Z})$, is surjective.*

The curve $A_1 \in \mathcal{A}$ is uniquely determined by these conditions and π_1 is determined up to sign. ∎

We refer to A_1 as the *optimal curve* in $\mathcal{A}$ and to $\pi_1 : X_1(N) \longrightarrow A_1$ as an *optimal parametrization*. From (1.5)(2) it follows

$$c(\pi_1) \cdot \mathcal{L}(f) = \mathcal{L}(A_1). \tag{1.6}$$

Proof of Theorem 1.4. Suppose first of all that Conjecture 2.3 of [9] is true: $\mathcal{L}(f) = \mathcal{L}(A_{min})$. Then from (1.6) we conclude $c(\pi_1) = \pm 1$ and $A_{min} = A_1$. Now let $A \in \mathcal{A}$ be arbitrary and let ω_A be a Neron differential on A. Since $A_1 = A_{min}$ there is an étale isogeny $\phi : A_1 \to A$. Then the modular parametrization $\pi = \phi \circ \pi_1 : X_1(N) \to A$ has $c(\pi) = \pm 1$. So Conjecture 1.3 is true.

Now suppose Conjecture 1.3 is true. Let $\pi : X_1(N) \to A_{min}$ be a parametrization of the minimal curve $A_{min} \in \mathcal{A}$ for which $c(\pi) = 1$. By the definition of optimality, $\pi = \phi \circ \pi_1$ for some isogeny $\phi : A_1 \to A_{min}$. If ω_{min}, ω_1 are Neron differentials on A_{min}, A_1 then $\phi^*\omega_{min} = n \cdot \omega_1$ for some integer $n \in \mathbf{Z}$. Since $1 = c(\pi) = n \cdot c(\pi_1)$ it follows that $n = c(\pi_1) = \pm 1$. From the equality $n = \pm 1$ we conclude that ϕ is an étale isogeny. Thus $A_1 \prec A_{min}$ and consequently $A_1 = A_{min}$. Now we use (1.6) and the equality $c(\pi_1) = \pm 1$ to conclude $\mathcal{L}(f) = \mathcal{L}(A_{min})$. ∎

§2 Values of L-functions.

Let A be a modular elliptic curve and let f be the associated weight 2 newform. If the q-expansion of f is given by $f = \sum_{n>0} a_n q^n$ then the L-series of A is represented by the Dirichlet series $L(A,s) = \sum_{n>0} a_n n^{-s}$. For $x \in \mathbf{Q}$ we define the partial L-series

$$L(A,x,s) = \sum_{n>0} a_n e^{2\pi i n x} n^{-s}.$$

These Dirichlet series define entire functions on the complex plane.

(2.1) Definition. The *modular symbol* associated to f is the function $[x]_f : \mathbf{P}^1(\mathbf{Q}) \longrightarrow \mathbf{C}$ defined by

$$[x]_f = \int_0^x f(q)\frac{dq}{q}$$

where the integral is over the geodesic in the upper half plane joining 0 to x. ∎

A simple calculation shows that the modular symbol and the partial L-series are related by the following formulas: $[\infty]_f = L(A,1)$, $[x]_f = L(A,1) - L(A,x,1)$, $x \in \mathbf{Q}$.

It follows from the Manin-Drinfeld theorem [3] that the values of the modular symbol lie in the $\mathbf{Q}$-span of the lattice of Neron periods of A: $[r]_f \in \mathcal{L}(A) \otimes \mathbf{Q}$, $r \in \mathbf{P}^1(\mathbf{Q})$. Thus the lattice $\mathcal{L}(A)$ provides a natural integral structure within which we can study the arithmetic properties of the modular symbol. Since Conjecture 2.3 of [9] asserts a connection between $\mathcal{L}(A)$ and the periods of ω_f, we expect Conjecture (1.3) to be closely tied to integrality properties of the modular symbol (2.1).

For example, we might inquire about the denominators in Stickelberger elements like those defined by Mazur and Tate [7]. For each positive integer M, let $F_M = \mathbf{Q}(e^{2\pi i/M})$, let $G_M = \mathrm{Gal}(F_M/\mathbf{Q}) \cong (\mathbf{Z}/M\mathbf{Z})^*$ and let $G_M^+ = \mathrm{Gal}(F_M^+/\mathbf{Q}) \cong (\mathbf{Z}/M\mathbf{Z})^*/(\pm 1)$ be the Galois group of the totally real subfield F_M^+ of F_M. The standard isomorphism $(\mathbf{Z}/M\mathbf{Z})^* \longrightarrow G_M$ is given by $a \mapsto (\sigma_a : e^{2\pi i/M} \mapsto e^{2\pi i a/M})$.

(2.3) Definition. To each integer $M > 0$ we define the *Stickelberger element of layer M* to be

$$\Theta_M = \sum_{a \in (\mathbf{Z}/M\mathbf{Z})^*} \left[\frac{a}{M}\right] \otimes \sigma_a \in \mathcal{L}(A) \otimes \mathbf{Q}[G_M].$$

The relation between these Stickelberger elements and those defined by Mazur and Tate is given in [10]. By analogy with what is known about Stickelberger elements attached to totally real number fields [2] we might hope that the following conjecture is true.

(2.4) Conjecture. *Let $J_M \subseteq \mathbf{Z}[G_M]$ be the annihilator of $A(F_M)_{tor}$. Then*

$$J_M \cdot \Theta_M \subseteq \mathcal{L}(A) \otimes \mathbf{Z}[G_M].$$

(2.5) Theorem. *If Conjecture 1.3 is true then so is Conjecture 2.4.* ∎

As another example, we let p be a prime of good ordinary reduction for the modular elliptic curve A and consider the p-adic L-functions defined by Mazur and Swinnerton-Dyer [6]. Let $\mathbf{C}_p$ be a fixed p-adic completion of an algebraic closure of $\mathbf{Q}_p$, let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and fix an imbedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}_p$. When we speak about p-adic properties of algebraic numbers, we are referring to the p-adic properties with respect to this fixed imbedding.

(2.6) Theorem. *Let α be the unit root of Frobenius at p and let Δ be a positive integer which is not divisible by p. Then there is a p-adic measure $\nu_{A,\Delta}$ with values in $\mathcal{L}(A) \otimes \mathbf{Q}_p$*

such that for every primitive Dirichlet character χ *of conductor* $p^n\Delta$, $n \geq 0$, *the p-adic Mellin-Mazur transform,* $L_p(A, \chi, s)$, *of the measure* $\nu_{A,\Delta}$ *satisfies*

(a) $L_p(A, \chi, 1) \in \mathcal{L}(A) \otimes \mathbf{Q}[\chi, \alpha]$;

(b) *Under the natural map,* $\mathcal{L}(A) \otimes \mathbf{Q}[\chi, \alpha] \to \mathbf{C}$, $(\lambda \otimes \beta \mapsto \lambda\beta)$ *we have*

$$L_p(A, \chi, 1) \mapsto \alpha^{-n} \cdot (1 - \chi(p)\alpha^{-1}) \cdot (1 - \overline{\chi}(p)\alpha^{-1}) \cdot \tau(\chi) L(A, \overline{\chi}, 1)$$

where $\tau(\chi)$ *is the Gauss sum associated to* χ. ∎

The Main Conjecture of Iwasawa Theory predicts the following (and much more).

(2.7) Conjecture. *For each* $A \in \mathcal{A}$ *the measures* $\nu_{A,\Delta}$ *take values in* $\mathcal{L}(A) \otimes \mathbf{Z}_p$.

(2.8) Theorem. *If* $p \neq 2$, *then Conjecture 1.3* $\Rightarrow$ *Conjecture 2.7.*

In [10] it is also shown how Conjecture 1.3 leads to lower bounds for the μ-invariants of these p-adic L-functions. These lower bounds are consistent with bounds proved by Greenberg for the μ-invariants of the characteristic power series on the other side of the Main Conjecture.

REFERENCES

[1] Deligne, P., Rapoport, M.: Les schémas de modules de courbes elliptiques. Modular Forms in One Variable II, *Lect. Notes in Math.* **349** (1973).

[2] Deligne, P., Ribet, K.: Values of Abelian *L*-functions at negative integers over totally real fields. *Invent. Math.* **59** (1980), 227-286.

[3] Drinfeld, V.G.: Two theorems on modular curves. *Functional Anal. App.* **7** (1973), 155-156.

[4] Katz, N.: *P*-adic *L*-functions via moduli of elliptic curves. Algebraic Geometry Arcata 1974, *Proc. of Symp. in Pure Math.* **29** (1975), 479-506.

[5] Manin, J.: Parabolic points and zeta functions of modular curves (Russian). *Izv. Akad. Nauk SSSR Ser. Mat. 36, No.1*, 19-65 (1972). Engl. transl. in *Math. USSR-Izv. 6*, No.1, 19-64 (1972).

