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ON THE SOLVABILITY OF THE OPERATOR EQUATION $A_1XA_2 - B_1XB_2 + XDX = Q$ WHEN Q HAS ONE DIMENSIONAL RANGE

Tapas Mazumdar

Presented by M.D. Choi, F.R.S.C.

ABSTRACT. We indicate here an analytic treatment of existence and uniqueness of solution X of the equation $A_1XA_2 - B_1XB_2 = Q$, when A_1, A_2, B_1, B_2, Q are appropriate linear operators, bounded or unbounded, in a suitable Hilbert space, Q having one dimensional range. This result can then be utilized to obtain existence of solution of the Riccati-type operator equation $A_1XA_2 - B_1XB_2 + XDX = Q$.

1. We will report here certain sufficient conditions under which solutions X of the equations

$$A_1XA_2 - B_1XB_2 = Q, \tag{1}$$

$$A_1XA_2 - B_1XB_2 + XDX = Q \tag{2}$$

exist, in which A_1, A_2, B_1, B_2, D, Q are given linear operators satisfying certain hypotheses. We need to begin by establishing notations and spaces we will work with. Accordingly, let $\mathbb{C}$ denote the set of complex numbers; let $\mathbb{N}$ denote the set of natural numbers. Let $\mathcal{H}$ denote a commplex Hilbert space with norm $|\cdot|_{\mathcal{H}}$. Let $\mathcal{V}^1$ be a complex Hilbert space with norm $\|\cdot\|_1$ such that $\mathcal{V}^1$ is a dense subspace of $\mathcal{H}$ with continuous inclusion injection from $\mathcal{V}^1$ into $\mathcal{H}$. Consequently, $\exists$ a constant $\gamma > 0$ such that

$$|v|_{\mathcal{H}} \leq \gamma\|v\|_1 \ \forall\ v \in \mathcal{V}^1. \tag{3}$$

We refer the reader to the reference [1] if rank $Q > 1$.

Let $\mathcal{V}^2$ be a complex separable pre-Hilbert space with norm $\|\cdot\|_2$.

Let $\mathcal{X} = \mathcal{L}(\mathcal{V}^2, \mathcal{H})$, $\mathcal{W} = \mathcal{L}(\mathcal{V}^2, \mathcal{V}^1)$ where, for example, $\mathcal{L}(\mathcal{V}^2, \mathcal{H})$ denotes the Banach space of bounded linear operators from $\mathcal{V}^2$ into $\mathcal{H}$, with the usual sup norm topology. Considered given in the problem of our concern are A_1, $B_1 \in \mathcal{L}(\mathcal{V}^1, \mathcal{H})$, $D \in \mathcal{L}(\mathcal{V}^1, \mathcal{V}^2)$, $Q \in \mathcal{X}$, and linear operators A_2, B_2: $\mathcal{V}^2 \to \mathcal{V}^2$. Because of (3), possibility is kept open that A_1, B_1 may be unbounded operators in $\mathcal{H}$. If $\mathcal{V}^2$ is a subspace of $\mathcal{H}$, then the rest of the operators above may also happen to be unbounded in $\mathcal{H}$.

With domains and ranges laid out as above, equations (1) and (2) are now well-defined. We now start listing the hypotheses we will work under.

(H1) Q has a one dimensional range; say, $Q[\mathcal{V}^2] = \{\alpha h : \alpha \in \mathbb{C}\}$, denoted by $[h]$, for some $h \in \mathcal{H}$ with $|h|_{\mathcal{H}} = 1$.

(H2) $[h] \subset \mathcal{V}^1$, $A_1[[h]] = [h] = B_1[[h]]$.

We will use the notations $A_{1,h}$, $B_{1,h}$ respectively for the restrictions of A_1, B_1 to $[h]$.

(H3) $\exists$ an orthonormal basis $\{b_i : i \in \mathbb{N}\}$ of $\mathcal{V}^2$ such that each b_i is an eigenvector of both A_2 and B_2.

We will denote by $\mathcal{V}^2_n$ the subspace of $\mathcal{V}^2$ generated by $\{b_1, \cdots, b_n\}$. $A_{2,n}$, $B_{2,n}$ are restrictions of A_2, B_2 to $\mathcal{V}^2_n$. $\mathcal{W}(n)$ is the set of restrictions to $\mathcal{V}^2_n$ of all those $X \in \mathcal{W}$ for which $Xb_i = 0 \ \forall \ i > n$, and $Xb_i \in [h] \ \forall \ i \leq n$. $\mathcal{W}(n)$ turns out to be isomorphic to $\mathcal{L}(\mathcal{V}^2_n, [h])$. Also, let $\mathcal{W}_h$ denote the space $\mathcal{L}(\mathcal{V}^2, [h])$. The natural topologies on $\mathcal{L}(\mathcal{V}^2_n, [h])$ and $\mathcal{W}_h$ are equivalent to their subspace topology of $\mathcal{W}$.

(H4) $\exists$ a constant $\beta>0$ such that $\forall$ nonzero $Y\in\mathcal{W}_h$, $\exists$ a $\Phi_Y\in\mathcal{V}^2$ satisfying the dominance relation:

$$|(A_{1,h}YA_2 - B_{1,h}YB_2)\ \Phi_Y|_{\mathcal{H}} > \beta\|Y\|_{\mathcal{W}}\|\Phi_Y\|_2.$$

(For examples with this condition satisfied, see [2,3]). This condition is an extension of the well known concept of ellipticity or coercivity, and may be called a one-sided coercivity condition. A direct consequence of (H4) is:

(H4)' $\exists$ a constant $\beta>0$ such that $\forall\ n\in\mathbb{N}$ and $\forall$ nonzero $Y\in\mathcal{W}(n)$, $\exists$ a $\Phi_Y\in\mathcal{V}_n^2$ satisfying the dominance relation,

$$|(A_{1,h}YA_{2,n} - B_{1,h}YB_{2,n})\Phi_Y|_{\mathcal{H}} > \beta\|Y\|_{\mathcal{W}}\ \|\Phi_Y\|_2.$$

We will use the notation $\alpha = (\gamma/\beta^2)\ \|Q\|\ \|D\|$, the norms of Q and D being in [illegible] and $\mathcal{X}(\mathcal{V}^1,\ \mathcal{V}^2)$ respectively.

(H5) $\exists$ a $\Delta > 0$ such that $1+\alpha < \Delta$ and $1+\alpha\Delta^2 < \Delta$.

Obviously, this condition is satisfied if $\Delta > 1$ and α is sufficiently small, i.e. the product $\|Q\|\ \|D\|$ is sufficiently small. Hypothesis (H5) points out that, in a sense, the part XDX in equation (2) is a perturbation of equation (1).

(H6) $\exists$ a fixed number $k_0 \geq 1$ such that (i) $k_0\alpha < 1$, (ii) $(2\alpha)/[1-k_0\alpha] + k_0\alpha^2 \leq k_0\alpha$, and (iii) if $\Delta > 2$, then $\alpha[1-k_0\alpha]^{-2} \leq 1$. (E.g., we may have $\alpha<(18)^{-1}$ and $k_0 = (2\alpha)^{-1}$).

We are now ready to state our main results.

Theorem 1. Under the hypotheses (H1) - H(4) above, $\exists$ a unique solution $X = X^{(Q)}\in\mathcal{W}$ of the equation

$$(A_1XA_2 - B_1XB_2)\Phi = Q\Phi \quad \forall\ \Phi\in\mathcal{V}^2. \tag{4}$$

Moreover, $X^{(Q)}$ has the same range as Q has, and $\|X^{(Q)}\|_{\mathcal{W}} \leq \frac{1}{\beta}\|Q\|$. ∎

This theorem can be proved by first establishing a version of it in a suitable finite dimensional form, and then adopting a Galerkin method of approach. If $X^{(n)}$ is the solution when ∇^2 is replaced by ∇_n^2, then the sequence $\{X^{(n)}\}_{n\in\mathbb{N}}$ is bounded in a suitable Hilbert space (this is where "rank $Q = 1$" plays a crucial role), and so has a weak limit. This weak limit turns out to be the solution $X^{(Q)}$ of (4). Details of the proof are given in [2].

Theorem 2. *Under the hypotheses (H1) - (H6) above, equation (2) has a solution* $X = X_Q \in \mathcal{W}$. ■

This theorem is proved by an iterative approach in which, at the nth stage,

$$A_1XA_2 - B_1XB_2 = Q - X_{n-1}DX_{n-1}$$

becomes a meaningful equation, and has the solution $X = X_n$ by virtue of Theorem 1. Hypotheses (H5) and (H6) ensure that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges weakly to a solution X_Q of (2). Details of the proof are found in [3].

It is shown in [2, 3] that our theorems show the existence of solution, in an appropriate space, of

$$\left(-a \frac{\partial^2}{\partial x^2} + k_1 I\right) X \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{2} I\right)$$
$$+ \left(b \frac{\partial^2}{\partial y^2}\right) X \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + k_2 I\right) + XDX = Q,$$

where a, k_1, b, k_2 are positive constants, I is the identity operator, and D, Q are to satisfy all the hypotheses made on them above.

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The Amalgam Structure of the Bianchi Groups

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Presented by K. Murasugi, F.R.S.C.

1. Introduction. If d is a positive square-free integer, the Bianchi group Γ_d is $PSL_2(\mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of integers in the quadratic imaginary number field $Q(\sqrt{-d})$. As a class the Bianchi groups have been extensively studied. A general method to generate presentations for the Γ_d was developed by Swan[7], while Fine[1],[2] showed that in the cases where $\mathcal{O}_d$ is Euclidean $\{d = 1,2,3,7,11\}$ Γ_d is either a non-trivial free product with amalgamation or an HNN group. Similar results in some additional cases were handled by Floge[4]. The purpose of this note is to announce that all the Bianchi groups with the sole exception of Γ_3 are amalgams. Specifically we prove that each Γ_d, $d \neq 3$, is decomposable as a non-trivial free product with amalgamation. Further in each of the non-Euclidean cases one of the factors in this decomposition can be taken as $PE_2(\mathcal{O}_d)$ - the projective elementary group over $\mathcal{O}_d$. It is known that for $d \neq 1,2,3,7,11$ the $PE_2(\mathcal{O}_d)$ are all isomorphic [1]. The proofs we employ are topological and depend upon the results of Swan. The same techniques yield a proof that each Γ_d with $d \neq 1,3$ is also an HNN group. It was communicated to us by A.Hatcher that a version of this last result was known but not published.

2. Main results. We first fix some notation. $G_1 \underset{H}{*} G_2$ will denote the free product with amalgamation of G_1 and G_2 amalgamated along the subgroup H. If G is a group with H,H_1 isomorphic subgroups of G then $HNN(t,G,H,H_1)$ will denote the HNN extension of G with free part t and associated subgroups H and H_1.

Our main result is the following.

Theorem 1. (1) *For each* $d \neq 3$ *the Bianchi group* Γ_d *is a non-trivial free product with amalgamation.*

(2) *If* $\mathcal{O}_d$ *is non-Euclidean* $d \neq 1,2,3,7,11$ *the above decomposition takes the form*

$$\Gamma_d = PE_2(\mathcal{O}_d) \underset{F}{\ast} G_d$$

where F is a subgroup of $PE_2(\mathcal{O}_d)$ *generated by two conjugates of the classical modular group* $PSL_2(Z)$ *and* G_d *is a group depending on d.*

The proof of theorem 1 in the non-Euclidean cases is essentially topological. For each d we construct a complex which has Γ_d as its fundamental group. The construction of this complex is based on the work of Swan. We then apply the Seifert-Van Kampen theorem. In order to show that this leads to a non-trivial amalgam we rely on the following lemma.

Lemma. *Let* $d \neq 1,2,3,7,11$ *and suppose* $1, w$ *constitute an integral basis for* $\mathcal{O}_d$. *Let* X_1, X_2, X_3, X_4 *be the projective matrices*

$$X_1 = \pm \begin{matrix} 1 & -1 \\ 1 & 0 \end{matrix} \qquad X_2 = \pm \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \qquad X_3 = \pm U X_1 U^{-1} \qquad X_4 = \pm U X_2 U^{-1}$$

where U is the projective matrix $U = \pm \begin{matrix} 1 & w \\ 0 & 1 \end{matrix}$.

Then the subgroup of $PE_2(\mathcal{O}_d)$ *generated by* X_1, X_2, X_3, X_4 *has the presentation*

$$< X_1, X_2, X_3, X_4 \ ; \ X_1^3 = X_2^2 = X_3^3 = X_4^2 = X_1X_2X_3X_4 = 1 >$$

The proof of the lemma is done algebraically and is based on work in [1].

The Euclidean cases are handled individually using presentations developed in [1]or [7] . In particular we obtain.

Theorem 2. (1) $\Gamma_1 = G_1 \underset{H}{\ast} G_2$ (This appeared in [2])

with $G_1 = S_3 \underset{Z_3}{\ast} A_4$

$$G_2 = S_3 \underset{Z_2}{*} D_2$$

$$H = PSL_2(Z)$$

(2) $\Gamma_2 = G_1 \underset{H}{*} G_2$ *where in this case*

$G_1 = HNN(t, D_2, Z_2, Z_2)$ - *that is* G_1 *is an HNN group whose base is the Klein 4-group* D_2 *with two 2-cycles as the associated subgroups.*

$G_2 = HNN(t, A_4, Z_3, Z_3)$ - G_2 is an HNN group whose base *is the alternating group* A_4 *with two 3-cycles associated.*

$$H = Z * Z_2$$

(3) $\Gamma_7 = G_1 \underset{H}{*} G_2$ *with*

$$G_1 = Z * Z_2$$

$G_2 = HNN(t, K, L_1, L)$ *with* $K = S_3 \underset{Z_2}{*} S_3$ *and* $L = L_1 = Z_3$.

G_2 *is an HNN group with base K and two 3-cycles associated.*

$$H = Z * Z_2 * Z_2$$

(4) $\Gamma_{11} = G_1 \underset{H}{*} G_2$ *with*

$$G_1 = Z * Z_3$$

$G_2 = HNN(t, K, L, L_1)$ *with* $K = A_4 \underset{Z_3}{*} A_4$ *with* $L = L_1 = Z_3$

$$H = Z * Z_3 * Z_3$$

It is known that the group Γ_3 does not decompose as either a free product with amalgamation or an HNN group [2] .

