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CONTENTS

A GENERALIZATION OF THE NOTION OF FREE TRIPLE

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Presented by J. Lambek, F.R.S.C.

Let $\mathbb{C}$ be a locally small category with finite coproducts and colimits of ω-chains. Let Q be an endofunctor preserving colimits of ω-chains. Let Q-Alg be the category whose objects are pairs (X,x) where $x:QX \to X$, and where a morphism $(X,x) \to (Y,y)$ is an $f:X \to Y$ such that $fx = y \circ Qf$. There is a forgetful functor $U_Q:Q\text{-Alg} \to \mathbb{C}$. A well known result (e.g. [1]) is the following: U_Q is tripleable and the triple thus obtained is the free triple generated by the endofunctor Q. Let us recall the construction of the left adjoint F_Q to U_Q. For $X \in \mathbb{C}$ define by induction a sequence

$$X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \cdots \to X_{n-1} \xrightarrow{x_n} X_n \to \cdots$$

where $X_0 = X$, $X_{n+1} = X + QX_n$, $x_1:X_0 \to X_1$ is the first coprojection $X \to X+QX$ and $x_{n+1}:X_n \to X_{n+1}$ is $X+Qx_n:X+QX_{n-1} \to X+QX_n$. Take X^*. to be $\mathrm{colim}(\ldots X_n \to X_{n+1} \ldots)$. Using the coprojections $QX_n \to X+QX_n = X_{n+1}$ one defines a natural transformation from the ω-chain

$$0 \xrightarrow{!} QX_0 \xrightarrow{Qx_1} QX_1 \xrightarrow{Qx_2} QX_2 \to \cdots \quad \text{to the } \omega\text{-chain} \quad X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} X_2 \to \cdots$$

and using the fact that Q preserves colimits of ω-chains one constructs $h:QX^* \to X^*$ and it is easy to check that the coprojection $X \to X^*$ is a universal arrow from X to the free algebra (X^*,h).

Notice that if $X = 0$, the initial object, then the ω-chain is simply

$$0 \xrightarrow{!} Q0 \xrightarrow{Q!} Q^2 0 \to \ldots \to Q^n 0 \xrightarrow{Q^n !} Q^{n+1} 0 \to \ldots$$

and the construction does not need other coproducts, i.e. for Q-Alg to have an initial object it suffices that $\mathbb{C}$ have an initial object along with colimits of ω-chains.

The aim of this note is to explore the following situation: what if $\mathbb{C}$ has finite *multicoproducts* instead of standard coproducts ? This situation arises naturally if one takes $\mathbb{C}$ to be the category of models for some simple first order theories [4]. Q can now be taken to be a definable functor and can sometimes be "very recursive". We assume that the notions of multiadjoint and multicolimit [2] are known to the reader. Let us fix some terminology. If $U:\mathbb{D}\to\mathbb{C}$ has a multiadjoint then we call an arrow $a:X\to UA$ in $\mathbb{C}$ which belongs to a multiuniversal family a *free object candidate* for $\mathbb{C}$. That is, a is an initial object in its component of (X/U). If $\mathbb{C}$ has finite multicoproducts and X, X' are objects of $\mathbb{C}$ then a *sum candidate* for X, X' is a cospan (discrete cocone) $y:X\to Y$, $y':X'\to Y$ which is initial in its component of $\mathrm{Cocone}(X,X')$. If $z:X\to Z$, $z':X\to Z'$ is any cospan then there is a unique (up to unique iso) sum candidate (Y,y,y') and a unique $f:Y\to Z$ such that $fy = z$ and $fy' = z'$. We say that y, y' is the sum candidate *determined by* z, z' and that f is the *factoring determined by* z, z'.

Theorem 1 Let $\mathbb{C}$ have multicoproducts for finite families and colimits of ω-chains. Let Q be an endofunctor preserving colimits of ω-chains. Then the forgetful functor U_Q:Q-Alg$\to\mathbb{C}$ has a multiadjoint.

The proof hinges on the following definition: if $X \in \mathbb{C}$ then an X-*train* $\mathcal{X}$ is a triple $((X_n)_n,(x_n)_n,(a_n)_n)$ where $(X_n)_{n\geq 0}$ is a sequence of objects of $\mathbb{C}$, with $X_0 = X$, along with morphisms $x_n:X_{n-1} \to X_n$, $a_n:QX_{n-1} \to X_n$ for $n\geq 1$ such that if we define $\tilde{x}_n:X \to X_n$ to be $\tilde{x}_n = x_n \circ x_{n-1} \circ \ldots \circ x_1$ the following conditions hold for $n\geq 1$

- $\tilde{x}_n$, a_n is a sum candidate
- $x_{n+1}:X_n \to X_{n+1}$ is the (necessarily unique) morphism making $a_{n+1} \circ QX_n = x_{n+1} \circ a_n$ and $x_{n+1} \circ \tilde{x}_n = \tilde{x}_{n+1}$.

If $\mathcal{X}' = (X'_n,x'_n,a'_n)_{n\geq 1}$ is another X-train a morphism $\mathcal{X} \to \mathcal{X}'$ is a sequence $(f_n:X_n \to X'_n)_{n\geq 0}$ of morphisms of $\mathbb{C}$ such that $f_0 = 1_X$, $x'_n \circ f_{n-1} = f_n \circ x_n$ and $a'_n \circ Qf_{n-1} = f_n \circ a_n$ for $n\geq 1$. If such a morphism exists then the multiuniversal properties force it to be unique and every f_n to be an isomorphism. Given $\mathcal{X}$ as above one defines a Q-algebra (X^*,h) and a morphism $c:X \to X^*$ as follows: X^* is $\mathrm{colim}(\ldots x_n:X_n \to X_{n+1} \ldots)$ with $(c_n:X_n \to X_{n+1})_{n\geq 0}$ the coprojections. Since $x_n \circ a_n = a_{n+1}$ and Q preserves ω-colimits, there exists a unique $h:QX^* \to X^*$ $h \circ Qc_n = c_{n+1} \circ a_n$. We take c to be c_0.