[6] Mazur, B., Swinnerton-Dyer, P.: Arithmetic of Weil curves. *Invent. Math.* **25**, 1-16 (1974).

[7] Mazur, B., Tate, J.: Refined conjectures of the "Birch and Swinnerton-Dyer type". *Duke Math. J.* **54**, No. 2, 711-750 (1987).

[8] Shimura, G.: Introduction to the arithmetic theory of automorphic forms. *Publications of the Mathematical Society of Japan* 11. Princeton: Princeton University Press 1971.

[9] Stevens, G.: On the periods of modular elliptic curves. *Comptes Rendus of the Royal Society of Canada.*

[10] Stevens, G.: Stickelberger elements and modular parametrizations of elliptic curves. To appear.

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ENERGY DENSITY IN CLASSICAL ELECTRODYNAMICS

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Once the Maxwell equations have been couched in terms of a system of first order partial time derivatives, a new type of energy density function presents itself. Analysis of some classical radiation and charge configurations suggests the new measures are interesting alternatives to the conventional Poynting measures.

We write the Maxwell equations in the form:

$$\partial A/\partial t = F \tag{1}$$

$$\partial F/\partial t = c^2\nabla^2 A + \frac{1}{\varepsilon_0} j \tag{2}$$

$$\partial \phi/\partial t = -c^2\nabla\cdot A \tag{3}$$

$$\partial \rho/\partial t = -\nabla\cdot j \tag{4}$$

together with the initial condition

$$\nabla^2\phi + \nabla\cdot F + \frac{1}{\varepsilon_0}\rho = 0 \tag{5}$$

Here A, c, ε_0, j, ϕ, and ρ are the standard magnetic potential vector field, speed of radiation in a vacuum, constant of proportionality, current density, electric potential scalar field, and charge density, all as in Feynman[1]. Equation (1) defines the vector field F. Equations (1) through (4) imply that if (5) holds initially, then (5) holds for all time.

The standard version of the Maxwell equations uses the vector fields $E = -\nabla\phi - F$ (in force/charge) and $B = \nabla \times A$ (in force/(charge×speed)). Our purposes are served better, however, by using ϕ (in energy/charge) and A (in energy/(charge×speed)).

We intend to describe new measures of electrodynamic energy density and electrodynamic power flux, alternatives to the conventional Poynting measures. In many instances the new measures and the Poynting measures have identical values, but generally they differ. We shall propose an experiment which might validate one system or the other.

In a vacuum equations (1) and (2) constitute an independent subsystem equivalent, of course, to the wave equations $\partial^2 A/\partial t^2 = c^2\nabla^2 A$. Likewise equations (3) and (5) can be rewritten as $\partial^2\phi/\partial t^2 = c^2\nabla^2\phi$.

In one-dimensional space the following abstract equations imply dependent variables u and v satisfy a wave equation:

$$\partial u/\partial t = \partial v/\partial x \qquad \partial v/\partial t = \partial u/\partial x \tag{6}$$

Suppose we regard a single closed segment R in one-dimensional space as partitioned into n adjacent small cells (segments) of lenth l. Using standard techniques[2], the following is a conversion of (6) to a 2n-dimensional dynamical system (of first order ordinary differential equations):

$$du^i/dt = (v^{i+1} - v^{i-1})/(2l) \qquad dv^i/dt = (u^{i+1} - u^{i-1})/(2l) \tag{7}$$

where u^0, u^{n+1}, v^0, v^{n+1} are all functions of t, $u^1,\ldots, u^n, v^1,\ldots, v^n$ determined by boundary conditions.

Let us write (7) as

$$du/dt = Mv + B_u \qquad dv/dt = Mu + B_v \tag{8}$$

where M is an n×n matrix and B_u and B_v are vector-valued functions of t, u, and v which amount to boundary conditions. Then M is a skew symmetric matrix with nonzero off-diagonal entries all equal in magnitude to $(2l)^{-1}$.

There are other spatial differences which approximate the derivatives $\partial u/\partial x$ and $\partial v/\partial x$ and some might be computationally more efficient in some circumstances. However, any M derived as a consistent difference of spatial values centered about its cells is skew symmetric.

Let us consider the abstract quadratic form

$Q(u,v) = \frac{1}{2}[(Mu)^*Mu + (Mv)^*Mv]l$

derived from any such M (* denotes transpose). Clearly Q(u,v) is positive unless Mu = Mv = 0. Furthermore the rate of change of Q along any trajectory of (7) is
dQ/dt = 0 + (boundary terms)
As cell size becomes smaller and n becomes larger, Q becomes closer to

$$\int_R \frac{1}{2}[(\partial u/\partial x)^2 + (\partial v/\partial x)^2]\,dx = \int_R \frac{1}{2}[(\partial u/\partial x)^2 + (\partial u/\partial t)^2]\,dx$$

These observations suggest a sort of partial measure of an energy density associated with any wave equation, namely, $.5[(\partial u/\partial x)^2+(\partial u/\partial t)^2]$. So let us consider the following generalizations of such an energy density for the wave equations satisfied by A and ø. We define

$$u_1 = \frac{1}{2}\varepsilon_0\left[F\cdot F + c^2\sum_{i,j=1}^{3}\left(\frac{\partial A_i}{\partial x_j}\right)^2\right] \tag{9}$$

and

$$u_2 = \frac{1}{2}\varepsilon_0\left[c^{-2}\left(\frac{\partial ø}{\partial t}\right)^2 + \nabla ø\cdot\nabla ø\right] = \frac{1}{2}\varepsilon_0\,[c^2(\nabla\cdot A)^2 + \nabla ø\cdot\nabla ø] \tag{10}$$

Using equations (1) through (5) we see that

$$\partial u_1/dt = -\nabla\cdot[-\varepsilon_0 c^2 D(A)^*F] + j\cdot F \tag{11}$$

where D(A) is the matrix $\left\{\frac{\partial A_i}{\partial x_j}\right\}$, so $[D(A)^*F]_i = \sum_{j=1}^{3}\frac{\partial A_j}{\partial x_i}F_j$. Also

$$\partial u_2/\partial t = -\nabla\cdot[-\varepsilon_0(\partial ø/\partial t)\nabla ø] + (c\rho)(c^{-1}\partial ø/\partial t) \tag{12}$$

We refer to

$$S_1 = -\varepsilon_0 c^2 D(A)^*F \tag{13}$$

and

$$S_2 = -\varepsilon_0(\partial ø/\partial t)\nabla ø = \varepsilon_0 c^2(\nabla\cdot A)\nabla ø \tag{14}$$

as power flux (power per area = momentum density) vector fields.

Forming the four-vectors (u_1, S_1) and (u_2, S_2) and their sum and difference and considering (10) and (11) lead to

$$\nabla_\mu\,(u_1+u_2, S_1+S_2) = (c\rho)(c^{-1}\partial ø/\partial t) + j\cdot F$$

and

$$\nabla_\mu(u_1-u_2, S_1-S_2) = -(c\rho)(c^{-1}\partial ø/\partial t) + j\cdot F = -(c\rho,j)\,\hat{\cdot}\,(c^{-1}\partial ø/\partial t, F)$$

where ∇_μ denotes Lorentz covariant differentiation and $\hat{\cdot}$ denotes Lorentz product. Of course, $(c\rho, j)$ and $(c^{-1}\partial ø/\partial t, F)$ are Lorentz four-vectors. Thus $\nabla_\mu(u_1-u_2, S_1-S_2)$ is independent of inertial coordinate frame.

In view of (8), (9), (12), and (13), the sums u_1+u_2 and S_1+S_2 are types of measures of energy density and power flux density which arise from certain mathematically pleasing considerations. We note immediately that if A and ø are constant with respect to time, then S = 0. By constrast the Poynting power flux measure $S_P = \varepsilon_0 c^2\, E\times B$ is, for example, generally nonzero throughout space around a fixed charge and a fixed bar magnet, a puzzling situation indeed[1]. It remains to establish the physical significance of u and S.

First let us consider the general forms for A and ø in one-dimensional waves traveling in a vacuum in the x_1 direction.: $A = (0, f(x_1-ct), g(x_1-ct))$ and $\phi = 0$. Thus $F = (0, -cf', -cg')$ where $'$ denotes differentiation with respect to the waveform parameter x_1-ct. Also,

$$D(A) = \begin{pmatrix} 0 & 0 & 0 \\ f' & 0 & 0 \\ g' & 0 & 0 \end{pmatrix}$$

It follows that $u_1 = \frac{1}{2}\varepsilon_0 c^2(f'^2+g'^2)$, $u_2 = 0$, $S_1 = \varepsilon_0 c^2([f'^2+g'^2], 0, 0)$, and $S_2 = (0,0,0)$. Thus $u = .5\varepsilon_0 c^2(f'^2+ g'^2)$ and $S = .5\varepsilon_0 c^3([f'^2+ g'^2],0,0)$.