Finally using the same topological constructs as in the proof of theorem 1 we are able to prove.

Theorem 3. *For* $d \neq 1,3$ Γ_d *is an HNN group. For each d, the associated subgroups are conjugates of the modular group* $PSL_2(Z)$ *while the base* K_d *depends on* d.

Theorem 3 has an interesting consequence in terms of the normal subgroup lattice of Γ_d, $d \neq 1,3$, which is in contrast to the situation in both the modular group $PSL_2(Z)$ and the Picard group Γ_1. In [3] it was shown that there are large gaps in the possible indices of normal subgroups in Γ_1. However if $d \neq 1$ or 3 we have.

Corollary. For $d \neq 1,3$, Γ_d contains at least one normal subgroup of index n for each positive integer n.

The work on this paper grew out of a seminar on the Bianchi groups held at the University of California Santa Barbara. In addition to the authors, Seymour Bachmuth and Morris Newman were active participants. Without their knowledge and enthusiasm this paper would not exist.

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UNITÉS RELATIVES À CERTAINS ORDRES

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Presented by P. Ribenboim, F.R.S.C.

I. Abstract

Given a unique factorization domain A, an element Δ integral over A and $f \in A$ irreducible, we consider the group of units $A[\Delta]^*$ (resp. $A[f\Delta]^*$) of the ring $A[\Delta]$ (resp. $A[f\Delta]$ and we want to study the quotient group $A[\Delta]^*/A[f\Delta]^*$.

In order to give information on the torsion of this group we introduce a further ring : $A + fA[\Delta]$ and we find that the torsion of $A[\Delta]^*/(A+fA[\Delta])^*$ and $(A+fA[\Delta])^*/A[f\Delta]^*$ are of different nature.

II. Etude algébrique

1. Anneaux des séries formelles à cœfficients dans un anneau factoriel

Soient :
- A un anneau factoriel
- A^* le groupe des éléments inversibles de A
- f un élément premier de A
- X une indéterminée
- $A[[X]]$ l'anneau des séries entières formelles à cœfficients dans A.

On pose :

$$G = A[[X]]^*$$
$$G^f = (A+fA[[X]])^*$$
$$G_f = A[[fX]]^*$$

Théorème 1.- Le groupe $G/_{G^f}$ est sans torsion.

Preuve : On montre que $G/_{G^f} \simeq F[[X]]^*/_{F^*}$ où $F = A/_{fA}$, et ce dernier groupe est sans torsion, d'après le lemme :

Lemme 1.- Soit I un anneau intègre, le groupe $I[[X]]^*/_{I^*}$ est sans torsion.

Théorème 2.- Si l'anneau A est de caractéristique nulle (resp. non nulle) et si l'on pose $\mathbf{Z} \cap fA = f_1.\mathbf{Z}$ où f_1 est 0 ou un nombre premier de $\mathbb{N}$ alors $G^f/_G$ est la somme directe d'un groupe sans torsion et d'un groupe de f_1-torsion (resp. sans torsion).

2. Extension des anneaux de valuation discrète

Soient :

$\mathcal{A}$ un anneau de valuation discrète d'idéal maximal $f\mathcal{A}$.

$\mathcal{A}^* = \mathcal{A} - f\mathcal{A}$ le groupe des unités de $\mathcal{A}$, $\mathcal{K}$ le corps de fractions de $\mathcal{A}$.

Δ un élément d'une clôture algébrique de $\mathcal{K}$ qui vérifie l'équation $G(\Delta) = \Delta^p - b_{p-1}\Delta^{p-1} - \ldots - b_o = 0$, où G est un polynôme irréductible à coefficients dans $\mathcal{A}$, de degré p non nécessairement premier.

$\mathcal{E} = \mathcal{K}(\Delta)$.

$\mathcal{O}$ l'anneau $\mathcal{A}[\Delta]$, $\mathcal{G}$ (resp. $\mathcal{U}$) les unités (resp. les unités de norme 1) de l'anneau $\mathcal{O}$.

$\mathcal{O}^f$ l'anneau $\mathcal{A} + f\mathcal{A}[\Delta]$, $\mathcal{G}^f$ (resp. $\mathcal{U}^f$) les unités (resp. les unités de norme 1) de l'anneau $\mathcal{O}^f$.

$\mathcal{O}_f$ l'anneau $\mathcal{A}[f\Delta]$, $\mathcal{G}_f$ (resp. $\mathcal{U}_f$) les unités (resp. les unités de norme 1) de l'anneau $\mathcal{O}_f$.

$F = \mathcal{A}/_{f\mathcal{A}}$.

s l'homomorphisme d'anneaux, $s : \mathcal{O} \longrightarrow \mathcal{O}/_{f\mathcal{O}}$.

Théorème 3.-

1) $s(\mathcal{G}^f) = F^*$ et $s(\mathcal{U}^f)$ est contenu dans le groupe des racines $p^{\text{ième}}$ de l'unité de F^*.

2) $\mathcal{G}/_{\mathcal{G}^f}$ s'injecte dans $\left(\mathcal{O}/_{f\mathcal{O}}\right)^* / F^*$

Preuve : 1) $s(\mathcal{G}^f) = F^*$ résulte de l'égalité $\mathcal{G}^f = \mathcal{A}^* + f\mathcal{A}[\Delta]$. Si $\varphi \in \mathcal{U}^f$, on a $s(N\varphi) = (s(\varphi))^p = 1$, d'où la deuxième assertion de 1.

2) L'homomorphisme s induit un homomorphisme $\tilde{s}$ de groupes :

$$\tilde{s} : \mathcal{G} \longrightarrow \left(\mathcal{O}/_{f\mathcal{O}}\right)^*$$

et on montre que $\tilde{s}^{-1}(F^*) = \mathcal{G}^f$.

Théorème 4.- Si l'anneau $\mathcal{A}$ est de caractéristique nulle (resp. non nulle ℓ) et si l'on pose $\mathbb{Z} \cap f\mathcal{A} = f_1.\mathbb{Z}$ où $f_1 = 0$ ou un nombre premier de $\mathbb{N}$ alors $\mathcal{G}^f/\mathcal{G}_f$ est un groupe de f_1-torsion (resp. ℓ-torsion). De plus, la torsion est bornée par f_1^{p-2} (resp. $\ell^{\{\mathrm{Log}_\ell(p-1)\}}$, où $\{\alpha\}$ désigne le plus petit entier supérieur ou égal à α).

Preuve : On la décompose en trois :

1) Caractéristique de $\mathcal{A}$ nulle et $f_1 = 0$

Soit $\varphi = u_o + u_1\Delta + \ldots + u_{p-1}\Delta^{p-1} \in \mathcal{G}^f$. On suppose $\varphi^q \in \mathcal{G}_f$ pour un certain entier $q \geqslant 1$. On montre alors par récurrence sur i la propriété $((f^j/u_j,\ \forall j,\ 1 \leqslant j \leqslant i)$ et $(f^i/u_j,\ \forall j \geqslant i))$ ce qui entraîne que $\varphi \in \mathcal{G}_f$.

2) Caractéristique de $\mathcal{A}$ nulle et $f_1 \neq 0$

Lemme 2.- Si on pose $\mathcal{G}(\alpha) = \mathcal{A}^* + f^\alpha \mathcal{A}[\Delta]$, $\alpha \in \mathbb{N}^*$ alors $\mathcal{G}(\alpha)$ est un groupe multiplicatif et $\mathcal{G}(\alpha)/\mathcal{G}(\alpha+1)$ est un $\mathbb{F}_{f_1}$-espace vectoriel.

Soit $\varphi \in \mathcal{G}^f = \mathcal{G}(1)$, il résulte du lemme 2 que $\varphi^{f_1^{p-2}} \in \mathcal{G}(p-1) \subset \mathcal{G}_f$.

3) Caractéristique de $\mathcal{A}$ non nulle

La démonstration est similaire à la précédente.

III. Applications

Hypothèses et notations

Soient : k un corps de caractéristique ℓ.

$A = \mathbb{Z}$ ou $k[X]$.

K le corps des fractions de A.

E une extension algébrique séparable de K de degré p, engendrée par un élément Δ entier sur A vérifiant l'équation :

$$\Delta^p = b_{p-1}\Delta^{p-1} + \ldots + b_o,\ b_i \in A,\ 0 \leqslant i \leqslant p-1.$$

B la clôture intégrale de A dans E.

f un élément irréductible de A.

G (resp. U) le groupe des unités (resp. unités de norme 1) de l'ordre $O = A[\Delta]$.

G^f (resp. U^f) le groupe des unités (resp. unités de norme 1) de l'ordre $O^f = A + fO$.

G_f (resp. U_f) le groupe des unités (resp. unités de norme 1) de l'ordre $O_f = A[f\Delta]$.

$fB = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ la décomposition de l'idéal fB en idéaux premiers distincts de B si $f \nmid \mathrm{Dis}_{E/K}(B)$,

$\beta_i = \left[{}^{B}/_{\mathfrak{p}_i} : {}^{A}/_{fA} \right]$,

s l'homomorphisme canonique d'anneaux $O \longrightarrow {}^{B}/_{fB} \simeq {}^{B}/_{\mathfrak{p}_1} \times \ldots \times {}^{B}/_{\mathfrak{p}_r}$,

$s_i = pr_i \circ s$, $pr_i : {}^{B}/_{fB} \longrightarrow {}^{B}/_{\mathfrak{p}_i}$.

1. Corps Globaux Généraux

Théorème 5.-

1. Pour tout f ne divisant pas $\mathrm{Disc}_{E/K}(\Delta)$

(i) ${}^{G}/_{G^f}$ s'injecte dans $({}^{B}/_{fB})^* / ({}^{A}/_{fA})^*$

(ii) ${}^{G}/_{G^f}$ est un groupe de torsion dont la torsion est première à $|f|$ et bornée par $|f|^{\gamma} - 1$, où $\gamma = \mathrm{ppcm}(\beta_i)$ et $|f|$ est la valeur absolue usuelle si $A = \mathbf{Z}$ et $q^{\deg f}$ si $A = \mathbb{F}_q[X]$

2. Pour tout élément irréductible f de A de caractéristique nulle (resp. ℓ non nulle), ${}^{G^f}/_{G_f}$ est un groupe de $|f|$ (resp. ℓ) torsion dont la torsion est bornée par $|f|^{p-2}$ (resp. $\ell^{\{\mathrm{Log}_{\ell}(p-1)\}}$).

Preuve :

1. On considère l'anneau $\mathcal{A} = A_{(f)}$ qui est le localisé de A par rapport à l'idéal premier fA. D'autre part pour tout $f \nmid \mathrm{Disc}_{E/K}(\Delta) / \mathrm{Disc}_{E/K}(B)$ on a $B_{(f)} = A_{(f)}[\Delta]$ (voir [2]), alors grâce au théorème 3.2) et à l'égalité $G^f = G \cap \mathcal{A}^f$, on obtient les monomorphismes multiplicatifs :

$$G/_{G^f} \hookrightarrow \mathcal{G}/_{\mathcal{G}^f} \hookrightarrow \left(B_{(f)}/_{fB_{(f)}}\right)^* \Big/ \left(A_{(f)}/_{fA_{(f)}}\right)^* \simeq \left(B/_{fB}\right)^* \Big/ \left(A/_{fA}\right)^*$$

2. On montre que $G_f = G^f \cap \mathcal{G}_f$ et on utilise le théorème 4.

2. Corps de fonctions d'une variable

Théorème 6.-

1. Pour tout f ne divisant pas $\mathrm{Disc}_{E/K}(\Delta)$

(i) U^f est non trivial si et seulement si il existe une unité non triviale φ de U telle que pour tout i $(1 \leqslant i \leqslant r)$, $s_i(\varphi)$ est une racine de l'unité du corps $B/_{\mathfrak{P}_i}$.

(ii) Si le rang de U est égal à un, si U^f est non trivial et si on pose

$$m_0 = \mathrm{Inf}\{m \in \mathbb{N}^*,\ s(\varphi_0^m) \in A/_{fA}\}$$

alors

$$m_0 = [U : U^f]$$

2. Si la caractéristique du corps k est nulle (resp. ℓ non nulle) alors le groupe $G^f/_{G_f}$ est sans torsion (resp. est un groupe de ℓ-torsion dont la torsion est bornée par $\ell^{\{\mathrm{Log}_\ell(p-1)\}}$).

La preuve de ce théorème est similaire à celle du théorème 5.

Dans le cas particulier où Δ vérifie une équation binôme $\Delta^p = D(X)$, nous pouvons compléter le théorème 6 par le suivant :

Théorème 7.- On suppose que Δ vérifie $\Delta^p = D$, $D \in k[X]$. Si $f|D$ et si la caractéristique du corps k est nulle (resp. ℓ non nulle) alors le groupe $G/_{G^f}$ est sans torsion (resp. de ℓ-torsion dont la torsion est bornée par $\ell^{\{\mathrm{Log}_\ell(p)\}}$).

L'idée de la démonstration, dans le cas de la caractéristique nulle, est la même que celle du théorème 4.

Corollaire.-

Soit D un polynôme de la forme X^3+aX+b avec a et b rationnels. Soit Δ un élément de $\mathbb{Q}((\frac{1}{X}))$ vérifiant $\Delta^3 = D$. Si on pose $A = k[X]$ et $0 = A[\Delta]$ alors le groupe des unités de l'ordre $0_{X-\alpha}$ est trivial quel que soit le rationnel α.

Preuve : Elle résulte des théorèmes 6 et 7 et du fait que l'on connait les couples (a,b) de rationnels [1] pour lesquels le groupe U des unités de norme 1 de 0 est non trivial.

Les théorèmes 6 et 7 sont une généralisation du théorème 4. II de [3].

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SUR LE GROUPE DES CLASSES DES ANNEAUX "D + M"

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Presented by G. deB. Robinson, F.R.S.C.

Cette note résume les résultats sans démonstration d'un article à paraître, extrait de la thèse de l'auteur.

Tous les anneaux sont intègres, commutatifs et unitaires.

