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CONTENTS

IRRATIONALITE DE $\zeta(s)$ DANS LE CORPS DES SERIES FORMELLES $\mathbb{F}_q((\frac{1}{t}))$

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Abstract

In [1] Y. Hellegouarch defined a zeta function in $\mathbb{F}_q((\frac{1}{x}))$ which is a generalization of Euler's zeta function but is not Artin's zeta function. The aim of this paper is to prove a result similar to those of Euler and Apéry concerning the values of this function at positive integers s such that $1 < s < q$.

1. Notations et définitions

Les notations sont celles de [1], c'est dire que l'on pose :

$$Z = \mathbb{F}_q[t], \quad Q = \mathbb{F}_q(t), \quad R = \mathbb{F}_q((\frac{1}{t}))$$

et, lorsque :

$$x = a_n t^n + \ldots + a_o + \frac{a_{-1}}{t} + \ldots + \frac{a_{-n}}{t^n} + \ldots$$

avec $n \in \mathbf{Z}$ et $a_n \neq 0$:

$$\sigma(x) = a_n \qquad \text{et} \qquad |x| = q^n.$$

Si l'on prolonge l'application $x \to |x|$ en écrivant $|0| = 0$, on obtient une valeur absolue ultramétrique sur R.

Pour s entier ≥ 1, on pose :

$$(1) \qquad \zeta(s) = \sum_{\substack{x \in Z \\ \sigma(x)=1}} \frac{1}{x^s} = \sum_{n=0}^{\infty} u_n(s)$$

avec :

$$u_n(s) = \sum_{\substack{x \in Z \\ \sigma(x) = 1 \\ |x| = q^n}} \frac{1}{x^s} = \frac{N_n(s)}{D_n(s)}$$

où :

$$\begin{cases} D_n(s) = \prod_{(a_o,\dots,a_{n-1})} (t^n + a_{n-1}t^{n-1} + \dots + a_o)^s \\ N_n(s) = \sum_{(a_o,\dots,a_{n-1})} \prod_{\substack{(b_o,\dots,b_{n-1}) \\ \neq (a_o,\dots,a_{n-1})}} (t^n + b_{n-1}t^{n-1} + \dots + b_o)^s \end{cases}$$

Remarque : Sauf mention expresse du contraire :

$$(a_o,\dots,a_{n-1}) \in \mathbb{F}_q^n,\ (b_o,\dots,b_{n-1}) \in \mathbb{F}_q^n, \text{ etc.}$$

2. Méthode et résultats

Carlitz [2] a donné l'expression suivante pour $D_n(s)$:

$$D_n(s) = \left[\prod_{i=1}^{n} (t^{q^i} - t)^{q^{n-i}}\right]^s$$

et on va montrer que :

$$N_n(s) = (-1)^{ns} \left[\prod_{i=1}^{n} (t^{q^i} - t)^{q^{n-i}-1}\right]^s \tag{2}$$

à l'aide des deux lemmes suivants :

Lemme 1.- X étant une indéterminée et s un entier tel que $1 < s < q$, on a :

$$\sum_{i \in \mathbb{F}_q} \prod_{\substack{j \neq i \\ j \in \mathbb{F}_q}} (X+j)^s = (-1)^s$$

La preuve consiste à remarquer que les fractions rationnelles $\frac{1}{(X-X^q)^s}$ et $\sum_{i\in\mathbb{F}_q} \frac{1}{(X+i)^s}$ ont les mêmes parties polaires.

<u>Lemme 2</u>.- <u>Si</u> λ <u>et</u> μ_k <u>désignent les applications</u> :

$$\begin{cases} R \xrightarrow{\lambda} R \\ x \longmapsto x^q - x \end{cases} \qquad \begin{cases} R \xrightarrow{\mu_k} R \\ x \longmapsto x/(t^{q^k}-t) \end{cases}$$

<u>on a</u> :

$$\mu_n \circ \lambda \circ \mu_{n-1} \circ \lambda \circ \dots \circ \mu_1 \circ \lambda\,(t^n) = 1$$

Un calcul direct de $N_n(s)$ nous donne alors l'identité remarquable suivante :

<u>Théorème 1</u>.- <u>Pour tout</u> s <u>tel que</u> $1 < s < q$, <u>on a</u> :

$$u_n(s) = \left[\frac{(-1)^n}{\prod_{i=1}^{n} (t^{q^i}-t)}\right]^s$$

d'où :

<u>Corollaire 1</u>.- <u>Pour tout</u> s <u>tel que</u> $1 < s < q$, <u>on a</u> :

$$\zeta(s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{ns}}{\prod_{i=1}^{n} (t^{q^i}-t)^s} \tag{3}$$

3. <u>Calcul de</u> $N_n(s)$

On pose, pour tout $x \in R$:

$$\begin{cases} H_{n,s}(x) = \sum\limits_{j \in \mathbb{F}_q} \prod\limits_{\substack{a_n \neq j \\ (a_o,\dots,a_n)}} (x + a_n t^n + \dots + a_o)^s \\ A_{n,s}(x) = \sum\limits_{(a_o,\dots,a_n)} \prod\limits_{\substack{(b_o,\dots,b_n) \\ \neq (a_o,\dots,a_n)}} (x + b_n t^n + \dots + b_o)^s \end{cases}$$

Le calcul de $H_{n,s}(x)$ et de $A_{n,s}(x)$ va nous montrer que ces expressions ne dépendent pas de x.

La valeur de $N_n(s)$ en découlera aussitôt :

$$N_n(s) = A_{n-1,s}(t^n)$$

a) On a :

$$H_{n,s}(x) = \sum_{j \in \mathbb{F}_q} \prod_{\substack{a_n \neq j \\ (a_1,\dots,a_n)}} (\lambda(x) + \lambda(a_n t^n) + \dots + \lambda(a_1 t))^s$$

$$= (t^q - t)^{q^{n-1}(q-1)s} \sum_{j \in \mathbb{F}_q} \prod_{\substack{a_n \neq j \\ (a_1,\dots,a_n)}} (\mu_1 \circ \lambda(x) + \mu_1 \circ \lambda(a_n t^n) + \dots + a_1)^s$$

En se servant de la linéarité de λ et μ et des lemmes 1 et 2, en itérant (n-1) fois et en utilisant à chaque fois le lemme 2, on obtient :

$$H_{n,s}(x) = \left[- \prod_{i=1}^{n} (t^{q^i} - t)^{(q-1)q^{n-i}} \right]^s . \tag{4}$$

b) (4) nous montre que $H_{n,s}(x)$ ne dépend pas de x ; nous allons utiliser cette propriété pour prouver que $A_{n,s}(x)$ ne dépend pas non plus de x en faisant un raisonnement par récurrence sur n.

Le lemme 1 prouve que la propriété est vraie pour $n = 0$.

Supposons maintenant que $A_{n-1,s}(x)$ ne dépende pas de x, on a alors :

$$A_{n,s}(x) = \sum_{(a_o,\ldots,a_n)} \prod_{\substack{(b_o,\ldots,b_n)\\ \neq(a_o,\ldots,a_n)}} (x+b_nt^n+\ldots+b_o)^s$$

$$= \sum_{a_n} \prod_{b_n\neq a_n} (x+b_nt^n+\ldots+b_o)^s\ A_{n-1,s}(x+a_nt^n)$$

Comme $A_{n-1,s}(x+a_nt^n)$ ne dépend pas de a_n, on a finalement :

$$A_{n,s}(x) = A_{n-1,s}(x)\ H_{n,s}(x)$$

Ce qui montre que $A_{n,s}(x)$ ne dépend pas de x d'après (4) et donne :

$$A_{n,s} = A_{o,s} \prod_{i=1}^{n} H_{i,s}$$

En utilisant (4) on trouve facilement (après simplifications) :

$$N_n(s) = A_{n-1,s}(t^n) = A_{n-1,s} = A_{o,s} \prod_{i=1}^{n-1} H_{i,s}$$

$$= (-1)^{ns}\left[\prod_{i=1}^{n} (t^{q^i}-t)^{q^{n-i}-1}\right]^s \qquad \square$$

4. Irrationalité de $\zeta(s)$

De l'identité (3) on tire :

Corollaire 2.- Si l'on pose :

$$1 + \sum_{\nu=1}^{n} \frac{(-1)^{\nu s}}{\prod_{i=1}^{\nu} (t^{q^i}-t)^s} = \frac{V_n(s)}{D_n(s)}$$

on a :

$$0 < \left|\zeta(s) - \frac{V_n(s)}{D_n(s)}\right| = \frac{C}{|D_n(s)|^q} < \frac{1}{|D_n(s)|^q}$$

où C désigne la constante q^{-qs}.

La méthode classique de Liouville ([3], p. 168) nous permet alors de conclure par :

Théorème 2.- Pour tout s tel que $1 < s < q$, $\zeta(s)$ n'est racine d'aucun polynôme non identiquement nul, à cœfficients dans Z et de degré $< q$.

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LA REGLE OPTIMALE DU TRAPEZE POUR L'INTEGRALE DE RIEMANN-STIELTJES D'UNE FONCTION DONNEE

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Résumé: Pour estimer $\int_{a,b} f(x)dg(x)$, nous cherchons la partition X à n+1 noeuds pour laquelle la règle composée du trapèze, T[f,g;X], s'approche le mieux de l'intégrale. Pour f convexe et g croissante, il s'agit d'étudier la minimisation de la fonctionnelle $X \longmapsto T[f,g;X]$. Il y a au moins une partition optimale; nous donnerons les conditions différentielles remplies par une telle partition optimale. Malheureusement, la partition optimale n'est pas toujours unique comme en fera foi un exemple.

1. La règle du trapèze pour l'intégrale de Riemann-Stieltjes.

Soit g(x) une fonction à variation bornée définie sur l'intervalle [a,b], soit $X = \{x_k\}_{0 \le k \le n}$ une partition de [a,b] en n sous-intervalles: $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, nous introduisons la règle du trapèze relative à cette partition. Comme en [2], nous définissons des poids partiels $p_{k-}[g]$ et $p_{k+}[g]$:

$$p_{k-}[g] = \int_{x_{k-1},x_k} (x-x_{k-1})/(x_k-x_{k-1})\, dg(x) \text{ où } 1 \le k \le n$$

$$p_{k+}[g] = \int_{x_k,x_{k+1}} (x_{k+1}-x)/(x_{k+1}-x_k)\, dg(x) \text{ où } 0 \le k \le n-1.$$

Par convention, nous posons $p_{0-}[g] = 0$ et $p_{n+}[g] = 0$. Partant de ces définitions, les poids de la règle du trapèze sont $p_k[g] = p_{k-}[g] + p_{k+}[g]$ où $0 \le k \le n$. Si f est une fonction définie sur [a,b], la règle du trapèze est

$$T[f,g;X] = \sum_{0 \le k \le n} p_k[g]\, f(x_k).$$

Lorsque f est une fonction convexe et que g est non-décroissante, il est facile de vérifier que $\int_{a,b} f(x)dg(x)$ est nécessairement majorée par T[f,g;X]. La question que nous voulons aborder est de déterminer la partition X pour laquelle T[f,g;X] s'approche le plus possible de $\int_{a,b} f(x)dg(x)$ alors que le nombre n+1 de noeuds de X est déterminé. Il s'agit d'étudier la minimisation de la fonctionnelle

$X \longmapsto T[f,g;X]$.

2. Partitions optimales.

Soit $\mathbf{P}_n$ l'ensemble des partitions qui se servent d'au plus n+1 points de [a,b], muni de la métrique de Hausdorff entre parties compactes, $\mathbf{P}_n$ est un espace compact et la fonctionnelle $X \longmapsto T[f,g;X]$ est continue.

Lemme 1. Soit n un entier au moins égal à 1, soient f une fonction continue sur [a,b] et dg une mesure signée sur [a,b], alors il existe une règle du trapèze $T[f,g;X_0]$ avec au plus n+1 noeuds telle que

$T[f,g;X_0] = \inf \{ T[f,g;X] : X \in \mathbf{P}_n \}$.

Une partition X de n+1 noeuds admet n-1 paramètres réels libres, x_1, x_2, ..., x_{n-1}. La fonctionnelle T[f,g;X] est donc une fonction de n-1 variables réelles. Nous présentons les formules des dérivées partielles de cette fonctionnelle. Désignons par $E_f(y,x)$ la quantité $f(y)-f(x)-(y-x)f'(x)$, c'est l'écart entre f(y) et l'ordonnée de la tangente à f au point x évaluée à l'abscisse y.

Théorème 2. Soit $X=\{x_0,x_1,\ldots,x_n\}$ une partition de $\mathbf{P}_n$ dont tous les noeuds sont fixes sauf le noeud x_k où $0 < k < n$, soient f une fonction dérivable sur [a,b] et g une fonction à variation bornée, alors la fonction $x_k \mapsto T[f,g;X]$ admet une dérivée sur (x_{k-1},x_{k+1}) dont la valeur est

(1) $\quad p_{k-}[g]\, E_f(x_{k-1},x_k) / (x_k-x_{k-1}) = p_{k+}[g]\, E_f(x_{k+1},x_k) / (x_{k+1}-x_k)$.

Corollaire 3. Si f est dérivable et si X est une partition qui minimise la fonctionnelle T[f,g;X], pour chaque valeur de k comprise entre 1 et n-1, la dérivée partielle de la fonctionnelle par rapport au noeud x_k est nulle.

3. Interprétation géométrique des conditions extrémales.

Dans cette section, nous présentons diverses représentations des conditions d'optimalité de l'intégration numérique de $\int_{a,b} f\, dg$. Une autre manière de regarder l'identité (1) est de dire ceci. Les écarts entre la pente $f'(x_k)$ de la tangente et les pentes respectives $(f(x_k)-f(x_{k-1})/(x_k-x_{k-1})$ et $(f(x_{k+1})-f(x_k)/(x_{k+1}-x_k)$ des cordes à gauche et à droite de x_k sont équilibrés par les poids partiels $p_{k-}[g]$ et $p_{k+}[g]$ à gauche et à droite de x_k.

Imaginons la situation où f admet la représentation intégrale suivante: $f(x) = \int_{a,x}(x-t)dm(t)$ où m est une fonction non-décroissante continue. On vérifie que $E_f(x_{k-1},x_k)/(x_k-x_{k-1}) = p_{k-}[m]$ et $E_f(x_{k+1},x_k)/(x_{k+1}-x_k) = p_{k+}[m]$. Les conditions d'optimalité peuvent alors s'écrire

(2) $\quad p_{k-}[m]\, p_{k-}[g] = p_{k+}[m]\, p_{k+}[g]$

C'est dire que les poids partiels des règles du trapèze pour l'intégrale de Riemann-Stieltjes relatives aux mesures dg et dm sont équilibrés en étant inversement proportionnels lorsque les conditions d'optimalité sont satisfaites.

