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CONTENTS

RECOVERY OF THE POTENTIAL FROM I-FUNCTION

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ABSTRACT.
Let $f'' + k^2 f - q(x) f = 0$, $x > 0$, $f \sim \exp(ikx)$ as $x \to +\infty$. Given the I-function $I(k) := f'(0,k) f^{-1}(0,k)$ for all $k > 0$ we recover $q(x)$ analytically, $f' := \partial f/\partial x$. A similar problem is treated for the equation $u'' - \alpha^2 u - q(x) u = 0$, $x > 0$, $u \sim \exp(-\alpha x)$ as $x \to +\infty$, $I(\alpha) := u'(0,\alpha) u^{-1}(0,\alpha)$, $\alpha > 0$.

In applications (seismic exploration, acoustic and electromagnetic prospecting) the problem of finding $q(x)$ from the knowledge of $I(k) := f'(0,k) f^{-1}(0,k)$ for all $k > 0$, where

$$f'' + k^2 f - q(x) f = 0, x > 0, f \sim \exp(ikx) \quad \text{as} \quad x \to +\infty \tag{1}$$

$$q \in Q := \{q : \int_0^\infty |q(x)|(1+x)dx < \infty, q = \overline{q}\} \tag{2}$$

is of interest. The bar denotes complex conjugate here and below. We refer to this problem as problem 1. A similar problem (problem 2) is to find $q(x)$ from the knowledge of $j(\alpha) := u'(0,\alpha) u^{-1}(0,\alpha), 0 < \alpha < \infty$, where

$$u'' - \alpha^2 u - q(x) u = 0, x > 0, \tag{3}$$

$$u \sim \exp(-\alpha x), x \to +\infty, \alpha > 0. \tag{4}$$

The value of $I(k)$ is the ratio of acoustic velocity to pressure in acoustic prospecting, or components of the magnetic and electric fields in electromagnetic prospecting. Thus $I(k)$ can be directly measured experimentally. We prove that $I(k)$ (or $j(\alpha)$) determine $q(x)$ uniquely and give an algorithm to recover $q(x)$.

The key observation is that

$$Im I(k) = (2i)^{-1}[I(k) - \overline{I(k)}] = |f(0,k)|^{-2} k \tag{5}$$

where the formulas $W[f, \overline{f}] = -2ik$ and $f(0,k) = \overline{f(0,-k)}$, $k > 0$, were used. It follows from (5) that $Im I(k)$, $k > 0$, determines uniquely the spectral function $\rho(\lambda)$ of the operator $\ell y = -d^2 y/dx^2 + q(x) y$, $x > 0$, $y(0, \sqrt{\lambda}) = 0$, where λ is the spectral parameter, $\lambda = k^2$,

$$d\rho(\lambda) = \begin{cases} \pi^{-1} |f(0,\sqrt{\lambda})|^{-2} \lambda^{1/2} d\lambda, & \lambda > 0 \\ \sum_{j=1}^N c_j \delta(\lambda + \kappa_j^2) d\lambda, & \lambda < 0 \end{cases} \tag{6}$$

$c_j = \| \phi_j \|^{-2}$ are the normalizing constants, $\ell\phi_j = -\kappa_j^2\phi_j$, $\phi_j(0) = 0$, $\phi_j'(0) = 1$, $\delta(\lambda + \kappa_j^2)$ is the delta-function, and N is the number of bound states.

In order to determine $\rho(\lambda)$ for $\lambda < 0$ and therefore to reduce the problem of finding $q(x)$ from the knowledge of $I(k)$ to the familiar inverse spectral problem, the one of finding $q(x)$ from the knowledge of $\rho(\lambda)$, one needs to know κ_j^2, $1 \le j \le N$ and c_j. Since $f(k) := f(0,k)$ is analytic in the region $Imk > 0$ and has simple zeros at the points $i\kappa_j$, $\kappa_j > 0$, which are not zeros of $f'(0,k)$ (since otherwise $f(x,k)$ should vanish by the uniqueness of the solution to the Cauchy problem), one can extend $I(k)$ analytically on the half-plane $Imk > 0$ and find the poles $i\kappa_j$ of $I(k)$ and the corresponding residues $f'(0, i\kappa_j)\ [\dot{f}(0, i\kappa_j)]^{-1}$ where prime and dot denote differentiation in x and k respectively. Then one can use formulas (7) and (8) below and find the spectral function. Let us prove that the normalizing constants are determined uniquely by $I(k)$. If this is true then $d\rho(\lambda)$ is uniquely determined by $I(k)$ and therefore $q(x)$ is uniquely determined by $I(k)$. The normalizing constants are known to be $c_j := -2i\kappa_j f'(0, i\kappa_j)[\dot{f}(i\kappa_j)]^{-1}$. Here and below $f(k) := f(0,k)$. One has

$$-2i\kappa_j \mathrm{Res}_{k=i\kappa_j} I(k) = -2i\kappa_j f'(0, i\kappa_j)[\dot{f}(i\kappa_j)]^{-1} = c_j. \tag{7}$$

From (5) and (6) it follows that

$$d\rho(\lambda) = \begin{cases} \pi^{-1} Im I(\sqrt{\lambda}) d\lambda & \lambda > 0 \\ \sum_{j=1}^{N} -2i\kappa \mathrm{Res}_{k=i\kappa_j} I(k)\delta(\lambda + \kappa_j^2) d\lambda & \lambda < 0. \end{cases} \tag{8}$$

Let us summarize the results.

THEOREM 1. *The knowledge of* $I(k)$ *for all* $k > 0$ *determines* $q(x) \in Q$ *uniquely. The spectral function corresponding to this* $q(x)$ *is given by formula (8). The algorithm for computing* $q(x)$ *is the Gel'fand-Levitan algorithm for computing* $q(x)$ *from the knowledge of* $d\rho(\lambda)$.

Let us recall for convenience of the reader the Gel'fand-Levitan algorithm (see e.g. [1]). Given $d\rho(\lambda)$ one defines the kernel

$$L(x,y) := \int_{-\infty}^{\infty} \phi_0(x,\lambda)\phi_0(y,\lambda) d\sigma(\lambda) \tag{9}$$

where (for the problem on the semiaxis $x > 0$ and the boundary condition $y(0,\lambda) = 0$):

$$\phi_0(x,\lambda) := (\sqrt{\lambda})^{-1} \sin(x\sqrt{\lambda}), d\sigma(\lambda) = d\rho(\lambda) - d\rho_o(\lambda) \tag{10}$$

and $d\rho_o(\lambda)$ is given by the formula

$$d\rho_o(\lambda) = 0 \quad \text{for} \quad \lambda < 0, d\rho_o(\lambda) = \pi^{-1}\lambda^{1/2} d\lambda \quad \text{for} \quad \lambda > 0. \tag{11}$$

Then one solves the (Gel'fand-Levitan) integral equation

$$K(x,y) + L(x,y) + \int_0^x L(t,y)K(x,t)dt = 0, 0 \le y \le x \tag{12}$$

for $K(x,y)$. Equation (12) is uniquely solvable, for example, if the set of growth points of $d\sigma$ has at least one finite limit point [1]. Therefore, if $ImI(k) \not\equiv 0$ is a continuous function, $|ImI(k)| \leq c(1+k^2)^{-a}$, $a > 1$, then (12) is uniquely solvable. If $K(x,y)$ is found then

$$q(x) = 2\frac{dK(x,x)}{dx}. \tag{13}$$

The function (13) produces the function $I(k) := f'(0,k)[f(k)]^{-1}$ which is the same as the function $I(k)$ which was the data. Necessary and sufficient conditions for $d\rho(\lambda)$ to be the spectral function corresponding to a potential $q(x)$ with m locally integrable derivatives are known [1]. Therefore one can give necessary and sufficient conditions for the given function $I(k)$ to be the I-function corresponding to $q(x)$ with these properties.

