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Sponsored by the departments of mathematics of Queen's University, the Universities of Toronto and Waterloo.
Subventionné par les départements de mathématiques des Universités Queen's, de Toronto et de Waterloo.

CLASSIFICATION OF C^*-CROSSED PRODUCTS ASSOCIATED WITH CHARACTERS ON FREE GROUPS

Hong-sheng Yin

Presented by D. Handelman, F.R.S.C.

Let G be a discrete (not necessarily abelian) group. A character χ on G is, by definition, just a group homomorphism from G to the one-dimensional torus T. There is a unique *-automorphism α_χ on the reduced group C^*-algebra $C_r^*(G)$ such that $\alpha_\chi(u_g) = \chi(g)u_g$ for any $g \in G$, where the u_g's are the canonical generators of $C_r^*(G)$. One can then form the C^*-crossed product $C_r^*(G) \times_{\alpha_\chi} Z$. A natural problem is to classify these crossed products up to *-isomorphism in terms of the characters. In the simplest case $G = Z$, these crossed products have been classified by combining the work of [3, 7, 2]. [6] considered the case $G = Z^n$ and χ being injective. In the present paper we consider the case $G = F_n$, the free group on n generators.

MAIN THEOREM : Let $G = F_n$ and χ_1, χ_2 be two characters on G. Then the following are equivalent :

(1). $C_r^*(G) \times_{\alpha_{\chi_1}} Z \simeq C_r^*(G) \times_{\alpha_{\chi_2}} Z$;

(2). $\tau_{1*}(K_0(C_r^*(G) \times_{\alpha_{\chi_1}} Z)) = \tau_{2*}(K_0(C_r^*(G) \times_{\alpha_{\chi_2}} Z))$ and

$t(C_r^*(G) \times_{\alpha_{\chi_1}} Z) = t(C_r^*(G) \times_{\alpha_{\chi_2}} Z)$,

where τ_{j*} is induced by the canonical tracial state τ_j on $C_r^*(G) \times_{\alpha_{\chi_j}} Z$ and $t(C_r^*(G) \times_{\alpha_{\chi_j}} Z)$ is a rational number defined in §2 below;

(3). $\chi_1(G) = \chi_2(G)$ and $t(\chi_1) = t(\chi_2)$,

where $t(\chi_j)$ is a rational number defined in §3 below;

(4). $\chi_1 = \chi_2 \circ \phi$ for some ϕ in $Aut(G)$;

(5). α_{χ_1} and α_{χ_2} are conjugate in $Aut(C_r^*(G))$;

(6). α_{χ_1} and α_{χ_2} are outer conjugate in $Aut(C_r^*(G))$.

In the following we indicate some of the basic ideas of the proof of our main theorem. For the details and more results see [8].

§1. Analysis.

Theorem 1. Let $\chi \in \hat{G}$. Then $C_r^*(G) \times_{\alpha_\chi} Z$ has unique tracial state if $\chi(G)$ is infinite and $C_r^*(\ker\chi)$ has unique tracial state.

We will also need to consider the case that $\chi(G)$ is finite. Then α_χ is a periodic *-automorphism. Suppose $\alpha_\chi^q = \text{id}$, but $\alpha_\chi^k \neq \text{id}$ for $0 < k < q$. Then the action α_χ induces an effective Z_q-action $\dot{\alpha}_\chi$ on $C_r^*(G)$.

Theorem 2. Let $\chi \in \hat{G}$. If $\chi(G)$ is finite with order q and $C_r^*(\ker\chi)$ has unique tracial state, then $C_r^*(G) \times_{\dot{\alpha}_\chi} Z_q$ has unique tracial state.

Using ideas of Elliott [1], we can prove

Theorem 3. Suppose A is a separable unital C^*-algebra, α is a *-automorphism of A with $\alpha^q =$ id, and $A \times_{\dot{\alpha}} Z_q$ has unique tracial state. Then all tracial states coincide on projections of $A \times_\alpha Z$, and moreover, they give the same map from $K_0(A \times_\alpha Z)$ to R.

Theorem 4. If G is an infinite-conjugacy-class group, $C_r^*(G)$ is simple and $\chi(G)$ is infinite, then $C_r^*(G) \times_{\alpha_\chi} Z$ is simple.

Corollary 5. $C_r^*(F_n) \times_{\alpha_\chi} Z$is a simple C^*-algebra with unique tracial state if $\chi(F_n)$ is infinite. If $\chi(F_n)$ is finite, $C_r^*(F_n) \times_{\alpha_\chi} Z$is no longer simple and has many tracial states, but all these tracial states give the same map from $K_0(C_r^*(F_n) \times_{\alpha_\chi} Z)$to R.

§2. K-Theory.

Let $\exp : R \longrightarrow T$ be the exponential map. The methods of Pimsner and Voiculescu [4,5] together with their computation of $K_*(C_r^*(F_n))$ enable us to get the following result.

Theorem 6. $\exp \circ \tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} Z)) = \chi(F_n)$,

where τ_* is induced from the canonical tracial state.

Now let $Q(C_r^*(F_n) \times_{\alpha_\chi} Z) = \{x \in K_0(C_r^*(F_n) \times_{\alpha_\chi} Z) : \tau_*(x) \in Q\}$.

Definition 7. If $Q(C_r^*(F_n)\times_{\alpha_\chi}Z)\neq Z^2$, define $t(C_r^*(F_n)\times_{\alpha_\chi}Z) = 0$; if $Q(C_r^*(F_n)\times_{\alpha_\chi}Z) = Z^2$, then one of its generators must be [1] (let the other generator be e), and define

$$t(C_r^*(F_n)\times_{\alpha_\chi}Z) = \mathrm{dist}(\tau_*(e),\ Z),$$

where dist denotes the usual distance on the real line.

Theorem 8. $t(C_r^*(F_n)\times_{\alpha_\chi}Z)$ is well-defined and is an isomorphism invariant for $C_r^*(F_n)\times_{\alpha_\chi}Z$.

Theorem 9. $t(C_r^*(F_n)\times_{\alpha_\chi}Z) = t(\chi)$, where $t(\chi)$ is defined in §3 below.

§3. Algebra.

Let $\hat{G}$ be the set of all characters on G. The automorphism group $Aut(G)$ acts on $\hat{G}$ via $\phi(\chi) = \chi\circ\phi^{-1}$ for $\phi\in Aut(G)$ and $\chi\in\hat{G}$. Since inner automorphism of G act trivially on $\hat{G}$, we get an action of $Out(G) = Aut(G)/Inn(G)$ on $\hat{G}$. We want to classify the orbits of this action for $G = Z^n$ and F_n.

Theorem 10. Given a character χ on Z^n, we can find a $\phi\in Aut(Z^n)$ such that $\chi\circ\phi(e_i) = e^{2\pi i\theta_i}, 0\leq\theta_i<1, i = 1,2,---,n$, with $\{1,\theta_1,---,\theta_k\}$ Z-linearly independent, $\theta_{k+1} = p/q, (p,q) = 1, 0\leq p\leq[q/2]$ and $\theta_{k+2} = ---- = \theta_n = 0$. Moreover, the number p/q only depends on χ.

Definition 11. For any character χ on Z^n, define $t(\chi) = p/q$ if $\chi(G)$ has torsion and its free rank $k=n-1$, where p/q is the number appearing in Theorem 10; and define $t(\chi) = 0$, otherwise.

Since any character on F_n factors through Z^n, we define

$$t(\chi) = t(\text{the quotient character on } Z^n),\ \chi\in\hat{F}_n.$$

Theorem 12. Suppose $G = F_n$ or Z^n. Two characters χ_1,χ_2 on G are in the same orbit of the $Out(G)$-action iff $\chi_1(G) = \chi_2(G)$ and $t(\chi_1) = t(\chi_2)$.

This announcement is part of the author's thesis work. He sincerely thanks his supervisor, Professor John Phillips, for suggesting the problem and for many helpful discussions. Most of this work was done while the author was visiting the University of Victoria, B.C., Canada, and it was completed in MSRI, Berkeley. This research was financially supported by a Killam Scholarship from the Dalhousie University, Halifax, N.S., Canada.

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Received July 5, 1985

C.R. Math. Rep. Acad. Sci. Canada - Vol. IX, No. 1, February 1987 février

ON SEQUENCES OF PROJECTIONS

Hans Wenzl

Presented by D. Handelman, F.R.S.C.

Abstract: Let e_1, e_2, ... be projections on a Hilbert space with relations (a) $e_i e_{i\pm 1} e_i = te_i$, $t \in \mathbb{R}$ and (b) $e_i e_j = e_j e_i$, if $|i-j| \geq 2$. Then if $4\cos^2(\pi/(m+1)) < 1/t < 4\cos^2(\pi/(m+2))$, there exist at most 2m-1 such projections. If one requires in (b) moreover that $e_i e_j = 0$, the upper bound is m.

We consider a sequence e_1, e_2, ... of orthogonal projections on a Hilbert space $\mathcal{H}$ with the following properties

(a) $e_i e_{i\pm 1} e_i = t\, e_i$, $t \in \mathbb{R}$,

(b) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

Sequences of projections with properties (a) and (b), together with a third condition, involving the trace, play an important role in Jones' analysis of subfactors of a II_1 factor (see (J), §3). It has been shown for this special case that one can only get an infinite sequence of such projections if $1/t \in I = \{4\cos^2(\pi/n),\ n = 3, 4, \ldots\} \cup \{x \in \mathbb{R},\ x \geq 4\}$. This was the key result in showing that the index of a subfactor has to be in the set I.

We will show among other things that the same conclusion holds for any sequence of projections with (a) and (b) without any additional assumptions.

So throughout this note e_1, e_2, ... will be projections on a

Hilbert space $\mathcal{H}$ with properties (a) and (b). Let us define polynomials $P_n(x)$ by

$$P_o(x) = P_1(x) = 1 \quad \text{and}$$
$$P_{n+1}(x) = P_n(x) - x\, P_{n-1}(x).$$

Furthermore, let for fixed $t \in \mathbb{R}$

$$f_o = 1,$$
$$(1) \qquad f_{n+1} = f_n - (P_n(t)/P_{n+1}(t))\, f_n e_{n+1} f_n,$$
$$\text{if } P_k(t) \neq 0 \quad \text{for } k = 1, 2, \dots n+1.$$

It follows by induction that f_n is of the form
$1 - \langle$linear combination of products on $e_1, e_2, \dots e_n\rangle$.
In particular, f_n commutes with $e_{n+2}, e_{n+3}, \dots$ by (b).

<u>Proposition 1</u>

<u>Assume</u> $P_k(t) \neq 0$ <u>for k = 1, 2, ... n+1. Then</u>

(i) $(e_{n+1}f_n)^2 = (P_{n+1}(t)/P_n(t))\, e_{n+1}f_n$,

(ii) f_n <u>is a projection</u>,

(iii) $(f_n e_{n+1} f_n)^2 = (P_{n+1}(t)/P_n(t))\, f_n e_{n+1} f_n$,

(iv) $f_{n+1} = 1 - e_1 \vee e_2 \vee \dots \vee e_{n+1}$.

<u>proof.</u>

(i) and (ii) can be shown by induction on n and (iii) follows from (i) and (ii). For (iv) note that $f_{n+1} \leq f_n$. Hence $e_k f_{n+1} = 0$, $1 \leq k \leq n$, by induction assumption, while $e_{n+1}f_{n+1}$ $= e_{n+1}f_n - (P_n(t)/P_{n+1}(t))\, e_{n+1}f_n e_{n+1} f_n = 0$ by (i). Hence $f_{n+1} \leq 1 - e_1 \vee \dots \vee e_{n+1}$. As $1 - f_{n+1} \in \langle e_1, e_2, \dots$

.. $e_{n+1}\rangle$, the other inclusion also holds.

Let now t be between ¼ and 1 such that $1/t \notin \{4\cos^2(\pi/n),$ $n = 3, 4, \ldots\}$. Then there exists an $m \in \mathbb{N}$ such that

$$(2) \quad 4\cos^2(\pi/(m+1)) < 1/t < 4\cos^2(\pi/(m+2)).$$

By (J),(4.2.5), $P_{m+1}(t) < 0$ and $P_k(t) > 0$ for $k = 1, 2, \ldots m$. (Note that P_n here is P_{n+1} in (J).)

In view of prop. 1, (iv), we can define $f_n = 1 - e_1 \vee \ldots \vee e_n$ without any assumptions on $P_k(t)$.

<u>Lemma 2</u>

<u>If t is as in (2), then there exists an $n \in \mathbb{N}$, $n \leq m$, such that</u> $f_n = f_{n+1}$.

<u>proof</u>.

By the remarks above, $P_{m+1}(t)/P_m(t) < 0$. Hence $(f_m e_{m+1} f_m)^2 \geq 0$ only if $f_m e_{m+1} f_m = 0$ by prop. 1, (iii).

<u>Proposition 3</u>

<u>Let t be as in (2) and $n \leq m$ such that</u> $f_n = f_{n+1}$. <u>Then</u>

(i) $e_{n+1} \leq 1 - f_{n-1}$,

(ii) $f_{n-2} e_i e_j = 0$ <u>for</u> $i,j \geq n$ <u>and</u> $|i-j| \geq 2$.

<u>proof</u>.

(i) We have $e_{n+1} f_n = e_{n+1} f_{n+1} = 0$. Hence $e_{n+1}(f_{n-1} - f_n)$ is a projection. By using (1), we get

$e_{n+1}f_{n-1} = e_{n+1}(f_{n-1} - f_n)e_{n+1} = (tP_{n-1}(t)/P_n(t))f_{n-1}e_{n+1}$.

If $e_{n+1}f_{n-1} \neq 0$, then $tP_{n-1}(t) = P_n(t)$, hence $P_{n+1}(t) = 0$.

But as $n \leq m$, this is a contradiction to (J),(4.2.5),(ii).

(ii) $e_{n+2}f_{n-1} = 1/t\ e_{n+2}e_{n+1}f_{n-1}e_{n+2} = 0$ by (i). Hence it can be shown as in (i) that

$e_{n+2}e_nf_{n-2} = (tP_{n-2}(t)/P_{n-1}(t))\,e_{n+2}e_nf_{n-2} = 0$.

It follows from relations (a) and (b) that

$u_{i,i+s} = (1/t)^{s/2}\ e_ie_{i+1} \ldots e_{i+s}$ is a partial isometry between e_i and e_{i+s}, which commutes with f_{n-2} whenever $i \geq n$. Using this, one shows that $e_je_nf_{n-2} = 0$ and $e_je_if_{n-2} = 0$ for $i \geq n+2$ and $j \geq i+2$.

Corollary 4

Let us replace (b) by (b') $e_ie_j = e_je_i = 0$ if $|i-j| \geq 2$. Then there exist at most m nonzero projections $e_1, e_2, \ldots$ with (a) and (b').

proof.

By lemma 2 and prop. 3 there exists a $k \leq m$ such that $e_{k+1} \leq 1 - f_{k-1} = e_1 \vee \ldots \vee e_{k-1}$. But as $e_{k+1}e_i = 0$ for $i = 1, 2, \ldots k-1$, $e_{k+1} = (1 - f_{k-1})e_{k+1} = 0$.

Remark

Corollary 4 can also be interpreted geometrically in the following way: For t as in (2), there are at most m lines $\ell_1, \ell_2, \ldots$ in an arbitrary Hilbert space such that $\cos^2(\measuredangle(\ell_i, \ell_{i+1})) = t$ and $\ell_i \perp \ell_j$ if $|i-j| \geq 2$.

Theorem 5

Let t be as in (2). Then there exist at most 2m-1 nonzero projections $e_1, e_2, \ldots$ with (a) and (b).

proof.

Suppose there are 2m nonzero projections fulfilling relations (a) and (b). By lemma 2, $f_k = f_{k+1}$ for some $k \leq m$. Choose k minimal. Then $f_{k-1}e_k f_{k-1} \neq 0$ by (1) and, as $f_{k-2} \geq f_{k-1}$,

(3) $f_{k-2}e_k \neq 0$.