Enfin on peut décrire d'une autre manière les conditions d'optimalité à l'aide de la fonction $h(x) = \int_{a,x} (x-t)dg(t)$:

(3) $E_f(x_{k-1},x_k)E_h(x_{k-1},x_k)/(x_k-x_{k-1})^2 = E_f(x_{k+1},x_k)E_h(x_{k+1},x_k)/(x_{k+1}-x_k)^2$

4. Dualité entre règles du trapèze.

Les conditions symétriques d'optimalité rencontrées dans la section précédente nous amènent à traiter simultanément de deux règles du trapèze. Soient f(x) et h(x) deux fonctions convexes dérivables, on peut trouver deux mesures dm(x) et dg(x) telles que

(4) $f(x) = f(a) + f'(a)(x-a) + \int_{a,x} (x-t)dm(t)$

(5) $h(x) = h(a) + h'(a)(x-a) + \int_{a,x} (x-t)dg(t)$

Le prochain théorème fait voir que si X est une partition de [a,b], les deux règles du trapèze T[f,g;X] et T[h,m;X] sont très fortement reliées indépendamment du choix de X.

Théorème 4. Si f et h sont deux fonctions convexes dérivables, si f,g,h et m satisfont aux relations (4) et (5), alors pour toute partition,

$$T[f,g;X]-T[h,m;X] = f(b)h'(b)-f'(b)h(b)-f(a)h'(a)+f'(a)h(a)$$

$$= \int_{a,b} f(x)dg(x) - \int_{a,b} h(x)dm(x)$$

Corollaire 5. Si X est une partition qui minimise T[f,g;X] dans la classe des partitions de P_n, alors la même partition X minimise T[h,m;X] parmi les partitions d'au plus n+1 points.

5. EXEMPLE.

Comme exemple, nous prenons pour intervalle [a,b] l'intervalle [0, 3] et nous choisissons f et g comme ce qui suit:

f(x) =	-5x	sur [0, 1]	g(x) = 0	sur [0,1),
	(x-6)	sur [1, 2]	1	sur [1, 2)
	-4(3-x)	sur [2, 3]	3	sur [2, 3].

Soit la partition dépendant d'un seul paramètre réel t: X_t = (0, t, 3), on pose u(t) = T[f,g;X_t]. Le poids p(t) qu'accorde la règle du trapèze au noeud (t) est 4/(3-t) si 0 ≤ t ≤ 1, 1/t + 2/(3-t) si 1 ≤ t ≤ 2, 5/t si 2 ≤ t ≤ 3.
u(t) = p(t)f(t). Sur les intervalles [0,1], [1,2] et [2,3], la fonction u(t) vaut respectivement: 20 - 60/(3-t) , -1 - 6/t - 6/(3-t) , 20 - 60/t. La fonction u(t) admet deux points de minimum: t = 1 et t = 2. De plus, t = 3/2 est un point de maximum local.

Cet exemple illustre la possibilité que deux partitions soient optimales. Il peut donc arriver que la partition optimale ne soit pas unique. Il est bon de remarquer que la fonction f n'est pas dérivable en t = 1 et en t = 2 et que la fonction g est discontinue en ce même lieu. Cependant, il est possible de perturber légèrement f et g pour faire disparaître ces irrégularités tout en s'assurant de l'existence de deux partitions optimales. De plus, il peut arriver que les conditions différentielles d'optimalité soient satisfaites alors que la partition considérée n'est pas l'endroit du minimum.

6. Conclusion.

Si la règle classique du trapèze a été fort bien étudiée, la règle du trapèze pour l'intégrale de Riemann-Stieltjes a attiré moins l'attention des chercheurs. Ce que nous avons exposé complète les travaux antérieurs [1] et [2]. Ce travail même provient d'une partie de la thèse de doctorat du second auteur, [3].

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NONLINEAR LASER OPTICS I: DUALITY OF SEMISIMPLE RINGS AND PHASE MATCHING

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Abstract. The breakthrough of representation theory of symmetric groups to applications on representations of general linear groups is due to I. Schur [6]. He recognized the fundamental duality which was elaborated and extended by H. Weyl [7]. - There has been considerable theoretical and experimental study of the nonlinear optical properties of anisotropic solids (crystals), liquids, and gases. In this first paper of a series of subsequent articles devoted to nonlinear laser optics we calculate group theoretically the phase matching conditions of n-th order harmonic generation of high-power laser radiation by the multiplicity formula valid for the tensor bimodule $E^{\otimes n}$ where E denotes the complex vector space spanned by the phases of the pump fields.

1. Dipole Suszeptibilities in the Frequency Domain

Nonlinear optical interactions are generally weak when compared to linear optical effects. Therefore experimental observation and verification of many nonlinear effects had to wait for the intense coherent fields that are generated by high-power lasers. In nonlinear optics the optical properties of materials are determined by the response of electrons and ions to incident laser light. The interaction between intense optical radiation and matter can be described conceptually as follows. Laser light that is incident on a medium induces in the electron or ion distribution an oscillating dipole moment. The induced dipole moment, or polarization, in turn radiates a second optical field that can interfere with the incident field. The secondary fields radiated by the induced polarization can contain frequencies that are different from those in the incident fields and give rise to nonlinear optical effects. Of course no physical significance can be attributed to the order in which the pump fields at different frequencies are enumerated. As a result, all the dipole suszeptibilities remain unchanged when their coordinates with respect to the pump fields and the corresponding frequencies are simultaneously interchanged. This property which is the key for the subsequent reasoning is generally referred to as intrinsic permutation symmetry (cf. Bloembergen [1]).

Let $n \geqq 1$ denote a natural number and $\mathfrak{S}_n$ the symmetric group acting by permutations on the sets of n letters. If E denotes the complex vector space of dimension $m \geqq 1$ spanned by the different phases of the pump fields then the intrinsic permutation symmetry takes the following form.

Theorem 1. The n-th order dipole suszeptibilities define bisymmetric transformations between the incident fields and the induced polarizations, i.e., linear mappings of the n-th tensor power $E^{\otimes n}$ into itself that centralize the linear permutation representation of the symmetric group $\mathfrak{S}_n$ in $E^{\otimes n}$.

Let $\mathbf{C}\mathfrak{S}_n$ denote the semisimple group algebra of $\mathfrak{S}_n$ having coefficients in the field $\mathbf{C}$ of complex numbers. Identify the complex algebras $(\mathrm{End}_{\mathbf{C}}(E))^{\otimes n}$ and $\mathrm{End}_{\mathbf{C}}(E^{\otimes n})$ by their natural $\mathbf{C}$-algebra isomorphism. Notice that the natural $\mathbf{C}$-algebra isomorphism is also a $\mathbf{C}\mathfrak{S}_n$-isomorphism of the algebras under consideration with respect to their natural right $\mathbf{C}\mathfrak{S}_n$-module structures. It follows

Corollary. The endomorphisms of $E^{\otimes n}$ that are defined by the n-th order dipole suszeptibilities belong to the subalgebra $S = \mathrm{End}_{\mathbf{C}\mathfrak{S}_n}(E^{\otimes n})$ of the complex algebra $(\mathrm{End}_{\mathbf{C}}(E))^{\otimes n}$.

2. The Multiplicity Formula

Let $\mathcal{P}_n$ denote the set of all partitions of $n \geq 1$. For any partition $\lambda \in \mathcal{P}_n$ let F_λ denote its frame and $|F_\lambda|$ the associated hook product, i.e., the product of the hook lengths of F_λ (see, for instance, Boerner [2] or James-Kerber [4]). Furthermore, for $\lambda \in \mathcal{P}_n$ let $\mathcal{T}_\lambda$ denote the set of all tableaux associated with F_λ, $T_\lambda \in \mathcal{T}_\lambda$ a standard tableau associated with F_λ, $\frac{1}{|F_\lambda|} e_{T_\lambda}$ the idempotent in $\mathbf{C}\mathfrak{S}_n$ associated with T_λ and B_λ the block $\sum_{T \in \mathcal{T}_\lambda} \mathbf{C}\mathfrak{S}_n . e_T = \sum_{T \in \mathcal{T}_\lambda} e_T . \mathbf{C}\mathfrak{S}_n$ in $\mathbf{C}\mathfrak{S}_n$. If $\varkappa(\lambda)$ denotes the number of rows of F_λ, then $R = \oplus \{ B_\lambda \mid \lambda \in \mathcal{P}_n, \varkappa(\lambda) \leq m \}$ is a semisimple ring and the annulator of $\mathbf{C}\mathfrak{S}_n$ in $E^{\otimes n}$ is given by the direct sum of blocks $\oplus \{ B_\lambda \mid \lambda \in \mathcal{P}_n, \varkappa(\lambda) > m \}$. Using the preceding notations, we have

Theorem 2. The n-th tensor power $E^{\otimes n}$ can be decomposed into a direct sum of tensor products of the simple left S-modules $E^{\otimes n}.e_{T_\lambda}$ with the corresponding simple right ideals $e_{T_\lambda}.R = e_{T_\lambda}.\mathbf{C}\mathfrak{S}_n$ $(\lambda \in \mathcal{P}_n, \varkappa(\lambda) \leq m)$ in R and $\mathbf{C}\mathfrak{S}_n$, respectively. Thus there is a (S,R)-bimodule isomorphism

$$E^{\otimes n} = \oplus \{ E^{\otimes n}.e_{T_\lambda} \otimes_{\mathbf{C}} e_{T_\lambda}.R \mid \lambda \in \mathcal{P}_n, \varkappa(\lambda) \leq m \}$$

$$= \oplus \{ E^{\otimes n}.e_{T_\lambda} \otimes_{\mathbf{C}} e_{T_\lambda}.\mathbf{C}\mathfrak{S}_n \mid \lambda \in \mathcal{P}_n, \varkappa(\lambda) \leq m \}.$$

For any partition $\lambda \in \mathcal{P}_n$ let $\chi_\lambda : \mathfrak{S}_n \longrightarrow \mathbf{C}$ denote the character of the simple right $\mathbf{C}\mathfrak{S}_n$-module $e_{T_\lambda}.\mathbf{C}\mathfrak{S}_n$. Then we have $f_\lambda := \chi_\lambda(1^n) \in \mathbf{N}$ and the identity

$$f_\lambda = \dim_{\mathbf{C}} e_{T_\lambda}.\mathbf{C}\mathfrak{S}_n = n!/|F_\lambda|$$

holds. Notice that f_λ is the number of standard tableaux associated with the frame F_λ and the degree of the irreducible linear representation of the group $\mathfrak{S}_n$ associated with F_λ.

Corollary. For any partition $\lambda \in \mathcal{P}_n$ such that $\varkappa(\lambda) \leq m$, the multiplicity of the simple left S-module $E^{\otimes n}.e_{T_\lambda}$ in $E^{\otimes n}$, i.e., the number of times $E^{\otimes n}.e_{T_\lambda}$ occurs in $E^{\otimes n}$, is given by f_λ.

3. The Phase Matching Conditions

The frequency up-conversion of high-power, high-energy, neodymium-doped glass laser beams for fusion applications has recently attracted considerable attention, because shorter wavelength radiation appears to offer substantial advantages in terms of increased absorption and improved energy coupling between the laser radiation and the target. Such laser beams have been frequency doubled, tripled, and quadrupled using, for instance, potassium dihydrogen phosphate (KDP) crystals.

KH_2PO_4 (KDP) is a negative uniaxial crystal belonging to the tetragonal crystal system. Its crystal class is the point group $\bar{4}2m$ (international notation) or D_{2d} (Schoenflies notation) and admits no inversion symmetry. The point group D_{2d} is a realization of the dihedral group D_4 of order 8 by rigid motions of the Euclidean space $\mathbb{R}^3$. There are four irreducible linear representations of degree 1 and one irreducible linear representation of degree 2 of D_4. The latter is induced by the character $id_{\mathbb{Z}_4}$ of the cyclic subgroup $\mathbb{Z}_4$ of D_4 and acts on the complex vector space E spanned by the phase of the pump field and the conjugate phase ("Principle of optical phase conjugation" [8]). Therefore we have $\dim_{\mathbb{C}} E = m = 2$. If $N_{o,\omega}$ denotes the refractive index of the ordinary wave, and $N_{e,\omega}$ the refractive index of the extraordinary wave of frequency ω in the crystal, the multiplicity formula of the Corollary of Theorem 2 supra yields a group theoretic approach to nonlinear optical harmonic generation.

Theorem 3. <u>The phase matching condition of the refractive index</u> $N_{e,n\omega}$ <u>producing the n-th order harmonic generation by uniaxial crystals without inversion symmetry expresses</u> $N_{e,n\omega}$ <u>as a weighted average of the refractive indices</u> $N_{o,k\omega}$ <u>and</u> $N_{e,k\omega}$ $(1 \leq k \leq n-1)$. <u>Their multiplicities</u> $f_\lambda = n!/|F_\lambda|$ <u>are determined by the hook products of the frames</u> F_λ <u>with</u> $\lambda \in \mathcal{P}_n$ <u>and</u> $\varkappa(\lambda) \leq 2$.

Let us compute explicitly the cases n = 2,3,4 for negative uniaxial crystals which have been checked by experimental measurements for KDP crystals [3].

