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AN ALTERNATIVE CONSTRUCTION OF CONWAY'S SURREAL NUMBERS

Norman L. Alling

Philip Ehrlich

*Presented by P. Ribenboim, F.R.S.C.*

**Abstract:** Another construction of the class No of all surreal numbers is given. In Section 1 No is built up, using more conventional mathematics than Conway used. Corollaries may be found in Section 2.

0. Background. The beginning of this paper is at most a slight reworking of ideas that can be found in Cuesta Dutari's paper [3]. Further important results are due to Harzheim [4, p. 119]. These ideas are then applied to some ideas developed by Conway [2, pp. 4-29]. On synthesizing all of this, we obtain our results.

1. Let  $T$  be an ordered set and let  $L$  and  $R$  be subclasses of  $T$ . We will write  $L < R$  if for all  $a \in L$  and  $b \in R$ ,  $a < b$ . Note that  $\emptyset < R$ ,  $L < \emptyset$  and  $\emptyset < \emptyset$ .

By a Cuesta Dutari cut in  $T$  will be meant a pair of subsets  $(L, R)$  of  $T$ , such that (i)  $L < R$  and (ii) the union of  $L$  and  $R$  is  $T$ . Note that  $L$  or  $R$  may be empty; thus  $(\emptyset, T)$  and  $(T, \emptyset)$  are Cuesta Dutari cuts in  $T$ . Let  $CD(T) = \{(L, R) \mid (L, R) \text{ is a Cuesta Dutari cut in } T\}$ ; then  $CD(T)$  is non-empty.

Let  $M$  be an ordered class that contains  $T$ , such that the order on  $M$  induces the original order on  $T$ . We will call  $M$  an extension of  $T$ . Let  $(L,R) \in CD(T)$ . An element  $x \in M$  will be said to fill  $(L,R)$  in  $M$  if  $L < \{x\} < R$ .

Let  $\chi(T)$ , the Cuesta Dutari completion of  $T$ , be  $T$  union  $CD(T)$ , ordered as follows. Let  $x, y \in \chi(T)$ . (i) If  $x$  and  $y$  are in  $T$ , let them be ordered as they were ordered as elements of  $T$ . (ii) If  $x \in T$  and  $y = (L,R) \in CD(T)$ , then  $x < y$  if  $x \in L$ , and  $y < x$  if  $x \in R$ . (iii) If  $x = (L,R)$  and  $y = (L',R')$  are in  $CD(T)$ , then we will write  $x < y$  if  $L$  is a proper subset of  $L'$ . Then the following hold.

- (0)  $\chi(T)$  is an ordered set.
- (1) (i) Given  $t_0 < t_1$  in  $T$ , there exists  $c \in CD(T)$  for which  $t_0 < c < t_1$ . (ii) Given  $c_0 < c_1$  in  $CD(T)$ , there exists  $t \in T$  for which  $c_0 < t < c_1$ . (iii)  $(\emptyset, T)$  is the least and  $(T, \emptyset)$  is the greatest element of  $\chi(T)$ .
- (2)  $c = (L,R) \in CD(T)$  fills the Cuesta Dutari cut  $(L,R)$  in  $\chi(T)$ .
- (3) Let  $V$  be an extension of  $T$  such that for each  $c = (L,R) \in CD(T)$  there is a  $\sigma(c) \in V$  that fills  $(L,R)$  in  $V$ . Then there exists an order-preserving map  $f$  of  $\chi(T)$  onto a subset  $U$  of  $V$  that contains  $T$ , such that  $f(t) = t$ , for all  $t \in T$ ; and for which  $f(c) = \sigma(c)$ , for all  $c \in CD(T)$ .

Let  $T_0$  be the empty set. Cuesta Dutari [3] then defined  $T_1$  to be  $\chi(T_0)$ , and noted that  $T_1 = \{(\emptyset, \emptyset)\}$ . Assume for some  $\tau$  in  $\mathbb{O}_n$ , the class of all ordinal numbers, that a family  $\{T_\nu\}_{\nu < \tau}$  has been defined. If there is a  $\nu \in \mathbb{O}_n$  such that  $\nu + 1 = \tau$ , then define  $T_\tau$  to be  $\chi(T_\nu)$ ; and if  $\tau$  is a non-zero limit

ordinal, let  $T_\tau$  be the union of  $(T_\nu)_{\nu < \tau}$  [3]. By induction,  $(T_\alpha)_{\alpha \in \text{On}}$  is defined. Let  $T(\tau)$  also be used to denote  $T_\tau$ . Note that  $T(\omega_0)$  is a countable  $\eta_0$ -set. Assume that  $\omega_\xi$  is regular. Harzheim proved that

(4)  $T(\omega_\xi)$  is an  $\eta_\xi$ -set [4, Satz 1, p.119].