Soient $A = K + M$ où K est un corps, M un idéal maximal de A et $R = D + M$ où D est un sous-anneau propre de K. On démontre que, si $K = \mathrm{Frac}(D)$, on a le diagramme commutatif suivant avec lignes et colonnes exactes :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Pic}(D) \arrow[r, "\alpha'"] \arrow[d] & \mathrm{Pic}(R) \arrow[r, "\beta'"] \arrow[d] & \mathrm{Pic}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C\ell(D) \arrow[r, "\alpha"] \arrow[d] & C\ell(R) \arrow[r, "\beta"] \arrow[d] & C\ell(A) \arrow[d] & \\
0 \arrow[r] & G(D) \arrow[r, "\alpha''"] \arrow[d] & G(R) \arrow[r, "\beta''"] \arrow[d] & G(A) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

où la suite $0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(A) \longrightarrow 0$ est scindée. On donne ensuite une condition nécessaire et suffisante pour que R soit pseudo-prüférien et on montre que les homomorphismes β et β'' sont alors surjectifs. Ces théorèmes permettent la généralisation de certains résultats classiques, la construction de nombreux exemples et contre-exemples et le calcul explicite du groupe des classes et du groupe local des classes d'anneaux tout à fait généraux.

§ 1. LE GROUPE DES CLASSES DES "D + M".

Soit A un anneau intègre. Comme dans [3] et [5], on note $T(A)$ le groupe des t-idéaux t-inversibles de A et on appelle *groupe des classes* (resp. *groupe local des classes*) de A, le groupe $C\ell(A) = T(A)/_{P(A)}$ (resp. $G(A) = T(A)/_{P(A)}$), de sorte que, si $\mathrm{Pic}(A) = \mathrm{Cart}(A)/_{P(A)}$ désigne *le groupe de Picard* de A, il existe une suite exacte de groupes :

$$0 \longrightarrow \mathrm{Pic}(A) \longrightarrow C\ell(A) \longrightarrow G(A) \longrightarrow 0 .$$

Un anneau A est *pseudo-prüférien* ou (PVMD) si l'ensemble $D_f(A)$ des idéaux divisoriels de type fini est un groupe pour la v-loi : $(I,J) \longmapsto (IJ)_v$. On dit qu'un anneau A est de *Gauss généralisé* (ou G - GCD domaine) [1], si tout idéal divisoriel de type fini est inversible.

Soit A un anneau pseudo-prüférien. Alors A est un anneau de Gauss (resp. de Gauss généralisé) si et seulement si $C\ell(A) = 0$ (resp. $G(A) = 0$) ([3], prop. 2).

LEMME (1.1). — Soient A *un anneau et B une extension plate de A. Alors, il existe un homomorphisme canonique de groupes* : $C\ell(A) \longrightarrow C\ell(B)$ *défini par* $[I] \longmapsto [IB]$.

THEOREMME (1.2). — Soient $A = K + M$ *où K est un corps, M un idéal maximal de A et* $R = D + M$ *où D est un sous-anneau propre de K. Alors,*

1) *Pour tout idéal fractionnaire I de D, on a* $(IR)_v = I_v R$.

2) Si $K = \mathrm{Frac}(D)$, *on a le diagramme commutatif suivant avec lignes et colonnes exactes* :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Pic}(D) \arrow[r, "\alpha'"] \arrow[d] & \mathrm{Pic}(R) \arrow[r, "\beta'"] \arrow[d] & \mathrm{Pic}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C\ell(D) \arrow[r, "\alpha"] \arrow[d] & C\ell(R) \arrow[r, "\beta"] \arrow[d] & C\ell(A) \arrow[d] & \\
0 \arrow[r] & G(D) \arrow[r, "\alpha''"] \arrow[d] & G(R) \arrow[r, "\beta''"] \arrow[d] & G(A) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

où la suite $0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(A) \longrightarrow 0$ *est scindée*.

3) L'anneau R est pseudo-prüférien si et seulement si A est pseudo-prüférien, $K = \mathrm{Frac}(D)$, D est pseudo-prüférien et A_M est de valuation.

4) Si R est pseudo-prüférien, les homomorphismes β et β'' sont surjectifs.

§ 2. APPLICATIONS ET EXEMPLES.

Les hypothèses sont celles du théorème (1.2).

PROPOSITION (2.1). — R est de Prüfer si et seulement si A est de Prüfer, $K = \mathrm{Frac}(D)$, D est de Prüfer et on a la suite exacte scindée :

$$0 \longrightarrow C\ell(D) \longrightarrow C\ell(R) \longrightarrow C\ell(A) \longrightarrow (0)\ .$$

PROPOSITION (2.2). — R est de Bezout si et seulement si A est de Bezout, $K = \mathrm{Frac}(D)$, et D est de Bezout.

PROPOSITION (2.3). — R est de Gauss (resp. de Gauss généralisé) si et seulement si A et D sont de Gauss (resp. de Gauss généralisé), $K = \mathrm{Frac}(D)$ et A_M est de valuation.

On retrouve, en particulier [6], Proposition 6, Théorèmes 5, 7 et 11, [1], Théorème 4.

PROPOSITION (2.4). — Si $K = \mathrm{Frac}(D)$ et si $C\ell(A) = 0$, alors on a la diagramme commutatif

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(D) \arrow[r] \arrow[d, "\alpha'"'] & C\ell(D) \arrow[r] \arrow[d, "\alpha"'] & G(D) \arrow[r] \arrow[d, "\alpha''"'] & 0 \\
0 \arrow[r] & \mathrm{Pic}(R) \arrow[r] & C\ell(R) \arrow[r] & G(R) \arrow[r] & 0
\end{tikzcd}$$

où α', α et α'' sont des isomorphismes.

EXEMPLES. — Ce résultat s'applique, en particulier, au cas où A est un anneau de Gauss (par exemple A est de valuation ou $A = K[X]$) avec $K = \mathrm{Frac}(D)$. Ainsi, le groupe des classes (resp. local des classes) de l'anneau $\mathbb{Z}[i\sqrt{5}] + (X_1,\dots,X_n)\,\mathbb{Q}(i\sqrt{5})[X_1,\dots,X_n]$ est $\mathbb{Z}/_{2\mathbb{Z}}$ (resp. 0).

PROPOSITION (2.5). — Tout groupe abélien G est le groupe des classes d'un anneau pseudo-prüférien de dimension 2 qui est ni de Krull ni noethérien.

On voit que, de même qu'il existe des anneaux de Dedekind aussi peu factoriels que possible, il existe des anneaux pseudo-prüfériens, non de Krull et non noethériens, aussi éloignés que l'on veut d'un anneau de Gauss.

D'une manière analogue on a,

PROPOSITION (2.6). — Tout groupe abélien G est le groupe local des classes d'un anneau pseudo-prüférien de dimension 2, ni de Krull, ni noethérien.

PROPOSITION (2.7). — Soit D un anneau de Krull tel que $0 \neq \mathrm{Pic}(D) \subsetneq C\ell(D)$. Alors, $R = D + (X,Y)(\mathrm{Frac}\ D)[X,Y]$ est un anneau vérifiant :

$$P(R) \subsetneq \mathrm{Cart}(R) \subsetneq T(R) \subsetneq D_f(R) .$$

EXEMPLE (2.8). — Considérons l'anneau de Krull gradué

$$D = \mathbb{Z}[i\sqrt{5}][U,V,W,T] / (UV - WT) .$$

On a,

$$\mathrm{Pic}(D) = \mathbb{Z}/_{2\mathbb{Z}} \quad \text{et} \quad C\ell(D) = \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z} .$$

Soient $K = \mathrm{Frac}(D)$, $A = K[X,Y] / (X^2 + Y^2 - 1) = K[x,y]$ et $R = D + (x,y-1)K[x,y]$.

Alors,

$$C\ell(R) = (\mathbb{Z}/_{2\mathbb{Z}})^2 \times \mathbb{Z} \quad \text{et} \quad G(R) = \mathbb{Z} .$$

R est un exemple d'anneau pseudo-prüférien que l'on peut qualifier de non "presque de Gauss" (resp. non "presque localement de Gauss"), puisque son groupe des classes (resp. local des classes) n'est pas de torsion.

EXEMPLE (2-9). — Soient $D = \mathbb{Z}[i\sqrt{5}][U,V,W,T]/(UV - WT)$, $K = \mathrm{Frac}(D)$, $A = K[X,Y]/(Y^2 - X^3) = K[x,y]$ et $R = D + (x,y)K[x,y]$.

Alors,

$$C\ell(R) = \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z} \times K \quad \text{et} \quad G(R) = \mathbb{Z} .$$

R est un exemple d'anneau non semi-normal, vérifiant les inclusions strictes :

$$P(R) \subsetneq \mathrm{Cart}(R) \subsetneq T(R) \subsetneq D_f(R) .$$

R est aussi un exemple d'anneau non pseudo-prüférien pour lequel l'homomorphisme $C\ell(R) \longrightarrow C\ell(S^{-1}R)$ est surjectif.

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DEFAUT DE COMPLETUDE ET MULTISECANTE

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Résumé

Soit X une courbe lisse de degré d de P^3(espace projectif de dimension 3 sur $\mathbb{C}$).La série linéaire découpée sur X par les surfaces de degré d-5(resp d-6) est incomplète si et seulement X a une (d-3)-sécante(resp (d-4)-sécante)

§0. Soit X une courbe lisse irréductible de degré d et de genre g de P^3.On note J_X l'idéal définissant X dans $\mathcal{O}_{P^3}$ .Si F est un faisceau cohérent ,on note $H^i(F)$ le ième $\mathbb{C}$-espace vectoriel de cohomologie et $h^i(F) = \dim_{\mathbb{C}} H^i(F)$ .La série linéaire découpée sur X par les surfaces de degré n est complète si $h^1(J_X(n)) = 0$. Dans[3] on montre que $h^1(J_X(n))=0$ pour $n \geqslant d-2$.Dans [6] on montre que $h^1(J_X(d-3)) > 0$ si et seulement si X admet une (d-1)-sécante.Dans [1]on montre que $h^1(J_X(d-4)) > 0$ si et seulement si X admet une (d-2)-sécante sauf pour trois valeurs du couple (d,g) .On s'interesse ici à la conjecture "$h^1(J_X(n) > 0$ si et seulement si X a une (n+2)-sécante" pour n=d-5 et n=d-6.

§1.

Proposition 1 : Soit $X \subset P^3$ une courbe lisse irréductible de degré d.On pose $M=\Omega^1{}_{P^3}(1) \otimes \mathcal{O}_X$.Si $h^1(J_X(n)) > 0$ pour $n > 2d/3 - 1$ alors X a une (n+2)-sécante ou M admet un sous-fibré de rang deux noté N ayant les propriétés:

a) Le degré de N est strictement supérieur à $2(n+1-d)$.

b) Tout quotient inversible de N a un degré strictement supérieur à $(n+1-d)$.

Démonstration: Voir [1] Proposition 1.1 C.Q.F.D

Si X vérifie les hypothèses de la proposition ,X a une $(n+2)$-sécante ou bien il existe N avec les propriétés indiquées.On pose $V=H^0(\mathcal{O}_{P^3}(1))$.On a la suite exacte

$$0\to M\to V\otimes\mathcal{O}_X\to\mathcal{O}_X(1)\to 0$$

On pose $E=V\otimes\mathcal{O}_X/N$,$\tilde{S}=P(E)$, $\pi:\tilde{S}\to X$ la projection et $\mathcal{O}_{\tilde{S}}(1)$ le quotient tautologique de π^*E .La flèche composée

$$V\otimes\mathcal{O}_{\tilde{S}}\longrightarrow\pi^*E\longrightarrow\mathcal{O}_{\tilde{S}}(1)\to 0$$

définit un morphisme $q:\tilde{S}\longrightarrow P^3$ de degré (deg E)=-deg N
La flèche naturelle $E\to\mathcal{O}_X(1)$ définit une section non nulle de $\pi^*(\Lambda^2E^{\vee}(1))\otimes\mathcal{O}_{\tilde{S}}(1)$ dont la courbe des zéros est isomorphe par π à l'inclusion de X dans P^3.X est donc tracée sur la surface réglée S de P^3,image de $\tilde{S}$ par q.
On associe à q un morphisme $\bar{q}$ de X dans la grassmannienne G des droites de P^3:à $x\in X$ correspond le point de G associé à la droite $\pi^{-1}(x)$ plongée par q dans P^3.
Le morphisme composé de $\bar{q}$ et de l'inclusion de G dans P^5 est défini par la flèche $\Lambda^2V\otimes\mathcal{O}_X\to\Lambda^2E$;il est de degré deg(E) .

§ 2.

Proposition 2:Soient S une surface de P^3 de degré s dont la normalisée $\tilde{S}$ est lisse,J_Γ le conducteur de $\mathcal{O}_{\tilde{S}}$ dans $\mathcal{O}_S$,Γ la courbe (non nécessairement réduite) de P^3 d'idéal J_Γ ,$\tilde{X}$ une courbe tracée sur $\tilde{S}$,section d'un $\mathcal{O}_{\tilde{S}}$-module L .

Soit X l'image de $\tilde{X}$ dans P^3. Notons J_X l'idéal de X dans P^3 et $J_{X/S}$ l'idéal de X dans S. Si $h^1(J_X(n)) > 0$ alors $h^1(L(s-n-4)) > 0$ ou $h^1(J_{X/S} \otimes \mathcal{O}_\Gamma(n)) > 0$.

Démonstration: Soit $p: \tilde{S} \longrightarrow S$ la normalisation. On a $J_\Gamma = \underline{\mathrm{Hom}}\ (p_* \mathcal{O}_{\tilde{S}}, \mathcal{O}_S)$. La dualité pour les morphismes finis donne $J_\Gamma = p_* \omega_{\tilde{S}} \otimes \omega_S{}^v$. La suite exacte $0 \rightarrow \mathcal{O}_{P^3}(-s) \rightarrow J_X \rightarrow J_{X/S} \rightarrow 0$ donne $H^1(J_X(n)) = H^1(J_{X/S}(n))$. La suite exacte

$$0 \rightarrow J_\Gamma \otimes J_{X/S}(n) \rightarrow J_{X/S}(n) \rightarrow J_{X/S} \otimes \mathcal{O}_\Gamma(n) \rightarrow 0$$

et la suite spectrale de p (dégénérée car p est fini) permettent de conclure.

