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ON ANTISYMMETRIC BI-ADDITIVE FUNCTIONS AND AN INTERESTING SYSTEM OF FUNCTIONAL EQUATIONS

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Presented by J. Aczél, F.R.S.C.

E. Vincze [3] asked, at the Tenth International Symposium on Functional Equations in 1972, whether the only function $H:R^2 \to R$ (R the reals) satisfying the system

$$(1)\qquad H(x+y,z) + H(x+z,y) + H(y+z,x) = 0,$$

$$(2)\qquad H(x,y) = -H(y,x),$$

$$(3)\qquad H(-x,y) = -H(x,y) = H(x,-y),$$

for all $x,y,z \in R$, is the zero function. In a remark at that same meeting, J. Rätz [2] provided an example which answered Vincze's question in the negative. Let $B = \{b_i \mid i \in I\}$ be a Hamel basis for R over Q (the rationals), with index set I ordered by <. The example provided by Rätz is the function $H:R^2 \to R$ defined by

$$H(x,y) = \sum_{\substack{i<j \\ i,j\in I}} (r_i s_j - r_j s_i) \quad [x = \sum_i r_i b_i,\ y = \sum_j s_j b_j].$$

(Although I is uncountable, all sums have only a finite number of nonzero terms; all $r_i, s_j \in Q$.) One can easily check that other examples of solutions of the system (1)-(3) are provided by maps H of the form

$$(4)\qquad H(x,y) = \sum_{i,j} a_{ij} r_i s_j \quad [a_{ij} = -a_{ji},\ x = \sum_i r_i b_i,\ y = \sum_j s_j b_j].$$

(Here the a_{ij} are elements of an arbitrary antisymmetric matrix, $a_{ij} = H(b_i, b_j)$; $r_i, s_j \varepsilon Q$.) As shown in [1: Corollary 1], this is the general form of any antisymmetric bi-additive function H. Clearly, Rätz' example is a special case of (4). One goal of this note is to show that the general solution of (1)-(3) is (4).

We begin by finding the general solution of (1) by itself. From this we determine the general solutions of subsystems of (1)-(3) which include (1). It is shown that (2) and parts of (3) are redundant. For this reason, we "split" (3):

(3) (a) $H(-x, y) = -H(x, y)$, (b) $H(x, -y) = -H(x, y)$,

(c) $H(x, -y) = H(-x, y)$,

for all $x, y \varepsilon R$.

Proposition 1. *The general solution of (1) is given by*

$$H(x, y) = F(x+y, y), \qquad (x, y) \varepsilon R^2, \tag{5}$$

where $F: R^2 \to R$ satisfies the conditions

$$y \to F(x, y) - F(x, 0) \text{ additive for each fixed } x \varepsilon R, \tag{6a}$$

$$F(x, x) = -2F(x, 0), \qquad x \varepsilon R. \tag{6b}$$

Proof: Suppose H satisfies (1). We define $F: R^2 \to R$ by $F(u, v) := H(u-v, v)$ on R^2, so (5) holds automatically. Substituti (5) into (1) gives

$$F(x+y+z, z) + F(x+y+z, y) + F(x+y+z, x) = 0 \tag{7}$$

for all $x, y, z \varepsilon R$. Letting $t = x+y+z$, we can express (7) as $F(t, y) + F(t, x) = -F(t, t-(x+y))$. So, if $u+v = x+y$, then $F(t, y) + F(t, x) = F(t, v) + F(t, u)$; in particular, if $u = x+y$ and $v = 0$, then

$$F(t,x) + F(t,y) = F(t,x+y) + F(t,0),$$

for all $t,x,y\in R$, which is condition (6a). Using (6a) twice on (7), we get $F(x+y+z, x+y+z) + 2F(x+y+z,0) = 0$ for all $x,y,z\in R$, which is condition (6b).

Conversely, if $F:R^2 \to R$ is any function satisfying (6a) and (6b), then the map H defined by (5) satisfies (1). This finishes the proof.

Remark 1. An alternate description of the general solution of (1) is $H(x,y) = G(x+y, 2y-x)$, where $G(x,y) := [F(x,y)-F(x,0)]/3$ defines G additive in its second variable.

Proposition 2. A map $H:R^2 \to R$ satisfies (1), (2) if and only if (5) holds with F satisfying

(8a) $F(x,\cdot)$ additive on R for each fixed $x\in R$,

(8b) $F(x,x) = 0$ $x\in R$.

Proof. Clearly, H defined by (5), with F satisfying (8a), (8b), is a solution of (1), (2).

Conversely, we have (5) with (6a) and (6b) by Proposition 1. Putting $z = 0$ in (1), we get $H(x+y,0) + H(x,y) + H(y,x) = 0$, which by (2) implies $H(x+y,0) = 0$ for all $x,y\in R$. Thus (6a) and (6b) become (8a) and (8b), respectively, and the proof is complete.

Proposition 3. The following are equivalent:

(i) H satisfies (1) and (3a),

(ii) H satisfies (1) and (3b),

(iii) H is given by (4).

Proof. We show (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii).

Part I [(iii) $\Rightarrow$ (i), (ii)]: Any H given by (4) is antisymmetric and bi-additive, so it satisfies (1)-(3).

Part II [(i) $\Rightarrow$ (iii)]: If H satisfies (1) and (3a), then (5) holds with (6a) and (6b) by Proposition 1. We shall use the facts (implied by (6a)) that

$$F(x,w-y) = F(x,w) - F(x,y) + F(x,0) \qquad x,w,y\in R, \tag{9}$$

$$F(x,t/2) = \tfrac{1}{2}[F(x,t) + F(x,0)], \qquad x,t\in R. \tag{10}$$

By (5) and (3a) we have $F(-x+y,y) = -F(x+y,y)$. Letting here $u = y-x$, $v = x+y$ (so that $y = \tfrac{1}{2}(u+v)$, $x = \tfrac{1}{2}(v-u)$) and expanding the resulting equation using (10) and (9), we get

$$F(u,u) + F(u,v) = -[F(v,u) + F(v,v)], \qquad u,v\in R. \tag{11}$$

Putting $v = u$, this implies $F(u,u) = -F(u,u) = 0$, so then (11) gives $F(u,v) = -F(v,u)$, i.e. F is antisymmetric. Now, $F(x,x) = 0$ and (6b) imply $F(x,0) = 0$, which by (6a) means that F is additive in its second variable. Together with antisymmetry, this gives the additivity of F also in its first variable. Finally, (5) now yields $H(x,y) = F(x+y,y) = F(x,y) + F(y,y) = F(x,y)$, showing that H is antisymmetric and bi-additive, and we're done.

Part III [(ii) $\Rightarrow$ (iii): If H satisfies (1) and (3b), then we have, by (5), $F(x-y,-y) = -F(x+y,y)$. Now, letting $u = x-y$, $v = x+y$, and using (10), (9) to expand the resulting equation, we get $F(u,u) - F(u,v) + 2F(u,0) = -[F(v,v)-F(v,u) + 2F(v,0)]$. By (6b), then, $-F(u,v) = F(v,u)$, i.e. F is again antisymmetric. The remainder of the proof is as in Part II, and we are finished.

Corollary. *H satisfies (1)-(3) if and only if H is given by (4)* (i.e., *H is antisymmetric and bi-additive*).

A remaining question is whether (1) and (3c) are sufficient to guarantee (4). The (negative) answer is provided by the following.

Proposition 4. $H:R^2 \to R$ *satisfies (1) and (3c) if and only if H has the form*

$$H(x,y) = G(x+y,\, 2y-x), \qquad x, y \in R, \tag{12}$$

for some map $G:R^2 \to R$ *which is additive in its second variable and satisfies*

$$G(u,v) = G(-u,-v), \qquad u, v \in R. \tag{13}$$

Proof. Certainly H given by (12), with G as described, satisfies (1) and (3c). Conversely, suppose H satisfies (1) and (3c). By Proposition 1 and Remark 1 following it, it suffices to show that the map G defined by (12) satisfies (13). But (3c) and (12) yield immediately $G(x-y, -2y-x) = G(-x+y, 2y+x)$, and with $u = x-y$, $v = -2y-x$, this is (13).

Remark 2. Such maps G as described in Proposition 4 can be constructed as follows. *Step 1*: For each $x>0$, choose an additive function $G(x,\cdot)$. *Step 2*: Define $G(0,y) := 0$ for all $y \in R$. *Step 3*: For $x<0$, define $G(x,y) := G(-x,-y)$ for all $y \in R$.

Remark 3. All propositions are correct also if $H:X^2 \to Y$, where X and Y are arbitrary abelian groups in which division by 2 and by 3 is possible, if the statement labeled (4) is replaced by the statement that H is antisymmetric and bi-additive.

Regular (in two variables) solutions of these systems can be determined by standard results on additive functions. For Propositions 1-4, respectively, they are: $H(x,y) = (2y-x)\,f(x+y)$; $H(x,y) = cy$ if $y = -x$, 0 otherwise; $H = 0$; and $H(x,y) = (2y-x)\,f(x+y)$ for odd f; where $f:R \to R$ is, except for regularity inherited from H, otherwise arbitrary.

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On Bernoulli Numbers, I

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1. Introduction. Let B_m ($m = 0, 1, 2, \ldots$) be the Bernoulli numbers defined by the formal power series expansion $x/(e^x - 1) = \sum_{m=0}^{\infty}(B_m/m!)x^m$, that is, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0, \ldots$.

It is easy to show that $B_m = 0$ if and only if $m \geq 3$ is odd. Also the non-zero Bernoulli numbers alternate in sign.

As a relation between Bernoulli numbers, it is well-known that if $l \geq 1$, then (*Euler's formula*; see, e.g., [2])

$$(1.\ 1) \qquad \sum_{i=0}^{l}\binom{l}{i}B_{l-i}B_i = (1 - l)B_l - lB_{l-1}.$$

The main purpose of this paper is to prove the following theorem which includes Euler's formula (1. 1) with $l \geq 2$ as a special case.

Theorem. If $l \geq 2$ and $m \geq 1$, then

$$(1.\ 2) \qquad m\sum_{i=1}^{l}\binom{l}{i}B_{l-i}B_{m-1+i} + l\sum_{j=1}^{m}\binom{m}{j}B_{m-j}B_{l-1+j} = -mlB_{m+l-2}.$$

Furthermore, we shall derive some basic properties of Bernoulli numbers from this theorem.

2. Some Lemmas. Let p be a prime number, $\mathbb{Z}_p$ be the ring of all rational numbers which are p-integral and $B_m(x) = \sum_{i=0}^{m}\binom{m}{i}B_{m-i}x^i$ ($m \geq 0$) be the Bernoulli polynomials. Also, denote by $[G(x)]_0^{(k)}$ the value of $d^k(G(x))/dx^k$ at $x = 0$ if $G(x)$ is a differentiable function of x.

First we shall give a recursion formula for Bernoulli numbers:

Lemma 1. *For* $m \ge 1$, $B_m(1) - B_m = \delta_{1m}$ *with Kronecker's* δ.

Proof. If $T(x) = x/(e^x - 1)$, then $T(x)e^x = T(x) + x$. Since $[T(x)e^x]_0^{(m)} = B_m(1)$ and $[T(x) + x]_0^{(m)} = B_m + \delta_{1m}$, the lemma holds.

The well-known theorem of von Staudt and Clausen completely describes the denominator of B_{2m}. That is, for $m \ge 1$

$$B_{2m} = \gamma_{2m} - \sum_{p-1|2m} \frac{1}{p}, \tag{2. 1}$$

where γ_{2m} is some integer and the sum is taken over all primes p such that $p-1|2m$. As a consequence of (2. 1) we see that

Lemma 2. *Let* $m \ge 1$. *If* $p-1\nmid 2m$, *then* $B_{2m} \in \mathbb{Z}_p$. *If* $p-1|2m$, *then* $pB_{2m} \in \mathbb{Z}_p$, *more precisely* $pB_{2m} \equiv -1 \pmod{p}$.

Let $p - 1 \ge k \ge 1$ for a prime p, $S_m(k, p) = k^m + (k + 1)^m + \cdots + (p - 1)^m$ and $S_m(p) = S_m(1, p)$.

Lemma 3. *Let* $m \ge 1$. *Then* $mS_{m-1}(k, p) = B_m(p) - B_m(k)$.

Proof. Using the identity $x(e^{kx} + e^{(k+1)x} + \cdots + e^{(p-1)x}) = (e^{px} - e^{kx})T(x)$, we have

$$\binom{m}{1}[x]_0^{(1)}\left[e^{kx} + e^{(k+1)x} + \cdots + e^{(p-1)x}\right]_0^{(m-1)} = \sum_{i=0}^{m}\binom{m}{i}\left[e^{px} - e^{kx}\right]_0^{(i)}\left[T(x)\right]_0^{(m-i)},$$

that is,

$$mS_{m-1}(k, p) = \sum_{i=0}^{m}\binom{m}{i}(p^i - k^i)B_{m-i} = B_m(p) - B_m(k).$$

Lemma 4. *Let* $m \ge 1$. *If* $p-1\nmid m-2$ *for* $m \ne 2$, *then*

$$mS_{m-1}(k, p) \equiv B_m + mpB_{m-1} - B_m(k) \pmod{p^2}.$$

Proof. By direct inspection we see that the lemma is true for $m = 1$ and 2. Assume here $m \ge 3$ and $p-1\nmid m-2$. By Lemma 2, $p^iB_{m-i} \equiv 0 \pmod{p^2}$ for $i = 2, 3, \ldots, m$, hence $B_m(p) \equiv B_m + mpB_{m-1} \pmod{p^2}$. So the congruence required in the lemma may be deduced from Lemma 3.

Lemma 5. *Let* $m \ge 1$. *If* $p-1 \nmid m-2$ *for* $m \ne 2$ *and* $p \nmid m$, *then*

$$S_{m-1}(p) \equiv pB_{m-1} - \delta_{1m} \pmod{p^2}.$$

Proof. Letting $k = 1$ in Lemma 4, it follows from Lemma 1 that $mS_{m-1}(p) \equiv mpB_{m-1} - \delta_{1m} \pmod{p^2}$. Since $p \nmid m$ and $\delta_{1m}/m = \delta_{1m}$, the lemma clearly holds.

We should add that Lemmas 1 and 3 are well-known identities (see, e.g., [1]).