Let $g_s = f_{k-2}e_{k+s-1}$. Then $g_1, g_2, \ldots$ fulfill conditions (a) and (b') by prop. 3, (ii). But then $g_{m+1} = 0$ by cor. 4. As $g_{m+1} \sim g_1$ (use the same partial isometries as in prop. 3, (ii)), this contradicts (3).

The existence of infinite sequences of projections with (a) and (b) for $t \in I$ has been shown in (J). Using the representations in (W), one can also show that the upper bounds in cor. 4 and theorem 5 are sharp.

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Received September 3, 1985

TWO INEQUALITIES FOR MEANS

Horst Alzer

Presented by P.G. Rooney, F.R.S.C.

Abstract. In this note we prove the inequalities

$$(G(x,y) I(x,y))^{1/2} < L(x,y) < \frac{1}{2}(G(x,y) + I(x,y))$$

for any positive x and y, $x \neq y$, where G, I, and L are defined by

$$G(x,y) = (xy)^{1/2}, \quad I(x,y) = \frac{1}{e}(x^x/y^y)^{1/(x-y)}, \text{ and}$$

$$L(x,y) = \frac{x - y}{\ln(x)-\ln(y)} .$$

1. Introduction. In 1975 K.B. Stolarsky [17] defined the so-called "generalized logarithmic mean" $L_r(x,y)$ of two distinct positive numbers x and y by

$$L_r(x,y) = \left[\frac{x^r - y^r}{r(x-y)}\right]^{1/(r-1)} \quad \text{for all real } r \neq 0,1. \tag{1}$$

If $r \to 0$ in (1) then $L_r(x,y)$ tends to the logarithmic mean

$$L(x,y) = \frac{x - y}{\ln(x) - \ln(y)} .$$

During the last years many interesting properties of the logarithmic mean have been published by several authors. In particular a lot of inequalities for L can be found in literature. (See the list of references.)

It is worth mentioning that this "little known 'average' " [15,p.99] has applications in some physical problems like heat transfer or fluid mechanics [16] and "somewhat surprinsingly" [15,p.99] in economical problems.

If we let $r \to 1$ in (1) then we get the identric mean

$$I(x,y) = \frac{1}{e}(x^x/y^y)^{1/(x-y)} .$$

(The name "identric mean" is chosen by E.B. Leach and M.C. Sholander [10], [11].)
This mean value "plays a central role" [11 ,p.209] within the family $L_r(x,y)$ because of the integral formula

$$L_r(x,y) = \exp \frac{1}{r-1} \int_1^r \frac{1}{t} \ln I(x^t,y^t)\, dt$$

which was discovered by Stolarsky [17] .
Inequalities for I can be found in [1], [3], [10], [11], [17].

For $x \neq y$, the function $L_r(x,y)$ is strictly increasing in r [10 ,p.89], [17,p.89]. Therefore the following inequalities hold:

$$G(x,y) < L(x,y) < I(x,y) < A(x,y) \;, \quad x \neq y,$$

where $G(x,y) = L_{-1}(x,y) = (xy)^{1/2}$ and $A(x,y) = L_2(x,y) = \frac{1}{2}(x+y)$ denote the geometric and the arithmetic mean of x and y.

The aim of this paper is to show how the inequalities

$$G(x,y) < L(x,y) < I(x,y) \;, \quad x \neq y,$$

can be sharpened. We shall prove that the logarithmic mean L separates the geometric and the arithmetic mean of G and I.

2. A double-inequality for the logarithmic mean.

<u>Theorem.</u> <u>If x and y are positive numbers with $x \neq y$ then</u>

$$(G(x,y)\, I(x,y))^{1/2} < L(x,y) < \frac{1}{2}(G(x,y) + I(x,y)). \quad (2)$$

<u>Proof.</u> Since the functions $G(x,y)$, $I(x,y)$, and $L(x,y)$ are symmetric it suffices to prove (2) for $x > y$.

We set $x=e^t$ and $y=e^{-t}$ with $t > 0$. Then we get

$$G(x,y) = 1,\ I(x,y) = \exp(t\coth(t)-1),\ L(x,y) = \sinh(t)/t,$$

and we shall show for all positive t:

$$\exp((t\coth(t)-1)/2) < \sinh(t)/t < (1+\exp(t\coth(t)-1))/2. \tag{3}$$

If we replace t by $\ln(x/y)/2$ in (3) and multiply by $(xy)^{1/2}$ then we obtain (2).

(This sort of trick occurs in [11] .)

First we prove the left-hand inequality of (3).

We define

$$f(t) = 2\ln(\sinh(t)) - 2\ln(t) - t\coth(t) + 1 \quad \text{for } t > 0,$$

$$f(0) = \lim_{t \to 0} f(t) = 0.$$

Differentiation yields for $t > 0$:

$$f'(t) = t(\coth(t))^2 + \coth(t) - t - 2/t$$

and

$$(\sinh(t))^2 f'(t) = \frac{1}{2}\sinh(2t) - \frac{1}{t}\cosh(2t) + t + \frac{1}{t}.$$

Now we expand sinh and cosh into power series then we get

$$(\sinh(t))^2 f'(t) = \sum_{n=2}^{\infty} 4^n \left[1 - \frac{2}{n+1}\right]\frac{t^{2n+1}}{(2n+1)!} > 0 \quad \text{for positive } t.$$

Therefore f is a strictly increasing function and we conclude

$$f(t) > f(0) = 0 \quad \text{for } t > 0$$

which is equivalent to the first inequality of (3).

Now we prove the right-hand inequality of (3).

We define

$$g(t) = \ln(t) + t\coth(t) - \ln(2\sinh(t)-t) - 1 \qquad \text{for } t > 0,$$

$$g(0) = \lim_{t\to 0} g(t) = 0.$$

Then we have

$$g'(t) = \coth(t) - t(\coth(t))^2 + t + \frac{1}{t} - \frac{2\cosh(t)-1}{2\sinh(t)-t}$$

and

$$(\sinh(t))^2(2\sinh(t)-t)\ g'(t)$$

$$= \frac{1}{2t}\sinh(3t) - \frac{t}{2}\sinh(2t) - \frac{3}{2t}\sinh(t) - 2t\sinh(t) + t^2.$$

We expand sinh into a power series then we obtain

$$(\sinh(t))^2(2\sinh(t)-t)\ g'(t)$$

$$= \sum_{n=2}^{\infty} \frac{27}{2}\left[9^n - \frac{4}{27}(4^n+2)(n+1)(2n+3) - \frac{1}{9}\right]\frac{t^{2n+2}}{(2n+3)!}\ .$$

A simple calculation yields that

$$9^n - \frac{4}{27}(4^n+2)(n+1)(2n+3) - \frac{1}{9}$$

is positive for all integers $n \geq 2$.

Therefore

$$g'(t) > 0 \quad \text{for } t > 0$$

and this implies

$$g(t) > g(0) = 0 \quad \text{for any positive } t.$$

The last inequality is equivalent to the second inequality of (3). Thus the theorem is proved.

3.Final remarks. In a recently published note [2] the double-inequality

$$G(x,y) < (\ L_r(x,y)\ L_{-r}(x,y)\ )^{1/2} < L(x,y)\ ,\ x\neq y,\ r\neq 0, \qquad (4)$$

has been proved. If we set $r=1$ in (4) then we obtain

$$G(x,y) < (G(x,y)\, I(x,y))^{1/2} < L(x,y) . \tag{5}$$

This means that the left-hand inequality of (2) is a special case of (4). We note that the proof we have given in this paper for (5) is new and easier than the proof given in [2].

In 1957 B. Ostle and H.L. Terwilliger [14] published the inequality

$$L(x,y) < A(x,y) , \quad x \neq y . \tag{6}$$

Since then a lot of new proofs and sharpenings have been discovered for (6). In [2] the following sharpening of inequality (6) has been conjectured:

$$L(x,y) < \tfrac{1}{2}(L_r(x,y) + L_{-r}(x,y)) < A(x,y) \quad \text{for all } r \neq 0. \tag{7}$$

Up to now neither a proof nor a disproof is known for this conjecture. At least we have shown in this paper that (7) is true for the special case $r=1$.

It is very easy to give a proof for the right-hand inequality of (7) if $r \in [-2,2]$:

Since $L_r(x,y)$ (with $x \neq y$) is strictly increasing in r we obtain

$$L_r(x,y) < L_2(x,y) \qquad \text{for } r < 2$$

$$L_{-r}(x,y) < L_2(x,y) \qquad \text{for } -2 < r,$$

and hence

$$\tfrac{1}{2}(L_r(x,y) + L_{-r}(x,y)) < L_2(x,y) \qquad \text{for } -2 \leq r \leq 2 .$$

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Received September 10, 1985

THE POLYNOMIAL SEPARATION PROBLEM IN $SPEC_r(A)$

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Abstract.-Given a noetherian ring A and $X,Y \subseteq \mathrm{Spec}_r(A)$, two constructible sets, we prove that if $\overline{X}^Z \cap \overline{Y} = \emptyset$ then X,Y can be separated by an element of the ring A.

Mostowski's Separation Lemma (c.f. [M]) states that any two disjoint closed semialgebraic subsets of $\mathbb{R}^n$ can be separated by a Nash function.However,he shows that polynomials are in general not enough to separate semialgebraic sets (c.f. [M] or also [B-C-R]).

Now,the polynomial separation problem consists in showing sufficient conditions to separate two closed disjoint semialgebraic sets by a polynomial.

Generalizing this situation to the real spectrum of a noetherian ring A,given two disjoint closed constructible subsets X,Y of $\mathrm{Spec}_r(A)$ we prove that if one of them is Zariski closed, X and Y can be separated by an element of the ring A.

As an immediate consequence we conclude that the same result holds for semialgebraic subsets of R^n,where R is any real closed field.

Notation follows that of [B-C-R].

I would like to thank professors T. Recio and M. Coste for their suggestions and helpful conversations.

(*) Partially supported by C.A.I.C.Y.T. 2280/83

Let A be a noetherian ring and let $Spec_r(A)$ be the quasicompact topological space of all the prime cones of A endowed with the Harrison topology (c.f.[C-R] or [B-C-R] Ch. 7 for definitions and basic properties).

For any subset $X \subseteq Spec_r(A)$ we shall define its associated ideal as

$$I(X) := \bigcap_{\alpha \in X} supp(\alpha)$$

Then we shall define the Zariski closure of X in $Spec_r(A)$ as

$\bar{X}^Z := \{ \alpha \in Spec_r(A) / f(\alpha)=0$ for every $f \in I(X) \}$

This definition gives raise to a topology on $Spec_r(A)$ which will be called the Zariski topology on $Spec_r(A)$.

Next,we observe that this topology is the inverse image topology on $Spec_r(A)$ induced by the mapping :

$$supp : Spec_r(A) \longrightarrow Spec(A)$$

where $supp(\alpha) := \alpha \cap (-\alpha)$ for every α in $Spec_r(A)$ and where Spec(A) is considered with its classical Zariski topology.

In the following Theorem, $\bar{Y}$ denotes the closure of Y in $Spec_r(A)$ for the Harrison topology.

Theorem

Let X,Y be two constructible subsets of $Spec_r(A)$.Then if $\bar{X}^Z \cap \bar{Y} = \emptyset$ there is $f \in A$ such that $f|_X > 0$ and $f|_Y < 0$.

In particular,given two disjoint closed constructible subsets of $Spec_r(A)$, if one of them is Zariski closed they can be separated by an element of the ring A.

Proof.-

First of all,let us consider $I(X) = (f_1,...,f_r)A$ and define $p=f_1^2+ ... + f_r^2 \in A$;then we have $\bar{X}^Z = \{ \alpha \in Spec_r(A) / p(\alpha) = 0 \}$ and $p(\beta) > 0$ for every $\beta \in Spec_r(A) \setminus \bar{X}^Z$.

On the other hand,let $\alpha \in Spec_r(A)$ be a prime cone,$k(\alpha)$ the real closure of the quotient field of $A/supp(\alpha)$ and B_α the semi-integral closure of $A/supp(\alpha)$ in $k(\alpha)$ .It is well-known that for any totally ordered field (K,P) and for any subring R of K the semi-integral closure of R in (K,P) is always a real valuation ring of K.Thus B_α is a real valuation ring of $k(\alpha)$ and its maximal ideal M_α is a $k(\alpha)^2 \cap B_\alpha$ - convex ideal.

If $\alpha \in \bar{Y}$ is a closed prime cone we claim that the quotient field of $A/ supp(\alpha)$ is contained in B_α and thus there is $g_\alpha \in A$ such that $(1/p(\alpha))^2 < (g_\alpha(\alpha))^2$.

In order to prove this claim let us observe that through the natural projection $\Pi : B_\alpha \longrightarrow B_\alpha/M_\alpha$ the ordering on $k(\alpha)$ induces a total order P_α on B_α/M_α and $\Pi^{-1}(P_\alpha)$ is an element of $Spec_r(B_\alpha)$ such that $M_\alpha = supp(\Pi^{-1}(P_\alpha))$.

Considering the morphism of rings $\pi:A \longrightarrow A/supp(\alpha)$ and $\lambda:A/supp(\alpha) \longrightarrow B_\alpha$ the morphism $\psi:\lambda\circ\pi :A \longrightarrow B_\alpha$ induces a prime cone $\beta:=\psi^{-1}(\Pi^{-1}(P_\alpha)) \in Spec_r(A)$ that contains the closed prime cone α. Therefore,$\beta = \alpha$ and $M_\alpha \cap (A/supp(\alpha)) = (0)$.

Then for every $b \in A$ such that $b(\alpha) \neq 0$ if $(1/b(\alpha)) \notin B_\alpha$ we would have $(1/b(\alpha))^2 > (a(\alpha))^2$ for every $a \in A$ and $b(\alpha) \in B_\alpha$.This would imply $b(\alpha) \in M_\alpha \cap (A/supp(\alpha))$ which is an absurd and the proof of the above claim is finished.

For every closed prime cone $\alpha \in \bar{Y}$ let us define the open neighborhood of α in $Spec_r(A)$:

$$V_\alpha := \{ \beta \in Spec_r(A) \ / \ 1 - p(\beta)^2 . g_\alpha(\beta)^2 < 0 \}$$

Next,for every $\beta \in \bar{Y}$ there is a closed prime cone $\alpha \in Spec_r(A)$ such that β is included in α (i.e. $\alpha \in \overline{\{\beta\}}$).This implies $\alpha \in \bar{Y}$ and $\beta \in V_\alpha$. Thus we conclude :

$$\bar{Y} = \bigcup_{\substack{\alpha \in \bar{Y} \\ \alpha \text{ closed}}} (\bar{Y} \cap V_\alpha)$$

Since Y is quasicompact $\bar{Y} = (V_{\alpha 1} \cap \bar{Y}) \cup .....\cup (V_{\alpha r} \cap \bar{Y})$, for some r, and then defining $f = 1 - p^2.(g_{\alpha 1}{}^2 + ... + g_{\alpha r}{}^2) \in A$ we finally obtain $f|_X > 0$ and $f|_Y < 0$.

Corollary

Let R be any real closed field and X,Y two semialgebraic subsets of R^n. Let $\bar{X}^Z$ denote the Zariski closure of X in R^n and $\bar{Y}$ the closure of Y in R^n for the euclidean topology.

If $\bar{X}^Z \cap \bar{Y} = \emptyset$ there is $p \in R[X_1,...,X_r]$ separating X and Y.

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Received August 1, 1986

A CHARACTERIZATION OF THE SIGNED HYPERBOLIC DISTANCE

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Abstract. In this note we characterize the signed hyperbolic distance by using that it is preserved by motions.

A representation of the hyperbolic plane on the complex plane is the unit disk $D=\{z\in\mathbb{C}:|z|<1\}$. A motion $g_{\alpha\beta}$ for D is described by the map $g_{\alpha\beta}: D\cup\partial D \to \mathbb{C}$

(1) $g_{\alpha\beta}(z)=\dfrac{\alpha z+\beta}{\bar\beta z+\bar\alpha}$ where $\alpha,\beta\in\mathbb{C}$, $|\alpha|^2-|\beta|^2=1$.

