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UNRAMIFIED DOUBLE COVERS OF HYPERELLIPTIC KLEIN SURFACES

Ernst J. Kani*

Presented by H.S.M. Coxeter, F.R.S.C.

Abstract

In their report [BEG2], Bujalance, Etayo and Gamboa study elliptic-hyperelliptic unramified double covers of a hyperelliptic Klein surface X with boundary and formulate a conjecture concerning all unramified double covers of X. The purpose of this note is to prove a more precise version of that conjecture, and to generalize their main result to arbitrary unramified double covers of X.

1. Results

Theorem 1. (a) Each unramified double covering $\pi: X' \to X$ of a hyperelliptic Klein surface (with non-empty boundary) of algebraic genus $p \geq 2$ is q-hyperelliptic (i.e. X' has an involution σ such that X'/σ has genus q) for a unique q with $0 \leq q \leq [\frac{p-1}{2}]$. (b) For $0 \leq q < \frac{p-1}{2}$ (resp., for $q = \frac{p-1}{2}$), the number of such q-hyperelliptic coverings of X is $\binom{p+1}{q+1}$ (resp. $\frac{1}{2}\binom{p+1}{q+1}$).

Part (b) of this theorem was conjectured by Bujalance et al. in their note [BEG2]. In the case $q \leq 1$, they deduce it from a more precise theorem which counts the number of q-hyperelliptic, unramified double covers $\pi: X' \to X$ with X' a given type; cf. [BEG1] for $q = 0$ and [BEG2] for $q = 1$. Now it turns out that the proof of Theorem 1 yields, as a by-product, also a count of the number of such coverings of a given type. (Recall from [BEG2] that if X is orientable (resp. non-orientable), has

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topological genus g , and its boundary ∂X has k components, then X is said to have type $(g,+,k)$ (resp. $(g,-,k)$).)

To state the result, we require a certain "refinement" $\begin{bmatrix} n \\ m \end{bmatrix}_k$ of the binomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}$:

<u>Notation</u>. Let C_n denote the cyclic graph with n vertices, and for integers $m, k \geq 0$, let $\begin{bmatrix} n \\ m \end{bmatrix}_k$ denote the number of induced subgraphs of C_n with m vertices and k components. (In this, the empty graph ϕ and C_n itself are viewed as having 0 components.) Note that we have

$$\sum_k \begin{bmatrix} n \\ m \end{bmatrix}_k = \begin{bmatrix} n \\ m \end{bmatrix} \quad . \tag{1}$$

<u>Theorem 2</u>. <u>Let</u> X <u>be a hyperelliptic Klein surface of type</u> $(g,\pm,k)$ <u>and of algebraic genus</u> $p = \alpha g + k - 1 \geq 2$ (where $\alpha = 2$ for "+" and $\alpha = 1$ for "-"), <u>and let</u> $\pi: X' \to X$ <u>be a</u> q-<u>hyperelliptic, unramified double covering of</u> X <u>with</u> $0 \leq q < \frac{p-1}{2}$ <u>(resp., with</u> $q = \frac{p-1}{2}$). <u>Put</u> $M = \min(q+1,[k/2])$ <u>and</u>

$$N(t) = \sum_m \begin{bmatrix} k \\ m \end{bmatrix}_t \begin{bmatrix} p+1-k \\ q+1-m \end{bmatrix} \quad . \tag{2}$$

(a) <u>If</u> X <u>is orientable and</u> $g \neq 0$, <u>then</u> X' <u>has type</u> $(2g+k-2,+,2)$ <u>if</u> p <u>is odd and</u> q <u>is even and type</u> $(2g-1,+,2k)$ <u>otherwise; the number of such coverings is</u> $\begin{bmatrix} p+1 \\ q+1 \end{bmatrix}$ (<u>resp.</u> $\frac{1}{2}\begin{bmatrix} p+1 \\ q+1 \end{bmatrix}$) <u>in both cases.</u>

(b) <u>If</u> X <u>is orientable and</u> $g = 0$, <u>then</u> X' <u>has type</u> $(t-1,+,2k-2t)$ <u>with</u> $1 \leq t \leq M$, <u>and the number of such coverings for each</u> t <u>(with</u> $1 \leq t \leq M$) <u>is</u> $\binom{p+1}{q+1}_t > 0$ (<u>resp.</u> $\frac{1}{2}\binom{p+1}{q+1}_t > 0$) .

(c) <u>If</u> X <u>is non-orientable and</u> $k \neq q + 1$, then X'

has type $(2(g+t-1),-,2k-2t)$, *where* $0 \le t \le M$, *if* $n \le p-q$, *and* $1 \le t \le M$ *otherwise, and the number of such coverings for each such* t *is* $N(t) > 0$ (*resp.* $\frac{1}{2}N(t) > 0$). *If, however,* $k = q+1$, *then there is exactly* 1 *covering of type* $(g-1,+,2k)$, *exactly* $\begin{bmatrix} p-q \\ q+1 \end{bmatrix}$ (*resp.*, 1) *coverings of type* $(2g-2,-,2k)$, *exactly* $N(t) > 0$ (*resp.* $\frac{1}{2}N(t) > 0$) *coverings of type* $(2(g+t-1),-,2k-2t)$ *with* $1 \le t \le M$ *and no others.*

Remark. Since $\begin{bmatrix} k \\ m \end{bmatrix}_1 = k$ and $\begin{bmatrix} k \\ m \end{bmatrix}_2 = \frac{k(m-1)(k-m-1)}{2}$ if $1 \le m \le k-1$, we obtain the following "explicit" formulae for $N(t)$ when $t \le 2$:

$$N(0) = \begin{bmatrix} p+1-k \\ q+1 \end{bmatrix} + \begin{bmatrix} p+1-k \\ q+1-k \end{bmatrix}$$

$$N(1) = k \sum_{m=1}^{k-1} \binom{p+1-k}{q+1-m}$$

$$N(2) = k \sum_{m=2}^{k-2} \frac{(m-1)(k-m-1)}{2} \begin{bmatrix} p+1-k \\ q+1-m \end{bmatrix} .$$

In particular, if $q = 1$ (resp. $q = 0$), then we see that $N(1) = k(p+2-k)$ and $N(2) = k(k-3)/2$ (resp. $N(1) = k$), so Theorem 2 reduces to that of [BEG2] (resp. [BEG1]) in this case.

2. Proofs.

For the proof of these theorems it is advantageous to use the language of *real function fields*, as in [GM]. Thus, let F denote the real function field associated to X and recall that by page 110 of [GM], the unramified double coverings $\pi: X' \to X$ correspond bijectively to the *totally real, unramified* field extensions F' of F of degree 2, i.e. to those of the form $F' = F(\sqrt{f})$, where

(a) (f) is a square in the divisor group $\mathrm{Div}(X)$ (i.e. f

has only poles and zeros of even order) but f is not a square in F ;

(b) $f \geq 0$ (and has no poles) on ∂X .

Moreoever, if ∂X has k components, then $\partial X'$ has

(3) $$k' = 2k - r(f)$$

components, where

$r(f) = \#\{\text{components } \Gamma_i \text{ of } \partial X\colon \deg(f)|_{\Gamma_i} \equiv 2 \pmod 4\}$.

Note that $r(f)$ and hence k' is always <u>even</u>; cf. [Kne], Theorem (3.4) or [GM], page 110.

<u>Proof of Theorem 1</u>.

a) Immediate by Castelnuovo's inequality (cf. [S] or [K]).

b) Since the total number of totally real, unramified quadratic extensions of F is $2^p - 1$ (cf. [GM]), it is by a) enough to <u>construct for each</u> q <u>with</u> $0 \leq q < \frac{p-1}{2}$ (<u>resp., with</u> $q = \frac{p-1}{2}$) <u>at least</u> $\binom{p+1}{q+1}$ (<u>resp</u>. $\frac{1}{2}\binom{p+1}{q+1}$) <u>distinct totally real, unramified,</u> q-<u>hyperelliptic extensions</u> F' <u>of</u> F <u>of degree</u> 2.

For this, note that $F = \mathbb{R}(x,y)$ where $y^2 = f(x)$ and $f(x) \in \mathbb{R}[x]$ is a monic polynomial of degree $2p + 2$ with distinct roots over $\mathbb{C}$. Write

(4) $$f(x) = (x-a_1)\ldots(x-a_{2n})\cdot(x-b_1)(x-\bar{b}_1)\ldots(x-b_m)(x-\bar{b}_m)$$

where $n, m \geq 0$, $n + m = p + 1$, $a_1 < \ldots < a_{2n} \in \mathbb{R}$ and $b_j \in \mathbb{C}\backslash\mathbb{R}$ $(1 \leq j \leq m)$. Put

$$f_i(x) = (x-a_{2i-1})(x-a_{2i})/h(x), \quad 1 \leq i \leq n$$

$$g_j(x) = (x-b_i)(x-\bar{b}_i)/h(x), \quad 1 \leq j \leq m$$

where $h(x) = f(x)$, if $n = 0$, and $h(x) = (x-c)^2$ with $a_1 < c < a_2$, if $n > 0$.