For all  $\alpha \in \text{On}$ , let  $O_\alpha$  (the set of "old numbers") be defined to be  $T_\alpha$ , let  $N_\alpha$  (the set of "new numbers") be defined to be  $T_{\alpha+1} - T_\alpha$ , and let  $M_\alpha$  (the set of "made numbers") be defined to be  $T_{\alpha+1}$  [2, p. 29]. Note that  $M_\alpha$  is the union of  $O_\alpha$  and  $N_\alpha$ . Finally, let  $\text{No}$  be defined to be the union of  $(O_\alpha)_{\alpha \in \text{On}}$  [2, p.4]. Note that  $\text{No}$  is also the union of  $(M_\alpha)_{\alpha \in \text{On}}$ . Further note that if  $\alpha < \beta$ , then  $M_\alpha$  is a proper subset of  $M_\beta$ . Given  $x \in \text{No}$ , there exists a unique least  $\beta \in \text{On}$  such that  $x \in M_\beta$ . Let  $b(x)$  be defined to be  $\beta$ . It will be called the birthday of  $x$ . Note:  $b(x)$  is the unique element in  $\text{On}$  such that  $x \in N_{b(x)}$ . The function  $b$  will be called the birth-order function on  $\text{No}$ .

Conway's Simplicity Theorem. Let  $L$  and  $R$  be subsets of  $\text{No}$  for which  $L < R$ . Let  $I = \{y \in \text{No} : L < \{y\} < R\}$ . Then (i)  $I$  is non-empty, and (ii) there exists a unique  $x \in I$  such that  $b(x) \leq b(y)$ , for all  $y \in I$ .

Proof. Let  $U$  be the union of  $L$  and  $R$ . Since  $U$  is a set, there exists a  $\xi \in \text{On}$  such that  $\omega_\xi$  is regular, and for which  $U$  is a subset of  $O_{\omega_\xi}$ . Since  $O_{\omega_\xi}$  is an  $\eta_\xi$ -set (4),  $I \cap O_{\omega_\xi}$  is non-empty.

- (5) Let  $J$  be a non-empty interval in  $No$ . There exists a unique  $x \in J$  such that  $y \in J$  implies  $b(x) \leq b(y)$ .

Indeed,  $b(J)$  is a non-empty subclass of  $On$ . Let  $\beta$  be its least element. Let  $x$  and  $x'$  be in  $J$  such that  $b(x) = \beta = b(x')$ . Assume for a moment that  $x < x'$ . Since  $b(x) = \beta = b(x')$ ,  $x$  and  $x'$  are in  $N_\beta$ , which was defined to be  $T_{\beta+1} - T_\beta$ . Since  $T_{\beta+1} = \chi(T_\beta)$ , we may apply (1,ii) and conclude that there is an element  $t \in T_\beta$  such that  $x < t < x'$ . Since  $t \in T_\beta$ ,  $b(t) < \beta$ . Since  $J$  is an interval that contains  $x$  and  $x'$ ,  $t \in J$ ; which is absurd, proving (5); and thus proving the Theorem.

2. Since  $x$  above is uniquely determined by  $L$  and  $R$ , we may denote  $x$  by the symbol  $\{L|R\}$ . (Cf. [2, p. 4].) Note that  $x$  is completely determined by  $I$ ; thus

- (0) Given any subsets  $L'$  and  $R'$  of  $No$  such that  $L' < R'$ , for which  $\{y \in No: L' < \{y\} < R'\} = I$ , then  $\{L'|R'\} = x = \{L|R\}$ .

We will say that  $L$  and  $L'$  (resp.  $R$  and  $R'$ ) are mutually cofinal (resp. mutually cointitial) if for all  $a \in L$  there exists  $a' \in L'$  such that  $a \leq a'$ , and for all  $a' \in L'$  there exists  $a \in L$  such that  $a' \leq a$  (resp. if for all  $c \in R$  there exists  $c' \in R'$  such that  $c' \leq c$  and for all  $c' \in R'$  there exists  $c \in R$  such that  $c \leq c'$ ). Clearly (0) implies the following:

- (1) If  $L$  and  $L'$  are mutually cofinal and  $R$  and  $R'$  are mutually cointitial, then  $\{L'|R'\} = \{L|R\}$ .