<u>Type II doubling</u>: $N_{e,2\omega} = \frac{1}{2}(N_{o,\omega}+N_{e,\omega})$

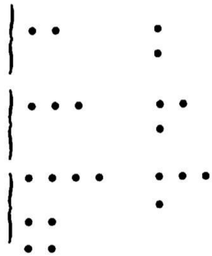
<u>Type I doubling</u>: $N_{e,2\omega} = N_{o,\omega}$

<u>Type II tripling</u>: $N_{e,3\omega} = \frac{1}{3}(2N_{o,2\omega}+N_{e,\omega})$

<u>Type I tripling</u>: $N_{e,3\omega} = \frac{1}{3}(2N_{o,2\omega}+N_{o,\omega})$

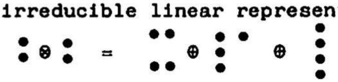
<u>Type II quadrupling</u>: $N_{e,4\omega} = \frac{1}{6}(N_{o,\omega}+3N_{e,3\omega}+2N_{e,2\omega})$

<u>Type I quadrupling</u>: $N_{e,4\omega} = \frac{1}{6}(N_{o,\omega}+3N_{e,3\omega}+2N_{o,2\omega})$



4. Bypass Phase Matching

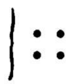
If we use the following rule for the decomposition of the tensor product of irreducible linear representations of $\mathfrak{T}_n$,

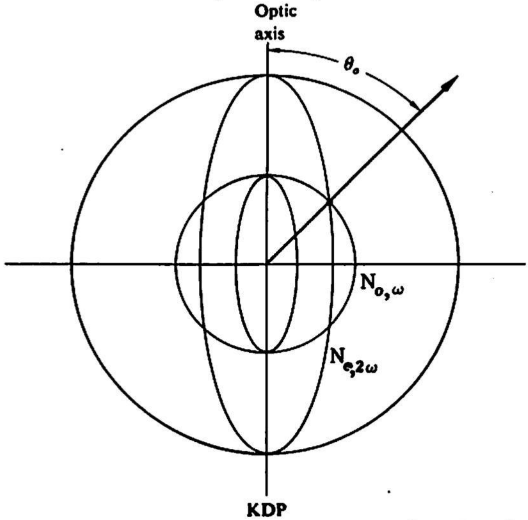


we get different phase matching conditions which read as follows:

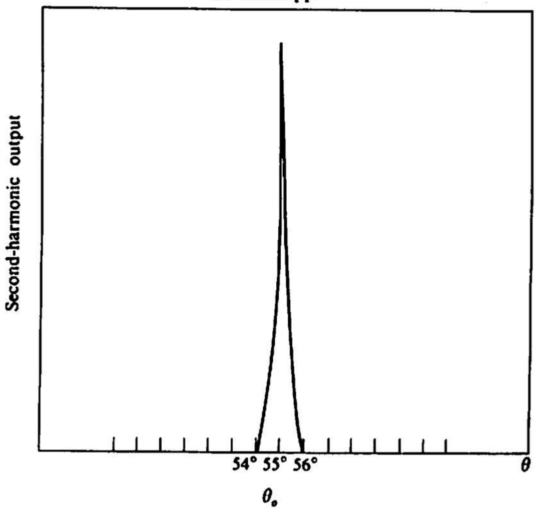
<u>Type II quadrupling</u>: $N_{e,4\omega} = \frac{1}{2}(N_{o,2\omega}+N_{e,2\omega})$

<u>Type I quadrupling</u>: $N_{e,4\omega} = N_{o,2\omega}$





Example. Type I doubling by focussing red (694.3 nm) ruby laser light onto a KDP crystal at the phase matching angle $\Theta_o = 55°$. The laser beam is incident at the polarization angle 45°; see [5].



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THE IDEAL STRUCTURE OF THE MULTIPLIER ALGEBRA OF AN AF ALGEBRA

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A description is given of the lattice of closed two-sided ideals of the multiplier algebra, M(A), of a separable approximately finite-dimensional C^*-algebra A. Namely, it is the same as the lattice of ideals of D(M(A)), the abelian local semigroup of equivalence classes of projections in M(A). D(M(A)) can be described in terms of D(A), the corresponding local semigroup for A. In the case that A is simple, without unit, the ideal lattice of D(M(A)) can be described solely in terms of the space of traces on A, normalized when finite.

1. Let A be an AF algebra, i.e. a separable approximately finite-dimensional C^*-algebra. Denote by M(A) the C^*-algebra of multipliers of A. (M(A) is the largest C^*-algebra containing A as an essential closed two-sided ideal.)

As pointed out in [4], additivity of the equivalence relation introduced by Murray and von Neumann for projections in a C^*-algebra means that the set of equivalence classes is an abelian local semigroup (addition of two classes being defined when they have orthogonal representatives). The main result of [4] was that for the AF algebra A, this local semigroup, D(A), – the range of the (abstract) dimension on A –, is a complete invariant.

It follows that the local semigroup D(A) determines the local semigroup D(M(A)), the range of the dimension on M(A). An explicit description of D(M(A)) in terms of D(A) was given in [7; Theorems 2.1 and 2.6]: it is the set of semicomplemented intervals in D(A), with addition where defined. By this is meant the following. We have D(A) $\subseteq$ D(M(A)), and, using the natural order in D(M(A)) (i.e., $x \leq x + y$), to each $x \in$ D(M(A)) we can associate the interval $[0,x]$, and the intersection $[0,x] \cap$ D(A). This intersection is an interval in D(A), i.e., an upward directed hereditary subset, and it determines x uniquely. An interval D_1 in D(A) is equal to $[0,x] \cap$ D(A) for some $x \in$ D(M(A)) precisely when it is semicomplemented, in the sense that for some interval D_2, the sum of D_1 and D_2 is defined and equal to D(A) (write $D_1 + D_2 =$ D(A)). That the sum of two intervals D_1 and

D_2 is defined means that all sums $g_1 + g_2$ with $g_i \in D_i$ are defined in D(A). The sum of two intervals in D(A), when defined, is always an interval (this follows from the Riesz decomposition property, which holds in D(A)), but the sum of two semicomplemented intervals is not necessarily semicomplemented. Nevertheless, the relation $x_1 + x_2 = x_3$ in D(M(A)) is equivalent to the relation $D_1 + D_2 = D_3$ among the corresponding semicomplemented intervals in D(A).

By an ideal of an abelian local semigroup let us understand a nonempty hereditary subset (i.e., an ideal with respect to the order structure) which is closed under addition, where defined. D(A), then, is an ideal of D(M(A)). (As we shall see, any ideal of D(M(A)) is upward directed.) Since any intersection of ideals is an ideal, the ideals of an abelian local semigroup form a complete lattice with respect to inclusion. It is known that the lattice of ideals of A is isomorphic to the lattice of ideals of D(A) (see e.g. [6], Section 5).

Theorem. *The lattice of ideals of* M(A) *is isomorphic to the lattice of ideals of* D(M(A)).

Proof. For each ideal I of M(A), it is clear that D(I) is an ideal of D(M(A)). For each ideal L of D(M(A)), denote by I(L) the ideal of M(A) generated by the projections with dimension (i.e., equivalence class) in L. Clearly the maps $I \mapsto$ D(I) and $L \mapsto$ I(L) both preserve inclusion. Let us show that these maps are inverse to each other.

Let I be an ideal of M(A). The argument in the first four paragraphs of the proof of Theorem 3.1 of [3] shows that if $a \in I$ then $a = a' + a_0 + a_1 + a_2$, with $a' \in I \cap A$, and with each a_i in I and commuting with some increasing approximate unit for A consisting of projections. From this it follows immediately that I is generated, as a C*- algebra, by its projections. In particular, I = I(D(I)).

Let L be an ideal of D(M(A)). To show that D(I(L)) = L, a somewhat indirect approach seems to be required. Denote by $\tilde{L}$ the ideal of D(M(A) ⊗ K) generated by L, where K is the C*-algebra of compact operators on l^2(N). We shall show that D(I(L) ⊗ K) = $\tilde{L}$, and hence that D(I(L)) = L.

To show that $D(I(L) \otimes K) = \tilde{L}$, we shall first prove that the set of projections in $M(A) \otimes K$ with dimension in $\tilde{L}$ is approximately upward directed. (The question whether this holds already for the set of projections in $M(A)$ with dimension in L seems to be difficult; it is to avoid this question that we have passed to matrices.) Let e and f be projections in $M(A) \otimes K$ with dimension in $\tilde{L}$, and let us show that there exists a projection g in $M(A) \otimes K$ with dimension in $\tilde{L}$ such that $ge - e$ and $gf - f$ are small. Replacing e and f by slightly different (necessarily equivalent) projections, we may suppose that e and f both belong to $M(A) \otimes p_n K p_n$ for some $n = 1,2,...$ where $p_n = e_{11} + \cdots + e_{nn} \in K$. In this case there exists a projection f' in $M(A) \otimes p_{2n} K p_{2n}$ orthogonal to e and equivalent to f. Set $e + f' = p$, and choose $u \in M(A) \otimes p_{2n} K p_{2n}$ with $u^* u = f$ and $uu^* = f'$. Denote by B the sub-C^*-algebra of $M(A) \otimes p_{2n} K p_{2n}$ generated by p and u. Clearly e and f belong to B, p is a full projection in B, and B is separable. Hence by Lemma 2.5 of [2], there exists $v \in M(B \otimes K)$ with $v^* v = 1$ and $vv^* = p \otimes 1$. This implies that $B \otimes K$ contains an approximate unit consisting of projections, each of which is a finite sum of projections equivalent to $p \otimes e_{11}$ (namely, $(v^*(p \otimes (e_{11} + \cdots + e_{kk}))v)_{k \geq 1}$). Since $B \subseteq M(A) \otimes p_{2n} K p_{2n}$, and K may be identified with $p_{2n} K p_{2n} \otimes K$, $B \otimes K$ may be identified with a subalgebra of $M(A) \otimes K$, in such a way that $B \otimes e_{11}$ becomes identified with $B \subseteq M(A) \otimes p_{2n} K p_{2n}$. Then the projections in the above approximate unit for $B \otimes K$ have dimension in $\tilde{L}$, and since $e, f \in B \otimes K$ we may choose one of these projections for g.

An equivalent formulation of the fact proved in the preceding paragraph is that

$$I(L) \otimes K = (\bigcup_{\dim(e) \in \tilde{L}} e(M(A) \otimes K)e)^- .$$

(Again, whether already $I(L) = (\bigcup_{\dim(e) \in L} eM(A)e)^-$ is a question we do not address.) To see that the left side is contained in the right side, note that the right side is an ideal (it is an algebra, and is closed under multiplication on the left and right by unitaries in the algebra $M(A) \otimes K$ with unit adjoined). To see the opposite inclusion, note that $D(I(L) \otimes K) \supseteq \tilde{L}$.

From this description of $\mathrm{I}(L) \otimes \mathrm{K}$ it follows by Lemma 2.1 of [3] that any projection in $\mathrm{I}(L) \otimes \mathrm{K}$ is equivalent to part of some projection with dimension in $\tilde{L}$. Hence $\mathrm{D}(\mathrm{I}(L) \otimes \mathrm{K}) = \tilde{L}$.

To show that $\mathrm{D}(\mathrm{I}(L)) = L$, it is now enough to show that $\tilde{L} \cap \mathrm{D}(\mathrm{I}(L) \otimes e_{11}) = L$. To show this, it is enough to prove that $\tilde{L}$ is the semigroup generated by L. Since, by definition, $\tilde{L}$ is the set of elements majorized by some element of this semigroup, it is enough to prove the Riesz decomposition property for $\tilde{L}$.

Let us prove the Riesz decomposition property for $\mathrm{D}(\mathrm{M}(A))$. (It follows for $\mathrm{D}(\mathrm{M}(A) \otimes \mathrm{K})$ and hence for $\tilde{L}$.) Given the inequality $g \le h + k$ in $\mathrm{D}(\mathrm{M}(A))$, we may replace [1] by $h + k$ and suppose that for some projection e in $\mathrm{M}(A)$, $h = \dim(e)$ and $k = \dim(1 - e)$. By Theorem 3.1 of [5], there exists an increasing approximate unit (e_n) for A consisting of projections commuting with e. If $g = \dim(f)$, and if f commutes with the increasing approximate unit (f_n) consisting of projections, then as shown in the proof of Theorem 2.4 of [3], we may transform f and (f_n) by a unitary in $\mathrm{M}(A)$ and suppose that, after replacing (e_n) and (f_n) by subsequences, we have $f_1 \le e_1 \le f_2 \le e_2 \le \cdots$. Using the Riesz decomposition property in $\mathrm{D}(A)$ we may replace each projection $(f_{n+1} - f_n)f$ by an equivalent projection (inside $f_{n+1} - f_n$) commuting with e_n. Then f, thus modified, also commutes with the sequence (e_n). The decomposition $g = \dim(f) = h_1 + k_1$ with $h_1 \le h$ and $k_1 \le k$ then follows from the Riesz decomposition property for $\mathrm{D}(A)$.

2. In general, it is still a problem to determine the ideal structure of $\mathrm{D}(\mathrm{M}(A))$. Let us consider the case that A is simple. Some results concerning the ideal structure (actually, of $\mathrm{M}(A)$) were given in this case in [3] and, more recently, in [9].

For any A, set $\mathrm{D}(\mathrm{M}(A)) \setminus \mathrm{D}(A) = \partial\mathrm{D}(A)$. Then $\partial\mathrm{D}(A)$ is a sub local semigroup of $\mathrm{D}(\mathrm{M}(A))$, and the ideals of $\partial\mathrm{D}(A)$ are in bijective correspondence with the ideals of $\mathrm{D}(\mathrm{M}(A))$ properly containing $\mathrm{D}(A)$. In the case that A is simple, these are exactly the ideals of $\mathrm{D}(\mathrm{M}(A))$ not equal to $\{0\}$ or $\mathrm{D}(A)$. In this case, it turns out that $\partial\mathrm{D}(A)$ can be canonically described in terms of the convex cone of lower semicontinuous semifinite traces on A – together with the norm function on the subcone of finite traces (and the topology of pointwise convergence) – as a certain local semigroup of affine functions.

Theorem. *Suppose that A is simple. Denote by $D(A)'$ the convex cone of nonzero additive maps $\tau: D(A) \to \mathbf{R}^+$, with the topology of pointwise convergence. For each $x \in \partial D(A)$, the function*

$$\hat{x}: D(A)' \ni \tau \mapsto \tau(x) = \sup \tau([0,x] \cap D(A)) \in \mathbf{R} \cup \{+\infty\}$$

is strictly positive, lower semicontinuous, and affine, and, furthermore, either $\hat{x} = \hat{1}$, or else $\hat{x} + g = \hat{1}$ for some strictly positive lower semicontinuous affine function $g: D(A)' \to \mathbf{R} \cup \{+\infty\}$. Conversely, any such function on $D(A)'$ is equal to $\hat{x}$ for a unique x in $\partial D(A)$.