The Poynting energy density and power flux are $u_p = \frac{1}{2}\varepsilon_0(E\cdot E+c^2 B\cdot B)$ and $S_p = \varepsilon_0 c^2 E\times B$.

It can be shown that $u_p = u$ and $S_p = S$, that is, the two measures of energy and power flux agree for one-dimensional waves.

Second let us consider a capacitor consisting of two parallel conducting plates separated by a vacuum and located at $x_3 = \pm h/2$. We use

$$A = \left(0, 0, -\frac{Vx_3^2}{2c^2hT} + \frac{V(x_1^2+x_2^2)}{4c^2hT}\right)$$

and

$$\phi = V\frac{x_3 t}{hT}$$

where V is a constant voltage (energy per charge) and T is a time constant. It follows that

$$u = \frac{1}{2}\varepsilon_0\left\{c^2\left(\frac{V}{c^2hT}\right)^2\left(\frac{x_1^2}{4}+\frac{x_2^2}{4}+2x_3^2\right)+\left(\frac{V}{hT}\right)^2 t^2\right\} \text{ so } \partial u/dt = \varepsilon_0\left(\frac{V}{hT}\right)^2 t. \text{ Also}$$

$$S = \varepsilon_0\left(0, 0, -\left(\frac{V}{hT}\right)^2 x_3 t\right).$$ Note that the direction of S is perpendicular to the plates and toward

the mid-plane, in contrast to the radial Poynting power flux $S_p = -\varepsilon_0\left(\frac{1}{2}\left(\frac{V}{hT}\right)^2 x_1 t, \frac{1}{2}\left(\frac{V}{hT}\right)^2 x_2 t, 0\right)$.

Also, $u_p = \frac{1}{2}\varepsilon_0\left(\left(\frac{V}{hT}\right)^2 t^2 + c^2\left(\frac{V}{c^2hT}\right)^2\left(\frac{x_1^2}{4}+\frac{x_2^2}{4}\right)\right) \neq u$. However, $\partial u/\partial t = \partial u_p/\partial t = \varepsilon_0\left(\frac{V}{hT}\right)^2 t$.

Third let us consider a wire with resistance and steady current. We assume the wire is cylindrical with radius a and has as its axis the x_1-axis. Inside the wire charges and currents are assumed to be continua; we take $j = (c[\rho(\rho+\rho_0)]^{.5}, 0, 0)$
where ρ is free charge density and ρ_0 is bound charge density, both constants. (Note that ρ and ρ_0 must be of the same sign and satisfy $\rho(\rho+\rho_0) < 1$.) We also take

$$A = \left(-\frac{(x_2^2+x_3^2)\, j_1}{4\varepsilon_0 c^2}, 0, 0 \right)$$

and

$$\phi = -\frac{j_1^2}{4\varepsilon_0 c^2 \rho}(x_2^2+x_3^2) - \frac{rj_1 x_1}{\rho}$$

where r is a constant resistance coefficient (in voltage×time per area). It follows that $u = u_p = \frac{1}{2}\varepsilon_0 r^2 j_1^2/\rho^2 + (\rho+\rho_0)(2\rho+\rho_0)(x_2^2+x_3^2)/(8\varepsilon_0)$. However, S = (0,0,0) while

$$S_p = \left(\frac{j_1^3}{4\varepsilon_0 c^2 \rho}(x_2^2+x_3^2), -\frac{rj_1^2}{2\rho}x_2, -\frac{rj_1^2}{2\rho}x_3 \right).$$

Note that the Lorentz force equation inside the wire reduces to
$0 = \rho E + j\times B - rj$
so free charges do not accelerate (and so do not radiate energy at the present level of approximation of currents in the wire).

Outside the same wire we take

$$A = \left(-\frac{a^2 j_1}{4\varepsilon_0 c^2} - \frac{a^2(\rho+\rho_0)}{4\varepsilon_0 c}\ln\left(\frac{x_2^2+x_3^2}{a^2}\right), 0, 0 \right)$$

and

$$\phi = -\frac{j_1^2 a^2}{4\varepsilon_0 c^2 \rho}\left(1 + \ln\left(\frac{x_2^2+x_3^2}{a^2}\right)\right) - \frac{rj_1 x_1}{\rho}$$

Thus at the surface of the wire $E = \left(r\frac{j_1}{\rho}, -\frac{(\rho+\rho_0)2x_2}{4\varepsilon_0}, -\frac{(\rho+\rho_0)2x_3}{4\varepsilon_0} \right)$, the conventional E vector directed almost parallel to the surface of the wire for large r and j_1. As inside the wire, u and u_p are equal; we find $u = \frac{1}{2}\varepsilon_0\frac{r^2 j_1^2}{\rho^2} + \frac{a^4(\rho+\rho_0)^2}{4\varepsilon_0(x_2^2+x_3^2)}$. Also S = (0,0,0) while

$$S_p = \frac{c^2 a^2(\rho+\rho_0)}{(x_2^2+x_3^2)}\left(-\frac{a^2(\rho+\rho_0)}{c\varepsilon_0}, -\frac{rj_1 2x_2}{4\rho c}, -\frac{rj_1 2x_3}{4\rho c} \right).$$

From the above examples a picture emerges of nonzero S associated with A or ϕ waves and vanishing S associated with constant (over time) A and linearly changing or constant ø. In particular the "obviously nuts"[1] radial power flux in the capacitor and the flow of power near the wire do not carry over from the Poynting measures.

We next consider waveguides of rectangular cross section and right rectangular resonant cavities.

Suppose the interior of a waveguide extends over $0 \le x_1 \le a$ and $0 \le x_2 \le b$ for positive numbers a and b. If m and n are positive integers and λ_g is a positive number, let λ_0 satisfy

$$\left(\frac{2}{\lambda_0}\right)^2 = \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{2}{\lambda_g}\right)^2$$

Let $\underline{ccc}$ denote the product $\cos\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right)\cos\left(2\pi\left[\frac{x_3}{\lambda_g} - \frac{ct}{\lambda_0}\right]\right)$, let $\underline{scc}$ denote the product $\sin\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right)\cos\left(2\pi\left[\frac{x_3}{\lambda_g} - \frac{ct}{\lambda_0}\right]\right)$, and so on through $\underline{sss}$. We take ø = 0 and

$$A = (Ana\underline{csc}, -Amb\underline{scc}, 0)$$

where A is a constant in energy/(charge×speed×distance). Thus

$$u = \frac{1}{2}\varepsilon_0 c^2 A^2 \pi^2 \left\{ \begin{array}{l} \left(\frac{na2}{\lambda_0}\right)^2 \underline{c}^2\underline{s}^2\underline{s}^2 + \left(\frac{mb2}{\lambda_0}\right)^2 \underline{s}^2\underline{c}^2\underline{s}^2 \\ + \left(\frac{nam}{a}\right)^2 \underline{s}^2\underline{s}^2\underline{c}^2 + \left(\frac{nan}{b}\right)^2 \underline{c}^2\underline{c}^2\underline{c}^2 + \left(\frac{na2}{\lambda_g}\right)^2 \underline{c}^2\underline{s}^2\underline{s}^2 + \\ + \left(\frac{mbm}{a}\right)^2 \underline{c}^2\underline{c}^2\underline{c}^2 + \left(\frac{mbn}{b}\right)^2 \underline{s}^2\underline{s}^2\underline{c}^2 + \left(\frac{mb2}{\lambda_g}\right)^2 \underline{s}^2\underline{c}^2\underline{s}^2 \end{array} \right\}$$

and

$$S = -\varepsilon_0 c^2 D(A)^*F = -\varepsilon_0 c^3 A^2 2\pi^2 \left\{ \begin{array}{l} \left(-\left(\frac{n^2 a^2 m}{a\lambda_0}\right) \underline{sc}\,\underline{s}^2\,\underline{sc} + \left(\frac{m^3 b^2}{a\lambda_0}\right) \underline{sc}\,\underline{c}^2\,\underline{sc}\right., \\ + \left(\frac{n^3 a^2}{b\lambda_0}\right) \underline{c}^2\,\underline{sc}\,\underline{sc} - \left(\frac{m^2 b^2 n}{b\lambda_0}\right) \underline{s}^2\,\underline{sc}\,\underline{sc}, \\ \left. -2\left(\frac{n^2 a^2}{\lambda_g\lambda_0}\right) \underline{c}^2\,\underline{s}^2\,\underline{s}^2 - 2\left(\frac{m^2 b^2}{\lambda_g\lambda_0}\right) \underline{s}^2\,\underline{c}^2\,\underline{s}^2 \right) \end{array} \right\}$$

Here $\underline{s}\underline{c}\ \underline{s}^2\ \underline{s}\underline{c}$ denotes

$$\left[\sin\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{m\pi x_1}{a}\right)\right]\left[\sin\left(\frac{n\pi x_2}{b}\right)\right]^2\left[\sin\left(2\pi\left[\frac{x_3}{\lambda_g}-\frac{ct}{\lambda_0}\right]\right)\cos\left(2\pi\left[\frac{x_3}{\lambda_g}-\frac{ct}{\lambda_0}\right]\right)\right]$$

and so on. The Poynting approach yields $u_P = u + \varepsilon_0 c^2 A^2 \pi^2 m^2 n^2 [\,\underline{c}^2\,\underline{c}^2 - \underline{s}^2\,\underline{s}^2]\,\underline{c}^2$; hence while u has, averaged over time, a maximal value on the midline of the waveguide, u_P is constantly zero on the midline. However, the integrals of u and u_P over a plane in the waveguide perpendicular to the midline are equal.