§ 3. $h^1(J_X(d-5)) > 0$

Soit X une courbe lisse irréductible de degré $d > 12$ telle que $h^1(J_X(d-5)) > 0$. X a une (d-3)-sécante ou il existe un morphisme $\bar{q}$ de degré inférieur ou égal à sept. Si $\bar{q}$ n'est pas birationnel le degré de la surface réglée S est inférieur ou égal à trois. Dans ce cas on sait que pour $n > d/2 - 1$ $h^1(J_X(n)) > 0$ si et seulement si il existe une (n+2)-sécante [2] . Si $\bar{q}$ est birationnel ,on distingue les cas suivants:

a) $\bar{q}(X)$ est plane: S est alors un cône de degré inférieur ou égal à sept ou une quadrique lisse selon que le plan de $\bar{q}(X)$ est ou n'est pas contenu dans G. Dans le premier cas X a un degré inférieur ou égal à huit. Dans le second cas une famille de génératrices de la quadrique est formée d'unisécantes; l'autre est donc formée de (d-1)-sécantes.

b) $\bar{q}(X)$ est non plane: Le degré de $\bar{q}(X)$ étant inférieur ou égal à sept on a $g \leq 6$ ([3])

§4. $h^1(J_X(d-6)) > 0$

Soit X une courbe lisse irréductible de degré $d > 15$ telle que $h^1(J_X(d-6)) > 0$ .X a une (d-4)-sécante ou il existe un morphisme $\bar{q}$ de degré ≤ 9 .Si $\bar{q}$ est birationnel on a 2 cas :

a) $\bar{q}(X)$ est plane:X a un degré inférieur ou égal à dix ou bien X a une famille de (d-1)-sécantes.

b) $\bar{q}(X)$ n'est pas plane: On a alors $g \leq 12$.

On suppose maintenant que $\bar{q}$ n'est pas birationnel.S est alors une surface de degré inférieur ou égal à quatre. Il reste donc à considérer le cas où $\bar{q}$ est un revêtement double d'une courbe de degré quatre.Une section plane de S est alors rationnelle ou elliptique .Dans le premier cas le lieu singulier de S est une cubique gauche (éventuellement dégénérée).Dans le second cas le lieu singulier est la réunion de deux droites disjointes (pouvant se spécialiser en une droite double).On utilise la proposition 2 dans le contexte suivant:Soit C une courbe lisse de genre p(p=0 ou p=1);E un $\mathcal{O}_C$-module localement libre de rang 2 et de degré s (s=4) .On pose $\tilde{S} = \mathbf{P}(E)$, $\pi : \tilde{S} \longrightarrow C$ la projection et $\mathcal{O}_{\tilde{S}}(1)$ le quotient tautologique de π^*E .On identifie $\mathrm{Pic}(X) \oplus \mathbb{Z}$ et $\mathrm{Pic}(\tilde{S})$ par la flèche $(A,k) \longrightarrow \pi^*A \otimes \mathcal{O}_{\tilde{S}}(k)$ .La matrice de la forme d'intersection est alors:

$$\begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix}$$

Le $\mathcal{O}_{\tilde{S}}$-module canonique est $\omega_{\tilde{S}} = \pi^*(\omega_C \otimes \Lambda^2 E) \otimes \mathcal{O}_{\tilde{S}}(-2)$
On note $\tilde{X}$ la transformée stricte de X dans $\tilde{S}$ et k le nombre de points d'intersection de X et d'une génératrice de S.

$$L = \pi^*((\Lambda^2 E^\vee)^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(1)) \otimes \mathcal{O}_{\tilde{S}}(k)$$

Dans notre cas k = 2.On vérifie que $h^1(L(6-d)) = 0$.
(on utilise la suite spectrale de Leray associée à π qui dégénère car C et les fibres de π sont de dimension un)
La transformée de Γ dans $\tilde{S}$ est une section de $\omega_{\tilde{S}}^\vee$.
Si S est une surface réglée elliptique ;chacune des droites du lieu singulier est une (d-4)-sécante car

$$(-2,1)\begin{pmatrix}0 & 1\\ 1 & 4\end{pmatrix}\begin{pmatrix}d-8\\ 2\end{pmatrix} = d-4$$

Si S est réglée rationnelle le lieu singulier est une cubique gauche.Si celle ci se décompose ,elle contient une droite section de $\mathcal{O}_{\tilde{S}}(-2,1)$ ([1]) . Si elle ne se décompose pas $J_{X/S} \otimes \mathcal{O}_\Gamma(d-6)$ est un $\mathcal{O}_\Gamma$-module inversible de degré $3(d-6)-(2d-4) = d-14$.On a donc $h^1(J_X(d-6)) = 0$ pour $d \geqslant 13$ dans le cas non dégénéré et une (d-4)-sécante dans le cas dégénéré.

§5.

On suppose maintenant que X a une (n+2)-sécante .D'après [6] J_X est alors (n+1)-irrégulier .Il en résulte ([7]) que $h^1(J_X(n)) > 0$ ou $h^1(\mathcal{O}_X(n-1)) > 0$ .Mais $h^1(\mathcal{O}_X(n-1))$ est nul pour $n > d/2 - 1$ ([4]).On a donc démontré le

Théorème: Soit X une courbe lisse irréductible de degré d et de genre g de P^3. Si $d \geqslant 13$ et $g \geqslant 7$, $h^1(J_X(d-5))>0$ si et seulement si X a une (d-3)-sécante. Si $d \geqslant 16$ et $g \geqslant 13$ $h^1(J_X(d-6)) > 0$ si et seulement si X a une (d-4)-sécante.

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AN INTERVAL OF FINITE CLONES ISOMORPHIC TO $(P(\mathbb{N}), \subseteq)$

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Presented by G. Grätzer, F.R.S.C.

Abstract

We exhibit an interval of the lattice of clones on a finite universe of cardinality $k > 2$ which is lattice isomorphic to the lattice P of subsets of $\mathbb{N}$.

Introduction

Let k be a positive integer and $\underline{k} = \{0, \ldots, k-1\}$. Let $\underset{\sim}{O}_k$ denote the set of all finitary operations on $\underline{k}$ (i.e. maps $\underline{k}^n \to k$, $n = 0, 1, \ldots$). A clone on $\underline{k}$ is a composition closed set of finitary operations on $\underline{k}$ containing all projections. It is known that the clones on $\underline{k}$, ordered by inclusion, form an algebraic lattice L_k in which an arbitrary meet is just the (set-theoretical) intersection. For $k = 2$ the lattice is countable and completely known (Post 1941 [6]), while for $k > 2$ it is largely unknown. An interval I of L_k is large if there is an order embedding of the lattice $\underset{\sim}{P}$ of subsets of $\mathbf{P} = \{1, 2, \ldots\}$ (ordered by conclusion) into I. If L_k has a large interval, then $|L_k| = 2^{\aleph_0}$ (as a collection of subsets of the countable set $\underset{\sim}{O}_k$ clearly $|L_k| \leq 2^{\aleph_0}$), in fact L_k contains even chains and antichains of cardinality $2^{\aleph_0}$.

In 1959 Janov and Mučnik [4] showed that for $k > 2$ the lattice L_k has a large interval.

In the sequel $k > 2$ and h is a positive integer. Let ρ be an h-ary relation on $\underline{k}$ (i.e. a subset of $\underline{k}^h$). We say that an n-ary operation f on $\underline{k}$ (i.e. a map $\underline{k}^n \to \underline{k}$) preserves ρ if $(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{h1}, \ldots, a_{hn})) \in \rho$ whenever $(a_{11}, \ldots, a_{h1}) \in \rho, \ldots, (a_{1n}, \ldots, a_{hn}) \in \rho$ (other names: f is compatible with ρ, f homomorphism $\rho^n \to \rho$, f invariant of ρ or ρ subuniverse of $< \underline{k}, f >^h$). The set of all $f \in \underset{\sim}{O}_k$ preserving ρ is denoted Pol ρ (which is always a clone). For $n = 1, 2, \dots$, put

$$\rho_n = (\underline{k}^n \setminus \{0, 1\}^n) \cup \{(x_1, \dots, x_n) \in \{0, 1\}^n : x_1 + \dots + x_n = 1\} \ ,$$

$E = \bigcap_{i=1}^{\infty} \mathrm{Pol}\, \rho_i$, $F = \bigvee_{i \geq 1}^{\infty} \mathrm{Pol}\, \rho_i$, $C = \{ \bigcap_{i \in X} \mathrm{Pol}\, \rho_i \ : X \subseteq P\}$. It is known that $|C| = 2^{\aleph_0}$, hence the interval $[E, F]$ is large ([4], cf [2]). The question arises whether C is a sublattice of L_k. To show that the answer is negative, we determine the structure of the interval $[E, F]$ of L_k. Another question arises whether there is an order isomorphism of P onto an interval of L_k, i.e. whether L_k has an interval which is a copy of P. We prove that this is the case, in fact, we find a clone T so that the interval $[T \cap E, \ T \cap F]$ is a copy of P.

An h-ary relation τ on $\underline{k}$ is <u>irredundant</u> if for all $1 \leq i < j \leq h$ there is $(a_1, \dots, a_h) \in \tau$ with $a_i \neq a_j$. Let τ be an h-ary irredundant relation on $\underline{k}$ and $\emptyset \neq I \subseteq P$ such that $\bigcap_{i \in I} \mathrm{Pol}\, \rho_i \subseteq \mathrm{Pol}\, \tau$. Put $H = \{1, \dots, h\}$. From the general theory (cf [5] p. 45-55) and the fact that $(a_1, \dots, a_n) \in \rho_n$ whenever at least one $a_i > 1$ it may be deduced that there exists a finite set Φ of maps $\varphi : \{1, \dots, a_\varphi\} \to H$ such that

$$\tau = \{(x_1, \dots, x_h) \in \underline{k}^h : (x_{\varphi(1)}, \dots, x_{\varphi(a\varphi)} \in \rho_{a\varphi} \text{ for all } \varphi \in \Phi\} \ .$$

For the ease of expression, we say that τ <u>belongs</u> to Φ. Due to the fact that all ρ_i are totally symetric (i.e. invariant under all permutations of coordinates or places), we may assume that all $\varphi \in \Phi$ are non-decreasing (i.e. $\varphi(1) \leq \dots \leq \varphi(a_\varphi)$). We say that $\varphi \in \Phi$ is <u>minimal</u> (for Φ) if

$$\operatorname{Im}\psi \subseteq \operatorname{Im}\varphi \Rightarrow \psi = \varphi$$

holds for all $\psi \in \Phi$. An injective and minimal (for Φ) map is said to be *intrinsic* (for Φ). We say that $\varphi \in \Phi$ is *almost injective* if $|\varphi^{-1}(i)| > 1$ for at most one $i \in H$. We need a few technical lemmas.

Lemma 1. Let τ *belong to* Φ. *Let* Λ *denote the set of intrinsic maps of* Φ *and let* $m = \max\{|\operatorname{Im}\varphi|, \varphi \in \Phi \setminus \Lambda\}$. *Then for each* $b > m$ *the relation* τ *belongs to a set* Ψ *of non-decreasing maps such that:*

i) Λ *is the set of intrinsic maps of* Ψ,

ii) *All maps from* $\Psi \setminus \Lambda$ *are almost injective maps from* $\{1, \dots, b\}$ *into* H *such that*

$$\{|\operatorname{Im}\varphi| : \varphi \in \Phi \setminus \Lambda\} = \{|\operatorname{Im}\psi| : \psi \in \Psi \setminus \Lambda\} .$$

For $h \geq 1$, put

$$\sigma_h = \{(x_1, \dots, x_h) : (x_1, x_1, \dots, x_h) \in \rho_{h+1}\} .$$

For a set P of positive integers let $\max P$ denote the greatest integer of P if $0 < |P| < \aleph_0$, $\max P = 0$ if $P = \emptyset$ and $\max P = \aleph_0$ if P is infinite.

Lemma 2. Let Φ *be a finite family of maps* $\varphi : \{1, \dots, a_\varphi\} \rightarrow H$, *(where all* a_φ *are positive integers) and let*

$$I = \{|\operatorname{Im} r| : r \text{ intrinsic for } \Phi\}$$

$$M = \{|\operatorname{Im} u| : u \text{ non-injective and minimal for } \Phi\}$$

$$\tau = \{(x_1, \dots, x_h) \in \underline{k}^h : (x_{\varphi(1)}, \dots, x_{\varphi(a_\varphi)}) \in \rho_{a_\varphi} \text{ for all } \varphi \in \Phi\} .$$

Then

$$\operatorname{Pol}\tau \subseteq \bigcap_{i \in I} \operatorname{Pol}\rho_i \quad . \qquad (*)$$

Moreover, if $M \neq \emptyset$ *and* $m = \operatorname{Max} M$, *then*

$$\mathrm{Pol}\,\tau \subseteq \mathrm{Pol}\,\sigma_m \ .$$

(For $I = \emptyset$, the right side of (*) equals $\underset{\sim}{O}_k$).

Corollary 3. Let Φ, I, M, m *and* τ *be as in Lemma 2. Put*

$i = |I|$, $n = \max\{|\mathrm{Im}\varphi| : \varphi \in \Phi, \varphi$ *non-injective*$\}$.

Then

$$1) \qquad i > n \Rightarrow \mathrm{Pol}\,\tau \subseteq \bigcap_{i\in I} \mathrm{Pol}\,\rho_i$$

$$2) \quad m = n > i \Rightarrow \mathrm{Pol}\,\tau = \mathrm{Pol}\,\sigma_m \cap \Big(\bigcap_{j\in I} \mathrm{Pol}\,\rho_j\Big) \ .$$

Lemma 4. *Let* Y *be an index set, let* $U_y \subseteq P$ *for all* $y \in Y$, *and let*

$U = \bigcap_{y\in Y} U_y$. *If* $\max U \geq \max U_z$ *for some* $z \in Y$, *then*

$$\bigvee_{y\in Y} \Big(\bigcap_{u\in U_y} \mathrm{Pol}\,\rho_u\Big) = \bigcap_{u\in U} \mathrm{Pol}\,\rho_u \ .$$

Consider the map χ from P into [E, F] defined by setting

$$\chi(X) = \bigcap_{i\in P\setminus X} \mathrm{Pol}\,\rho_i \quad \text{for all} \quad X \subseteq P \ .$$

χ is an order embedding of P into the interval [E, F]. However χ is not surjective because of the three strict inclusions:

$$\mathrm{Pol}\,\rho_3 \cap \mathrm{Pol}\,\rho_2 \subset \mathrm{Pol}\,\sigma_2 \cap \mathrm{Pol}\,\rho_2 \subset \mathrm{Pol}\,\rho_2 \ ,$$

therefore $\chi(P)$ is not a sublattice of $L_{\underline{k}}$ and [E, F] is not a copy of P. We shall remedy to this by modifying χ:

For a clone T, put $\chi_T(X) = T \cap \chi(X)$ for each $X \subseteq \mathbf{P}$. We have

Theorem 5. Let $E = \bigcap_{i=1}^{\infty} \mathrm{Pol}\,\rho_i$, $T = \bigcap_{m=1}^{\infty} \mathrm{Pol}\,\sigma_m$ *and put*

$\chi_T(X) = T \cap \Big(\bigcap_{x\in P\setminus X} \mathrm{Pol}\,\rho_x\Big)$ *for all* $X \subseteq \mathbf{P}$. *Then* χ_T *is a lattice isomorphism from the lattice* $(\mathbf{P}, \subseteq)$ (of the subsets of $\mathbf{P}$) *onto the interval* [E, T] *of the lattices of clones* (i.e. [E, T] is a copy of P).