Proof. We must show that if $f: D(A)' \to \mathbf{R} \cup \{+\infty\}$ is strictly positive, lower semicontinuous, and affine, and if $f + g = \hat{1}$ for another such function g, then $f = \hat{x}$ for a unique $x \in \partial D(A)$. Fix $0 \neq u \in D(A)$, and denote by S_u the compact base $\{\tau \in D(A)'; \tau(u) = 1\}$. Proposition 4.1 of [8] yields intervals D_1 and D_2 in $D(A)$ with $f(\tau) = \sup \tau(D_1)$ and $g(\tau) = \sup \tau(D_2)$, $\tau \in S_u$, and it follows from Theorem 5.1 of [8] (with $G_2 = \mathbf{Z} \oplus \mathbf{Z}$) that, with D_1 and D_2 as constructed in 4.1 of [8], $D_1 + D_2$ is defined and equal to $D(A)$. By the description of $D(M(A))$ in Section 1, x exists as desired. Uniqueness holds by Proposition 7.7 of [8].

Remark. Using the Riesz decomposition property, it can be shown that the ideal structure of $(\partial D(A))^{\wedge}$ (i.e. of $\partial D(A)$) is the same as that of the semigroup of affine functions it generates. Note, however, that the ideal structure is with respect to the order in the semigroup. It is not enough to take the pointwise order, unless the norm function $\hat{1}$ is finite-valued and continuous. (And then $\partial D(A)$ is simple – see [9].)

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HARMONIC GEODESIC SYMMETRIES

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Presented by G. de B. Robinson, F.R.S.C.

We prove that a Riemannian manifold is locally symmetric if and only if all local geodesic symmetries are harmonic.

1. INTRODUCTION

Let (M,g) be a smooth n-dimensional Riemannian manifold and m a point of M. Consider an orthonormal basis, $\{e_1,\dots,e_n\}$, of T_mM and denote by $(x^1,\dots,x^n)$ the system of normal coordinates centered at the point m with $\frac{\partial}{\partial x^i}(m) = e_i$, $i = 1,\dots,n$.

If ξ is a unit tangent vector at m, and γ is the geodesic $r \longmapsto \exp_m(r\xi)$ through $m = \gamma(0)$ with tangent vector $\xi = \gamma'(0)$ and arc length r, we define the map

$$\varphi_m : \exp_m(r\xi) \longrightarrow \exp_m(-r\xi) : (x^i) \longmapsto (-x^i) .$$

For each m there exists a neighborhood of m such that φ_m is a local diffeomorphism and in what follows we will always restrict to such a domain. The map φ_m is called the local geodesic symmetry centered at m.

This map may be used to define special classes of Riemannian manifolds. The classical example is that of a locally symmetric space. Indeed, it is well-known that a Riemannian manifold is locally symmetric if and only if each local geodesic symmetry is an isometry. If ∇ denotes the Riemannian connection and R the associated curvature tensor, then the above condition is equivalent to $\nabla R = 0$. A very useful criterion is given in the following

LEMMA 1. (M,g) is locally symmetric if and only if

$$\nabla_X R_{XYXY} = 0$$

for all tangent vectors X,Y.

For a proof of this lemma we refer to [1],[5],[7].

Next, let (M,g) and (N,h) be two Riemannian manifolds with metrics g and h and let $f : (M,g) \longrightarrow (N,h)$ be a smooth map. The pullback f^*h is a semidefinite symmetric covariant tensor of order two, called the first fundamental form. Further, the covariant differential $\nabla(df)$ is a symmetric tensor of order two with values in $f^{-1}(TN)$, called the second fundamental form of f (see [2],[3]). A map with vanishing second fundamental form is said to be totally geodesic.

The trace of $\nabla(df)$ is denoted by $\tau(f)$ and is called the tension field of f. A map with vanishing tension field is called a harmonic map.

If $\mathcal{U} \subset M$ is a domain with coordinates $(x^1,\ldots,x^m)$ and $\mathcal{V} \subset N$ is a domain with coordinates $(y^1,\ldots,y^n)$ such that $f(\mathcal{U}) \subset \mathcal{V}$, then f can be locally represented by $y^\alpha = f^\alpha(x^1,\ldots,x^m)$, $\alpha = 1,\ldots,n$. The metric tensor g is represented by $g(x) = g_{ij}(x)dx^i dx^j$, $i,j = 1,\ldots,m$, and similarly we have $h(y) = h_{\alpha\beta}(y)dy^\alpha dy^\beta$, $\alpha,\beta = 1,\ldots,n$: df(x) is represented by the matrix $\left(\frac{\partial f^\alpha}{\partial x^i}\right)$. In this case we have

$$(f^*h)_{ij} = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta} ,$$

$$(\nabla(df))^\gamma_{ij} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} ,$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^k_{ij}$ are the Christoffel symbols for (M,g) and (N,h) respectively.

It follows that f is harmonic if and only if

$$g^{ij}(\nabla(df))^\gamma_{ij} = 0 . \tag{1}$$

Finally, we get easily from (1)

LEMMA 2. For the geodesic symmetry φ_m we have

$$\nabla(d\varphi_m)^k_{ij}(p) = {}^M\Gamma^k_{ij}(p) + {}^N\Gamma^k_{ij}(-p) , \tag{2}$$

$$\tau(\varphi_m)^k(p) = g^{ij}(p)\{{}^M\Gamma^k_{ij}(p) + {}^N\Gamma^k_{ij}(-p)\} . \tag{3}$$

2. HARMONIC GEODESIC SYMMETRIES

To prove our main result we need the expressions for g_{ij} and g^{ij} with respect to normal coordinates $(x^1,\dots,x^n)$. Let $p = \exp_m(r\xi)$ where ξ is a unit vector. Then we have (see [1],[4],[6]) :

$$g_{ij}(p) = \delta_{ij} - \frac{r^2}{3} R_{\xi i \xi j}(m) - \frac{r^3}{6} \nabla_\xi R_{\xi i \xi j}(m) + O(r^4) , \tag{4}$$

$$g^{ij}(p) = \delta_{ij} + \frac{r^2}{3} R_{\xi i \xi j}(m) + \frac{r^3}{6} \nabla_\xi R_{\xi i \xi j}(m) + O(r^4) . \tag{5}$$

Also we note that

$$^M\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right) . \tag{6}$$

Now we are ready to prove

THEOREM. (M,g) <u>is locally symmetric if and only if each geodesic symmetry</u> φ_m, $m \in M$, <u>is harmonic</u>.

<u>Proof</u>. First, suppose that (M,g) is locally symmetric. Then each φ_m is isometric and hence harmonic.

To prove the converse we write (4) as follows :

$$g_{ij}(p) = \delta_{ij} - \frac{1}{3} x^k x^\ell R_{ki\ell j}(m) - \frac{r^3}{6} x^k x^\ell x^s \nabla_k R_{\ell i s j}(m) + O(r^4) .$$

Then a straightforward computation, using (6), shows that

$$\begin{aligned} ^M\Gamma^k_{ij}(p) = & -\frac{r}{3} (R_{ik\xi j} + R_{jk\xi i})(m) \\ & - \frac{r^2}{12} (\nabla_i R_{\xi k \xi j} + \nabla_j R_{\xi i \xi k} + 2\nabla_\xi R_{ik\xi j} + 2\nabla_\xi R_{\xi ijk} - \nabla_k R_{\xi i \xi j})(m) \\ & + O(r^3) . \end{aligned} \tag{7}$$

Using the Bianchi identities we get from (7)

(8) $${}^{M}\Gamma_{ij}^{k}(p) + {}^{M}\Gamma_{ij}^{k}(-p) = -\frac{r^2}{6}(3\nabla_\xi R_{ik\xi j} + 3\nabla_\xi R_{\xi ijk} + \nabla_k R_{\xi j\xi i})(m) + O(r^4) .$$

Moreover, it is easily seen that the left hand side is an even function of r. Hence we may put

$${}^{M}\Gamma_{ij}^{k}(p) + {}^{M}\Gamma_{ij}^{k}(-p) = r^2\alpha_{2ij} + r^4\alpha_{4ij} + O(r^6) .$$

Finally, using (1), (3) and (5) we get the following condition for a harmonic geodesic symmetry φ_m :

$$\sum_{i,j=1}^{n} (\delta_{ij} + \frac{r^2}{3} R_{\xi i\xi j}(m) + \frac{r^3}{6}\nabla_\xi R_{\xi i\xi j}(m) + O(r^4))(r^2\alpha_{2ij} + r^4\alpha_{4ij} + O(r^6)) = 0 .$$

Hence we have the following necessary conditions :

(9) $$\begin{cases} \sum_i \alpha_{2ii} = 0 , \\ \sum_{i,j} R_{\xi i\xi j}(m)\alpha_{2ij} + 3\sum_i \alpha_{4ii} = 0 , \\ \sum_{i,j} \nabla_\xi R_{\xi i\xi j}(m)\alpha_{2ij} = 0 . \end{cases}$$

Using (8), the third condition in (9) becomes

(10) $$\sum_{i,j} \nabla_\xi R_{\xi i\xi j}(m)(3\nabla_\xi R_{ik\xi j} + 3\nabla_\xi R_{\xi ijk} + \nabla_k R_{\xi j\xi i})(m) = 0 .$$

Since e_k is arbitrary we may put $e_k = \xi$. Then, from (10) we obtain

$$\sum_{i,j} (\nabla_\xi R_{\xi i\xi j})^2(m) = 0$$

and hence

$$\nabla_\xi R_{\xi i\xi j} = 0 .$$

Now the result follows at once from Lemma 1.

As a special case we have

COROLLARY. (M,g) is locally symmetric if and only if each local geodesic symmetry is a totally geodesic map.

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LOCALIZATION AND SPECTRAL TRANSFORM FOR THE HALF LINE

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Abstract. Significant spectral quantities for half line impedance problems are displayed and studied as functions of the appropriate potential. Localizations (Frechet derivatives) are obtained in terms of products of eigenfunctions; a systematic development of Marčenko (M) equations is given with recovery formulas for potentials via spectral traces of transmutation kernels containing appropriate spectral data; a spectral trace deduced from calculations with Gelfand-Levitan (G-L) kernels containing suitable spectral data leads to formulas for a kind of spectral transform (IST) extending the Sine transform with products of eigenfunctions in the kernels. The details will appear in [10].

1. Background. In a full line scattering theory for (*) $\psi'' - q\psi = -k^2\psi$ (see [2;5;18-20;23;24;26;30]) one constructs Jost solutions $f_+ = f_1$ and $f_- = f_2$ with (•) $f_1 \sim e^{ikx}$, $f_2 \sim (1/T)e^{-ikx} + (R/T)e^{ikx}$ as $x \to \infty$ and $f_1 \sim (1/T)e^{ikx} + (R_2/T)e^{-ikx}$, $e^{-ikx} \sim f_2$ as $x \to -\infty$. For $W(f,g) = fg' - f'g$ one has $W(f_\pm(k,x), f_\pm(-k,x)) = \pm 2ik$ with (■) $f_2(k,x) = (1/T)f_1(-k,x) + (R/T)f_1(k,x)$ and $f_1(k,x) = (1/T)f_2(-k,x) + (R_2/T)f_2(k,x)$. The Zakharov-Shabat systems are also relevant to this paper but we omit discussion here (cf. [1;28;29;31]). For half line problems one has natural points of departure in geophysics, radial quantum mechanical scattering, or transmission line problems for example (cf. [4;6-9;14-1820;21;23-27;30]). It is standard to begin with an impedance problem (▲) $(Av_y)_y/A = Q(D_y)v = v_{tt}$ (A = impedance and y = travel time) or a system (♠) $w_y + w_t = -ru$; $u_y - u_t = -rw$ (with $r \sim (1/2)D_y\log A \sim$ reflectivity). The associated second order problem for $A \in C^2$, which we assume for convenience (along with (♦) $A(0) = 1$, $A'(0) = 0$ - for simplicity, $0 < \alpha \le A \le \beta < \infty$, and $A \to A_\infty$ rapidly as $y \to \infty$) involves eigenfunctions $\dot\varphi$ and $\dot\theta$ of (★) $\dot Q\psi = \psi'' - \dot q\psi = -\lambda^2\psi$ (cf. (*) and note $k \sim \lambda$) where $\dot\varphi(0) = 1$, $\dot\varphi'(0) = 0$, $\dot\theta(0) = 0$, and $\dot\theta'(0) = 1$. When we want to distinguish operators P and Q (~ P and Q) we write e.g. $\dot\varphi^P_\lambda$ or $\dot\varphi^P$ or $\dot\varphi^P(\lambda,x)$. Jost solutions $\Phi^P_{\pm\lambda} = f^P_1(\pm\lambda,y) \sim e^{\pm i\lambda y}$ as $y \to \infty$ are determined as before with (••) $\varphi = cf_1 + c^- f_1^-$ ($f_1^- \sim f_1(-\lambda,y)$) and $\theta = (F^- f_1 - F f_1^-)/2i\lambda$. Writing $d\omega = d\lambda/2\pi|c|^2$ and $d\nu = 2\lambda^2 d\lambda/\pi|F|^2$ the basic orthogonality relations are

$$(1.1) \qquad \int_0^\infty \dot\varphi^Q_\lambda(x)\dot\varphi^Q_\lambda(y)\,d\omega_Q = \delta(x-y) = \int_0^\infty \dot\theta^Q_\lambda(x)\dot\theta^Q_\lambda(y)\,d\nu_Q$$

Using results from [8;9;14] one can write (▲▲) $1/T = c^- + (1/2)F$, $R/T = c - (1/2)F^-$, and $R_2/T = (1/2)F - c^-$ $(2c^- = (1-R_2)/T$ and $F = (1+R_2)/T)$. Here R, R_2 are the full line coefficients for $\dot{Q}$ where $f_1^Q = (1/T)e^{i\lambda y} + (R_2/T)e^{-i\lambda y}$ and $f_2^Q = e^{-i\lambda y}$ for $y \leq 0$ ($\dot{\varphi} = \text{Cos}\lambda y$ and $\dot{\theta} = \text{Sin}\lambda y/\lambda$ for $y \leq 0$). In addition (♣♣) $f_1 = F\dot{\varphi} + 2i\lambda c^-\dot{\theta}$ and $f_2 = \dot{\varphi} - i\lambda\dot{\theta}$ (all y) and

$$(1.2) \qquad (1/2\pi)\int_{-\infty}^{\infty} Tf_1(\lambda,x)f_2(\lambda,y)d\lambda = \delta(x-y)$$

We remark also (from [6;7]) that (♦♦) $W(\dot{\varphi},f_1)(0) = 2i\lambda c^-$, $W(f_1,\dot{\theta})(0) = F$, and $Fc + F^-c^- = 1$.