An analogous configuration in a right rectangular prismatic cavity yields $u_P = u + \varepsilon_0 c^2 A^2 \pi^2 m^2 n^2 [\underline{c}^2\underline{c}^2 - \underline{s}^2\underline{s}^2]\underline{s}^2\underline{s}^2$; hence u and u_P generally differ throughout the cavity. While u has, averaged over time, a positive value at the midpoint of the cavity, u_P is constantly zero at the midpoint. However, the integrals of u and u_P over any plane in the cavity perpendicular to the x_3 axis are equal.

Using these calculations, assuming one of the measures is correct, and assuming that the heating effect of radiation on matter is proportional to local energy density, we can envisage an experiment to decide which approach is correct as follows. Let us assume that $m = n = l = 1$ so at the middle of the cavity $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}\lambda_g)$ the Poynting approach gives $u_p = 0$ for all time. By contrast u at the midpoint is $\varepsilon_0 c^2 A^2 \pi^2 \sin^2(\pi t c/\lambda_0)$ with an average value over time of $.5\varepsilon_0 c^2 A^2 \pi^2$. Thus it seems that acceptance of the Poynting energy density measure implies a droplet of water at the middle of a microwave oven operated with pure half-waves would not be heated. Acceptance of alternative energy density measure evidently implies the water would be heated. A similar discrepancy between the two densities is prominent at the corners of the plane $x_3 = .5\lambda_g$. However, for any fixed time t and fixed plane x_3 = constant, the integrals of u and u_p over the plane are equal.

We now assume the position of an isolated charge is (f(t), 0, 0) with f(0) = 0 and $0 < df/dt << c$. We also assume d^2f/dt^2 is suitably small. Thus the Liénard-Wiechart potentials at a point (x_1, x_2, x_3) not in the particle are approximately $A = (\,q/(4\pi\varepsilon_0 c^2[\ \cdot]^{.5})\, df/dt, 0, 0)$ and $\phi = q/(4\pi\varepsilon_0[\ \cdot]^{.5})$ where q is the charge of the particle and $[\ \cdot] = [(x_1 - f(t))^2 + x_2^2 + x_3^2]$. It follows that

$$S \cong \varepsilon_0 c^2 q^2 (4\pi\varepsilon_0 c^2)^{-1} [\cdot]^{-3} \{c^2 \dot{f}(t)(x_1 - f(t)) + [\cdot]\ddot{f}(t)\,\dot{f}(t) + \dot{f}(t)^3 (x_1 - f(t))\}\,(x_1 - f(t), x_2, x_3)$$

The outward pointing normal vector n at a point (x_1, x_2, x_3) on a sphere of radius a centered at (f(t),0,0) is $[\ \cdot]^{-.5}(\,x_1 - f(t), x_2, x_3)$. Thus integrating S·n over the sphere of radius a centered (f(t), 0, 0) yields a power emission rate of $q^2/(4\pi\varepsilon_0 c^2 a) \cdot df/dt \cdot d^2f/dt^2$.

When an external force such as that due to a gravitational field acts along the x_1-axis on a Newtonian particle of mass m at position (f(t), 0, 0), the increase in kinetic energy of the particle is $m \cdot df/dt \cdot d^2/dt^2$. The theory of scattering of one-dimensional waves (for which $u = u_P$ and $S = S_p$) predicts a radius value a and mass value for the electron. It is interesting that the classical mass value is the same as the above value, $q^2(4\pi\varepsilon_0 c^2 a)^{-1}$. By comparison the integral of $S_p \cdot n$ over the sphere of radius a yields 2/3 of the classical mass value[1].

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REFERENCES

[1]Feynman,R. P., Leighton, R. B., & Sands, M. *The Feynman Lectures on Physics*, Volume II, 27-8,28-2 (Addison-Wesley, Reading, Massachusetts, 1964).

[2]Duchateau, P. &Zachmann, D. W. *Partial Differential Equations* 124-128 (McGraw-Hill, New York, 1986).

[3]Jeffries, C. Energy density in classical electrodynamics, *Proc. 20th Southeastern Symposium on System Theory*, IEEE 13-18 (1988).

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SIMPLIFICATION DES MODULES QUADRATIQUES SUR L'ANNEAU DES ENTIERS RELATIFS

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Presented by P. Ribenboim, F.R.S.C.

RESUME: Dans ce travail on prouve que sur les anneaux de fractions de l'anneau des entiers relatifs Z on peut simplifier l'isométrie des modules quadratiques : $\langle a, a\rangle \perp (M,q) \cong \langle b_1, b_2\rangle \perp (M,q)$.
Ceci est une généralisation du théorème de simplification de A.Roy [2] dans le cas où $A=S^{-1}Z$ et le module quadratique est de la forme $\langle a, a\rangle$ .On donne un contre-exemple (1.4) montrant que la simplification n'est pas possible si le module quadratique diagonal de rang 2 est de la forme : $\langle a_1, a_2\rangle$ où $a_1 \neq a_2 \bmod U(A)^2$; ainsi notre résultat est, dans un certain sens, le meilleur possible.

0-INTRODUCTION :

Nous désignerons par Z l'anneau des entiers relatifs. Soit A un anneau commutatif unitaire et dans lequel 2 est inversible. On note le groupe des éléments inversibles de A par U(A).

•Un A-module quadratique est un couple (M,q) où M est A module projectif de type fini et q est une forme quadratique non-dégénérée sur M. Si M admet une base orthogonale $\{e_i\}_{i=1,n}$, on note alors (M,q) par $\langle a_1,\ldots,a_n\rangle$ où $q(e_i)=a_i$ pour $i=1,n$.

•Une isométrie entre deux A-modules quadratiques (M,q) et (M',q') est un isomorphisme f des A-modules M et M' préservant les formes (i.e pour tous x appartenant à M on a $q(x)=q'(f(x))$) on note alors $(M,q) \cong (M',q')$.

•On note par $M_2(A)$ l'anneau des matrices 2x2 de A et $GL_2(A)$ le groupe des matrices inversibles de $M_2(A)$.

•Pour les termes inexpliqués on pourrait voir [1] et [2].

1. SIMPLIFICATION SUR L'ANNEAU DES ENTIERS RELATIFS

PROPOSITION 1.1: Soit un entier n, l'équation $X^2 + Y^2 = n$ admet une solution dans Z x Z si et seulement si, dans la décomposition de n en facteurs premiers, les entiers premiers de la forme (4k+3) ont des puissances paires.

PREUVE.-Voir([3] théorème.366).

LEMME 1.2. Soit un entier n, les propositions suivantes sont équivalentes :

1) l'équation $X^2 + Y^2 = n.T^2$ admet une solution (X',Y',T') dans Z^3.

2) l'équation $x^2 + y^2 = n$ admet une solution (x',y') dans Z^2.

PREUVE. -

2) ⇒ 1). Le triplet $(x',y',1) \in Z^3$ est une solution de l'équation $X^2 + Y^2 = n.T^2$.

1) ⇒ 2). Supposons qu'il n'existe pas de couple (x',y'), dans Z^2, solution de l'équation $x^2+y^2=n$ en vertu de (1.1) il existe un facteur premier de n, de la forme $(4k+3)$, ayant une puissance impaire.
Pour tout entier relatif T, le terme $n.T^2$ possède alors un facteur premier $(4k+3)$ de puissance impaire, donc l'équation $X^2 + Y^2 = n.T^2$ n'admet pas de solution (d'après (1.1)) d'où le résultat.