Given  $x \in \mathbb{N}_0$ , it may be written in many ways as  $\{L|R\}$ . Each such  $\{L|R\}$  will be called a representation of  $x$ . Conway also writes  $\{L|R\}$  as  $\{x^L|x^R\}$ , where  $x^L$  is a typical element in  $L$ , and  $x^R$  is a typical element in  $R$  [2, p.4]. A representation  $\{L|R\}$  of  $x$  will be called timely if  $L$  and  $R$  are subsets of  $O_b(x)$ .

(2) Each  $x \in \mathbb{N}_0$  has a timely representation.

Indeed,  $\{\{y \in O_b(x) : y < x\} | \{y \in O_b(x) : y > x\}\}$ , which we will call the Cuesta Dutari out representation of  $x$ , is a timely representation of  $x$  [2, p. 29]. Unless stated to the contrary we will assume, as Conway does (except in Theorem 13, (iii) and (iv) [2, p. 24]), that all representations that we consider are timely. Let  $x, y \in \mathbb{N}_0$ , let  $x = \{x^L|x^R\}$  and let  $y = \{y^L|y^R\}$ .

(3) Let  $\{L|R\}$  and  $\{L'|R'\}$  be (timely) representations of  $x \in \mathbb{N}_0$ ; then  $L$  and  $L'$  are mutually cofinal and  $R$  and  $R'$  are mutually cointial.

(4)  $x \leq y$  iff  $x < y^R$  for all  $y^R$ , and  $x^L < y$  for all  $x^L$ . (Cf.[2, p.4].)

The contrapositive of (4) is the following:

(5)  $x < y$  iff  $x \leq y^L$ , for some  $y^L$ , or  $x^R \leq y$ , for some  $x^R$ .

3. It is not difficult to see that the following holds:

Theorem. There exists a unique order-preserving mapping of  $\mathbb{No}$  onto Conway's class of all surreal numbers, that preserves birth-order.

This result will be expanded on and proved in a subsequent publication.

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Department of Mathematics  
University of Rochester  
Rochester, NY 14627  
U.S.A.

Department of Philosophy  
Northwestern University  
Evanston, IL 60201  
U.S.A.

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Received 25 June, 1985

THE EQUATION  $F(x) + M(x) F(x^{-1}) = 0$   
FOR ADDITIVE F AND MULTIPLICATIVE M  
ON THE POSITIVE CONE OF  $R^n$

Bruce R. Ebanks

*Presented by J. Aczél, F.R.S.C.*

**Abstract.** We present the general solution of the problem in the title and the related problem with  $F$  affine.

Our purpose is to find all solutions of the system

$$\begin{aligned} (1_n) \quad & F(x) + M(x) F(x^{-1}) = 0 & x \in P^n, \\ (2_n) \quad & F(x+y) = F(x) + F(y), & x, y \in P^n, \\ (3_n) \quad & M(xy) = M(x) M(y), & x, y \in P^n, \end{aligned}$$

(all operations componentwise) for functions  $F, M: P^n \rightarrow R$  (reals) where  $P^n = \{x = (\xi_1, \dots, \xi_n) \mid \xi_i > 0, i = 1, \dots, n\}$  is the positive cone of  $R^n$ . Motivation for studying the system  $(1_n) - (3_n)$  comes from two sources. On one hand, this system arises in the process of solving the  $n$ -dimensional fundamental equation of information of multiplicative type. (This connection will be explained elsewhere.) On the other hand, for  $n = 1$  the system  $(1_1) - (3_1)$  is connected with (functionally) homogeneous biadditive forms and has been solved by C.T. Ng [2]. We extend his result to higher dimensions.