Every motion is conformal and maps D and $\partial D=B=\{z\in\mathbb{C}:|z|=1\}$ onto D and B, respectively. A horocycle is a circle in D tangential to the boundary B. The motions map horocycles into horocycles. For all fixed $z\in D$ and $w\in B$ consider the unique horocycle ξ through z in D tangential to B at w. The real number $d(z,w)$ will be called the signed hyperbolic distance from z to w if

(2) $d(z,w)<0$ if 0 lies inside the horocycle ξ and

(3) $d(g_{\alpha\beta}(z),\, g_{\alpha\beta}(w)) = d(z,w) + d(g_{\alpha\beta}(0),\, g_{\alpha\beta}(w))$

for all motions $g_{\alpha\beta}$ given by (1).

In this note we determine all functions $d: D\times B\to\mathbb{R}$ satisfying the functional equation (3) and then we give all signed hyperbolic distance functions. Equation (3) says that the hyperbolic distance is preserved by motions. Moreover the function $d: D\times B\to\mathbb{R}$ given by

$$d(z,w)=\frac{1}{2}\ln\frac{1-|z|^2}{|w-z|^2}$$

has the properties (2) and (3). (see [1]).

Our main result is contained in the following

THEOREM. The function $d : D \times B \to \mathbb{R}$ is a solution of the functional equation (3) if and only if there exists $\ell :]0,+\infty[\to \mathbb{R}$ such that

$$(4) \qquad d(z,w) = \ell\left(\frac{1-|z|^2}{|w-z|^2}\right) \qquad (z,w) \in D \times B$$

and

$$(5) \qquad \ell(pq) = \ell(p) + \ell(q) \qquad p,q \in]0,+\infty[.$$

(i.e. all solutions of (3) can be expressed as compositions of the Poisson kernel and of a solution of the Cauchy functional equation (5).)

Proof. First we show that, if $d : D \times B \to \mathbb{R}$ satisfies (3) for all $z \in D$, $w \in B$ and $g_{\alpha\beta}$ given by (1), then there exists $\varphi : D \to \mathbb{R}$ such that

$$(6) \qquad d(z,w) = \varphi(z\overline{w}) \qquad (z,w) \in D \times B$$

and

$$(7) \qquad \varphi\left(\frac{t(1-s)+s(1-\overline{s})}{t\overline{s}(1-s)+1-\overline{s}}\right) = \varphi(t) + \varphi(s) \qquad t,s \in D.$$

Indeed, let $(z,w) \in D \times B$, $\beta = 0$ and $\alpha \in \mathbb{C}$ so that $\alpha^2 = \overline{w}$. Then, from (3), we get

$$(8) \qquad d(z\overline{w},1) = d(z,w) + d(0,1).$$

With the substitutions $z = 0$, $w = 1$ this imlies that $d(0,1) = 0$ thus, by (8), (6) is satisfied by the function $\varphi(z) = d(z,1)$, $z \in D$.

According to (6), equations (3) and (1) imply that

$$(9)\qquad \varphi\left(\frac{\alpha z+\beta}{\bar\beta z+\bar\alpha}\;\frac{\bar\alpha\bar w+\bar\beta}{\beta\bar w+\alpha}\right)=\varphi(z\bar w)+\varphi\left(\frac{\beta}{\alpha}\;\frac{\bar\alpha\bar w+\bar\beta}{\beta\bar w+\alpha}\right)$$

holds for all $(z,w)\in D\times B$, $\alpha,\beta\in\mathbb{C}$, $|\alpha|^2-|\beta|^2=1$.

Let now $t,s\in D$. Then, with the substitutions

$$\alpha=\frac{1}{\sqrt{1-|s|^2}},\quad \beta=\alpha s,\quad w=\frac{1-s}{1-\bar s},\quad z=tw,$$

(9) implies (7).

Define the set $A=\{u\in\mathbb{C}:\ \mathrm{Re}\,u>0\}$ and the function ψ on A by

$$(10)\qquad \psi(u)=\varphi\left(\frac{u-1}{u+1}\right).$$

If $u,v\in A$ and $t=\frac{u-1}{u+1}$, $s=\frac{v-1}{v+1}$ then $t,s\in D$ and (7) goes over into

$$(11)\qquad \psi(u\,\mathrm{Re}\,v+i\,\mathrm{Im}\,v)=\psi(u)+\psi(v)\qquad u,v\in A.$$

Now we verify that

$$(12)\qquad \psi(u)=\psi(\mathrm{Re}\,u)\qquad u\in A.$$

Indeed, let $u\in A$, $u=x+iy$. Then, by (11),

$\psi(1+2iy)=\psi((1+iy)\cdot 1+iy)=\psi(1+iy)+\psi(1+iy)=2\psi(1+iy).$

On the other hand, again by (11),

$\psi(1+2iy)=\psi(2)+\psi(1+2iy)-\psi(2)=\psi(2+2iy)-\psi(2)=$

$=\psi((1+iy)\cdot 2)-\psi(2)=\psi(1+iy)+\psi(2)-\psi(2)=\psi(1+iy).$

Thus $\psi(1+iy)=0$ and it follows from (11) that

$\psi(u)=\psi(x+iy)=\psi(x\cdot 1+iy)=\psi(x)+\psi(1+iy)=\psi(x)=\psi(\mathrm{Re}\,u).$

Finally, let ℓ be the restriction of Ψ to $]0,+\infty[$. Then (5) directly follows from (11) and, by (6), (10) and (12), we get

$$d(z,w) = \varphi(z\bar{w}) = \Psi\left(\frac{1+z\bar{w}}{1-z\bar{w}}\right) = \Psi\left(\mathrm{Re}\,\frac{1+z\bar{w}}{1-z\bar{w}}\right) = \Psi\left(\frac{1-|z|^2}{|w-z|^2}\right) = \ell\left(\frac{1-|z|^2}{|w-z|^2}\right)$$

for all $(z,w) \in D \times B$.

The converse is an easy computation.

COROLLARY. <u>All signed hyperbolic distance functions</u> d <u>are of the form</u>

$$d(z,w) = c\,\ln\frac{1-|z|^2}{|w-z|^2} \qquad (z,w) \in D \times B$$

<u>where</u> c <u>is a positive real constant.</u>

<u>Proof</u>. The assumption (2) implies that $d(z,w) < 0$ if $(z,w) \in D \times B$ and $|z-w|^2 + |z|^2 > 1$. In particular, $d(tw,w) < 0$ if $w \in B$ and $t \in]-1,0[$. Applying our theorem, we have from (4) that $\ell(p) < 0$ if $p \in]0,1[$. Therefore there exists $c > 0$ such that $\ell = c \cdot \ln$ thus, because of (4), the proof is complete.

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Received August 14, 1986

WHEN IS A BEZOUT DOMAIN A KRONECKER FUNCTION RING?(*)

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Abstract. Various characterizations are given for a Kronecker function ring in one variable. All such subrings of $\mathbb{Q}(x)$ are classified.

1. Introduction. Throughout, K denotes a field and X denotes an indeterminate over K . A domain S with quotient field K(X) is a Kronecker function ring (with respect to K and X) , written KFR, in case there exist a domain R with quotient field K and an endlich arithmetisch brauchbar (e.a.b.) *-operation, * , on the set of nonzero fractional ideals of R such that S coincides with $R^* = \{0\} \cup \{f/g: f, g \in R[X]\setminus\{0\}$ and $c(f)^* \subset c(g)^*\}$. Background on Kronecker function rings appears in [4] and [3, sections 32-34]. Note that any R admitting an e.a.b. * as above must be integrally closed.

There are several reasons for interest in Kronecker function rings. First, if T is any domain then X(T) , the abstract Riemann surface of T , is homeomorphic to Spec(R*) with the Zariski topology for a suitable R and e.a.b. *: see [1, Theorem 2]. (Here, X(T) is the collection of all valuation overrings of T.) Secondly, each KFR is a Bézout domain. Finally (cf. [1, Lemma 6 (c)]), if T is any treed domain, then Spec(T) is order-isomorphic to Spec(R*), for a suitable R and e.a.b. *.

(*)Each author gratefully acknowledges partial support from his coauthor's institution and NATO Collaborative Research Grant N.85/0035.

However, KFR's form a proper subclass of all Bézout domains having rational function quotient fields. For instance, no polynomial ring can be a KFR (see Proposition 2.3 (a)). Section 2 is devoted to such "rarity" results and actually addresses scarcity for the more intrinsic concept of a kfr. We shall say that S is a <u>kfr</u> (with respect to K and X) if $S = R^*$ where R is a subring of $K(X)$ having quotient field F, $K(X) = F(Y)$ for some indeterminate Y over F, $*$ is an e.a.b. $*$-operation on the nonzero fractional ideals of R, and R^* is constructed with respect to the variable Y. Each KFR is a kfr ; however, Example 2.2 shows the converse is false.

Section 3 characterizes KFR's (and, by varying K and X, thus characterizes kfr's). Specifically, Theorem 3.2 shows that a subring S of $K(X)$ is a KFR if and only if S is integrally closed, K is the quotient field of $S \cap K$, and $(W \cap K)^* = W$ for each $W \in \mathcal{X}(S)$. (Here, V^* denotes the trivial extension of V to $K(X)$ via the inf-extension).

Proofs of these results will appear elsewhere.

2. <u>Rarity</u>. We first collect some useful facts.

LEMMA 2.1. <u>Let $S = R^*$ be a kfr with respect to K and X, where R has quotient field F, $K(X) = F(Y)$, and R^* is constructed using the e.a.b. $*$-operation $*$ with respect to the variable Y. Then:</u>

(a) <u>S is a Bézout domain; $S \cap F = R$; the quotient field of $S \cap F$ is F ; R is integrally closed; and $(W \cap F)^*$, the</u>

trivial extension of $W \cap F$ to $F(Y)$, coincides with W for each $W \in X(S)$.

(b) S is a field if and only if R is a field; that is, $S = K(X)$ if and only if $S \cap F = F$.

Let b denote the *-operation called completion. Then $R^b \subset R^*$ for each e.a.b. $*$; $R(X) \subset R^b$, where the Nagata ring $R(X)$ is $R[X]_S$ for $S = \{f \in R[X]: c(f) = R\}$; and $R(X) = R^b$ if and only if R is a Prüfer domain.

Lemma 2.1 implies that if a ring S is contained properly between K and $K(X)$, then S cannot be a KFR. However, such an S can be a kfr.

EXAMPLE 2.2. If $K = \mathbb{Q}(Y)$, then $S = \mathbb{Q}[X](Y)$ is a kfr but not a KFR. If R denotes the domain $\mathbb{Q}[X]$ and the ring R^b is constructed with respect to the variable Y, then $S = R^b$, a kfr (with respect to K and X). However, Lemma 2.1 (b) shows that S is not a KFR (with respect to K and X).

Despite the preceding result, not every Bézout domain with quotient field $K(X)$ is a kfr. For instance, we have

PROPOSITION 2.3. (a) No polynomial ring is a kfr.

(b) If T is an indeterminate over $\mathbb{Q}$, and $f \in \mathbb{Q}[T]$ is irreducible, then the DVR, $\mathbb{Q}[T]_{(f)}$, is not a kfr.

REMARK 2.4. (a) Proposition 2.3 (a) may also be proved using the ideas in [2].

(b) No formal power series ring $A[[Y_1, \ldots, Y_n]]$ can be a kfr. However, the DVR, $F[T]_{(f)}$, can be a Kronecker function ring for suitable F and f . Let T and U be algebraically independent indeterminates over a field k , set $F = k(U)$, and set $S = F[T]_{(T)}$. Then S is an overring of $k[T](U)$; hence S is a kfr with respect to $K = k(T)$ and $X = U$.

Our final "rarity" result is

THEOREM 2.5. <u>Let L be a field of positive characteristic which is algebraic over its prime subfield, and let T be an indeterminate over L . If a subring S of $L(T)$ is a kfr, then S is a field.</u>

3. <u>Characterizations</u>. If $K(X)$ contains a domain S expressed as $S = \cap W_i$ for some set $\mathbf{W} = \{W_i\}$ of valuation overrings W_i of S , let $\mathbf{W}_K = \{W_i \cap K\}$. In this setting, the elements of $\mathbf{W}_K$ need not be overrings of $S \cap K$. Now suppose that $R = S \cap K$ has quotient field K . Then each $W_i \cap K$ is a valuation overring of R . Let w_K denote the w-operation on the nonzero fractional ideals of R induced by $\mathbf{W}_K$. If $\mathbf{W} = X(S)$, then w_K will be denoted by b_K .

THEOREM 3.2. <u>Let S be an integrally closed subring of $K(X)$; set $R = S \cap K$. Then the following conditions are equivalent:</u>

(1) S is a KFR (with respect to K and X) ;

(2) K is the quotient field of R , and $S = R^{b_K}$;

(3) Each overring of S is a KFR (with respect to K and X);

(4) K is the quotient field of R ,and $R^{b_K} \subset S$;

(5) K is the quotient field of R , and there exists $\mathbf{W} = \{W_i\} \subset X(S)$ such that $S = \cap W_i$ and $R^{w_K} \subset S$;

(6) K is the quotient field of R , and there exists $\mathbf{W} = \{W_i\} \subset X(S)$ such that $S = \cap W_i$ and $(W_i \cap K)^* = W_i$ for each i ;

(7) K is the quotient field of R , and $(W \cap K)^* = W$ for each $W \in X(S)$.

Next, we describe a field (cf. Theorem 2.5) all of whose kfr subrings can be listed.

PROPOSITION 3.3. Let T be an indeterminate over $\mathbb{Q}$. Then the set of all the Kronecker function subrings of $\mathbb{Q}(T)$ is

$$\{\mathbb{Z}_S((aT+b)/(cT+d)) : S \text{ is a multiplicatively closed subset of } \mathbb{Z} \text{ and } a, b, c, d \in \mathbb{Z} \text{ satisfy } ad - bc \neq 0\} .$$

PROPOSITION 3.4. Let (W, M) be a valuation ring of $K(X)$; set $R = W \cap K$. Then the following conditions are equivalent:

(1) W is a KFR;

(2) $W = R^b$;

(3) $W = R^*$, that is, W is the trivial extension of R to $K(X)$;

(4) $W_P = (W_P \cap K)^*$, <u>for each $P \in \text{Spec}(W)$</u>

(5) $R^* \subset W$;

(6) $R(X) \subset W$;

(7) $R(X) = W$;

(8) <u>The canonical map $X(W) \to X(R)$ is bijective, with inverse map $X(R) \to X(W)$ given by $V \to V^*$</u> .

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Received August 14, 1986

VOLUME-PRESERVING φ-GEODESIC SYMMETRIES

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Presented by G. de B. Robinson, F.R.S.C.

1. INTRODUCTION

It is well-known that the local geodesic symmetries on a locally Riemannian symmetric space are isometries and hence they are volume-preserving local diffeomorphisms. However, there are many Riemannian manifolds all of whose local geodesic symmetries are volume-preserving but which are not locally symmetric (see for example [5],[6],[10]). All the known examples are locally homogeneous and to our knowledge, it is not known if this is a property which extends to the whole class. It is shown in [4] that this is indeed the case for three-dimensional manifolds. Further, K. Sekigawa and the second author showed in [8] that any four-dimensional Kähler manifold with volume-preserving geodesic symmetries is locally symmetric. This property extends to arbitrary dimensional Kähler manifolds when the local geodesic symmetries are assumed to be symplectic or holomorphic [7].

For Sasakian manifolds it is more natural to consider the so-called local φ-geodesic symmetries. T. Takahashi [9] used them to define the (locally) φ-symmetric spaces which seem to be the analogues of the (locally) Hermitian symmetric spaces. These symmetries and these spaces are also studied in [2], [3],[11].

The main purpose of this paper is to study Sasakian spaces M such that all local φ-geodesic symmetries are volume-preserving. For dim M = 3 they have been classified in [2]. In this paper we concentrate on the five-dimensional case and prove that such spaces are locally φ-symmetric. As for the Kähler and Riemannian manifolds, the problem for higher dimensions seems to be much more difficult.