PROPOSITION 1. *The function $ImI(k)$ corresponds to an I-function associated with a $q(x)$ which has m locally integrable derivatives iff the function $\rho(\lambda)$ defined by (8) has the properties*

- *(i_1) if $h(x) \in L^2(0,\infty)$ and has compact support, $E(\lambda) := \int_0^\infty h(x)\sin(x\sqrt{\lambda})dx$ and $\int_{-\infty}^{\infty} |E(\lambda)|^2 d\rho(\lambda) = 0$ then $h(x) = 0$, and*
- *(i_2) the limit $\Phi(x) = \lim_{N\to\infty} \int_{-\infty}^{N} \cos(\sqrt{\lambda}x)d\sigma(\lambda)$ exists and is bounded on any finite interval of the x-axis and $\Phi(x)$ has $m+1$ locally integrable derivatives.*

This proposition follows from Theorem 1.5.1 in [1] and formula (8).

Problem 2 reduces to problem 1 by setting $\alpha = -ik$. The half-plane $Re\alpha > 0$ corresponds to the half-plane $Imk > 0$. The set $\alpha > 0$ corresponds to the set $\{Imk > 0, Rek = 0\}$. Since $u(x,\alpha)$ is analytic in the region $Re\alpha > 0$, the function $j(\alpha)$ is meromorphic in this region and can be continued analytically on the half-plane $Re\alpha > 0$, so that the values $j(i\alpha)$ are uniquely determined. Therefore, by Theorem 1, $q(x)$ is uniquely determined.

LEMMA 1. *The knowledge of $j(\alpha)$ for all $\alpha > 0$ (in fact, even for a countable set $\{\alpha_m\}$, $\alpha_m > 0$, which has a finite limit point α_∞) determines $q(x)$ uniquely.*

Note that $\mathrm{Res}_{\alpha=\kappa} j(\alpha) = -i\mathrm{Res}_{k=i\kappa} I(k)$, where $\alpha = -ik$.

Let us briefly discuss the practical question of finding the poles and residues of $I(k)$ on the imaginary axis from the knowledge of $I(k)$ on the real axis. This is an unstable problem in the sense that one can add a term $a(k-i\kappa)^{-1}$ with large $\kappa > 0$ and small a without changing much $I(k)$ on the real axis. The poles of $I(k)$ are simple and $I(k)$ is meromorphic in the half-plane $Imk > 0$. At infinity $I(k)$ behaves as follows

$$\begin{aligned} I(k) &= [ik - A(0,0) + \int_0^\infty A_x(0,t)\exp(ikt)dt][1 + \int_0^\infty A(0,t)\exp(ikt)dt]^{-1} \\ &= ik + o(1) \quad \text{as} \quad k \to \pm\infty \end{aligned} \tag{14}$$

where we used the well-known formula

$$f(x,k) = \exp(ikx) + \int_x^\infty A(x,t)\exp(ikt)dt \tag{15}$$

and the fact that $\int_0^\infty \{|A_t(0,t) + |A_x(0,t)|\}dt < \infty$ if $q \in Q$ [3]. Thus

$$I(k) = ik + \sum_{j=1}^N a_j(k - i\kappa_j)^{-1} + \int_0^\infty a(t)\exp(ikt)dt \tag{16}$$

where $a(t) \in L^1$. Define

$$h(x) := (2\pi)^{-1}\int_{-\infty}^\infty [I(k) - ik]\exp(-ikx)dk. \tag{17}$$

Then

$$h(x) = i\sum_{j=1}^N a_j \exp(\kappa_j x)\theta(-x) + a(x)\theta(x),$$

$$\theta(x) := \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \tag{18}$$

Therefore the constants a_j and κ_j are uniquely determined by $I(k)$. Note that (16) and (17) imply that

$$\mathrm{Res}_{k=i\kappa_j} I(k) = a_j = c_j(-2i\kappa_j)^{-1}. \tag{19}$$

Let us summarize the method for recovery of $q(x)$ from $I(k)$:

1. Given $I(k)$ find $Im\ I(k)$, 2) compute $h(x)$ by formula (17), (18) and find constants a_j and κ_j from the asymptotic behavior of $h(x)$ as $x \to -\infty$.
2. Construct $d\rho(\lambda)$ by formulas (8), (19), and $L(x,y)$ by formulas (9), (10). Solve equation (12) for $k(x,y)$. (This equation is uniquely solvable if L is given by formulas (9), (10)).
3. Construct $q(x)$ by formula (13).

REMARK 1. *The constructed q generates by the formula $I(k) = f'(0,k)/f(0,k)$ the same function $I(k)$ with which we started. Here $f(x,k)$ is the solution to (1) with $q(x)$ given by (13).*

We can reformulate Proposition 1 so that all the assumptions will be given in terms of $I(k)$.

PROPOSITION 1'. *A function $I(k)$ is the I-function corresponding to a $q \in L^m_{1\,loc}$ such one that zero is not a resonance iff the following conditions hold: (1) $I(-k) = \overline{I(k)}$ for k real, $I(k)$ is the value on the real axis of a meromorphic in $\mathbb{C}_+ := \{k : Imk > 0\}$ function with only a finite number of simple poles $i\kappa_j$, $\kappa_j > 0$, and the quantities $-iRes_{k=i\kappa_j} I(k) > 0$ for $\kappa_j > 0$;*

(2) The following limit exists and is a $L^{m+1}_{1\,loc}$ function:

$$\Phi_1(x) := \lim_{N\to\infty}\int_0^N [k^{-1}Im\ I(k) - 1]\cos(kx)k^2dk \in L^{m+1}_{1\,loc}.$$

REMARK 2. *The point $k=0$ is called a resonance if $f(0,0)=0$ and $f(x,0)\notin L^2$. It is well known that if $f(0,0)=0$ and $f(x,0)\in L^2$ then $f(x,0)=0$ provided that $\int_0^\infty (1+x)|q(x)|dx<\infty$ (see e.g. [4] where also the three dimensional case is discussed). If zero is a resonance, that is a zero of $f(k):=f(0,k)$, then it is a simple zero of $f(k)$, it is not a zero of $f'(0,k)$, and it is a simple pole of $Im\ I(k)$ in the sense that $\lim_{k\to 0} k\ Im\ I(k)=|\dot f(0)|^{-2}\neq 0$. If one drops the assumption that zero is not the resonance in Proposition 1', then the conclusion still holds if one adds in condition (1) that $\kappa_0=0$ may be a simple pole of $I(k)$, and the residue at this pole is ir, $r>0$.*

Let us prove that $r>0$. One has $Res_{k=0}I(k)=f'(0,0)/\dot f(0)$. The Wronskian $f(0,k)f'(0,-k)-f'(0,k)f(0,-k)=-2ik$. Divide this formula by k and pass to the limit $k\to 0$ to get $2\dot f(0)f'(0,0)=-2i$. Therefore $f'(0,0)/\dot f(0)=-i[\dot f(0)]^{-2}:=ir$. From the well known formula $f(k)=1+\int_0^\infty \exp(iky)A(0,y)dy$, where $A(0,y)$ is a real-valued function, one derives $\dot f(0)=i\int_0^\infty yA(0,y)dy$. Thus $[\dot f(0)]^2<0$ and $r>0$.