Put $I = \{1,\ldots,n\}$, $J = \{1,\ldots,m\}$ and $S = I \,\dot{\cup}\, J$; thus

$\#S = p + 1$. For each subset $T \subset S$ with $\#T = q + 1$ (where $q \geq 0$ is fixed) put

$$f_T(x) = \prod_{i \in I \cap T} f_i(x) \cdot \prod_{j \in T \cap J} g_j(x) ,$$

and let $E_T = \mathbb{R}(x, z_T)$ be the function field defined by $z_T^2 = f_T(x)$. Then f_T satisfies (a) and (b) above, and hence

(1) $F_T = E_T \cdot F = F(\sqrt{f_T})$ is a totally real, unramified extension of F of degree 2;

(2) E_T has genus q $(= \#T - 1)$, hence F_T is q-hyperelliptic.

Now observe that if $T_1 \neq T_2$, then $E_{T_1} \neq E_{T_2}$ (because E_{T_1} and E_{T_2} are ramified at different places). Thus, if $q < \frac{p-1}{2}$, then also $F_{T_1} \neq F_{T_2}$ (by Castelnuovo) and hence we obtain in this case $\binom{p+1}{q+1} = \#(\text{choices of } T \subset S \text{ with } \#T = q + 1)$ distinct q-hyperelliptic (etc.) extensions F_T , as claimed.

If, however, $q = \frac{p-1}{2}$, then E_T and $E_{S \setminus T}$ both have genus q and $F_T = F_{S \setminus T}$. Since E_T, $E_{S \setminus T}$ and F are the only fields between F_T and $F_0 = \mathbb{R}(x)$, we obtain exactly $\frac{1}{2}\binom{p+1}{q+1}$ such extensions.

Remark. Essentially the same proof shows that the corresponding statement of Theorem 1 is also true for Riemann surfaces in place of Klein surfaces; in that case, however, the numbers in (b) are $\binom{2p+2}{2q+2}$ (resp. $\frac{1}{2}\binom{2p+2}{2q+2}$). For another (more complicated) proof of this fact see [B].

Proof Sketch of Theorem 2. We begin with the observation (which complements [GH], Proposition 6.3 and which follows easily from [GM]) that the type of X is determined by $n = \frac{1}{2}\#(\text{real roots of } f(x))$ (and by p):

Fact 1. $n = 0$, p odd $\Rightarrow X$ has type $(\frac{p-1}{2}, +, 2)$

$n = 0$, p even $\Rightarrow X$ has type $(p/2,+,1)$

$n = p + 1$ $\Rightarrow X$ has type $(0,+,p+1)$

$1 \le n \le p$ $\Rightarrow X$ has type $(p+1-n,-,n)$.

Recall that by the proof of Theorem 1, each unramified double covering X' of X is of the form $X' = X'_T$, where X'_T is the Klein surface corresponding to F_T .

From [GM], Satz 2a) and Lemma 3 one easily obtains:

Fact 2. (a) If X is orientable, then so is X'_T .

(b) If X is nonorientable, then X'_T is orientable if and only if $T = I$ (or $T = J$, if $n = \frac{p-1}{2}$).

To determine $k(X'_T)$ = # components of $\partial X'_T$, identify $I = \{1,\ldots,n\}$ with the vertices of the cyclic graph C_n and $T \cap I$ with the corresponding induced subgraph of C_n . It is then easy to see that (if $n > 0$) $r(f_T) = c(T\cap I)$, the number of components of $T \cap I$, and hence we obtain by (3):

Fact 3. $k(X'_T) = \begin{cases} 2 , & \text{if } n=0,\ p\equiv 1(2),\ q\equiv 0(2) \\ 2k-2c(T\cap I), & \text{otherwise} \end{cases}$.

From Facts (1), (2) and (3), Theorem 2 follows immediately.

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ON ADDITION THEOREMS

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ABSTRACT. In this note polynomially additive functions are defined and characterized.

Heuristically, an addition theorem is a relation between the function values $f(x)$, $f(y)$, $f(x+y)$ of some function f, which shows how to calculate the value of f at $x+y$ knowing its values at x and y. More generally, we may call a relation an addition theorem, if it expresses the value of a function f at $x+y$ using the values of some other functions $g,h,\ldots$ at x and y. For instance, addition theorems of this type for the trigonometric functions are standard subjects of undergraduate courses in mathematics. On the other hand, these types of addition theorems can be considered as functional equations if the question is : which functions have a given addition theorem ? From this point of view the simplest addition theorems are expressed by the classical Cauchy-equations

(1) $$a(x+y) = a(x) + a(y)$$

and

(2) $$m(x+y) = m(x)m(y) \quad .$$

Solutions of (1) are called additive, and those of (2) are called exponential functions.

A common property of the above mentioned addition theorems is that for the computation of the function value at $x+y$ one uses only two operations : addition and multiplication, that is, $f(x+y)$ is a polynomial of the values of some fixed functions at x and at y. There are other classical and "nice" addition theorems lacking this property : for instance, addition theorems for tangent and cotangent, and for some elliptic functions. It is natural to pose the problem : determine all functions which possess a "polynomial addition theorem" in the above mentioned sense.

First we need a definition. $\mathbb{C}$ denotes the set of complex numbers.

Definition. Let G be an abelian group and $f: G \to \mathbb{C}$ a function. We say that f is polynomially additive, if there exist nonnegative integers n, m, a complex polynomial P in $n+m$ variables, and functions $g_1,\dots,g_n,h_1,\dots,h_m : G \to \mathbb{C}$ such that

$$f(x+y) = P(g_1(x),\dots,g_n(x),h_1(y),\dots,h_m(y)) \tag{3}$$

holds for all x,y in G. A functional equation of the form (3) is called a polynomial addition theorem.

Now our problem is the following : given G, determine all polynomially additive functions. Trivially, all additive and exponential functions are polynomially additive. But we can go one step further : let k,l be nonnegative integers, Q a complex polynomial in $k+l$ variables and $a_1,\dots,a_k,m_1,\dots,m_\ell : G \to \mathbb{C}$ functions, where a_i is additive and m_j is exponential ($i = 1,\dots,k$; $j = 1,\dots,\ell$). Then the function $f: G \to \mathbb{C}$ defined by

$$f(x) = Q(a_1(x),\dots,a_k(x),m_1(x),\dots,m_\ell(x)) \tag{4}$$

is polynomially additive.

The proof is trivial by easy calculation. Functions of the form (4) are called exponential polynomials. Hence we have the following.

THEOREM 1. On an arbitrary abelian group any complex valued exponential polynomial is polynomially additive.

The complete solution of our above mentioned problem is that the converse statement is also true.

THEOREM 2. On an arbitrary abelian group any complex valued polynomially additive function is an exponential polynomial.

Proof. Suppose, that $f: G \to \mathbb{C}$ is polynomially additive, that is (3) holds for all x,y in G. This means, that f satisfies a functional equation of the form

$$f(x+y) = \sum_{i=1}^{N} G_i(x)H_i(y)$$

with some functions $G_i, H_i : G \to \mathbb{C}$ ($i = 1,\ldots,N$).Hence, f belongs to a finite dimensional translation invariant subspace of all complex valued functions on G ; it follows from the results of [2] (see also [1] , [3]) that f is an exponential polynomial.

We remark finally that the notion of "polynomially additive function" can be modified by replacing P in (3) by elements of other function classes. For instance, in the real case it would be interesting to determine all "rationally additive functions", where P in (3) is a rational function. In the complex case the same can be studied concerning entire functions. In the general case, the problem can be modified by asking for the solutions of (3) where P is a generalized polynomial.

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THE GENERALIZED PSEUDOPRIME CONGRUENCE $a^{n-k} \equiv b^{n-k} \pmod{n}$

Wayne L. McDaniel

Presented by P. Ribenboim, F.R.S.C.