In other words, each clone C between E and T is of the form

$T \cap (\bigcap_{x \in X} \mathrm{Pol}\, \rho_x)$, for some $X \subseteq \mathbf{P}$.

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ERDÖS-MORDELL'S AND RELATED INEQUALITIES

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In memory of Professor Paul R. Beesack

Presented by H.S.M. Coxeter, F.R.S.C.

Abstract. In this paper we give some generalizations of the important Erdős-Mordell inequality.

1. In a triangle ABC, let $R_1, R_2, R_3, r_1, r_2, r_3$ denote the distances of a point P from the vertices and sides, as in [1]. We shall prove the following generalization of the inequalities 12.16 of [1, p. 107]:

THEOREM 1. If t is a real number, we have

$$(1)\qquad \Sigma R_1{}^t \geqq 2^{t-1}\Sigma((b/c)^t+(c/b)^t)r_1{}^t \geqq 2^t\Sigma r_1{}^t \qquad (0<t\leqq 1),$$

$$(2)\qquad \Sigma R_1{}^t > \Sigma((b/c)^t+(c/b)^t)r_1{}^t \geqq 2\Sigma r_1{}^t \qquad (t>1).$$

PROOF. It is known that

$$(3)\qquad R_1 \geqq (c/a)r_2+(b/a)r_3,\ \text{etc.}$$

with equality in all inequalities only if P is the circumcentre.

To prove the first inequality in (1) we shall apply to (3) the elementary inequality: $(\frac{u+v}{2})^t \geqq \frac{u^t+v^t}{2}$ $(u,v>0,\ 0<t\leqq 1)$; equality occurring only if $u=v$ (for $t\neq 1$ of course).

Thus, for $0<t\leqq 1$, (3) yields

$$R_1{}^t \geqq 2^{t-1}((c/a)^t r_2{}^t+(b/a)^t r_3{}^t),\ \text{etc.}$$

By adding these inequalities we obtain the first inequality in (1).

Since $(u/v)+(v/u)\geqq 2$ $(u,v>0)$, with equality for $u=v$, we get the second inequality in (1).

Similarly, using the elementary inequality: $(u+v)^t > u^t+v^t$ $(t>1)$, we obtain (2).

Remark: 1° Theorem 1 is a refinement of the inequalities 12.24 of [1, p. 110] in the case $t>0$. Note that the case $t<0$ of 12.24 follows from the case $t>0$ by reciprocation (see [3]).

There is an analogous generalization of 12.31 of [1, p. 112]:

THEOREM 2.

$$(4)\quad \Sigma(r_1R_1)^t \geqq 2^{t-1}\Sigma((b/c)^t+(c/b)^t)(r_2r_3)^t \geqq 2^t\Sigma(r_2r_3)^t \quad (0<t\leqq 1),$$

$$(5)\quad \Sigma(r_1R_1)^t > \Sigma((b/c)^t+(c/b)^t)(r_2r_3)^t \geqq 2\Sigma(r_2r_3)^t \quad (t>1).$$

PROOF. From (3) we get: $R_1r_1 \geqq (c/a)r_2r_1+(b/a)r_3r_1$, etc., so, similarly to the proof of Theorem 1 we obtain Theorem 2.

THEOREM 3.

$$(6)\quad \Sigma(r_1R_1)^t \geqq 2^t\Sigma(r_2r_3)^t \quad (0<t\leqq 1),$$

$$(7)\quad \Sigma(r_1R_1)^t \leqq 2^t\Sigma(r_2r_3)^t \quad (-1\leqq t<0),$$

$$(8)\quad \Sigma(r_1R_1)^t > 2\Sigma(r_2r_3)^t \quad (t>1)$$

and the reverse inequality for $t<-1$.

PROOF. (6) and (8) are given in Theorem 2. The reverse results follow from these inequalities by isogonal conjugate transformations ([3]).

Remark: 2° The case $0<t\leqq 1$, i.e. the inequality (6) is given in [1, 12.33]. For $t=-1$ we have the inequality 12.32 of [1].

THEOREM 4.

$$(9)\quad \Sigma(R_2R_3)^t \geqq 2^t\Sigma(r_1R_1)^t \quad (0<t\leqq 1),$$

$$(10)\quad \Sigma(R_2R_3)^t \leqq 2^t\Sigma(r_1R_1)^t \quad (-1\leqq t<0),$$

$$(11)\quad \Sigma(R_2R_3)^t > 2\Sigma(r_1R_1)^t \quad (t>1)$$

and the reverse inequality for $t<-1$.

PROOF. This follows from the inequalities 12.24 of [1, p. 110] by inversion ([3]).

Remark: 3^o For $t=1$ we get the inequality 12.30 of [1] and for $t=-1$ the inequality 12.34.

A simple consequence of Theorems 3 and 4 is the following generalization of the inequalities 12.21 and 12.35 of [1]:

THEOREM 5.

$$\Sigma(R_2R_3)^t \geqq 4^t\,\Sigma\,(r_2r_3)^t \quad (0<t\leqq 1), \tag{12}$$

$$\Sigma(R_2R_3)^t \leqq 4^t\,\Sigma\,(r_2r_3)^t \quad (-1\leqq t<0), \tag{13}$$

$$\Sigma(R_2R_3)^t > 4\,\Sigma\,(r_2r_3)^t \quad (t>1) \tag{14}$$

and the reverse inequality for $t<-1$.

2. Recently, as answer to the problem of G.Tsintsifas, M.S.Klamkin [5] proved the following generalization of the Erdős-Mordell inequality:

THEOREM 6. Let P_j $(j=1,\dots,n)$ denote any set of n points lying in the interior or on the boundary of a given triangle ABC, and let R_{1j} and r_{1j} denote the distance from P_j to the vertice A and to the side a (similarly we define $R_{2j}, r_{2j}, R_{3j}, r_{3j}$). If λ_j $(j=1,\dots,n)$ are positive numbers with $\sum\limits_{j=1}^{n}\lambda_j=1$, and $R_i=\prod\limits_{j=1}^{n}R_{ij}^{\lambda_j}$ and $r_i=\prod\limits_{j=1}^{n}r_{ij}^{\lambda_j}$ $(i=1,2,3)$, then

$$\Sigma R_1 \geqq 2\,\Sigma r_1 \tag{15}$$

with equality only if the triangle is equilateral and all the points P_j coincide with its center.

Now, we shall prove the following extension of this result:

THEOREM 7. With the same notations as in Theorem 6, Theorems 1 and 2 are also valid.

PROOF. In his proof of Theorem 6, M.S.Klamkin proved that (3) is also valid (but now R_1, r_1, etc., are defined as in Theorem 6). So, the proofs of Theorems 1 and 2 are valid in our case, too.

Remark: 4° M.S.Klamkin [5] also noted that the following generalization of Theorem 6 is also valid:

Consider n triangles $A_{1j}A_{2j}A_{3j}$ of sides a_{1j}, a_{2j}, a_{3j}, and n points $P_1,\dots,P_n$ where P_j is an interior or boundary point of triangle $A_{1j}A_{2j}A_{3j}$ for each j. Then if R_{ij} and r_{ij} denote the distances from P to the vertex A_{ij} and the side a_{ij}, respectively, and if λ_j, R_i, r_i are defined as in Theorem 6, inequality (15) is still valid.

Of course, we can show that Theorem 7 is also valid, but in this case we should put $a = \prod_{j=1}^{n} a_{1j}^{\lambda_j}$, etc.

3. Let w_1 be angle-bisector of the angle BPC ($= 2\delta_1$), etc., where P is an internal point of the triangle ABC. Of course, $w_1 \geqq r_1$, etc., so the following inequality of D.F.Barrow ([1, 12.48])

$$\Sigma R_1 \geqq 2\,\Sigma w_1$$

is better than the Erdös-Mordell inequality.

A.Oppenheim [4] gave a generalization of this inequality. Here, we shall give an extension of his result:

THEOREM 8. Let x,y,z be real numbers. Then

$$\Sigma w_1(R_2^{-1} + R_3^{-1})yz \leqq \Sigma x^2 \tag{16}$$

with equality only if

$$x/\sin\delta_1 = y/\sin\delta_2 = z/\sin\delta_3. \tag{17}$$

PROOF. As in [4], we have

$$w_1(R_2 + R_3) = 2R_2R_3\cos\delta_1$$

i.e.

$$w_1(R_2^{-1} + R_3^{-1}) = 2\cos\delta_1. \tag{18}$$

Hence,

$$\Sigma w_1(R_2^{-1} + R_3^{-1})yz = 2\,\Sigma\, yz\cos\delta_1 \leqq \Sigma x^2,$$

where we used the asymmetric trigonometric inequality of J.Wolstenholme [7] (see also [2]), because $\delta_1 + \delta_2 + \delta_3 = \pi$.

Remark: 5° Oppenheim proved (16) only for positive numbers x, y, z. He didn't give necessary and sufficient condition for equality, i.e. (17). He only noted that equality in (16) is valid if $x = y = z$ and $\delta_1 = \delta_2 = \delta_3$, which is a trivial case.

THEOREM 9. Let $\varrho_1 = \sqrt{R_2 R_3} \cos \delta_1$, *etc. Then*

$$\Sigma x^2 R_1 \geqq 2\, \Sigma yz \varrho_1 \tag{18}$$

with equality only if

$$x\sqrt{R_1}/\sin\delta_1 = y\sqrt{R_2}/\sin\delta_2 = z\sqrt{R_3}/\sin\delta_3. \tag{19}$$

PROOF. Using the substitutions $x \to x\sqrt{R_1}$, etc. and identity (18) we get Theorem 9 from Theorem 8.

THEOREM 10. If $x, y, z \geqq 0$, *then*

$$\Sigma x^2 R_1 \geqq 2\, \Sigma yz w_1 \tag{20}$$

with equality only if P *is the circumcentre and (17) is valid.*

PROOF. Since $\varrho_1 \geqq w_1$ with equality only if $R_2 = R_3$, we get (20) from (18).

THEOREM 11. If $x, y, z \geqq 0$, *then*

$$\Sigma x^2 R_1 \geqq 2\, \Sigma yz r_1 \tag{21}$$

with equality only if the triangle is equilateral, P *is its center and* $x = y = z$.

The following two theorems are equivalent to Theorem 8:

THEOREM 12. Let x, y, z *be real numbers. Then*

$$\Sigma x^2 w_1^2 \geqq (\Pi w_1)\, \Sigma yz (R_2^{-1} + R_3^{-1}) \tag{22}$$

with equality only if

$$xw_1/\sin\delta_1 = yw_2/\sin\delta_2 = zw_3/\sin\delta_3. \tag{23}$$

THEOREM 13. Let x, y, z *be real numbers. Then*

$$\Sigma x^2 w_1^2 R_1^2 \geqq (\Pi w_1)\, \Sigma yz (R_2 + R_3) \tag{24}$$

with equality only if

(25) $$xw_1R_1/\sin\delta_1 = yw_2R_2/\sin\delta_2 = zw_3R_3/\sin\delta_3.$$

Remark: 6° Theorems 12 and 13 are generalizations of some results from [4]. The case $x=y=z=1$ of Theorem 9 is generalized for polygons in [6].

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LARGE DEVIATIONS AND TUNNELLING FOR PARTICLE SYSTEMS WITH MEAN FIELD INTERACTION

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Presen~~ted~~ by ~~Lee~~ Lorch, F.R.S.C.

ABSTRACT

A system of N diffusions with mean field interaction is studied in the limit as $N \to \infty$. In particular, results are obtained on equilibrium large deviations, an analogue of the results of Freidlin and Wentzell on large deviations from the limiting dynamics and a result on tunnelling from the domain of attraction of one stable equilibrium to another.

1. INTRODUCTION

The objective of this research is to study the long time behavior of a system of N interacting particles on R^d when N is large. The system is given by the solution to the martingale problem associated to the Itô equations:

$$dx_k = -[\nabla \mathcal{U}(x_k) + \frac{\theta}{N}\sum_{\ell=1}^{N}(x_k - x_\ell)]dt + \sigma dw_k, \quad k=1,\ldots,N, \tag{1.1}$$

where $w_1,\ldots,w_N$ are independent d-dimensional Wiener processes and θ and σ denote positive constants. The potential $\mathcal{U} : R^d \to R$ is assumed to satisfy:

($\mathcal{U}$1) $\mathcal{U}$ is twice continuously differentiable, $\mathcal{U} \geq 0$, and

$$\lim_{|x|\to\infty} \mathcal{U}(x)/|x|^2 = \infty.$$

($\mathcal{U}$2) There exist constants $\kappa \in (0,1)$, $c > 0$ and a non-decreasing convex function $\psi : [0,\infty) \to (0,\infty)$ with $\int^{\infty} dx/\psi(x) < \infty$ such that

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$$\sigma^2 \Delta \mathcal{U} - (1-\kappa)|\nabla \mathcal{U}|^2 \leq c - \psi(\mathcal{U}) \quad \text{on } \mathbf{R}^d.$$

($\mathcal{U}$3) There exists a constant C such that $-(\nabla\mathcal{U}(x)-\nabla\mathcal{U}(y),x-y)$ $\leq C|x-y|^2$ for all $x,y \in \mathbf{R}^d$.