There are a number of natural spectral ingredients associated to the half line theory and we mention in particular c^-, F, $S = F^-/F$, and $\mathcal{S} = c/c^-$, already isolated in geophysics and quantum scattering problems (cf. [4;8;9;14;21;25;30]). We want to indicate here another quantity $\mathcal{R}_2 = F/2c^-$ of both practical and theoretical significance which apparently has not been studied explicitly before (although it arises implicitly in [16]). The quantities $\mathcal{S}$ and S also have theoretical significance as in [7;11;13;20] and indicated later in §2. We note from (♠♠) that (♠♣) $\mathcal{R}_2 = F/2c^- = (1+R_2)/(1-R_2)$ and to see how it arises one considers e.g. an impulse response problem for (♠) with say $v(y,0) = \delta_+(y)$ and solution $v(y,t) = \langle\varphi_\lambda^Q(y),\text{Cos}\lambda t\rangle_\omega$ (see [7;8;14] for half line and full line delta functions). Then the readout (impulse response) is $v(0,t) = G(t) = \delta_+(t) + g(t) = \langle 1,\text{Cos}\lambda t\rangle_\omega = \int_0^\infty \hat{\omega}\text{Cos}\lambda t d\lambda$ ($d\omega = \hat{\omega}d\lambda$, $\hat{\omega} = 1/2\pi|c|^2$). From this one can recover $\hat{\omega}$ (and thence c^- by the Poisson-Jensen formula) and invoke the G-L machinery to determine A (cf. [6;7;11;15-17]). However for the IST these are simply not "natural" ingredients and we proceed a little differently to obtain $\mathcal{R}_2 = F/2c^-$ directly in terms of readout. Thus we define (•♣) $G_1(t) = (1/2\pi)\int_{-\infty}^{\infty} (F/2c^-)\exp(-i\lambda t)d\lambda$ and then, using (••), (♦♦), and properties of analyticity etc. for $F/2c^-$ from [7], $G_1(t) = 0$ for $t < 0$ and $G_1(t) = G(t)$ for $t > 0$ (recall also for impedance problems as indicated there are no bound states). An adjustment at $t = 0$ has to be made to accomodate δ functions and one obtains

<u>THEOREM 1.1.</u> <u>Under the hypotheses indicated</u> $\delta + g(t) = G_1(t)$ <u>for</u> $t \geq 0$ <u>from which</u> $\mathcal{R}_2 = F/2c^- = \int_{-\infty}^{\infty} G_1(t)\exp(i\lambda t)dt$.

2. <u>Spectral ingredients. G-L and M equations</u>. For the full line theory based on (*) with no bound states one thinks of a transmutation $\check{B}$: $f_1^Q \to f_1^P$ with kernel (•♠) $\check{\beta}(x,y) = \delta(x-y) + K(x,y) = (1/2\pi)\int_{-\infty}^{\infty} T_Q f_1^P(\lambda,x)f_2^Q(\lambda,y)d\lambda$ having the triangularity property $\check{\beta}(x,y) = 0$ for $y < x$. An inverse kernel is then $\check{\gamma}(y,x) = \delta(x-y) + L(y,x) = (1/2\pi)\int_{-\infty}^{\infty} T_P f_1^Q(\lambda,y)f_2^P(\lambda,x)d\lambda$ with $\check{\gamma}(y,x) = 0$ for $x < y$ and using (▪) one obtains,

$$(2.1)\qquad K(x,y) = (1/2\pi)\int_{-\infty}^{\infty} (R_Q - R_P)f_1^Q(\lambda,y)f_1^P(\lambda,x)d\lambda \quad (Q = D^2-q,\ P = D^2-p)$$

for $y > x$. Standard techniques based on the transmutation property $P\check{B} = \check{B}Q$ give (●♦) $(q-p)(x) = 2D_xK(x,x)$ and specializing to $Q = D^2$ one obtains the classical IST

$$(2.2)\qquad p = (1/\pi)D_x\int_{-\infty}^{\infty} R_P e^{i\lambda x}f_1^P(\lambda,x)d\lambda;\ S = 2i\lambda(R/T) = \int_{-\infty}^{\infty} pe^{-i\lambda x}f_2^P(\lambda,x)dx$$

(see Remark 3.3 for discussion - the last formula is immediate from construction of the Jost solutions via variation of parameters and asymptotic properties). In a somewhat more general spirit one has (cf. [5] for some of this)

$$(2.3)\qquad S^Q - S^P = \int_{-\infty}^{\infty} (q-p)f_2^Q(\lambda,x)f_1^P(-\lambda,x)dx;\ S_2^Q - S_2^P = \int_{-\infty}^{\infty} (q-p)f_1^Q(\lambda,x)f_2^P(-\lambda,x)dx;$$

$$(1/T_Q) - (1/T_P) = \int_{-\infty}^{\infty} (q-p)f_1^P(\lambda,x)f_2^Q(\lambda,x)dx/(-2i\lambda)$$

where we have written $S_2 = 2i\lambda(R_2/T)$. Such formulas lead to localization results for the Frechet (F) derivative of the map $p \to S(p)$ for example, in suitable spaces, of the form (cf. [5;22;28;29])

$$(2.4)\qquad d_pS(v) = \int_{-\infty}^{\infty} f_1^P(-\lambda,x)f_2^P(\lambda,x)v(x)dx$$

(or $\partial S/\partial p = f_1^- f_2$). Thus the products of eigenfunctions arise in an interesting way and in fact one has an inverse for d_pS in the form (✱✱) $(d_pS)^{-1}(\sigma) = \int_{-\infty}^{\infty} \alpha(\lambda)\sigma(\lambda) D_x[f_1^P(\lambda,x)f_2^P(-\lambda,x)]d\lambda$ where $\alpha(\lambda) = TT^-/2\pi i\lambda$ (cf. [22]). One can also establish (2.4), as in [22], by noting that F derivatives of any spectral quantity determined by Wronskians of solutions $f_i(\lambda,x,p)$ of (✱) can be computed via F derivatives of the f_i which in turn are calculable from the differential equation (✱) and variation of parameters.

Let us now develop the half line situation using $\dot{Q}$ in (✱) with (▲▲), (♠♠), etc. First to see how the quantities S and $\mathcal{S}$ arise naturally in M equations one can provide a unified derivation as follows (for $\mathcal{S}$ this represents a new version, for half line problems, with a new derivation, of the author's M equation in [7;8;13] which supplies some additional perspective about such equations - a derivation based on methods in [7;13] leading to operator factorizations etc. is also possible and instructive; it will appear in [10] - the approach of [32] can also be envisioned here). Thus rewriting the orthogonality relations (1.1) in the author's generalized Kontorovič-Lebedev (K-L) format (cf. [7;8;12]) and using the kernels $\check{\beta}$ and $\check{\gamma}$ above (based now on $\dot{Q}$) there results

<u>THEOREM 2.1</u>. <u>For</u> $\Xi = \mathcal{S}$ <u>or</u> $-S$ <u>one has an M equation</u> $(y > x)$, $0 = K(x,y) + \Upsilon(x,y) + \int_x^{\infty} K(x,\xi)\Upsilon(\xi,y)d\xi$ <u>where</u> $\Upsilon(x,y) = (1/2\pi)\int_{-\infty}^{\infty} (\Xi_P - \Xi_Q)f_1^Q(\lambda,x)f_1^Q(\lambda,y)d\lambda$. <u>Further for</u>

$y > x$, $K(x,y) = (1/2\pi)\int_{-\infty}^{\infty} (R_Q - \Xi_P) f_1^P(\lambda,x) f_1^Q(\lambda,y) d\lambda$ and $\dot{q}-\dot{p} = (1/\pi) D_x \int_{-\infty}^{\infty} (\Xi_Q - \Xi_P) f_1^P(\lambda,x) f_1^Q(\lambda,x) d\lambda$.

Thus in view of (2.1) $\Xi \to p$ plays a role not unlike $R \to p$. However there is no direct formula $p \to \Xi$ but rather (as for $p \to S$) only formulas $p \to c = c^- \mathcal{S}$ or $p \to F = F^- S$. These can be written down directly via ($\star$) for $\dot{\varphi}$ or $\dot{\theta}$ and asymptotics in the form (cf. also [7] - set $\Delta F = F_Q - F_P$ and $\Delta\dot{p} = \dot{q}-\dot{p}$)

$$(2.5) \qquad 2i\lambda(c_Q^- - c_P^-) = -\int_0^{\infty} (\dot{q}-\dot{p})\dot{\varphi}_\lambda^Q(x) f_1^P(\lambda,x)dx; \ \Delta F = \int_0^{\infty} \Delta\dot{p}\dot{\theta}_\lambda^Q(x) f_1^P(\lambda,x)dx$$

For localizations one can also use the definition of $\mathfrak{C} = 2i\lambda c^- = W(\dot{\varphi},f_1)$ and $F = W(f_1,\dot{\theta})$ to obtain

THEOREM 2.2. The formulas (2.5) hold along with $\partial\mathfrak{C}/\partial\dot{p} = -\dot{\varphi} f_1$, $\partial F/\partial\dot{p} = \dot{\theta} f_1$, $\partial\mathcal{S}/\partial\dot{p} = (1/2i\lambda c^{-2})\dot{\varphi}\dot{\varphi}$, $\partial S/\partial\dot{p} = -(2i\lambda/F^2)\dot{\theta}\dot{\theta}$, and $\partial R_2/\partial\dot{p} = (1/4i\lambda c^{-2}) f_1^2$.

REMARK 2.3. In particular one could find a right inverse for $\partial R_2/\partial\dot{p}$ in the manner of [22] and this is further evidence that R_2 is indeed a "natural" ingredient. We note also that $\dot{\varphi}\dot{\varphi} + \lambda^2\dot{\theta}\dot{\theta} = f_2 f_2^-$ and $\lambda\dot{\varphi}\dot{\theta} = (i/4)(f_2(\lambda,x)^2 - f_2(-\lambda,x)^2)$.

3. Half line IST type formulas. In order to produce recovery formulas of the type ($\bullet\blacklozenge$) via kernels directly involving $\dot{\varphi}$ and $\dot{\theta}$ we consider transmutations $\dot{B}$: $\dot{\varphi}^P \to \dot{\varphi}^Q$ and $\hat{B}$: $\dot{\theta}^P \to \dot{\theta}^Q$ via kernels ($\star\bullet$) $\dot{\beta}(y,x) = \langle\dot{\varphi}_\lambda^Q(y),\dot{\varphi}_\lambda^P(x)\rangle_P$ and $\hat{\beta}(y,x) = \langle\dot{\theta}_\lambda^Q(y),\dot{\theta}_\lambda^P(x)\rangle_P^\wedge$ where $\langle f,g\rangle_P \sim \int fg d\omega_P$ and $\langle f,g\rangle_P^\wedge \sim \int fg d\nu_P$. These will have inverses ($\star\blacktriangle$) $\gamma(x,y) = \langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_Q$ and $\gamma(x,y) = \langle\theta_\lambda^P(x),\theta_\lambda^Q(y)\rangle_Q$. Standard triangularity holds in the form $\dot{\beta}(y,x) = 0$ for $x > y$ and $\dot{\gamma}(x,y) = 0$ for $y > x$ for example (cf. [7]) and one will have $\dot{\beta}(y,x) = \delta(x-y) + \dot{K}(y,x)$ with $\dot{\gamma}(x,y) = \delta(x-y) + \dot{L}(x,y)$ etc. The standard recovery formulas are $2D_x\dot{K}(x,x) = \dot{q}-\dot{p} = 2D_x\hat{K}(x,x)$ and we can write for $x < y$ ($\star\blacklozenge$) $\dot{K}(y,x) = \int_0^{\infty} \dot{\varphi}_\lambda^Q(y)\dot{\varphi}_\lambda^P(x)[d\omega_P - d\omega_Q]$ or $\hat{K}(y,x) = \int_0^{\infty} \dot{\theta}_\lambda^Q(y)\dot{\theta}_\lambda^P(x)[d\nu_P - d\nu_Q]$. Now rewrite these kernels in K-L form, e.g. ($\star\clubsuit$) $\dot{\beta}(y,x) = (1/2\pi)\int_{-\infty}^{\infty} \dot{\varphi}_\lambda^Q(y)[f_1^P(\lambda,x)/c_P^-]d\lambda$ and $\hat{\beta}(y,x) = (-i/\pi)\int_{-\infty}^{\infty} \lambda\dot{\theta}_\lambda^Q(y)[f_1^P(\lambda,x)/F]d\lambda$ and expand f_1 via ($\clubsuit\clubsuit$) to obtain formulas of the type ($\clubsuit\blacklozenge$) $\dot{\beta}(y,x) = (1/\pi)\int_{-\infty}^{\infty} [\dot{\varphi}_\lambda^Q(y)\dot{\varphi}_\lambda^P(x)R_2^P + i\lambda\dot{\varphi}_\lambda^Q(y)\dot{\theta}_\lambda^P(x)]d\lambda$ and $\hat{\beta}(y,x) = (-i/\pi)\int_{-\infty}^{\infty} \lambda[\dot{\theta}_\lambda^Q(y)\dot{\varphi}_\lambda^P(x) + (i\lambda/R_2^P)\dot{\theta}_\lambda^Q(y)\dot{\theta}_\lambda^P(x)]d\lambda$ for example. There will be 4 such formulas; combine them and use the recovery formula above to get 2 formulas for $\dot{q}-\dot{p} = \Delta\dot{p}$ in terms of integrals involving $R_2^Q - R_2^P$ or $(1/R_2^Q) - (1/R_2^P)$. Adding those 2 formulas gives us the desired combination of $\dot{\varphi}$ and $\dot{\theta}$ as displayed in

THEOREM 3.1. Under the hypotheses indicated

(3.1) $\dot{q} - \dot{p} = \Delta\dot{p} =$

$$(1/\pi)D_x\int_{\infty}^{\infty} [\dot{\varphi}_\lambda^P(x)\dot{\varphi}_\lambda^Q(x)(R_2^P - R_2^Q) - \lambda^2\dot{\theta}_\lambda^P(x)\dot{\theta}_\lambda^Q(x)((1/R_2^Q) - (1/R_2^P))]d\lambda$$