Let us summarize the method for solving the problem of finding $q(x)$ given $j(\alpha)$. Note that

$$j(\alpha)=-\alpha+o(1) \quad \text{as} \quad \alpha\to+\infty, \tag{20}$$

$j(\alpha)$ is meromorphic in the region $Re\alpha>0$ with a finite number of simple poles at the points $\alpha_n=\kappa_n>0$, and $Res_{\alpha=\kappa_n}j(\alpha)=(2\kappa_n)^{-1}c_n$, where $c_n>0$ are the same as in (7). Define

$$J(\alpha)=j(\alpha)+\alpha-\sum_{n=1}^{N}\frac{c_n(2\kappa_n)^{-1}}{\alpha-\kappa_n}, \tag{21}$$

and

$$b(t):=Lap^{-1}J(\alpha), J(\alpha):=\int_0^\infty \exp(-\alpha t)b(t)dt=Lapb(t). \tag{22}$$

Let

$$I(k):=J(-ik)+ik+\sum_{n=1}^{N}c_n(2\kappa_n)^{-1}(-ik-\kappa_n)^{-1}. \tag{23}$$

The inversion procedure is as follows:

1. *Given $j(\alpha)$ find κ_n and c_n, then $b(t)$ by formula (22);*
2. *find $I(k)$ by formulas (22) and (23), compute $Im\ I(k)$ and find $d\rho$ by formula (6);*
3. *given $d\rho$ find $q(x)$ as above.*

REMARK 3. The constructed q generates $j(\alpha)$ by the formula $j(\alpha)=u'(0,\alpha)u^{-1}(0,\alpha)$, where u is the solution to (3), (4), the same function which was used as the data. We do not give details here.

REMARK 4. In [5] some analytical formulas are given for the potential under the assumption that the Weyl function is a rational function of k which is given for all k. The connection between Weyl's function and the spectral function one can find e.g. in [6] p. 25-34.

REMARK 5. If one considers the scattering problem on the whole line by a $q(x)$ which is equal to $q(x)$ in (1) for $x > 0$ and vanishes for $x < 0$, then the reflection coefficient $r(k)$ defined by the formula $f(x,k) = \exp(ikx) + r(k)\exp(-ikx)$ for $x < 0$, can be related to $I(k)$ by the formula $I(k) = ik(1-r(k))(1+r(k))^{-1}$, $r = (ik-I(k))(ik+I(k))^{-1}$. If $q(x) = 0$ for $x < 0$ then the solution $f \sim \exp(ikx)$ as $x \to +\infty$ can be written as $T^{-1}(k)[\exp(ikx) + r(k)\exp(-ikx)]$ for $x < 0$. Here $T(k)$ and $r(k)$ are the transmission and reflection coefficients respectively. The bound states $i\kappa_j$, $\kappa_j > 0$, are the poles of $T(k)$. Therefore, given $I(k)$ for all $k > 0$, one can uniquely determine $r(k)$, bound states and normalizing constants (provided that $q(x) = 0$ for $x < 0$) and then use the Marchenko equation to recover $q(x)$. This gives an alternative approach to the problem.

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AN EXPLICIT FORMULA COUNTING NONCOMMUTATIVE CLASSICAL INVARIANTS

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1. Introduction

Let k be an algebraically closed field of characteristic 0. Then the group G=SL(2,k) is reductive, hence every finite-dimensional G-module is completely reducible. For every integer $d\geq 0$ there is exactly one irreducible G-module of dimension d+1. We denote this module by R_d. The module R_d can be described as follows: define an action of G on the polynomial algebra $R=k[X,Y]$ by the formulas $g^{-1}.X=aX+bY$, $g^{-1}.Y=cX+dY$, where $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is an element of G. Then R_d is the submodule of R consisting of homogeneous polynomials of degree d, i.e., $R_d=\{a_0X^d+\ldots+a_dY^d; a_i\in k\}$. For the details and the proofs, we refer to [4], Ch. 3. Let $T^m(R_d)$ be the mth tensor power and $S^m(R_d)$ the mth symmetric power of R_d. We also put $I_d^m=S^m(R_d)^G$, $I_d=\bigoplus_{m\geq 0} I_d^m$, $\tilde{I}_d^m=T^m(R_d)^G$, and $\tilde{I}_d=\bigoplus_{m\geq 0}\tilde{I}_d^m$. Then I_d ($\tilde{I}_d$) is the classical ring of invariants in commuting (noncommuting) variables of a binary form of degree d.

<u>Definition</u>: The Hilbert series of I_d and $\tilde{I}_d$ are the formal power series $H(I_d,t)=\sum_{m\geq 0}(\dim_k I_d^m)t^m$ and $H(\tilde{I}_d,t)=\sum_{m\geq 0}(\dim_k \tilde{I}_d^m)t^m$.

It is well-known that $H(I_d,t)$ represents a rational function ([4] p. 25 and 28), and it is at least known that $H(\tilde{I}_d,t)$ is algebraic ([2] p. 207). In [6], p. 15, Teranishi considers the series $\sum_{d\geq 0}(\dim_k \tilde{I}_d^m)t^{md}$, and shows that it is rational.

Here we will consider the series $F_m(t)=\sum_{d\geq 0}(\dim_k \tilde{I}_d^{\,m})t^d$, and show that $F_m(t)$ is much simpler than $H(\tilde{I}_d,t)$, by writing down a fairly explicit expression for it, which in particular will show that it is rational.

Before we formulate our theorem, we need the following

<u>Definition</u>: Consider the field extension $\mathbb{C}(t^n)\to\mathbb{C}(t)$, which is Galois with Galois group generated by $t\mapsto e^{2\pi i/n}t$. Define the Reynolds operator $\phi_n:\mathbb{C}(t)\to\mathbb{C}(t)$ by

$$(\phi_n f)(t^n)=n^{-1}\sum_{k=0}^{n-1} f(e^{2\pi ik/n}t).$$

<u>Theorem</u>: <u>Let</u> $m\geq 3$ <u>and define</u> $h_m(t)=(t/(1-t^2))^{m-2}$. <u>Then</u> $F_2(t)=(1-t)^{-1}$

<u>and</u> $$F_m(t)=\frac{1}{2t}\sum_{0\leq j<\frac{1}{2}m}\binom{m}{j}(-1)^{j+1}(\phi_{m-2j}h_m)(t) \quad \text{for } m\geq 3.$$

It is interesting to note that this formula bears a striking resemblance to the formula for $H(I_d,t)$ obtained by Springer (see [1] p. 340, [3] and [5]).

2. The proof

Let $\rho_{d,m}$ be the representation of G on $T^m(R_d)$. Then

$\chi_{d,m}(\xi)=\mathrm{Tr}(T^m(R_d),\rho_{d,m}\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix})=(\mathrm{Tr}(R_d,\rho_{d,1}\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}))^m=$
$=(\xi^d+\xi^{d-2}+\ldots+\xi^{-d})^m$ (we need only consider $\rho_{d,m}\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$, see [4] p. 50).

By complete reducibility we can write $\chi_{d,m}(\xi)=\sum_{i\geq 0}a(d,m,i)\chi_i(\xi)$, where $\chi_i=\chi_{i,1}$, and the coefficients $a(d,m,i)$ are non-negative integers (and only finitely many are non-zero). Then

$\sum_{d\geq 0}\chi_{d,m}(\xi)t^d=\sum_{i\geq 0}(\sum_{d\geq 0}a(d,m,i)t^d)\chi_i(\xi)=\sum_{i\geq 0}F_{m,i}(t)\chi_i(\xi)$, where $F_{m,0}(t)=F_m(t)$, since $a(d,m,0)=\dim_k\tilde{I}_d^{\,m}$.