In particular, the conditions ($\mathcal{U}$1)-($\mathcal{U}$3) are satisfied in the one dimensional case for $\mathcal{U}(x) = x^4/4 - x^2/2$, which arises in statistical physics as a continuous model of ferromagnetism. For $R < \infty$ let $M_R = \{\nu : \langle\nu,\mathcal{U}\rangle \leq R\}$ where $\langle\nu,\mathcal{U}\rangle = \int \mathcal{U} d\nu$ and equip M_R with the weak topology. Let $M_\infty = \{\nu : \langle\nu,\mathcal{U}\rangle < \infty\}$ and equip it with the strongest topology that induces the weak topology on M_R for all R. $M^{(N)}$ denotes the subspace of M_∞ consisting of empirical measures associated with N-particle systems. Let $C_{s,t} = C([s,t];M_\infty)$ furnished with the strongest topology which induces on $C([s,t];M_R)$ the topology of uniform convergence. Let $C_\infty = C([0,\infty);M_\infty)$ furnished with the weakest topology for which the projections onto all $C_{0,T}$, $0 < T < \infty$, are continuous.

Denote by $\{P^{(N)}_{\nu,t} : (\nu,t) \in M^{(N)} \times \mathbf{R}_+\}$ the family of probability laws on C_∞ induced by the empirical measure process

$$X_N(t) = N^{-1} \sum_{k=1}^{N} \delta_{x_k(t)} , \quad t \geq 0,$$

associated to (1.1). Let $\nu_N \to \nu$ in M_∞. The <u>law of large numbers</u> (cf. Gärtner [5]) states that the sequence $P^{(N)}_{\nu_N,0}$ converges to $\delta_{\mu(.;\nu)}$ in the sense of weak convergence of probabilities on C_∞, where $\mu(.;\nu) \in C_\infty$ is the unique weak solution of the McKean-Vlasov equation

(1.2) $$\frac{d}{dt}\langle\mu(t),f\rangle = \langle\mu(t),L(\mu(t))f\rangle , \quad t \geq 0,$$

$$\mu(0) = \nu ,$$

for all $f \in \mathcal{D}$, where

$$L(\mu)f = \frac{\sigma^2}{2}\Delta f - (\nabla\mathcal{U},\nabla f) - \theta(x-m(\mu),\nabla f),$$

$$m(\mu) = \int_{\mathbf{R}^d} x\,\mu(dx),$$

and $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support on $\mathbf{R}^d$.

2. LARGE DEVIATIONS AND QUASIPOTENTIAL

For each N the empirical measure process $X_N(.)$ has a unique equilibrium law π_N on M_∞ which is induced by the equilibrium distribution for the system (1.1). The latter is given by

$$p_N(dx_1,\ldots,dx_N) = Z_N^{-1}\exp\{-\frac{\theta}{2\sigma^2}N^{-1}\sum_{k,\ell=1}^{N}|x_k-x_\ell|^2\}\,\mu_\sigma(dx_1)\ldots\mu_\sigma(dx_N),$$

where

$$\mu_\sigma(dx) = Z^{-1}\exp\{-\frac{2}{\sigma^2}\mathcal{U}(x)\}dx ,$$

and Z_N and Z are normalizing constants. The sequence of measures π_N is relatively compact and any limit point is a measure on M_∞ concentrated on the minima of the equilibrium action functional (cf. Ellis and Newman [3], and Léonard [6]). The equilibrium action functional is defined by

$$I(\nu) = <\nu,\log\frac{d\nu}{d\mu_\sigma}> + \frac{\theta}{2\sigma^2}<\nu\otimes\nu,|x-y|^2> - p,\quad \nu\in M_\infty,$$

where p is a constant called the Gibbs free energy defined by

$$p = \inf\{<\nu,\ \log\frac{d\nu}{d\mu_\sigma}> + \frac{\theta}{2\sigma^2}<\nu\otimes\nu,|x-y|^2> : \nu\in M_\infty\}.$$

In addition the equilibrium large deviation behavior can be determined as follows.

THEOREM 1. Let π_N be defined as above.

(i) If G is an open subset of M_∞, then

$$\liminf_{N\to\infty} N^{-1}\log \pi_N(G) \geq -\inf_{\mu\in G} I(\mu).$$

(ii) If F is a closed subset of M_∞, then

$$\limsup_{N\to\infty} \log N^{-1}\pi_N(F) \leq -\inf_{\mu\in F} I(\mu).$$

In order to investigate the corresponding large deviation questions for the dynamical behavior it is necessary to develop an infinite dimensional version of the large deviation theory of Freidlin and Wentzell [4]. The following result of this type is proved in Dawson and Gärtner [1].

THEOREM 2. Given $\nu_N \in M^{(N)}$ suppose that $\nu_N \to \nu$ in M_∞. Let A be a Borel subset of $C_{s,t}$. Then

$$\inf_{\mu\in A^o,\mu(s)=\nu} S_{s,t}(\mu) \leq \liminf_{N\to\infty} N^{-1}\log P^{(N)}_{\nu_N,s}(A)$$

$$\leq \limsup_{N\to\infty} N^{-1}\log P^{(N)}_{\nu_N,s}(A) \leq -\inf_{\mu\in\overline{A},\mu(s)=\nu} S_{s,t}(\mu).$$

where A^o and $\overline{A}$ denote the interior and closure of A respectively.

The action functional $S_{s,t}$ is defined by

$$S_{s,t}(\mu(.)) = \int_s^t \|\dot{\mu}(u) - L(\mu(u))^*\mu(u)\|^2_{\mu(u)}\, du$$

if $\mu(.) \in C_{s,t}$ is absolutely continuous in the distribution sense.

Otherwise $S_{s,t}(\mu(.)) = \infty$. In the above

$$\|\phi\|^2_\mu = \frac{1}{2\sigma^2} \sup_{f\in D} \frac{|<\phi,f>|^2}{<\mu,|\nabla f|^2>}, \quad \phi \in D'.$$

Let ρ be an arbitrary equilibrium for the McKean-Vlasov dynamics. Then the quasipotential Q_ρ is defined by:

$$Q_\rho(\nu) = \inf \{S_{s,t}(\mu(.)) : -\infty < s < t < \infty,\ \mu(s) = \rho,\ \mu(t) = \nu\}.$$

The main result of this section is to show that the quasipotential Q_ρ and the equilibrium action functional agree up to an additive constant in the domain of attraction of ρ.

THEOREM 3. Suppose that $\nu \in M_\infty$ belongs to the domain of attraction of ρ (with respect to the McKean-Vlasov dynamics). Then

$$Q_\rho(\nu) = I(\nu) - I(\rho) .$$

Furthermore if $I(\nu) < \infty$, then the infimum in the definition of Q_ρ is attained at $\overline{\mu}(u) = \mu(-u;\nu)$, $u \in (-\infty,0]$, that is, the time reverse of the McKean-Vlasov trajectory. Up to a time shift, this is the only path at which this infimum is attained.

The proofs of Theorems 1 and 3 are given in Dawson and Gärtner [2].

3. TUNNELLING

In this section we assume that the McKean-Vlasov dynamics has exactly three equilibria ν_-, ν_0, ν_+, that ν_- and ν_+ are stable and that ν_0 is unstable in the topology of M_∞. We assume that there exists a path of the McKean-Vlasov dynamics connecting ν_0 with ν_+ and a path connecting ν_0 with ν_- (on $(-\infty,+\infty)$). Let D_-, D_0, D_+ denote the domains of attraction of ν_-, ν_0, ν_+, respectively. According to the law of large numbers the N particle system, for large N, follows the McKean-Vlasov trajectory on time intervals of fixed length. However in a long time scale the N-particle system can exhibit tunnelling, that is, it can escape from the domain of attraction of the stable equilibrium ν_- and end up near the stable equilibrium ν_+. This idea is made precise in the next theorem.

Let V_+ be an open neighbourhood of ν_+ contained in D_+, and

$$\tau_+(\mu(.)) = \inf\{t \geq 0: \mu(t) \in V_+\},$$

$$\sigma_-(\mu(.)) = \inf\{t \geq 0: \mu(t) \notin D_-\}.$$

Let $\Delta I = I(\nu_0) - I(\nu_-)$.

THEOREM 4. Given $\nu_N \in \mathcal{M}^{(N)}$, $\nu \in D_-$, *suppose that* $\nu_N \to \nu$ in $\mathcal{M}_\infty$. *Then*

$$\lim_{N\to\infty} P^{(N)}_{\nu_N}\left(e^{N(\Delta I-\delta)} < \sigma_- \leq \tau_+ < e^{N(\Delta I+\delta)}\right) = 1$$

for each $\delta > 0$. *Moreover*

$$\lim_{N\to\infty} N^{-1} \log E^{(N)}_{\nu_N}\sigma_- = \lim_{N\to\infty} N^{-1} \log E^{(N)}_{\nu_N}\tau_+ = \Delta I,$$

where $E^{(N)}_{\nu_N}$ *denotes expectation with respect to* $P^{(N)}_{\nu_N}$.

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ANNEAUX DE POLYNOMES SUR UN ANNEAU DE MORI

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Presented by P. Ribenboim, F.R.S.C.

Rappelons qu'un anneau de Mori est un anneau intègre dans lequel les idéaux divisoriels entiers satisfont à la condition de chaine ascendante. Tous les anneaux de Mori actuellement connus se construisent à partir des anneaux de Krull et des anneaux noethériens (qui sont évidemment des anneaux de Mori particuliers) en utilisant essentiellement les résultats suivants :

1) Toute intersection à caractère fini d'anneaux de Mori est un anneau de Mori .(2).

2) Tout "pullback" d'un anneau de Mori local est un anneau de Mori (1)

3) Tout anneau généralisé de fractions d'un anneau de Mori est un anneau de Mori (6).

Maintenant la question : " est-ce qu'un anneau de polynômes sur un anneau de Mori est un anneau de Mori ? " est une conjecture qui a été résolue par l'affirmative dans le cas où l'anneau de base est supposé être , de plus , intégralement clos (5).

Un anneau de Mori A sera dit transcendant si $A[X]$ est un anneau de Mori . Nous montrerons que tous les anneaux construits à partir d'anneaux de Mori transcendants par l'un des trois procédés ci-dessus ,le sont aussi .

1) ETUDE DE CERTAINES CLASSES D'IDEAUX DIVISORIELS DE A [X]

Proposition 1-1

Soient A un anneau de Mori et $\mathfrak{a}$ un idéal divisoriel entier de $A[X]$ de trace non nulle $\mathfrak{a}_0$ sur A . Alors $\mathfrak{a}_0 A[X] \subsetneq \mathfrak{a} \subsetneq \sqrt{\mathfrak{a}_0}\, A[X]$

Corollaire 1-2

Dans un anneau de Mori , tout idéal divisoriel $\mathfrak{a}$ de $A[X]$ dont la trace $\mathfrak{a}_0$ est un idéal radical non nul , vérifie $\mathfrak{a} = \mathfrak{a}_0 A[X]$.

Proposition 1-3

Soient A un anneau de Mori et Q un idéal premier divisoriel de $A[X]$ ayant une trace non nulle Q_0 . Alors ,

1) $Q = Q_0 A[X]$.

2) Q est maximal (dans l'ensemble des idéaux divisoriels de $A[X]$) si et seulement s'il en est de même de sa trace .

Remarque 1-4

Si Q a une trace nulle , alors il existe un polynôme f de Q irréductible dans $K[X]$,(K corps des fractions de A),tel que $Q = f K[X] \cap A[X]$ ceci etant vrai même si Q est seulement supposé idéal premier de $A[X]$. Mais inversement , il n'est pas certain que l'intersection avec $A[X]$ d'un idéal $f K[X]$ où f est un polynôme irréductible de $K[X]$, soit un idéal divisoriel de $A[X]$. C'est cependant vrai si A est intégralement clos (5)

Corollaire 1-5

Soit A un anneau de Mori . Alors $A[X]$ satisfait à la condition de chaîne ascendante sur les idéaux premiers divisoriels.

Remarque 1-6

Dans un anneau de Mori A, les idéaux divisoriels entiers principaux de $A[X]$ satisfont à la condition de chaîne ascendante.

2) INTERSECTIONS A CARACTERE FINI D'ANNEAUX DE MORI TRANSCENDANTS

Défintion 2-1

On appellera anneau de Mori transcendant, un anneau de Mori A tel que $A[X]$ soit un anneau de Mori.

Proposition 2-2

Soit Σ un système multiplicatif d'idéaux entiers de A. Si A est un anneau de Mori transcendant, alors l'anneau généralisé de fractions A_{Σ} l'est aussi.

Nous allons étudier dans ce paragraphe la transcendance des intersections à caractère fini d'anneaux de Mori transcendants. Remarquons tout de suite que, si A est une intersection à caractère fini d'anneaux A_{λ}, $\lambda \in \Lambda$, alors bien sur $A[X] = \bigcap_{\Lambda} A_{\lambda}[X]$, mais cette intersection n'est jamais à caractère fini. Nous allons mettre en oeuvre un procédé qui va nous permettre de considérer $A[X]$ comme une intersection à caractère fini.

Proposition 2-3

Soit P un idéal premier d'un anneau intègre A. Alors, si K est le corps des fractions de A, $(A_P[X])_{PA_P[X]} \cap K[X] = A_P[X]$.

Proposition 2-4

Soit A un anneau de Mori. Alors $A[X] = \bigcap_{\mathcal{M}^{d}(A)} A_P[X] = \bigcap_{\mathcal{M}^{d}(A)} (A[X])_{PA[X]} \cap K[X]$

De plus, la famille $\left\{ \left\{ (A[X])_{PA[X]} \right\}_{\mathcal{N}_0^d(A)} , \left\{ K[X] \right\} \right\}$ est à caractère fini

Proposition 2-5

Soit $\{A_\lambda\}_\Lambda$ une famille à caractère fini d'anneaux de Mori transcendants tous inclus dans un anneau intègre B. Alors $A = \bigcap_\Lambda A_\lambda$ est un anneau de Mori transcendant.