the proof of theorem 1 is now the verification that for every X-train $\mathcal{X}$ the construction above yields a morphism $c:X \to U_Q(X^*,h)$ which is a free algebra candidate, and that every Q-algebra (Y,y) and morphism $f:X \to Y$ give rise naturally to an X-train $\mathcal{X}$ such that there is a morphism $\overline{f}:(X^*,h) \to (Y,y)$ satisfying $\overline{f}c = f$. A question that arises naturally now is: can some X-trains be uniformly obtained, so that the algebras they determine are essentially recursive ? First recall [3] that if $\mathbb{D}$ is some category then finite multicoproducts exist in the functor category $\mathbb{C}^{\mathbb{D}}$ and are calculated pointwise, i.e. a sum candidate $\alpha:F\to H$, $\beta:G\to H$ in $\mathbb{C}^{\mathbb{D}}$ is exactly a diagram where for every $X\in\mathbb{D}$ $(\alpha X,\beta X)$ is a sum candidate in $\mathbb{C}$ (here we ignore questions of size in the families involved).

Definition: A Q-*constructor* on $\mathbb{C}$ is a sum candidate $\eta:I\to T$, $\alpha:Q\to T$ in $End(\mathbb{C})$, where I is the identity.

Theorem 2: Given a Q-constructor as above there exists a functor $(T^*,h):\mathbb{C}\to Q\text{-}Alg$ (i.e. $T^*:\mathbb{C}\to\mathbb{C}$ and $h:QT^*\to T^*$) and a natural transformation $c:I\to T^*$ such that for any $X\in\mathbb{C}$ the arrow $cX:X\to U_Q(T^*X,hX)$ is a free algebra candidate and such that given a Q-algebra (Y,y) and a morphism $f:X\to Y$ in $\mathbb{C}$ a sufficient condition for there being $\bar{f}:(T^*X,hX)\to(Y,y)$ with $\bar{f}\circ cX=f$ is that the sum candidate determined by f,y be the same as the one determined by $\eta Y\circ f,\alpha Y$.

Therefore a Q-constructor yields free Q-algebra candidates which obey a universal property. The proof is obtained by working in the category $X/\mathbb{C}$ and defining an endofunctor Q^* over that category that sends the object $f:X\to Y$ of $X/\mathbb{C}$ to $w:X\to W$ where w,r is the sum candidate determined by the cospan $\eta Y\circ f,\alpha Y$ (so $r:QA\to W$). Then a Q^*-algebra $Q^*f\to f$ is exactly the same as a cospan $f:X\to Y$, $y:QY\to Y$ that satisfies the sufficiency condition of the theorem. It is easy to show that $X/\mathbb{C}$ has colimits of ω-chains and that Q^* preserves them. Since $X/\mathbb{C}$ has an initial object $0 = 1_X$ there is an initial Q^*-algebra which we notate as the cospan $cX: X \to T^*X$, $hX: QT^*X \to T^*X$. Examination of the construction of $\mathrm{colim}(\ldots Q^{n-1}0 \to Q^n 0 \ldots)$ shows that T^*X, hX, cX are obtained from an X-train by the construction of Theorem 1 , which guarantees that they form a free algebra candidate. It is easy to check that the whole process is functorial when X varies, so that T^* is indeed a functor and h, c natural transformations.

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LUCAS NUMBERS OF THE FORM px^2, WHERE p=3, 7, 47, OR 2207

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Abstract: Using elementary methods, we find all solutions to equations of the form $L_n = px^2$, where L_n is the n^{th} Lucas number and $p = L_{2^m}$ and is prime. Our results apply to the cases $m = 1, 2, 3, 4$.

INTRODUCTION. Let L_n denote the n^{th} Lucas number where $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. In [1], J.H.E. Cohn proved that the only solutions for the equation $L_n = x^2$ are $(n,x^2) = (1,1)$ and $(3,4)$. Cohn also showed that the only solutions for $L_n = 2x^2$ are $(n,x^2) = (0,1)$ and $(\pm 6,9)$. Adapting Cohn's methods, and clarifying some of his results via Lemma 1, we find solutions to all equations of the form $L_n = px^2$, where $p = L_{2^m}$ and is prime.

PRELIMINARIES. Let n,m,k denote integers not necessarily positive. The following formulae are elementary and/or are stated in [1]:

(1) $2L_{m+n} = 5F_mF_n + L_mL_n$

(2) $F_{2m} = F_mL_m$

(3) $L_{2m} = L_m^2 - 2(-1)^m$

(4) If $n \geq 2$, then $L_n | L_m$ if and only if $m = kn$, with k odd.

(5) $2 | L_m$ if and only if $3 | m$.

(6) $L_k \equiv 3 \pmod 4$ if $2 | k$, $3 \nmid k$.

(7) $L_{-n} = (-1)^n L_n$

THE MAIN RESULTS

Lemma 1. If $j = 3^r k$ for $r \geq 0$, $3\nmid k$, $2|k$, then $L_{m+2j} \equiv -L_m \pmod{L_k}$.

Proof: By (1), $2L_{m+2j} = 5F_m F_{2j} + L_m L_{2j}$. By (2) and (3), $2L_{m+2j} \equiv -2(-1)^j L_m \pmod{L_j}$. By hypothesis, $2|j$, which implies that $2L_{m+2j} \equiv -2L_m \pmod{L_j}$, so $L_{m+2j} \equiv -L_m \pmod{L_j/(2,L_j)}$. If $r = 0$, then $j = k$ and $3\nmid k$ implies $2\nmid L_k$ by (5), so $L_{m+2j} \equiv -L_m \pmod{L_k}$. If $r > 0$, then $2|L_j$, so $L_{m+2j} \equiv -L_m \pmod{\frac{L_j}{2}}$. By (4), $j = 3^r k$ implies $L_k|L_j$. Also, by (5), $3\nmid k$ implies $2\nmid L_k$. Therefore, $L_k|\frac{L_j}{2}$, so $L_{m+2j} \equiv -L_m \pmod{L_k}$.