ABSTRACT. We prove that $a^{n-k} \equiv b^{n-k} \pmod{n}$ has infinitely many composite solutions for all triples (a,b,k) of integers $a > b \geq 1$, $(a,b) = 1$, $k \geq 1$, with the exception of (2,1,4), (3,2,3), (7,3,3), $(2^u+1, 2^u-1, 3)$ for $u \geq 2$, and (b+1,b,2) and (b+3,b,2) for $b \geq 2$.

1. INTRODUCTION. Let $a > b \geq 1$, $(a,b) = 1$, and $k \geq 1$. The congruence

$$a^{n-k} \equiv b^{n-k} \pmod{n} \tag{1}$$

has been shown to have infinitely many composite solutions n if (a,b,k) = (2,1,1) by Cipolla [1], for all a if b = k = 1 by Steuerwald [9], and for all pairs a and b if k = 1 by Rotkiewicz [5]. The solutions are known, respectively, as pseudoprimes, pseudoprimes with respect to a, and pseudoprimes with respect to a and b.

For $k > 1$, the congruence has been shown to have infinitely many solutions n if $(a,3) = 1$, $b = 1$, $k = 3$ by Morrow [4], if $(a,k) = 1$ and $b = 1$ by Makowski [2], if $b = 1$ and $k = 3$ for all $a > 1$ by Rotkiewicz [6], and if (a,b,k) = (2,1,2) by Rotkiewicz [7].

It is our purpose here to announce the following general result:

THEOREM 1. *The congruence* $a^{n-k} \equiv b^{n-k} \pmod{n}$ *has infinitely many composite solutions* n *for all triples* (a,b,k) *with the possible exception of* (2,1,4), (3,2,3), (7,3,3), $(2^u+1, 2^u-1, 3)$ *for* $u \geq 2$, *and* (b+1,b,2) *and* (b+3,b,2) *for* $b \neq 1$.

2. DEVELOPMENT OF THE PROOF. Our proof is based on an examination of the divisibility properties of the cyclotomic polynomial

$$F_m(a,b) = \prod_{d|m} (a^d - b^d)^{\mu(m/d)}$$

which is related to our problem through the relationship

$$a^m - b^m = \prod_{d|m} F_d(a,b).$$

Zsigmondy [10] showed that $F_m(a,b)$ has at least one divisor of the form jm + 1 ($j \geq 1$) except when $F_m(a,b) = F_1(a,a-1)$, $F_2(a,2^u-a)$ for $u > 1$, and $F_6(2,1)$. Such divisors are called primitive divisors.

Our first step in proving Theorem 1 involves establishing the following generalization of Makowski's result to which we referred in the Introduction:

THEOREM 2. *If* $k \geq 2$ *and* (ab,k) = 1, *there exist infinitely many solutions* n *of* (1).

To prove Theorem 2, we show that if m is the product of the distinct prime factors of k, and $t > \max\{a,b,k,6\}$, then $n = pk$ is a solution for p any primitive divisor of $F_{t\phi(m)}(a,b)$. (ϕ is the Euler-phi function.)

Rotkiewicz has shown that if f(n) satisfies certain conditions and n_o is a composite solution of $a^{f(n)} \equiv b^{f(n)} \pmod{n}$ such that $2 < f(n_o) \geqslant n_o/2$, then this congruence has infinitely many solutions ([6], Thm. 31). In our third theorem, we establish that the condition that n_o be composite is not necessary when $f(n) = n - k$. The proof shows that if N is a solution of (1) and p is a primitive divisor of $F_{N-k}(a,b)$ then $n = pN$ is also a solution of (1).

THEOREM 3. *If there exists a solution* $n > 2k - 1$ *of* (1), *then the congruence has infinitely many composite solutions.*

We next find a "source" of solutions of (1):

THEOREM 4. *If* $k \geqslant 2$, *the primitive divisors of* $F_{k-1}(a,b)$ *satisfy* (1).

The proof is immediate: $n = j(k - 1) + 1 \Rightarrow (k - 1)(j - 1) = n - k$ which implies (1), if $n \mid F_{k-1}(a,b)$.

Theorems 3 and 4, then, prove Theorem 1 for all k for which $F_{k-1}(a,b)$ has a primitive divisor $> 2k - 1$. Now, $F_m(a,b)$ is bounded below by $(a - b)^{\phi(m)}$; it is not difficult to show that with the exception of the cyclotomic polynomials $F_m(a,b)$ which by Zsigmondy's theorem have no primitive divisors, and those for which $a - b = 1$, there exist only a finite number of polynomials $F_m(a,b)$ which do not have a primitive divisor greater than $2m + 1$. Since our Theorem 2 proves (1) for all triples (a,b,k) such that $(ab,k) = 1$, independent of whether $F_{k-1}(a,b) > 2k - 1$, the remainder of the proof of Theorem 1 involves reducing the set of triples (a,b,k) for which

(i) $F_{k-1}(a,b)$ has not been shown to have a primitive divisor greater than $2k - 1$, and

(ii) $(ab,k) \neq 1$

to the set identified in the statement of Theorem 1. This is accomplished with the aid of Lemma 1 and Theorem 5:

LEMMA 1. *Let* $m \geq 2$.

(i) $F_m(a,b) > a^{\phi(m)} m^{-d(m)/2}$, *where* $d(m)$ *is the number of positive divisors of* m;

(ii) *If* $m = pM$, p *prime and* $M > 1$, $F_m(a,b) > a^{(p-2)\phi(M)}$;

(iii) *If* $m = pM$, p *prime and* p *a factor of* M,

$$F_m(a,b) > a^{p\phi(M)} M^{-d(M)/2}.$$

THEOREM 5. *If* $m > 2$, $m \equiv 2 \pmod 4$ *and* $(ab,m+1) > 1$, $a^m - b^m$ *has a prime factor* $p = jm + 1$ *with* $j \geq 3$.

The proofs of the above Theorems appear in our paper [3].

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On the Dirichlet-Poisson problem for Schrödinger operators

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Abstract. For a given domain D in $\mathbb{R}^d$ and a bounded, Borel-measurable function q on $\mathbb{R}^d$ we study the Schrödinger operator $-\frac{1}{2}\Delta+q$ on D. In particular we are interested in conditions which imply that the following Dirichlet-Poisson problem

$$(-\frac{1}{2}\Delta+q)u=g \quad \text{in } D, \qquad u=f \quad \text{on } \partial D$$

is uniqely solvable and that $f\geq 0$ and $g\geq 0$ yield $u\geq 0$.

1. Preliminaries. In what follows, D will denote a domain in $\mathbb{R}^d$ $(d\geq 1)$, q a bounded, Borel-measurable, real-valued function on $\mathbb{R}^d$ and Q a real number (sometimes regarded as constant function on $\mathbb{R}^d$) satisfying $Q\geq 0$ and $Q\geq q$. Let ∂D be the boundary of D and ∂D_{reg} be the set of regular boundary points of D (with respect to the Laplace equation $\Delta u=0$). $B(E)$ ($C^*(E)$, $C(E)$) denotes the set of Borel-measurable (lower semicontinuous, continuous) real-valued functions on a subset E of $\mathbb{R}^d$. The subsets of bounded resp. nonnegative functions are indicated by subscripts b and $+$.

D is called *q-quasiregular* if for every $f\in C_b(\partial D)$ and $g\in B_b(D)$ there exists a unique solution $u\in C_b(D)$ of the Dirichlet-Poisson problem (D,q,f,g) and if $f\geq 0$ and $g\geq 0$ imply $u\geq 0$. Here the function $u\in C_b(D)$ is called a *solution of the Dirichlet-Poisson problem* (D,q,f,g) iff

$$(-\frac{1}{2}\Delta+q)u=g \qquad \text{in } D \qquad \text{and}$$
$$\lim_{x\to z}u(x)=f(z) \qquad \text{for all } z\in\partial D_{reg}.$$

Solutions or supersolutions of differential equations are understood in the sense of distributions. The main object of this article is to give conditions for D to be q-quasiregular.