3. Proof of the Theorem. Here, we shall prove the Theorem. Let $m \ge 1$. Then we have the identity

$$(T(x)e^{x})\left\{\sum_{k=1}^{p-1} k^{m-1}e^{kx}\right\} = T(x)\left\{\sum_{k=1}^{p-1} k^{m-1}e^{(k+1)x}\right\}$$

$$= x\left\{\sum_{k=1}^{p-1} k^{m-1}\,\frac{e^{(k+1)x} - 1}{e^{x} - 1}\right\} + S_{m-1}(p)T(x)$$

$$= x\left\{\sum_{k=1}^{p-1} k^{m-1}\left(\sum_{i=0}^{k} e^{ix}\right)\right\} + S_{m-1}(p)T(x)$$

$$= x\left\{S_{m-1}(p) + \sum_{k=1}^{p-1} S_{m-1}(k,\, p)e^{kx}\right\} + S_{m-1}(p)T(x).$$

Now let $\ell \ge 2$ and p be a prime such that $p - 1 \ge m + \ell$. Also, let $\varepsilon_1 = -1$ and $\varepsilon_i = 1$ for $i \ne 1$. Noting that $[T(x)e^{x}]_0^{(i)} = \varepsilon_i B_i$ and $p \nmid m(m+1)\cdots(m+\ell)$, we have, from Lemma 5

$$\left[(T(x)e^{x})\left\{\sum_{k=1}^{p-1} k^{m-1}e^{kx}\right\}\right]_0^{(\ell)} = \sum_{i=0}^{\ell}\binom{\ell}{i}\varepsilon_{\ell-i}B_{\ell-i}S_{m-1+i}(p)$$

$$\equiv \sum_{i=0}^{\ell}\binom{\ell}{i}\varepsilon_{\ell-i}B_{\ell-i}\left\{pB_{m-1+i} - \delta_{1m+i}\right\} \pmod{p^2}.$$

On the other hand, using Lemmas 3, 4 and 5

$$\left[x\left\{S_{m-1}(p) + \sum_{k=1}^{p-1} S_{m-1}(k,\, p)e^{kx}\right\} + S_{m-1}(p)T(x)\right]_0^{(\ell)}$$

$$= \binom{\ell}{1}\left\{\sum_{k=1}^{p-1} k^{\ell-1}S_{m-1}(k,\, p)\right\} + S_{m-1}(p)B_{\ell}$$

$$= l\ \Big\{\sum_{k=1}^{p-1} k^{l-1}\ \frac{B_m(p) - B_m(k)}{m}\Big\} + S_{m-1}(p)B_l$$

$$= \frac{l}{m}\ \Big\{S_{l-1}(p)B_m(p) - \sum_{k=1}^{p-1} k^{l-1}B_m(k)\Big\} + S_{m-1}(p)B_l$$

$$= \frac{l}{m}\ \Big\{S_{l-1}(p)B_m(p) - \sum_{j=0}^{m}\binom{m}{j}B_{m-j}S_{l-1+j}(p)\Big\} + S_{m-1}(p)B_l$$

$$\equiv \frac{l}{m}\ \Big\{(pB_{l-1} - \delta_{1l})(B_m + pB_{m-1}) - \sum_{j=0}^{m}\binom{m}{j}B_{m-j}(pB_{l-1+j} - \delta_{1\,l+j})\Big\}$$
$$+ (pB_{m-1} - \delta_{1m})B_l$$

$$\equiv \frac{l}{m}\ \Big\{pB_{l-1}B_m - p\sum_{j=0}^{m}\binom{m}{j}B_{m-j}B_{l-1+j}\Big\} + (pB_{m-1} - \delta_{1m})B_l \pmod{p^2}.$$

Hence, it follows that

$$p\ \Big\{\sum_{i=0}^{l}\binom{l}{i}\varepsilon_{l-i}B_{l-i}B_{m-1+i}\Big\} - (\varepsilon_l B_l)\delta_{1m}$$

$$\equiv \frac{lp}{m}\ \Big\{B_{l-1}B_m - \sum_{j=0}^{m}\binom{m}{j}B_{m-j}B_{l-1+j}\Big\} + (pB_{m-1} - \delta_{1m})B_l \pmod{p^2},$$

i.e.,

$$\text{(3. 1)}\quad \sum_{i=1}^{l}\binom{l}{i}\varepsilon_{l-i}B_{l-i}B_{m-1+i} + \frac{l}{m}\ \sum_{j=1}^{m}\binom{m}{j}B_{m-j}B_{l-1+j} \equiv 0 \pmod{p}.$$

This congruence holds for any integers l and m such that $l \geq 2$, $m \geq 1$ and $p - 1 \geq m + l$. Also the left hand side of (3. 1) does not depend on p. Therefore, in (3. 1) we may replace p by any prime p' with $p' > p$, which shows that the equality (1. 2) holds. Here, note that $\varepsilon_1 B_1 = B_1 + 1$.

The formula (1. 2) is of no interest if l and m have the same parity. In this case $B_i B_{m+l-1-i} = 0$ for $i \neq 1,\ m + l - 2$, hence (1. 2) becomes $m\binom{l}{l-1}B_1 B_{l+m-2} + l\binom{m}{m-1}B_1 B_{m+l-2} = -mlB_{m+l-2}$, however this is trivial. On the other hand, if l and m have different parity and $l + m \geq 5$, then $B_{l+m-2} = 0$, hence we may rewrite the equality (1. 2) as follows:

$$m\sum_{i=1}^{l}\binom{l}{i}B_{l-i}B_{m-1+i} + l\sum_{j=1}^{m}\binom{m}{j}B_{m-j}B_{l-1+j} = 0.$$

In particular, if we take $m = 1$ in (1. 2), then for $l \geq 2$

$$\sum_{i=1}^{l}\binom{l}{i}B_{l-i}B_{i} + l\binom{1}{1}B_0B_l = -lB_{l-1},$$

which is exactly Euler's formula (1. 1) with $l \geq 2$. Hence the equality (1. 2) is a generalization of (1. 1).

4. Applications. As applications, we can derive two well-known and important properties of Bernoulli numbers from the Theorem.

(P1) (*Kummer's congruence*). *Let* $p \geq 5$ *be a prime and* m *be an even integer with* $p - 3 \geq m \geq 2$. *Then*

$$\frac{B_m}{m} \equiv \frac{B_{m+p-1}}{m+p-1} \pmod{p}.$$

Proof. Letting $l = p$ in (1. 2) we have

$$m\sum_{i=1}^{p}\binom{p}{i}B_{p-i}B_{m-1+i} + p\sum_{j=1}^{m}\binom{m}{j}B_{m-j}B_{p-1+j} = -mpB_{m+p-2}.$$

By Lemma 2, $B_i \in \mathbb{Z}_p$ for $i = 0, 1, \ldots, p-2$ and for $i = p, p+1, \ldots, p+(m-1)$. Since $p \mid \binom{p}{i}$ unless $i = 0$ and p, from Lemma 2

$$m\left\{\binom{p}{1}B_{p-1}B_m + \binom{p}{p-m}B_mB_{p-1} + \binom{p}{p}B_0B_{m-1+p}\right\}$$

$$\equiv m\left\{p + \binom{p}{m}\right\}B_mB_{p-1} + mB_{m-1+p} \equiv \left\{m + \binom{p-1}{m-1}\right\}B_m(pB_{p-1}) + mB_{m+p-1}$$

$$\equiv -\left\{m+(-1)^{m-1}\right\}B_m + mB_{m+p-1} \equiv -m(m+p-1)\left\{\frac{B_m}{m} - \frac{B_{m+p-1}}{m+p-1}\right\} \equiv 0 \pmod{p}.$$

Since $p \nmid m$ and $p \nmid m+p-1$, we obtain Kummer's congruence.

(P2) *Let* $p \geq 3$ *be a prime*. *Then*

$$B_{2(p-1)} - 2B_{p-1} + 1 \equiv \frac{1}{p} \pmod{p}.$$

Proof. Take $l = p$ and $m = p - 1$ in (1. 2). By Lemma 2,

$$\begin{aligned}(p-1)\sum_{i=1}^{p}\binom{p}{i}B_{p-i}B_{p-2+i} &+ p\sum_{j=1}^{p-1}\binom{p-1}{j}B_{p-1-j}B_{p-1+j}\\ &\equiv (p-1)\left\{\binom{p}{1}B_{p-1}^2+\binom{p}{p}B_0B_{2(p-1)}\right\}+p\binom{p-1}{p-1}B_0B_{2(p-1)}\\ &\equiv (p-1)pB_{p-1}^2+(2p-1)B_{2(p-1)} \equiv -(p-1)pB_{2p-3} \equiv 0 \pmod{p}.\end{aligned}$$

When $p-1|m$ for $m \geq 1$, write $B_m = -1/p + \alpha_m$ with $\alpha_m \in Z_p$. Then $pB_{p-1}^2 \equiv 1/p - 2\alpha_{p-1} \pmod{p}$, so $pB_{p-1}^2 \equiv -2B_{p-1} - 1/p \pmod{p}$. By substituting this into the above congruence we can deduce the congruence stated in (P2).

By (P2) we see that $\alpha_{2(p-1)} - 2\alpha_{p-1} + 1 \equiv 0 \pmod{p}$, i.e.,

$$\gamma_{2(p-1)} - 2\gamma_{p-1} + 1 \equiv \sum_{\substack{q-1|2(p-1)\\ q-1\nmid p-1\\ q\neq p}} \frac{1}{q} - \sideset{}{'}\sum_{\substack{r-1|p-1\\ r\neq p}} \frac{1}{r} \pmod{p},$$

where the γ's are as in (2. 1), and the sums $\sum$ and $\sum'$ are taken over all primes q and r, respectively, satisfying the conditions indicated there.

We note that (P1) and (P2) are also easy consequences of the theory of p-adic L-functions (see, e.g., [3]).

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THE BAUM-CONNES CONJECTURE, THE COMMUTATOR THEOREM, AND RIEFFEL PROJECTIONS

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Presented by G.A. Elliott, F.R.S.C.

Abstract. For a discrete group G, a homomorphism from G into the geometric group $K^1(\cdot,G)$ is given, which coincides with the canonical map $G \to K_1(C^*(G))$ via the index map $K^1(\cdot,G) \to K_1(C^*(G))$. The injectivity of the induced map $H_1(G) \to K^1(\cdot,G)$ and of the index map is discussed.

1. The Baum-Connes conjecture. Let G be a Lie group, not necessarily connected, operating by diffeomorphisms on a smooth manifold X. In [1], P. Baum and A. Connes defined the geometric group $K^*(X,G)$ in a topological spirit by using what they call geometric K-cycles. They constructed a map μ, called the index map, from $K^*(X,G)$ into the K-groups of the reduced crossed product C*-algebra $C_0(X)\mathrm{x}G$. If X is compact, $C_0(X)$ is the C*-algebra $C(X)$ of all continuous complex-valued functions on X.

<u>Conjecture 1.1 (Baum-Connes)</u>. The index map μ is always an isomorphism.

2. The Bott periodicity map. In the case where G is discrete, there exist the canonical maps, both denoted by i, from $K^*(X)$ and $K_*(C_0(X))$ into $K^*(X,G)$ and $K_*(C_0(X)\mathrm{x}G)$ respectively, and the following diagram is commutative:

$$\begin{tikzcd}
K^*(X) \arrow[r] \arrow[d, "i"] & K_*(C_0(X)) \arrow[d, "i"] \\
K^*(X,G) \arrow[r] & K_*(C_0(X)\mathrm{x}G)
\end{tikzcd} .$$

The upper horizontal arrow is the standard isomorphism valid for any locally compact Hausdorff space.

Put
$$\bar{K}^*(X,G) = K^*(X,G)/i(K^*(X)), \text{ and}$$
$$\bar{K}_*(C_0(X)\mathrm{x}G) = K_*(C_0(X)\mathrm{x}G)/i(K_*(C_0(X)).$$

By the commutativity of the above diagram, we get a map

$$\bar{\mu}: \bar{K}^*(X,G) \longrightarrow \bar{K}_*(C_0(X)\times G).$$

In [3] when a torsion-free discrete group G acts on an oriented connected manifold X by orientation preserving diffeomorphisms, a homomorphism β_a, called the Bott periodicity map, from G into $\bar{K}_{\dim X+1}(C_0(X)\times G)$ is constructed. In the present note we shall show the following theorem.

Theorem 2.1. There exists a homomorphism $\beta_t: G \to \bar{K}^{\dim X+1}(X,G)$ such that $\bar{\mu}\beta_t = \beta_a$.

We shall call β_a (resp. β_t) the analytical (resp. the topological) Bott periodicity map.

In [3] we studied a class C^r of torsion-free discrete groups, for which the kernel of β_a is just the commutator subgroup. It is shown there that C^r contains $\mathbb{Z}$ and is closed under the operations of direct sum, direct limit, and free product. The definition of C^r leads to the following.

Corollary 2.2. Assume $G \in C^r$. Then $\bar{K}^{\dim X+1}(X,G)$ contains a subgroup H isomorphic to the abelianization $G/[G,G]$ of G, on which $\bar{\mu}$ is injective. Consequently, μ is injective on the inverse image of H under the canonical map from $K^{\dim X+1}(X,G)$ onto $\bar{K}^{\dim X+1}(X,G)$.

Remark 2.3. It is not known whether the Baum-Connes conjecture is valid for an arbitrary transformation group (X,G) with $G \in C^r$. For instance, it is not known whether the conjecture is affirmative for $(X,G_1\times G_2)$ if the conjecture is true for arbitrary transformation groups (X_1,G_1) and (X_2,G_2). Thus Corollary 2.2 gives us some information about the injectivity of the index map.

It is also proved in [3] that for any torsion-free discrete group G, not necessarily a member of C^r, the canonical map $G/[G,G] \to K_1(C^*(G))$ induced from the analytical Bott periodicity map is rationally injective. In particular, if $H_1(G) = G/[G,G]$ is torsion free, then the above map is injective. This together with Theorem 2.1 yields the following.

Theorem 2.4. For any torsion-free discrete group G, if $H_1(G)$ is also torsion free, then $K^1(\cdot,G)$ contains a subgroup isomorphic to $H_1(G)$ and the index map μ is injective there.

3. Proof of Theorem 2.1.

For a group G operating on X, denote by XG the homotopy quotient $[EG \times X]/G$, and by π the natural projection $XG \to BG$.

Since G is torsion free, the geometric group $K^*(X,G)$ is isomorphic to the twisted K-homology group $K_*^\tau(XG)$, where τ is the vector bundle over XG arising from the G-vector bundle TX over X (see, for details, [1]).