Theorem. If $L_{2^m} = p$ (a prime), then $L_n = px^2$ if and only if $n = \pm 2^m$ and $x^2 = 1$.

Proof. $L_{2^m} = p$ and $L_n = px^2$ imply $L_{2^m}|L_n$. This implies, by (4), that for $m \geq 1$, $n = 2^m j$ where j is odd. Therefore, $n \equiv \pm 2^m \pmod{2^{m+2}}$. If $n = \pm 2^m$, by (7), $L_n = px^2$ and $x^2 = 1$. If $n \neq \pm 2^m$, then $n = \pm 2^m + 2.3^r k$ where $2^{m+1}|k$, $3\nmid k$, $r \geq 0$. By Lemma 1, $L_n \equiv -L_{\pm 2^m} \pmod{L_k}$. Therefore, $px^2 \equiv -p \pmod{L_k}$. $2^{m+1}|k$ implies that $k = 2^{m+1}z = (2^m)2z$ which implies, by (4), that $L_{\pm 2^m}\nmid L_k$ or $p\nmid L_k$. Therefore, $x^2 \equiv -1 \pmod{L_k}$. By (6), $2|k$, $3\nmid k$ imply $L_k \equiv 3 \pmod 4$. This implies that there exists a prime q, $q \equiv 3 \pmod 4$ such that $q|L_k$. This implies that $x^2 \equiv -1 \pmod q$, an impossibility.

Corollary. If $L_n = px^2$ where $p = 3, 7, 47$, or 2207, then $x^2 = 1$ and $n = 2, 4, 8$, or 16 respectively.

Proof. This is implied by the theorem, since $L_2 = 3$, $L_4 = 7$, $L_8 = 47$, and $L_{16} = 2207$, all primes.

Remark. L_{2^m} is composite for $5 \le m \le 9$.

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SYSTÈMES HYPERBOLIQUES À COEFFICIENTS DISCONTINUS: SOLUTIONS GÉNÉRALISÉES ET UNE APPLICATION À L'ACOUSTIQUE LINÈAIRE

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Résumé - On montre l'existence et l'unicité des solutions généralisées dans l'algèbre de Colombeau pour les systèmes hyperboliques linèaires à deux variables, dont les coefficients sont des fonctions généralisées. Pour un problème de l'acoustique linèaire dans un milieu discontinu, où apparait une multiplication "formelle" de distributions, on prouve que les solutions généralisées sont associées à des fonctions classiques, satisfaisant aux conditions de passage qui sont les conditions physiquement attendues.

INTRODUCTION - Dans la première partie de cette note nous considérons le problème de Cauchy global pour un système hyperbolique linèaire à deux variables indépendantes

$$(1)\quad \begin{aligned} (\partial_t + \Lambda(x,t)\partial_x)\, V &= F(x,t)\, V + G(x,t) \quad \text{pour } (x,t) \in \mathbb{R}^2 \\ V(x,0) &= A(x) \quad \text{pour } x \in \mathbb{R} \end{aligned}$$

où Λ et F désignent des matrices $(n\times n)$ dont les éléments sont des fonctions discontinues; la matrice Λ est sous forme diagonalisée et à valeurs réelles. Plus précisément, on s'intéresse au cas de $\Lambda \in L^\infty(\mathbb{R}^2)$, $F \in W^{-1,\infty}_{\ell oc}(\mathbb{R}^2)$. Il est bien connu que le système (1) - même sous forme conservative - n'aura pas de solutions distributions en général, voir [2]. Nous étudions le système (1) dans l'algèbre de Colombeau ([1]). On montre que ce système admet des solutions uniques dans $G(\mathbb{R}^2)$ pour des données initiales quelconques $A \in G(\mathbb{R})$, si Λ, F, G sont des éléments de $G(\mathbb{R}^2)$ satisfaisant à certaines conditions de croissance. Dans le cas où $\Lambda \in L^\infty(\mathbb{R}^2)$, $F \in W^{-1,\infty}_{\ell oc}(\mathbb{R}^2)$ on trouve des éléments de $G(\mathbb{R}^2)$ qui satisfont à ces conditions et sont associés (voir [1]) à Λ et F.

La seconde partie de cette note est consacrée à l'application de ces résultats à un problème de l'acoustique linéaire dans un milieu discontinu. On considère le système suivant:

$$(2)\qquad \begin{aligned} \partial_t\rho + \rho_0(x)\partial_x u &= 0 \\ \rho_0(x)\partial_t u + \partial_x p &= 0 \qquad \text{pour } (x,t) \in \mathbb{R}^2 \\ p &= c_0^2(x)\rho \end{aligned}$$

Cet système décrit la propagation des ondes acoustiques en variables de Lagrange; ρ_0 est la densité, c_0 la vitesse du son dans le milieu au repos. Les quantités acoustiques u, p, ρ sont une approximation du premièr ordre aux perturbations de la vitesse, pression et densité, voir Poirée ([5], [6]). Nous supposons que les fonctions ρ_0, c_0 sont strictement positives et constantes dans les demi-plans $\{x < 0\}$ et $\{x > 0\}$. Les quantités (u,p,ρ) sont connues en un point $x_0 < 0$ pour tout $t \in \mathbb{R}$. On cherche les valeurs de (u,p,ρ) globalement et notamment au voisinage de la discontinuité en $x = 0$. Supposons que les données $(\overline{u}(t),\overline{p}(t),\overline{\rho}(t))$ de (u,p,ρ) en point x_0 sont continuement diffèrentiables. Alors, on peut trouver une solution classique sur $\{x < 0\}$ et $\{x > 0\}$, si on impose une condition de passage en $x = 0$. La condition physiqement correcte est la continuité de u et p, voir ([5], [6]). Nous appellerons "solution classique par raccordement" la solution obtenue de cette façon.