We define the set of *q-harmonic* functions on D:

$${}^qH(D)=\{u\in C(D):\ (-\frac{1}{2}\Delta+q)u=0 \quad \text{in } D\ \}.$$

$({}^qH, \mathbb{R}^d)$ turns out to be a Brelot harmonic space. Let ${}^qS(D)$ denote the corresponding set of q-superharmonic functions on D. It can be shown that

$$ {}^qS_b(D) = \{u \in C_b^*(D) : \ (-\frac{1}{2}\Delta + q)u \geq 0 \quad \text{in } D \}. $$

Next consider the set

$$ {}^qH_0(D) = \{u \in {}^qH_b(D) : \ \lim_{x \to z} u(x) = 0 \quad \text{for all } z \in \partial D_{reg}\}. $$

By $\mathrm{spec}_D(-\frac{1}{2}\Delta + q)$ we denote the point spectrum of $-\frac{1}{2}\Delta + q$ on D, that is,

$$ \mathrm{spec}_D(-\frac{1}{2}\Delta + q) = \{\lambda \in \mathbb{R} : \ {}^{q-\lambda}H_0(D) \neq \{0\}\}. $$

The basic idea now is to define suitable Poisson and Green kernels for the Schrödinger operator $-\frac{1}{2}\Delta + q$ with the help of corresponding kernels for the operator $-\frac{1}{2}\Delta + Q$. Suppose that $Q > 0$ and define

$$ {}^QGu(x) = \int_{\mathbb{R}^d} u(y) \int_0^\infty \exp(-Qt) \cdot (2\pi t)^{-d/2} \cdot \exp(-\frac{|x-y|^2}{2t}) \ dt \ dy $$

for $u \in B_+(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ ($| \cdot |$ being the Euclidean norm in $\mathbb{R}^d$). Let ${}^QP_D$ be the balayage kernel for ${}^cD = \mathbb{R}^d \setminus D$ associated with the strong harmonic space $({}^QH, \mathbb{R}^d)$ (cf. [2]) and define ${}^QG_D = {}^QG - {}^QP_D \cdot {}^QG$. Finally, consider the (Q,q)-transformation kernel

$$ {}^{Q,q}T_D = \sum_{n=0}^{\infty} {}_{Q-q}^{\ \ Q}G_D^{\ n} , $$

where ${}_{Q-q}^{\ \ Q}G_D u = {}^QG_D((Q-q) \cdot u)$ (*for* $u \in B_+(\mathbb{R}^d)$). The kernels

$$ {}^qP_D = {}^{Q,q}T_D \cdot {}^QP_D \qquad \text{and} \qquad {}^qG_D = {}^{Q,q}T_D \cdot {}^QG_D $$

are called *q-Poisson* and *q-Green kernels* for D. They turn out to be independent of Q. However, it may happen that $\|{}^qP_D\| = \|{}^qG_D\| = \infty$, even if D and q are bounded. Here we write $\|K\| = \sup\{K1(x) : \ x \in \mathbb{R}^d\}$ for a kernel K on $\mathbb{R}^d$.

Of particular interest is the behaviour of the kernel ${}^0G_D$ which is exactly the classical Green kernel for D. If this kernel, considered as an operator on $B_b(\mathbb{R}^d)$, is bounded (compact), we call D *Green-bounded (Green-compact)*. These notions will be discussed later.

2. Quasiregularity. We give several equivalent conditions for D to be q-quasiregular.

2.1. Theorem ("Gauge Theorem").

1. *Assume $Q > 0$. Then the following statements are equivalent:*

   a) *D is q-quasiregular;*

   b) $\|{}^qG_D\| < \infty$;

   c) $\|{}_{Q-q}^{\;Q}G_D{}^{\,n}\| < 1$ *for some* $n \in \mathbf{N}$;

   d) *there exist a function* $u \in {}_+C_b^*(D)$ *and an* $\epsilon > 0$ *such that* $(-\frac{1}{2}\Delta + q)u \geq \epsilon$ *in* D.

2. *If D is Green-bounded, the above remains true without the additional assumption $Q > 0$. Moreover, the statements are equivalent to:*

   e) $\|{}^qP_D\| < \infty$;

   f) *there exist a function* $u \in {}^q_+S_b(D)$ *and an* $\epsilon > 0$ *such that* $\liminf_{x\to z} u(x) \geq \epsilon$ *for all* $z \in \partial D_{reg}$.

3. *If D is Green-compact, the above statements are further equivalent to:*

   g) ${}^q_+S(D) \setminus {}^qH_0(D) \neq \emptyset$;

   h) *there exists a function* $f \in B_+(\mathbf{R}^d)$ *such that in D neither* ${}^qP_Df \equiv 0$ *nor* ${}^qP_Df \equiv \infty$;

   i) *there exists a function* $g \in B_+(\mathbf{R}^d)$ *such that in D neither* ${}^qG_Dg \equiv 0$ *nor* ${}^qG_Dg \equiv \infty$;

   j) $\inf \operatorname{spec}_D(-\frac{1}{2}\Delta + q) > 0$;

   k) *D is a minimum principle set for* $-\frac{1}{2}\Delta + q$, *that is: for every function* $u \in {}^qS(D)$ *that is bounded below and satisfies* $\liminf_{x\to z} u(x) \geq 0$ *for all* $z \in \partial D_{reg}$, *we have* $u \geq 0$ *in* D.

If D is q-quasiregular, the unique solution $u \in C_b(D)$ *of the Dirichlet-Poisson problem* (D, q, f, g) *for* $f \in C_b(\partial D)$ *and* $g \in B_b(D)$ *is given by* $u = {}^qP_Df + {}^qG_Dg$.

2.2. Remark. The proof of the preceding theorem is purely analytic. It depends essentially on two facts.

1. For $Q > 0$ every domain in $\mathbf{R}^d$ is Q-quasiregular.
2. Let $f \in C_b(\partial D)$ and $g \in B_b(D)$. A function $u \in C_b(D)$ is a solution of the Dirichlet-Poisson problem (D, q, f, g) if and only if $v = u - {}_{Q-q}^{\;Q}G_Du$ is a solution of the Dirichlet-Poisson problem (D, Q, f, g).

2.3. Remarks. 1. In general (the finiteness of) the quantity $\|{}^qG_D\|$ serves as a *gauge* of the q-quasiregularity of D. This quantity can be replaced by $\|{}^qP_D\| = \sup\{{}^qP_D1(x) : x \in D\}$ if D is Green-bounded, or even by $\inf\{{}^qP_D1(x) : x \in D\}$ if D is Green-compact.

2. Suppose now that D is Green-compact. The most interesting condition for q-quasiregularity is condition g). It says that D is q-quasiregular if there exists at least one nonnegative q-superharmonic function u on D which is either unbounded or fails to be q-harmonic (i.e. the distribution $(-\frac{1}{2}+q)u$ is not $=0$ in D) or does not vanish identically on the regular part of the boundary (i.e. $\limsup_{x\to z} u(x) > 0$ for some $z \in \partial D_{reg}$). Each of the conditions d), f), h), and i) ensures the existence of a special function of this type.

3. The identity ${}^qP_Df \equiv 0$ in D is equivalent to $f = 0$ a.e. on ∂D with respect to ${}^0P_D(x, .)$, the classical harmonic measure for D in an arbitrary point $x \in D$.

4. The identity ${}^qG_Dg \equiv 0$ in D is equivalent to $g = 0$ a.e. on D with respect to Lebesgue measure on $\mathbb{R}^d$.

5. Condition h) (i)) is obviously fulfilled if ${}^qP_D1 \not\equiv \infty$ (${}^qG_D1 \not\equiv \infty$) in D.

3. Green-bounded and Green-compact domains. Since by definition D is Green-bounded iff $\|{}^0G_D\| < \infty$, the first part of the above theorem yields a characterization of Green-bounded domains. However, these results can be improved. We note that the following two propositions are valid for every $Q \geq 0$ (not depending on any q).

3.1. Proposition. *The following statements are equivalent:*

a) *D is Green-bounded;*

b) *D is 0-quasiregular;*

c) $\|{}^QG_D\| < \frac{1}{Q}$ *(with $\frac{1}{0} = \infty$);*

d) *there exist a function $u \in {}_+C_b^*(D)$ and an $\epsilon > 0$ such that $-\frac{1}{2}\Delta u \geq \epsilon$ in D;*

e) *for all $q \in B_b(\mathbb{R}^d)$:* ($\|{}^qP_D\| < \infty \Longleftrightarrow \|{}^qG_D\| < \infty$).

3.2. Proposition. *The following statements are equivalent:*

a) *D is Green-compact;*

b) *${}^QG_D$ is compact (considered as an operator on $B_b(\mathbb{R}^d)$);*

c) $\lim_{x\to\infty} {}^QG_D1(x) = 0$;

d) *there exist a function* $u \in {}_+C_b^*(D)$ *and an* $\epsilon > 0$ *such that* $(-\frac{1}{2}\Delta + Q)u \geq \epsilon$ *in* D *and* $\lim_{x\to\infty} u(x) = 0$.