As $\pi_1(BG) = G$, each $g \in G$ can be considered as the homotopy class of a map from S^1 into BG. Take a representative of g. Denote this map by the same letter g. There exists a map $\tilde{g}: S^1 \to XG$ which covers g. Since the action of G on X preserves the orientation, $\tilde{g}^*\tau$ is an oriented trivial bundle over S^1. This orientation along with that of TS^1 determines an orientation of $\tilde{g}^*\tau \oplus TS^1$. Thus $\tilde{g}$ is K-oriented. Consider a K-cycle $(S^1,\tilde{g},\theta)$, where θ is the trivial complex line bundle over S^1. By bordism arguments it is easy to see that the map β_t which assigns to g the class of $(S^1,\tilde{g},\theta)$ is a well-defined homomorphism from G into $\tilde{K}^\tau_{\dim X+1}(XG) := K^\tau_{\dim X+1}(XG)/i(K^*(X))$.

It remains to show that $\bar{\mu}\,\beta_t = \beta_a$. Let $g \in G$ be fixed, and consider the $\mathbb{Z}$-action defined by g. The following diagram is commutative:

$$\begin{tikzcd}
K_*(X\mathbb{Z}) \arrow[r, "\mu"] \arrow[d] & K_*(C_0(X)\times\mathbb{Z}) \arrow[d] \\
K_*(XG) \arrow[r, "\mu"] & K_*(C_0(X)\times G),
\end{tikzcd}$$

where both vertical arrows are the canonical maps. Hence it is enough to show the equality $\bar{\mu}\,\beta_t = \beta_a$ for $G = \mathbb{Z}$.

The group $\mathbb{Z}$ acts on $\mathbb{R}$ by translations. This action is proper in the sense of [1]. There exists a $\mathbb{Z}$-equivariant smooth map $f'\colon \mathbb{R} \to X$. Put $\tilde{f}(x) = (f'(x), x)$ to get a $\mathbb{Z}$-equivariant map from $\mathbb{R}$ to $X \times E\mathbb{Z}$, where $\mathbb{Z}$ acts diagonally on $X \times E\mathbb{Z}$. The map $\tilde{f}$ is pushed down to be a map $f\colon S^1 \to X\mathbb{Z}$. Obviously f is a degree one map $S^1 \to B\mathbb{Z} = S^1$, and therefore is the canonical generator of $\pi_1(S^1) = \mathbb{Z}$. The class of (S^1, f, θ) in $\bar{K}^{\dim X+1}(X,\mathbb{Z})$ expresses the image of the generator $1 \in \mathbb{Z}$ under β_t.

Let (M,ϕ,e) be a K-cycle for $(X,\mathbb{Z})$. We get in a natural way a smooth map $\Phi\colon M\mathbb{Z} \to X\mathbb{Z}$. The $\mathbb{Z}$-vector bundle e gives rise to a vector bundle E over $M\mathbb{Z}$. Moreover $\mathbb{R}$ acts on both E and $M\mathbb{Z}$, so that Φ is $\mathbb{R}$-equivariant and E is an $\mathbb{R}$-vector bundle over $M\mathbb{Z}$. The map assigning the class of $(M\mathbb{Z}, \Phi, E)$ to (M,ϕ,e) defines a map from $K^*(X,\mathbb{Z})$ to $K^*(X\mathbb{Z},\mathbb{R})$. We have the following commutative diagram:

$$\begin{tikzcd}
K^*(X,\mathbb{Z}) \arrow[r] \arrow[d] & K_*(C_0(X)\times\mathbb{Z}) \arrow[d] \\
K^*(X\mathbb{Z},\mathbb{R}) \arrow[r] & K_*(C_0(X\mathbb{Z})\times\mathbb{R}),
\end{tikzcd}$$

where the left vertical arrow is the map described above, and the right one is the isomorphism coming from the fact that $C_0(X)\times\mathbb{Z}$ and $C_0(X\mathbb{Z})\times\mathbb{R}$ are strongly Morita equivalent. Putting together this picture and the proposition of [1, §4] we get the following commutative diagram:

$$\begin{tikzcd}
K^*(X,\mathbb{Z}) \arrow[r] \arrow[d] & K_*(C_0(X)\times\mathbb{Z}) \arrow[d] \\
K^{*+1}(X) \arrow[r] & K_{*+1}(C_0(X)).
\end{tikzcd}$$

The right vertical arrow is the connecting map of the Pimsner-Voiculescu exact sequence, and the left vertical arrow is given by taking the intersection of a K-cycle with the fibre X of $X\mathbb{Z} \to B\mathbb{Z}$.

By definition, $\beta_a(1)$ is given by an element of $K_{\dim X+1}(C_0(X)\times\mathbb{Z})$ which is transferred to the class $[X]$ in $K_{\dim X}(C_0(X))$ by the connecting map of the Pimsner-Voiculescu exact sequence. Therefore by the commutativity of the diagram above, in order to show that $\overline{\mu}(\beta_t(1)) = \beta_a(1)$, it is sufficient to show that the class of (S^1, f, θ) constructed above is in fact mapped to the class $[X] \in K^{\dim X}(X)$. This follows from the definition of the class $[X]$ and the description of the map $K^*(X,\mathbb{Z}) \to K^{*+1}(X)$ above.

That is the end of the proof of Theorem 2.1.

4. The Chern character. Let us consider the Chern character ch: $K^*(X,G) \to H_*(XG,\mathbb{Q})$ (see [2]). It is obvious that π_* ch $= 0$ on $i(K^{\dim X+1}(X)) \subset K^{\dim X+1}(X,G)$. Hence we get a homomorphism $\overline{\mathrm{ch}}$: $\overline{K}^{\dim X+1}(X,G) \to H_{odd}(BG,\mathbb{Q})$. Denote by ϕ the canonical map $G \to G/[G,G] = H_1(G)$ followed by $H_1(G) \to H_1(G,\mathbb{Q})$.

Theorem 4.1. We have the equality $\overline{\mathrm{ch}}\,\beta_t = \phi$.

The proof is given by the definition of β_t and the formula given in Proposition 6.3 of [2].

Corollary 4.2. If $H_1(G)$ is torsion free, then $\ker \beta_t = [G,G]$.

Corollary 4.3. Assume that the Baum-Connes conjecture is valid for (X,G). Then $\ker \beta_a = [G,G]$ if $H_1(G)$ is torsion free.

Let now X be a point. In this situation, the rational injectivity of the analytical Bott periodicity map ([3]) and the rational bijectivity of the Chern character ([1]), by the help of Theorems 2.1 and 4.1, yields the following.

Theorem 4.4. If $H_{2n+1}(BG,\mathbb{Q}) = 0$ for $n \geq 1$, then the index map μ: $K^1(\cdot,G) \to K_1(C^*(G))$ is rationally injective.

Example 4.5. The assumption of Theorem 4.4 is satisfied for the discrete group $\mathbb{Q}$.

Example 4.6. Let G be the group of all compactly supported homeomorphisms of $\mathbb{R}$. The group $\mathbb{Q}$ acts on $\mathbb{R}$ by translations, consequently acts on G by conjugation. The assumption of Theorem 4.4 is fulfilled for the semidirect product group $G\times\mathbb{Q}$.

5. Rieffel projections.

Via strong Morita equivalence between $C(S^1)$ and $C_0(\mathbb{R})\times\mathbb{Z}$, the generator $[\theta]\in K^0(S^1)$ corresponds to the generator $[e]$ of $K_0(C_0(R)\times Z)$ given by a Rieffel projection e. A Z-equivariant K-oriented map $f': R\to X$ gives rise to a push-forward map $f'!: K_0(C_0(R)\times Z)\to K_{\dim X+1}(C_0(X)\times\mathbb{Z})$, and we get $\mu([S^1,f,\theta]) = f'!([e])$ (see [2]). Thus Theorem 2.1 and the construction of β_t given in §3 say that the image of G under β_a is given by push-forward images of a Rieffel projection.

When $X = S^1$ or $\mathbb{R}$, as pointed out in [3], $\beta_a(g)$ is in fact given by a Rieffel projection, provided that g acts nontrivially on X.

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On degrees of irreducible representations of Lie superalgebras

by

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1. Introduction. The algebraic study of Lie superalgebras began quite recently (see e.g. Kac (1977) and Scheunert (1979)] after they had become of interest in theoretical physics. It should be noted, however, that long before that Lie superalgebras had been considered in algebraic topology. At present we have quite a few fundamental results on finite-dimensional Lie superalgebras, the most prominent of them, probably, being the classification of simple superalgebras over an algebraically closed field of characteristic zero [Kac(1977)]. There are also many nice results on representations of finite dimensional Lie superalgebras [see, e.g., Leites(1985)]. One of the basic questions on representations of arbitrary, not necessarily finite-dimensional Lie superalgebras, is to describe those of them for which all irreducible representations have finite bounded degrees. As it is pointed out in our paper [Bahturin(1985b)] this question for a Lie superalgebra L over an algebraically closed field is connected with that of the existence of a nontrivial polynomial identity in the universal enveloping algebra U(L) (all definitions see below).

Proposition 1.1. If U(L) is a PI-algebra, L a Lie superalgebra over an algebraically closed field, then the degrees of all irreducible finite-dimensional representations of L are totally bounded.

As for the description of Lie superalgebras L with U(L) a PI-algebra, it has been obtained in [Bahturin (1985b)] in the case of fields of characteristic zero (it is of interest to remark that the proof uses the techniques developed in our paper [Bahturin(1974)] for Lie algebras over prime characteristic fields).

Theorem 1.2. Let K be a field of characteristic zero, $L = L_0 \oplus L_1$ a Lie superalgebra over K, $U = U(L)$ its universal enveloping algebra. Then U(L) is a PI-algebra if and only if L_0 is an abelian Lie algebra, L_1 contains an L_0-submodule M of finite codimension and an L_0-submodule N of finite dimension such that $[M,M] = 0$ and $[L_0,M] \subseteq N$.

While in the Lie algebra case similar result has almost immediately led to a satisfactory description of Lie algebras all of whose irreducible representations are of finite bounded degree [Bahturin(1979)], the behaviour of Lie superalgebras is by no means so smooth. The main rift between the two cases lies in the fact that the Jacobson radical of the universal enveloping algebra in the superalgebra case need not be trivial as it happens in the case of ordinary Lie algebras [cf.Bahturin(1985a), chapter 6].

The aim of this paper is to compute the Jacobson radical for the universal enveloping algebras in some cases connected with the problem of degrees of irreducible representations (Theorem 4.1.) and to give an application to this latter problem (Theorem 4.2).

2.1.Some background. If G is an additive group, F a field then a G-graded algebra is an algebra of the form $A = \oplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{g+h}$. A Z_2-graded algebra is also called a superalgebra. If A is an associative superalgebra then a new superalgebra [A] can be be defined on the same space if we put for $a\in A_\alpha$, $b\in A_\beta$

$$[a,b] = ab - (-1)^{\alpha\beta} ba. \qquad (1)$$

Then the following identities will be satisfied ($a\in A_\alpha, b\in A_\beta, c\in A_\gamma$):

$$[a,b] = -(-1)^{\alpha\beta}[b,a] \qquad (2)$$

$$[a,[b,c]] = [[a,b],c] - (-1)^{\beta\gamma}[[a,c],b] \qquad (3)$$

It is easy to see that the elements of A_0 then form a Lie subalgebra of [A].Now, by the way of abstraction , we call any superalgebra $L = L_0 \oplus L_1$ with bracket operation a Lie superalgebra if (2) and (3) are satisfied in L. If L = [A] then L is called a Lie superalgebra of an associative superalgebra A.

All subalgebras and homomorphisms considered in the sequel are homogeneous in the sense that such a subalgebra has the form $B = (B\cap A_0)\oplus(B\cap A_1)$, and under such homomorphism we have $\phi(A_0)\subseteq B_0$, $\phi(A_1)\subseteq B_1$.

The above way of constructing a Lie superalgebra is universal in the sense that for each Lie superalgebra there exists an associative superalgebra $A = A_0\oplus A_1$ such that L is a subalgebra of a Lie superalgebra [A]. We define the universal enveloping algebra U = U(L) for a Lie superalgebra L in the same way as in the case of ordinary Lie algebras, that is, U is an associative superalgebra with 1 and with a (Lie superalgebra) homomorphism $\phi: L \rightarrow [U]$ such that if A is another associative superalgebra with 1 and ψ is a (Lie superalgebra homomorphism $\psi: L \rightarrow [A]$ then there exists a unique homomorphism $\varepsilon: U \to A$ of associative superalgebras with 1 such that $\varepsilon\phi=\psi$. The existence and the uniqueness of the universal enveloping superalgebras can be proved by standard arguments (see, e.g., Scheunert(1979)). The analogue of Poincaré - Birkhoff - Witt's Theorem for Lie superalgebras reads as follows.

Theorem 2.1. *Let* $L = L_0 \oplus L_1$ *be a Lie superalgebra over a field* K, $E = (e_\alpha)_{\alpha\in A}, F = (f_\beta)_{\beta\in B}$ *be linear bases for* L_0 *and* L_1 *respectively, A and B being totally ordered sets. Then the above map* ε *is injective and one can choose a basis for* $U(L)$ *as the set of all monomials of the form*

$$\varepsilon(e_{\alpha_1})\varepsilon(e_{\alpha_2})\dots\varepsilon(e_{\alpha_s})\varepsilon(f_{\beta_1})\varepsilon(f_{\beta_2})\dots\varepsilon(f_{\beta_t}) \tag{4}$$

where $\alpha_1 \le \alpha_2 \le \dots \le \alpha_s, \beta_1 < \beta_2 < \dots < \beta_t, s \ge 0, t \ge 0.$

It is conventional to omit ε's in (4), so that the basis takes the form

$$e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_s} f_{\beta_1} f_{\beta_2} \dots f_{\beta_t}. \tag{4'}$$

Clearly, in this case L is identified with a subalgebra of U under the bracket operation introduced by formula (1).

If $V = V_0 \oplus V_1$ is a superspace [see e.g. Berezin (1983)] then the algebra $E = \mathrm{End}\, V$ is an associative superalgebra if one sets

$$E_0 = \{\mathcal{A} \mid \mathcal{A}(V_i) \subseteq V_i\} \text{ and } E_1 = \{\mathcal{A} \mid \mathcal{A}(V_i) \subseteq V_{i+1}\}.$$

Any homomorphism $\phi: L \to [E]$ is called a *representation* of a Lie superalgebra L in a superspace V. We also say that V is an *L-module*. A representation is called *irreducible* if it contains no proper $\phi(L)$-invariant subspaces.

An associative algebra A is called a *PI-algebra* if it satisfies a nontrivial polynomial identity.

3.Some necessary conditions. A fairly simple observation as concerns the degrees of irreducible representations is the following.