PROPOSITION 1.3. - Soit S une partie multiplicative de Z, $A=S^{-1}Z$; $\langle a, a\rangle$, $\langle b_1, b_2\rangle$ et (M,q) des A-modules quadratiques tels-que :

$$\langle a, a\rangle \perp (M,q) \cong \langle b_1, b_2\rangle \perp (M,q) \quad . \text{ Alors } \langle a, a\rangle \cong \langle b_1, b_2\rangle .$$

PREUVE. - On a par hypothèse $\langle a, a\rangle \perp (M,q) \cong \langle b_1, b_2\rangle \perp (M,q)$, en "tensorisant" par le corps des rationnels Q, on obtient aussi une isométrie :

$$(\langle a, a\rangle \perp (M,q))\otimes Q \cong (\langle b_1,b_2\rangle \perp (M,q))\otimes Q ,$$

en vertu de ([1].4.3 p82), les modules quadratiques $\langle a, a\rangle$ et $\langle b_1, b_2\rangle$ sont isométriques sur Q il existe alors une matrice U appartenant à $GL_2(Q)$,

$$U = \begin{pmatrix} x/x_1 & , y/y_1 \\ z/z_1 & , t/t_1 \end{pmatrix} \text{ est telle-que } \begin{pmatrix} b_1 & , 0 \\ 0 & , b_2 \end{pmatrix} = U.\begin{pmatrix} a & , 0 \\ 0 & , a \end{pmatrix}.{}^tU$$

On obtient entre autres : $(x/x_1)^2 a + (y/y_1)^2 a = b_1$, posons

$$X = x.y_1 , Y = y.x_1 \text{ et } T = x_1.y_1 \quad ; \text{ on a donc } X^2 + Y^2 = (b_1/a).T^2 ;$$

en vertu de (1.2), il existe un couple $(z,t) \in Z^2$ tel que $z^2 + t^2 = b_1/a$ d'où $z^2 a + t^2 a = b_1$.
Soit $\{e_1, e_2\}$ la base orthogonale du A-module quadratique $\langle a, a\rangle$, le vecteur $e' = z.e_1 + t.e_2$ a une norme égale à b_1 qui est inversible dans A, d'où le A-module libre $(A.e')$ est un facteur direct du A-module quadratique $\langle a, a\rangle$.
Soit $A.f = (A.e')^{\perp}$, la norme du vecteur f est égale $c \in U(A)$, on a alors $\langle a, a\rangle \cong \langle b_1, c\rangle$,

$$\langle a, a\rangle \otimes Q \cong \langle b_1, c\rangle \otimes Q \quad , \text{sachant que } \langle a, a\rangle \otimes Q \cong \langle b_1, b_2\rangle \otimes Q ,$$

on a donc l'isométrie $\langle b_1 , c\rangle \otimes Q \cong \langle b_1, b_2\rangle \otimes Q$ et en appliquant une nouvelle fois ([1].4.3 p82) on obtient donc $\langle c \rangle \otimes Q \cong \langle b_2 \rangle \otimes Q$, ceci veut dire qu'il existe un élément u non-nul appartenant à Q tel que $u^2.c=b_2$, l'anneau A étant intégralement clos ,il existe alors un élément inversible v appartenant à A tel que $v^2.c=b_2$.

Donc $\langle c \rangle \cong \langle b_2 \rangle$ et $\langle a , a\rangle \cong \langle b_1 , c\rangle \cong \langle b_1, b_2\rangle$, ce qui achève la démonstration □

•on montre dans le prochain exemple qu'on ne peut pas simplifier si le module quadratique de rang 2 est de la forme $\langle a_1 , a_2\rangle$ où $a_1 \not\equiv a_2 \mod U(A)^2$. Pour tout A-module projectif de type fini M, soit $\mathbb{H}(M)$ le A-module quadratique $(M\oplus M^*, q^M)$ où $q^M(x+f)=f(x)$ pour tout (x,f) appartenant à $M\oplus M^*$, le A-module quadratique $\mathbb{H}(M)$ est dit hyperbolique ([1] p. 15).

<u>EXEMPLE 1.4</u>. Soit $S=\{82^n / n \in Z\}$, $A=S^{-1}Z$, $\langle 1 , 41\rangle$ et $\langle 2 , 2/41^3\rangle$ des A-modules quadratiques

On a $\langle 1 , 41\rangle \perp \mathbb{H}(A^n) \cong \langle 2 , 2/41^3\rangle \perp \mathbb{H}(A^n)$,mais $\langle 1 , 41\rangle \not\cong \langle 2 , 2/41^3\rangle$.

PREUVE. - Soit :

$$U=\begin{pmatrix} 3/5 , & 1/5 \\ -1/(5\times 41) , & 3/(5\times 41^2) \end{pmatrix} \in GL_2(Q) .$$

On a

$$\begin{pmatrix} 2 , & 0 \\ 0 , & 2/41^3 \end{pmatrix} = U . \begin{pmatrix} 1 , & 0 \\ 0 , & 41 \end{pmatrix} . {}^tU ,$$

donc les matrices $\begin{pmatrix} 2 , & 0 \\ 0 , & 2/41^3 \end{pmatrix}$ et $\begin{pmatrix} 1 , & 0 \\ 0 , & 41 \end{pmatrix}$ sont équivalentes dans $M_2(Q)$.

Les modules quadratiques $\langle 1 , 41\rangle$ et $\langle 2 , 2/41^3\rangle$ sont alors isométriques sur Q .
L'anneau A étant principal , l'homomorphisme des anneaux de Witt $W(A) \hookrightarrow W(Q)$ est injectif d'après ([1].4.10 ,p20) , ceci veut dire que $[\langle 1 , 41\rangle] = [\langle 2 , 2/41^3\rangle]$ dans W(A), les crochets indiquent la classe d'équivalence du module quadratique considéré (voir [1] p17-20);

il existe, alors, un entier n tel que $\langle 1 , 41\rangle \perp \mathbb{H}(A^n) \cong \langle 2 , 2/41^3\rangle \perp \mathbb{H}(A^n)$.

<u>Montrons que les A-modules quadratiques $\langle 1 , 41\rangle$ et $\langle 2 , 2/41^3\rangle$ ne sont pas isométriques sur A</u>

Pour prouver ceci ,il suffit de montrer que pour tout couple $(n,m) \in \mathbb{N}^2$ l'équation :
$X^2 + 41.Y^2 = 2^n.41^m$ n'a pas de solution dans Z x Z

. Si $m > 1$, X est, alors, un multiple de 41, on peut écrire $X=41.X_1$, d'où : $Y^2+41.X_1^2=2^n.41^{m-1}$. En itérant ce procédé ; il existe, donc, $(X_s, Y_s) \in Z\times Z$ tel-que : $Y_s^2+41.X_s^2=2^n$.

. Si l'entier X_s est pair, Y_s est alors nécéssairement pair, d'où en posant $X_s=2.X_{s+1}$, $Y_s=2.Y_{s+1}$, on obtient $(Y_{s+1})^2+41.(X_{s+1})^2=2^{n-2}$.

En itérant ce procédé ; il existe un couple d'entiers impairs $(X_k, Y_k) \in Z\times Z$ tel que :

$$Y_k^2+41.X_k^2=2^{\ell},\ \ell \in \mathbb{N}.$$

En posant $X_k=2.X'_k+1$ et $Y_k=2.Y'_k+1$, on obtient

$$41.X_k^2=41(X'^2_k+4.X'_k+1)=1 \bmod [4],$$

$$Y_k^2=(4.Y'^2_k+4Y'_k+1)=1 \bmod [4] \text{ d'où}$$

$$Y'^2_k+41.X'^2_k=2 \bmod [4].$$

on aura donc, nécéssairemment, $\ell=1$ et sachant que l'équation $Y_k^2+41.X_k^2=2$ n'a pas de solution dans $Z\times Z$ on en déduit qu'il n'existe pas de couple $(X,Y)\in Z\times Z$ tel que :

$X^2+41.Y^2=2^n.41^m$ □

BIBLIOGRAPHIE :

[1] R.BAEZA, "Quadratic forms over semi-local rings". L.N.M n°655, Springer Verlag, New york, (1978).

[2] A.ROY, "Cancellation of quadratic forms over commutative rings". J.of Algebra 10,116-132, (1968)

[3] E.M. WRIGHT and G.H. HARDY, "An introduction of the theory of numbers". Clarendon Press-Oxford, (1965).

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MODULI OF CONTINUITY FOR SOME HILBERT SPACE VALUED ORNSTEIN-UHLENBECK PROCESSES

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Abstract: We establish Hölder continuity for some vector valued Ornstein-Uhlenbeck processes.

Let $\{X_k(\cdot)\}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, i.e. $X_k(\cdot)$ is a stationary, mean zero Gaussian process with $E(X_k(t)X_k(s)) = \gamma_k/\lambda_k \exp(-\lambda_k|t-s|)$. Suppose we want to establish the continuity in ℓ^2 of the vector valued process

$$(1) \qquad X(\cdot) = (X_1(\cdot), X_2(\cdot), \ldots, X_k(\cdot), \ldots) .$$

To begin with, in order that $X(t) \in \ell^2$ at fixed times we must assume that $E\|X(t)\|^2 = \sum \gamma_k/\lambda_k < \infty$. This does not guarantee that $X(t) \in \ell^2$ for all t however ([13]), to conclude that $X(\cdot)$ is an ℓ^2 valued process requires further growth conditions on γ_k. A natural first step would be to try and bound the increment moments $E\|X(t)-X(s)\|^{2m}$ and use Kolmogorov's criterion. Failing that, since we have a stationary, mean zero Gaussian process, we'd expect to get more refined results by using Fernique's condition. That is, if we can show

$$E\|X(t) - X(s)\|^2 \le \phi^2(|t-s|)$$

where $\phi(u)/u(\log(1/u))^{1/2}$ is integrable at zero, then continuity follows. If the vector space were finite dimensional this condition would be necessary as well ([9]) , but for an infinite dimensional space this is not true. A counterexample due to D.A. Dawson ([7]) is found by letting $\gamma_k \equiv 1$ and $\lambda_k = k(\log k)^2$.