2. SASAKIAN MANIFOLDS AND φ-SYMMETRIC SPACES

A C^∞ manifold M^{2n+1} is said to be an almost contact manifold if the structural group of its tangent bundle is reducible to $\mathcal{U}(n) \times 1$. It is well-known that such a manifold admits a tensor field φ of type (1,1), a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \quad , \quad \varphi^2 = -I + \eta \otimes \xi \ .$$

These conditions imply that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, M admits a

Riemannian metric g satisfying

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any tangent vector fields X and Y. Note that this implies that $\eta(X) = g(X,\xi)$. M together with these structure tensors is said to be an <u>almost contact metric manifold</u>. If now these structure tensors satisfy

$$(1) \qquad (\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X ,$$

where ∇ denotes the Riemannian connection of g, M is said to be a <u>Sasakian manifold</u>. It is easy to see from (1) that

$$(2) \qquad \nabla_X\xi = -\varphi X$$

from which it follows that ξ is a Killing vector field. The curvature tensor

$$R_{XY}Z = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]}Z$$

of a Sasakian manifold satisfies

$$(3) \qquad R_{X\xi}Y = \eta(Y)X - g(X,Y)\xi .$$

For a general reference to the above ideas see [1],[12].

A geodesic γ on a Sasakian manifold is said to be a <u>φ-geodesic</u> if $\eta(\gamma') = 0$. From (2) it is easy to see that a geodesic which is orthogonal to ξ remains orthogonal to ξ. A local diffeomorphism s_m of M, $m \in M$, is said to be a <u>φ-geodesic symmetry</u> if its domain U is such that, for every φ-geodesic $\gamma(s)$ such that $\gamma(0)$ lies in the intersection of U with the integral curve of ξ trough m

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in U$, s being the arc length. At the point m the differential s_{m*} of s_m is given by

$$s_{m*}(m) = -I + 2\eta \otimes \xi .$$

In [9] Takahashi introduced the notion of a <u>locally φ-symmetric space</u> by requiring that

$$\varphi^2(\nabla_V R)_{XY}Z = 0$$

for all vector fields V,X,Y,Z orthogonal to ξ. On the other hand he defined a globally φ-symmetric space by requiring that any φ-geodesic symmetry be extendable to a global automorphism of M and that the Killing vector field ξ generate a global one-parameter subgroup of isometries.

Let $\tilde{\mathcal{U}}$ be a neighborhood on M on which ξ is regular. Then, as is well-known, the fibration $\pi : \tilde{\mathcal{U}} \longrightarrow \mathcal{U} = \tilde{\mathcal{U}}/\xi$ gives a Kähler structure (J,G) on the base manifold $\mathcal{U}$. Further we have

$$\bar{s}_{\pi(m)} \circ \pi = \pi \circ s_m \tag{4}$$

where $\bar{s}_{\pi(m)}$ denotes the geodesic symmetry on $\mathcal{U}$ with center $\pi(m)$.

We shall need the following propositions :

Proposition 1 [9] : A Sasakian manifold is a locally φ-symmetric space if and only if each Kähler manifold, which is the base space of a local fibering is a Hermitian locally symmetric space.

Proposition 2 [8] : Let M be a connected four-dimensional Kähler manifold. Then M is a Hermitian locally symmetric space if and only if each local geodesic symmetry is volume-preserving.

3. VOLUME-PRESERVING φ-GEODESIC SYMMETRIES

Before proving our main results we prove a useful property.

Lemma 3. We have $s_{m*}\xi = \xi$ on a general Sasakian manifold M.

Proof. Let f_t denote the one-parameter group of isometries generated by ξ. Let γ be a φ-geodesic through a point $m \in M$ and consider the action of f_t on γ. We then note that $\gamma_t = f_t \circ \gamma$ is also a φ-geodesic and that the integral curves of ξ are equidistant curves. Further we have $s_{m*}\xi = \lambda\xi$. Clearly at m, $s_{m*}\xi = \xi$ and we shall show that λ is constant along γ. γ_t is a variation of γ through geodesics by the action of f_t, and hence the tangent field to this variation is a Jacobi field collinear with ξ, say $f\xi$ for some function f. Thus letting U denote the unit tangent field of γ, we have

$$\nabla_U\nabla_U f\xi - R_{Uf\xi}U = 0 \ .$$

Expanding the first term and using (1) and (3) we obtain

$$0 = \nabla_U((Uf)\xi - f\varphi U) + f\xi$$

$$= (UUf)\xi - 2(Uf)\varphi U - f(\nabla_U\varphi)U + f\xi$$

$$= (UUf)\xi - 2(Uf)\varphi U ,$$

from which we see that f is constant along γ. Thus the curves γ_t are equidistant curves and hence $s_{m*}\xi = \xi$.

We now give the general result.

Theorem 4. Let M^{2n+1} be a Sasakian manifold. Then the φ-geodesic symmetries s_m are volume-preserving if and only if with respect to local fibrations $\tilde{\mathcal{U}} \longrightarrow \mathcal{U} = \tilde{\mathcal{U}}/\xi$ the geodesic symmetries $\bar{s}_m$ on $\mathcal{U}$ are volume-preserving.

Proof. Recall that on a contact metric manifold the volume element is $\eta \wedge (d\eta)^n$ to within a constant factor depending only on the dimension. Thus s_m is volume-preserving if and only if $s_m^*(\eta \wedge (d\eta)^n) = \eta \wedge (d\eta)^n$ and similarly, $\bar{s}_m$ is volume-preserving if and only if $\bar{s}_m^*\Omega^n = \Omega^n$, where Ω denotes the Kähler form on $\mathcal{U}$. Note that $d\eta = \pi^*\Omega$.

Now let $X_1,\ldots,X_{2n}$ be a local basis on $\mathcal{U}$ and let X_i^* denote the horizontal lift of X_i with respect to the connection form η. Since $s_{m*}\xi = \xi$, $\eta(\xi) = 1$, $d\eta(\xi,X) = 0$ and $\bar{s}_{\pi(m)*}\pi_* = \pi_* s_{m*}$ we have

$$s_m^*(\eta \wedge (d\eta)^n)(X_1^*,\ldots,X_{2n}^*,\xi) = (\eta \wedge (d\eta)^n)(s_{m*}X_1^*,\ldots,s_{m*}X_{2n}^*,s_{m*}\xi)$$

$$= (d\eta)^n(s_{m*}X_1^*,\ldots,s_{m*}X_{2n}^*)$$

$$= (\pi^*\Omega)^n(s_{m*}X_1^*,\ldots,s_{m*}X_{2n}^*)$$

$$= \Omega^n(\bar{s}_{\pi(m)*}X_1,\ldots,\bar{s}_{\pi(m)*}X_{2n}) .$$

In the same manner we have that

$$(\eta \wedge (d\eta)^n)(X_1^*,\ldots,X_{2n}^*,\xi) = \Omega^n(X_1,\ldots,X_{2n}) .$$

Hence $s_m^*(\eta \wedge (d\eta)^n) = \eta \wedge (d\eta)^n$ if and only if $\bar{s}_{\pi(m)}^*\Omega^n = \Omega^n$.

For low dimensional Sasakian manifolds this result gives rise to stronger consequences.

Theorem 5. Let M be a five-dimensional Sasakian manifold such that all φ-geodesic symmetries s_m are volume-preserving. Then M is locally φ-symmetric and conversely.

Proof. The converse is easy since the s_m are isometries on a φ-symmetric space. Therefore, let M be such that the s_m are volume-preserving. Then Theorem 4 implies that on the Kähler manifold $\mathcal{U}$ obtained from M by a local fibration, the geodesic symmetries $\bar{s}_m$ are also volume-preserving. Now Proposition 2 implies that $\mathcal{U}$ is locally symmetric and hence, by Proposition 1, M is locally φ-symmetric.

Remark. For three-dimensional manifolds, one may easily derive the same result from Theorem 4. This has also been done in [2] but in a different way. There we gave also a complete classification of connected, simply connected, complete three-dimensional φ-symmetric spaces.

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Received September 2, 1986

THE SEMINORMALIZATION OF A UNION OF LINES

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Abstract Let A be the homogeneous coordinate ring of a union of lines in projective space. I investigate the degree in which A becomes equal to its seminormalization.

Let A be the homogeneous coordinate ring of a union X of s straight lines in $\mathbb{P}^n_k$ = Proj R (k a field, $R = k[x_0,\ldots,x_n]$). Alternatively one can think of A as the affine coordinate ring of a union of s planes in $\mathbb{A}^{n+1}_k$, all of which contain the origin. Let $\tilde{I}_i$ be the ideal in R of the i^{th} line, and I_i the canonical image of $\tilde{I}_i$ in A. Then $R/\tilde{I}_i \cong A/I_i \cong k[t_i,u_i]$, and the integral closure of A in its total ring of fractions is $B = \Pi^s_{i=1}A/I_i \cong \Pi^s_{i=1}k[t_i,u_i]$. Let ^+A be the seminormalization of A in B. (^+A is also called simply the seminormalization of A, and one says that A is seminormal if $A = {}^+A$.) By Theorem 1.2 of [1] $^+A = \{(\bar{f}_1,\ldots\bar{f}_s)\in B \mid f_i \equiv f_j \bmod \tilde{I}_i+\tilde{I}_j\}$ (where $f_i \in R$ and $\bar{f}_i$ is the canonical image of f_i in $R/\tilde{I}_i$). The rings A, ^+A, and B are all graded in positive degrees. For any integer $i \geq 0$, and any commutative ring S graded in positive degrees let S_i denote the degree i part of S. Then it was proved in [2, Theorem 20] that if the inclusion $A_i \longrightarrow {}^+A_i$ is an isomorphism for all $i \leq s-1$, then A is seminormal (i.e. $A \cong {}^+A$, under the canonical inclusion). By Corollary 4.2 of [3] $\dim_k({}^+A/A) < \infty$ if and only if the directions of the lines are linearly independent at each intersection point. The purpose of this note is to tie these two results together as

Theorem 1 *Let A be the homogeneous coordinate ring of a union X of*

<u>s straight lines in $\mathbb{P}^n_k$, such that the directions of the lines at each intersection point are linearly independent, and let ^+A be the seminormalization of A. Then the inclusion $\iota_i : A_i \to {}^+A_i$ is an isomorphism for $i \geq s-1$. If X is connected then ι_{s-2} is also an isomorphism.</u>

<u>Proof</u> We can assume that $s \geq 3$, since A is always seminormal if $s \leq 2$. First I prove the the homomorphism $\tau_d : A_d \to \Pi_{i<j} A/(I_i+I_j)_d$ is onto for $d \geq s-2$. Note that $A/(I_i+I_j) \cong k[v_k]$ if the lines ℓ_i and ℓ_j intersect in a point P_k, if not then $A/(I_i+I_j) \cong k$. Thus it suffices, for each point of intersection P, to find an element $F \in R$, of degree $\leq s-2$, such that F does not vanish at P, but F does vanish on all lines which do not contain P. There are at most s-2 lines which do not contain P. For each such line ℓ_i let $F_i \in R$ be the equation of some hyperplane that contains ℓ_i but not P. We can take F to be the product of all such F_i.

Now consider the homomorphism $\iota_r : A_r \to {}^+A_r$ for $r \geq s-1$. By the first part of the proof it suffices to show that the image of ι_r contains all elements $(\bar{f}_j) \in {}^+A_r$, where $\bar{f}_j \in k[t_j, u_j]$ vanishes at all points of intersection on ℓ_j. Then (after renumbering the lines) it suffices to find $a = (\bar{f}_j) \in A_r$, such that $\bar{f}_1 \in k[t_1, u_1]$ is arbitrary (except that $\bar{f}_1$ vanishes on all points of intersection on ℓ_1) and $\bar{f}_j = 0$, for $j \geq 2$. Suppose that there are d points of intersection P_i on ℓ_1 ($1 \leq i \leq d$, $d \geq 0$). Let $H \in k[t_1, u_1]$ be a non-zero element of degree d that is 0 on all points of intersection on ℓ_1 (H is unique up to multiplication by a unit in k). Then $\bar{f}_1 = Hh$, for some $h \in k[t_1, u_1]$. Let $G_i \in R$ be the equation of a hyperplane that contains all lines except ℓ_1 that pass through P_i (such a hyperplane exists because of our assumption that the directions of the lines through each intersection point are linearly independent). Then the image in $k[t_1, u_1]$ of $\tilde{H} = \Pi_{i=1}^d G_i \in R$ can be taken as our element H. Now let ℓ_j be a line that does not intersect ℓ_1. Because ℓ_1 and ℓ_j do not intersect, $\tilde{I}_1+\tilde{I}_j$ = $(x_0,\ldots x_n)$, so $\tilde{I}_j \to R/\tilde{I}_1$ is onto in degree 1. Thus there exist elements $t_{j1}, t_{j2} \in \tilde{I}_j$ which map to t_1, u_1 respectively. By taking a suitable sum of products of the t_{j1}, t_{j2} one obtains an element $b \in R$ such that b maps to h in $k[t_1,u_1]$, and to 0 in A/I_j for all j such that $\ell_1 \cap \ell_j = \phi$ (this is possible because h is of degree $r-d \geq s-d-1$ (≥ 0), and there are at most s-d-1 lines that do not intersect ℓ_1. Thus each monomial in the t_{gk} can include either t_{j1}, or t_{j2} at least once, for all j such that $\ell_1 \cap \ell_j = \phi$). We can take a to be the canonical image in A of $\tilde{H}b$, completing the proof of Theorem 1, for $r \geq s-1$.

Now assume that X is connected, and $r=s-2$. As above, it suffices to find $a=(\bar{f}_j) \in A_r$ such that $\bar{f}_1 \in k[t_1,u_1]$ is arbitrary, of degree s-2, except for vanishing on all d intersection points P_i on ℓ_1, and $\bar{f}_j=0$ for $j>1$. If $d=s-1$ then $\bar{f}_1=0$,so take $a=0$. If $d<s-1$, write $\bar{f}_1=Hh$, as above. This time h is of degree s-d-2. If at least three lines intersect in one of the P_i, then there are at most s-d-2 lines not intersecting ℓ_1, and the proof can be completed as in the $r \geq s-1$ case. If $d<s-1$ and only two lines (i.e. ℓ_1 and one other) intersect in each P_i, then if $\ell_j \cap \ell_1 = \phi$, then there exists ℓ_i such that $\ell_j \cap \ell_i \neq \phi$, $\ell_i \cap \ell_1 = P_i$, and we can take G_i to be a hyperplane containing ℓ_i and ℓ_j. Then H vanishes on all d lines that intersect ℓ_1 , and on ℓ_j. Since h is of degree s-d-2 and there are s-d-2 lines other than ℓ_j that do not intersect ℓ_1, we can again complete the proof as in the $r \geq s-1$ case.

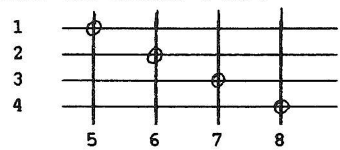
The following example shows that the bounds s-1 and s-2 in Theorem 1 are in general the best possible.

Example 2 Let X consist of s skew lines in a quadric surface in $\mathbb{P}^3_k$ $(s \geq 3)$. Then ι_r is not onto for $1 \leq r < s-1$. If X consists of s-1 lines in one ruling system, and 1 in the other, $(s \geq 4)$, then ι_r is not onto for $1 \leq r < s-2$.

<u>Proof</u> This follows from the explicit calculations in [4,Theorem 3] and [3,Example 1.9].

Often the bound $s-2$ can be improved. One way to do this is (in the proof of Theorem 1) to choose (if possible) the F_i to contain more than one line that does not pass through P, and (as in the $r=s-2$ case of Theorem 1) to choose the G_i to contain some line that does not pass through P_i. Quadric hypersurfaces could be useful also. It seems difficult to formulate a general theorem. However I will illustrate the idea by considering the double 4.