We also have $\sum_{d\geq 0} \chi_{d,m}(\xi) t^d = \sum_{d\geq 0} ((\xi^{d+1}-\xi^{-d-1})/(\xi-\xi^{-1}))^m t^d =$

$$=(\xi-\xi^{-1})^{-m} \sum_{d\geq 0}\sum_{j=0}^{m} \binom{m}{j}(-1)^j \xi^{(d+1)(m-2j)} t^d = (\xi-\xi^{-1})^{-m} \sum_{j=0}^{m}\binom{m}{j}(-1)^j \frac{(-1)^m}{\xi^{m-2j}-t}.$$

(to get the last equality we sum over m-j instead). If $\xi=e^{ix}$, then

$$\frac{(-1)^m}{(2i)^m \sin^{m-1}x} \sum_{j=0}^{m}\binom{m}{j}(-1)^j \frac{1}{e^{(m-2j)ix}-t} = \sum_{i\geq 0} F_{m,i}(t)\sin(i+1)x.$$

If we multiply by sin x and integrate from 0 to 2π we get

$$F_m(t)=F_{m,0}(t)=\frac{(-1)^m}{(2i)^m \pi}\int_0^{2\pi} \frac{1}{\sin^{m-2}x} \sum_{j=0}^{m}\binom{m}{j}(-1)^j \frac{1}{e^{(m-2j)ix}-t}\,dx,$$

since $\int_0^{2\pi} \sin x \sin nx\,dx=0$ if $n\geq 2$.

Here the integrand looks dangerous with $\sin^{m-2}x$ in the denominator, which has zeros of order m-2 at $x=0,\pi$, and 2π. But the sum in the integrand has zeros of order m-1 there by the following

<u>Lemma</u>: $\sum_{j=0}^{m}\binom{m}{j}(-1)^j(m-2j)^\nu=0$ <u>for</u> $\nu\leq m-1$.

<u>Proof</u>: $\sum_{j=0}^{m}\binom{m}{j}(-1)^j(m-2j)^\nu = \frac{d^\nu}{dt^\nu}\left(\sum_{j=0}^{m}\binom{m}{j}(-1)^j e^{(m-2j)t}\right)\Big|_{t=0} =$

$=2^m \frac{d^\nu}{dt^\nu}(\sinh^m t)\Big|_{t=0}=0$ if $\nu\leq m-1$. Q.E.D.

(See also [4], p. 63.)

The integral above can be evaluated by putting $z=e^{ix}$ and using residue theory. We get

$$F_m(t)=\frac{(-1)^m}{(2i)^m \pi}\int_C \left(\frac{2iz}{z^2-1}\right)^{m-2} \sum_{j=0}^{m}\binom{m}{j}(-1)^j \frac{1}{z^{m-2j}-t}\frac{dz}{iz} =$$

$$=\frac{(-1)^{m+1}}{4\pi i}\int_C \frac{z^{m-3}}{(z^2-1)^{m-2}} \sum_{j=0}^{m} \binom{m}{j}(-1)^j \frac{1}{z^{m-2j}-t}\, dz,$$

where C is the unit circle $|z|=1$. For the same reason as above, the integrand is a meromorphic function in the unit disc with simple poles at $e^{2\pi ik/(m-2j)}t^{1/(m-2j)}$, $k=1,2,\dots,m-2j$, $0\leq j<\frac{1}{2}m$, and at 0 if $m=2$, if we consider t as a real variable with $0<t<1$. By the residue theorem we have

$$F_m(t)=\frac{(-1)^{m+1}}{4\pi i}\cdot 2\pi i(\text{sum of residues})=\frac{(-1)^{m+1}}{2}(\text{sum of residues}).$$

The case $m=2$ has to be handled separately:

$$F_2(t)=-\frac{1}{4\pi i}\int_C \frac{1}{z}\left(\frac{1}{z^2-1}-\frac{1}{1-t}+\frac{1}{z^{-2}-t}\right) dz=-\tfrac{1}{2}\left(-\frac{1}{t}-\frac{1}{1-t}+\frac{1}{\sqrt{t}\cdot 2\sqrt{t}}+\frac{1}{\sqrt{t}\cdot 2\sqrt{t}}\right)=\frac{1}{1-t}.$$

If $m\geq 3$ we obtain

$$F_m(t)=\frac{(-1)^{m+1}}{2}\sum_{j,k}\left(\frac{(e^{2\pi ik/(m-2j)}t^{1/(m-2j)})^{m-3}}{(e^{4\pi ik/(m-2j)}t^{2/(m-2j)}-1)^{m-2}}\binom{m}{j}(-1)^j\cdot\right.$$

$$\left.\cdot\frac{1}{(m-2j)e^{2\pi ik(m-2j-1)/(m-2j)}t^{(m-2j-1)/(m-2j)}}\right)=$$

$$=\frac{(-1)^{m+1}}{2}\sum_j \frac{1}{m-2j}\binom{m}{j}(-1)^j\,\frac{1}{t}\,\sum_k\left(\frac{e^{2\pi ik/(m-2j)}t^{1/(m-2j)}}{e^{4\pi ik/(m-2j)}t^{2/(m-2j)}-1}\right)^{m-2},$$

where the sums are over $0\leq k\leq m-2j-1$, $0\leq j<\frac{1}{2}m$. Using the definition of ϕ_n the theorem is proved. Q.E.D.

<u>Corollary 1</u>: $F_m(1/t)=(-1)^m t^2 F_m(t)$ <u>if</u> $m\geq 3$.

<u>Proof</u>: If $f(t)\in\mathbb{Q}(t)$, let $\tilde{f}(t)=f(1/t)$. Then

$$(\phi_n\tilde{f})(t^n)=n^{-1}\sum_{k=0}^{n-1} f(1/e^{2\pi ik/n}t)=n^{-1}\sum_{k=0}^{n-1} f(e^{2\pi ik/n}/t)=(\phi_n f)^{\sim}(t^n).$$

Hence $F_m(1/t)=-\frac{1}{2}t\sum_{0\leq j<\frac{1}{2}m}\binom{m}{j}(-1)^j(\phi_{m-2j}h_m)(1/t)=$

$=-\frac{1}{2}t\sum_{0\leq j<\frac{1}{2}m}\binom{m}{j}(-1)^j(-1)^m(\phi_{m-2j}h_m)(t)=(-1)^m t^2F_m(t)$, since $h_m(1/t)=(-1)^m h_m(t)$. Q.E.D.

Corollary 2: If $m\geq 3$, then $F_m(t)=g_m(t)/(1-t^2)^{m-2}$, where $g_m(t)$ is a symmetric polynomial of degree $2(m-3)$.

Proof: Write $h_m(t)=(t(1+t^2+\ldots+t^{2(m-2j-1)})/(1-t^{2(m-2j)}))^{m-2}$. Then, if $g_{m,j}(t)=t^{2(j-1)}(1+t^2+\ldots+t^{2(m-2j-1)})^{m-2}$, we get $F_m(t)=-\frac{1}{2}(1-t^2)^{-(m-2)}\sum_j \binom{m}{j}(-1)^j(\phi_{m-2j}g_{m,j})(t)$. Now $(\phi_{m-2j}g_{m,j})(t)$ is a polynomial if $j>0$, and since $(\phi_m f)(t)=0$, where $f(t)=1/t^2$, if $m\geq 3$, $(\phi_m g_{m,0})(t)$ is also a polynomial. Hence $F_m(t)$ has the form $g_m(t)/(1-t^2)^{m-2}$. The assertions about $g_m(t)$ follow immediately from Corollary 1 if one notes that $\dim_k \tilde{I}_0{}^m=1$. Q.E.D.