On outre, si les données $(\bar{u},\bar{p},\bar{\rho})$ sont des fonctions généralisées appartenant à $G(\mathbb{R})$, nous verrons qu'il existe une solution unique $(u,p,\rho) \in G(\mathbb{R}^2)$ du problème (2), sans que nous ayons besoin des conditions de passage. En effet, on montre que la solution $(u,p,\rho) \in G(\mathbb{R}^2)$ admet une distribution associée qui est égale à la solution classique par raccordement. Cette assertion est déduite des résultats d'existence et d'unicité dans l'espace de Sobolev $H^1_{\ell oc}(\mathbb{R}^2)$ pour le système

$$(3) \qquad \begin{aligned} c_0(x)^{-2}\,\rho_0(x)^{-1}\,\partial_t p + \partial_x u &= 0 \\ \rho_0(x)\,\partial_t u + \partial_x p &= 0 \end{aligned}$$

qui équivaut au système (2) au sens de $G(\mathbb{R}^2)$.

Nous utilisons la notion de l'algèbre $G(\mathbb{R}^m)$ de Colombeau et la notion de solution dans $G(\mathbb{R}^m)$ comme définie dans [3]. Les details se trouvent dans [4].

<u>1. EXISTENCE ET UNICITÉ</u> - Soit $u \in E_M[\mathbb{R}^m]$. Considérons les propriétés suivantes:

(4) Il existe $N \in \mathbb{N}$ tel que pour toute $\varphi \in A_N(\mathbb{R}^m)$ il y a $C > 0$, $\eta > 0$ tels que $\sup_{y \in \mathbb{R}^m} |u(\varphi_\varepsilon, y)| \leq C$ pour $0 < \varepsilon < \eta$.

(5) Pour toute partie compacte $K \subset \mathbb{R}^m$ il existe $N \in \mathbb{N}$ tel que pour toute $\varphi \in A_N(\mathbb{R}^m)$ il y a $C > 0$, $\eta > 0$ tels que $\sup_{y \in K} |u(\varphi_\varepsilon, y)| \leq N \log \frac{C}{\varepsilon}$ pour $0 < \varepsilon < \eta$.

Une fonction généralisée $U \in G(\mathbb{R}^m)$ admetant un représentant u avec la propriété (4) (respectivement (5)) est dite globalement bornée (respectivement à croissance logarithmique locale).

Théorème 1 - Supposons que Λ, F, G appartiennent à $\mathcal{G}(\mathbb{R}^2)$, que Λ soit globalement borné et à valeurs réelles, et que F et $\partial_x \Lambda$ soient à croissance logarithmique locale. Alors, pour toute fonction généralisée $A \in \mathcal{G}(\mathbb{R})$, le système (1) admet une solution généralisée unique $V \in \mathcal{G}(\mathbb{R}^2)$.

Pour prouver le théorème, on utilise des arguments similaires à [3, Prop. 2]. La borne globale de Λ implique que les courbes caractéristiques existent globalement, la croissance logarithmique entre dans un argument du type Gronwall.

Remarquons que si les conditions du Théorème 1 ne sont pas remplies, l'existence ou bien l'unicité n'est pas valable en général. Pour obtenir la solution généralisée dans le cas où $\Lambda \in L^\infty(\mathbb{R}^2)$, $F \in W^{-1,\infty}_{\ell oc}(\mathbb{R}^2)$ on utilise le résultat suivant.

Proposition - (a) Soit $w \in W^{-1,\infty}_{\ell oc}(\mathbb{R}^m)$. Alors il existe $U \in \mathcal{G}(\mathbb{R}^m)$ telle que U est associée à w et U est à croissance logarithmique locale.

(b) Soit $w \in L^\infty(\mathbb{R}^m)$. Alors il existe $U \in \mathcal{G}(\mathbb{R}^m)$ telle que U est associée à w, U est globalement bornée, et $\partial^\alpha U$ est à croissance logarithmique pour $|\alpha| = 1$.

2. UNE APPLICATION À L'ACOUSTIQUE LINÈAIRE - Supposons que ρ_0, c_0 sont des fonctions strictement positives, constantes sur les demi-plans $\{x < 0\}$ et $\{x > 0\}$. Alors, on peut trouver des fonctions généralisées dans $\mathcal{G}(\mathbb{R}^2)$, qui sont associées à ρ_0 et

c_0 , et qui sont inversibles dans l'algèbre $G(\mathbb{R}^2)$. Il résulte que le système (2) est équivalent au système (3) au sens de $G(\mathbb{R}^2)$, si on prend en compte $\rho = c_0^{-2} p$. Mais le système (3) peut être transformé en forme diagonale de sorte que les hypothèses du Théorème 1 sont remplies à condition de permuter x et t . D'où le résultat suivant.

Théorème 2 - Soit ρ_0, $c_0 \in G(\mathbb{R}^2)$ comme ci dessus; soit $x_0 < 0$ et $(\bar{u},\bar{p},\bar{\rho}) \in G(\mathbb{R})$ avec $\bar{p} = c_0^2(x_0)\,\bar{\rho}$. Alors, le système (2) admet une solution unique $(u,p,\rho) \in G(\mathbb{R}^2)$ qui coincide avec $(\bar{u},\bar{p},\bar{\rho})$ en point x_0. Au delà, (u,p) est aussi la solution unique de (3).

Supposons maintenant que $(\bar{u},\bar{p})$ appartient à l'espace de Sobolev $W^{1,\infty}_{\ell oc}(\mathbb{R})$. La fonction $(\bar{u},\bar{p})$ peut être interprétée comme élément de $G(\mathbb{R})$, défini par l'inclusion canonique de $\mathcal{D}'(\mathbb{R})$ dans $G(\mathbb{R})$, voir [1]. Evidement la forme du système (3) permet de définir des solutions faibles au sens des distributions. Ces solutions sont uniques. Pour obtenir le résultat de coherence suivant, on construit des solutions régularisées et on utilise un argument de compacité.