3.3. Examples.

a) *Every domain of finite Lebesgue measure is Green-compact.*

b) *Every Green-compact domain is Green-bounded; and so is every domain whose projection on a suitable subspace* ($\neq \{0\}$) *of* $\mathbb{R}^d$ *is (Green-)bounded.*

c) *The set* $\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1\}$ *is Green-bounded but not Green-compact.*

d) *The set* $\{(x_1, x_2) \in \mathbb{R}^2 : |x_1 \cdot x_2| < 1\}$ *is Green-compact but has infinite Lebesgue measure.*

4. Probabilistic Interpretation. The kernels used in the preceding sections can also be defined by probabilistic tools: Let $(P^x, X_t : x \in \mathbb{R}^d, t \geq 0)$ be standard Brownian motion in $\mathbb{R}^d$ and $\tau(D) = \inf\{t > 0 : X_t \notin D\}$ be the first exit time of D.

4.1. Proposition. *Let* $u \in B_+(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$. *Then*

$$ {}^qP_Du(x) = E^x\left[u(X_{\tau(D)}) \cdot \exp\left(-\int_0^{\tau(D)} q(X_s)\, ds\right) \cdot 1_{\{\tau(D)<\infty\}}\right], $$

$$ {}^qG_Du(x) = E^x\left[\int_0^{\tau(D)} u(X_t) \cdot \exp\left(-\int_0^t q(X_s)\, ds\right)\, dt\right]. $$

This allows a probabilistic interpretation of the preceding results and a comparison with results in [1] and [3]-[6].

Suppose D to be Green-compact, which is to say in the probabilistic language,

$$ \lim_{x\to\infty} E^x[\tau(D)] = 0. $$

The implications h) $\Rightarrow$ e) $\Rightarrow$ a) of our theorem generalize the gauge theorem of Chung and Rao (cf. [3]-[5]) which we obtain if we put $f = 1$ in h):

$$ {}^qP_D1 \not\equiv \infty \text{ in } D \quad \Longrightarrow \quad \|{}^qP_D\| < \infty \quad \Longrightarrow \quad D \text{ is } q\text{-quasiregular}. $$

Next assume $Q = 0$, $q \leq 0$, and D to be Green-bounded, that is to say,

$$ \sup_{x\in\mathbb{R}^d} E^x[\tau(D)] < \infty. $$

The probabilistic version of the equivalence of the statements 2.1.c) and e) yields the following sharpening of Khas'minskii's lemma ([1], 1.2.):

$$\sup_{x\in\mathbf{R}^d} E^x\left[\frac{1}{n!}\left(-\int_0^{\tau(D)} q(X_s)\,ds\right)^n\right] < 1 \quad \text{for some } n \in \mathbf{N}$$

$$\Longleftrightarrow \quad \sup_{x\in\mathbf{R}^d} E^x\left[\exp\left(-\int_0^{\tau(D)} q(X_s)\,ds\right)\right] < \infty.$$

If q is a Kato function or a suitable measure, related results have been obtained by Hansen and Hueber [7].

The present article is a comprehensive version of the author's diploma thesis [8]. The author is grateful to Professor H. Bauer for his guidance and encouragement in writing this note.

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USING INTEGRAL CLOSURE TO CHARACTERIZE GOING-DOWN DOMAINS

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Abstract. Let R be an integral domain with integral closure R'. It is proved that R is a quasilocal going-down domain if and only if, for each prime ideal P of R, R' is the intersection of those valuation overrings V of R which have a prime lying over P. Attention is then paid to analogues in which restrictions are placed on the Krull dimension of the rings R, V.

Let R be an integral domain, with integral closure R' and quotient field K. Let $X(R)$ denote the set of valuation overrings of R, i.e., the valuation domains contained between R and K. One reason for the importance of $X(R)$ is surely Krull's result (cf. [7, Corollary 19.9]) that if (R,M) is quasilocal, then R' is the intersection of those $V \in X(R)$ which are centred on M, i.e., those $V \in X(R)$ for which M is in the image of the canonical contraction map $f_V = f_V^R : \text{Spec } V \to \text{Spec } R$. It thus seems natural to ask which R have the following property, dubbed (*) for convenience:

(*) For each $P \in \text{Spec }(R)$, $R' = \cap\{V \in X(R) : P \in \text{im}(f_V)\}$.

It follows easily from Krull's result that arbitrary valuation domains and arbitrary quasilocal integral domains of (Krull) dimension at most 1 satisfy (*). This suggests our main result:

THEOREM 1. <u>An integral domain R satisfies (*) if and only if R is a quasilocal going-down domain</u>.

Recall from [3], [6] that an integral domain R is called a going-down domain in case $R \subset T$ satisfies GD (the going-down property) for each integral domain T containing R; and that one may restrict attention to test extensions T belonging to X(R). As the property of being a going-down domain is a local one, there has been some interest in finding characterizations of quasilocal going-down domains (cf. [5, Proposition 2.1]). Theorem 1 adds one more.

Next, recall that the most natural examples of going-down domains are Prüfer (i.e., locally valuation) domains, integral domains of dimension at most 1, and certain D + M constructions. The first and second of these were used above to motivate the statement of Theorem 1. After interplay with valuative dimension, complete integral closure, and universal catenarity, our work will lead naturally back to Prüfer domains in Remark 5(c). Moreover, Example 4 will depend on a going-down domain of D + M type.

<u>Proof of Theorem 1</u>. Assume that (R, M) is a quasilocal going-down domain. By [3, Theorem 2.2], the primes of R are linearly ordered via inclusion. We proceed to show that R satisfies (*). Let $P \in \text{Spec } R$ and consider an element $u \in \cap\{V \in X(R) : P \in \text{im}(f_V)\}$. As $R' = \cap\{W : W \in X(R)\}$, it will suffice to show that u is a member of each $(W, N) \in X(R)$. Put $Q = N \cap R$. By the above appeal

to [3], there are two cases: either $P \subset Q$ or $Q \subset P$. If $P \subset Q$, the fact that $R \subset W$ satisfies GD yields $N_1 \in \text{Spec } W$ such that $N_1 \subset N$ and $P = N_1 \cap R$; thus, $P = f_W(N_1) \in \text{im}(f_W)$, whence $u \in W$, as desired. In the remaining case, $Q \subset P$. By appeal to [7, Corollary 19.7], W contains some $(V, N_2) \in X(R)$ having a chain of primes $N \subset N_3 \subset N_2$ that contracts to the chain $Q \subset P \subset M$. As $P = N_3 \cap R = f_V(N_3) \in \text{im}(f_V)$, we have $u \in V$. Since $V \subset W$, it follows that $u \in W$, as desired.

Conversely, assume that R satisfies (*). By [5, Proposition 2.1], in order to prove that R is a quasilocal going-down domain, we need only prove, for each $P \in \text{Spec } R$, that each element of PR_P is integral over R. Suppose this fails. Then there exist $a \in P$, $b \in R \backslash P$ such that $v = ab^{-1} \notin R'$. Since R satisfies (*), we can now find $(W, N) \in X(R)$ such that $v \notin W$ and $P \in \text{im}(f_W)$. Accordingly, choose $Q \in \text{Spec } W$ so that $Q \cap R = P$. As W is a valuation domain, v^{-1} is a nonunit of W; i.e., $v^{-1} \in N$. Moreover, as $bv = a \in PW \subset Q$ and $b \notin Q$, it follows that $v \in Q$. (More than just the primeness of Q in W is involved here, for v belongs only to the quotient field of W. One must recognize, in the notation of [9, Proposition 1.1], that Q is strongly prime in the (pseudo)valuation domain W.) Thus $v \in W$, the desired contradiction. The proof is complete.

<u>REMARK 2</u>. As an easy corollary of Theorem 1, we obtain a new proof of [4, Corollary 2.5]: if R' is a valuation domain, then each overring of R is a (necessarily quasilocal) going-down domain.

For each positive integer n, and integral domain R, set

$X_n(R) = \{W \in X(R) : \dim(W) \leq n\}$. One may ask whether R satisfies

$(*)_n$: For each $P \in \text{Spec}(R)$, $R' = \cap\{V \in X_n(R) : P \in \text{im}(f_V)\}$.

The next two results address the question of finding a $(*)_n$-theoretic analogue of Theorem 1.

PROPOSITION 3. *Let R be an integral domain and n a positive integer. If R satisfies $(*)_n$, then R is a quasilocal going-down domain of dimension at most n.*

Proof. Since $(*)_n$ implies $(*)$, an application of Theorem 1 yields the first assertion. It remains only to prove that $\dim(R) \leq n$. Suppose this fails. Accordingly, one may find a chain

$$P_1 \subset \ldots \subset P_n \subset P_{n+1}$$

of $n + 1$ distinct nonzero primes of R. As R satisfies $(*)_n$, there exist $V \in X_n(R)$ and $Q \in \text{Spec } V$ such that $P_{n+1} = f_V(Q) = Q \cap R$. Next, as $R \subset V$ satisfies GD, one has a chain of $n + 1$ (necessarily distinct and nonzero) primes of V descending from Q and contracting to the displayed chain in Spec R. Hence $\dim(V) \geq n + 1$, contradicting $V \in X_n(R)$. The proof is complete.