<u>Example 3</u> Consider the double 4 in $\mathbb{P}^3$



where as usual lines drawn parallel do not intersect, and the circles also denote nonintersection. The configuration is symmetric in the lines and intersection points, so in order to prove that τ is surjective in degree d it suffices to find a form of degree d that vanishes on all intersection points but one. Let Π_{ij} be the equation of the plane spanned by lines i and j. Then $\Pi_{18}\Pi_{27}\Pi_{36}$ vanishes on all intersection points except $\ell_4 \cap \ell_5$, so τ is onto in degrees ≥ 3. Similarly one can take $\tilde{H}=\Pi_{36}\Pi_{47}\Pi_{28}$, which vanishes on all lines but ℓ_1 and ℓ_5. Let Π_5 and Π_5' be two planes containing ℓ_5 whose restrictions to ℓ_1 are linearly independent. Then $\tilde{H}\Pi_5$ and $\tilde{H}\Pi_5'$ vanish on all lines but ℓ_1 and their canonical images Ht_{51} and Ht_{52} in $k[t_1,u_1]$ span those elements of $k[t_1,u_1]$ in degree 4 that vanish on the intersection points on ℓ_1. Thus ι is onto in degrees ≥ 4 for any double 4 contained in $\mathbb{P}^3_k$ (rather than 6, as given by Theorem 1). According to the calculations in

[4], and [3,section 1] this is the exact bound. Similarly, for the double 5 configuration, one obtains that ι is onto in degrees ≥ 4 (the exact bound) if one takes $\tilde{H}=Q_{789}\Pi_{2,10}\Pi_{3,11}$, where the lines are numbered as in [2,page 110] and Q_{789} is an equation for the unique quadric containing ℓ_7, ℓ_8, and ℓ_9.

The ideas used in the proof of Theorem 1 were implicit in the proofs of [2, Lemma 5 and Theorem 20], but Theorem 1 was not a natural result in the context of [2] because at that time we were not thinking explicitly in terms of the seminormalization ^{+}A, and also because the importance of having linearly independent directions at intersection points was not clearly understood.

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Received September 19, 1986

SUR UN THEOREME D'EULER

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The starting point of this paper was Euler's classical result concerning his totient function φ. This result is extended here, in a natural way, to $r \times r$ invertible matrices with entries in $\mathbb{Z}/(n)$ (last paragraph).

But our purpose is quite different inasmuch as it concerns the entire ring of matrices whereas Euler's result is restricted to its group of invertible elements. Applications to $M_r(Z)$ are given.

1. Généralités sur les idempotents d'un anneau unitaire

Soit un anneau unitaire R $(1 \neq 0)$ on rappelle qu'un idempotent de R est une racine du polynôme X^2-X.

Soit $\mathcal{J}(R)$ l'ensemble des idempotents, $\mathcal{J}(R)$ est muni d'une structure de sous-anneau partiel de R (cela veut dire que si x et $y \in \mathcal{J}(R)$, $x+y$ et xy ne sont pas nécessairement dans $\mathcal{J}(R)$).

Définition 1.- Soient x et $y \in R$. Si $xy = yx = 0$, on dit que x et y sont orthogonaux.

Définition 2.- Soient x et $y \in \mathcal{J}(R)$. Si $xy = yx = y$, on dit que $x \geqslant y$.

Remarquons que $\geqslant$ est une relation d'ordre sur $\mathcal{J}(R)$ pour laquelle 1 est un maximum et 0 un minimum.

La proposition suivante est immédiate.

Proposition 1

1) L'application $x \xrightarrow{\sigma} (1-x)$ est une involution de $\mathcal{J}(R)$.
2) Si $x \in \mathcal{J}(R)$, x et $\sigma(x)$ sont orthogonaux.

3) Si x et y sont dans $\mathfrak{I}(R)$ et si x et y sont orthogonaux :

$$\sigma(x+y) = \sigma(x)\sigma(y)$$

4) Si x et y sont dans $\mathfrak{I}(R)$ et si $\sigma(x)$ et $\sigma(y)$ sont orthogonaux :

$$\sigma(xy) = \sigma(x) + \sigma(y)$$

5) σ est décroissante.

6) σ n'admet pas de point fixe.

Corollaire.- Si $\mathfrak{I}(R)$ est fini, $\# \mathfrak{I}(R)$ est pair.

Définition 3.- Soit $x \in \mathfrak{I}(R)$, on dit que :

1) x est additivement irréductible ssi :

$x \neq 0$ et $x = y+z$ avec y et $z \in \mathfrak{I}(R)$, y et z orthogonaux, entraîne y ou $z = 0$.

2) x est multiplicativement irréductible ssi :

$x \neq 1$ et $x = yz$ avec y et $z \in \mathfrak{I}(R)$, $\sigma(y)$ et $\sigma(z)$ orthogonaux, entraîne y ou $z = 1$.

Proposition 2.- Les conditions suivantes sont équivalentes :

1) x est additivement irréductible.

2) $\sigma(x)$ est multiplicativement irréductible.

Théorème 1.- On suppose que $\mathfrak{I}(R)$ ne contient pas de suites strictement décroissantes infinies, alors tout élément de $\mathfrak{I}(R)$ est somme d'un nombre fini d'éléments additivement irréductibles orthogonaux.

Si de plus R est commutatif (donc $\mathfrak{I}(R)$ est un monoïde multiplicatif) la décomposition est unique.

La preuve de ce résultat suit les schémas classiques.

2. Définitions

On dira qu'un anneau R est un *anneau de torsion* ssi pour tout $x \in R$, il existe un entier $n \geq 1$, tel que $x^n \in \mathfrak{I}(R)$.

Le plus petit n possédant la propriété ci-dessus sera appelé l'exposant de x et sera noté $e(x)$.

Théorème 2.- *On suppose que* R *est un anneau de torsion et on se donne* $x \in R$.

1) *L'ensemble des* $n \geq 1$ *tels que* $x^n \in \mathfrak{I}(R)$ *est un semi-groupe* $E(x) \subset \mathbb{N}$.

2) *Pour tout* $n \in E(x)$, $x^n = x^{e(x)}$ *et on note* $\theta(x) = x^{e(x)}$. *Alors* θ *est une surjection* $R \longrightarrow \mathfrak{I}(R)$.

3) *Si* $R \xrightarrow{f} R'$ *est un épimorphisme*, R' *est un anneau de torsion et* :

$$\theta' \circ f = f \circ \theta$$

4) *Si* $S = \varprojlim_i (R_i)$ *où les* R_i *sont des anneaux de torsion, il existe une application* $\theta : S \longrightarrow \mathfrak{I}(S)$ *telle que si* ϖ *désigne la projection canonique* $S \longrightarrow R_i$, *on ait* :

$$\varpi \circ \theta = \theta_i \circ \varpi$$

Remarque : Si x et y commutent $\theta(xy) = \theta(x)\theta(y)$.

Définition.- On appellera *ordre de* x le p.g.c.d. des éléments de $E(x)$.

Proposition 3.- *On suppose que* R *est un anneau de torsion, on se donne* $x \in R^*$ *(x inversible) et on désigne son ordre par* ν, *alors* $\nu = e(x) \in E(x)$.

Contre-exemple : Si $R = \mathbf{Z}/(24)$ et $x = \overline{2}$, on a :

$$E(x) = \{n \; ; \; n \geq 3, \; n \text{ pair}\} = \{4,6,8 \ldots\}$$

donc $\nu = 2$ et $2 \notin E(x)$, $e(x) = 4$.

Définition.- Un anneau de torsion R sera dit harmonieux s'il existe $n \geq 1$ tel que pour tout $x \in R$, $x^n \in \mathfrak{I}(R)$.

Le plus petit n possédant cette propriété sera appelé l'*exposant* de R.

Proposition 4.- On suppose que R est harmonieux.

1) L'ensemble des $n \geq 1$ tels que $x^n \in \mathfrak{I}(R)$ pour tout $x \in R$, est un semi-groupe $E(R) \subset \mathbb{N}$.

2) R^* est un groupe de torsion admettant un exposant, et l'exposant de R^* divise tous les éléments de $E(R)$.

Exemples :

1) On verra (paragraphe 4) que $M_r(\mathbb{F}_q)$ est harmonieux.

2) Il est clair que $\varinjlim_n M_r(\mathbb{F}_{q^n}) \cong M_r(\overline{\mathbb{F}}_q)$ (où $\overline{\mathbb{F}}_q$ désigne une clôture algébrique de $\mathbb{F}_q$) est un anneau de torsion, mais il n'est pas harmonieux.

3. $M_r(Z/(n))$ est harmonieux

Dorénavant, $Z = \mathbb{Z}$ ou $\mathbb{F}_q[t]$ et n désigne un élément "normalisé" de Z, i.e. $n > 0$ si $Z = \mathbb{Z}$ et n est unitaire si $Z = \mathbb{F}_q[t]$.

Finalement $M_r(Z/(n))$ désigne l'anneau des matrices $r \times r$ à cœfficients dans $Z/(n)$.

Si l'on décompose n en produit de facteurs premiers normalisés :

$$n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$$

le "théorème chinois" dit que :

$$M_r(Z/(n)) \cong M_r(Z/(p_1^{\alpha_1})) \times \ldots \times M_r(Z/(p_s^{\alpha_s}))$$

Théorème 3.- $M_r(Z/(n))$ est harmonieux.

Preuve : On va définir une fonction ψ_r tels que pour tout $A \in M_r(Z/(n))$:

$$A^{\psi_r(n)} \in \mathfrak{I}_r(Z/(n))$$

1) Si $n = p$ premier, on prend :

$$\psi_r(p) = e_1 \ell^{h_1}$$

où ℓ désigne la caractéristique de l'anneau $Z/(p)$, q son cardinal, où e_1 est le p.p.c.m. de $q-1, q^2-1,\ldots,q^r-1$ et où h_1 désigne le plus petit entier h tel que $\ell^h \geqslant r$. Avec les notations du corollaire 2 du théorème 2 de [1] on voit que $e_o\ell^{h_o}$ divise $e_1\ell^{h_1}$.

2) Si $n = p^\alpha$, on prend :

$$\psi_r(p^\alpha) = \ell^h\psi_r(p)$$

avec $h = \alpha-1$ si $Z = \mathbb{Z}$ et $-h = [\mathrm{Log}_\ell\, \alpha^{-1}]$ si $Z = \mathbb{F}_q[t]$.

3) Si $n = p_1^{\alpha_1}\ldots p_s^{\alpha_s}$ on prend :

$$\psi_r(n) = [\psi_r(p_1^{\alpha_1}),\ldots,\psi_r(p_s^{\alpha_s})] = \text{p.p.c.m. } \{\psi_r(p_1^{\alpha_1}),\ldots,\psi_r(p_s^{\alpha_s})\}$$

<u>Corollaire 1</u>.- $\psi_r(n) \in E_r(Z/(n))$ <u>et</u> $\psi_r(p_1\ldots p_s)$ <u>divise tous les éléments de</u> $E_r(Z/(n))$.

4. <u>Application à</u> $M_r(Z)$

On se propose de donner des conditions nécessaires pour que deux matrices A et $B \in M_r(Z)$ soient semblables modulo $GL_r(Z)$.

Soit $\hat{Z}$ l'anneau de Prüfer de Z, c'est-à-dire :

$$\hat{Z} = \varprojlim_n Z/(n) \cong \prod_p Z_p$$

où Z_p désigne l'anneau des entiers p-adiques.

On désigne par ϖ_n la projection $\hat{Z} \longrightarrow \hat{Z}/(n) \cong Z/(n)$ et par θ_n, l'application : $M_r(Z/(n)) \longrightarrow \mathcal{J}_r(Z/(n))$.

<u>Corollaire 2</u>

1) <u>Il existe une surjection</u> $\theta : M_r(\hat{Z}) \longrightarrow \mathcal{J}_r(\hat{Z})$ <u>telle que</u> :

$$\varpi_n \circ \theta = \theta_n \circ \varpi_n$$

2) <u>Si</u> A <u>et</u> B <u>commutent, on a</u> : $\theta(AB) = \theta(A)\theta(B)$.

Corollaire 3.- Soient A et $B \in M_r(\hat{Z})$ telles qu'il existe $P \in GL_r(Z)$ vérifiant $B = PAP^{-1}$, alors :

1) Pour tout n on a : $E[\varpi_n(B)] = E[\varpi_n(A)]$

2) $\theta(A)$ et $\theta(B)$ sont conjugués modulo $GL_r(Z)$.

Théorème 4.- Soit $A \in M_r(\hat{Z}) \cong \prod_p M_r(Z_p)$ et soit $\theta(A) = (\theta_p(A)) \in \prod_p M_r(Z_p)$. Alors $_p(A)$ est une matrice idempotente dont le rang est égal au nombre de valeurs propres de A (comptées avec leur multiplicité) dont la norme n'est pas divisible par p.

5. Rapport avec la fonction d'Euler

Définition.- Soit un anneau unitaire fini R dont le groupe des éléments inversibles est R^*, on appelle fonction d'Euler de R le cardinal de R^*, et on le note $\varphi(R)$.

Théorème d'Euler

1) Pour tout $a \in R^*$, on a $a^{\varphi(R)} = 1$

2) $\varphi(R_1 \times R_2) = \varphi(R_1)\varphi(R_2)$.

Finalement, on peut montrer que $\psi_r(n)$ divise $\varphi[M_r(Z/(n))]$, mais Jack Lescot a prouvé le résultat plus général suivant :

Théorème 5.- On suppose que R est fini. Alors pour tout $a \in R$, on a $a^{\varphi(R)} \in \mathfrak{J}(R)$.

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Received September 19, 1986

CLASSIFICATIONS AND BASE ENUMERATIONS OF THE MAXIMAL SETS OF THREE-VALUED LOGICAL FUNCTIONS

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Abstract. Functional completeness theory of P_k involves classifying functions of a closed set of P_k by using all its maximal sets. This also divides all its bases into finite equivalence classes. This paper presents classifications and enumerations of all bases for the set P_3 and all its 18 maximal sets.

1. Introduction. The set of k-valued logical functions, i.e. the union of all the functions $\{f \mid E_k^n \to E_k,$ for $E_k=\{0,1,\dots,k-1\}$ and $n=0,1,2,\dots\}$ is denoted by P_k. A subset F of P_k is said to be closed if it contains all superpositions of its members (cf. [6, 23]). For closed sets F and H such that $F\subset H$ (proper inclusion), F is H-maximal set if there is no closed set G such that $F\subset G\subset H$. A subset X of H is complete in H if H is the least closed set containing X. If the number m of H-maximal sets is finite then a subset of functions in H is complete in H if and only if it is not contained in any one H-maximal set (completeness condition)(cf. [6]). Investigations of completeness and related topics, which are usualy called functional completeness problems are directly related to logical circuit design, and they have a wide area of applications in addition to their mathematical importance.

A complete set X in H is called base of H if no proper subset of X is complete in H. A set of functions $\{f_1,\dots,f_s\}$ is called pivotal in H, if for each i, $1\le i\le s$, there exists an H-maximal set H_i which does not contain f_i while all the other functions f_j $(j=1,\dots,s, j\neq i)$ are elements of H_i (pivotalness condition). From these definitions it follows that a complete pivotal set is a base. The rank of a base (pivotal set) is the number of its elements.

We classify the set H of functions into nonempty equivalence classes by using all its maximal sets as indicated below. Then we can discuss the completeness in H in terms of these classes instead of individual functions: if a set is complete, then by replacing a function in the set by any function in the corresponding equivalence class yields another complete set.

The characteristic vector of $f \in H$ is $a_1 \ldots a_m$, where $a_i=0$ if $f \in H_i$ and $a_i=1$ otherwise($1 \leq i \leq m$). All functions $f \in H$ with the same characteristic vector form a class of functions. For a given set $F \subseteq H$ the class of F is the set of classes of $f \in F$. The conditions of completeness and pivotalness of F can be conveniently checked by using characteristic vectors corresponding to the class of F.

If we have a complete list of characteristic vectors for nonempty classes of a set, we can enumerate all its bases (pivotal sets). All bases (pivotal sets) with the same class form a class of bases (pivotal sets).

We use the notation of functions preserving a relation to describe H-maximal sets [cf. 23]. An h-ary relation ρ on E_k, $h \geq 1$, is a subset of E_k^h whose elements are written as columns

$$(a_1, \ldots, a_h)^T \in \rho \Leftrightarrow (a_{1i}, \ldots, a_{hi})^T \in \rho \text{ for all } i,\ 1 \leq i \leq n,$$
$$\text{where } a_i=(a_{i1}, \ldots, a_{in}).$$

The relation ρ is written as a matrix whose columns are elements of the relation ρ.