In recent years a number of approaches have evolved for studying infinite dimensional diffusions, both of Ornstein-Uhlenbeck type and also more generally. See, for example, [1-8, 10-14]. For the process (1) it is known ([7], [13]) that if $\sum \gamma_k^2/\lambda_k < \infty$, then $X(\cdot)$ is a continuous ℓ^2-valued process. This condition is sufficient but "far" from necessary ([6]) , and in fact gives more than "just" continuity, as we see in Proposition 1.

To prove (i) in the following proposition we simply use Kolmogorov's criterion. In (ii) we see that if this (or even Fernique's condition) fails one can still show continuity by considering the norm squared process $\|X(\cdot)\|^2 = \sum X_k^2(\cdot)$.

<u>Proposition 1</u>:

(i) <u>If</u> $\sum \gamma_k/\lambda_k^{1-\epsilon} < \infty$ <u>for some</u> $0<\epsilon\le 1$, <u>then</u> $X(\cdot)$ <u>is μ-Hölder continuous in</u> ℓ^2 <u>for any</u> $\mu < \epsilon/2$.

(ii) <u>If</u> $\sum \gamma_k^2/\lambda_k^{2-\epsilon} < \infty$ <u>for some</u> $0<\epsilon\le 1$, <u>then</u> $\|X(\cdot)\|^2$ <u>is μ-Hölder continuous for any</u> $\mu < \epsilon/2$.

(ii)(b) <u>If</u> $\sum \gamma_k^2/\lambda_k < \infty$ <u>and</u> $\sum \gamma_k^2/\lambda_k^{1-\epsilon} < \infty$ <u>for some</u> $0<\epsilon\le 1$, <u>then</u>

$\|X(\cdot)\|^2$ <u>has Lévy's Hölder modulus</u> $(h \log(1/h))^{1/2}$.

<u>Comment</u>: $\|X(t) - X(s)\|^2 = (\|X(t)\|^2 - \|X(s)\|^2) - 2\langle X(t) - X(s), X(s)\rangle$. One can use standard Gaussian techniques ([9]) to give a Hölder modulus for the processes $\langle X(\cdot),x\rangle;\ x \in \ell^2$. This and the equation above show that under (ii) or (ii)(b) $X(\cdot)$ is Hölder continuous in ℓ^2 .

<u>Proof</u>: (i) Under the given condition we show that $E\|X(t) - X(s)\|^{2m} = O(|t-s|^{m\varepsilon})$ for every $m \in N$. It suffices to show $\sum_k (X_k(t) - X_k(s))^{2m}$ is $O(|t-s|^{m\varepsilon})$ for every $m \in N$.

But $\sum_k E(X_k(t) - X_k(s))^{2m}$

$$= \sum_k c(m)\left[\gamma_k/\lambda_k\left(1-e^{-\lambda_k|t-s|}\right)\right]^m$$

$$= c(m)\left\{\sum_k\left(\gamma_k/\lambda_k{}^{1-\varepsilon}\right)^m\left(1-e^{-\lambda_k|t-s|}/\ \lambda_k{}^{\varepsilon}|t-s|^{\varepsilon}\right)^m\right\}|t-s|^{m\varepsilon}$$

$$= O\left(|t-s|^{m\varepsilon}\right) ,$$

the term in braces being uniformly bounded in $|t-s|$.

(ii) Since $X_k(\cdot)$ is stationary, $E(X_k{}^2(t) - X_k{}^2(s)) = 0$. Therefore $X_k^2(t) - X_k^2(s) = (X_k(t) - X_k(s))(X_k(t) + X_k(s))$ is a decomposition into the product of two independent mean zero Gaussian random variables.

$E[X_k^2(t) - X_k^2(s)]^2$

$$= E(X_k(t) - X_k(s))^2 \cdot E(X_k(t) + X_k(s))^2$$

$$= 2(\gamma_k/\lambda_k)\left(1-e^{-\lambda_k|t-s|}\right)\cdot 2(\gamma_k/\lambda_k)\left(1+e^{-\lambda_k|t-s|}\right)$$

$$= 4(\gamma_k/\lambda_k)^2\left(1-e^{-2\lambda_k|t-s|}\right).$$

Using a similar argument for higher moments and taking the sum we get

$$\sum_k E[X_k^2(t) - X_k^2(s)]^{2m}$$

$$= c(m) \sum_k (\gamma_k/\lambda_k)^{2m} \left(1-e^{-2\lambda_k|t-s|}\right)^m$$

$$= c(m) \left\{ \sum_k \left(\gamma_k^2/\lambda_k^{2-\varepsilon}\right)^m \left(1-e^{-2\lambda_k|t-s|}/\lambda_k^{\varepsilon}|t-s|^{\varepsilon}\right)^m \right\} |t-s|^{m\varepsilon}$$

$$= O\left(|t-s|^{m\varepsilon}\right).$$

The remainder of the proof is as in (i). Part (b) of (ii) can be established as in [14] by using Fukushima's decomposition $\|X(t)\|^2 - \|X(0)\|^2 = M(t) + N(t)$. Under the stated condition one can show $M(\cdot)$ has the required modulus, while $N(\cdot)$ is Hölder continuous with exponent strictly greater than 1/2.

REFERENCES

[1] A. Antoniadis and R. Carmona, "Eigenfunction expansions for infinite dimensional Ornstein-Uhlenbeck processes". Probab. Theory Related Fields. 74, 1987, 31-54.

[2] R. Carmona, "Measurable norms and some Banach space valued Gaussian processes". Duke Math J. 44, No. 1, 1977, 109-127.

[3] M. Csorgo and Z.Y. Lin, "Path properties of infinite dimensional Ornstein-Uhlenbeck processes". Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 111, Carleton University - University of Ottawa, Ottawa.

[4] G. DaPrato, S. Kwapien and J. Zabczyk, "Regularity of solutions of linear stochastic equations in Hilbert space". Stochastics 23, No. 1, 1987, 1-23.

[5] D.A. Dawson, "Stochastic evolution equations". Math. Biosci. 15, 1972, 287-316.

[6] I. Iscoe, M. Marcus, D. McDonald and J. Zinn, "Continuity of ℓ^2-valued Ornstein-Uhlenbeck processes". Manuscript.

[7] I. Iscoe and D. McDonald, "Continuity of ℓ^2-valued Ornstein-Uhlenbeck processes". Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 68, Carleton University - University of Ottawa, Ottawa.

[8] K. Ito, "Infinite Dimensional Ornstein-Uhlenbeck processes". Taniguchi Symp. SA Katata, 1982, 197-224.

[9] N. Jain and M. Marcus, "Continuity of subgaussian processes". Adv. Probab. Related Topics, Vol. 4, 1978, 81-196.

[10] T. Koski, "Some metric order of entropy properties of an infinite dimensional Ornstein-Uhlenbeck process". Acta Acad. Aboensis. Ser. B Vol. 45 No. 4, 1985.

[11] P. Kotelenez, "Continuity properties of Hilbert space valued martingales". Stoch. Proc. Appl. 17, 1984, 115-125.

[12] P. Kotelenez, "A maximal inequality for stochastic integrals on Hilbert spaces and space-time regularity of linear stochastic partial differential equations". Stochastics 21, No. 4, 1987, 345-358.

[13] B. Schmuland, "Some regularity results on infinite dimensional diffusions via Dirichlet forms". To appear.

[14] B. Schmuland, "Hölder continuity for some infinite dimensional Ornstein-Uhlenbeck processes". Tech. Rep. 88.05, University of Alberta, Edmonton, 1988.

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ON MODULI OF CONTINUITY FOR GAUSSIAN AND χ^2 PROCESSES GENERATED BY ORNSTEIN-UHLENBECK PROCESSES

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Presented by D.A. Dawson, F.R.S.C.

Abstract: We prove large deviation inequalities and moduli of continuity for infinite series of independent Ornstein-Uhlenbeck processes and for those of the squares of the latter processes.