EXAMPLE 4. Let n be a positive integer. Then there exists a quasilocal n-dimensional going-down domain R which does not satisfy $(*)_n$. It can be arranged that $\dim_v(R)$, the valuative dimension of R, is $n + 1$.

To find such an R, we begin with a nonalgebraic field extension F/k. Next, consider any n-dimensional valuation domain (V, M) of the form $V = F + M$. Then $R = k + M$ has the asserted properties.

Indeed, since k is a going-down domain, [6, Corollary] yields that R is, too. The lore of the $D + M$ construction (cf. [1, Theorem 2.1]) shows that R is n-dimensional and quasilocal; and that $\dim(W) \geq n + 1$ for each valuation overring W of R which is properly contained in V. Since [1, Theorem 3.1] assures that each member of $X(R)$ compares with V under inclusion, it follows that V is contained in <u>any</u> intersection of members of $X_n(R)$. Let L denote the algebraic closure of k in F. As $R' = L + M \subsetneq V$, R does not satisfy $(*)_n$. Finally, if F is taken as $k(X)$, one has $\dim_v(R) = n + 1$: just note that $\sup(\{\dim A : A$ is a valuation ring of $k(X)$ containing $k\}) = 1$.

<u>REMARK 5</u>. (a) In view of Example 4, one might conjecture the following partial converse of Proposition 3: if R is a quasilocal going-down domain and $n = \dim(R) = \dim_v(R) < \infty$, then R satisfies $(*)_n$. This assertion is easily proved by appeal to Theorem 1, for the present hypotheses yield $X(R) = X_n(R)$, so that $(*)$ reduces to $(*)_n$ in this case.

(b) The converse of the result in (a) is false, even for $n = 1$. In other words, if R satisfies $(*)_1$, it need not be the case that $\dim_v(R) \leq 1$. To see this, let (R, M) be Ribenboim's example [10] of a quasilocal one-dimensional completely integrally closed integral domain which is not a valuation domain. Showing that R satisfies $(*)_1$ amounts to showing that $R(= R') = \cap \{V : V \in X_1(R)\}$. However, since R is an integrally closed G-domain, a result of Gilmer and Heinzer [8] shows that this intersection is the complete integral closure of R, which <u>is</u> R. Finally, $\dim_v(R) > 1$, essentially because R is not a Prüfer domain (cf. [7, Corollary 30.12 and Proposition 30.14]).

(c) In view of (b), one should ask: which integral domains satisfy the hypothesis of the result in (a)? Here is the answer. Let R be a quasilocal going-down domain of dimension $n < \infty$; then $\dim_v(R) = n$ if and only if R' is a Prüfer domain. Note that the "if" assertion is standard; and the "only if" assertion is a consequence of some recent work [2, Theorem 6.2] on universal catenarity.

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A REDUCTION PROCESS VIA TILTING THEORY *

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Presented by V. Dlab, F.R.S.C.

Abstract: Let A be a finite dimensional algebra over an arbitrary field $\mathbf{k}$. We associate with certain modules S_A a finite dimensional $\mathbf{k}$-algebra $C(S)$ which enjoys the following properties: $C(S)$ has less simple modules than A; its global dimension is bounded by the global dimension of A and its module category is fully and exactly embedded in the module category of A. $C(S)$ is constructed by using the theory of tilting modules. A main result for a wild hereditary algebra A and an indecomposable regular module S_A without self-extensions is that the algebra $C(S)$ is a wild hereditary algebra. This implies that, if the endomorphism algebra of a tilting module T_A with A a wild hereditary algebra is not of wild representation type, then T_A has non-zero preprojective as well as non-zero preinjective direct summands.

1. Notation and basic definitions. Let A be a finite dimensional $\mathbf{k}$-algebra. Denote by *mod A* the category of all right A-modules which are finite dimensional over $\mathbf{k}$.

For $X \in mod\ A$, denote by $\mathcal{G}(X)$ and $\mathcal{A}(X)$ the full subcategories of *mod A*, where $\mathcal{G}(X) = \{Y \in mod\ A : Ext^1_A(X,Y) = 0\}$ and $\mathcal{A}(X) = \{Y \in mod\ A : Ext^t_A(X,Y) = 0 \text{ for all } t \geq 0\}$. If $p.d.X \leq 1$, then $\mathcal{A}(X)$ is closed under the formation of images, kernels, cokernels and extensions in *mod A*. In particular, $\mathcal{A}(X)$ is an abelian subcategory of *mod A*.

For $X \in mod\ A$, the numerical invariant $n(X)$ of X is defined by a decomposition $X \cong \bigoplus_{i=1}^{n(X)} X_i^{x_i}$, where $X_1, X_2, \ldots, X_{n(X)}$ are indecomposable and pairwise non-isomorphic modules, while $x_i \geq 1$.

Recall that a module T_A is called a **tilting module**, if $p.d.T \leq 1$, $Ext^1_A(T,T) = 0$ and $n(T_A) = n(A_A)$; see [HR 1] and [B] for the theory of tilting modules.

2. The tilting module T(S) and the algebra C(S). Following an idea of W. Geigle [G] and

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modifying some arguments of K. Bongartz [B], we have the following

Theorem 1. Let A be a finite dimensional **k**-algebra and $n = n(A_A)$. Suppose $S = \bigoplus_{i=1}^{s} S_i \in mod\ A$ satisfies the following conditions : i) $p.d.S \leq 1$; ii) $Ext^1_A(S,S) = 0$; iii) each $End\ S_i$ is a division algebra; iv) $Hom_A(S_i, S_j) = 0$ for all $i \neq j$ and v) $Hom_A(S, A) = 0$. Then there exist, up to an isomorphism, uniquely determined non-zero, indecomposable and pairwise non-isomorphic modules $V_1, V_2, \ldots, V_{n-s} \in \mathcal{A}(S)$ with the following properties for $V := \bigoplus_{i=1}^{n-s} V_i$ and $T(S) := V \oplus S$:

a) V is a generator of $\mathcal{A}(S)$.

b) $T(S)$ is a tilting module with $\mathcal{G}(T(S)) = \mathcal{G}(S)$.

c) Let $B(S) = End\ T(S)_A$ and $C(S) := End\ V_A$. Note that $n(C(S)_{C(S)}) = n - s$ and $B(S) \cong \begin{pmatrix} C(S) & 0 \\ * & End\ S \end{pmatrix}$. In particular, there exists a full and exact embedding $mod\ C(S) \rightarrow mod\ B(S)$. The categories $\mathcal{A}(S)$ and $mod\ C(S)$ are equivalent via the restriction of the tilting functors $F_S := Hom_A(T(S), -)$ and $G_S := - \otimes_{B(S)} T(S)$.

Moreover, $gl.dim.C(S) \leq sup\{p.d.X : X \in \mathcal{A}(S)\} \leq gl.dim.A$.

The V_i's are characterized by the following

Proposition 1. For a non-zero indecomposable $U \in mod\ A$, the following statements are equivalent:

i) $U \cong V_i$ for some $1 \leq i \leq n - s$.

ii) $U \in \mathcal{A}(S)$, $p.d.U \leq 1$ and $Ext^1_A(U, V) = 0$.

iii) $U \in \mathcal{A}(S)$ and $Ext^1_A(U, X) = 0$ for all $X \in \mathcal{A}(S)$, i.e. U is Ext-projective in $\mathcal{A}(S)$ in the sense of [AS].

Note that in an Auslander-Reiten sequence (see [AR])

$$0 \rightarrow \tau_A S_i \rightarrow Y_i \rightarrow S_i \rightarrow 0$$

the middle term Y_i belongs to $\mathcal{A}(S)$. Set $S^* := \bigoplus_{i=1}^{s} Y_i$. Theorem 1 allows us to reduce the construction of certain tilting modules over A to tilting modules over the "smaller" algebra $C(S)$ via

Proposition 2. Let $T = T' \oplus S$ be a tilting module satisfying $Hom_A(S, T') = 0$. Then $F_S(T')$ is a tilting module over $C(S)$, satisfying $Ext^1_{C(S)}(F_S(T'), F_S(S^*)) = 0$. Conversely, if U is a tilting module over $C(S)$ satisfying $Ext^1_{C(S)}(U, F_S(S^*)) = 0$, then $G_S(U) \oplus S$ is a tilting module over A.