Let us finish with some examples: we have already proved that $F_2(t)=1/(1-t)$, and Corollary 2 gives $F_3(t)=1/(1-t^2)$. By Corollary 2 again, $F_4(t)=(1+at+t^2)/(1-t^2)^2$ for some a, and since $\dim_k \tilde{I}_1{}^4=\frac{1}{3}\binom{4}{2}$, by [2], we must have $a=2$ and $F_4(t)=1/(1-t)^2$. For $m=5$ we get $F_5(t)=(1+at+bt^2+at^3+t^4)/(1-t^2)^3$, and since $a=\dim_k \tilde{I}_1{}^5=0$ and $b+3=\dim_k \tilde{I}_2{}^5=6$, by [2] again, we get $F_5(t)=(1+3t^2+t^4)/(1-t^2)^3$ (cf. [6], p. 17-18).

Remark: One can of course study the analogue of $F_m(t)$ in the commutative case, but this gives the ordinary Hilbert series $H(I_m,t)$, since $\dim_k I_d{}^m=\dim_k I_m{}^d$, by Hermite's reciprocity law.

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REGULAR SIMPLEXES WITH INTEGRAL VERTICES

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1. For which values of n does there exist a regular n-simplex S (i.e., with all its edges of equal length) in $\mathbb{R}^n$ with all its vertices integral (i.e., in $\mathbb{Z}^n$)?

This question was answered fifty years ago by I. J. Schoenberg [1] who showed that the necessary and sufficient conditions on n arethe following :- either (i) n is even and n + 1 is a perfect square; or (ii) $n \equiv 3 \pmod{4}$; or (iii) $n \equiv 1 \pmod{4}$ and n + 1 is not divisible to an odd exponent by any prime number of the form 4k + 3.

The integers n in case (iii) are, as Schoenberg remarks, precisely those for which n + 1 is the sum of two odd squares. As to case (ii), since every positive integer N is the sum of three triangular numbers, say

$$N = \sum_{i=1}^{3} \tfrac{1}{2} m_i (m_i + 1),$$

we have

$$8N + 4 = 1 + \sum_{i=1}^{3} (2m_i + 1)^2$$

and

$$8N + 8 = 1 + 1 + 1 + 1 + 1 + \sum_{i=1}^{3} (2m_i + 1)^2,$$

so that every multiple of 4 is the sum of either 4 or 8 odd squares. Thus we may restate Schoenberg's result as follows :-

THEOREM. <u>There exists a regular</u> n-<u>simplex in</u> $\mathbb{R}^n$ <u>with vertices in</u> $\mathbb{Z}^n$ <u>if and only if</u> n + 1 <u>is the sum of</u> 1, 2, 4 <u>or</u> 8 <u>odd squares.</u>

In this note we shall first give a variant of Schoenberg's proof, and then for each allowable value of n we shall explicitly construct a regular n-simplex S with vertices in $\mathbb{Z}^n$.

2. We may assume that the origin O is one vertex of S. If the other vertices are $x_i \in \mathbb{Z}^n$ $(1 \leq i \leq n)$, we must have

$$|x_i|^2 = |x_i - x_j|^2 = 2c$$

for some c and all $i \neq j$, where $|x|$ is the Euclidean norm of the vector x, and therefore

$$x_i.x_i = 2c, \quad x_i.x_j = c \text{ if } i \neq j,$$

where x.y is the standard scalar product on $\mathbb{R}^n$. Hence if X is the matrix with rows $x_1, \ldots, x_n$, we have

(2.1) $$XX^t = c(1_n + E)$$

where E is the n x n matrix in which every entry is 1.

Since E has rank 1 we have $\det(1_n + E) = 1 + \text{trace } E = n + 1$, and therefore by taking determinants in (2.1) we obtain

$$c^n(n + 1) = (\det X)^2$$

so that $c^n(n + 1)$ is a square.

The easiest case is that in which n is even. Then n + 1 must be a square, say $n + 1 = r^2$. Conversely, if $n + 1 = r^2$ it is easy to verify that the simplex with vertices $(r - 1)e_i$ $(1 \leq i \leq n)$ and $e_1 + \ldots + e_n$ (where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$) is regular.

Suppose now that n is odd. From (2.1) it follows that the quadratic forms

(2.2) $$\sum_{i=1}^{n} x_i^2$$

(2.3) $$2c \sum_{i \leq j} x_i x_j$$

are rationally equivalent. By the usual process of completing the square we find that

$$2 \sum_{i \leq j} x_i x_j = \sum_{i=1}^{n} \frac{i+1}{i} \left(x_i + \frac{1}{i+1} \sum_{j>i} x_j \right)^2$$

and hence that (2.3) is rationally equivalent to

(2.4) $$c \sum_{i=1}^{n} i(i+1) x_i^2.$$

Thus we seek the values of n such that the forms (2.2) and (2.4) are rationally equivalent, for some value of c.

3. The necessary and sufficient conditions for the rational equivalence of two diagonal quadratic forms are most conveniently expressed in terms of the Hilbert symbols $(a,b)_p$. The facts we require may all be found in Serre's book [2].

The Hilbert symbol $(a,b)_p$ is defined for p-adic numbers $a, b \neq 0$ as follows :- $(a,b)_p = 1$ or -1 according as the equation $ax^2 + by^2 = z^2$ does or does not have a solution $(x,y,z) \neq (0,0,0)$ in $\mathbb{Q}_p^3$. It has the following properties [2, p. 38] :

(3.1) $(a,b)_p = 1$ if either a or b or a + b is a square in $\mathbb{Q}_p$.
(For example, if $a = u^2$ we can take $(x,y,z) = (1,0,u)$.)

(3.2) $(a,b)_p$ is a symmetric bilinear function of a and b, so that

$$(a_1a_2,b)_p = (a_1,b)_p(a_2,b)_p.$$

(3.3) $(a,a)_p = (-1,a)_p = (a-1,a)_p$.
(For $(a,a)_p(-1,a)_p = (-a,a)_p = 1$, by (3.2) and (3.1), and likewise $(-1,a)_p(a-1,a)_p = (1-a,a)_p = 1$.)

Now let $Q = a_1x_1^2 + \dots + a_nx_n^2$ be a diagonal quadratic form with positive rational coefficients a_i. Then [2, p. 70] Q is rationally equivalent to $x_1^2 + \dots + x_n^2$ if and only if (a) $a_1 \dots a_n$ is a square and (b) for each prime p,

$$\varepsilon_p := \prod_{i<j} (a_i,a_j)_p = 1.$$

In our case we have $a_i = ci(i+1)$, and n is odd. Hence $a_1 \dots a_n = c^n n!(n+1)! = c(n+1) \times$ square, and therefore $c(n+1)$ must be a square, so that for any a we have

(3.4) $$(c,a)_p = (n+1, a)_p.$$

We have to compute ε_p. Let us temporarily drop the suffix p from the Hilbert symbol, and let $b_i = i(i+1)$. Then by (3.2) we have

$$(a_i,a_j) = (cb_i,cb_j) = (c,c)(c,b_i)(c,b_j)(b_i,b_j)$$

and therefore

$$\varepsilon_p = \prod_{i<j}(a_i,a_j) = (c,c)^{\frac{1}{2}n(n-1)}\prod_{i=1}^{n}(c,b_i)^{n-1}\cdot\prod_{i<j}(b_i,b_j)$$
$$= (-1,n+1)^{\frac{1}{2}n(n-1)}\prod_{i<j}(b_i,b_j)$$

since n is odd and $(c,c) = (n+1,n+1) = (-1,n+1)$ by (3.4) and (3.3).