Next one goes to (2.5) and writes out f_1 via (♣♣). Similarly one has equations of the form (♣▲) $2i\lambda[(1/2) + (c_Q^- R^P - c_Q)/T_P] = -\int_0^\infty (\dot{q}-\dot{p})\dot{\varphi}_\lambda^Q(x)f_2^P(\lambda,x)dx$ and $1 - (F_Q^- + F_Q R^P)/T_P = -\int_0^\infty (\dot{q}-\dot{p})\dot{\theta}_\lambda^Q(x)f_2^P(\lambda,x)dx$. Combining all these and writing $\Delta\dot{p} = \dot{q}-\dot{p}$, $\Delta c^- = c_Q^- - c_P^-$, etc. one obtains 4 integrals of the form $\int_0^\infty \Delta p$[products $\dot{\varphi}^Q\dot{\theta}^P$, $\dot{\varphi}^Q\dot{\varphi}^P$, $\dot{\theta}^Q\dot{\theta}^P$, or $\dot{\theta}^Q\dot{\varphi}^P$]dx = spectral quantity. Various combinations are then interesting. In particular one has ($R_2^{-P}(\lambda) = R_2^P(-\lambda)$)

THEOREM 3.2. Under the hypotheses indicated

(3.2)
$$\int_0^\infty \Delta\dot{p}[\dot{\varphi}^Q\dot{\theta}^P + \dot{\theta}^Q\dot{\varphi}^P]dx = [(c_Q^-/c_P^-)(R_2^Q - 1) + 2c_Q^- c_P(1-R_2^{-P})(R_2^P + R_2^Q) + 2c_P^- c_Q(R_2^{-Q} - R_2^P)(1 + R_2^P)]/(1 + R_2^P)$$

If we set now $\dot{P} = D^2$ with $\dot{p} = 0$ and $R_2^P = 1$ then (3.2) and (3.1) may be interperted as a kind of half line IST

(3.3)

$$(1/\pi)D_x\int_{\infty}^{\infty} [\text{Cos}\lambda x\dot{\varphi}_\lambda^Q(x)(1 - R_2^Q) - \lambda\text{Sin}\lambda x\dot{\theta}_\lambda^Q(x)((1/R_2^Q) - 1)]d\lambda = \dot{q}$$

$$\int_0^\infty \dot{q}[\text{Sin}\lambda x\dot{\varphi}_\lambda^Q(x) + \lambda\text{Cos}\lambda x\dot{\theta}_\lambda^Q(x)]dx = \lambda c_Q^-(R_2^Q - 1) + \lambda c_Q(R_2^{-Q} - 1)$$

REMARK 3.3. For $\dot{q}$ small with $R_2^Q \to 1$, $c_Q \to 1/2$, $\dot{\varphi}_\lambda^Q \sim \text{Cos}\lambda x$, and $\dot{\theta}_\lambda^Q \sim \text{Sin}\lambda x/\lambda$ these equations (3.3) have the form ($R_2^{-Q} \sim R_2^Q$ and $1/R_2^Q - 1 \sim 1 - R_2^Q$) $\int_0^\infty \dot{q}\text{Sin}2\lambda x dx \sim \lambda(R_2^Q - 1)$ and $\dot{q} \sim (4/\pi)\int_0^\infty \lambda(R_2^Q - 1)\text{Sin}2\lambda x d\lambda$ (note here $\lambda(R_2^Q - 1)$ will be odd). Thus $2\lambda(R_2^Q - 1)$ is the Sine transform of $\dot{q}(\xi/2)$ and this is analogous to (2.2) where for p small with $T \sim 1$, $f_2 \sim e^{-i\lambda x}$, and $f_1 \sim e^{i\lambda x}$ one has $p \sim (1/\pi)\int_{-\infty}^\infty 2i\lambda Re^{2i\lambda x}d\lambda$ and $R \sim (1/2i\lambda)\int_{-\infty}^\infty pe^{-2i\lambda x}dx$ (so $4i\lambda R$ is the Fourier transform of $p(\xi/2)$). We remark also that KdV type theory for radial problems is possible as in [27] and we anticipate some applications of the present theory in that direction.

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SETS WITH ELEMENTS SUMMING TO SQUAREFREE NUMBERS

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Abstract: In this note, we prove that for n sufficiently large, every subset A of $\{1,2,\ldots,4n\}$ with $n+1$ elements must contain two elements a and b with $a+b$ squarefree.

Recently, Erdös and Freud considered the problem of showing that if n is a positive integer, $A \subseteq \{1,2,\ldots,4n\}$, and $|A| \geq n+1$, then there are squarefree numbers which can be expressed as sums of elements of A (cf. [1],[2],[3]). Freiman [2] then showed that in fact if n is sufficiently large, then there are $\ll \log n$ elements of A summing to a squarefree number. Later, Erdös, Nathanson, and Sárközy [1] considered infinite analogs to this problem, and Nathanson and Sárközy [3] subsequently showed that if n is sufficiently large, there exist $\gg \sqrt{n}$ squarefree numbers each of which can be expressed as a sum of ≤ 21 elements of A. In this note, we show that for n sufficiently large there are in fact $\gg n$ squarefree numbers each of which is a sum of 2 elements of A. (We comment, however, the previous papers apply to more general questions than the one considered here.)

The condition that $|A| \geq n+1$ is necessary for any of the above results. There are examples of $A \subseteq \{1,2,\ldots,4n\}$ such that $|A| = n$ and such that no squarefree number is a sum of elements of A. Two such examples are $A = \{4,8,12,\ldots,4n\}$ and $A = \{2,6,10,\ldots,4n-2\}$. We note that for n sufficiently large, these are the only such examples. Our results follow easily from the

Theorem. *Let* $A_n \subseteq \{1,\dots,4n\}$ *be of maximal size such that*

(i) *there exists an* $a \in A_n$ *such that* $a \not\equiv 0 \pmod 4$,

(ii) *there exists an* $a \in A_n$ *such that* $a \not\equiv 2 \pmod 4$, *and*

(iii) *if* $a, b \in A_n$ *with* $a \neq b$, *then* $a + b$ *is not squarefree*.

Then

$$\frac{4}{9} \le \liminf_{n\to\infty} \frac{|A_n|}{n} \le \limsup_{n\to\infty} \frac{|A_n|}{n} \le 4 - \frac{32}{\pi^2}.$$

Before proving the above result, we pose the problem of determining if $\lim\limits_{n\to\infty}(|A_n|/n)$ exists. If so, what is its value?

Proof of Theorem.

The lower bound follows immediately by considering $A_n' = \{a \in \mathbb{Z}: a \equiv 0 \pmod 9\} \cap [1,4n]$. For $n \ge 9$, (i), (ii), and (iii) hold, and $\lim\limits_{n\to\infty}(|A_n'|/n) = \lim\limits_{n\to\infty}((1/9)4n/n) = 4/9$.

For the upper bound, let $k := k(n) := |A_n|$ and fix $\varepsilon > 0$. We assume throughout that n is sufficiently large (depending on ε) and consider 3 cases.

Case 1. A_n contains an even element and an odd element.

Write $A_n = B \cup C$ where $B = \{a \in A_n: 2 \mid a\}$ and $C = \{a \in A_n: 2 \nmid a\}$. Since $|B| + |C| = |A_n| = k$, $B \neq \emptyset$, and $C \neq \emptyset$, we get $B + C$ has at least $k-1$ elements (because if $B = \{b_1,\dots,b_r\}$ and $C = \{c_1,\dots,c_s\}$, with $b_1 < \cdots < b_r$ and $c_1 < \cdots < c_s$, then $b_1 + c_1 < b_1 + c_2 < \cdots < b_1 + c_s < b_2 + c_s$

$< \cdots < b_r + c_s)$. Also, $B \cap C = \emptyset$, and all the elements of $B + C$ are odd and $\leq 8n$. Since the number of odd positive integers $\leq 8n$ that are not squarefree is

$$\leq (\frac{1}{2}(1 - \prod_{p \geq 3}(1 - \frac{1}{p^2}))8 + \frac{\varepsilon}{2})\, n = (4 - \frac{32}{\pi^2} + \frac{\varepsilon}{2})\, n,$$

we get from (iii) that $k-1 \leq (4 - \frac{32}{\pi^2} + \frac{\varepsilon}{2})\, n$ so that

$$\frac{|A_n|}{n} \leq 4 - \frac{32}{\pi^2} + \varepsilon .$$

<u>Case 2</u>. A_n has only even elements.

Write $A_n = B \cup C$ where $B = \{a \in A_n : a \equiv 0 \pmod 4\}$ and $C = \{a \in A_n : a \equiv 2 \pmod 4\}$. By (i) and (ii), $B \neq \emptyset$ and $C \neq \emptyset$. Here, $|B| + |C| = k$ and $B \cap C = \emptyset$. Again, we get $B + C$ has at least $k-1$ elements. But all the elements of $B + C$ are $\leq 8n$ and are congruent to 2 modulo 4. The number of positive integers $\leq 8n$ that are congruent to 2 modulo 4 and not squarefree is

$$\leq (\frac{1}{4}(1 - \prod_{p \geq 3}(1 - \frac{1}{p^2}))8 + \frac{\varepsilon}{2})\, n = (2 - \frac{16}{\pi^2} + \frac{\varepsilon}{2})\, n .$$

Thus, in this case, we get $k-1 \leq (2 - \frac{16}{\pi^2} + \frac{\varepsilon}{2})\, n$ so that

$$\frac{|A_n|}{n} \leq 2 - \frac{16}{\pi^2} + \varepsilon \leq 4 - \frac{32}{\pi^2} + \varepsilon .$$

<u>Case 3</u>. A_n has only odd elements.

Write $A_n = B \cup C$ where $B = \{a \in A_n : a \equiv 1 \pmod 4\}$ and $C = \{a \in A_n : a \equiv 3 \pmod 4\}$. Either $|B| \geq k/2$ or $|C| \geq k/2$ since $|B \cup C| = k$. Let D be B if $|B| \geq k/2$, and otherwise, let D be C. Since $|D| \geq k/2$, there exist at least $k - 3$ numbers of the form $d_1 + d_2$ where $d_1, d_2 \in D$ and $d_1 \neq d_2$ (because if $D = \{u_1,\ldots,u_r\}$ with $u_1 < \cdots < u_r$, then $u_1 + u_2 < u_1 + u_3 < \cdots < u_1 + u_r < u_2 + u_r < \cdots < u_{r-1} + u_r$). All such sums $d_1 + d_2$ are $\equiv 2 \pmod 4$ and $\leq 8n$. The number of nonsquarefree positive integers $\leq 8n$ that are congruent to 2 modulo 4 was discussed in Case 2. Thus, we get by (iii), $k-3 < (2 - \frac{16}{\pi^2} + \frac{\varepsilon}{2})\, n$. Hence,

$$\frac{|A_n|}{n} \leq 2 - \frac{16}{\pi^2} + \varepsilon \leq 4 - \frac{32}{\pi^2} + \varepsilon .$$

Combining the above three cases completes the proof.

In conclusion, the author thanks Professor Pomerance and Professor Selfridge for their comments.

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NEUMANN-COMPLETE FORMAL POWER SERIES FIELDS

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Abstract: Let K be a field, let G be an (additive) ordered Abelian group, and let $K((G)) = F^{\wedge}$ be the field of all formal power series with coefficients in K and exponents in G. Let $M^{\wedge}$ be the maximal ideal of the valuation ring of $F^{\wedge}$. By Neumann's Theorem, one can evaluate any $f \in K[[X_1, \ldots, X_n]]$ on $M^{\wedge n}$. Let F be a subfield of $F^{\wedge}$, and let M be the intersection of $M^{\wedge}$ and F. Let Σ be a subset of the union of $(K[[X_1, \ldots, X_n]] \times 2^{M^n})_{n \in N}$, where 2^{M^n} denotes the set of all subsets of M^n. F will be called Neumann-complete relative to Σ if for every $(f, M(f)) \in \Sigma$, $f(M(f))$ is contained in F. This idea is investigated, and a few applications given. Numerous further applications are planned.

Text: Continuing with the conventions and notation introduced above, let $v^{\wedge}$ be the formal power series valuation on $F^{\wedge}$, and let $A^{\wedge}$ be its valuation ring; then $M^{\wedge}$ is the maximal ideal of $A^{\wedge}$. Given $g \in G$, let T^g be the element in $F^{\wedge}$ that has support $\{g\}$, for which $T^g(g) = 1$. In 1948, B.H. Neumann proved a vast generalization of the following deep and very useful theorem [2, 4.7].

NEUMANN'S THEOREM. Let $(k_n)_{n<\omega}$ be a sequence in K and let x be in $M^\wedge$; then $\sum_{n=0}^{\infty} k_n \cdot x^n$ is a well-defined element of $F^\wedge$.

For $n \in N$, let $K[[X_1, \ldots, X_n]]$ denote the ring of all formal power series in n indeterminates, with coefficients in K.

(0) For all $(x_1, \ldots, x_n) \in M^{\wedge n}$, there exists a K-homomorphism taking $f(X_1, \ldots, X_n) \in K[[X_1, \ldots, X_n]]$ to $f(x_1, \ldots, x_n) \in F^\wedge$, defined in [1, 7.41].

Let $m < n$ in N, and let $K[[X_1, \ldots, X_m]]$ be identified with the image in $K[[X_1, \ldots, X_n]]$, under the inclusion mapping. Let the union of $(K[[X_1, \ldots, X_n]])_{n \in N}$ be defined to be the K-algebra $K[[X_1, \ldots]]$.

Let B be a subset of $F^\wedge$, and let M (resp. A) be the intersection of $M^\wedge$ (resp. $A^\wedge$) and B. If $F^\wedge$ is an ordered field, let $B^+ = \{b \in B: b > 0\}$. Let Σ be contained in the union Ω of $(\{(f, M(f)): f \in K[[X_1, \ldots, X_n]]$, with $M(f)$ a subset of $M^n\})_{n \in N}$.

B will be called _Neumann-complete relative to_ Σ if for all $(f, M(f)) \in \Sigma$, and for all $p \in M(f)$, $f(p)$ is in B. B will be called _absolutely Neumann-complete_ if it is Neumann-complete relative to Ω. Clearly (0) is equivalent to the following statement:

(1) _$F^\wedge$ is absolutely Neumann-complete._

Let ξ be an ordinal number for which $\xi > 0$ and ω_ξ is regular. Let $\xi F^\wedge = \{f \varepsilon F^\wedge: |\mathrm{supp}(f)| < \omega_\xi\}$, where supp(f) denotes $\{g \varepsilon G: f(g) \neq 0\}$. It is well-known that $\xi F^\wedge$ is a subfield of $F^\wedge$.

(2) <u>$\xi F^\wedge$ is absolutely Neumann-complete</u>.