Proposition 3.1. *Let K be an arbitrary field,* $L = L_0 \oplus L_1$ *a Lie superalgebra over K with all* irreducible representations of finite bounded degree. Then $U(L_0)$ is a PI-algebra.

From this proposition using Bahturin(1985a), chapter 6, we obtain the following.

Corollary 3.2. Let K be a field of characteristic zero and $L = L_0 \oplus L_1$ a Lie superalgebra over K with all irreducible representations of finite bounded degree. Then L_0 is abelian and $\dim L_0 < \operatorname{Card} K$ (so that if K is countable then L_0 is of finite dimension).

Corollary 3.3. Let K be a field of characteristic $p > 0$ and $L = L_0 \oplus L_1$ a Lie superalgebra over K with all irreducible representations of finite bounded degree. Then L_0 has an abelian subalgebra of finite codimension and for some fixed n every inner derivation ad x is annihilated by a nonzero polynomial of degree at most n.

It should be emphasized here that, in distinction to Lie algebra case,under our general hypothesis on the boundedness of degrees of irreducible representations of a Lie superalgebra L its universal enveloping algebra U(L) need not be a PI-algebra.

Proposition 3.4. Let K be an algebraically closed field, $L_0 = Kx$, $L_1 = Ky_1 \oplus Ky_2 \oplus .. \oplus Ky_n \oplus ...$, $L = L_0 \oplus L_1$ a Lie superalgebra over K with $[L_1,L_1] = 0$ and $[x, y_i] = y_{i+1}$, $i = 1, 2, ...$ Then all irreducible representations of L are 1-dimensional, whereas U(L) is not a PI-algebra.

Proof. From Theorem 1.2 [if char(K) = $p > 0$ one should consult the proof of this theorem in Bahturin (1985b)] it follows that $U = U(L)$ is not a PI-algebra. Now from Theorem 4.1 below it follows that $U/J(U) \cong K[x]$, the polynomial ring in one variable x. If $V = V_0 \oplus V_1$ is an irreducible L-module then V_i, $i = 1,2$, is either zero or it is an irreducible U_0-module. In the latter case this is, in fact, an irreducible K[x]-module, i.e. $\dim V_i = 1$. So $\dim V \leq 2$ and the proof is complete.

4. Some sufficient conditions. As we saw just above, an important role in the study of representations of Lie superalgerbas is played by the Jacobson radical of the universal enveloping algebra. We compute this in an important particular case.

Theorem 4.1. Let $L = L_0 \oplus L_1$ be a Lie superalgebra over K, M an L_0-submodule in L_1 such that $[L_1,M] = 0$. Then U(L)M is contained in the Jacobson radical J(U) of $U = U(L)$. If

$[L_1,L_1] = 0$ then $J(U) = UL_1$.

Outline of the proof. We set $R = U(L)M$. It is sufficient to prove that R is a locally nilpotent associative algebra. This can be done in the following way. It is sufficient to consider a subalgebra generated by finitely many elements of the form (4'), say $u_1,\dots,u_n$, with $t \neq 0$. Let q be the number of different elements of the form e_α involved in $u_1,\dots,u_n$, c the maximum of s for all these monomials [see (4')]. Then the nilpotency class of the subalgebra under consideration is at most $(d_q(1)+\dots+d_q(m))nc$ where $d_q(k)$ is the dimension of the space of homogeneous polynomials of degree k in q variables and m is such that

$$d_q(1) + 2d_q(2) +\dots+ md_q(m) > (d_q(1) + d_q(2) +\dots+ d_q(m))nc.$$

Now $U(L)/U(L)L_1 \cong U(L_0)$. Since this latter is semisimple the proof is complete.

Theorem 4.2. Let K be an algebraically closed field, $L = L_0 \oplus L_1$ a Lie superalgebra over K with an L_0-submodule M in L_1 of finite codimension such that $[L_1,M] = 0$. Then all irreducible representations of L are of finite bounded degree if and only if this is true for L_0.

Corollary 4.3. Let K be an algebraically closed field of characteristic zero, $L = L_0 \oplus L_1$ a Lie superalgebra over K with an L_0-submodule M in L_1 of finite codimension such that $[L_1,M] = 0$. Then all irreducible representations of L are of finite bounded degree if and only if $L_0 <$ Card K.

One could also formulate a similar corollary for the case of prime characteristic fields.

Remark 4.4. Let K be a field, $L = L_0 \oplus L_1$ a Lie superalgebra over K with $[L_1,L_1] = 0$. If L_0 is a finitely generrated Lie algebra and L_1 a finitely generated L_0-module then U(L) is a finitely generated associative algebra with locally nilpotent Jacobson's radical. If $\dim L_1 = \infty$ then it is not nilpotent since it contains Grassmann's algebra of infinite-dimensional vector space L_1. Thus we arrive at a regular way of producing finitely generated associative algebras with locally nilpotent but non-nilpotent Jacobson's radical.

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ON TWO VARIABLE FUNCTIONAL INEQUALITIES

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ABSTRACT

It is shown for a large class of functional inequalities with two variables that all continuous solutions can be determined from particular solutions of the corresponding functional equations.

Often functional equations and the corresponding inequalities play equally important roles. Additivity and subadditivity, Jensen-equation and Jensen-convexity are the simplest examples.

In what follows we deal with the following functional inequality:

$$f(n(x,y))+f(m(x,y)) \leqq f(x)+f(y), \quad x,y \in I \tag{1}$$

where I is a proper real interval, $n,m:I\times I \rightarrow I$ are given two place functions and $f:I \rightarrow \mathbb{R}$ is an unknown function.

The following theorem shows that the corresponding functional equation

$$\phi(n(x,y))+\phi(m(x,y)) = \phi(x)+\phi(y), \quad x,y \in I \tag{2}$$

with the unknown function $\phi:I \rightarrow \mathbb{R}$ is closely related to (1).

THEOREM. *Let $n,m:I\times I \rightarrow I$ be continuous functions and mean values, i.e. $m(x,y)$ and $n(x,y)$ are strictly between $\min(x,y)$ and $\max(x,y)$ for all $x \neq y$ in I. If there exists a continuous strictly increasing solution $\phi:I \rightarrow \mathbb{R}$ of (2), then a continuous function $f:I \rightarrow \mathbb{R}$ satisfies (1) if and only if $f \circ \phi^{-1}$ is a convex function on $\phi(I)$.*

Note: By the continuity and mean value property, $m(x,x) = x = n(x,x)$ $(x \in I)$.

Proof. *Only if*: Suppose that ϕ is a continuous solution of (2) and f is a continuous solution of (1). We are going to show that

$$2f\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right) \leqq f(x)+f(y) \tag{3}$$

holds for all $x,y \in I$. With $x = \phi^{-1}(t)$, $y = \phi^{-1}(s)$ this yields the Jensen convexity of $f \circ \phi^{-1}$, which implies convexity since $f \circ \phi^{-1}$ is continuous.

To prove (3), let $x,y \in I$ be arbitrary. Without loss of generality we can assume that $x \leqq y$.

Define

$$x_1 := \min[m(x,y),n(x,y)] \ ,$$
$$y_1 := \max[m(x,y),n(x,y)] \ .$$

Then, by the mean value property of m and n, we have $x \leqq x_1 \leqq y_1 \leqq y$. Further, (1) and (2) imply that $f(x_1)+f(y_1) \leqq f(x)+f(y)$ and $\phi(x_1)+\phi(y_1) = \phi(x)+\phi(y)$. Using the recursive definitions

$$x_{k+1} := \min[m(x_k,y_k),n(x_k,y_k)] \ , \tag{4}$$
$$y_{k+1} := \max[m(x_k,y_k),n(x_k,y_k)] \ ,$$

we obtain two sequences (x_k) and (y_k) such that

$$x \leqq x_1 \leqq \cdots \leqq x_k \leqq y_k \leqq \cdots \leqq y_1 \leqq y \ .$$

Further

$$f(x_k)+f(y_k) \leqq f(x)+f(y), \ \ \phi(x_k)+\phi(y_k) = \phi(x)+\phi(y) \tag{5}$$

hold for all $k \geqq 1$.

Now, (x_k) is increasing, (y_k) is decreasing and $x_k \leqq y_k$, so both sequences have a limit and

$$\lim_{k\to\infty} x_k := x_0 \leqq y_0 := \lim_{k\to\infty} y_k \ .$$

We show that $x_0 = y_0$. Let $k \to \infty$ in equation (4). Then, by the continuity of m and n, we get

$$x_0 = \min[m(x_0,y_0),n(x_0,y_0)] \ . \tag{6}$$

If $x_0 < y_0$, then the properties of m and n yield $x_0 < m(x_0,y_0)$ and $x_0 < n(x_0,y_0)$ which contradict (6). Therefore $x_0 = y_0$.

Let $k \to \infty$ in (5). Then we get

$$f(x_0)+f(y_0) \leqq f(x)+f(y) \tag{7}$$

and

$$\phi(x_0)+\phi(y_0) = \phi(x)+\phi(y) \ . \tag{8}$$

Since $x_0 = y_0$, it follows from (8) that

$$x_0 = y_0 = \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) \ .$$

Substituting this into (7) we obtain (3), which was to be proved.

If: Assume that ϕ satisfies (2), $f \circ \phi^{-1}$ is a convex function on $\phi(I)$ and let $x,y \in I$ be arbitrary.

The mean value property of m yields the existence of a value $\lambda \in [0,1]$ such that

$$\phi(m(x,y)) = \lambda\phi(x)+(1-\lambda)\phi(y) \ . \tag{9}$$

(λ may depend upon x,y). Then, using (2), we find that

$$\phi(n(x,y)) = (1-\lambda)\phi(x)+\lambda\phi(y) \ . \tag{10}$$

By the convexity of $f \circ \phi^{-1}$ we obtain from (9) that

$$\begin{aligned} f(m(x,y)) &= f(\phi^{-1}(\phi(m(x,y)))) = f(\phi^{-1}(\lambda\phi(x)+(1-\lambda)\phi(y)) \\ &\leqq \lambda f(\phi^{-1}(\phi(x)))+(1-\lambda)f(\phi^{-1}(\phi(y))) = \lambda f(x)+(1-\lambda)f(y) \ . \end{aligned}$$

Similarly, (10) implies

$$f(n(x,y)) \leqq (1-\lambda)f(x)+\lambda f(y) \ .$$

Adding these inequalities we get (2), which was to be proved. □

Finally, we give a few examples. The first two are applications of the Theorem. Others show that conditions in the Theorem are essential.

1. Consider the functional inequality

$$f\left(\frac{2x+y}{3}\right)+f\left(\frac{2y+x}{3}\right) \leqq f(x)+f(y), \ x,y \in I \ . \tag{11}$$

Since $\phi(x) = x$ is a solution of the corresponding functional equation, therefore, by the Theorem, a continuous function $f:I \to \mathbb{R}$ is a solution of (11) if and only if it is convex.

2. Consider the functional inequality

$$f\left(\frac{2xy}{x+y}\right)+f\left(\frac{x+y}{2}\right) \leqq f(x)+f(y), \ x,y > 0 \ . \tag{12}$$

Then clearly $\frac{2xy}{x+y}$ and $\frac{x+y}{2}$ are continuous means of x and y. Furthermore $\phi(x) := \ln x$ is a solution of the corresponding functional equation. Therefore a continuous function $f:\mathbb{R}_+ \to \mathbb{R}$ is a solution of (12) if and only if the function

$$x \to f(e^x), \ x \in \mathbb{R}$$

is convex.

3. The functional inequality

$$f(\sqrt{xy})+f\left(\frac{x+y}{2}\right) \leqq f(x)+f(y), \ x,y > 0$$

cannot be handled with our Theorem because it can be proved that the corresponding functional equation has no nonconstant continuous solutions. Nevertheless, the inequality has nonconstant solutions, for instance $f(x) = x$.

4. Consider Hosszú's functional inequality [1]:

$$f(xy)+f(x+y-xy) \leqq f(x)+f(y), \quad x,y \in]0,1[\ . \tag{13}$$

The corresponding functional equation is satisfied by $\phi(x) = x$, however xy and $x+y-xy$ are not mean values of x and y. Thus the Theorem cannot be applied. However, it turns out that any concave function fulfils (13), but there are also nonconcave solutions. For instance $f(x) = -x^4+2x^3-\frac{23}{16}x^2$ (see [1]).

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THEOREMES DES ZEROS REELS ET DES ELEMENTS POSITIFS AVEC PARAMETRES

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ABSTRACT: We study in this note the real nullstellensatz and the positivstellensatz, when the coefficients of the polynomials in the data vary.

Presented by P. Ribenboim, F.R.S.C.

INTRODUCTION

Rappelons tout d'abord l'énoncé du théorème des zéros réels : soit R un corps réel clos. Soit $f, g_j \in R[x_1,\dots x_n]$ avec $j=1,\dots,s$ tels que $\mathcal{Z}(f) \supset \mathcal{Z}(g_1,\dots g_s)$ (où $\mathcal{Z}$ désigne l'ensemble des zéros réels). Alors il existe un entier m tel que

$$f^{2m} + \sum_{i=1}^{r} h_i^2 = \sum_{j=1}^{s} q_j g_j \text{ avec } h_i, q_j \in R[x_1,\dots x_n]$$

Dans ce travail on s'intéresse à la situation où les coefficients des données f et g_j varient. Il existe des bornes uniformes pour l'entier m, le nombre et le degré des h_i en fonction du degré de f et des g_j. Ces résultats peuvent s'obtenir par des méthodes logiques (ultraproduits). Cependant on utilisera ici les outils de la théorie du spectre réel et des familles semi-algébriques avec les notations de [2] ; en particulier celles de l'opération tilda et du corps réels clos $k(\alpha)$. On obtient en plus des renseignements sur la variation des coefficients des h_i et q_j en fonction des paramètres.

§ 1. THEOREMES DES ZEROS REELS AVEC PARAMETRES.

a) Rappels concernant les familles semi-algébriques.

Soit X un ensemble semi-algébrique de $R^n \times R^p$ où R est un corps réels clos quelconque. On peut considérer X comme une famille semi-algébrique de sous-ensembles de R^n paramétrée par R^p .

La fibre de X au point t de R^p est $X = \{x\in R^n ; (x,t)\in X\}$. Si $f : X \to R$ est une fonction semi-algébrique, la fibre f en t est la fonction semi-algébrique $f_t : X_t \to R$ définie par $f_t(x)=y \Leftrightarrow f(x,t)=(y,t)$.