**Theorem.** The general solution of  $(1_n) - (3_n)$  for  $n \geq 1$  is given by one of

$$\begin{aligned} (I_n) \quad & F(\xi_1, \dots, \xi_n) = \delta(\xi_k), \quad M(\xi_1, \dots, \xi_n) = \xi_k^2 \quad \text{for fixed } k; \text{ or} \\ (II_n) \quad & F(\xi_1, \dots, \xi_n) = a \operatorname{Im} \phi(\xi_k), \quad M(\xi_1, \dots, \xi_n) = |\phi(\xi_k)|^2 \\ & \text{for fixed } k; \text{ or} \end{aligned}$$

$$(III_n) \quad F(\xi_1, \dots, \xi_n) = a(\xi_k - \xi_j), \quad M(\xi_1, \dots, \xi_n) = \xi_j \xi_k$$

for fixed  $k, j$  ( $k \neq j$ ); or

$$(IV_n) \quad F = 0, \quad M \text{ an arbitrary solution of } (3_n);$$

where  $a(\neq 0)$  is a constant,  $\delta: R \rightarrow R$  is a nontrivial real derivation (i.e.  $\delta$  is additive, and  $\delta(x^{-1}) = -x^{-2}\delta(x)$ ), and  $\phi: R \rightarrow C$  is a nontrivial (field) embedding of the reals into the complex field  $C$ . Obviously,  $(III_n)$  occurs only if  $n \geq 2$ .

The proof of this Theorem will be given after the establishment of a preliminary result. A map  $F: P^n \rightarrow R$  is called affine if

$$(2'_n) \quad F(x+y) = F(x) + F(y) + b, \quad x, y \in P^n,$$

for some constant  $b$ .

Lemma. The general solution of  $(1_1)$ ,  $(2'_1)$ ,  $(3_1)$  with  $F \neq 0$  is given by

$$(I_1) \quad F(s) = \delta(s), \quad M(s) = s^2 \quad (s > 0); \text{ or}$$

$$(II_1) \quad F(s) = a \operatorname{Im} \phi(s), \quad M(s) = |\phi(s)|^2 \quad (s > 0); \text{ or}$$

$$(V_1) \quad F(s) = b(s-1), \quad M(s) = s \quad (s > 0);$$

with constants  $a \neq 0$ ,  $b \neq 0$ , and maps  $\delta$  and  $\phi$  as in the Theorem.

Proof of Lemma: First, observe that  $(2'_1)$  implies  $F(u-v) = F(u) - F(v) - b$  for  $u > v > 0$ . Also,  $F \neq 0$  implies  $M \neq 0$  by  $(1_1)$ , and so  $(3_1)$  and  $(1_1)$  give  $M(1) = 1$ ,  $F(1) = 0$ .

As in [2], we consider the identity  $(1-p)^{-1}(1-q)^{-1} - (1-p)^{-1} = pq(1-p)^{-1}(1-q)^{-1} + q(1-q)^{-1}$  and apply  $F$  to it, keeping  $p$  and  $q$  in  $]0, 1[$ . Multiplying both sides of the resulting equation by  $M((1-p)(1-q))$  and expanding by  $(2'_1)$ ,  $(3_1)$  and  $(1_1)$ , we find after some computation that

$$\begin{aligned}
 (4) \quad & F(p)[1 + M(q) - M(1-q)] + F(q)[1 - M(p) - M(1-p)] - 2F(pq) \\
 & + b[1 - 2M((1-p)(1-q)) - M(pq)] \\
 & = b[M(1-q) + M(q(1-p))], \quad p, q \in ]0,1[.
 \end{aligned}$$

Since the left side of (4) is symmetric in  $p$  and  $q$ , the right side must be also. Thus (4) yields (again using  $(3_1)$ )

$$(5) \quad bM(1-q)[1-M(p)] = bM(1-p)[1-M(q)], \quad p, q \in ]0,1[.$$

If  $b = 0$ , then  $(2'_1)$  becomes  $(2_1)$  and the solution is obtained from Corollary 2.3 in [2] by standard extension techniques. In this case,  $F$  and  $M$  are given by  $(I_1)$  or  $(II_1)$ .

Now suppose  $b \neq 0$ . It is well-known that  $M \neq 0$  for  $M$  satisfying  $(3_1)$  (or  $(3_n)$  for any  $n$ ) implies that  $M$  is never zero and is, in fact, a positive function. Thus from (5) we get

$$[1-M(p)]M(1-p)^{-1} = [1-M(q)]M(1-q)^{-1} = k, \quad p, q \in ]0,1[,$$

for some constant  $k$ . Now the positivity of  $M$  gives

$$(6) \quad kM(1-q) = 1-M(q) < 1, \quad q \in ]0,1[.$$

If  $k = 0$ , then (6) shows that  $M = 1$ . Next (4) and  $(2'_1)$  give  $F(p+q-2pq) = 3b$ ; i.e.  $F = 3b$  on  $]0,1[$  in contradiction to  $(2'_1)$ .