Théorème 3 - Soit $(\bar{u},\bar{p}) \in W^{1,\infty}_{\ell oc}(\mathbb{R})$. Alors:

(a) Le système (3) admet une solution $(u_1,p_1) \in H^1_{\ell oc}(\mathbb{R}^2)$, dont le trace à $x = x_0$ est égale à $(\bar{u},\bar{p})$.

(b) La solution $(u,p) \in G(\mathbb{R}^2)$ du système (3) construite dans le Théorème 2 admet (u_1,p_1) comme distribution associée.

Corollaire - Soit $(\bar{u},\bar{p},\bar{\rho}) \in C^1(\mathbb{R})$ avec $\bar{p} = c_0^2(x_0)\,\bar{\rho}$. Alors la solution $(u,p,\rho) \in G(\mathbb{R}^2)$ de (2) construite dans le Théorème 2 est associée à la solution classique par raccordement.

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EMPIRICAL SADDLEPOINT CONVERGENCE

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ABSTRACT. The uniform consistency, moment structure, and weak convergence to normality of the empirical moment generating function and empirical cumulant generating function and also of the arbitrary derivatives of these processes is established and used to investigate the properties of the saddlepoint approximation in the case that the required cumulant generating function is obtained empirically.

1. INTRODUCTION. If $X_1, X_2, \cdots, X_n$ are iid with density $f(x)$, moment generating function $M(t) = \int e^{tx} f(x)\,dx$ assumed finite in an interval I about the origin, and cumulant generating function $K(t) = \log M(t)$, then the saddlepoint approximation (see Daniels, 1954, 1980; Reid, 1988) for the density of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is given by

$$f_n(x) = \left(\frac{n}{2\pi K''(t)}\right)^{1/2} \exp\left[n\{K(t) - t\cdot x\}\right] \tag{1}$$

where $t = t(x)$ is the unique real root of $K'(t) = x$. Our object is to study the consequence of replacing $K(t)$ in (1) by its sample version $K_n(t) = \log M_n(t)$ where $M_n(t) = \frac{1}{n}\sum_{i=1}^{n} e^{tX_i}$. The importance of this modification stems from the fact that the analytic form of $M(t)$ often is not tractable; a similar situation arises also when $f(x)$ itself is not available, but where a sample may be obtained. Another important application is given in Davison and Hinkley (1988) where saddlepoint approximations are applied in the context of the bootstrap and other resampling schemes.

2. MAIN RESULTS. In order to study the consequences of replacing $K(t)$ in (1) by $K_n(t)$ it is necessary to understand the sampling properties of the transforms $M_n(t)$ and $K_n(t)$. Letting D^ℓ denote differentiation applied ℓ times we have:

(A) $\sup_{a\le t\le b} |D^\ell M_n(t) - D^\ell M(t)| \to 0$ a.s., $\ell = 0,1,2,\cdots$;

(B) $\sup_{a\le t\le b} |D^\ell K_n(t) - D^\ell K(t)| \to 0$ a.s., $\ell = 0,1,2,\cdots$;

(C) $E D^\ell M_n(t) = D^\ell M(t), \quad \ell = 0,1,2,\cdots$;

(D) $n\cdot\text{cov}(D^\alpha M_n(s), D^\beta M_n(t)) = D^{\alpha+\beta}M(s+t) - D^\alpha M(s)\cdot D^\beta M(t)$

for $s, t, s+t \in I$, and integers $\alpha \ge 0$, $\beta \ge 0$; and

(E) If $Y_n(t) = \sqrt{n}\,(M_n(t) - M(t))$ and $Z_n(t) = \sqrt{n}\,(K_n(t) - K(t))$ then $D^\ell Y_n(t)$ and $D^\ell Z_n(t)$ converge weakly, in the space of continuous functions on $[a, b]$ under the supremum norm, for $\ell = 0,1,2,\cdots$ to zero mean Gaussian processes having covariance structures respectively given by (D) and by

$$\underline{\text{asymp cov}}\,(D^\alpha Z_n(t), D^\beta Z_n(t)) = D_s^\alpha D_t^\beta \left[\frac{M(s+t)}{M(s)M(t)} - 1 \right]$$

where $\alpha \ge 0$, $\beta \ge 0$ are integers and subscripts on D denote variables of partial differentiation.

In (A) and (B) we require $a, b \in I$, and for $\ell > 0$ we require a, b to be interior. In (E) we require $[a, b] \subseteq I/2$, and for $\ell > 0$ we require a,b to be interior. The proof of (A) and (B) involves the strong law of large numbers and convexity, while a proof for (E) may be based on Theorem 12.3 of Billingsley (1968) and Taylor expansion arguments. Related results in a different context are given in Ghosh (1987). The proofs of (A) - (E) will be given elsewhere.

Now let $\hat{f}_n(x)$ denote the ***empirical saddlepoint approximation*** given by expression (1) except with K_n replacing K and $\hat{t}$ replacing t throughout, where $\hat{t} = \hat{t}(x)$ is defined by $K_n'(\hat{t}) = x$; let $g_n(x)$ denote the saddlepoint approximation for the normalized variable $\sqrt{n}\cdot(\overline{X} - \mu)$ where $\mu = EX$; and let $\hat{g}_n(x)$ denote the correspondingly normalized empirical saddlepoint approximation, but centered now at $\overline{X}$ instead of μ. Then we have

$$\frac{\hat{g}_n(x)}{g_n(x)} = 1 + O_P(\frac{1}{\sqrt{n}}) \tag{2}$$

where the error term is best possible in powers of n, and uniform over finite intervals.