Soit $\alpha \in \widetilde{R^p}$. On peut définir aussi la fibre de X au point α : si X est défini par une formule $\Phi(x,t)$ du premier ordre à variables libres $x=(x_1,\dots,x_n)$ et $t=(t_1,\dots,t_p)$ alors

$$X_\alpha = \{x\in k(\alpha)^n ; \Phi(x,T(\alpha))\} \subset k(\alpha)^n$$

où $T(\alpha)$ est le p-uplet image des coordonnées $T = (T_1,\dots,T_p)$ par l'homomorphisme canonique $R[T] \to k(\alpha)$. La fibre $f_\alpha : X_\alpha \to k(\alpha)$ est la fonction semi-algébrique dont le graphe est la fibre en α du graphe de f.

La philosophie dans l'étude des fibres de familles d'ensembles et de fonctions semi-algébriques est le fait qu'une propriété d'une fibre en $\alpha \in \widetilde{R^p}$ reste valable sur un sous-ensemble semi-algébrique $S \subset R^p$ tel que $\alpha \in \widetilde{S}$. Rappelons quelques propriétés qui nous seront utiles:

Proposition 1. [cf. [2] ; 7.2.2 (iii)].

Soit une formule du premier ordre du langage des corps ordonnés $\Phi(t)$ *avec* $t=(t_1,\dots,t_p)$ *et soit* $\alpha \in \widetilde{R^p}$. *Alors* $\Phi(T(\alpha))$ *est vraie dans* $k(\alpha)$ *si et seulement s'il existe un semi-algébrique* S *tel que* $\alpha \in \widetilde{S}$ *et que pour tout* $t \in S$, $\Phi(t)$ *est vraie dans* R.

Proposition 2. [cf. [2] ; 8.8.3].

Soit $a \in k(\alpha)$. *Alors il existe une sous-variété de Nash* S^α *de* R^p, $\alpha \in \widetilde{S^\alpha}$, *et une fonction de Nash* $\varphi : S^\alpha \to R$ *telle que* $\varphi(\alpha)=a$.

b) Théorème des zéros réels avec paramètres.

Théorème 3.

Soient $f_t(x), g_{1,t}(x),\dots g_{s,t}(x)$ *des polynômes en* $x=(x_1,\dots,x_n)$ *dont les coefficients sont des fonctions semi-algébriques de* $t\in R^p$, *soit* $S=\{t\in R^p\,;\,\mathfrak{Z}(f)\supset\mathfrak{Z}(g_{1,t},\dots g_{s,t})\}$. *Alors* S *est une réunion disjointe d'un nombre fini de sous-variétés de Nash* S_l *de* R^p *et sur chaque* S_l *il existe des polynômes* $h_{1,t}(x),\dots,h_{r,t}(x), q_{1,t}(x),\dots,q_{s,t}(x)$ *dont les coefficients sont des fonctions de Nash en t sur* S_l, *et un entier m tels que*

$$f_t^{2m}+\sum_{i=1}^{r} h_{i,t}^2=\sum_{j=1}^{s} q_{j,t}q_{j,t} \quad \underline{\text{pour tout}} \quad t\in S_l$$

Démonstration.

On pose $f_t(x)=\sum_\lambda a_\lambda(t)x^\lambda$ où λ est un multi-indice et a_λ est une fonction semi-algébrique en t. Soit $\alpha\in\widetilde{S}\subset\widetilde{R^p}$ fixé alors $f_\alpha(x)=\sum_\lambda a_\lambda(\alpha)x^\lambda$ où $a_\lambda(\alpha)\in k(\alpha)$. De même on obtient $g_{1,\alpha}(x),\dots g_{1,\alpha}(x)\in k(\alpha)[x]$. Ainsi, on a, par hypothèse et d'après la proposition 1, $\mathfrak{Z}(g_{1,\alpha},\dots g_{s,\alpha})\subset\mathfrak{Z}(f_\alpha)$.

Alors en appliquant le théorème classique des zéros réels sur le corps $k(\alpha)$ on obtient

$$f^{2m}+\sum_{i=1}^{r} H_i^{\,2}=\sum_{j=1}^{s} Q_j g_{j,\alpha} \quad \text{où} \quad H_i\,,\,Q_j\in k(\alpha)[x]\,.$$

Ainsi $H_i(x)=\sum_\lambda b_\lambda x^\lambda$ où $b_\lambda\in k(\alpha)$. Alors, d'après la proposition 2, il existe une sous-variété de Nash S^α et une fonction de Nash $\varphi_\lambda : S^\alpha\rightarrow R$ avec $\alpha\in\widetilde{S^\alpha}$ telle que $b_\lambda=\varphi_\lambda(\alpha)$; puisqu'il n'y a qu'un nombre fini de λ qui figure dans les H_i, on peut choisir S^α qui convient pour tous les λ. On a $H_i(x)=\sum_\lambda \varphi_\lambda(x)x^\lambda$. On peut prendre alors $h_{i,t}(x)=\sum_\lambda \varphi_\lambda(x)x^\lambda$, $t\in S^\alpha$ et donc $H_i=h_{i,\alpha}$. De la même manière on peut trouver $q_{j,t}$ tel que $Q_j=q_{j,\alpha}$.

Maintenant, quitte à restreindre S^α, on a d'après la proposition 1, que pour tout $t\in S^{\alpha}$, $f^{2m}+\sum h_{i,t}^2=\sum q_{j,t}g_{j,t}$.

Enfin $\widetilde{S}$ est recouvert par les $\widetilde{S^{\alpha}}$ et d'après la compacité de $\widetilde{S}$ pour la topologie constructible, S est une réunion disjointe d'un nombre fini de sous-variétés de Nash S_l et chacune d'elles sont définies les coefficients (fonctions de Nash en t sur S_l) des polynômes $h_{i,t}$ et $q_{j,t}$ de la formule ci-dessus.

Théorème 4.

Soit $f_t(x)$ un polynôme en $x=(x_1,\dots,x_n)$ dont les coefficients sont des fonctions semi-algébriques de $t\in R^p$. Soit $S=\{t\in R^p;\forall x\in R^n, f(x)>0\}$. Alors S est une réunion disjointe d'un nombre fini de sous-variétés de Nash S_l de R^p et sur chaque S_l il existe des polynômes : $h_{1,t}(x),\dots,h_{r,t}(x),q_{1,t}(x),\dots,q_{s,t}(x)$ dont les coefficients sont des fonctions de Nash en t sur S_l tels que

$$f_t(\sum_{i=1}^{r} h_{i,t}^2)=1+\sum_{j=1}^{s} q_{j,t}^2 \quad \forall t\in S_l .$$

La démonstration est analogue à celle du théorème 3 précédent.

Comme application, substituons aux coefficients des polynômes des germes de fonctions analytiques à l'origine $\mathcal{O}_n(\mathbb{R})$, alors on a :

Théorème 5.

Soient $f, g_1,\dots g_s$ des polynômes en $x=(x_1,\dots,x_n)$ à coefficients dans $\mathcal{O}_n(\mathbb{R})$ et soit S le germe semi-analytique de l'ensemble $\{t\in U;\mathcal{Z}(f_t)\supset\mathcal{Z}(g_{1,t},\dots g_{s,t})\}$ avec U un voisinage ouvert de $0\in\mathbb{R}^p$ où les coefficients de $f, g_1,\dots g_s$ sont définis. Alors S est une réunion finie de germes semi-analytiques $S_1,\dots,S_k$ et sur chaque germe S_l il existe des polynômes $h_1,\dots,h_r,q_1,\dots,q_s$ à coefficients des fonctions semi- analytiques continues sur S_l et un entier m tels que

$$f^{2m}+\sum_{i=1}^{r} h_i^2=\sum_{j=1}^{s} q_j g_j.$$

Théorème 6.

Soit f un polynôme en $x=(x_1,\dots,x_n)$ à coefficients dans $\mathcal{O}_p(\mathbb{R})$ et soit S le germe semi-analytique de $\{t\in U;\ f_t>0$ sur $\mathbb{R}\}$ avec U un voisinage ouvert de $0\in\mathbb{R}^p$. Alors S est une réunion finie de germes semi-analytiques $S_1,\dots,S_k$ et sur chaque germe S_l il existe des polynômes $h_1,\dots,h_r, q_1,\dots,q_s$ dont les coefficients sont des fonctions semi-analytiques continues sur S_l tels que

$$f\Big(\sum_{i=1}^{r} h_i^2\Big)=1+\sum_{j=1}^{s} q_j^2.$$

§ 2. VERSION ABSTRAITE DU THEOREME DES ZEROS REELS AVEC PARAMETRES.

a) Fonctions semi-algébriques abstraites continues.

Soient A un anneau quelconque, $S\subset Spec_rA$ un ensemble constructible et $\alpha\in S$. D'après [1], l'anneau des fonctions semi-algébriques continues abstraites sur S, noté $\mathfrak{R}(S)$ est l'anneau des sections $s:S\rightarrow Spec_rA[T]$ de la projection $\pi:Spec_rA[T]\rightarrow Spec_rA$ telles que s est continue et que $s(S)$ est constructible et fermé dans $\pi^{-1}(S)$. En particulier, pour tout $\alpha\in S$, $s(\alpha)$ est un point constructible de $\pi^{-1}(S)\simeq Spec_rk(\alpha)[T]$, et s'identifie donc à un élément de $k(\alpha)$.

Alors on a les propriétés suivantes :

Proposition 7.

Soit $\alpha\in Spec_rA$ et soit $a\in k(\alpha)$. Alors il existe un constructible $S\subset Spec_rA$, $\alpha\in S$, et $f\in\mathfrak{R}(S)$ tel que $f(\alpha)=a$.

Proposition 8 [cf. [3], p. 57].

Soit $\Phi(X_1,\dots,X_p)$ une formule sans quantificateur, φ_i $(i=1,\dots,p)$ des fonctions *semi-algébriques abstraites continues sur un constructible* $S \subset \mathrm{Spec}_r A$ *et soit* $\alpha \in S$. *Alors* $\Phi(\varphi_1(\alpha),\dots,\varphi_p(\alpha))$ *est vraie dans* $k(\alpha)$ *si et seulement s'il existe* $T \subset S$ *un constructible tel que* $\alpha \in T$ *et* $\Phi(\varphi_1,\dots,\varphi_p)$ *est vraie dans* $\mathfrak{R}(T)$.

b) Théorème des zéros réels avec paramètres.

Théorème 9.

Soient $f, g_1,\dots,g_s \in \mathfrak{R}(S)[x_1,\dots,x_n]$, *où* $S \subset \mathrm{Spec}_r A$ *est un constructible, tels que* $\forall \alpha \in S$, $\mathcal{Z}(f_\alpha) \supset \mathcal{Z}(g_{1,\alpha},\dots,g_{s,\alpha})$ *où* $f_\alpha, g_{1,\alpha},\dots,g_{s,\alpha} \in k(\alpha)[x_1,\dots,x_n]$. *Alors il existe un recouvrement fini* $S = \bigcup_l S_l$ *où les* S_l *sont des constructibles et pour tout* l *des polynômes* $h_i, q_j \in \mathfrak{R}(S_l)[x_1,\dots,x_n]$ $i=1,\dots,r\,;\, j=1,\dots,s$ *et un entier* m *tels que*

$$f^{2m} + \sum_{i=1}^{r} h^2_i = \sum_{j=1}^{s} q_j g_j \text{ dans } \mathfrak{R}(S_l)[x_1,\dots,x_n].$$

La démonstration est analogue à celle du théorème 3 précédent.

Théorème 10.

Soient $f \in \mathfrak{R}(S)[x_1,\dots,x_n]$ *où* $S \subset \mathrm{Spec}_r A$ *est un constructible. On suppose que* $\forall \alpha \in S$, $f_\alpha > 0$ *sur* $k(\alpha)^n$. *Alors* S *est une réunion disjointe d'un nombre fini de constructibles* S_l *et sur chaque* S_l *il existe* $h_1(x),\dots,h_r(x), q_1(x),\dots,q_s(x)$ *à coefficients dans* $\mathfrak{R}(S_l)$ *où* $x = (x_1,\dots,x_n)$ *tels que*

$$f\Big(\sum_{i=1}^{r} h_i^2\Big) = 1 + \sum_{j=1}^{s} q_j^2 \quad .$$

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A NOTE ON A CLASS OF p-VALENTLY α-CONVEX FUNCTIONS

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Presented by J. Aczél, F.R.S.C.

ABSTRACT.

The object of the present paper is to prove some interesting properties of a class of p-valently α-convex functions in the unit disk.

1. INTRODUCTION.

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1,2,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

A function $f(z)$ belonging to the class A_p is said to be p-valently starlike of order α if and only if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for some α $(0 \leq \alpha < p)$ and for all $z \in U$. We denote by $S_p^*(\alpha)$ the subclass of A_p consisting of functions which are p-valently starlike of order α in U.

A function $f(z)$ belonging to A_p is said to be p-valently α-convex if and only if it satisfies

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \tag{1.3}$$

for some α $(\alpha \geq 0)$ and for all $z \in U$. Also we denote by $M_p(\alpha)$ the subclass of A_p consisting of such functions. Note that

$$M_p(0) = S_p^*(0) \equiv S_p^*.$$

The class $M_p(\alpha)$ was recently introduced by Owa and Ren [5]. In particular, the class $M_1(\alpha)$ when $p = 1$ was studied by Mocanu [4], Miller [1], and Miller, Mocanu and Reade ([2], [3]).

2. SOME PROPERTIES OF THE CLASS $M_p(\alpha)$.

In order to derive our results for the class $M_p(\alpha)$, we have to recall here the following lemmas.

LEMMA 1. (Owa and Ren [5]) The function $f(z)$ belongs to the class $M_p(\alpha)$, $\alpha > 0$, if and only if there exists a function $F(z)$ belonging to the class S_1^* such that

$$f(z) = \left\{ \frac{p}{\alpha} \int_0^z \frac{F(t)^{p/\alpha}}{t} \, dt \right\}^{\alpha}. \tag{2.1}$$

LEMMA 2. (Ren [6]) If $f(z) \in S_1^*$, then, for $r_1 \leq r_2 < 1$, $\mu \geq 0$, we have

$$|f(-r_1)|^{\mu} + |f(r_2)|^{\mu} \leq \left(\frac{r_1}{(1 + r_1)^2} \right)^{\mu} + \left(\frac{r_2}{(1 - r_2)^2} \right)^{\mu}. \tag{2.2}$$

Equality in (2.2) holds for the Koebe function

$$k(z) = \frac{z}{(1 - z)^2} . \tag{2.3}$$

LEMMA 3. (Yang and Liu [7]) Suppose the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.4}$$

is analytic and univalent in the unit disk $\mathbb{U}$. Let

$$\left(\frac{f(z)}{z} \right)^{\lambda} = \sum_{n=0}^{\infty} D_n(\lambda) z^n, \quad D_0(\lambda) = 1, \ \lambda > 0 \tag{2.5}$$

and

$$\left(\frac{k(z)}{z} \right)^{\lambda} = \sum_{n=0}^{\infty} d_n(2\lambda) z^n, \quad d_0(2\lambda) = 1, \ \lambda > 0, \tag{2.6}$$

where

$$d_n(x) = \frac{x(x + 1) \cdots (x + n - 1)}{n!} \qquad (n \in \mathbb{N}) \tag{2.7}$$

and $k(z)$ is given by (2.3). Then, if $\lambda \geq 1$,

$$|D_n(\lambda)| \leq d_n(2\lambda) \qquad (n \in \mathbb{N}). \tag{2.8}$$

Equalities in (2.8) hold if and only if $f(z)$ is the rotation of the Koebe function $k(z)$.