Thus  $k \neq 0$ ,  $M \neq 1$ , and (6) shows that  $M$  is either bounded from above (by  $k^{-1}$ , if  $k > 0$ ) or bounded away from zero from below (by 1, if  $k < 0$ ) on  $]0,1[$ . Now  $(3_1)$  yields  $M(q) = q^c$  for some  $c$ , and (5) requires  $c = 1$ . This form of  $M$  extends to all  $q > 0$  via  $(3_1)$ . Substituting

$$(7) \quad M(s) = s, \quad s > 0,$$

into (4), we find that the map  $G: P^1 \rightarrow R$  defined by

$$(8) \quad G(s) := F(s) + b(1-s), \quad s > 0,$$

satisfies  $(1_1)$ . Moreover  $(8)$  and  $(2'_1)$  show that  $G$  is  $(2_1)$  additive. Therefore, by our argument in the case  $b = 0$ , we have  $G, M$  given by  $(I_1)$  or  $(II_1)$  if  $G \neq 0$ . But  $(7)$  excludes both  $(I_1)$  and  $(II_1)$ . Thus  $G = 0$ , whereupon  $(8)$  and  $(7)$  give  $(V_1)$ .

Proof of Theorem.

Solution  $(IV_n)$  is obvious. Also, for  $n = 1$  the result  $((I_1), (II_1)$  or  $(IV_1))$  follows from the Lemma. Henceforth, we require  $F \neq 0$  and  $n \geq 2$ .

It is well-known [1] that  $F$  and  $M$  satisfying  $(2_n)$  and  $(3_n)$ , respectively, have (unique) representations of the form

$$(9) \quad F(\xi_1, \dots, \xi_n) = \sum_{i=1}^n f_i(\xi_i), \quad M(\xi_1, \dots, \xi_n) = \prod_{i=1}^n m_i(\xi_i)$$

for all  $(\xi_1, \dots, \xi_n) \in P^n$ , where each  $f_i: P^1 \rightarrow R$  is  $(2_1)$  additive and each  $m_i: P^1 \rightarrow R$  is  $(3_1)$  multiplicative ( $1 \leq i \leq n$ ). Note that  $F \neq 0$  induces  $M \neq 0$  via  $(1_n)$ , so  $(3_n)$  and  $(1_n)$  yield  $M(1, \dots, 1) = 1$  and  $F(1, \dots, 1) = 0$ , whence by (9) we get  $m_i(1) = 1$  ( $1 \leq i \leq n$ ). Now, choosing  $k \in \{1, \dots, n\}$  arbitrarily, substituting (9) into  $(1_n)$  with  $\xi_k = s \in P^1$  and all other  $\xi_j = 1$  ( $j \neq k$ ), we obtain

$$(10) \quad [f_k(s) - f_k(1)] + m_k(s)[f_k(s^{-1}) - f_k(1)] = 0, \quad s > 0.$$

Fixing  $k$  temporarily, defining  $f: P^1 \rightarrow R$  by

$$(11) \quad f(s) := f_k(s) - f_k(1), \quad s > 0,$$

and writing  $m$  for  $m_k$ , we see by (10) that  $f$  and  $m$  satisfy (1<sub>1</sub>) and (3<sub>1</sub>). Moreover, by (11) and the additivity of  $f_k$ , we find that  $f$  is (2'<sub>1</sub>) affine, with  $b = f_k(1)$ .

Therefore, by our Lemma and (11), each pair  $f_k, m_k$  ( $1 \leq k \leq n$ ) must be of one of the forms

$$(I_1) f_k = \delta_k, m_k(s) = s^2; \quad (II_1) f_k = a_k \operatorname{Im} \phi_k, m_k(s) = |\phi_k(s)|^2;$$

$$(III'_1) f_k(s) = b_k s, m_k(s) = s; \quad (IV_1) f_k = 0, m_k \text{ arbitrary};$$

with  $a_k, b_k \neq 0$ ,  $\delta_k$  (derivation) and  $\phi_k$  (embedding) as before. By detailed but straightforward calculations (using (1<sub>n</sub>), (9), (10)), we discover the following facts.

- (i) Two distinct component functions  $f_k$  and  $f_j$  can both be nonzero if and only if  $f_k(s) = as = -f_j(s)$  and  $m_j(s) = m_k(s) = s$ .
- (ii)  $f_i = 0$  for some  $i$  entails  $m_i = 1$ .