To see this note that

$$g_n(x) = n^{-1/2} f_n(t(\mu + n^{-1/2}x)) \quad \text{and} \quad \hat{g}_n(x) = n^{-1/2}\hat{f}_n(\hat{t}(\overline{X} + n^{-1/2}x))$$

so that

$$\frac{\hat{g}_n(x)}{g_n(x)} = \left(\frac{K''(t)}{K_n''(\hat{t})}\right)^{1/2} \exp\left[n\left\{(K_n(\hat{t}) - K(t) - \overline{X}\hat{t} + \mu t) - (\hat{t} - t)\cdot\frac{x}{\sqrt{n}}\right\}\right] \tag{3}$$

where now $K'(t) = \mu + n^{-1/2}x$ and $K_n'(\hat{t}) = \overline{X} + n^{-1/2}x$. The fractional term on the right in (3) is $1 + O_P(n^{-1/2})$, and since $K(t) = \mu t + ½\sigma^2 t^2 + O(n^{-3/2})$ and $K_n(\hat{t}) = \overline{X}\hat{t} + ½S^2\hat{t}^2 + O_P(n^{-3/2})$ where σ^2, S^2 are the variances of the actual and empirical distributions, the exponent in (3) equals

$$n\left[\frac{1}{2}(S^2\hat{t}^2 - \sigma^2 t^2) - (\hat{t} - t)\frac{x}{\sqrt{n}}\right] + O_P(\frac{1}{\sqrt{n}}). \tag{4}$$

But since $\mu + n^{-1/2}x = K'(t) = \mu + \sigma^2 t + O(n^{-1})$ and $\overline{X} + n^{-1/2}x = K_n'(\hat{t}) = \overline{X} + S^2\hat{t} + O_P(n^{-1})$ then $t = O(n^{-1/2})$ and $\hat{t} = O_P(n^{-1/2})$ and therefore we find, in turn, $S^2\hat{t} - \sigma^2 t = O_P(n^{-1})$, $\hat{t} - t \equiv \sigma^{-2}[(S^2\hat{t} - \sigma^2 t) + (\sigma^2 - S^2)\hat{t}] = O_P(n^{-1})$ and

$S^2\hat{t}^2 - \sigma^2 t^2 = O_P(n^{-3/2})$. Consequently the exponent term (4) is $O_P(n^{-1/2})$. The normalization is required for (2) to hold, while studentizing does not improve the order of convergence. For the nonnormalized case we state the result

$$\frac{\hat{f}_{m,n}(x)}{f_n(x)} = 1 + O_P\left(\frac{n}{\sqrt{m}}\right) \tag{5}$$

where $\hat{f}_{m,n}(x)$ dnotes the saddlepoint approximation $f_n(x)$ of (1) but based now on the sample cumulant function $K_m(t)$ from a sample of size m. The error term is uniform over any interval of x-values corresponding to an interval of t-values interior to the domain on which M(t) is finite. The case $n = 1$ in (5) in conjunction with higher terms in the saddlepoint approximation leads to some interesting new possibilities for nonparametric density estimation. A fuller analysis (Feuerverger, 1988) will be given elsewhere.

Essentially similar analyses may be carried out for empirical versions of the tail area saddlepoint approximation (Lugannani and Rice, 1980; Daniels, 1987), for Edgeworth expansions (Feller, 1971, Theorem 2, page 535; Barndorff-Nielsen and Cox, 1979), and also for quantities such as the Chernoff index $\inf_z e^{-z(t+\mu)}M(z)$ for large deviation probabilities (Serfling, 1980, chapter 10).

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ELEMENTARY HOLOGRAMS AND 3-ORBIFOLDS

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Presented by G. de B. Robinson, F.R.S.C.

Abstract. A p-orbifold |1| is a topological space with a local modelling on (open set in R^p)/(finite group action), $p \geq 1$. The Euclidean orientable 3-orbifolds of crystallographic grids associated with elementary planar holograms are identified. As applications, the optical phase conjugation by degenerate four-wave mixing, and in the vein of synergetics also various neurophysiological realizations of planar holographic grids are considered.

1. Elementary Planar Holograms

Let U_1 denote the linear Schrödinger representation |8| of the three-dimensional real Heisenberg nilpotent Lie group $A(\mathbf{R})$ with one-dimensional center Z. Suppose that the holographic plane $A(\mathbf{R})/Z$ has been identified with the real plane $\mathbf{R} \oplus \mathbf{R}$ or the complex plane $\mathbf{C}$. Then for all pairs (x,y) of the holographic plane $\mathbf{R} \oplus \mathbf{R}$ the holographic Fourier transform $(x,y) \rightsquigarrow H(u,v;x,y)$ is given by the coefficient function of U_1:

$$H(u,v;x,y) = \langle U_1(x,y,0)u|v\rangle ,$$

where u, v belonging to $L^2(\mathbf{R})$ are the waves of the two coherent writing beams, x = Re z denotes their path difference (off-axis holography), and y = Im z their phase difference |12| which both are simultaneously 'frozen' in the intensity-interference pattern forming the planar hologram. Notice that $H(u,v;.,.) \in L^2(\mathbf{R} \oplus \mathbf{R})$ does merely depend upon the equivalence class of U_1, i.e., upon the element of the unitary dual of $A(\mathbf{R})$ which is represented by U_1.

Let H_n denote the harmonic oscillator wave function of degree $n \geqq 0$. The image of the holographic plane $A(\mathbf{R})/Z$ under the holographic Fourier transform $z \longrightarrow H(H_m, H_n; z)$ is called an elementary planar hologram. Explicitly we have for all $z = x+iy \in \mathbf{C}$ the identity

$$H(H_m, H_n; z) = \sqrt{\frac{n!}{m!}}(\sqrt{\pi} z)^{m-n} L_n^{(m-n)}(\pi |z|^2) \quad (m \geqq n \geqq 0)$$

where $L_n^{(\alpha)}$ denotes the Laguerre function of degree n and order α. Based on the preceding formula, the coaxial coupling coefficients of quantized transverse eigenmodes have been calculated by symplectic transformations of the holographic plane $A(\mathbf{R})/Z$ to the coupling plane |11|, |13|.