3. Hereditary algebras. In what follows, assume that A is a hereditary algebra. Fix a decomposition $A_A = \bigoplus_{i=1}^{n} P_i^{a_i}$ with $P_1, \ldots, P_n$ indecomposable and pairwise non-isomorphic projective modules. Set $E_i = P_i / rad\, P_i$ and denote by I_i the injective hull of E_i.

The **valued graph** of A is given by the following data: The set of vertices is $\{1, 2, \ldots, n\}$ and there is a valued edge $i \overset{(d_{ij}, d_{ji})}{\longrightarrow} j$ whenever $Ext^1_A(E_i, E_j) \neq 0$ with $d_{ij} := dim_{\,End\, E_j} Ext^1_A(E_i, E_j)$ and $d_{ji} := dim\, Ext^1_A(E_i, E_j)_{End\, E_i}$. Since $Ext^1_A(E_i, E_j) \neq 0$ implies that $Ext^1_A(E_j, E_i) = 0$ for a finite dimensinal hereditary algebra, this defines a valued graph in the sense of [DR]. The fact that $Ext^1_A(E_i, E_j) \neq 0$ is expressed by an arrow $i \longrightarrow j$, which defines an orientation on the graph. Set $f_i := [End\, E_i : \mathbf{k}]$. The **quadratic form** q_A on Z^n is defined by

$$q_A(\underline{x}) := \sum_{i=1}^{n} f_i x_i^2 - \sum_{i-j} d_{ij} f_j x_i x_j$$

for $\underline{x} = (x_1, x_2, \ldots, x_n) \in Z^n$.

An algebra B is called a **reflection of** A [APR], if there exists a simple projective, non-injective module P_i such that $B \cong End(\bigoplus_{\substack{j=1 \\ j \neq i}}^{n} P_j \oplus \tau_A^{-1} P_i)$. Note that a reflection B is again hereditary and that the valued graphs of A and B coincide. The orientations however are different. i is a sink in the valued graph of A, i.e there are no arrows of the form $i \longrightarrow j$. Arrows of the form $j \longrightarrow i$ correspond to arrows $j \longleftarrow i$ in the valued graph of B, while the orientations of the other arrows are not changed.

The construction in §2 is well-suited for the hereditary case. The assumptions of Theorem 1 are already satisfied if the module $S = \bigoplus_{i=1}^{s} S_i$ satisfies i) $Ext^1_A(S, S) = 0$; ii) $Hom_A(S_i, S_j) = 0$ for $i \neq j$ and iii) each S_i is indecomposable and non-projective, see [HR 1, Lemma 4.1]. Moreover, any non-projective tilting module T_A has an indecomposable non-projective direct summand S, such that $T = T' \oplus S^\alpha$ for some $\alpha \geq 1$ and $Hom_A(S, T') = 0$, see [HR 1, Corollary 4.2]. The algebra $C(S)$ is compatible with Auslander-Reiten translations according to the following

Theorem 2. Let A be a hereditary algebra and $S = \bigoplus_{i=1}^{s} S_i \in mod\ A$ satisfy the following conditions: i) $Ext^1_A(S,S) = 0$; ii) $Hom_A(S_i, S_j) = 0$ for all $i \neq j$ and iii) each S_i is indecomposable and neither S_i nor $\tau_A S_i$ is projective. Then $C(\tau_A S)$ is obtained from $C(S)$ by a finite sequence of reflections. In particular, the valued graphs of $C(S)$ and $C(\tau_A S)$ coincide, while the orientations may be different.

Proposition 3. If either $S = \tau_A^{-t} P_i$ for some $t \geq 1$ or $S = \tau_A^t I_i$ for some $t \geq 0$, then the valued graph of $C(S)$ is obtained from the valued graph of A by removing the vertex i and all edges adjacent to i. Moreover, if $T(S) = \bigoplus_{t=1}^{n-1} V_t \oplus S$, then there exists a bijection $\pi : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, i-1, i+1, \ldots, n\}$, such that $End\, V_t \cong End\, P_{\pi(t)}$.

Recall that an indecomposable module X is called **regular**, if $\tau_A^z X$ is neither projective nor injective for every integer z. We refer to [R 2] for properties of regular modules.

Theorem 3. Let S be an indecomposable regular module of quasi-length q with $Ext^1_A(S,S) = 0$. Let

$$S_1 \to S_2 \to \ldots \to S_{q-1} \to S_q = S$$

be a chain of irreducible monomorphisms. Then $T(S) = \bigoplus_{t=1}^{n-q} V_t \oplus \bigoplus_{i=1}^{q} S_i$. Moreover

$$Hom_A(\bigoplus_{t=1}^{n-q} V_t, \bigoplus_{i=1}^{q-1} S_i) = 0 = Hom_A(\bigoplus_{i=1}^{q} S_i, \bigoplus_{t=1}^{n-q} V_t).$$

$End \bigoplus_{i=1}^{q} S_i$ is isomorphic to the full lower triangular matrix ring over the division algebra $End\, S$. $C^w(S) := End \bigoplus_{t=1}^{n-q} V_t$ is an indecomposable hereditary algebra. The definiteness of the quadratic form $q_{C^w(S)}$ coincides with that of q_A; $mod\, C^w(S)$ and $\mathcal{A}(\bigoplus_{i=1}^{q} S_i)$ are equivalent via the restriction of the tilting functors F_S and G_S.

Remark 1. Parts of the proof of Theorem 3 were inspired by a paper of M. Hoshino [H]. His result, that the quasi-length of an indecomposable regular module without self-extensions is bounded by $n-2$, follows easily from Theorem 3.

Remark 2. For A a tame hereditary algebra and S_A a quasi-simple module without self-extensions, the algebra $C(S)$ is studied in [S] as the universal localization A_α, where α is given by a projective resolution $0 \to P \xrightarrow{\alpha} Q \to S \to 0$ of S.

4. Ringel's reduction of wild hereditary algebras to wild bimodules. In [R 1] C. M. Ringel reduced the proof of the fact that a **k**-species with indefinite quadratic form is wild, to the case of a bimodule with indefinite quadratic form. Theorem 3 provides an alternative reduction, using the following lemma, which seems to be well-known:

Lemma. Let A be an hereditary algebra of infinite representation type with $n(A_A) \geq 3$. Then there exists an indecomposable regular module S with $Ext^1_A(S,S) = 0$.

Corollary. If A is an indecomposable hereditary algebra with q_A indefinite and $n(A_A) \geq 3$, then there exist a hereditary algebra H, with $n(H_H) = 2$ and q_H indefinite, and a full and exact embedding $\Phi : mod\ H \longrightarrow mod\ A$ such that Φ has no non-zero preprojective module in its image.

5. Reduction of tilting modules without preinjective direct summands. By repeated use of the algebra $C(S)$ and Theorem 3 we get

Theorem 4. Let A be an indecomposable wild hereditary algebra and T_A be a tilting module without non-zero preinjective direct summands. Then, there exists a decomposition $T_A = T' \oplus T''$ which satisfies

i) $Hom_A(T'', T') = 0$. Thus, $B := End\ T_A \cong \begin{pmatrix} End\ T' & 0 \\ * & End\ T'' \end{pmatrix}$.

ii) There exist a wild hereditary algebra H and a preprojective tilting module U_H such that $End\ T' \cong End\ U_H$.

iii) The preprojective component of the algebra $End\ T'$ is a full component in the Auslander-Reiten quiver of B.

Theorem 5. Let A be a wild hereditary algebra and T_A be a tilting module. If $B := End\ T_A$ is not of wild representation type, then T_A has non-zero preprojective as well as non-zero preinjective direct summands.

Theorem 5 extends a result of [HR 2] which asserts such a condition for B of finite representation type.

Most of the presented material is part of my Ph. D. thesis. I would like to express my sincere gratitude to Professor Vlastimil Dlab for guiding me to the subject as well as for his valuable comments and suggestions.

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Stability Index and the u-Invariant

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Abstract. Let F be a formally real Pythagorean field with $|\dot{F}/\dot{F}^2| < \infty$. Then $st(F) = \log_2(u(F(\sqrt{-1})))$, where $st(F)$ is the stability index of F and $u(F(\sqrt{-1}))$ is the u-invariant of the field $F(\sqrt{-1})$.

§1. Introduction. This paper is meant as a continuation of the paper [12]. Therefore we keep to the notation in [12]. All notions and basic definitions are also in [5], [6], [7], [8], [9], [10], [11].