PROOF. Let $B = \xi F^\wedge$, let $f \varepsilon K[[X_1, \ldots, X_n]]$, and let $p = (x_1, \ldots, x_n) \varepsilon \dot{M}^n$. By (0), f(p) is a well-defined element y in $F^\wedge$. Let S be the union of the support sets of $x_1, \ldots, x_n$. Since each $x_j \varepsilon \xi F^\wedge$, $|\mathrm{supp}(x_j)| < \omega_\xi$; hence $|S| < \omega_\xi$. Let $\omega \cdot S$ be the union of $((n \cdot S))_{n \varepsilon N}$. Since $\xi > 0$, $|\omega \cdot S| < \omega_\xi$. By [1, 7.41], supp(y) is contained in $\omega \cdot S$. □

A subset Σ of Ω will be called <u>consolidated</u> if (f, M_0) and (f, M_1) in Σ implies $M_0 = M_1$. Let $c\Sigma$, the <u>consolidation</u> of Σ, be $\{(f, cM(f)) \varepsilon \Omega: f \varepsilon \pi_1(\Sigma)$ and $cM(f)$ is the union $\pi_2((\pi_1|\Sigma)^{-1}(f))\}$. Clearly $c\Sigma$ is consolidated. Let Ξ be the set $\{(f, M(f)) \varepsilon \Omega$: if $p \varepsilon M(f)$, then f(p) is in B$\}$. Ξ will be called the <u>set of all hyper-convergent functions</u> over B.

(3) (i) If B is Neumann-complete relative to Σ, then B is Neumann-complete relative to $c\Sigma$.

(ii) B is Neumann-complete relative to Ξ.

(iii) If B is Neumann-complete relative to Σ, then Σ is contained in Ξ.

(iv) If B is Neumann-complete relative to Σ, then it is Neumann-complete relative to any subset of Σ.

(4) Assume that B is Neumann-complete relative to $\{(\sum_{n=0}^{\infty} (-1)^n \cdot X^n, M_0)\}$; then, for all $\mu \in M_0$, $(1 + \mu)^{-1}$ is in B. (Cf. [2, 4.9].)

(5) Let K be of characteristic 0, and let B be Neumann-complete relative to the union of $((\sum_{n=0}^{\infty} \binom{1/m}{n} \cdot X^n, M_0))_{m\in N}$. For all $\mu \in M_0$, there exists $b_m \in B$, with $b_m^{\,m} = 1 + \mu$. (Cf. [2, 4.91].)

Assume that B is a subring of $F^{\wedge}$ that contains K and T^G.

(6) Let x be in B^* (resp. B^+). There are unique elements $k \in K^*$ (resp. K^+), $g \in G$, and $\mu \in M$, such that $x = T^g \cdot k \cdot (1 + \mu)$.

PROOF. Let $g = v^{\wedge}(x)$. Since $v^{\wedge}(T^{-g} \cdot x) = 0$, $T^{-g} \cdot x$ is a unit in $A^{\wedge}$. Since K is the residue class field of $A^{\wedge}$, there exists a unique $k \in K^*$ (resp. $k \in K^+$), such that $T^{-g} \cdot x - k$ is in $M^{\wedge}$. Define $k^{-1} \cdot T^{-g} \cdot x - 1$ to be μ, an element in M. Then we see that $x = T^g \cdot k \cdot (1 + u)$. Since g and k are uniquely determined, so is μ. □

(7) B is Neumann-complete relative to $\{(\sum_{n=0}^{\infty} (-1)^n \cdot X^n, M)\}$ if and only if B is a field. (See (4) and (6). Cf. [2, 4.9].)

Let $T(\Sigma)B$ (or TB, if Σ is understood) be defined to be the subring of $F^{\wedge}$ that is generated by B and $\{f(p): (f, M(f)) \in \Sigma$ and p in $M(f)\}$. Let $T^{\omega}B$ be the union of $(T^n(B))_{n\in N}$.

(8) (i) $T^{\omega}B$ is the smallest subring of $F^{\wedge}$ containing B that is Neumann-complete relative to Σ.

(ii) If Σ_0 is contained in Σ, then $T^{\omega}(\Sigma_0)B$ is contained in $T^{\omega}(\Sigma)B$. Further, $T^{\omega}(c\Sigma)B = T^{\omega}(\Sigma)B$.

(iii) If $\{(\sum_{n=0}^{\infty} (-1)^n \cdot X^n, M)\}$ is in $c\Sigma$, then $T^{\omega}B$ is the smallest subfield of $F^{\wedge}$ containing B which is Neumann-complete relative to Σ. (See (7).)

$T^{\omega}(\Sigma)B$ will be called the <u>Neumann-completion of B with respect to Σ</u>, and $T^{\omega}(\Omega)B$ the <u>absolute Neumann-completion of</u> B.

Assume now that F (= B) is a subfield of $F^{\wedge}$ that contains K and T^G. v (= $v^{\wedge}|F$) is a valuation on F having value group G, valuation ring A, maximal ideal M, and residue class field K. It is easily seen that $(1 + M, \times)$ is a group. (5) then implies the following:

(9) If K is of characteristic 0, and if F is Neumann-complete relative to the union of $((\sum_{n=0}^{\infty} \binom{1/m}{n} \cdot X^n, M))_{m \in N}$, then $(1 + M, \times)$ is a divisible group. (Cf. [2, 4.91].)

(10) If K is of characteristic 0, and if F is Neumann-complete relative to $\{(\sum_{n=0}^{\infty} X^n/n!, M), (\sum_{n=1}^{\infty} (-1)^{n-1} \cdot X^n/n, M)\}$, then $\mu \in M \to \sum_{n=0}^{\infty} \mu^n/n!$ is a monomorphism of $(M, +)$ onto $(1 + M, \times)$. (See [1, 7.36].)

In subsequent publications these ideas will be used, for example, to formulate necessary and sufficient conditions under which formal power series fields are Pythagorean or Euclidean. Further, application will be made of these ideas to the study of trigonometry over certain formal power series fields; curves over surreal and surcomplex number fields; ... .

In many planned applications, the field of constants K is either R or C, while G may be very large ordered group; thus the full field of formal power series $F^\wedge$ is often very much larger than some other interesting and useful subfields F, that contain K and T^G. One way to pick out such a subfield F is to require that it be Neumann-complete relative to $\{(f_0, M(f_0)), (f_1, M(f_1)), \ldots\}$, such that $f_0, f_1, \ldots$ are formal power series that we want to evaluate over the subsets $M(f_0)$, $M(f_1)$, ... of M; and we further require that $f_0(M(f_0))$, $f_1(M(f_1))$, ... be subsets of F.

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STABILITY INDICES AND POINCARÉ POLYNOMIALS

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Abstract: Let F be a field, $T_F = \Sigma F^2$, $\dot{F} = F - \{0\}$, $\dot{T}_F = T_F - \{0\}$. Suppose we know that $[\dot{F}:\dot{T}_F] = 2^n$ and that F has $k > 0$ orderings. Then we shall determine all possible reduced stability indices which the field F can have. This answers a question considered by J. Merzel. In the proofs we use some properties of Poincaré polynomials.

§1. This paper is meant as a continuation of the paper [9]. Therefore we shall use freely notations and notions from that paper. The only exception is that now we do not assume that our field is a Pythagorean field. Instead throughout the paper we shall only assume that F is a formally real field with $[\dot{F}:\dot{T}_F] < \infty$.

Definition. Let F be any field with space of orderings X_F. We define $P_F(t)$ to be the Poincaré polynomial of the field F as defined by the formula in Theorem 1 in [9].

For the reader's convenience we recall here the basic properties of Poincaré polynomials which depend only on order spaces and not on the cohomology of groups.

Theorem 1. $P_F(1) = N(F)$, $P_F(0) = 1$, $P_F'(0) + 1 = \dim_{\mathbb{Z}/2\mathbb{Z}} \dot{F}/\dot{T}_F$, $\deg P_F(t) = st(F)$. (Here $N(F)$ is number of orderings of F, and $st(F)$ is the reduced stability index of F). Moreover $P_F(t)$ can be written uniquely in the form

$$\text{(P)} \qquad P_F(t) = (1+t)^{s-1} + t((1+t)^{s-1}a_{s-1} + \cdots + a_0) \ ,$$

<u>where</u> $0 \leq a_0, a_1, \ldots, a_{s-1}, 1 \leq s,\ a_{s-1}$.

Note that from Theorem 1 we get

$$(*) \qquad \begin{aligned} k = N(F) &= 2^s + 2^{s-1}d_{s-1} + \cdots + d_0 \\ \log_2|\dot{F}/\dot{T}_F| &= s+1+d_{s-1} + \cdots + d_0 \\ st(F) &= s \\ 0 \leq d_0, d_1, &\ldots, d_{s-1} \in \mathbb{Z} \ . \end{aligned}$$

In [7], Chapter 5, Merzel investigated the relationship between k, n and s. Some interesting partial results were obtained, but as was already observed in Remark 5.10 of that paper the estimates failed to be faithful. Our main goal in this paper is to give precise estimates. Other references for motivation, notions and methods are e.g. [1], [2], [3], [4], [5], [6].

§2. <u>Results and sketches of proofs</u>

As before we let $k = N(F)$, $n = \log_2|\dot{F}/\dot{T}_F|$ and $s = st(F)$. Furthermore we shall always write $k = 2^{a_1} + \cdots + 2^{a_t}$, $0 \leq a_1 < \cdots < a_t$, and let i be such that $a_{i-1} < s \leq a_i$ where we put $a_0 = -1$ ($s = 0$ iff $k = 1$ and $n = 1$ so <u>we can exclude this case</u>). Furthermore we define functions g_k and G_k: $\{1,2,\ldots,a_t\} \to \mathbb{N}$ as follows:

$$G_k(s) = k - 2^s + s + 1$$

$$g_k(s) = \begin{cases} s + i - 3 + [\frac{k}{2^{s-1}}] & \text{if } a_{i-1} = s-1 \\ s + i - 2 + [\frac{k}{2^{s-1}}] & \text{if } a_{i-1} < s-1 \ . \end{cases}$$

Here [ ] indicates the integer part function.

Theorem 2. Let F be a field with k, n, s defined as before. Then $n \in \{g_k(s), g_k(s)+1, \dots, G_k(s)\}$.

Moreover if k, s and n are given such that $k, s, n \in \mathbb{N}$, $1 \le s \le [\log_2 k]$, $2 \le k$, $n \in \{g_k(s), g_k(s)+1, \dots, G_k(s)\}$, then there exists a field F such that $k = N(F)$, $n = \log_2 |\dot{F}/\dot{T}_F|$, $s = st(F)$.

We shall only sketch the proof: Write $k = 2^{a_1} + \cdots + 2^{a_t}$. Then gradually replacing the powers 2^b by $2^b = 2^c 2^d$, $c+d = b$, where 2^c is considered as a coefficient we can write k in the form (*). We see that $d_0 + d_1 + \cdots + d_{s-1}$ is maximal if k is written in the form

$$(1) \qquad k = 2^s + (k - 2^s)$$

and $d_0 + d_1 + \cdots + d_{s-1}$ is minimal if

$$k = 2^s + [\frac{k-2^s}{2^{s-1}}] 2^{s-1} + d_{s-2} 2^{s-2} + \cdots + d_0$$

where $\{d_{s-2}, \dots, d_0\} \subseteq \{0,1\}$.

From (1) we get that the largest possible n is $G_k(s)$ and the smallest possible n is $g_k(s)$. This proves the first part of the theorem.

The second part of the theorem is a consequence of the fact that the polynomial P(t) written in the form (P) is realizable as the Poincaré polynomial of some field F and that any number $n \in \{g_k(s), \dots, G_k(s)\}$ can be written in the form (*). □

Noticing that $G_k(s)$, $g_k(s)$ are decreasing functions and $G_k(s+1) + 1 \ge g_k(s)$, $1 \le s < [\log_2 k]$, we get, as a consequence of Theorem 2, the following Theorem 3.

<u>Theorem 3</u>. Let F <u>be a field</u>, $k = N(F)$, $2^n = |\dot{F}/\dot{T}_F|$. <u>Then</u>

$$s \in \{m(k,2^n), m(k,2^n)+1, \dots, M(k,2^n)\} ,$$

<u>where</u> $m(k,2^n)$ <u>is the smallest</u> $r \in \{1,\dots,[\log_2 k]\}$ <u>such that</u> $g_k(r) \leq n$, <u>and</u> $M(k,2^n)$ <u>is the biggest</u> $r \in \{1,\dots,[\log_2 k]\}$ <u>such that</u> $n \leq G_k(r)$.

<u>Moreover if the numbers</u> k, n, s <u>are given as before and</u> $t+a_t \leq n \leq k$, $2 \leq k$, <u>then there exists a field</u> F <u>such that</u> $k = N(F)$, $2^n = |\dot{F}/\dot{T}_F|$, $s = st(F)$.

<u>Remark</u>. We can redefine $M = M(k,2^n)$ in the following way:

$$\text{(A)} \qquad 2^M - M - 1 \leq k - n < 2^{M+1} - M - 2 .$$

Note that condition (A) is identical with the condition on page 215, Remark 5.9 in [7]. This proves that the upper estimate in [7] is the best possible.

The condition $a_t + t \leq n \leq k$ can be also derived from Theorem 2. This condition is equivalent to the condition $k \in O(n)$, where $O(n)$ is a Bröcker set. (See e.g. [1], [7], [8].)

<u>Example</u>. Let $k = 10$. Then we have

$$g_{10}(1) = G_{10}(1) = 10$$
$$g_{10}(2) = 6, \quad G_{10}(2) = 9$$
$$g_{10}(3) = 5, \quad G_{10}(3) = 6 .$$

Hence if

$$s = 1 \text{ then } n = 10$$
$$s = 2 \text{ then } n \in \{6,7,8,9\}$$
$$s = 3 \text{ then } n \in \{5,6\} .$$

Furthermore $m(10,2^{10}) = 1 = M(10,2^{10})$, $m(10,2^9) = m(10,2^8) = m(10,2^7) = m(10,2^6) = 2$, $M(10,2^9) = M(10,2^8) = M(10,2^7) = 2$, $M(10,2^6) = 3$, $m(10,2^5) = M(10,2^5) = 3$. Therefore if $n = 6$, $s \in \{2,3\}$.

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ISOMORPHISMS AND UNITS IN ALTERNATIVE LOOP RINGS

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ABSTRACT

Three well-known open problems in the theory of integral group rings are settled for alternative loop rings; namely, the isomorphism problem, a conjecture of Sehgal concerning normalized automorphisms and the Zassenhaus conjecture about torsion units.