Now, we derive

THEOREM 1. *If* $f(z) \in \mathcal{M}_p(\alpha)$, $\alpha > 0$, *then, for* $r_1 \leq r_2 < 1$, *we have*

$$(2.9)\qquad |f(-r_1)|^{1/\alpha} + |f(r_2)|^{1/\alpha} \leq (-1)^{p/\alpha}K(p,\alpha,-r_1)^{1/\alpha} + K(p,\alpha,r_2)^{1/\alpha},$$

where

$$(2.10)\qquad K(p,\alpha,r) = \left\{ \frac{p}{\alpha}\int_0^r t^{p/\alpha-1}(1-t)^{-2p/\alpha}dt \right\}^{\alpha}.$$

Equality in (2.9) holds for the function $K(p,\alpha,z)$.

PROOF. With the aid of Lemma 1 and Lemma 2, we have, for $r_1 \leq r_2 < 1$,

$$\begin{aligned}(2.11)\qquad \frac{\alpha}{p}\left(|f(-r_1)|^{1/\alpha} + |f(r_2)|^{1/\alpha}\right) &= \left|\int_0^1 \frac{F(-r_1t)^{p/\alpha}}{t}dt\right| + \left|\int_0^1 \frac{F(r_2t)^{p/\alpha}}{t}dt\right| \\ &\leq \int_0^1\left\{\left(\frac{r_1t}{(1+r_1t)^2}\right)^{p/\alpha} + \left(\frac{r_2t}{(1-r_2t)^2}\right)^{p/\alpha}\right\}\frac{dt}{t} \\ &= (-1)^{p/\alpha}\int_0^{-r_1}\left(\frac{t}{(1-t)^2}\right)^{p/\alpha}\frac{dt}{t} \\ &\qquad + \int_0^{r_2}\left(\frac{t}{(1-t)^2}\right)^{p/\alpha}\frac{dt}{t}.\end{aligned}$$

This implies that the inequality (2.9) holds true.

Next, we prove

THEOREM 2. *Suppose the function* $f(z)$ *defined by (1.1) is in the class* $\mathcal{M}_p(\alpha)$, $\alpha > 0$. *Let*

$$(2.12)\qquad \left(\frac{f(z)}{z^p}\right)^{1/\alpha} = \sum_{n=0}^{\infty}\hat{D}_n(\alpha,p)z^n$$

and

$$(2.13)\qquad \frac{1}{(1-z)^{2p/\alpha}} = \sum_{n=0}^{\infty}d_n(2p/\alpha)z^n.$$

If $p \geq \alpha$ *and* $m = [\alpha]$ ([ ] means the Gauss symbol), *then*

$$(2.14)\qquad |\hat{D}_n(\alpha,p)| \leq A_n(\alpha,p) \equiv \frac{p}{p+n\alpha}d_n(2p/\alpha)$$

for $n \in \mathbb{N}$, and

$$(2.15) \qquad |a_{p+k}| \le \alpha A_k(\alpha,p) + \binom{\alpha}{2}\sum_{j_1+j_2=k} A_{j_1}(\alpha,p)A_{j_2}(\alpha,p)$$

$$+ \binom{\alpha}{3}\sum_{j_1+j_2+j_3=k} A_{j_1}(\alpha,p)A_{j_2}(\alpha,p)A_{j_3}(\alpha,p) + \cdots$$

$$+ \binom{\alpha}{k}\sum_{j_1+\cdots+j_{k-1}=k}\left\{\prod_{i=1}^{k-1} A_{j_i}(\alpha,p)\right\}$$

$$+ (A_1(\alpha,p))^k$$

for $k = 1,2,\cdots,m+1$, where

$$(2.16) \qquad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \qquad (k \in \mathbb{N}).$$

Equalities in (2.14) and (2.15) hold for the function $f(p,\alpha,z)$ given by

$$(2.17) \qquad f(p,\alpha,z) = \left\{ \frac{p}{\alpha}\int_0^z t^{p/\alpha-1}(1-t)^{-2p/\alpha}dt \right\}^{\alpha}.$$

PROOF. It follows from Lemma 1 that

$$(2.18) \qquad \left\{\frac{f(z)}{z^p}\right\}^{1/\alpha} = \frac{p}{\alpha z^{p/\alpha}}\int_0^z \left\{\frac{F(t)}{t}\right\}^{p/\alpha} t^{p/\alpha-1}dt$$

$$= \sum_{n=0}^{\infty} \frac{p}{p+n\alpha} D_n(p/\alpha)z^n,$$

where

$$(2.19) \qquad \left\{\frac{F(t)}{t}\right\}^{p/\alpha} = \sum_{n=0}^{\infty} D_n(p/\alpha)z^n, \quad D_0(p/\alpha) = 1.$$

Since the function $F(z)$ is in the class S_1^*, $F(z)$ is analytic and univalent in the unit disk $\mathbb{U}$ and $F(0) = F'(0) - 1 = 0$. Therefore, using Lemma 3, we have

$$(2.20) \qquad \hat{D}_n(\alpha,p) = \frac{p}{p+n\alpha} D_n(p/\alpha)$$

and

$$|\hat{D}_n(\alpha,p)| \le A_n(\alpha,p) = \frac{p}{p+n\alpha} d_n(2p/\alpha)$$

for $n \in \mathbb{N}$ which proves (2.14). Further, the equality in (2.14) is attained for the function $f(p,\alpha,z)$ given by (2.17).

Next, by (2.18) and

$$(1+z)^{\alpha} = 1 + \alpha z + \binom{\alpha}{2}z^2 + \cdots + \binom{\alpha}{n}z^n + \cdots ,$$

we have

$$(2.21)\qquad \frac{f(z)}{z^p} = 1 + \alpha\left(\sum_{n=1}^{\infty}\hat{D}_n(\alpha,p)z^n\right) + \binom{\alpha}{2}\left(\sum_{n=1}^{\infty}\hat{D}_n(\alpha,p)z^n\right)^2$$

$$+ \cdots + \binom{\alpha}{k}\left(\sum_{n=1}^{\infty}\hat{D}_n(\alpha,p)z^n\right)^k + \cdots \quad .$$

Comparing the coefficients of z^k in both sides of (2.21), we have

$$(2.22)\qquad a_{p+k} = \alpha\hat{D}_k(\alpha,p) + \binom{\alpha}{2}\sum_{n_1+n_2=k}\hat{D}_{n_1}(\alpha,p)\hat{D}_{n_2}(\alpha,p) + \cdots$$

$$+ \binom{\alpha}{j}\sum_{n_1+\cdots+n_j=k}\left(\prod_{i=1}^{j}\hat{D}_{n_i}(\alpha,p)\right) + \cdots$$

$$+ \binom{\alpha}{k}\left(\hat{D}_1(\alpha,p)\right)^k .$$

Therefore, (2.15) follows from (2.14), (2.21) and (2.22). Also the equality in (2.15) holds for the function $f(p,\alpha,z)$ given by (2.17). Thus we complete the proof of Theorem 2.

REMARK. Taking $k = 1, 2$ in Theorem 2, we have

$$(2.23)\qquad |a_{p+1}| \le \frac{2p^2}{p+\alpha}$$

and

$$(2.24)\qquad |a_{p+2}| \le \frac{p^2}{p+2\alpha}\left(\frac{2p}{\alpha} + 1\right) + \frac{2(\alpha-1)p^4}{\alpha(p+\alpha)^2}$$

for $\alpha > 0$ and $p \in \mathbb{N}$.

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REMARKS ON SOME DETERMINANTAL INEQUALITIES

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ABSTRACT. In this paper we gave some remarks about some inequalities related to important Andréief's identity for determinants.

We shall use the following notation

$$\left|\int_a^b p(x)f_i(x)g_j(x)\,dx\right|_n = \begin{vmatrix} \int_a^b p(x)f_1(x)g_1(x)\,dx & \dots & \int_a^b p(x)f_1(x)g_n(x)\,dx \\ \vdots & & \\ \int_a^b p(x)f_n(x)g_1(x)\,dx & \dots & \int_a^b p(x)f_n(x)g_n(x)\,dx \end{vmatrix}$$

and

$$|f_i(x_i)|_n = \begin{vmatrix} f_1(x_1)\dots f_1(x_n) \\ \vdots \\ f_n(x_1)\dots f_n(x_n) \end{vmatrix}.$$

In 1883 C.Andréief showed that for arbitrary continuous functions $f_1,\dots,f_n,g_1,\dots,g_n$ and p the following identity is valid

$$(1)\quad \left|\int_a^b p(x)f_i(x)g_j(x)\,dx\right|_n = \frac{1}{n!}\int_a^b\dots\int_a^b |f_i(x_i)|_n|g_i(x_i)|_n p(x_1)\dots p(x_n)dx_1\dots dx_n.$$

The simple consequence of (1) is the following result:

Let p be a nonnegative function. If

$$|f_i(x_i)|_n|g_i(x_i)|_n \geqq 0 \quad (\forall x_1,\dots,x_n \in [a,b])$$

then

$$(2)\qquad \left|\int_a^b p(x)f_i(x)g_j(x)\,dx\right|_n \geqq 0.$$

Simple consequences of (2) are Čebyšev, Cauchy and some other inequalities. Of course, Gram's inequality

$$(3) \qquad \left| \int_a^b p(x)f_i(x)f_j(x)\,dx \right|_n \geq 0$$

is an obvious consequence of (2).

Identity (1) for Stieltjes' integral is given in [3]. A generalization of (1) for functions of several variables is also valid (the case $p(x) \equiv 1$ is given in [7]). Of course, results analogous to (2) and (3) are also valid in these cases.

An interpolation of (3) in the case $p(x) \equiv 1$ is given in [7]. Here, we shall note that with the same idea of the proof we can give the weighted version of this result.

Let $\phi_1(x), \phi_2(x), \dots, \phi_n(x), \dots$ be a system of normalized orthogonal functions with respect to p, i.e.

$$\int_a^b p(x)\phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n$$
$$= 1, \quad m = n.$$

If we denote by $a_i(f_i)$ the following

$$a_i(f_j) = \int_a^b p(x)f_j(x)\phi_i(x)\,dx \qquad (j=1,\dots,p;\ i=1,2,\dots),$$

then,

$$(4) \quad \left| \int_a^b p(x)f_i(x)f_j(x)\,dx \right|_p \geq \Sigma \begin{vmatrix} a_{m_1}(f_1) \dots a_{m_p}(f_1) \\ \vdots \\ a_{m_1}(f_p) \dots a_{m_p}(f_p) \end{vmatrix}^2 ,$$

where Σ denotes the summation with respect to $m_1, \dots, m_p$ such that $1 \leq m_1 < m_2 < \dots < m_p \leq k$; k being any positive integer which is not smaller than p.

Of course, similar result can be formulated for Stieltjes' integral, too.

Discrete versions of (1) are also valid. For example, we have

$$\begin{vmatrix} \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}f_{ij}^{1}g_{ij}^{1} & \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}f_{ij}^{1}g_{ij}^{2} \\ \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}f_{ij}^{2}g_{ij}^{1} & \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}f_{ij}^{2}g_{ij}^{2} \end{vmatrix}$$

$$= \frac{1}{2}\sum_{i,r=1}^{n}\sum_{j,s=1}^{m} a_{ij}a_{rs}\begin{vmatrix} f_{ij}^{1} & f_{rs}^{1} \\ f_{ij}^{2} & f_{rs}^{2} \end{vmatrix}\begin{vmatrix} g_{ij}^{1} & g_{rs}^{1} \\ g_{ij}^{2} & g_{rs}^{2} \end{vmatrix}.$$

If we put: $f_{ij}^{1} = x_i$, $f_{ij}^{2} = y_i$, $g_{ij}^{1} = z_j$, $g_{ij}^{2} = u_j$, after some simple manipulations, we get the following identity from [10]:

$$\begin{vmatrix} \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_i z_j & \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_i u_j \\ \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}y_i z_j & \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}y_i u_j \end{vmatrix}$$

$$= \sum_{1 \leq i < j \leq n}\ \sum_{1 \leq r < s \leq n} \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}\begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix}\begin{vmatrix} z_r & u_r \\ z_s & u_s \end{vmatrix}.$$

As an immediate consequence of the above identity we can formulate the following ([10]):

If, for all positive integers i,j,r,s, such that $1 \leq i < j \leq n$ and $1 \leq r < s \leq m$, we have

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}\begin{vmatrix} z_r & u_r \\ z_s & u_s \end{vmatrix} \geq 0 \quad \text{and} \quad \begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \geq 0,$$

then the following inequality holds

$$\begin{vmatrix} \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_i z_j & \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_i u_j \\ \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}y_i z_j & \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}y_i u_j \end{vmatrix} \geq 0.$$

This inequality is also a generalization of Čebyšev's and Cauchy's inequality (see for example [6]).

Finally, we shall give an interesting consequence of (2) for generalized convex functions.