In the paper [12] it was observed that st(F) equals the cohomological dimension of a certain Galois group. We shall connect st(F) with another invariant of the field, namely the u-invariant. We shall recall its definition. (See [8], Chapter 11, Definition 4.1).

Let K be a field. Then

$$u(K) = \min\{n \mid \text{forms of dimension} > n \text{ over F are isotropic}\}$$
$$= \min\{n \mid \text{forms of dimension} \geq n \text{ over F are universal}\}$$

In general there are big gaps in our knowledge of the u-invariant. One of the most interesting open questions is (see [8], Chapter 11, §6, Question 2):

If u(K) is finite, is it always a power of 2?

For more information about the u-invariant and its generalisations the reader may consult e.g. [2], [3], [4], [6], [7], [14], etc.

Although in our special case the situation turned out to be rather simple, it seems that it did not occur explicitly in the literature.

§2. <u>Theorem</u>. <u>Let</u> F <u>be a formally real Pythagorean field with finite chain length</u> cl(F). <u>Then</u>

(1) $|\dot{F}/\dot{F}^2| < \infty \Rightarrow st(F) = \log_2 u(F(\sqrt{-1}))$

(2) $|\dot{F}/\dot{F}^2| = \infty \Rightarrow st(F) = \infty = u(F(\sqrt{-1}))$.

<u>Proof</u>. (1) Suppose that $|\dot{F}/\dot{F}^2| < \infty$. We shall prove our theorem by induction on cl(F).

(A) If cl(F) = 1, then F is Euclidean field and $F(\sqrt{-1})$ is a quadratically closed field. (See [1]). Thus $u(F(\sqrt{-1})) = 1$ and st(F) = 0.

(B) Suppose that

$$X_F = X_1 \cup \cdots \cup X_s, \quad 2 \leq s ,$$

where $X_1,\ldots,X_s$ are the connected components of X. By [5] §5, pp. 261, 262, lemma 7 and lemma 8, there exist Pythagorean fields $F_1,\ldots,F_s \subset F(2)$ such that

$$F = F_1 \cap \cdots \cap F_s, \quad X_{F_i} \cong X_i \quad \text{for} \quad i \in \{1,\ldots,s\}$$

and if ϕ is a quadratic form over $K = F(\sqrt{-1})$, $2 \leq \dim \phi$, then ϕ represents $b \in K$ iff ϕ represents b in $K_i = F_i(\sqrt{-1})$, $i = 1,\ldots,s$. Furthermore the natural map $\dot{K}/\dot{K}^2 \longrightarrow \dot{K}_1/\dot{K}_1^2 \oplus \cdots \oplus \dot{K}_s/\dot{K}_s^2$ is surjective.

Consequently we find that ϕ is an universal form over K iff ϕ is an universal form over each K_i, $i = 1,\ldots,s$. Since $cl(F_i) < cl(F)$ by the induction hypothesis we may assume that $u(K_i) = 2^{st(F_i)}$, $i = 1,\ldots,s$. Thus

$$\begin{aligned} u(K) &= \max\{2, u(K_i),\ i \in \{1,\ldots,s\}\} \\ &= \max\{2, 2^{st(F_i)},\ i \in \{1,\ldots,s\}\} \\ &= 2^{st(F)} \end{aligned}$$

(C) Suppose that X_F is an indecomposable space with $|X_F| > 2$ and that our theorem is true for every field L with $cl(L) < cl(F)$. According to Theorem 2.8 in [5] there exists a 2-Henselian valuation V on F such that $|\dot{F}/\dot{F}^2 U_V| = 2^\ell > 1$. We may further assume that X_{F_V} is a decomposable order space. Then there exists a unique extension W of V from the field F to the field $K = F(\sqrt{-1})$. W is 2-Henselian valuation on K. Since $F_V \subset K_W$ and char $F_V \neq 2$ we see that we can use Springer theory (see [12] or [7] §4). Note also that

$$\dot{F}/\dot{F}^2 U_V \cong \dot{K}/\dot{K}^2 U_W \quad \text{and} \quad K_W = F_V[\sqrt{-1}] .$$

Let $f_1, \dots, f_{2^\ell} \in \dot{F}$ be representatives of the classes of the factor-group $\dot{F}/\dot{F}^2 U_V$. Then by Springer theory we can associate to each quadratic form ϕ over K, the residue forms $\phi_1, \dots, \phi_{2^\ell}$ over K_W. Then ϕ is isotropic over K iff one of its residue forms is isotropic over K_W. From the induction hypothesis and point (B) we get $\mu(K_W) = 2^{st(F_V)}$. Thus

$$\begin{aligned} u(K) &= 2^\ell u(K_W) \\ &= 2^\ell 2^{st(F_V)} \\ &= 2^{st(F)} \end{aligned}$$

which completes the proof of the claim (1).

Claim (2). We have $st(F) = \infty \Leftrightarrow |\dot{F}/\dot{F}^2| = \infty$ (see [9], Theorem 13.9). But from the proof of (1) we see that

$$st(F) = \infty \Leftrightarrow \mu(F(\sqrt{-1})) = \infty .$$

Our theorem is proved.

Remark. As a consequence of the theorem, we get $u(F(\sqrt{-1})) = |\dot{F}(\sqrt{-1})/\dot{F}(\sqrt{-1})^2| < \infty$ iff F is a superpythagorean field with $|\dot{F}/\dot{F}^2| < \infty$. This and other facts concerning the equality $|\dot{K}/\dot{K}^2| = u(K)$ were obtained in [6] and [14]. (See also [9], Chapter 11).

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A N N O U N C E M E N T

XVIIIth International Congress of the History of Science

Hamburg and Munich

Federal Republic of Germany

Presented by G. de B. Robinson, F.R.S.C.

The 18th International Congress of the History of Science is scheduled for 1 to 9 August 1989. It will take place in two cities of the Federal Republic of Germany, in Hamburg and Munich. From Tuesday, 1 August, until Saturday, 5 August, the venue will be the 'Congress Center' in Hamburg (CCH). On Sunday, 6 August, the Congress will transfer to Munich and will be continued in the 'Deutsches Museum von Meisterwerken der Naturwissenschaft und Technik' until Wednesday, 9 August 1989.

The *general theme* of this Congress will be

Science and Political Order

(Wissenschaft und Staat).

This theme is to comprise all facets of the relations between science - here always understood to include technology and medicine - and the numerous forms of political order. Political order, in this context, is to be interpreted in a broad sense: from the various philosophies about society and state to the actual realizations they have found in past and present in all parts of the world.

Thus the term 'state' is meant to include not only the organization of governmental power (and the processes of political decision making) at various times in history, but also all forms of governmental and semi-governmental institutions that have influenced the growth of science in one way or another. The relations between science, technology and political systems are, however, not only to be studied in one direction. The congress theme should also direct attention to the response of science to the political order: response in the form of organisation and management of science, in the choice of research topics, in the ways in which science, medicine and technology have been applied to meet the needs of the state in peace-time and in war, and the actions these disciplines have taken at various times to bring their interests to bear on state and government. Last but not least the responsibility of science and the scientists towards the state and its various forms of political activities under which science is undertaken in daily research, teaching, planning etc. should be topics of reflection and discussion at this Congress.

As usual, the Congress will consist of *Symposia* which will address themes of special interest, and *Scientific Sections* devoted to the various branches and periods of the history of science and technology. As a new departure, we propose to introduce *Poster Sessions*. Colleagues availing of this facility will be allocated space on a poster board, and one morning or afternoon session will be reserved for discussion. During this period (or at any additional time they may wish to announce on the board) they may be contacted by other congress participants, explain their research projects and discuss in an informal way problems and results of their work.

Chairman of the *National Program Committee* is Prof. Fritz Krafft (Fachbereich Mathematik, Staudinger Weg 7, D-6500 Mainz, F. R. of Germany), chairman of the *Organizing Committee* is Prof. Christoph J. Scriba (Institut für Geschichte der Naturwissenschaften, Bundesstr. 55, D-2000 Hamburg 13, F. R. of Germany). Chairpersons of the various Commissions and Committees of the Division of History of Science of the International Union for the History and Philosophy of Science (IUHPS/DHS) who are interested in organizing special symposia are invited to contact Prof. Krafft in the near future.

The first detailed circular will be distributed by the National Committees of the IUHPS/DHS, or may be requested from Prof. Scriba (Bundesstr. 55, IGN, D-2000 Hamburg 13, F. R. of Germany). This circular should be ready for distribution in late summer 1987. The second circular will be mailed to all colleagues who by returning the entry-form express interest in further information.

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