Next, we have

$$\prod_{i=1}^{j-1}(b_i,b_j) = (b_1\ldots b_{j-1}, b_j)$$
$$= ((j-1)!j!, j(j+1))$$
$$= (j,j(j+1))$$
$$= (-1,j(j+1)) \quad \text{since } (-j,j(j+1)) = 1 \text{ by (3.1)}$$
$$= (-1,j)(-1,j+1)$$

and therefore

$$(b_i,b_j) = \prod_{j=2}^{n}(-1,j)(-1,j+1)$$
$$= (-1,2)(-1,n+1) = (-1,n+1)$$

since $(-1,2) = 1$ by (3.1). Hence finally we obtain

$$\varepsilon_p = (-1,n+1)_p^{\frac{1}{2}n(n-1)-1}.$$

Suppose first that $n \equiv 3 \pmod 4$. Then $\frac{1}{2}n(n-1)$ is odd, so that $\varepsilon_p = 1$ for all p, and therefore the forms (2.2) and (2.4) are equivalent (provided that $(n+1)c$ is a square).

Suppose next that $n \equiv 1 \pmod 4$. Then $\frac{1}{2}n(n-1)$ is even and therefore we must have $(-1,n+1)_p = 1$ for all primes p, that is to say the equation $-x^2 + (n+1)y^2 = z^2$ must have a nontrivial solution in $\mathbb{Q}_p$ for all primes p. Since this equation also has solutions in $\mathbb{R}$, it follows [2, p. 73] that it has solutions in $\mathbb{Q}$, i.e., $n+1 = (x/y)^2 + (z/y)^2$ is the sum of two <u>rational</u> squares and therefore is also the sum of two <u>integral</u> squares (necessarily odd, since $n \equiv 1 \pmod 4$). This completes the proof of the theorem.

4. We shall now exhibit a regular n-simplex S with vertices in $\mathbb{Q}^n$ for each allowable value of n. (A suitable integer multiple kS of S will then have integral vertices.) Let

$$n + 1 = \sum_{i=1}^{d} r_i^{\,2}$$

where the r_i are positive odd integers and d = 1, 2, 4 or 8. Let $V_i = Q^{r_i^2}$ with standard basis e_{ij} $(0 \leq j \leq r_i^{\,2} - 1)$, and let $V = \bigoplus_{i=1}^{d} V_i$, so that $\dim V = n+1$. Define

$$e'_{i0} = r_i^{-1}\sum_j e_{ij},$$
$$e'_{ij} = e_{ij} + (r_i - 1)^{-1}(e_{i0} - e'_{i0}) \qquad (1 \leq j \leq r_i^2 - 1).$$

Then the e'_{ij} for fixed $\underline{i}$ form an orthonormal basis of $\underline{V}_i$, and hence the e'_{ij} for all i and all j are the vertices of a regular n-simplex Δ' in V.

Let K be the subspace of V spanned by $e_{10}, \ldots, e_{d0}$. Since d = 1, 2, 4 or 8 we can make K into a $\mathbb{Q}$-algebra with identity element e_{10} and a bilinear multiplication such that $e_{i0}^2 = -e_{10}$ for $2 \leq i \leq d$, and $e_{i0}e_{j0} = \pm e_{k0}$ for $2 \leq i, j \leq d$, $i \neq j$ and $k \neq 1$, and such that $|xy| = |x||y|$ for x, y in K, where as before $|x|$ is the Euclidean norm. (When d = 4 (resp. 8), K is the rational quaternion (resp. octonion) algebra.)

The orthogonal complement V'_i of e_{i0} in V_i has dimension $r_i^2 - 1$, which is divisible by 8 and is therefore a multiple of d. We may therefore regard V'_i (relative to the basis (e_{ij})) as a free left K-module : $V'_i = K^{s_i}$ where $s_i = (r_i^2 - 1)/d$. Thus $V = K^s$ where $s = 1 + \sum s_i$.

Now let

$$u = r_1 e_{10} - \sum_{i=2}^{d} r_i e_{i0} \in K.$$

Then $\Delta = u\Delta'$ is a regular n-simplex in V, with vertices

$$ue'_{ij} = u(e'_{ij} - r_i^{-1}e_{i0}) + r_i^{-1}ue_{i0}.$$

The vector $e'_{ij} - r_i^{-1}e_{i0}$ is orthogonal to e_{i0}, and hence $u(e'_{ij} - r_i^{-1}e_{i0}) \in V'_i$. It follows that the e_{10}-coordinate of ue'_{ij} is equal to the e_{10}-coordinate of $r_i^{-1}ue_{i0}$, hence is equal to 1. Hence finally $S = \Delta - e_{10}$ is a regular n-simplex with vertices in $\mathbb{Q}^n$.

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CODAZZI TENSORS AND INTEGRAL FORMULAE

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§1. INTRODUCTION. The concept of a Codazzi tensor originated in the Weingarten map of a codimension-one isometric immersion of a manifold in an euclidean space. The concept has been generalized from tensors of type (1,1) such as the Weingarten map to tensors of type (p,p) (cf. [1]). Furthermore it has been shown that such tensors form an algebra under the "dot" product of Mixed Exterior Algebra (cf. [5]).

The purpose of this paper is to announce a very general formula for the exterior derivative of an (n-1)-form, involving Codazzi tensors, on an orientable n-manifold. If the manifold is compact, Stokes' theorem then yields an integral formula. Proofs will appear elsewhere.

§2. CODAZZI TENSORS. We suppose that M is a smooth n-manifold with a connection ∇. If U is a tensor field on M of type (p,p), skew-symmetric in both its co and contravariant parts, then ∇U is of type (p+1, p) and, skew-symmetrizing its covariant part, we obtain a tensor field $\pi(\nabla U)$.

DEFINITION. U is called ∇-Codazzi of type (p,p) if $\pi(\nabla U) = 0$.

§3. THE MAIN THEOREM. We now assume the M is oriented by an n-form Δ, that Z is a vector field on M and that U is a tensor field of type $(n-1, n-1)$ on M, skew in both parts. Define a field e^* on M by

$$< e^*, X_1 \wedge \ldots \wedge X_n > \; = \; \Delta(X_1, \ldots, X_n) \; ;$$

and define an (n-1)-form ϕ on M by

$$\Phi(X_1, \ldots, X_{n-1}) = \langle e^*, Z \wedge U(X_1 \wedge \ldots \wedge X_{n-1}) \rangle$$

(here, $X_1, \ldots, X_n$ are vector fields on M). Finally, let $\hat{Z}$ denote the (1,1) tensor field on M given by $X \mapsto \nabla_X Z$. Then we have

THEOREM. *If* (i) *the torsion of* ∇ *is* 0, (ii) U *is* ∇-Codazzi of type (n−1, n−1) *and* (iii) ∇ *is* ∇*-parallel, then*

$$\delta\Phi = \mathrm{tr}(\hat{Z} \cdot U)\Delta .$$

COROLLARY: *If, in addition,* M *is compact, then*

$$\int_{\partial M} \Phi = \int_M \mathrm{tr}(\hat{Z} \cdot U) .$$

§4. **REMARKS.** Many examples of Codazzi tensors of type (1,1) are given in [3]. An example of type (2,2) is the curvature tensor of a Riemannian manifold. Any ∇-parallel U is, of course, ∇-Codazzi. If we combine these facts with the result that the (mixed exterior) product of ∇-Codazzi tensors is ∇-Codazzi we obtain a large family of integral formulae. This latter fact was shown in [1], where it is also shown how to specialize a particular case of the above (Z comes from an isometric immersion of M in an euclidean space) to obtain the integral formulae of [2], [4], [6] and [7].