Let f:(a,b)->R be a convex function with respect to an ETC-system of functions $\{u_i\}_0^n$ (see [5]). Then, in symbols, we write

$$f \in C(u_0, u_1, \dots, u_n).$$

The following result is a simple consequence of (2):

Let $f \in C(u_0, u_1, \dots, u_n)$ and $g \in C(v_0, v_1, \dots, v_n)$, and let p be a nonnegative integrable function. Then

$$\begin{vmatrix} \int_a^b pu_0v_0\,dx & \dots & \int_a^b pu_0v_n\,dx & \int_a^b pu_0g\,dx \\ \vdots & & & \\ \int_a^b pu_nv_0\,dx & \dots & \int_a^b pu_nv_n\,dx & \int_a^b pu_ng\,dx \\ \int_a^b pfv_0\,dx & \dots & \int_a^b pfv_n\,dx & \int_a^b pfg\,dx \end{vmatrix} \geqq 0. \tag{5}$$

If $f,g:(a,b)\to R$ are two $(n+1)$-convex functions, i.e. if $f,g \in C(1,t,\dots,t^n)$, then (5) becomes (for $p(x) \equiv 1$, of course)

$$\begin{vmatrix} b-a & \dots & \dfrac{b^{n+1}-a^{n+1}}{n+1} & \int_a^b g\,dx \\ \vdots & & & \\ \dfrac{b^{n+1}-a^{n+1}}{n+1} & \dots & \dfrac{b^{2n+1}-a^{2n+1}}{2n+1} & \int_a^b x^n g\,dx \\ \int_a^b f\,dx & \dots & \int_a^b x^n f\,dx & \int_a^b fg\,dx \end{vmatrix} \geqq 0.$$

For $n = 1$, we get that for convex functions f and g the following inequality is valid:

$$\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b g(x)\,dx$$

$$\geqq \frac{12}{(b-a)^3}\int_a^b \left(x-\frac{a+b}{2}\right)f(x)\,dx \int_a^b \left(x-\frac{a+b}{2}\right)g(x)\,dx.$$

It is a result from [4], and a generalization of a result from [2]. For some related results see [8,9,11].

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ON THE STACK COMPLETION OF A CATEGORY OBJECT IN A PRETOPOS

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Let $\mathbb{P}$ be a pretopos, and $\mathbb{C}$ a category object in $\mathbb{P}$. We give an elementary construction of the stack completion for the finite coverings topology of $[\mathbb{C}]$, the fibration naturally generated by $\mathbb{C}$.

Let $\mathbb{E}$ be an elementary topos. For a number of years, the study of "categories living in the universe $\mathbb{E}$" has been ascribed to the study of $\mathbb{E}$-indexed categories, or equivalently fibrations above $\mathbb{E}$ [2],[5]. This identification is less than totally satisfying: since internal semantics in a topos is governed by Kripke-Joyal semantics for the finite coverings topology, by extrapolation one would want the fibrations in question to have the glueing (descent) conditions for finite coverings, that is, to be stacks for that topology. This leads to the following difficulty: if $\mathbb{C}$ is a category object in $\mathbb{E}$ the fibration $[\mathbb{C}]$ it gives rise to is not a stack. In this note we give a construction for the free stack generated by $[\mathbb{C}]$, which is elementary in comparison with the construction that has appeared in the literature [3], and works if $\mathbb{E}$ is replaced by any pretopos $\mathbb{P}$.

Let $\mathbb{C},\mathbb{D}$ be two small categories in *Set*. A *concept of functor* $F:\mathbb{C}\to\mathbb{D}$ is a profunctor $F:\mathbb{C}^{op}\times\mathbb{D}\to$ *Set* such that for every object $X\in\mathbb{C}$ the functor $F(X,-)$ is representable. This notion was developed

[1],[4] in order to give rigor to the idea of a functor $\mathbb{C}\rightarrow\mathbb{D}$ defined only "up to isomorphism". If now $\mathbb{C}$ and $\mathbb{D}$ are in topos $\mathcal{E}$ and F is a profunctor $\langle F,F'\rangle:\mathbb{F}\rightarrow\mathbb{C}\times\mathbb{D}$ the representability condition can be defined using universal elements: for x a variable in $\mathbb{F}_0$ define

$$U(x) \quad \text{iff} \quad (\forall y\in\mathbb{F}_0)\big(F_0(y)=F_0(x) \Rightarrow (\exists!f:x\rightarrow y)(F_1(f)=1_{F_0(x)})\big) .$$

F is now a concept of functor iff $(\forall z\in\mathbb{C}_0)(\exists x\in\mathbb{F}_0)\, U(x)\wedge F_0(x)=z$. We claim that the stack completion of $[\mathbb{C}]$ for the finite coverings topology is given by the (pseudo)functor $[[\mathbb{C}]]:\mathcal{E}^{op}\rightarrow$ *Cat* which sends X in $\mathcal{E}$ to the full subcategory of profunctors $X\rightarrow\mathbb{C}$ given by the concepts of functors, where X is considered as a discrete category. $[[\mathbb{C}]]$ acts on morphisms by the usual pullback. In order to make this construction work in any pretopos $\mathfrak{P}$, where the comprehension scheme is much more restricted, we have to restrict $\langle F,F'\rangle$ to the full subcategory of $\mathbb{F}$ of universal elements:

Definition: Let $\mathfrak{P}$ be a pretopos, $\mathbb{C}$ a category object in $\mathfrak{P}$, X an object of $\mathfrak{P}$. A *loose* X-*object* of $\mathbb{C}$ is a triple $(\mathbb{A},A,A')$ where $\mathbb{A}$ is a category object such that $\langle d_0,d_1\rangle:\mathbb{A}_1\rightarrow\mathbb{A}_0\times\mathbb{A}_0$ is an equivalence relation, $A:\mathbb{A}_0\rightarrow X$ a regular epi which admits $\langle d_0,d_1\rangle$ as kernel pair, $A':\mathbb{A}\rightarrow\mathbb{C}$ a functor which necessarily factors through the underlying groupoid $\mathrm{Iso}(\mathbb{C})\rightarrowtail\mathbb{C}$ and is a discrete fibration above $\mathrm{Iso}(\mathbb{C})$. Let $(\mathbb{B},B,B')$ be another loose X-object of $\mathbb{C}$, and let $p_A:P\rightarrow\mathbb{A}_0$, $p_B:P\rightarrow\mathbb{B}_0$ be the pullback of A and B . A *morphism* $F:(\mathbb{A},A,A')\rightarrow(\mathbb{B},B,B')$ of loose X-objects of $\mathbb{C}$ is a morphism $F:P\rightarrow\mathbb{C}_1$ in $\mathfrak{P}$ such that $d_0F=A'p_A$ and $d_1F=B'p_B$. One can check that loose X-objects of $\mathbb{C}$ form a category in a natural way. If $f:Y\rightarrow X$ is some morphism, it is easy to see that pulling back gives a loose Y-object of $\mathbb{C}$ (f^*A, f^*A, f^*A') and that f^* is functorial. Therefore we have a pseudofunctor $[[\mathbb{C}]]:\mathbb{C}^{op}\rightarrow \mathcal{C}at$.

Theorem: If $\mathbb{C}$ is any category object of $\mathbb{P}$ then $[[\mathbb{C}]]$ is the stack completion of $[\mathbb{C}]$ for the finite coverings topology. The proof is rather straightforward.

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SOUS-HARMONICITE DE L'ANGLE SPECTRAL

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Summary. Given an analytic family of operators we prove several results on subharmonicity of the angle under which are seen the spectra of these operators. We apply these results to the holomorphic variation of the isolated spectral values and to the distribution of values of the convex hulls of spectra.

0. Introduction

Si l'on regarde dans [1], page 17, la démonstration du théorème de variation holomorphe des points isolés du spectre, on s'aperçoit qu'elle repose essentiellement sur le théorème non trivial de caractérisation des fonctions holomorphes par les fonctions sous-harmoniques (Théorème 17 de l'Appendice II).

Dans la première partie, nous montrons qu'on peut se dispenser d'utiliser ce résultat, donc qu'on peut démontrer la variation holomorphe élémentairement (voir Corollaire 1.3), en utilisant le Théorème 1.1. Ce dernier théorème peut être étendu sans aucun problème à la situation plus générale des fonctions analytiques multiformes.

Dans la deuxième partie, nous montrons que pour une fonction analytique multiforme $\lambda \to K(\lambda)$, avec $K(\lambda)$ toujours convexe (ce qui est le cas de l'enveloppe convexe du spectre d'une famille analytique d'opérateurs), l'angle spectral est une fonction sous-harmonique. Cela nous permet d'améliorer le Théorème 2.1 de [3], avec le Théorème 2.2.

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Nos résultats découlent des deux théorèmes suivants, d'ailleurs assez faciles à démontrer.

Théorème 0.1: *Soit* $\lambda \to f(\lambda)$ *une fonction analytique définie sur un domaine* D *de* $\mathbb{C}$ *à valeurs dans une algèbre de Banach. Posons*

$$u(\lambda) = \mathrm{Max}\,\{\mathrm{Re}\, u : u \in \mathrm{Sp}\, f(\lambda)\},$$

$$v(\lambda) = \mathrm{Min}\,\{\mathrm{Re}\, u : u \in \mathrm{Sp}\, f(\lambda)\}.$$

Alors on a $u(\lambda) = \mathrm{Log}\, \rho(e^{f(\lambda)})$ *et* $v(\lambda) = -\mathrm{Log}\, \rho(e^{-f(\lambda)})$. *De plus* u *et* $-v$ *sont sous-harmoniques sur* D.

Cela résulte du théorème de Vesentini. Voir l'argument du Théorème 2, page 11 de [1].

Théorème 0.2: *Soit* $\lambda \to f(\lambda)$ *une fonction analytique définie sur un domaine* D *de* $\mathbb{C}$ *à valeurs dans une algèbre de Banach. Supposons que* $\mathrm{Sp}\, f(\lambda) \subset \mathbb{R}$ *pour tout* $\lambda \in D$. *Alors* $\mathrm{Sp}\, f(\lambda)$ *est constant sur* D.

On démontre le résultat localement en appliquant à $g(\lambda) = e^{if(\lambda)}$ un autre théorème de Vesentini ([7], Proposition 2) sur la constance du spectre périphérique.

Pour plus de détails, voir par exemple [4].

1. Sous-harmonicité angulaire

Théorème 1.1: *Soient* $0<r<s$, $0<\theta_2-\theta_1<2\pi$, $\Omega = \{z : |z|>s,\ \theta_1<\mathrm{Arg}\, z < \theta_2\}$ *et soit* $\lambda \to f(\lambda)$ *une fonction analytique définie sur un domaine* D *de* $\mathbb{C}$, *à valeurs dans une algèbre de Banach telle que* $\mathrm{Sp}\, f(\lambda) \subset \Omega \cup B(0,r)$, *pour tout* $\lambda \in D$.

Posons

$$\mu(\lambda) = \mathit{Max}\,\{\operatorname{Arg} z \colon z \in \operatorname{Sp} f(\lambda) \cap \Omega\},$$

$$\nu(\lambda) = \mathit{Min}\,\{\operatorname{Arg} z \colon z \in \operatorname{Sp} f(\lambda) \cap \Omega\}.$$

Alors μ *et* $-\nu$ *sont sous-harmoniques sur* D.

Démonstration. Sans perte de généralité on peut supposer que $\operatorname{Sp} f(\lambda) \cap \Omega \neq \emptyset$, pour tout $\lambda \in D$. Sur Ω on considère la branche du logarithme $\operatorname{Log} z = \operatorname{Log}|z| + i \operatorname{Arg} z$ et on définit

$$h(\lambda) = \begin{cases} -i \operatorname{Log} z, & \text{sur } \Omega \\ \alpha \quad , & \text{sur } B(0,r) \end{cases}$$

où $\alpha < \theta_1$ est un nombre réel fixé. D'après le calcul fonctionnel holomorphe on a

$$\operatorname{Sp} h(f(\lambda)) \subset \{-i \operatorname{Log} z \colon z \in \operatorname{Sp} f(\lambda) \cap \Omega\} \cup \{\alpha\}.$$

Donc $\mu(\lambda) = \operatorname{Max}\,\{\operatorname{Re} z \colon z \in \operatorname{Sp} h(f(\lambda))\}$. On applique alors le théorème 0.1 à la fonction analytique $\lambda \to h(f(\lambda))$. Pour $-\nu$ l'argument se fait de façon semblable. □

De là découle le résultat suivant:

Théorème 1.2: *Soit* $\lambda \to f(\lambda)$ *une fonction analytique définie sur un domaine* D *de* $\mathbb{C}$ *à valeurs dans une algèbre de Banach. Supposons que* $\operatorname{Sp} f(\lambda) = \{0,\alpha(\lambda)\}$, *pour tout* $\lambda \in D$ *(ou bien* $\operatorname{Sp} f(\lambda) = \{\alpha(\lambda)\}$, *pour tout* $\lambda \in D$). *Alors* α *est holomorphe sur* D.

Démonstration. Etudions le premier cas, le second se faisant de la même façon. Le théorème de Newburgh (Corollaire 7, page 8 de [1]) implique que α est continue sur D. Soit $D' = D \backslash \{\lambda \colon \alpha(\lambda) = 0\}$. Si D' est vide c'est terminé. Supposons donc D' non vide. D'après le théorème d'extension de Radó, il suffit de prouver que α est localement holomorphe sur D'. Soit $\lambda_0 \in D'$, il existe $\delta > 0$ tel que pour $|\lambda-\lambda_0|<\delta$ on est dans la situation du Théorème 1.1 et dans ce cas $\mu(\lambda) = \nu(\lambda)$. Donc μ et ν sont harmoniques sur $B(\lambda_0,\delta)$, aussi il existe h holomorphe sur ce disque telle que $\mu(\lambda) = \operatorname{Arg} \alpha(\lambda) = \operatorname{Im} h(\lambda)$. Prenant $g(\lambda) = e^{-h(\lambda)}f(\lambda)$, qui est analytique sur D, on a $\operatorname{Sp} g(\lambda) \subset \mathbb{R}$ sur ce disque, donc $\operatorname{Sp} g(\lambda)$ constant d'après le Théorème 0.2. Ainsi $\alpha(\lambda) = ce^{h(\lambda)}$, pour une constante c, ce qui prouve que α est localement holomorphe. □

Corollaire 1.3 (Théorème de variation holomorphe des points isolés du spectre): *Soit* $\lambda \to f(\lambda)$ *une fonction analytique définie sur un domaine* D *de* $\mathbb{C}$ *à valeurs dans une algèbre de Banach. Supposons qu'il existe* $\lambda_0 \in D$, $\alpha_0 \in \operatorname{Sp} f(\lambda_0)$ *et* $r, \delta > 0$ *tels que* $|\lambda-\lambda_0| < \delta$ *implique que* $\lambda \in D$ *et* $\operatorname{Sp} f(\lambda) \cap B(\alpha_0,r)$ *contient seulement un point* $\alpha(\lambda)$. *Alors* α *est holomorphe dans un voisinage de* λ_0.