Then set of functions preserving ρ (denoted by Pol ρ) is defined by

$$\text{Pol } \rho = \{ f \mid (a_1, \ldots, a_h)^T \in \rho \Rightarrow (f(a_1), \ldots, f(a_h))^T \in \rho \}.$$

Theorem 1([6]) P_3 has exactly the following 18 maximal sets:

T_0=Pol(0), T_1=Pol(1), T_2=Pol(2),

T_{01}=Pol(0 1), T_{02}=Pol(0 2), T_{12}=Pol(1 2),

$B_0=\text{Pol}\begin{pmatrix} 0&1&2&0&1&0&2 \\ 0&1&2&1&0&2&0 \end{pmatrix}$, $B_1=\text{Pol}\begin{pmatrix} 0&1&2&0&1&1&2 \\ 0&1&2&1&0&2&1 \end{pmatrix}$, $B_2=\text{Pol}\begin{pmatrix} 0&1&2&0&2&1&2 \\ 0&1&2&2&0&2&1 \end{pmatrix}$,

$U_0=\text{Pol}\begin{pmatrix} 0&1&2&1&2 \\ 0&1&2&2&1 \end{pmatrix}$, $U_1=\text{Pol}\begin{pmatrix} 0&1&2&0&2 \\ 0&1&2&2&0 \end{pmatrix}$, $U_2=\text{Pol}\begin{pmatrix} 0&1&2&0&1 \\ 0&1&2&1&0 \end{pmatrix}$,

$M_0=\text{Pol}\begin{pmatrix} 0&1&2&2&2&0 \\ 0&1&2&0&1&1 \end{pmatrix}$, $M_1=\text{Pol}\begin{pmatrix} 0&1&2&0&0&1 \\ 0&1&2&1&2&2 \end{pmatrix}$, $M_2=\text{Pol}\begin{pmatrix} 0&1&2&1&1&2 \\ 0&1&2&2&0&0 \end{pmatrix}$,

$L=\text{Pol}(\{ (a,b,c)^T \in E_3^3 \mid c=2(a+b) \pmod 3 \})$, $S=\text{Pol}\begin{pmatrix} 0&1&2 \\ 1&2&0 \end{pmatrix}$,

$T=\text{Pol}(\{ (a,b,c)^T \in E_3^3 \mid a=b \text{ or } a=c \text{ or } b=c \})$.

2. Classifications of functions. Determination of maximal sets for the set P_k and its closed sets has been subject of investigation in growing number of papers ([20, 5, 6, 21, 22, 12, 2, 3, 10, 11]). Next step, description of classes of functions and classes of bases was done first for the set P_2([5, 4, 8]). First attempt to derive classes of functions of P_3 was done in [13]. This paper also give the notion of pivotal sets as necessary conditions for a set to be base. But, it counted several characteristic vectors twice as different classes, consequently the number of bases reported in [14] was incorrect; this was corrected in [24]. The following table present the number of maximal sets and the number of classes of functions for the sets P_2, P_3 and all P_3-maximal sets. Several classification results exist for some of closed sets of P_k [26, 29, 30, 19].

	P_2	P_3	B_0	M_1	T_0	U_2	T_{01}	T	L	S
maximal sets	5 [20, 5]	18 [6]	7 [10]	13 [12]	12 [10]	13 [10]	15 [10]	5 [10]	5 [2]	2 [3]
classes of functions	15 [5, 4, 8]	406 [13, 24]	54 [15]	88 [25]	253 [17]	383 [27]	607 [28]	6 [16]	10 [16]	4 [16]

3. Enumerations of bases. Two algorithms for the enumeration of bases and pivotal sets are given: [14, 18, 34] and [24, 18, 34]. They are compared in [18, 34].

The numbers of classes of bases and pivotal incomplete sets for the same sets as in the former table are shown in the following two tables. There are several results about maximal rank of a base of P_3 [9, 14] and two proofs that maximal rank of a base of P_3 is 6: computational [14] and theoretical [36].

classes of bases

rank	P_2 [4, 8]	P_3 [24]	B_0 [15]	M_1 [25]	T_0 [17]	U_2 [27]	T_{01}	T [16]	L [16]	S [16]
1	1	1	-	-	1	1	1	-	-	1
2	17	8265	28	-	4492	4344	12259	-	18	1
3	22	794256	999	1514	234031	680285	2580026	6	6	-
4	2	4612601	2831	40104	552927	7300491	38508259	-	-	-
5	-	810474	724	75209	91377	7627060	53641851	-	-	-
6	-	14124	17	1916	892	944257	7545748	-	-	-
7	-	-	-	1	-	15804	35616	-	-	-
Σ	42	6239721	4599	118744	883720	16572242	102323760	6	24	2

pivotal incomplete sets

rank	P_2 [26]	P_3	B_0	M_1	T_0	U_2	T_{01}	T	L	S
1	13	404	53	87	251	381	605	5	9	2
2	31	60335	931	3153	21363	57284	147266	10	10	-
3	7	1418970	3678	37946	202689	1594342	6385808	-	-	-
4	-	2677899	2240	96323	149804	5057975	32278690	-	-	-
5	-	176187	168	15087	6595	1911408	18947380	-	-	-
6	-	1368	1	55	8	96464	1198502	-	-	-
7	-	9	-	-	-	240	648	-	-	-
Σ	51	4335172	7071	152651	380710	8718094	58958899	15	19	2

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Received October 14, 1986

ON SOME APPLICATIONS OF HERMITE'S INTERPOLATION POLYNOMIAL

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ABSTRACT. In this paper we obtained generalizations of two integral inequalities of G.S.Mahajani by using some results for Hermite's interpolation polynomial.

1. Using some geometrical arguments, G.S.Mahajani [2] (see also [4, pp. 297-298]) proved the following results:

1° If f fas a bounded derivative on $[a,b]$, i.e. if $|f'(x)| \leq M$ $(M>0)$ and if $\int_a^b f(x)dx = 0$, then for $x \in [a,b]$

(1) $$\left|\int_a^x f(t)\,dt\right| \leq \frac{M(b-a)^2}{8};$$

2° If, besides the conditions given in 1°, $f(a)=f(b)=0$, then

(2) $$\left|\int_a^x f(t)\,dt\right| \leq \frac{M(b-a)^2}{16}.$$

Analytic proofs of these results were given by P.R.Beesack [1]. In this paper we shall give generalizations of these results.

2. Let us define the two-parametar class of polynomials $P_n^{(m,k)}$ $(0 \leq m \leq k < n;\ m,k,n \in N)$ by means of

$$P_n^{(m,k)}(x) = P_n^{(m,k)}(x;a,b)$$

$$= \frac{(-1)^{n-k}(n-m)!}{m!(k-m)!(n-k-1)!} \sum_{i=0}^{k-m} \frac{(b-a)^{m-n+i}}{n-m-i} \binom{k-m}{i} (x-a)^m (x-b)^{n-m-i}$$

where a and b are real parametars.

If the values of derivatives of function F in $x=a$ and $x=b$ are known, using polynomials $P_n^{(m,k)}$, Hermite's interpolation polynomial can be represented in the following form ([3]):

$$S_{n,k}(x) = \sum_{m=0}^{k-1} P_{n-1}^{(m,k-1)}(x;a,b)F^{(m)}(a)$$

$$+ \sum_{m=0}^{n-k-1} P_{n-1}^{(m,n-k-1)}(x;b,a)F^{(m)}(b),$$

and if $|F^{(n)}(x)| \leq M \quad (\forall x \in (a,b))$, then

$$|F(x) - S_{n,k}(x)| \leq \frac{M}{n!}|(x-a)^k(x-b)^{n-k}|. \tag{3}$$

Remark. For $k=0$ ($k=n$) the first (second) sum is not exists, i.e. we have Taylor's formula.

Now, we shall give the following generalization of 1°:

THEOREM 1. Let $x \mapsto f(x)$ be a n-times differentiable function such that $|f^{(n)}(x)| \leq M$ $(\forall x \in (a,b))$ and $\int_a^b f(t)\,dt = 0$, then

$$\left|\int_a^x f(t)dt - \bar{S}_{n,k}(x)\right| \leq \frac{M}{(n+1)!}(x-a)^k(b-x)^{n+1-k} \leq \frac{k^k(n-k+1)^{n-k+1}}{(n+1)^{n+1}(n+1)!}M(b-a)^{n+1} \tag{4}$$

where

$$\bar{S}_{n,k}(x) = \sum_{m=1}^{k-1} P_n^{(m,k-1)}(x;a,b)f^{(m-1)}(a) + \sum_{m=1}^{n-k} P_n^{(m,n-k)}(x;b,a)f^{(m-1)}(b)$$

(for $k=1$ ($k=n$) the first (second) sum does not exist)

PROOF. Using the substitutions $n \to n+1$, $F(x) = \int_a^x f(t)\,dt$, we get the first inequality in (4) from (3). For the second inequality we should only observe that the function $x \mapsto (x-a)^k(b-x)^{n-k+1}$ has maximum for $x = (kb+(n+1-k)a)/(n+1)$.

COROLLARY 1. If, besides the conditions given in Theorem 1, $f^{(i)}(a) = 0$ $(i = 0,1,\dots,k-2)$, $f^{(i)}(b) = 0$ $(i=0,1,\dots,n-k-1)$ (for $k=1$ ($k=n$) the first (second) condition does not exist), then

$$\left|\int_a^x f(t)\,dt\right| \leq \frac{M}{(n+1)!}(x-a)^k(b-x)^{n-k+1} \leq \frac{k^k(n-k+1)^{n-k+1}}{(n+1)^{n+1}(n+1)!}M(b-a)^{n+1}. \tag{5}$$

In a special case, if the conditions from 1° are fulfiled, then

$$\left|\int_a^x f(t)\,dt\right| \leq \frac{M}{2}(x-a)(b-x) \leq \frac{M(b-a)^2}{8}, \tag{6}$$

what is a refinement of (1).

Now, we shall prove the following generalization of 2^{o}:

THEOREM 2. Let $x \mapsto f(x)$ be n-times differentiable function such that $|f^{(n)}(x)| \leq M$ $(\forall x \in (a,b))$, $\int_a^b f(t)\,dt = 0$ and $f^{(i)}(a) = f^{(i)}(b) = 0$ $(i=0,1,\dots,n-2)$. Then

$$(7)\qquad \left|\int_a^x f(t)\,dt\right| \leq \frac{M(b-a)^{n+1}}{2^{n+1}n(n+1)!}.$$

PROOF. In the proof we shall use the following result from [3] which is also a consequence of (3):

Let $x \mapsto f(x)$ be n-times differentiable function such that $|f^{(n)}(x)| \leq M$ $(\forall x \in (a,b))$, $f^{(i)}(a) = 0$ $(i=0,1,\dots,k-1)$ and $f^{(i)}(b) = 0$ $(i=0,1,\dots,n-k-1)$. Then

$$(8)\qquad \int_a^b |f(x)|\,dx \leq \frac{k!(n-k)!}{n!(n+1)!}M(b-a)^{n+1}.$$

For the proof of Theorem 2 we may assume that $f(c) = 0$ for some $c \in (a,b)$. Moreover, by symmetry we may assume that $a < c \leq (a+b)/2$. We may also assume that c is the largest zero of f on $(a,(a+b)/2]$. For $a \leq x \leq c$, (8) for $b = c$, $k = n-1$, implies that

$$\left|\int_a^x f(t)\,dt\right| \leq \int_a^c |f(t)|\,dt \leq \frac{M(b-a)^{n+1}}{2^{n+1}n(n+1)!} \quad (\equiv T).$$

If $f(x) \neq 0$ for $c < x < b$, then $|G(x)| = \left|\int_a^x f(t)\,dt\right|$ would be decreasing on $[c,b]$, so that $|G(x)| \leq |G(c)| \leq T$ would follow. We may thus assume that $f(c_1) = 0$ for some $c_1 \in (c,b)$, hence for some $c_1 \in [(a+b)/2,b)$. Now we may assume that c_1 is the least zero of f on this interval, and with no loss of generality suppose $f(x) > 0$ for $c < x < c_1$. Then

$$|G(x)| = \left|\int_x^b f(t)\,dt\right| \leq \int_{c_1}^b |f(t)|\,dt \leq T$$

if $c_1 \leq c \leq b$ by (8) in the case $a = c_1$, $k = 1$. So it only remains to consider the case $c < x < c_1$. On this interval $G'(x) = f(x) > 0$ so $G(x)$ increases on (c,c_1). It follows that

$$\max_{c \leq x \leq c_1} |G(x)| = \max\{|G(c)|, |G(c_1)|\} \leq T,$$

completing the proof of (7).

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Received November 5, 1986

A METRIC VECTOR SPACE PROOF OF MIQUEL'S THEOREM

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Presented by H.S.M. *Coxeter*, F.R.S.C.

Miquel's theorem states the following:

Let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_8$ *denote distinct points in the (real) Möbius plane, and assume that the following quadruples of points are concyclic:*

$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$; $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_5, \mathbf{p}_6$; $\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_6, \mathbf{p}_7$;
$\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_7, \mathbf{p}_8$; $\mathbf{p}_4, \mathbf{p}_1, \mathbf{p}_8, \mathbf{p}_5$.

Then the points $\mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7$ and $\mathbf{p}_8$ *are also concyclic.*

As is well-known, this theorem can be reformulated (via stereographic projection) as follows:

Let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_8$ *denote distinct points on an unruled quadric in three-dimensional real projective space, and assume that the following quadruples of points are coplanar:*

$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$; $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_5, \mathbf{p}_6$; $\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_6, \mathbf{p}_7$;
$\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_7, \mathbf{p}_8$; $\mathbf{p}_4, \mathbf{p}_1, \mathbf{p}_8, \mathbf{p}_5$.

Then the points $\mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7$ and $\mathbf{p}_8$ *are also coplanar.*

In this note, we give a short metric vector space proof of the latter version of the theorem.

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Choose the plane through $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ to be the plane at infinity; then the quadric becomes a hyperboloid of two sheets in the corresponding affine space. We may take the affine space to be $\mathbb{R}^3$, and the hyperboloid to have equation $(\mathbf{r},\mathbf{r}) = -1$, where $(\ ,\ )$ is the metric defined on $\mathbb{R}^3$ by

$$(\mathbf{r}_1,\mathbf{r}_2) := x_1x_2 + y_1y_2 - z_1z_2$$

for all $\mathbf{r}_1 := (x_1, y_1, z_1)$ and $\mathbf{r}_2 := (x_2, y_2, z_2)$ in $\mathbb{R}^3$. (We thus have a real metric vector space of signature (2,1).)

If $\mathbf{p}_5$, $\mathbf{p}_6$, $\mathbf{p}_7$ or $\mathbf{p}_8$ is infinite, then all eight points are coplanar and we are done, so assume otherwise. As a line in $\mathbb{R}^3$, the line $\mathbf{p}_1\mathbf{p}_5$ intersects the hyperboloid in exactly one point and is not tangent to it. The following lemma shows that any such line must be null, i.e., must have a null direction vector.

Lemma: Let a line in $\mathbb{R}^3$ having equation $\mathbf{s}(\lambda) = \mathbf{s}_0 + \lambda\mathbf{v}$ $(\mathbf{v} \neq \mathbf{0})$ intersect the hyperboloid $(\mathbf{r},\mathbf{r}) = -1$ at the point $\mathbf{s}_0$ only, and assume the line is not tangent to the hyperboloid. Then $(\mathbf{v},\mathbf{v}) = 0$.