0. INTRODUCTION

If L is a loop and R an associative ring with identity, the loop ring of L over R, denoted RL, is defined precisely as is the group ring; namely,

$$RL = \{ \sum_{g \in L} \alpha_g g \mid \alpha_g \in R, \alpha_g = 0 \text{ for almost all } g \}$$

with componentwise addition and multiplication given by extending that in L via distributivity and linearity. Loop rings are integral, rational, or complex according as the coefficient ring R is the ring $\mathbf{Z}$ of integers, the ring $\mathbf{Q}$ of rational numbers or the ring $\mathbf{C}$ of complex numbers, respectively. If L is not a group, a loop ring is typically a highly nonassociative ring: one generally cannot even expect powers of elements to be well-defined. Nevertheless, in recent years, it has been shown that for certain loops and almost all rings, RL is an alternative (but not associative) ring [1]. Alternative rings, those defined by the laws $x(xy) = x^2y$ and $(yx)x = yx^2$, not only include but are closely related to associative rings in many ways. Thus it is felt that many properties of group rings should hold also for alternative loop rings. We refer the reader to a previous article in these Comptes Rendus

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for an indication of some results in this direction [2]. In this paper, we consider in the context of alternative loop rings several important conjectures concerning isomorphisms and units which have been formulated for integral group rings.

Loops whose loop rings are alternative are interesting objects. If L is such a loop, then L contains a subloop G of index 2 which is actually a group. Elements of the loop ring ZL can be written in the form x+yu where x and y belong to the group ring ZG and u is an element of L but not G. Suppose N is a normal subloop of L. Then the natural homomorphism $L \rightarrow L/N$ extends to a ring homomorphism $\omega: ZL \rightarrow Z(L/N)$. Fundamental to the work herein described are the two results which follow.

Proposition. Suppose N is a normal subloop of L contained in G and ω is the map defined above. An element $x+yu \in ZL$ belongs to the kernel of ω if and only if x and y belong to the ideal $\Delta(G{:}N)$ of the group ring ZG generated by the set $\{n-1 \mid n \in N\}$.

Corollary. If $x+yu \in ZL$ is a torsion unit (that is, a unit of finite order), then either

$$x \equiv \pm g \mod \Delta(G)\Delta(G') \quad \text{and} \quad y \equiv 0 \mod \Delta(G{:}G')$$

or

$$x \equiv 0 \mod \Delta(G{:}G') \quad \text{and} \quad y \equiv \pm g \mod \Delta(G)\Delta(G')$$

for some $g \in G$ which is uniquely determined.

(Here, $\Delta(G)$ (respectively $\Delta(G')$) is used for $\Delta(G,G)$ (respectively $\Delta(G'{:}G')$, G' denoting the commutator subgroup of G. These are the augmentation ideals of G and G'.) The preceding results allow us to compute modulo augmentation ideals in very much the same way as is done in group rings.

1. ISOMORPHISMS

The so-called isomorphism problem for group rings, first posed and solved for abelian groups by Graham Higman [5,6], is perhaps the most famous problem in group rings. It has been settled for some special classes of finite groups such as metabelian and circle groups but is still open in general. We have been able to solve it in the alternative case.

Theorem 1. Let L_1 and L_2 be loops, one of them a torsion loop (all elements of finite order), such that the alternative (but not associative) loop rings ZL_1 and ZL_2 are isomorphic. Then $L_1 \cong L_2$.

Let $\epsilon: ZL \rightarrow Z$ denote the augmentation function defined by $\epsilon(\sum \alpha_g g) = \sum \alpha_g$. An automorphism $\Theta: ZL \rightarrow ZL$ is called normalized if it preserves augmentation: $\epsilon(\Theta(r)) = \epsilon(r)$ for all $r \in ZL$. S. K. Sehgal has conjectured that every normalized automorphism of an integral group ring ZG is the composition of an inner automorphism of the rational group ring QG and an automorphism of G and shown this to be the case for finite nilpotent class 2 groups [9]. We are able to establish the analogous result for alternative rings after suitably modifying the concept of inner automorphism. (The "associative reader" is cautioned that rarely is a map of the sort $r \rightarrow x^{-1}rx$ an automorphism of an alternative ring.)

Definition. An inner automorphism of an alternative ring A is an automorphism of A contained in the group generated the right and left multiplication maps, $R(x): a \rightarrow ax$ and $L(x): a \rightarrow xa$, as x ranges over the units of A.

It is reassuring to note that if A happens to be an associative ring and ψ an automorphism of A which is inner in the above alternative sense, then in fact there exists a unit $x \in A$ such that $\psi(a) = x^{-1}ax$ for all $a \in A$.

Theorem 2. Let L be a torsion loop, ZL an alternative loop ring (which is not associative) and Θ a normalized automorphism of ZL. Then there exists an inner automorphism ψ of

QL and an automorphism σ of L such that $\Theta = \psi o \sigma$.

2. THE ZASSENHAUS CONJECTURE

A unit r in a loop (or group) ring is called normalized if $\epsilon(r) = 1$, ϵ the augmentation map defined in Section 1. What is the nature of the torsion units in the group ring of a finite group G? Certainly $\pm g$ for any $g \in G$ are torsion units. A second gem in Higman's classical paper [6] is his proof that if G is abelian, such trivial units are the only possibilities; equivalently, the elements of G are the only normalized torsion units. When G is not abelian, an obvious way to exhibit more torsion units is to consider conjugates of trivial ones by units of ZG, or more generally, by units in the rational group algebra QG. H. J. Zassenhaus has conjectured that all torsion units can be constructed this way (in the case G finite). Known to hold for certain families of groups (see, for example, [7]), the conjecture in general remains very much open. It is true, however, for alternative loop rings.

Theorem 3. Let L be a finite loop and r a normalized torsion unit in the alternative loop ring ZL. Then there exists a unit $\alpha \in QL$ and an element $t \in L$ such that $\alpha^{-1} r \alpha = t$.

Actually more can be said: Theorem 3 is constructive in the following sense. According to the Corollary in Section 0, if $r = x + yu$ is a normalized torsion unit in ZL and G is the group mentioned in the introduction, there exists an element $g \in G$ such that either $x \equiv g \bmod \Delta(G)\Delta(G')$ or $y \equiv g \bmod \Delta(G)\Delta(G')$. We are able to show that either $\alpha^{-1} r \alpha = g$ or $\alpha^{-1} r \alpha = gu$ correspondingly. The proof begins by extending to alternative rings a result of Polcino Milies and Sehgal [8, Lemma 5] which shows that it is sufficient to prove the desired conjugacy in the complex loop algebra CL. The problem is then reduced to establishing conjugacy in each simple component of CL. Here rather delicate calculations based upon the fact that the square of a torsion unit is central and hence in each component the square of a scalar, complete the proof.

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ON MILLER'S GENERALIZED DIHEDRAL GROUP

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For a long time, a certain group of order $2pr$ has been known as $[p, 2, r]'$ or $[p, 2, r]^+$, and one of order $2abcd^2$ as $((bcd, cad, abd; 2))$. But the isomorphism

$$((bcd, cad, abd; 2)) \cong [abcd, 2, d]^+$$

has apparently escaped detection for nearly fifty years!

1. A group of order $2pr$

G.A. Miller [7, p. 168] derives a "generalized dihedral group H", of order $2pr$, from two cyclic groups C_p and C_r. Beginning with their direct product

$$T^p = U^r = T^{-1}U^{-1}TU = E,$$

he adjoins an involutory element R which transforms both the generators T and U into their inverses, so that

$$R^2 = (RT)^2 = (RU)^2 = E.$$

Since these relations imply $(RTU)^2 = T^{-1}R \cdot RU^{-1}TU = T^{-1}U^{-1}TU$, H is the special case $q = 2$ of the "rotation group"

$$[p, q, r]^+$$

[4, p. 568; 6, p. 125], which has the presentation

$$T_1^p = T_2^q = T_3^r = (T_1T_2)^2 = (T_1T_2T_3)^2 = (T_2T_3)^2 = E. \qquad 1.1$$

See also [5, p. 124, Section (iii)]. Thus

$$\text{H} \cong [p, 2, r]^+ \qquad 1.2$$

When $\sin(\pi/p)\sin(\pi/r) > \cos(\pi/q)$, $[p, q, r]^+$ is the rotatory symmetry group of the regular spherical honeycomb $\{p, q, r\}$ [3, p. 30]. Clearly

$$[p, q, r]^+ = [r, q, p]^+ ,$$

although the reciprocal honeycombs $\{p, q, r\}$ and $\{r, q, p\}$ are different (unless $p = r$).

Returning to the case $q = 2$, we notice that

$$[p, 2, 1]^+ \cong D_p , \quad [p, 2, 2]^+ \cong D_p \times C_2 ,$$

and if p and r are coprime,

$$[p, 2, r]^+ \cong D_{pr} .$$

2. A group of order $2abcd^2$

Many interesting groups can be presented in the form $(l, m, n; k)$, meaning

$$R^l = S^m = T^n = (RST)^k = TSR = E$$

or

$$R^l = S^m = (RS)^n = (R^{-1} S^{-1} RS)^k = E$$

[6, p. 96]. The three periods l, m, n can be freely permuted, but their values are somewhat restricted; for instance, relations with $l = m = 2, n = 3$ and $k = 1$ imply $R = S$, causing the period of RS to be 2 instead of 3. In fact [1, pp. 87-88], when $k = 1$, the numbers l, m, n must be of the form bcd, cad, abd, where a, b, c are mutually coprime:

$$(b, c) = (c, a) = (a, b) = 1 .$$

This group, which is abelian since $k = 1$, has the presentation

$$R^{bcd} = S^{cad} = (RS)^{abd} = E , \quad RS = SR , \tag{2.1}$$

It is known to be the direct product of two cyclic groups:

$$(bcd, cad, abd; 1) \cong C_{abcd} \times C_d . \tag{2.2}$$

These cyclic groups are generated by

$$T = R^{\beta a} S^{\alpha b} \quad \text{and} \quad U = T^{(\alpha' \alpha b - \beta' \beta a)c} RS , \tag{2.3}$$

where $\alpha, \beta, \alpha', \beta'$ are (positive or negative) integers such that

$$\alpha b - \beta a = 1, \quad \gamma a - \alpha' c = 1, \quad \beta' c - \gamma' b = 1 .$$

Adjoining to this group 2.1, of order $abcd^2$, an involutory element R_1, which transforms both R and S into their inverses, so that

$$R_1^2 = (R_1 R)^2 = (R_1 S)^2 = E ,$$

and then defining $R_2 = R_1 S$ and $R_3 = RR_1$, we obtain an extended group

$$((bcd, cad, abd; 2))$$

of order $2abcd^2$ [1, p. 143] with the presentation

$$R_1^2 = R_2^2 = R_3^2 = (R_3 R_1)^{bcd} = (R_1 R_2)^{cad} = (R_2 R_3)^{abd} = (R_1 R_2 R_3)^2 = E . \qquad 2.4$$

We see from 2.3 that the involutory element R_1, which transforms R and S into their inverses, also transforms T and U into their inverses. Hence

$$((bcd, cad, abd; 2)) \cong [abcd, 2, d]^+ : \qquad 2.5$$

an instance of Miller's generalized dihedral group 1.2. It reduces to D_{abc} if $d = 1$, and to $D_{2abc} \times C_2$ if $d = 2$.

The theory of "twisted honeycombs" [2, p.26] begins with the observation that

$$[p, q, r]^+ \cong ((p, h, r; q)), \qquad 2.6$$

where h is the period of $T_1 T_3$ in 1.1. In particular,

$$[p, 2, p]^+ \cong ((p, p, p; 2)),$$

which is 2.5 with $a = b = c = 1$.

In Table 1 of *Generators and Relations* [6, p. 134], one of the groups of order 18 appears as $((3, 3, 3; 2))$. We could now replace this by the simpler symbol $[3, 2, 3]^+$.

In certain cases [2, p. 28], the group 2.6 has a factor group $((p, t, r; q))$, where t is a suitable divisor of h. In particular, the group 2.5, of order $2abcd^2$, may be regarded as a factor group of $[bcd, 2, abd]^+$, of order $2ab^2cd^2$ (in which the period of $T_1 T_3$ is the least common multiple of bcd and abd, namely $abcd$). In the special case when $b = 1$, this group $((cd, cad, ad; 2))$ is the whole group 2.6; so, if $(c, a) = 1$, we have the isomorphism

$$[acd, 2, d]^+ \cong [cd, 2, ad]^+ .$$

Setting $d = 2$, we deduce that, if c and a are coprime,

$$[2c, 2, 2a]^+ \cong D_{2ac} \times C_2 . \qquad 2.7$$

3. A group of order $2(b^2 + bc + c^2)$

It has been observed [6, p. 109 (8.48)] that the vertices and edges of the regular map $\{6, 3\}_{b,c}$ provide a Cayley graph for the group

$$T_1^2 = T_2^2 = T_3^2 = (T_1 T_2 T_3)^2 = (T_2 T_3)^b (T_2 T_1)^c = (T_2 T_3)^{-c} (T_2 T_1)^{b+c} = E , \qquad 3.1$$

generated by three half-turns T_1, T_2, T_3. Its abelian subgroup of index 2, generated by translations

$$X = T_1 T_2 \quad \text{and} \quad Y = T_2 T_3 ,$$

is the direct product

$$C_{t/d} \times C_d$$

[6, p. 5 (1.33)], where

$$t = b^2 + bc + c^2 \quad \text{and} \quad d = (b, c) = \gamma b - \beta c ,$$

with the presentation

$$X^c = Y^b = X^{-b} Y^{-c}, \quad XY = YX .$$

Here the $C_{t/d}$ is generated by Y, and the C_d by

$$XY^{\{\beta b + \gamma(b+c)\}/d} .$$

Izak Bouwer has pointed out that the book suffers from a misprint: $\{\beta b + \gamma(b+c)\}$ appears as $(\beta b + \gamma c)$.

This time it is T_2 that transforms both X and Y into their inverses, and we conclude that the group 3.1, of order $2t$, is simply

$$[t/d, 2, d]^+,$$

reducing to $[t] = D_t$ when $d = 1$, and to $D_{t/2} \times C_2$ when $d = 2$.

Georg Günther has noticed another misprint [6, p. 105 (8.34)]: $T_3\,T_1$ should be $T_3\,T_4$, twice.

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