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SOME EXAMPLES IN PROJECTIVE CONVEXITY

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A subset B of a real projective three-space P^3 is convex in P^3 if there is a plane $\alpha \subset P^3$ disjoint from B and $B = \text{conv}_\alpha(B)$, the convex hull of B in the affine restriction $P^3 \setminus \alpha$; cf. [2]. A family $F = \{B_1, \ldots, B_n\}$ of $n \geq 2$ convex sets in P^3 is affinely embeddable if each $B_i \in F$ is convex in the same affine restriction of P^3.

THEOREM ([1]). If F is a family of $n \geq 5$ mutually disjoint closed convex sets in P^3 such that (i) any line meets at most two sets of F and (ii) no set of F is contained in a convex hull of any other three sets of F, then F is affinely embeddable.

In this note, we present examples which show that neither (i) (Figure 1) nor (ii) (Figure 2) may be dropped from the THEOREM.

0. Let A and B be closed convex sets in P^3. We recall that $A \cap B$ is convex if and only if $A \cap B$ is connected. If $A \cap B$ is not convex, then any plane in P^3 meets $A \cup B$ and $\{A, B\}$ is not affinely embeddable. Let A and B be disjoint. Then (cf. [3]) the set of planes in P^3, which do not meet $A \cup B$, is not connected and has two components. Thus $A \cup B$ has two convex hulls.

Let P^3 be coordinatized by $\mathbb{R}$, $\{\lambda,\mu\} \subset \mathbb{R}$ and denote the points and planes of P^3 by p, q, r and α, β, γ, respectively. For a subset $X \subset P^3$, $\langle X \rangle$ denotes the flat spanned by the points of X .

We remark that as our examples are algebraic, we only list observations and indicate arguments.

1. Let $p_1 = (1, 0, 0, 0)$, $q_1 = (0, 1, 0, 2)$, $p_2 = (0, 1, 0, 0)$, $q_2 = (1, 0, 2, 0)$, $p_3 = (0, 0, 1, 0)$, $q_3 = (2, 0, 0, 1)$, $p_4 = (0, 0, 0, 1)$, $q_4 = (0, 2, 1, 0)$ $p_5 = (1, 1, 2, 2)$, $q_5 = (-2, -2, -1, -1)$ and

$$B_i = \{\lambda p_i + \mu q_i \mid \lambda, \mu \geq 0, \lambda + \mu > 0\}; \quad i = 1, 2, 3, 4, 5 .$$

Let $T = \{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{R}, x_i \geq 0\}$ and $\gamma \equiv x_1 + x_2 = 0$. Then $\gamma \cap T = \phi$ and $T = \text{conv}_\gamma (\{p_1, p_2, p_3, p_4\})$ is a 3-simplex containing $B_1 \cup B_2 \cup B_3 \cup B_4$; cf. Figure 1.

We note the following properties of $F_5 = \{B_1, \dots B_5\}$.

1.1 $T = \text{conv}_\gamma (B_1 \cup B_2 \cup B_3 \cup B_4)$ and $p_l \notin \text{conv}_\gamma (B_i \cup B_j \cup B_k)$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

1.2 Let $\{i, j\} = \{1, 2\}$ and $\{k, l\} = \{3, 4\}$. Then $\text{conv}_\gamma (B_i \cup B_k)$ meets B_j, B_l and B_5, and $B_5 \cap \text{conv}_\gamma (B_j \cup B_k)$ is connected.

1.3 Let $\{i, j\} = \{1, 3\}$. Then $(B_i \cup B_{i+1}) \cap \text{conv}_\gamma (B_j \cup B_{j+1}) = \phi$ and $B_5 \cap \text{conv}_\gamma (B_j \cup B_{j+1})$ is not connected.

1.4 The plane $x_1 - 2x_2 - x_3 + 2x_4 = 0$ is disjoint from $B_1 \cup B_2 \cup B_5$ and separates B_1 and B_2 in T, and the plane $x_1 - 2x_2 + 2x_3 - x_4 = 0$ is disjoint from $B_3 \cup B_4 \cup B_5$ and separates B_3 and B_4 in T.

1.5 By 1.1 to 1.4, any three element subset of F_5 is affinely embeddable.

1.6 Let $\{i, j\} = \{1, 3\}$ and β strictly separate B_i and B_{i+1} in T. Then $B_j \cap \text{conv}_\beta (B_i \cup B_{i+1})$ and $B_{j+1} \cap \text{conv}_\beta (B_i \cup B_{i+1})$ are not connected.

1.7 By the preceding, T is the unique convex hull of $B_1 \cup B_2 \cup B_3 \cup B_4$ and $T \cap B_5$ is not connected. Thus F_5 is not affinely embeddable.

We claim that <u>no set of</u> F_5 <u>is contained in a convex hull of any other three sets of</u> F_5.

Let B_i, B_j and B_k be three sets of F_5. By 1.5, there is an α disjoint from $B_i \cup B_j \cup B_k$. Let $H = \text{conv}_\alpha (B_i \cup B_j \cup B_k)$. By 1.1 and 1.7, we may assume that $H \neq T$ and that $B_5 \subset H$. Thus if a fourth set is in H, then $B_1 \cup B_2 \subset H$ or $B_3 \cup B_4 \subset H$.

If $B_1 \cup B_2 \subset H$, then $B_5 \cup \text{conv}_\alpha (B_1 \cup B_2)$ is connected. Thus α separates B_1 and B_2 in H by 1.3, and neither B_3 nor B_4 is in H by 1.6. A similar argument yields that there is no fourth set in H when $B_3 \cup B_4 \subset H$.

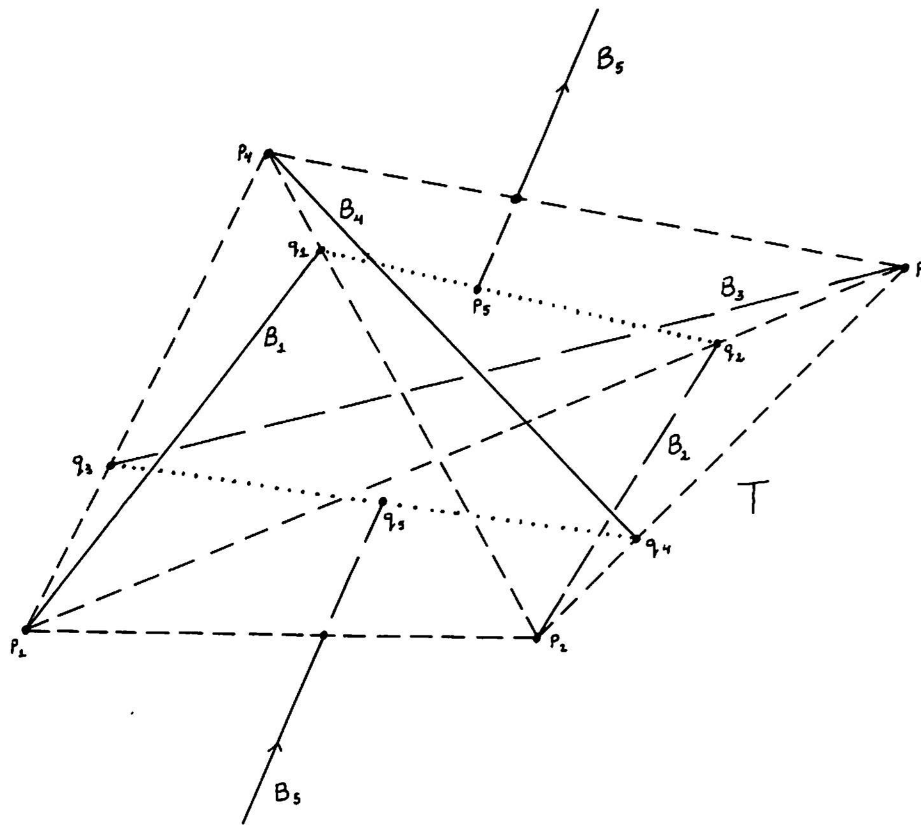

Figure 1

2. Let $p_1 = (0, 0, 1, 0)$, $q_1 = (0, 0, 0, 1)$, $p_2 = (0, 1, 1, 0)$, $q_2 = (0, 1, 0, 1)$, $p_3 = (1, 0, 1, 0)$, $q_3 = (1, 0, 0, 1)$,