2. The Weyl Filter

Form the complex Hilbert space $L^2(\mathbf{R} \oplus \mathbf{R})$ with respect to Lebesgue measure of $\mathbf{R} \oplus \mathbf{R}$. The square integrability of the linear Schrödinger representation U_1 of $A(\mathbf{R})$ mod Z implies that the family $\{H(H_m, H_n; .,.) \mid m \geqq 0, n \geqq 0\}$ forms a Hilbert basis of $L^2(\mathbf{R} \oplus \mathbf{R})$. It follows from the Stone-von Neumann-Segal theorem |15| that the mapping

$$u \otimes v \longrightarrow H(u, v; .,.)$$

extends to an isometric isomorphism of $L^2(\mathbf{R} \oplus \mathbf{R})$ onto the complex Hilbert space of Hilbert-Schmidt operators with kernel functions $k_f \in L^2(\mathbf{R} \oplus \mathbf{R})$ acting on $L^2(\mathbf{R})$. For $f \in L^2(\mathbf{R} \oplus \mathbf{R})$ define $sf: (x,y) \longrightarrow f(x-y, y)$. Then the kernel function of the Weyl filter acting on $L^2(\mathbf{R})$ is given by

$$k_f = s\bar{\mathscr{F}}_2 f$$

where $\bar{\mathscr{F}}_2$ denotes the Fourier cotransform with respect to the second variable.

3. The 3-Orbifold of Holographic Grids

Let $L_n = L_n^{(0)}$ for short, and let $\mathbf{Z}|i|$ denote the grid in $\mathbf{C}$ formed by the Gaussian integers. For $m \geqq n \geqq 0$ we infer from harmonic analysis on the compact Heisenberg nilmanifold (a circle bundle over the compact two-dimensional torus $\mathbf{T}^2$ |8|)

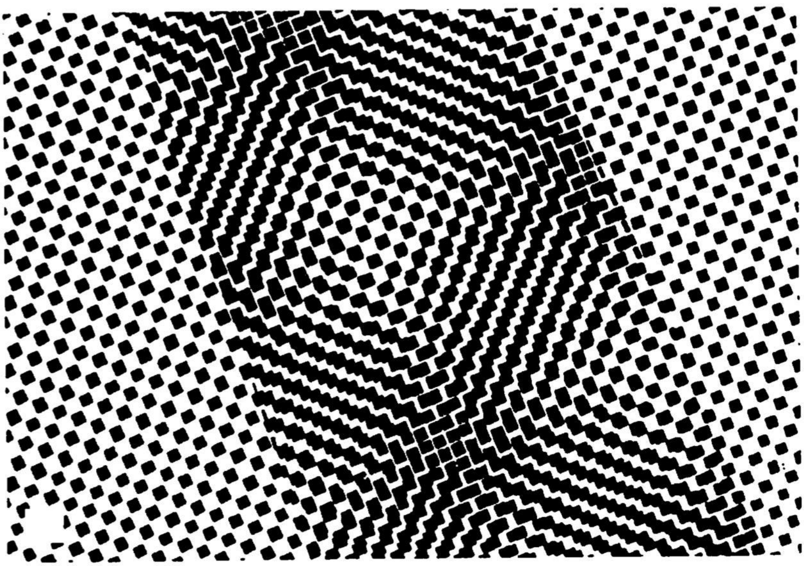
the identities between signal terms and interference cross terms

$$\sum_{\mu \in Z|i|} L_m(\pi|\mu|^2)L_n(\pi|\mu|^2)=\frac{n!}{m!}\,\pi^{m-n}\sum_{\mu \in Z|i|}(|\mu|^2)^{m-n}(L_n^{(m-n)}(\pi|\mu|^2))^2.$$

The cyclic groups $C_k = Z/kZ$, $k\in\{1,2,3,4,6\}$ (crystallographic condition) define as subgroups of the compact torus group **T** the orientation preserving invariants of the preceding identity |10|. It follows

Theorem. There are five Euclidean orientable 3-orbifolds of planar holographic grids which correspond to the orientation preserving crystallographic groups C_k, $k\in\{1,2,3,4,6\}$. All of them are $\mathbf{T}^2$-bundles over **T** and the monodromies are the identity and periodic homeomorphisms of period 2, 3, 4, and 6, respectively.

The following figure illustrates the coherent superposition of two elements of the Euclidean 3-orbifold of square holographic grids (k=2) in the holographic plane **R** ⊕ **R**.



4. Optical Phase Conjugation

Firstly observe that the planar holographic grids have the dihedral Coxeter groups D_k ($k \in \{1,2,3,4,6\}$; Schoenflies notation) of order 2k as their full symmetry groups and the cyclic groups C_k as index 2 subgroups of orientation-preserving isometries fixing the crystallographic grid. A reflection through an axis of mirror in the holographic plane **C** sends as an orientation-reversing involution the corresponding Euclidean 3-orbifold into the spherical tangent bundle of the 3-orbifold under its opposite orientation.

The classical Fourier inversion formula $\mathcal{F} \circ \bar{\mathcal{F}} = id_{L^2(\mathbf{R})}$ shows that the inverse of the Fourier cotransform is obtained by a time reversal procedure. The inverse of the isometric isomorphism $f \rightarrow k_f$ is given by the mapping $\mathcal{F}_2 s^{-1}$. In the conjugation invariant geometry of degenerate four-wave mixing (Feinberg |3|), the Weyl filter generates without losses or gains the phase conjugate of the antiparallel reading beam which is scattered from the planar holographic grid.