1. Introduction. Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, i.e., $X_k(\cdot)$ is a Gaussian process with $EX_k(t) = 0$ and $EX_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|)$, $(\gamma_k, \lambda_k > 0, k = 1,2,\ldots)$. The process $Y(\cdot)$ was first studied by Dawson (1972) as the stationary solution of the infinite array of stochastic differential equations

$$dX_k(t) = -\lambda_k X_k(t)dt + (2\gamma_k)^{1/2}dW_k(t) \quad (k = 1,2,\ldots), \tag{1.1}$$

where $\{W_k(t), -\infty < t < \infty\}$ are independent Wiener processes (cf. also Dawson (1975), Walsh (1981), and Antoniadis and Carmona (1987)). If we assume

$$\sum_{j=1}^{\infty} \gamma_j/\lambda_j < \infty, \tag{1.2}$$

then $Y(t)$ is almost surely (a.s.) an ℓ^2-valued Ornstein-Uhlenbeck process $(E\|Y(t)\|_{\ell^2}^2 = \sum_{j=1}^{\infty}\gamma_j/\lambda_j)$. Dawson (1972) showed that under the condition (1.2) $Y(\cdot)$ is weakly continuous. Assuming also that for large j we have also $cj^{1+\delta} \le \lambda_j \le dj^{1+\delta}$ for some $c > 0$, $d > 0$ and $\delta > 0$, then Dawson (1972) showed $Y(\cdot)$ in ℓ^2 to be a.s. continuous as well. Iscoe and McDonald (1986, 1987) established the a.s. continuity of $Y(\cdot)$ in ℓ^2 under the weaker additional (to (1.2)) condition

$$\Gamma_2 = \sum_{j=1}^{\infty} \gamma_j^2/\lambda_j < \infty, \tag{1.3}$$

by studying and solving the equivalent problem of a.s. continuity of the process

$$\{\chi^2(t), -\infty < t < \infty\} = \left\{\sum_{k=1}^{\infty} X_k^2(t), -\infty < t < \infty\right\} \tag{1.4}$$

via powerful large deviation theorems for the latter stochastic process. Iscoe, Marcus, McDonald and Zinn (1987) dealt with the strong continuity of $Y(\cdot)$ in ℓ^2 by using Fernique-type techniques concerning a.s. continuity of stationary Gaussian processes. Schmuland (1986, 1987) used Dirichlet forms to study a class of infinite dimensional processes, modelled on the process $Y(\cdot)$. His general results can be also used to recover state space and continuity results of the so far mentioned papers. Schmuland (1987), as well as Csörgő and Lin (1988b), also proved various laws of the iterated logarithm for $Y(\cdot)$ in terms of the two-time parameter stochastic process $\{X(t,n), -\infty < t < \infty, n = 1,2,\ldots\}$, $X(t,n) = \sum_{k=1}^{n} X_k(t)$, $X(t,0) = 0$ for all $t \in \mathbb{R}$. Csörgő and Lin (1988a) also established P. Lévy type moduli of continuity and large increment rates results for the latter two-time parameter process $X(\cdot,\cdot)$ along the lines of Sections 1.12 - 1.15 and Supplementary remarks of Chapter 1 in Csörgő and Révész(1981), where the a.s. behaviour of some path increments of the Wiener sheet and the Kiefer process is established. While the two- time parameter increments results of Csörgő and Lin (1988a) for $X(\cdot,\cdot)$ are of interest on their own, of course, they do not imply any immediate, direct a.s. moduli of continuity for the process $X(t,\infty)$ in t. In this note we assume (1.2), and establish P. Lévy-type moduli of continuity results for $\chi^2(\cdot)$, as well as for $X(\cdot,\infty) = X(\cdot)$, namely also for the process

$$\{X(t), -\infty < t < \infty\} = \left\{\sum_{k=1}^{\infty} X_k(t), -\infty < t < \infty\right\}, \tag{1.5}$$

under further global conditions on the coefficients γ_k and λ_k of the Ornstein-Uhlenbeck processes of $Y(\cdot)$. Clearly, in addition to (1.2), finiteness of Γ_2 of (1.3) will play a role in dealing with the process $\chi^2(\cdot)$ and, as we will see, a similar additional role will be played by the condition

$$\Gamma_1 = \sum_{j=1}^{\infty} \gamma_j < \infty, \tag{1.6}$$

when studying the just introduced process $X(\cdot)$.

In Section 2 we state and comment on our results which we then prove in Section 3.

2. Large deviations and moduli of continuity. First we state our large deviation results.

LEMMA 2.1. *Assume (1.2), and for any $\epsilon > 0$ let $N(s) = \min\{N : \sum_{j=N}^{\infty} \gamma_j/\lambda_j \le \epsilon s\}$ and $M(s) = \min\{N : \sum_{j=N}^{\infty} (\gamma_j/\lambda_j)^2 \le \epsilon s\}$. There exist $h(\epsilon) > 0$ and $C = C(\epsilon) > 0$ such that for any $T \ge h(\epsilon)$, $h \le h(\epsilon)$, and $v > 0$, we have*

$$P\left\{\sup_{|t|\le T} \sup_{0\le s\le h} |X(t+s) - X(t)| \ge v(2h\Gamma_1)^{1/2}\right\} \le (CT/h)\exp\left(-\frac{v^2}{2+\epsilon}\right),$$

if $\Gamma_1 < \infty$ and

$$\left(\max_{1\le j\le N(s)} \lambda_j\right) s \to 0 \quad (s \to 0), \tag{2.1}$$

and we have

$$P\left\{\sup_{|t|\le T} \sup_{0\le s\le h} |\chi^2(t+s) - \chi^2(t)| \ge v(8h\Gamma_2)^{1/2}\right\} \le (CT/h)\exp\left(-\frac{v^2}{2+\epsilon}\right)$$

if $\Gamma_2 < \infty$ and

$$\left(\max_{1\le j\le M(s)} \lambda_j\right) s \to 0 \quad (s \to 0). \tag{2.2}$$

THEOREM. *Suppose that condition (1.2) is satisfied, and assume also that $T_h \uparrow \infty$ continuously as $h \to 0$. Then, if $\Gamma_1 < \infty$, and condition (2.1) is satisfied, we have*

$$\lim_{h\to 0} \sup_{|t|\le T_h} \frac{|X(t+h) - X(t)|}{(2h\Gamma_1 \log(T_h/h))^{1/2}} = 1 \quad \text{a.s.}, \tag{2.3}$$

$$\lim_{h\to 0} \sup_{|t|\le T_h} \sup_{0\le s\le h} \frac{|X(t+s) - X(t)|}{(2h\Gamma_1 \log(T_h/h))^{1/2}} = 1 \quad \text{a.s.}, \tag{2.4}$$

and, if $\Gamma_2 < \infty$, and condition (2.2) is satisfied, we have

$$\lim_{h\to 0} \sup_{|t|\le T_h} \frac{|\chi^2(t+h) - \chi^2(t)|}{(8h\Gamma_2 \log(T_h/h))^{1/2}} = 1 \quad \text{a.s.}, \tag{2.5}$$

$$\lim_{h\to 0} \sup_{|t|\le T_h} \sup_{0\le s\le h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8h\Gamma_2 \log(T_h/h))^{1/2}} = 1 \quad \text{a.s.}. \tag{2.6}$$

Taking $T_h = 1/h$ then, apart from their respective constants, the norming functions of (2.3) - (2.6) look like those of the classical P. Lévy moduli of continuity for Brownian motion (cf., e.g., Theorem 1.1.1 and Remark 1.1.2 in Csörgő and Révész(1981)).

In addition to Lemma 2.1, the proof of Theorem will also utilize the next, well known result.

LEMMA 2.2. (Slepian 1962). *Let $G(t)$ and $G^*(t)$ be Gaussian processes on $\mathbb{R}^+$, possessing continuous sample path functions, with $EG(t) = EG^*(t) = 0$, $EG^2(t) = EG^{*2}(t) = 1$, and let $\rho(s,t)$ and $\rho^*(s,t)$ be their respective covariance functions. Suppose that we have $\rho(s,t) \geq \rho^*(s,t)$, $s,t \in \mathbb{R}^+$. Then*

$$P\left\{\sup_{0\leq t\leq T} |G(t)| \leq u\right\} \geq P\left\{\sup_{0\leq t\leq T} |G^*(t)| \leq u\right\}.$$

3. Proofs.

PROOF OF LEMMA 2.1. By definition we have

$$X(t+s) - X(t) \stackrel{\mathcal{D}}{=} N\left(0, 2\sum_{j=1}^{\infty}(\gamma_j/\lambda_j)(1-e^{-\lambda_j s})\right),$$

where $N(\cdot,\cdot)$ stands for a normal random variable with (mean, variance) as indicated inside brackets. Also,

$$E\chi^2(t) = \sum_{j=1}^{\infty}\gamma_j/\lambda_j, \quad E\left(\chi^2(t) - \sum_{j=1}^{\infty}\gamma_j/\lambda_j\right)^2 = 2\sum_{j=1}^{\infty}\gamma_j^2/\lambda_j^2,$$

and

$$E(\chi^2(t+s) - \chi^2(t))^2 = 4\sum_{j=1}^{\infty}(\gamma_j^2/\lambda_j^2)(1-e^{-2\lambda_j s}).$$

Under condition (2.1), as $s \to 0$, we have

$$\left(\sum_{j=1}^{\infty}(\gamma_j/\lambda_j)(1-e^{-\lambda_j s})\right)/(s\Gamma_1) \to 1,$$

similarly, under condition (2.2), as $s \to 0$, we have

$$\left(\frac{1}{2}\sum_{j=1}^{\infty}(\gamma_j^2/\lambda_j^2)(1-e^{-2\lambda_j s})\right)/(s\Gamma_2) \to 1.$$

From here on the proof of this lemma is similar to that of (1.1) of Lemma 1.3 of Csörgő and Lin (1988a). Hence we omit the details.