3) PULLBACKS D'ANNEAUX DE MORI TRANSCENDANTS

Soit S un anneau local, d'idéal maximal $M \neq (0)$. Soient $k(S) = S/M$ et $\varphi : S \longrightarrow k(S)$ la surjection canonique. Si D est un sous-anneau de $k(S)$, posons $R = \varphi^{-1}(D)$. Alors on sait que R est un anneau de Mori si et seulement si S est un anneau de Mori et D un corps. (1). R est appelé pullback de l'anneau local de Mori S.

Proposition 3-1

Soient un anneau intègre S, P un idéal premier de S. Considérons le diagramme

$$\begin{tikzcd} S \arrow[r, "\varphi"] & S/P \\ R = \varphi^{-1}(D) \arrow[u, hook] \arrow[r] & D \arrow[u, hook] \end{tikzcd}$$

D sous-anneau de S/P

Alors, si S est un anneau de Mori et si D est de la forme $S/P \cap k$ où k est un sous-corps du corps S_P/PS_P, R est un anneau de Mori.

Proposition 3-2

Soit S un anneau local de Mori transcendant et soit R un pullback de S. Alors R est un anneau de Mori transcendant.

Ces deux dernières propositions ont été trouvées avec V.Barucci en nov. 85

Les résultats du paragraphe 2 permettent de "globaliser" la construction précédente : partant d'un anneau de Mori A admettant donc la décomposition à caractère fini $A = \bigcap_{\mathcal{M}^d(A)} A_P$, pour chaque $P \in \mathcal{M}^d(A)$, posons $k(P) = A_P / PA_P$ et soit $A'(P)$ un pullback de l'anneau A_P.

Proposition 3_3

Soit A un anneau de Mori. Avec les notations précédentes, si on pose $A' = \bigcap_{\mathcal{M}^d(A)} A'(P)$, alors la famille $\{A'(P)\}_{\mathcal{M}^d(A)}$ est à caractère fini et l'anneau A' est de Mori. Si, de plus, A est un anneau de Mori transcendant, il en est de même de l'anneau A'.

Exemple 3-4 :

1) Si A est un anneau de Mori transcendant de caractéristique $p \neq 0$, alors A contient le corps Z/pZ. Prenons, pour tout $P \in \mathcal{M}^d(A)$, le pullback de A_P correspondant au sous-corps premier $k_0(P) = Z/pZ$ du corps résiduel k(P). Alors l'anneau de Mori transcendant de la construction précédente est $A' = \bigcap_{P \in \mathcal{M}^d(A)} (Z/pZ + PA_P)$. Plus généralement si A contient un corps k, $A' = \bigcap_{\mathcal{M}^d(A)} (k + PA_P)$ est un anneau de Mori transcendant.

2) Si A est un anneau de Mori transcendant de caractéristique 0, alors $Z \subset A$. Posons $\mathcal{M}_1^d(A) = \{ P \in \mathcal{M}^d(A),\ PA_P \cap Z = pZ \neq \{0\} \}$, $p \in Z$ et $\mathcal{M}_2^d(A) = \{ P \in \mathcal{M}^d(A),\ PA_P \cap Z = \{0\} \}$. Prenons, pour tout $P \in \mathcal{M}_1^d(A)$, le pullback de l'anneau A_P correspondant au sous-corps premier $k_0(P) = Z/pZ$ et, pour tout $P \in \mathcal{M}_2^d(A)$, le pullback de l'anneau A_P correspondant au sous-corps premier $k_0(P) = Q$. Alors l'anneau de Mori

transcendant A' de la construction précédente est :

$$A' = \bigcap_{\mathcal{M}_1^d(A)} (Z_{pZ} + PA_p) \bigcap_{\mathcal{M}_2^d(A)} (Q + PA_p)$$

<u>Remarque 3-5</u> La construction précédente , efféctuée à partir d'un anneau de Krull ou d'un anneau noethérien , donne un anneau de Mori transcendant qui n'a apparemment aucune raison d'être un anneau de Krull ou noethérien.

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POINT SPECTRUM FOR THE ALMOST MATHIEU EQUATION

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Presented by G.A. Elliott, F.R.S.C.

ABSTRACT

We show that the operator $(h\xi)_n = \xi_{n+1} + \xi_{n-1} + 2\cos(2n\pi\alpha + \theta)\xi_n$ has no eigenvectors in $\ell^2(\mathbb{Z})$ for all constants $\alpha, \theta \in \mathbb{R}$.

1. We consider the almost Mathieu operator $(h\xi)_n = \xi_{n+1} + \xi_{n-1} + 2\beta\cos(2\pi n\alpha + \theta)\xi_n$, $\xi \in \ell^2(\mathbb{Z})$, and show that for $\beta = 1$ point spectrum does not occur, thus answering in part problem 4 in [2]. Our approach heavily relies on the methods developed in [1]. We briefly review the material from [1] which we shall need in the sequel.

Since it follows from Floquet theory that the operator h does not have eigenvectors for rational α we may assume henceforth that α is irrational. We consider the unitary operators u and v which are defined as follows

$$\left.\begin{aligned}(u\xi)_n &= \xi_{n+1}\\ (v\xi)_n &= e^{-2\pi i\alpha}\xi_n\end{aligned}\right\} \quad n \in \mathbb{Z}\,,\ \xi \in \ell^2(\mathbb{Z})\,.$$

and the C*-algebra A_α generated by u and v. The assignment

$$\rho : \begin{cases} u \to v^* \\ v \to u \end{cases}$$

determines an automorophism of A_α. We set $\sigma = \rho^2$. The operator h is a fixed point of σ if and only if $\theta = 0$, and h is a fixed point of ρ if and only if $\theta = 0$ and $\beta = 1$. A state ϕ on A_α is called an eigenstate of h for $\chi \in Sp(h)$ if

$$\phi(ha) = \chi\phi(a) \quad \text{for every} \quad a \in A_\alpha .$$

We set $\lambda = e^{\pi\alpha i}$ and for $p,q \in \mathbb{Z}$

$$S_{pq} = \lambda^{-pq}(u^p v^q + u^{-p} v^{-q})$$
$$T_{pq} = \lambda^{-pq} i(u^p v^q - u^{-p} v^{-q}) \;;$$

$$W_{pq} = S_{pq} + S_{-q,p}$$
$$Z_{pq} = S_{pq} - S_{-q,p}$$

For some eigenstate ϕ we set

$s_{pq} = \phi(S_{pq})$, $t_{pq} = \phi(T_{pq})$, $w_{pq} = \phi(W_{pq})$, $z_{pq} = \phi(Z_{pq})$.
We have shown in [1] that the s_{pq} and t_{pq} are real numbers solving the following system of linear equations in the variables x_{pq}

$$(1.1) \quad \begin{cases} \cos(\pi\alpha q)(x_{p-1,q}+x_{p+1,q}) + \beta\cos(\pi\alpha p)(x_{p,q-1}+x_{p,q+1}) = \chi x_{pq} \\ \sin(\pi\alpha q)(x_{p-1,q}-x_{p+1,q}) - \beta\sin(\pi\alpha p)(x_{p,q-1}-x_{p,q+1}) = 0 \end{cases}$$

Henceforth we assume that $\beta = 1$. Then the w_{pq} and z_{pq} are real and solve the system (1.1) also. We have shown in [1], section 2 that the linear space of solutions of (1,1) has dimension five and all solutions are determined by the values of $x_{00}, x_{10}, x_{01}, x_{11}, x_{-1,0}$. Moreover, we have the following (see [1] Lemma 2.5)

(1.2) (i) If $\{x_{pq}\}$ solves (1.1) and $x_{00} = x_{01} = x_{11} = 0$, $x_{10} \neq 0$, then $x_{pp} = 0$ and $x_{p+1,p} = x_{10}(-1)^p$ for all $p \geq 0$.

(ii) If $\{x_{pq}\}$ solves (1.1) and $x_{00} = x_{10} = x_{11} = 0$, $x_{01} \neq 0$, then $x_{pp} = 0$ and $x_{p,p+1} = x_{01}(-1)^p$ for all $p \geq 0$.

2. We shall prove the following

2.1. Theorem. For any constant θ the operator h does not have eigenvectors.

The following lemma lists elementary properties of vector states.

2.2. Lemma. Let $\xi \in \ell^2(\mathbb{Z})$ and let ϕ_ξ be the vector state on A_α associated with ξ. Then

(i) $\lim_{|p|\to\infty} \phi_\xi(u^p) = 0$,

(ii) $\lim_{|p|\to\infty} \phi_\xi(u^p v^{p+k}) = 0$ for all $k \in \mathbb{Z}$,

(iii) $\{\phi_\xi(v^q)\}$ does not converge to zero as $q \to \infty$ or $q \to -\infty$.

The next lemma is an immediate consequence of 2.2 (i), (iii).

2.3. Lemma. There is no ρ-invariant vector state ϕ_ξ on A_α.

Proof of Theorem 2.1: Suppose that ϕ_ξ is a vector state on A_α, i.e.

$$\phi_\xi(a) = \langle a\xi, \xi\rangle \quad \text{for all} \quad a \in A_\alpha \; ; \quad \xi \in \ell^2(\mathbb{Z})$$

which is also an eigenstate of h for some $x \in Sp(h)$. We distinguish two different cases:

Case 1: $\theta \notin 2\pi\alpha\mathbb{Z}$.

It follows from [1], 3.2 that ϕ_ξ is not a fixed point under σ. Thus $\psi = \phi_\xi - \phi_\xi \circ \sigma$ is non-zero. Since the $t_{pq} = \psi(T_{pq})$ solve the system (1.1) it follows that $x_{10} \neq 0$ and (1.2)(i) applies or $x_{01} \neq 0$ and (1.2)(ii) applies. In both cases the conclusions in (1.2) contradict 2.2 (ii).

Case 2: $\theta \in 2\pi\alpha\mathbb{Z}$.

Replacing h by $u^{-k}hu^k$ for some suitably chosen integer k and transforming the automorphisms ρ and σ accordingly, if necessary, we may assume that $\theta = 0$. If ϕ_ξ is not a fixed point under σ then the same reasoning as in case 1 can be applied. Therefore we assume that ϕ_ξ is a fixed point under σ. By 2.3 the vector state ϕ_ξ can not be a fixed point under ρ. Thus $\psi = \phi_\xi - \phi_\xi \circ \rho$ is non-zero. The $z_{pq} = \psi(Z_{pq})$ solve the system (1.1) and we have $z_{00} = z_{01} = z_{11} = 0$, $z_{10} = z_{01} \neq 0$. Hence (1.2) once again shows we have reached a contradiction with 2.2 (ii).

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A FOUR DIMENSIONAL PROJECTION OF THE POLYTOPE 2_{21}

Asia Ivić Weiss

Presented by H.S.M. Coxeter, F.R.S.C.

Abstract

New coordinates are found for the 27 vertices of the 6-dimensional polytope 2_{21} and it is shown that this polytope can be projected to the 4-dimensional 24-cell $\{3,4,3\}$.

A 6-dimensional semi-regular polytope 2_{21} , first discovered by Gosset [2,p. 164], can easily be described by the Coxeter graph

Each vertex of the graph, including the "distinguished" one, represents a reflection. The product of two reflections is of order 3 whenever the two vertices are connected otherwise it is of order 2 . The six reflections generate the symmetry group of 2_{21} . The vertices of the polytope coincide with the orbit of the point belonging to all planes of the generating reflections except the "distinguished" plane. For a more detailed account of Coxeter graphs and 2_{21} the reader could consult [2, pp. 187-203; 3].

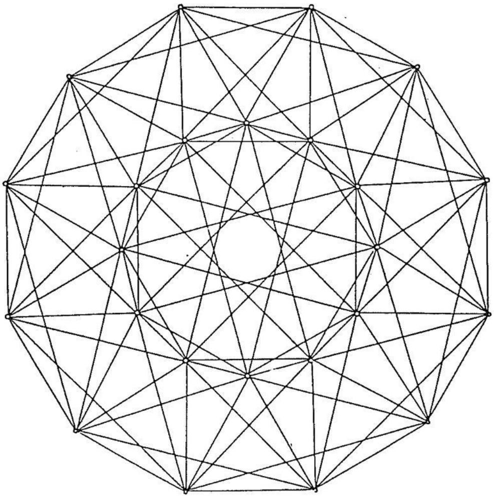

Figure 1

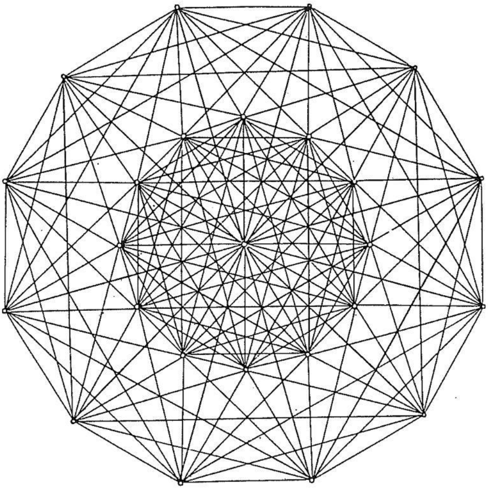

Figure 2

The polytope 2_{21} has 27 vertices and 5-dimensional cells of two kinds: 72 simplexes 2_{20} and 27 cross-polytopes 2_{11}. Figure 2 shows 2_{21} in Coxeter's dodecagonal plane projection [1, pp. 461,463]. Figure 1 is the dodecagonal plane projection of the regular 4-dimensional polytope {3,4,3} or 24-cell. We note that the vertices and edges of the projection of a 24-cell are subsets of the vertices and edges of the projection of 2_{21}, suggesting that the 24-cell is a 4-dimensional projection of 2_{21}.

{3,4,3} is a self-dual polytope with 24 cells (hence the name) and same number of vertices. The 24 vertices of {3,4,3}, can be given the following coordinates:

$$\begin{aligned} A'_k &= (a \cos k\theta,\ a \sin k\theta,\ b \cos 5k\theta,\ b \sin 5k\theta),\quad k = 0,2,\dots,22\,, \\ B'_k &= (b \cos k\theta,\ b \sin k\theta,\ a \cos 5k\theta,\ a \sin 5k\theta),\quad k = 1,3,\dots,23\,, \end{aligned} \tag{1}$$

where $\theta = \pi/12$, a and b are positive roots of the equation

$$6x^4 - 6x^2 + 1 = 0 \tag{2}$$

[2, p. 245].