One of the most fascinating and useful applications of holography is that optical phase conjugate mirrors can undistort a distorted image provided the scatterer is linear and non-absorbing. Stated somewhat differently, under the above conditions any distortions introduced by the scatterer are completely eliminated by the "healing effect" of optical phase conjugation.

For a group theoretical treatment of optical third harmonic generation, which can be regarded as a special case of four-wave mixing, see |9|.

5. Holographic Grids in Neurophysiology

Finally, let us point out the following instances where planar holographic grids occur in the retino-cortical channel transmitting visual information to the cerebral cortex in pulse code modulation form:

(i) The hexagonal grid in the extra-foveal retina of primates (see Koenderink-van Dorn |5|, Kronauer-Zeevi |6|, and Rodieck |7|). The non-uniformity can be explained by the logarithmic transformation of the visual field into the cortex.

(ii) The hexagonal presynaptic vesicular grid of the cleft of LGN (corpus geniculatum laterale) synapses forms a "paracrystalline" plate which acts probabilistically in quantal release (see Eccles |2|).

(iii) The hexagonal excitation patterns in the area 17 of the visual cortex (cf. von Seelen |16|) which have been discovered by Hubel and Wiesel in their microelectrode studies of the striate cortex.

(iv) The hexagonal symmetry of the patterns underlying visual hyperacuity (cf. Hirsch-Hylton |4|).

These experimental results combined with the preceding theoretical results support the holographic as well as the quantum hypothesis of neurophysiology |12|. It follows that harmonic analysis on the Heisenberg nilpotent Lie group $A(\mathbf{R})$ which underlies the Heisenberg uncertainty principle and the superposition principle of quantum mechanics, is at the basis of neuroscience, too |14|.

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ON THE BOMBIERI-VINOGRADOV THEOREM

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In this note we announce some recent results obtained in joint work with E. Bombieri and H. Iwaniec. Details of proof will appear elsewhere [5]. For ease of exposition we do not state the results in their most general form.

We denote as usual by $\pi(x;q,a)$ the number of primes not exceeding x and in the residue class a modulo q, that is

$$\pi(x;q,a) = \sum_{\substack{p\le x \\ p\equiv a (\mathrm{mod}\ q)}} 1 .$$

We let $\mathrm{li}(x) = \int_2^x dt/\log t$ and $L = \log x$.

We are interested in estimates for $\pi(x;q,a)$ on average with respect to q. The prototype for this is

Theorem 1: (Bombieri [2], Vinogradov [10]). Let $a\neq 0$ be an integer and let A and ϵ be positive reals. Then, provided that $Q < X^{\frac{1}{2}-\epsilon}$ we have

$$\sum_{\substack{Q<q\le 2Q \\ (q,a)=1}} \left| \pi(x;q,a) - \frac{\mathrm{li}(x)}{\phi(q)} \right| \ll xL^{-A}$$

where the constant implied in the symbol << may depend on A and ϵ.

We remark that in fact the Theorem of Bombieri-Vinogradov is stronger than that stated above and especially that, in the form given by Bombieri [2], the restriction $Q < x^{\frac{1}{2}-\epsilon}$ may be weakened to $Q < x^{\frac{1}{2}}L^{-B(A)}$ for some constant B depending on A.

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Theorem 1 has many applications and for most of these it would be of great interest to ease the restriction on the size of Q. Despite the conjecture of Elliott and Halberstam [6] that Theorem 1 should hold for $Q < x^{1-\epsilon}$, the most important progress [9,8,7,3] has dealt with the ingenious devising of ways (largely due to Iwaniec) to avoid the need, at least for the sake of most applications, to improve Theorem 1 directly.

Nevertheless with Bombieri and Iwaniec we were able [4] to move a little beyond $x^{\frac{1}{2}}$ and prove

Theorem 2. Let $a \neq 0$, $3 < y < x^{\frac{1}{4}}$, $Q < (xy)^{\frac{1}{2}}$. For some B we have

$$\sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \left| \pi(x;q,a) - \frac{\mathrm{li}(x)}{\phi(q)} \right| \ll \frac{x}{L} \left(\frac{\log y}{L} \right)^2 \log^B L.$$

Remark: This is non trivial in a range $Q < x^{\frac{1}{2}}\exp(L/\log^B L)$.

More recently, again with Bombieri and Iwaniec, we improved this to

Theorem 3. Let $a \neq 0$, $Q = x^\theta$. Then

$$\sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \left| \pi(x;q,a) - \frac{\mathrm{li}(x)}{\phi(q)} \right| \leq K(\theta - \tfrac{1}{2})^2 \frac{x}{L} + K^* \frac{x}{L^3} (\log L)^2,$$

where K is absolute and K^* depends on a.

This gives the expected asymptotic formula

$$\pi(x;q,a) \sim \frac{\mathrm{li}(x)}{\phi(q)}$$

for 'most' q with $(q,a) = 1$, $q < x^{\frac{1}{2}}h(x)$ for any function h(x) satisfying $\log h(x) = o(L)$. This wider range just misses by the narrowest possible margin giving the asymptotics (on average) for a fixed $\theta > \frac{1}{2}$. Moreover, unlike Theorem 2, the new result is non-trivial for $\theta > \frac{1}{2}$ fixed, giving (upper and lower) Brun-Titchmarsh average estimates which, for θ near $\frac{1}{2}$, improve earlier results and which as $\theta \to \frac{1}{2}$ approach the expected result. Here we are not able however to get bounds on the exceptional set of q and by most q in (Q,2Q] we mean apart from a set of q for which $\sum_q \frac{1}{\phi(q)} = o(1)$.

In conclusion we mention a Corollary of Theorem 3 which follows from the argument of Balog [1].

<u>Corollary</u>. <u>Let $a \neq 0$ and</u> $\alpha > \frac{1}{2\sqrt{e}} + \epsilon$. <u>Then for</u> $\gg \frac{x}{L^2}$ <u>primes $p \leq x$, all prime factors of p-a are</u> $\leq p^{\alpha}$.

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