PROOF OF THEOREM. For a given $\epsilon > 0$, let h_n be such that

$$\sum_{n=1}^{\infty}(h_{n-1}/T_{h_n})^{\epsilon} < \infty \tag{3.1}$$

and, as $n \to \infty$,

$$(h_{n-1}/T_{h_n})/(h_n/T_{h_{n+1}}) \to 1.$$

By Lemma 2.1 we have

$$P\left\{\sup_{|t|\leq T_{h_n}}\sup_{0\leq s\leq h_{n-1}}|X(t+s)-X(t)| \geq (1+\epsilon)(2h_{n-1}\Gamma_1\log(T_{h_n}/h_{n-1}))^{1/2}\right\}$$
$$\leq (CT_{h_n}/h_{n-1})\exp\left(-\frac{2(1+\epsilon)^2}{2+\epsilon}\log(T_{h_n}/h_{n-1})\right) \leq C(h_{n-1}/T_{h_n})^{\epsilon}.$$

The latter combined with (3.1) yields

$$\limsup_{n\to\infty}\sup_{|t|\leq T_{h_n}}\sup_{0\leq s\leq h_{n-1}}\frac{|X(t+s)-X(t)|}{(2h_{n-1}\Gamma_1\log(T_{h_n}/h_{n-1}))^{1/2}} \leq 1+\epsilon \quad \text{a.s.},$$

which, by condition (3.2), now results in

$$\limsup_{h\to 0}\sup_{|t|\leq T_h}\sup_{0\leq s\leq h}\frac{|X(t+s)-X(t)|}{(2h\Gamma_1\log(T_h/h))^{1/2}} \leq 1+\epsilon \quad \text{a.s.} \tag{3.3}$$

Next we prove

(3.4) $$\liminf_{h\to 0} \sup_{|t|\le T_h} \frac{|X(t+h)-X(t)|}{(2h\Gamma_1 \log(T_h/h))^{1/2}} \ge 1-\epsilon \quad \text{a.s.}$$

To this end, define h_n such that $T_{h_{n-1}}/h_n = (\log n)^{3/\epsilon}$. For $i<j$ we have

$$E(X((i+1)h_n)-X(ih_n))(X((j+1)h_n)-X(jh_n))$$
$$= \sum_{\ell=1}^{\infty}(\gamma_\ell/\lambda_\ell)e^{-\lambda_\ell j h_n}(2e^{\lambda_\ell i h_n} - e^{\lambda_\ell(i-1)h_n} - e^{\lambda_\ell(i+1)h_n}) < 0.$$

Thus, by using Lemma 2.2, we obtain

$$P\left\{\max_{|k|\le [T_{h_{n-1}}/h_n]} |X((k+1)h_n)-X(kh_n)| \le (1-\epsilon)(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}\right\}$$
$$\le (P\{|X(h_n)| \le (1-\epsilon)(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}\})^{2[T_{h_{n-1}}/h_n]+1}$$
$$\le \left(1-\frac{1}{6(\log(T_{h_{n-1}}/h_n))^{1/2}}\exp(-(1-\frac{\epsilon}{2})\log(T_{h_{n-1}}/h_n))\right)^{2T_{h_{n-1}}/h_n}$$
$$\le \exp(-2(T_{h_{n-1}}/h_n)^{\epsilon/3}) \le 1/n^2.$$

Hence,

$$\liminf_{n\to\infty} \max_{|k|\le [T_{h_{n-1}}/h_n]} \frac{|X((k+1)h_n)-X(kh_n)|}{(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}} \ge 1-\epsilon \quad \text{a.s.}$$

which in turn implies

(3.5) $$\liminf_{n\to\infty} \sup_{|t|\le T_{h_{n-1}}} \frac{|X((t+h_n)-X(t)|}{(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}} \ge 1-\epsilon \quad \text{a.s.}$$

Obviously, for $h_n \le h < h_{n-1}$, as $n\to\infty$, we have

(3.6) $$h_n\log(T_{h_{n-1}}/h_n)/(h\log(T_h/h)) \to 1.$$

Furthermore, we have

(3.7)
$$\sup_{h_n\le h<h_{n-1}} \sup_{|t|\le T_{h_{n-1}}} \frac{|X((t+h)-X(t+h_n)|}{(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}}$$
$$\le \sup_{0\le s\le h_{n-1}-h_n} \sup_{|t|\le T_{h_{n-1}}+h_n} \frac{|X((t+s)-X(t)|}{(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}}.$$

By definition of h_n, as $n\to\infty$, $(h_{n-1}-h_n)/h_n \to 0$ and we have also

(3.8) $$(h_{n-1}-h_n)\log((T_{h_{n-1}}+h_n)/(h_{n-1}-h_n))/(h_n\log(T_{h_{n-1}}/h_n)) \to 0.$$

Moreover, by (3.3),

$$\limsup_{n\to\infty} \sup_{|t|\le T_{h_{n-1}}+h_n} \sup_{0\le s\le h_{n-1}-h_n} \frac{|X(t+s)-X(t)|}{(2(h_{n-1}-h_n)\Gamma_1\log((T_{h_{n-1}}+h_n)/(h_{n-1}-h_n)))^{1/2}} \le 1+\epsilon \quad \text{a.s.}$$

Combining the latter with (3.7) and (3.8), we get

(3.9) $$\limsup_{n\to\infty} \sup_{h_n\le h<h_{n-1}} \sup_{|t|\le T_{h_{n-1}}} \frac{|X(t+h)-X(t+h_n)|}{(2h_n\Gamma_1\log(T_{h_{n-1}}/h_n))^{1/2}} = 0 \quad \text{a.s.},$$

and now (3.5), (3.6) and (3.9) yield

(3.10) $$\liminf_{h\to 0} \sup_{|t|\le T_h} \frac{|X(t+h)-X(t)|}{(2h\Gamma_1\log(T_h/h))^{1/2}} \ge 1-\epsilon \quad \text{a.s.}$$

Consequently, by (3.3) and (3.10) we have (2.3) and (2.4).

The proof of (2.5) and (2.6) is similar, and hence the details are omitted.

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REFERENCES

[1] ANTONIADIS, A. and CARMONA, R. (1987). Eigenfunction expansions for infinite dimensional Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields* 74 31-54.

[2] CSÖRGŐ, M. and LIN, Z.Y. (1988a). Path properties of infinite dimensional Ornstein-Uhlenbeck processes. *Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 111*, Carleton University - University of Ottawa, Ottawa.

[3] CSÖRGŐ, M. and LIN, Z.Y. (1988b). A law of the iterated logarithm for infinite dimensional Ornstein-Uhlenbeck processes. *Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 111*, Carleton University - University of Ottawa, Ottawa.

[4] CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics.* Academic, New York.

[5] DAWSON, D.A. (1972). Stochastic evolution equations. *Math. Biosciences* 15 287-316.

[6] DAWSON, D.A. (1975). Stochastic evolution equations and related measure processes. *J. Multivariate Anal.* 5 1-52.

[7] ISCOE, I., MARCUS, M., McDONALD, D. and ZINN, J. (1987). Continuity of ℓ^2-valued Ornstein-Uhlenbeck processes. Manuscript.

[8] ISCOE, I. and McDONALD, D. (1986). Continuity of ℓ^2-valued Ornstein-Uhlenbeck processes. *Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 68*, Carleton University - University of Ottawa, Ottawa.

[9] ISCOE, I. and McDONALD, D.,(1987). Large deviations for ℓ^2-valued Ornstein-Uhlenbeck processes. *Ann. Probab.* To appear.

[10] SCHMULAND, B. (1986). Dirichlet forms and infinite dimensional stochastic processes. *Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 95*, Carleton University - University of Ottawa, Ottawa.

[11] SCHMULAND, B. (1987). *Dirichlet Forms and Infinite Dimensional Ornstein-Uhlenbeck Processes.* Ph.D. Dissertation, Carleton University, Ottawa.

[12] WALSH, J.B. (1981). A stochastic model of neural response. *Adv. Appl. Probab.* 13 231-281.

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