We proceed to find coordinates for 2_{21}. The product of the six generating reflections for the symmetry group of 2_{21} is an isometry of period 12 which permutes the 27 vertices in two cycles of 12 and one cycle of 3 [1,pp. 461,463]. More precisely, it is a triple rotation through angles $\pi/6$, $5\pi/6$ and $2\pi/3$ in three completely orthogonal planes [2, pp. 218, 234]. One cycle of 12, say $A_0A_2\ldots A_{22}$, might be called a 'tetrahelix' [4, p. 385] because every 4 consecutive vertices belong to a regular tetrahedron. By analogy with (1), we can take coordinates for these points to be

$$\begin{aligned} A_k = (&a \cos k\theta,\ a \sin k\theta,\ b \cos 5k\theta,\ b \sin 5k\theta, \\ &c \cos 4k\theta,\ c \sin 4k\theta),\quad k \text{ even}, \end{aligned} \tag{3}$$

where a, b and c remain to be determined.

If we take the edges of 2_{21} to be of length $\sqrt{2}$ we obtain

$$a^2 + b^2 = 1 \quad \text{and} \quad c^2 = 1/3 .$$

Since $A_0A_2A_4$ is an equilateral triangle we get

$$a/b = (\sqrt{3} + 1)/\sqrt{2} ,$$

and hence

$$2a^2 = 1 + 1/\sqrt{3} , \quad 2b^2 = 1 - 1/\sqrt{3} ,$$

so that a and b satisfy equation (2). We can assume that a, b

and c are all positive.

Let B_1 be the vertex which makes, with the tetrahedron $A_{22}A_0A_2A_4$, a simplex α_4 of type 2_{01}, that is, the interface between two facets $\beta_5 = 2_{11}$, not belonging to a facet $\alpha_5 = 2_{20}$. Then we can find the coordinates for B_1 to be

$$(\tfrac{1}{2}a,\ \tfrac{1}{2}(2-\sqrt{3})a,\ \tfrac{1}{2}b,\ \tfrac{1}{2}(2+\sqrt{3})b,\ -\tfrac{1}{2}c,\ -\tfrac{1}{2}c)$$
$$= (b \cos\theta,\ b\sin\theta,\ a\cos 5\theta,\ a\sin 5\theta,\ -c\cos 4\theta,\ -c\sin 4\theta)\ .$$

And, in general, we can name the points of 2_{21} to obtain

$$B_k = (b\cos k\theta,\ b\sin k\theta,\ a\cos 5k\theta,\ a\sin 5k\theta,\ -c\cos 4k\theta,\ -c\sin 4k\theta)\ ,\quad k \text{ odd}\ . \tag{4}$$

Hence the orthogonal projection of the 6-dimensional polytope 2_{21} on the 4-dimensional subspace $x_5 = x_6 = 0$ is the 4-dimensional regular polytope $\{3,4,3\}$. The three vertices of 2_{21}, corresponding to the central point of the dodecagonal projection of 2_{21}, project to $(0,0,0,0)$ and have the following coordinates

$$C_k = (0,0,0,0,\ -2c\cos 8k\theta,\ -2c\sin 8k\theta),\quad k = 0,1,2\ . \tag{5}$$

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Poincaré polynomials and ordered fields

by Jan Mináč

Presented by P.Ribenboim, F.R.S.C

Abstract: Let F be a formally real pythagorean field with $\mid \dot{F}/\dot{F}^2 \mid < \infty$. Let $F(2)$ be the maximal 2-extension of F and H the Galois group of automorphisms of the field extension $F(2)/F(\sqrt{-1})$. We shall determine Poincaré polynomial $P_H(t)$ of the group H and call attention to some interesting properties of $P_H(t)$. This gives also a new insight into sets $O(n)$ defined by Bröcker.

§1. *Introduction.* The aim of this paper is to announce some results in the paper [12]. We define F to be a formally real pythagorean field. $\dot{F}$ is the multiplicative group of F. We shall always assume that $\mid \dot{F}/\dot{F}^2 \mid < \infty$.

Put $h_i = dim_{\mathbb{Z}/2\mathbb{Z}} H^i(H,2), 0 \leq i < \infty$, where $H^i(H,2)$ is the i-th cohomological group of the group H with coefficients in the two element field. Then $P_H(t) = \sum_{i=0}^{\infty} h_i t^i$ is the Poincaré series of the group H. Since the cohomological dimension d(H) = d = st(F) = stability index of $F, P_H(t)$ is a polynomial. (See [10]). We shall use freely notation, definitions and Theorems in [4], [5], [6], [7]. We shall recall here just the most essential notation and definitions. $N(F)$-the number of orderings of the field F; F is of type $(k, 2^n)$ if $N(F) = k$ and $\mid \dot{F}/\dot{F}^2 \mid = 2^n$; V a valuation on F; A_V a valuation ring corresponding to V; U_V the group of units of A_V; M_V the maximal ideal of A_V; V is fully compatible with $\dot{F}^2$ iff $1 + M_V \subset \dot{F}^2$; (X, J) the abstract Marshall's space of orderings ; (we shall identify isomorphic order spaces) ; $(X_F, \dot{F}/\dot{F}^2)$, (or $(X, \dot{F}/\dot{F}^2)$, or simply X) space of orderings of the field F ;

$c\ell(X)$ chain length of X ; $|T|$ the number of elements of a finite set T ; $\chi(H)$ the Euler-Poincaré characteristic of the group H ; $G = \mathrm{Gal}(F(2)/F)$; $S_G(t)$ the Poincaré series of the group G .

For extensions, sums of order spaces and their valuation theoretic interpretation we refer the reader to the papers [4], [6]. (*We shall always assume that in the decomposition of order spaces into sums all summands are indecomposable order spaces*).

Following [2], [3], we shall attach to any order space $(X_F, \dot{F}/\dot{F}^2)$ the graph $Gr(X_F)$:

1) *The set of vertices*, $V(X_F)$, *of the graph* $Gr(X_F)$ is defined inductively

1A) $X_F \in V(X_F)$

1B) If $Y \in V(X_F)$, $Y = Z \times H$, where H is a 2-elementary group and Z is a decomposable space, then $Z \in V(X_F)$.

1C) If $W \in V(X_F)$ and $W = W_1 \oplus \ldots \oplus W_d$, then $W_i \in V(X_F)$, $i = 1,\ldots,d$.

2) *Edges*: Two vertices Y,Z are connected with oriented edge (Y,Z) if

2A) $Y = Z \times H$, for some 2-elementary group H, $H \neq \{1\}$, and Z is decomposable order space.

2B) $Y = Z \oplus Z_2 \oplus \ldots \oplus Z_d$, $2 \leq d$.

We shall define functions $n,m,s\colon V(X_F) \longrightarrow \mathbb{N} \cup \{0\}$.

Let $Y = Y_1 \oplus \ldots \oplus Y_d \in V(X_F)$. Then we put $n(Y) = d$. If $Y = Z \times H \in V(X_F)$, and Z is not an extension of any proper quotient space of Z , or $|Z| = 2$, we put $m(Y) = \log_2 |H|$.

For every vertex $Y \in V(X_F)$ there exists a unique shortest path $Y = Y_0$, Y_1 , ..., $Y_k = X_F$, where (Y_i, Y_{i+1}) , $i = 0,1,\ldots,k-1$, belong to the set of edges of $Gr(X_F)$. Then we put

$$s(Y) = \sum_{i=1}^{k} m(Y_i) .$$

We shall also define the function

$$R : V(X_F) \to \mathbb{Z}[t]$$
$$R(Y) = (1-b) + t(d-1) ,$$

where $Y = Y_1 \oplus \ldots \oplus Y_a \oplus Z_1 \oplus \ldots \oplus Z_b$, $|Y_1| = \ldots = |Y_a| = 1$, $1 < |Z_1| , \ldots , |Z_b|$, $d = a + b$ and t is indeterminate.

§2. Some Results

Theorem 1. *If* $N(F) \neq 1$, *then* $P_F(t) = \sum_{\substack{C \in V(X_F) \\ n(C) \geq 2}} (1+t)^{s(C)} R(C)$.

If $N(F) = 1$, *then* $P_F(t) = 1$.

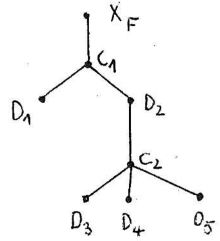
Theorem 2. $P_F(1) = N(F)$; $P_F(0) = 1$;
$P_F(-1) = \chi(H) = 2 - a - 2b$; $P'_F(0) + 1 = \dim_{\mathbb{Z}/2\mathbb{Z}} \dot{F}/\dot{F}^2$,
$\deg P_F(t) = st(F)$.

Theorem 3. *Denote* $V_{(1+t)}$ *the valuation on the field* $\mathbb{Q}(t)$ *determined by the polynomial* $(1+t)$. *If* F *is not a superpythagorean field, then*

$$V_{(1+t)} P_F(t) = \max\{\dim_{\mathbb{Z}/2\mathbb{Z}} \dot{F}/\dot{F}^2 U_V \text{ , } V \text{ is a valuation on } F \text{ fully compatible with } \dot{F}^2\} .$$

If F *is a superpythagorean field, then* $\max\{\dim_{\mathbb{Z}/2\mathbb{Z}} \dot{F}/\dot{F}^2 U_V$, V *is a valuation on* F *fully compatible with* $\dot{F}^2\} \in$
$\in \{(V_{(1+t)} P_F(t)) - 1 , V_{(1+t)} P_F(t)\}$.

Example. Let F be a field of the type $(14, 2^6)$. Then $Gr(X_F)$ is



$X_F = C_1 \times H_1$; $|H_1| = 2$
$D_2 = C_2 \times H_2$; $|H_2| = 2$

(See [9])

$$\begin{aligned} P_F(t) &= (1+t)\, R(C_1) + (1+t)^2\, R(C_2) \\ &= (1+t)t + (1+t)^2\,(2t+1) \\ &= 1 + 5t + 6t^2 + 2t^3 \end{aligned}$$

Hence $st(F) = 3$, $\chi(H) = P_F(-1) = 0$, $V_{(1+t)}\, P_F(t) = 1$ and there exists a valuation V on F fully compatible with $\dot F^2$ such that $|\dot F/\dot F^2 U_V| = 2$, $P_F(1) = 14$, $P'_F(0) + 1 = 6$. Which polynomials are Poincaré polynomials of some field F? To give the answer define for each $n \in \mathbb{N}$ the set $B(n)$:

$B(1) = \{1\}$

$B(2) = \{(1+t)\}$

$B(3) = \{(1+2t), (1+2t+t^2)\}$

$B(4) = \{(1+3t), (1+3t+t^2), (1+3t+2t^2), (1+3t+3t^2+t^3)\}$

...

$B(n) = \{B(n-1)+t\} \cup \{B(n-1)(1+t)\}, \ 2 \leq n$

Note that by putting $t = 1$ we get sets $O(n)$. ([1]).

Theorem 4. Let F *be a field with* $|\dot F/\dot F^2| = 2^n$. *Then* $P_F(t) \in B(n)$. *Each polynomial from the set* $B(n)$ *is a Poincaré polynomial of some field* F.

Theorem 5. If $2 \leq n$, *then* $|B(n)| = 2^{n-2}$. *The polynomial* $g(t) \in B(n)$ *iff*

$$g(t) = (1+t)^{s-1} + t((1+t)^{s-1}a_{s-1} + \ldots + a_0),$$

where $0 \leq a_0, a_1, \ldots, a_{s-1} \in \mathbb{Z}$, $1 \leq s$, a_{s-1} *and*

$$a_0 + a_1 + \ldots + a_{s-1} + s = n.$$

From Theorem 5 we can get also a description of $O(n)$ which is essentially equivalent to the description in the Theorem in [11]. (See also [8]. Proposition 2.8 and Propositin 2.A.4).

Theorem 6. Let Y *be a class of all order spaces with given Poincaré polynomial* $P(t)$. *Then there exists a unique order space* $X \in Y$ *such that*

$$c\ell(X) = \max\{c\ell(Z),\ Z \in Y\}.$$

Call such space Poincaré order space. Hence there is 1-1 correspondence between the sets of Poincaré polynomials and Poincaré order spaces.

EXERCISE (EASY) Characterise Poincaré order spaces.

Theorem 7.

$$S_G(t) = \frac{P_F(t)}{1-t}\ . \quad \text{Hence}$$

$$\lim_{t \to 1} (1-t)S_G(t) = N(F)\ .$$

In the paper [12] a complete characterisation of the cohomology rings $H^*(G,2)$, $H^*(H,2)$ is given. Here we shall restrict ourselves to the following observation.

Theorem 8. Both rings $H^*(G,2)$, $H^*(H,2)$ are local Cohen-Macaulay rings. Each of them, considered as graded ring, determines the order space X_F. $\dim H^*(G,2) = 1$, $\dim H^*(H,2) = 0$ (dim means Krull dimension). $H^*(H,2)$ is a Gorenstein ring iff the field F is a superpythagorean field.

For other properties of Poincaré polynomials, cohomology rings, stability indices, etc. see the paper [12]. That paper also gives references to all literature needed in our proofs.

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PERIODICITE DES PUISSANCES D'UNE MATRICE DONT LES CŒFFICIENTS APPARTIENNENT A UN CORPS FINI. APPLICATION

ERRATA

Yves HELLEGOUARCH

L'énoncé du théorème 3, p. 188 (C.R. Math. Rep. Acad. Sci. Canada, vol. VIII n° 3, June 1986) est incorrect.

Dans le 2), il faut remplacer "$q > 2$" par "$\alpha \geqslant 2$".

Si $\alpha = 1$ et $q > 2$, il y a un nombre fini d'exceptions du type $N = P^2$ avec degré $P = 1$.

Si $q = 2$, il y a une infinité d'exceptions du type

$$N = P_1^h P_2 \ldots P_s$$

avec degré $P_1 = 1$, $0 \leqslant h \leqslant 3$, et $(d_i, d_j) = 1$ pour tous les i et j distincts.

La preuve du théorème 4, p. 189 contient une faute :

Dans 2) et 3) l'expression $q^{[d_1,\ldots,d_s]}-1$ doit être remplacé par $[q^{d_1}-1,\ldots,q^{d_s}-1]$.

Finalement, page 188, il faut lire "endomorphisme" de Frobenius à la place de "automorphisme".

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