$$B_i = \{\lambda p_i + \mu q_i \mid \lambda, \mu \geq 0, \lambda + \mu > 0\}, \quad i = 1, 2, 3,$$

$$B_4 = \{(1, 2, 0, 0)\} \quad \text{and} \quad B_5 = \{\sum_{i=1}^{8} \lambda_i r_i \mid \lambda_i \geq 0, \sum_{i=1}^{8} \lambda_i > o\}$$

where $r_1 = (1, -1, 1, -2)$, $r_2 = (1, -4, 2, -4)$, $r_3 = (-1, -1, 1, -2)$, $r_4 = (-4, 1\ 2, -4)$, $r_5 = (-1, 1, 1, -2)$, $r_6 = (1, 1, 2, -4)$, $r_7 = (-1, -1, 0, -3)$ and $r_8 = (1, 1, 3, 0)$.

We list some properties of this F_5, and remark that we exclude B_4 from Figure 2 for simplicity.

2.1 $< B_1 \cup B_2 > \equiv x_1 = 0$, $< B_1 \cup B_3 > \equiv x_2 = 0$ and $< B_2 \cup B_3 > \equiv x_1 + x_2 - x_3 - x_4 = 0$. Thus any line meets at most two sets of $\{B_1, B_2, B_3, B_4\}$.

2.2 $\alpha = < q_1, q_2, q_3 > \equiv x_3 = 0$, $\alpha' = < p_1, p_2, p_3 > \equiv x_4 = 0$, $B_4 \subset \alpha \cap \alpha'$, $\alpha \cap B_5 = \{r_7\}$ and $\alpha \cap B_5 = \{r_8\}$.

2.3 Let P and Q be the closed half-spaces of P^3 determined by α and α'. Then, say, $B_1 \cup B_2 \cup B_3 \subset P$, $B_5 \subset Q$ and the intersection of B_5 and the convex hull of $B_1 \cup B_2 \cup B_3$, which is in P, is not connected.

2.4 $L = < B_4 \cup \{r_7\} > \equiv x_3 = 6x_1 - 3x_2 - x_4 = 0$, $M = < B_4 \cup \{r_8\} > \equiv x_4 = 6x_1 - 3x_2 - x_3 = 0$ and $(L \cup M) \cap (B_1 \cup B_2 \cup B_3) = \phi$. Thus no line meets B_4, B_5 and $B_1 \cup B_2 \cup B_3$, and B_5 is disjoint from the convex hull $B_j \cup B_k$, which is in P; $1 \leq j \neq k \leq 3$.

2.5 $\gamma_1 = < p_3, q_1, q_2 > \equiv x_1 - x_3 = 0$, $\gamma_2 = < q_2, p_1, p_3 > \equiv x_2 - x_4 = 0$, $\gamma_3 = < p_1, q_2, q_3 > \equiv x_1 + x_2 - x_4 = 0$, $\gamma_4 = < q_3, p_1, p_2 > = x_1 - x_4 = 0$, $\gamma_5 = < p_2, q_1, q_3 > \equiv x_2 - x_3 = 0$, $\gamma_6 = < q_1, p_2, p_3 > \equiv x_1 + x_2 - x_3 = 0$ and $\gamma_i \cap B_5 = \{r_i\}$; $i = 1, \ldots, 6$.

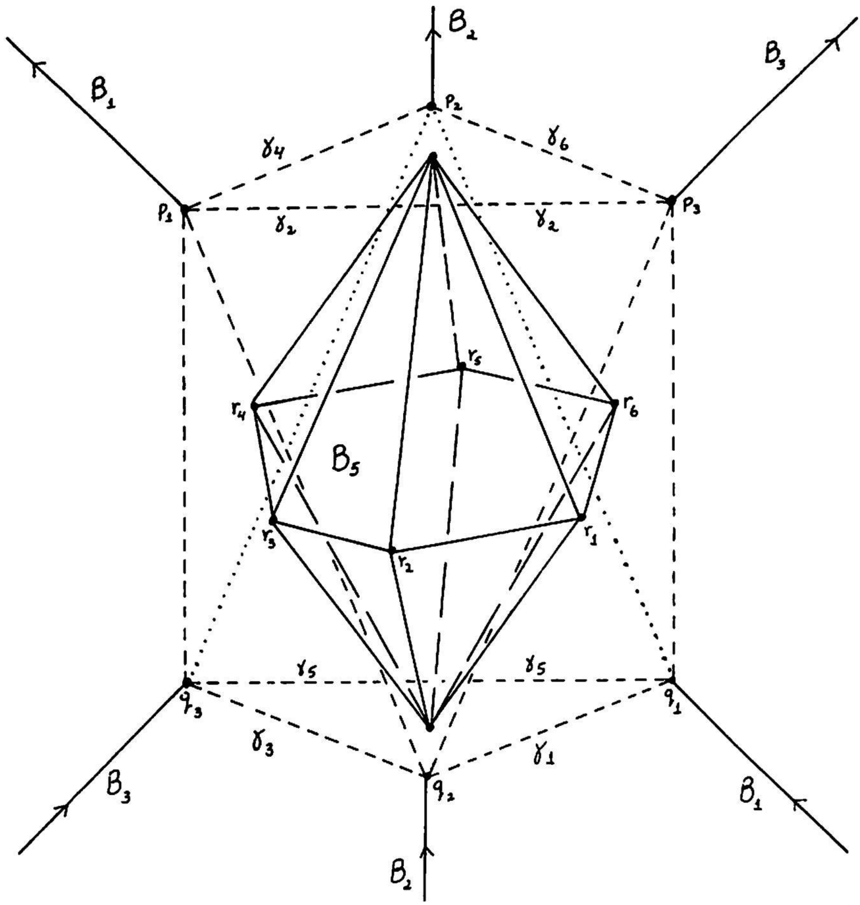
B2
B1
B3
p2
γ4
γ6
p1
γ2
γ2
p3
r5
r4
r6
B5
r3
r1
r2
γ5
γ5
q3
q1
γ3
γ1
q2
B3
B1
B2

Figure 2

2.6 Let P_i and Q_i be the closed half-spaces of P^3 determined by γ_i and γ_{i+3}, $i = 1, 2, 3$. Then, say, $B_1 \cup B_2 \cup B_3 \subset P_i$, $B_5 \subset Q_i$ and the intersection of B_5 and the convex hull of $B_1 \cup B_2 \cup B_3$, which is in P_i, is not connected. Thus with 2.3, $\{B_1, B_2, B_3, B_5\}$ is not affinely embeddable.

2.7 Let $\{i, j, k\} = \{1, 2, 3\}$. Then P_i contains the convex hull of $B_j \cup B_k$, which is not in P, and $\{r_i, r_{i+3}\} \cap < B_j \cup B_k > = \phi$. Thus B_5 is also disjoint from this convex hull of $B_j \cup B_k$.

2.8 By 2.4 and 2.7, any line meets at most two sets of $\{B_1, B_2, B_3, B_5\}$.

2.9 By 2.1, 2.4 and 2.8, any line meets at most two sets of F_5.

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