Démonstration. Si $\operatorname{Sp} f(\lambda_0) = \{\alpha_0\}$ on peut supposer sans perte de généralité que $\operatorname{Sp} f(\lambda) = \{\alpha(\lambda)\}$, pour $|\lambda-\lambda_0| < \delta$, d'après la semi-continuité supérieure du spectre. Alors on applique le Théorème 1.2. Si $\operatorname{Sp} f(\lambda_0) \neq \{\alpha_0\}$, pour les mêmes raisons on peut supposer que $\operatorname{Sp} f(\lambda_0) \cap \{z: |z-\alpha_0| > r\} \neq \emptyset$. D'après le théorème de Newburgh, on peut supposer que pour $|\lambda-\lambda_0| < \delta$ on a $\operatorname{Sp} f(\lambda)$ qui est la réunion de $\alpha(\lambda)$ et d'un morceau non vide inclus dans $\{z: |z-\alpha_0|>r\}$. On introduit la fonction holomorphe h définie par $h(z) = z$ sur $B(\alpha_0,r)$ et $h(z) = 0$ sur $\{z: |z-\alpha_0| > r\}$. Alors $\operatorname{Sp} h(f(\lambda)) = \{0,\alpha(\lambda)\}$, pour $|\lambda-\lambda_0| < \delta$. Ensuite on applique le Théorème 1.2. □

Il est connu que la fonction multiforme $\lambda \to \operatorname{Sp} f(\lambda)$ est un cas particulier de fonction analytique multiforme (pour plus de détails voir

[2], [3], [5], [6]). Il est facile de voir, en paraphrasant les arguments, que le Théorème 1.1 se généralise au cas des fonctions analytiques multiformes.

2. *Fonctions analytiques multiformes convexes*

Soit $\lambda \to K(\lambda)$ une fonction analytique multiforme définie sur D telle que $K(\lambda)$ soit convexe pour tout $\lambda \in D$. Supposons que $\alpha \notin \bigcup_{\lambda \in D} K(\lambda)$. L'angle $\theta(\lambda)$ sous lequel on voit $K(\lambda)$ de α est alors parfaitement défini et l'on a $0 \leq \theta(\lambda) < \pi$. D'après le Théorème 1.1 appliqué à $K(\lambda)-\alpha$ on obtient alors immédiatement le résultat suivant:

Théorème 2.1: *La fonction "angle spectral"* $\lambda \to \theta(\lambda)$ *est sous-harmonique sur* D.

Ce théorème ainsi qu'un résultat de T.J. Ransford ([6], Theorem 6.1) permettent d'améliorer le Théorème 2.1 de [3] de la façon suivante:

Théorème 2.2: (Théorème de Picard pour les fonctions analytiques multiformes convexes): *Soit* $\lambda \to K(\lambda)$ *une fonction analytique multiforme telle que* $K(\lambda)$ *soit convexe pour tout* $\lambda \in \mathbb{C}$. *On a les trois cas possibles:*

a) si $\bigcup_{\lambda \in \mathbb{C}} K(\lambda)$ *évite deux points de* $\mathbb{C}$ *alors* $K(\lambda)$ *est constant sur* $\mathbb{C}$*;*

b) si $\bigcup_{\lambda \in \mathbb{C}} K(\lambda)$ *évite un point* $\alpha \in \mathbb{C}$ *alors* $K(\lambda)$ *est de la forme* $K(\lambda) = \alpha + e^{h(\lambda)}K_0$, *ou* h *est entière et où* K_0 *est un compact convexe fixe;*

c) $\bigcup_{\lambda \in \mathbb{C}} K(\lambda)$ *recouvre tout le plan complexe.*

Démonstration. Si $K(\lambda)$ est convexe alors $K(\lambda) = K(\lambda)^\wedge$ et $K(\lambda)$ est connexe. Donc d'après le théorème de T.J. Ransford il suffit d'étudier la situation b). Introduisons la fonction analytique multiforme. $L(\lambda) = K(\lambda)-\alpha$ et l'angle $\theta(\lambda)$ sous lequel on voit $L(\lambda)$ de 0. Comme θ est sous-harmonique sur $\mathbb{C}$ et majorée par π, cette fonction est constante sur $\mathbb{C}$, d'après le théorème de Liouville pour les fonctions sous-harmoniques. Appelons θ_0 cette constante. D'après le Théorème 1.1 on a $\mu = \nu+\theta_0$ sur $\mathbb{C}$, autrement dit μ est simultanément sous-harmonique et sur-hamonique sur $\mathbb{C}$, donc harmonique. Ainsi il existe h entière telle que $\mu(\lambda) = \operatorname{Im} h(\lambda)$. Considérons $M(\lambda) = e^{-h(\lambda)}L(\lambda)$ qui est une fonction analytique multiforme. On a toujours $M(\lambda) \subset \{z: -\theta_0 \leqslant \operatorname{Arg} z \leqslant 0\}$ donc, d'après le Théorème 1.7 de [3], $M(\lambda)$ est constante et égale à un compact convexe fixe K_0, d'où le résultat. □

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A FAMILY OF HOMOGENEOUS SASAKIAN STRUCTURES ON $S^2 \times S^3$

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Presented by G. de B. Robinson, F.R.S.C.

1. Introduction.

In Sasakian geometry, typical examples are Sasakian manifolds with constant ϕ-holomorphic sectional curvature(cf. S. Tanno[8]). T. Takahashi[7] generalized this notion to Sasakian ϕ-symmetric spaces, whose geometric properties have been studied by himself, Y. Watanabe[9], D.E. Blair and L. Vanhecke[3], [4], [5] and so on. In this note we show that $S^2 \times S^3$ has a family of Sasakian ϕ-symmetric structures, which are not of constant ϕ-holomorphic sectional curvature. Since $S^3 \times S^3/S^1$ is diffeomorphic to $S^2 \times S^3$(see D. Barden[1]), our result is suggested by T. Takahashi[7] and Y. Watanabe[10].

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2. Sasakian ϕ-symmetric spaces.

Let (M,g,ϕ,ξ,η) be a $(2n+1)$-dimensional Sasakian manifold with structure tensors g, ϕ, ξ and η satisfying the following conditions :

$$(2.1) \qquad \phi^2 X = -X + \eta(X) \ , \qquad \eta(\xi) = 1 \ ,$$

$$(2.2) \qquad g(X,\xi) = \eta(X) \ , \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \ ,$$

$$(2.3) \qquad d\eta(X,Y) = -g(\phi X, Y) \ , \qquad (\nabla_X \phi)Y = -\eta(Y)X + g(X,Y)\xi \ ,$$

for any vector fields X, Y on M, where ∇ denotes the Levi-Civita connection. It is easy to see from (2.3) that

$$(2.4) \qquad \nabla_X \xi = -\phi X \ ,$$

from which it follows that ξ is a unit Killing vector field.

The curvature tensor $R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$ of a Sasakian manifold satisfies

$$(2.5) \qquad R(X,\xi)Y = -\eta(Y)X + g(X,Y)\xi \ .$$

Conversely, it is well known that if a Riemannian manifold (M,g) admitting a unit Killing vector field ξ satisfies the condition (2.5), then M is a Sasakian manifold(cf. D.E. Blair[2]).

A geodesic γ on a Sasakian manifold (M,g,ϕ,ξ,η) is said to be a ϕ-geodesic if $\eta(\gamma')=0$. A local diffeomorphism s_m of M, $m \in M$, is said to be a ϕ-geodesic symmetry if its domain U is such that, for every ϕ-geodesic $\gamma(s)$ such that $\gamma(0)$ lies in the intersection of U with the integral curve of ξ through m, $(s_m \circ \gamma)(s) = \gamma(-s)$, for all s with $\gamma(\pm s) \in U$, s being the arc length(cf. [7]). A Sasakian manifold is said to be a globally ϕ-symmetric space if any ϕ-geodesic symmetry is extendable to a global automorphism of M and the Killing vector field ξ generates a global one-parameter subgroup of isometries.

On the other hand, the curvature tensor of a naturally reductive Riemannian homogeneous space is well known(cf. S. Kobayashi and K. Nomizu[6]). Let $M = G/H$ be a naturally reductive Riemannian homogeneous space, and $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be an orthogonal decomposition. Then $\mathfrak{m}$ may be considered as the tangent space of M at $\pi(e)$, where π denotes the canonical projection $G \longrightarrow G/H$ and e is the unit element of G. Now we have for $X, Y, Z \in \mathfrak{m}$

$$(2.5) \qquad R(X,Y)Z = [[X,Y]_{\mathfrak{h}},Z] + (1/2)[[X,Y]_{\mathfrak{m}},Z]_{\mathfrak{m}} + (1/4)[[Y,Z]_{\mathfrak{m}},X]_{\mathfrak{m}} + (1/4)[[Z,X]_{\mathfrak{m}},Y]_{\mathfrak{m}},$$

where $[X,Y]_{\mathfrak{h}}$ (resp. $[X,Y]_{\mathfrak{m}}$) denotes the $\mathfrak{h}$ (resp. $\mathfrak{m}$)-component of $[X,Y]$.

A Sasakian manifold (M,g,ϕ,ξ,η) which is also a Riemannian homogeneous space is said to be a homogeneous Sasakian space if the structure (g,ϕ,ξ,η) is invariant by the group of the isometries acting transitively on M.

3. $S^3 \times S^3/S^1$.

$S^3 \times S^3/S^1$ may be considered as G/H, where

$$(3.1) \qquad G=\left\{\begin{pmatrix} SU(2) & 0 \\ 0 & SU(2) \end{pmatrix}\right\}, \qquad H=\left\{\begin{pmatrix} e^{-iv} & & & 0 \\ & e^{iv} & & \\ & & e^{iv} & \\ 0 & & & e^{-iv} \end{pmatrix}, v \in R\right\}.$$

Then G is a connected compact Lie group with the Lie algebra $\mathfrak{g}$. The $ad(G)$-invariant positive definite symmetric bilinear form $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow R$ is taken $-(1/4)\mathrm{trace}(\ \ )$ on $\mathfrak{g} = su(2) + su(2)$ An orthogonal basis for $\mathfrak{g}$ with respect to this form is given by

(3.2)
$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} -i & & & 0 \\ & i & & \\ & & -i & \\ 0 & & & i \end{pmatrix}, \quad e_6 = \begin{pmatrix} -i & & & 0 \\ & i & & \\ & & i & \\ 0 & & & -i \end{pmatrix}.$$

First we list the multiplication table for completeness :

(3.3)
$$\begin{aligned} &[e_1,e_2] = -(e_5+e_6), && [e_1,e_3] = 0, && [e_1,e_4] = 0, && [e_1,e_5] = 2e_2, \\ &[e_1,e_6] = 2e_2, && [e_2,e_3] = 0, && [e_2,e_4] = 0, && [e_2,e_5] = -2e_1, \\ &[e_2,e_6] = -2e_1, && [e_3,e_4] = -(e_5-e_6), && [e_3,e_5] = 2e_4, && [e_3,e_6] = -2e_4, \\ &[e_4,e_5] = -2e_3, && [e_4,e_6] = 2e_3, && [e_5,e_6] = 0 . \end{aligned}$$

We take a basis $\{e_6\}$ for the Lie algebra $\mathfrak{h}$, and set $\mathfrak{m}$ = the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, spanned by $\{e_1,e_2,e_3,e_4,e_5\}$. Let us consider the orthogonal decomposition

(3.4)
$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h} ,$$

with $[\mathfrak{h},e_5] = 0$. It is seen from (3.3) that

(3.5)
$$[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m} .$$

Moreover, by direct calculation using (3.3) we have

(3.6)
$$<[Z,X]_{\mathfrak{m}},Y> + <X,[Z,Y]_{\mathfrak{m}}> = 0 ,$$

for all $X, Y, Z \in \mathfrak{m}$. Since H is connected, by (3.4), (3.5) and (3.6) we have the following

LEMMA 1. G/H is a naturally reductive Riemannian homogeneous space.

REMARK. This follows also easily from the fact that G/H is a normal homogeneous space(cf. S. Kobayashi and K. Nomizu[6]).

Now we shall calculate the curvature tensor of G/H at $\pi(e)$. Using the multiplication table (3.3) and the formula (2.5), we have

$R(e_1,e_2)e_1 = (5/2)e_2$, $R(e_1,e_2)e_2 = -(5/2)e_1$, $R(e_1,e_2)e_3 = -e_4$,
$R(e_1,e_2)e_4 = e_3$, $R(e_1,e_3)e_2 = -(1/2)e_4$, $R(e_1,e_3)e_4 = (1/2)e_2$,
$R(e_1,e_4)e_2 = (1/2)e_3$, $R(e_1,e_4)e_3 = -(1/2)e_2$, $R(e_1,e_5)e_1 = (1/2)e_5$,
$R(e_1,e_5)e_5 = -e_1$, $R(e_2,e_3)e_1 = (1/2)e_4$, $R(e_2,e_3)e_4 = -(1/2)e_1$,
$R(e_2,e_4)e_1 = -(1/2)e_3$, $R(e_2,e_4)e_3 = (1/2)e_1$, $R(e_2,e_5)e_2 = (1/2)e_5$,
$R(e_2,e_5)e_5 = -e_2$, $R(e_3,e_4)e_1 = -e_2$, $R(e_3,e_4)e_2 = e_1$,
$R(e_3,e_4)e_3 = (5/2)e_4$, $R(e_3,e_4)e_4 = -(5/2)e_3$, $R(e_3,e_5)e_3 = (1/2)e_5$,
$R(e_3,e_5)e_5 = -e_3$, $R(e_4,e_5)e_4 = (1/2)e_5$, $R(e_4,e_5)e_5 = -e_4$,
others $= 0$.

Now e_5 defines a left invariant vector field on G . Since $[e_5,e_6] = 0$ holds, $d\pi(e_5)$ defines a vector field on G/H, which will be denoted by ξ , and also a 1-form η on G/H . Moreover, if we put $\phi = -\nabla\xi$, then the tensors ϕ , ξ and η defined above satisfy the conditions (2.1), (2.2) and (2.5). Thus we have

LEMMA 2. (g,ϕ,ξ,η) *is a homogeneous Sasakian structure on* G/H .

By Lemmas 1 and 2, Theorem 7 in [4] implies the following

THEOREM 3. $(G/H,g,\phi,\xi,\eta)$ *is a globally* ϕ*-symmetric space*.

By deforming (g,ϕ,ξ,η) by D-homothety of S. Tanno[8], we obtain the following

THEOREM 4. *There exists a family of Sasakian globally* ϕ*-symmetric structures on* $S^3{\times}S^3/S^1$.

REMARK. $S^3{\times}S^3/S^1$ may be regarded as a principal circle bundle over the complex quadric of complex dimension 2(cf. Watanabe[10]). Therefore, it seems that Theorems 3 and 4 are also shown by using the results of T. Takahashi[7] and Y. Watanabe[10].

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