Proof: All points $\mathbf{s}(\lambda)$ of the line which lie on the hyperboloid satisfy

$$-1 = (\mathbf{s}(\lambda),\mathbf{s}(\lambda)) = -1 + 2\lambda(\mathbf{s}_0,\mathbf{v}) + \lambda^2(\mathbf{v},\mathbf{v}),$$

whence

$$\lambda\{\lambda(\mathbf{v},\mathbf{v}) + 2(\mathbf{s}_0,\mathbf{v})\} = 0.$$

Assume that $(\mathbf{v},\mathbf{v}) \neq 0$; then, for a unique solution, $(\mathbf{s}_0,\mathbf{v}) = 0$. Thus, for all points $\mathbf{s}(\lambda) \neq \mathbf{s}_0$ on the line, the quantity

$$(\mathbf{s}(\lambda),\mathbf{s}(\lambda)) + 1 = \lambda^2(\mathbf{v},\mathbf{v})$$

has the same sign, so the line lies completely on one side of the hyperboloid $(\mathbf{r},\mathbf{r}) = -1$. It is thus a tangent, contrary to our hypothesis.

Thus $(\mathbf{v},\mathbf{v}) = 0$. ■

We thus have that the lines $\mathbf{p}_1\mathbf{p}_5$, $\mathbf{p}_2\mathbf{p}_6$, $\mathbf{p}_3\mathbf{p}_7$ and $\mathbf{p}_4\mathbf{p}_8$ are all null. Now, since the points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_5$ and $\mathbf{p}_6$ are coplanar, the lines $\mathbf{p}_1\mathbf{p}_5$ and

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p_2p_6 intersect: set $\mathbf{a} := p_1p_5 \cap p_2p_6$. Similarly, there exist points $\mathbf{b} := p_2p_6 \cap p_3p_7$, $\mathbf{c} := p_3p_7 \cap p_4p_8$ and $\mathbf{d} := p_4p_8 \cap p_1p_5$. The points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are all finite. Furthermore, since $p_1, p_2, \ldots, p_8$ are distinct and non-coplanar, it is easily seen that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are not distinct, then they all coincide. In this latter case, p_5, p_6, p_7 and p_8 all satisfy the equations $(\mathbf{r},\mathbf{r}) = -1$ and $(\mathbf{r}-\mathbf{a}, \mathbf{r}-\mathbf{a}) = 0$ (since they lie on null lines through $\mathbf{a}$). By subtraction, they satisfy $(\mathbf{r},\mathbf{a}) = \frac{1}{2}\{(\mathbf{a},\mathbf{a}) - 1\}$, the equation of a plane, which completes the proof of the theorem in this special case.

We may thus assume that the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are all distinct. Then the lines $\mathbf{ab} = p_2p_6$, $\mathbf{bc} = p_3p_7$, $\mathbf{cd} = p_4p_8$ and $\mathbf{da} = p_1p_5$ are all null, and for some scalars α, β, γ and δ,

$$p_6 = \alpha\mathbf{a} + (1-\alpha)\mathbf{b}, \qquad p_7 = \beta\mathbf{b} + (1-\beta)\mathbf{c},$$
$$p_8 = \gamma\mathbf{c} + (1-\gamma)\mathbf{d}, \qquad p_5 = \delta\mathbf{d} + (1-\delta)\mathbf{a}.$$

From $(\mathbf{a}-\mathbf{b}, \mathbf{a}-\mathbf{b}) = 0$ follows $2(\mathbf{a},\mathbf{b}) = (\mathbf{a},\mathbf{a}) + (\mathbf{b},\mathbf{b})$, from which the relation $(p_6,p_6) = -1$ simplifies to

$$\alpha\{(\mathbf{b},\mathbf{b}) - (\mathbf{a},\mathbf{a})\} = (\mathbf{b},\mathbf{b}) + 1.$$

Since $(\mathbf{b},\mathbf{b}) \neq (\mathbf{a},\mathbf{a})$ (else both would equal -1, whence $\mathbf{a}$ and $\mathbf{b}$ would be coincident points on the hyperboloid), we have

$$\alpha = \{(\mathbf{b},\mathbf{b}) - (\mathbf{a},\mathbf{a})\}^{-1}\{(\mathbf{b},\mathbf{b}) + 1\}, \qquad 1-\alpha = \{(\mathbf{a},\mathbf{a}) - (\mathbf{b},\mathbf{b})\}^{-1}\{(\mathbf{a},\mathbf{a}) + 1\}.$$

Similarly,

$$\beta = \{(\mathbf{c},\mathbf{c}) - (\mathbf{b},\mathbf{b})\}^{-1}\{(\mathbf{c},\mathbf{c}) + 1\}, \qquad 1-\beta = \{(\mathbf{b},\mathbf{b}) - (\mathbf{c},\mathbf{c})\}^{-1}\{(\mathbf{b},\mathbf{b}) + 1\},$$
$$\gamma = \{(\mathbf{d},\mathbf{d}) - (\mathbf{c},\mathbf{c})\}^{-1}\{(\mathbf{d},\mathbf{d}) + 1\}, \qquad 1-\gamma = \{(\mathbf{c},\mathbf{c}) - (\mathbf{d},\mathbf{d})\}^{-1}\{(\mathbf{c},\mathbf{c}) + 1\},$$
$$\delta = \{(\mathbf{a},\mathbf{a}) - (\mathbf{d},\mathbf{d})\}^{-1}\{(\mathbf{a},\mathbf{a}) + 1\}, \qquad 1-\delta = \{(\mathbf{d},\mathbf{d}) - (\mathbf{a},\mathbf{a})\}^{-1}\{(\mathbf{d},\mathbf{d}) + 1\}.$$

Now consider the following system of homogeneous equations in λ, μ, ν and ω.

$$\begin{aligned}
\lambda\alpha \qquad\qquad\qquad\qquad\qquad + \omega(1-\delta) &= 0,\\
\lambda(1-\alpha) + \mu\beta \qquad\qquad\qquad\qquad &= 0,\\
\mu(1-\beta) + \nu\gamma \qquad\qquad &= 0,\\
\nu(1-\gamma) + \omega\delta &= 0.
\end{aligned}$$

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The determinant of this system is

$$\alpha\beta\gamma\delta - (1-\alpha)(1-\beta)(1-\gamma)(1-\delta),$$

which, using the above relations for α, β, γ and δ, is 0. The system thus has a non-trivial solution, so there exist scalars λ, μ, ν and ω, not all zero, satisfying

$$\{\lambda\alpha + \omega(1-\delta)\}\mathbf{a} + \{\lambda(1-\alpha) + \mu\beta\}\mathbf{b} + \{\mu(1-\beta) + \nu\gamma\}\mathbf{c} + \{\nu(1-\gamma) + \omega\delta\}\mathbf{d} = \mathbf{0}.$$

But this equation can be rearranged to read

$$\lambda\mathbf{p}_6 + \mu\mathbf{p}_7 + \nu\mathbf{p}_8 + \omega\mathbf{p}_5 = \mathbf{0},$$

so since $\lambda + \mu + \nu + \omega = 0$ (add the equations of the system), the points $\mathbf{p}_5$, $\mathbf{p}_6$, $\mathbf{p}_7$ and $\mathbf{p}_8$ are coplanar.

This completes the proof of the theorem.

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Received November 9, 1986

A PARAMETRIX FOR THE $\bar{\partial}$-NEUMANN PROBLEM ON A STRONGLY PSEUDOCONVEX DOMAIN

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Presented by P. C. Greiner, F.R.S.C.

Abstract. In this paper we sketch a construction of an approximate Neumann kernel on a bounded strongly pseudoconvex domain with smooth boundary. We use the method of osculating the domain near the boundary by the generalized upper half space (the Siegel domain) to approximate the Neumann kernel on the domain by the Neumann kernel on the Siegel domain.

Introduction. The $\bar{\partial}$-Neumann problem on a domain $\mathcal{D} \subset C^{n+1}$ consists of the second-order equation $\Box u \equiv (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\, u = f$ in $\mathcal{D}$ subject to the $\bar{\partial}$-Neumann boundary conditions: $u \in$ Domain of $\bar{\partial}^*$ and $\bar{\partial}u \in$ Domain of $\bar{\partial}^*$ (we rewrite them as $B_{\bar{\partial}}u = 0$ on $b\mathcal{D}$). Since the $\bar{\partial}$-Neumann problem was solved by Kohn by exploiting strong pseudoconvexity to establish subelliptic estimates, there has been considerable interest in a concrete description of the Neumann operator for this problem, see [3]-[8]. The first concrete description of an approximate Neumann operator on a bounded strongly pseudoconvex domain was given in [3], using reduction to the boundary.

Here we announce a more direct approach to a construction of an approximate Neumann kernel on a bounded strongly pseudoconvex domain with smooth boundary $b\mathcal{D}$, by imitating the method developed in [2] for the study of the $\Box_b$ problem. We assume that is equipped with a Levi metric, and $n>1$. Our construction then consists of three major parts.

Research partially supported by NSF grant MCS-82-01599

Part 1. We examine the Neumann kernel on the Siegel domain $\mathbf{D}$.

Part 2. In a neighborhood of a point on $b\mathcal{D}$ we construct an appropriate coordinate system so that we have a good comparison between vector fields on and those on $\mathbf{D}$. We remark that unlike the boundary version $\square_b$ one cannot approximate $\square$ on $\mathcal{D}$ by $\square$ on $\mathbf{D}$ in such a way that all the error terms are negligible. However, this difficulty will be overcome by constructing an appropriate correction operator.

Part 3. We transplant kernels on $\mathbf{D}$ into $\mathcal{D}$ via the coordinate system of Part 2, and then evaluate integral operators on $\mathcal{D}$ with transplanted kernels.

After our work was completed, we learned that at about the same time Phong and Stein [7] constructed a parametrix for the $\bar\partial$-Neumann problem in the same setting. Both they and we employ a method of osculation in principle, but there are two major differences. In Part 1 our model is the exact Neumann kernel $\underline{N}$ on $\mathbf{D}$ found in [8]. We observe that the crucial part of the kernel can be viewed as a Heisenberg convolution of two kernels of different types - one is of Heisenberg type, the other of Euclidean type. Thus in Part 3 we are led to study integral operators whose kernels are expressed via a coordinate system as a Heisenberg convolution of these two types of kernels. Phong and Stein study integral operators whose kernels are described as a product of two kernels, one with Heisenberg homogeneity and another with Euclidean homogeneity, due to the parametrix on $\mathbf{D}$ in [6]. Secondly, their construction of a coordinate system for osculation follows the spirit of [2], §14, based on an argument on integral curves, whereas ours follows the spirit of [3], §4 , and then uses very elementary argument about a change of variables. Our construction is concrete and explicit.

The Neumann kernel on the Siegel domain. Let $\mathbf{D}$ be the Siegel domain $\mathbf{D} = H_n \times \mathbf{R}^+ = \{(z,t,\rho);\ (z,t) \in H_n,\ \rho > 0\}$ where H_n is the Heisenberg group. The metric on $\mathbf{D}$ is given so that $\omega_j = dz_j$, $j=1,\dots,n$, and $\omega_{n+1} = \sqrt{2}\,\partial\rho$ form an orthonormal basis for (1,0) forms on $\mathbf{D}$. We call their duals $Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$, $j=1,\dots,n$, and $Z_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \rho} + i\frac{\partial}{\partial \rho}\right)$ vector fields of H-type.

The expression for the Neumann kernel $\underline{N}$ on $\mathbf{D}$ in [8] essentially involves two kernels $e^\alpha \in C^\infty(H_n \times \mathbf{R}\setminus\{0\})$ and $q \in C^\infty(H_n \times \bar{\mathbf{R}}^+\setminus\{0\})$:

$$N((x,\rho),(y,r)) = \sum_{j=1}^{n} \{g^{n-2}(y^{-1}x,\rho,r) + q^+(y^{-1}x,\ \rho + r)\}\bar{\omega}_j \otimes \omega_j$$

$$+ g^n(y^{-1}x,\rho,r)\bar{\omega}_{n+1} \otimes \omega_{n+1}, \text{ for } (x,\rho),\ (y,r) \in H_n \times \mathbf{R}^+,$$

where $g^\alpha(x,\rho,r) = e^\alpha(x,\rho-r) - e^\alpha(x,\rho+r)$, and $q^+(x,\rho) = 2e^{n-2}(x,\rho) + q(x,\rho)$. Let D^j (X^j, resp.) stand for any j product of Z_k, $\bar{Z}_k$, $k=1,\dots,n+1$ ($k=1,\dots,n$, resp.). $\|\ \|$ denotes the Euclidean norm, and $|\ |$ the Heisenberg norm.

Lemma. For any compact K in $H_n \times \mathbf{R}$ ($H_n \times \overline{\mathbf{R}^+}$, resp.) there exists a constant C_K such that in K,

$$|D^j e^\alpha(x,\rho)| \le C_K(\|x\|^2 + \rho^2)^{-n-\frac{j}{2}}$$

$$(|X^j q(x,\rho)| \le C_k(\|x\|^2 + \rho^2)^{-n+1-\frac{j}{2}}(|x|^2 + \rho)^{-2}, \text{ resp.}).$$

Moreover, $\bar{Z}_{n+1}\, q = \sqrt{2}\, i \frac{\partial}{\partial t} e^{n-2}$ and $\mathcal{L}_{n-2}\, q = 0$ in $H_n \times \mathbf{R}^+$, where $\mathcal{L}_\alpha = \frac{1}{2}\sum_{j=1}^{n} (Z_j\bar{Z}_j + \bar{Z}_j Z_j) + i\alpha\frac{\partial}{\partial t}$. Thus $\bar{Z}_{n+1}\, q$ is a kernel of Euclidean type, and $q(x,\rho) = (\sqrt{2}\, i\, Z_{n+1} \frac{\partial}{\partial t} e^{n-2})*_{H_n} \Phi_{n-2}$ where Φ_α is the fundamental solution for $\Box_b$ in [1].

<u>An admissible coordinate system</u>. Using a partition of unity we may restrict our attention to a small neighborhood U of a point on $b\mathcal{D}$. Let ρ denote the geodesic distance from $b\mathcal{D}$ with respect to our fixed metric, so that $\rho>0$ in $\mathcal{D}\cap U$. We choose an orthonormal basis $\{\omega_j\}_{j=1}^{n+1}$ with $\omega_{n+1} = \sqrt{2}\,\partial\rho$ for (1,0) forms on U, and denote the dual basis by $\{Z_j\}_{j=1}^{n+1}$. Then

<u>Proposition</u>. <u>If U is sufficiently small, there exists a smooth mapping $\Theta: U \times U \to H_n$ with the following properties.</u>

(i) $\Theta(\xi,\xi) = 0$ <u>for</u> $\xi \in U$. <u>For each</u> $\xi \in U$, $(\Theta(\xi,\eta), \rho(\eta)) \in H_n \times \mathbf{R}$ <u>gives a coordinate representation of</u> $\eta \in U$.

(ii) <u>For the coordinates</u> $(z,t,\rho) = (z(\xi,\cdot), t(\xi,\cdot), \rho(\cdot))$

$$Z_j = Z_j^H + \sum_{k=1}^{n} (\gamma_{jk} Z_k^H + \gamma_{j\bar{k}} \bar{Z}_k^H) + \gamma_{j0} \frac{\partial}{\partial t}, \; j=1,\dots,n+1,$$

<u>where</u> $\gamma_{jk}, \gamma_{j\bar{k}} = O^1$, γ_{j0}, $\gamma_k = \gamma_{(n+1)k}$, $\gamma_{\bar{k}} = \gamma_{(n+1)\bar{k}} = O^{1,2}$ $j,k=1,\dots,n$, <u>and</u> $\gamma_0 = \gamma_{(n+1)0} = -\sqrt{2}i\, \tau_0(\xi)[\rho - \rho(\xi) + it] + O^{2,3}$ (τ_0 <u>is real and</u> C^∞ <u>in</u> U). Z_j^H, $j=1,\dots,n+1$, <u>are vector fields of</u> H-<u>type with respect to</u> (z,t,ρ). $f \in C^\infty(U\times U)$ <u>is</u> O^j <u>if</u>

$$|f(\xi,\eta)| \le C(\|\Theta(\xi,\eta)\|^2 + |\rho(\eta)-\rho(\xi)|^2)^{\frac{j}{2}}.$$

Also f is $O^{j,k}$ if f is O^j and, in addition, satisfies

$$|f(\xi,\eta)| \le C(|\Theta(\xi,\eta)|^2 + |\rho(\eta)| + |\rho(\xi)|)^{\frac{k}{2}}.$$

An approximate Neumann operator. For $\xi \in U$ and $f \in C^\infty(\bar{\mathcal{D}})$, supported in U, we define

$$G^\alpha[f](\xi) = \int_{\mathcal{D}} g^\alpha(\Theta(\eta,\xi), \rho(\xi), \rho(\eta)) f(\eta)\, dV(\eta).$$

Besides G^α, the operator $\mathcal{O}p[ak]$ associated with the kernel $k \in C^\infty(H_n \times R^+ \setminus \{0\})$ with the coefficient $a \in C^\infty(U \times U)$ is defined to be

$$\mathcal{O}p[ak]\, f(\xi) = \int_{\mathcal{D}} a(\xi,\eta) k(\Theta(\eta,\xi), \rho(\xi) + \rho(\eta)) f(\eta) dV(\eta).$$

Theorem. A local approximate Neumann operator $\underline{N}$ is given by

$$\underline{N} = \begin{pmatrix} G^{n-2} & & 0 \\ & \ddots G^{n-2} & \\ 0 & & G^n \end{pmatrix} + \mathcal{O}p\left[\begin{pmatrix} q^+ & & 0 \\ & \ddots & q^+ \\ 0 & & 0 \end{pmatrix}\right.$$

$$\left. + \sqrt{2}\,\rho(\xi) \begin{pmatrix} -\bar{A}^H q^+ & & 0 \\ & \ddots\ -\bar{A}^H q^+ & \\ 0 & & 0 \end{pmatrix} + \sqrt{2}\,\rho(\xi) \begin{pmatrix} \bar{S}^k_{j(n+1)} & 0 \\ & \vdots \\ \frac{i}{\sqrt{2}} c_{k(n+1)}(\xi) q^+ & 0 \end{pmatrix}\right]$$

Here $A^H = \sum_{j=1}^{n} (\gamma_j Z_j^H + \gamma_{\bar{j}} \bar{Z}_j^H) + \gamma_0 \frac{\partial}{\partial t}$, $\bar{\partial}\, \bar{\omega}_k = \sum_{j<\ell} \bar{S}^k_{j\ell}\, \bar{\omega}_j \wedge \bar{\omega}_\ell$

and $c_{k(n+1)} = \langle [Z_j , \bar{Z}_{n+1}] , T \rangle$. Then

$$\Box \underline{N} = I - R \text{ in } \mathcal{D} \cap U , \quad B_{\bar{\partial}} \underline{N} = 0 \text{ on } b\mathcal{D} \cap U .$$

Here R is a smoothing operator so that $R: S^m \to S^{m+1}$

where $S^m = \{f \in L^2 : x^j \left(\frac{\partial}{\partial \rho}\right)^k f \in L^2 \text{ for } j + 2k \le m\}$, $m=0,1,2\ldots$.

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Received November 12, 1986

Analytic And Numerical Results In Random Fields Estimation Theory

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Presented by M. Shinbrot

Abstract.

Let $u(x) = s(x) + n(x)$, $x \epsilon R^r$, be a random field observed in a domain $D \subset R^r$. Let $Lu = \int_D h(x,y) u dy$ be a linear estimate of As, where A is a given operator. The problem is to find the estimate optimal in the following sense $\overline{(Lu - As)^2} = min$. Here the bar denotes mean value, $s(x)$ is a useful signal, $n(x)$ is noise, $\bar{s} = \bar{n} = 0$, and the covariance functions $\overline{u^*(x)u(y)} = R(x,y)$ and $\overline{u^*(x)s(y)} = f(x,y)$ are known. No assumptions about distributions of random fields are made. The optimal estimate is defined by the function $h(x,y)$. This function is a solution to the multidimensional integral equation (1) $\int_D R(x,z)h(z,y)dz = f(x,y), x, y \epsilon D \cup \partial D$, if $A = I$, i.e. if we deal with the filtering problem. A theory of equation (1) is given. Numerical methods for solving (1) are suggested.

I. INTRODUCTION

Let $u(x) = s(x) + n(x)$ be a random field, $s(x)$ be a useful signal, $n(x)$ be noise, $\overline{s(x)} = \overline{n(x)} = 0$, where the bar denotes mean value. We observe $u(x)$ in a domain $D \subset R^r$ with the boundary Γ and want to estimate some signal As, where A is a known operator. For example, if $As = s$, then we talk about filtering problem. The estimate

$$Lu = \int_D h(x,y) u(y) dy \tag{1}$$

is optimal in the sense that

$$\overline{(Lu - As)^2} = min. \tag{2}$$

For simplicity we discuss the case $A = I$. A necessary condition for the optimal estimate is the equation

$$\int_D R(x,y) h(y,z) dy = f(x,z), \quad x, z \epsilon D \cup \Gamma \tag{3}$$

where

$$R(x,y) = \overline{u*(x)u(y)}, \quad f(x,y) = \overline{u*(x)s(y)}. \tag{4}$$

We do not assume anything about the statistical distribution of $u(x)$, $u(x)$ is neither Gaussian nor Markovian. The theory is developed entirely within the correlation theory. The covariance functions (4) are all we need. Knowledge of these functions is necessary for the formulation of the filtering problem within the framework of the correlation theory, because in this theory one uses covariance functions and mean values only. We can assume without loss of generality that the mean values are zeros, since otherwise we can subtract the mean values from the signals. The variable z enters as a parameter in (3). Thus, mathematically one can study the equation

$$Rh = \int_D R(x,y)h(y)dy = f(x), \quad x\epsilon\overline{D} = D \cup \Gamma. \tag{5}$$

We restrict the discussion by assuming that D is a finite domain with a smooth boundary.

Equation (5) does not have solutions in $L^2(D)$, generally speaking.

EXAMPLE. Equation

$$\int_{-1}^{1} \exp(-|x-y|)hdy = f(x), \quad -1 \le x \le 1 \tag{6}$$

has a solution of minimal order of singularity [1]

$$h(x) = \frac{1}{2}(-f'' + f) + \frac{\delta(x+1)}{2}(-f'(-1) + f(-1)) + \frac{\delta(x-1)}{2}(f'(1) + f(1)) \tag{7}$$

It is clear from (7) that h is a distribution and $h\epsilon L^2$ iff $f(-1) = f'(-1)$ and $f(1) = -f'(1)$.

We define a class of random fields, in other words, a class $\mathcal{R}$ of kernels $R(x,y)$, such that the following questions will be answered:

1. In what functional space should one look for a solution of (5)? What is the order of singularity of the solution to (5) which solves the statistical problem? What is the singular support of the solution?

2. Is the solution stable towards small perturbations of the data? The notions of the stability and smallness should be specified.

3. Can the solution be obtained analytically? Numerically?

All these questions we answer in detail for the following class $\mathcal{R}$ of kernels. We say that $R(x,y)\epsilon\mathcal{R}$ if there exists a self-adjoint elliptic operator ℓ in $L^2(\mathbb{R}^r)$ such that

$$R(x,y) = \int_\Lambda P(\lambda)Q^{-1}(\lambda)\Phi(x,y,\lambda)d\rho(\lambda) \tag{8}$$

Here $\Lambda, \Phi(x,y,\lambda)d\rho(\lambda)$ are the spectrum, spectral kernel and spectral measure of ℓ, and P and Q are positive on Λ polynomials.

Note that positivity of PQ^{-1} guarantees that $R(x,y)$ is a non-negative definite kernel, so that a necessary condition for a covariance function is satisfied.

Let $s =$ *order* of ℓ, $p = degP$, $q = degQ$, $q > p$, and $\alpha = s(q-p)/2$. Let $H^{\alpha}(D)$ be the Sobolev space $W^{2,\alpha}(D)$, and $\dot{H}^{-\alpha}(D)$ be its dual space with respect to the inner product in $L^2(D)$. The space $\dot{H}^{-\alpha}(D)$ consists of the elements of $H^{-\alpha}(R^r)$ with support in $\overline{D}$. We assume that $f \epsilon H^{\alpha}(D)$.

In section II we formulate the results, in section III we give some examples and point out some of the many applications.

The theory of the equations of class $\mathcal{R}$ has been developed in [1]-[4], where one can find further references.

II. Basic results

1. Theorem 1. *The mapping $R : \dot{H}^{-\alpha}(D) \to H^{\alpha}(D)$ is an isomorphism. The solution of equation (5) of minimal order of singularity exists, is unique, and solves the estimation problem (2) (with $A = I$). The minimal order of singularity $\leq \alpha$, the singular support of the solution to (5) with $f \epsilon H^{\alpha}(D)$ is Γ. The solution can be calculated analytically by the formula*

$$h = Q(\ell)G, \tag{9}$$

where

$$G = \begin{cases} g_0 + v & in \quad \overline{D} \\ u & in \quad \Omega = R^r \setminus D, \end{cases} \tag{10}$$

and $g_0 \epsilon H^{(p+q)s/2}$ *is any particular solution to the equation*

$$P(\ell)g_0 = f \quad in \quad \overline{D}, \tag{11}$$

while u and v solve the interface problem

$$Q(\ell)u = 0 \quad in \quad \Omega, \quad P(\ell)v = 0 \quad in \quad D, \quad u(\infty) = 0, \tag{12}$$

$$\partial_N^j u = \partial_N^j (g_0 + v) \quad on \quad \Gamma, \quad 0 \leq j \leq \frac{1}{2}s(p+q) - 1, \tag{13}$$

where ∂_N is the normal derivative on Γ, N points into Ω.

Theorem 1 answers questions (1)-(3) in section I, except the question of numerical solution of the equation (5). Indeed, one sees from (8) that the order of singularity of $h \leq \alpha$, that the singular support of h is $\Gamma = \partial D$, that the solution h is stable in the sense that small perturbations of f in $H^{\alpha}(D)$

lead to small perturbations of h in the norm of $\dot{H}^{-\alpha}(D)$, and small perturbations $R_\delta(x,y)$ of $R(x,y)$ such that the corresponding perturbed operator R_δ satisfies the inequality

$$\| R - R_\delta \|_{\dot{H}^{-\alpha}(D)\to H^{\alpha}(D)} \leq \delta$$

with

$$\delta \| R^{-1} \|_{H^{\alpha}(D)\to \dot{H}^{-\alpha}(D)} < 1,$$

lead to a small perturbation of h in $\dot{H}^{-\alpha}(D)$. The solution is obtained analytically by formula (9).

It solves the estimation problem: all other solutions to (5) have the order of singularity $> \alpha$ and give infinite value to the variance $\epsilon = \overline{(Lu-s)^2}$.

Indeed, if $Rh = f$, one has

$$\epsilon = (Rh, h) - 2Re(h, f) + \overline{|s|^2} = \overline{|s|^2} - (Rh, h) \tag{14}$$

Therefore ϵ is finite iff (Rh, h) is finite. If $h\epsilon\dot{H}^{-\beta}(D)$ with $\beta > \alpha$, then $Rh\epsilon\dot{H}^{-\gamma}(D)$, $\gamma = -\beta + 2\alpha < \beta$. Therefore the expression (Rh, h) is not finite. One can easily understand the situation if one considers the familiar case when $r = 1$ and $R(x,y) = R(x-y)$, $\tilde{R}(\lambda) = P(\lambda)Q^{-1}(\lambda)$, where $\tilde{R}$ is the Fourier transform of $R(x)$. In this case

$$(Rh, h) = \int_{-\infty}^{\infty} P(\lambda)Q^{-1}(\lambda)|\tilde{h}|^2 d\lambda,$$

$$\ell = -i\frac{d}{dx}, s = 1, \alpha = \frac{1}{2}(q-p),$$

p and q are even, $P(\lambda)Q^{-1}(\lambda) \sim |\lambda|^{-2\alpha}$ as $|\lambda| \to \infty$, $D = [0, T]$. If $\tilde{h}\epsilon\dot{H}^{-\beta}(D)$ and $\beta > \alpha$, then $\int_{-\infty}^{\infty}(1+\lambda^2)^{-\alpha}|\tilde{h}|^2 d\lambda = \infty$, so that (Rh, h) is infinite. Formula (7) is a particular case of (9). In (7) one has $r = 1$, $\ell = -i\frac{d}{dx}$, $D = [-1, 1]$, $P(\lambda) = 1$, $Q(\lambda) = \frac{\lambda^2+1}{2}$, $\Phi(x, y, \lambda)d\rho(\lambda) = (2\pi)^{-1}\exp i\lambda(x-y)d\lambda$, $p = 0$, $q = 2$, $\alpha = 1$.

2. Numerical solution of integral equations of estimation theory in distributions.

Usually it is assumed that integral equations of the first kind have L^2 solution and this solution can be found numerically by a regularization method. This is not the case with the integral equations of estimation theory. If the noise is colored, i.e. $R(x,y)$ does not contain a delta-function term, in other words if $q > p$, then equation (5) has no solution in $L^2(D)$, generally speaking. Therefore, regularization procedures are useless. In fact, the problem of finding the solution of (5) in $\dot{H}^{-\alpha}(D)$ is well-posed as follows from Theorem 1. The numerical solution of such integral equations was discussed for the first time in [3]. Here we outline a projection method and a choice of basis functions for solving equation (5) in the space of distributions $\dot{H}^{-\alpha}(D)$. This method converges.

The basic idea is to use the analytical structure of the solution. Let us illustrate the idea by an example. Consider equation (6). Let us look for the approximate h of the form

$$h_n(x) = \sum_{j=1}^{n} c_j^{(n)} h_j(x) + A_n \delta(x-1) + B_n \delta(x+1) \tag{15}$$

where $c_j^{(n)}$, A_n and B_n are constants and system $\{h_1, \dots h_n(x), \exp(x), \exp(-x)\}$ is linearly independent in $H^1([-1,1])$. The form (15) is suggested by formula (9): according to this formula the singular support of h consists of two points $x = \pm 1$ and the order of singularity of h is 1. Let us determine the coefficients $c_j^{(n)}$, A_n, B_n from the requirement

$$\| Rh_n - f \|_1 = min \tag{16}$$

where $\| \cdot \|_\alpha$ is the norm in $H^\alpha(D)$, $\overline{D} = [-1,1]$. This leads to the problem

$$\epsilon \equiv \int_{-1}^{1} \left\{ \left| \sum_{j=1}^{n} c_j^{(n)} h_j + A_n \exp(x) + B_n \exp(-x) - f \right|^2 + \right.$$

$$\left. \left| \sum_{j=1}^{n} c_j^{(n)} h_j' + A_n \exp(x) - B_n \exp(-x) - f' \right|^2 \right\} dx = min. \tag{17}$$

One has the linear system for finding the coefficients $c_j^{(n)}$, A_n, B_n:

$$\frac{\partial \epsilon}{\partial c_j} = 0, \quad \frac{\partial \epsilon}{\partial A_n} = 0, \quad \frac{\partial \epsilon}{\partial B_n} = 0. \tag{18}$$

The matrix of the system is positive definite because the system $\{h_j, \exp(x), \exp(-x)\}$ is linearly independent. So, the system (18) is uniquely solvable and h_n is uniquely defined. Assume additionally that the system $\{h_j\}$ is complete in $H^1(D)$. Then

$$\| Rh_n - f \|_1 \to 0 \quad \text{as} \quad n \to \infty. \tag{19}$$

Since $R : \dot{H}^{-1}(D) \to H^1(D)$ is an isomorphism, (19) implies that

$$\| h_n - R^{-1} f \|_{\dot{H}_{-1}} \to 0 \quad \text{as} \quad n \to \infty. \tag{20}$$

This proves convergence of the method (16). Practically, the regular part of h_n, namely

$$h_{nreg} = \sum_{j=1}^{n} c_j^{(n)} h_j$$

converges to the regular part of h, and the singular part of h_n, namely $A_n\delta(x-1) + B_n\delta(x+1)$ converges to the singular part of h, which is $A\delta(x-1) + B\delta(x+1)$; in particular, $A_n \to A$, $B_n \to B$ as $n \to \infty$.

A similar but technically more complicated argument is valid for the multidimensional equation (5). The set of the basis functions will include the delta functions and its derivatives on Γ. These singular functions are suggested by formula (9).

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Received November 20, 1986

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