The result follows immediately, by (9).

By similar methods, we can extend the Lemma also to  $p^n$  for  $n \geq 2$ .

Corollary. The general solution of (1<sub>n</sub>), (2'<sub>n</sub>), (3<sub>n</sub>) for  
 $n \geq 1$  is given by one of (I<sub>n</sub>) - (IV<sub>n</sub>) or

$$(V_n) F(\xi_1, \dots, \xi_n) = b(\xi_k - 1), M(\xi_1, \dots, \xi_n) = \xi_k \text{ for fixed } k,$$

with  $a, b, \delta, \phi$  as before.

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Received 25 March, 1986

Department of Mathematics  
University of Louisville  
Louisville, Kentucky 40292

ON INTEGRABLE SOLUTIONS TO THE BARON-BOYARSKY FUNCTIONAL  
EQUATION II.

Preben Alsholm

*Presented by J. Aczél, F.R.S.C.*

In [1] the present author showed that the functional equation

$$(1) \quad f(x) + f(1-x) = 2r \left| \frac{1}{2} - x \right| f(g_r(x)) \quad \text{for all } x \in [0,1], \text{ where}$$

$$(2) \quad g_r(x) = rx(1-x) \quad (r \in [0,4])$$

has no non-negative Riemann integrable solutions other than  $f=0$  (a.e.) when  $r \in S$ , a set of positive Lebesgue measure contained in  $[3.57,4]$ .

It is essential for this result that the equation (1) be satisfied for all  $x \in [0,1]$ . Indeed, although  $r = 4 \in S$ , it is immediately verified that  $f(x) = (x(1-x))^{-1/2}$ ,  $0 < x < 1$ , satisfies (1) for  $0 < x < \frac{1}{2}$  and for  $\frac{1}{2} < x < 1$ .

In [1] the author did not notice that the equation (1) is in fact the equation satisfied for almost every  $x \in [0,1]$  by the density function for a measure  $\nu$  which is absolutely continuous w.r.t. Lebesgue measure and invariant under the mapping  $g_r$

$$(i.e. \nu(g_r^{-1}(E)) = \nu(E) \text{ for all Lebesgue measurable sets } E).$$

In this context the more natural problem is to solve (1) for a.e.  $x \in [0,1]$  (problem (1a.e.)) instead of the problem treated in [1], i.e. solving (1) for all  $x \in [0,1]$  (problem (1a)). Problem (1a.e.) is precisely the problem solved by M.V. Jakobson (see the references in [1]). He has shown that if  $S$  is defined as the set of parameter values  $r \in [0,4]$  for which  $g_r$  has a non-trivial, absolutely continuous (w.r.t. Lebesgue measure) invariant measure then  $S$  has positive Lebesgue measure. We state the consequences of this for the problems (1a) and (1a.e.). By positive we shall mean non-negative and different from zero on a set of positive Lebesgue measure.

Proposition 1. The problem (1a.e.) has positive Lebesgue integrable solutions iff  $r \in S$ .

Proposition 2. The problem (1a) has no positive Riemann integrable solutions for  $r \in S$  and no positive Lebesgue integrable solutions for  $r \notin S$ .

Thus the result in [1] can be strengthened considerably:

Corollary. The problem (1a) has no positive Riemann integrable solutions for any  $r \in [0,4]$ .

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Received 2 April, 1986

Matematisk Sektion  
Danish Engineering Academy  
DIA-K, Bygning 376  
DK 2800 Lyngby  
Denmark

## POINCARÉ GROUPS AND ORDERED FIELDS

Ján Mináč

*Presented by P. Ribenboim, F.R.S.C.*

**Abstract:** Let  $F$  be a formally real pythagorean field with a finite number of orderings. Let  $G_{F(\sqrt{-1})}$  be the Galois group of the extension  $F(2)|F(\sqrt{-1})$ , where  $F(2)$  is the maximal 2-extension of the field  $F$ . Then the group  $G_{F(\sqrt{-1})}$  is a Poincaré pro-2-group (or 2-adic analytic group) if and only if  $F$  is a superpythagorean field.

**§1. Introduction.** It is well known that  $G_{F(\sqrt{-1})}$  is an abelian pro-2-group if and only if  $F$  is a superpythagorean field. ([1]). In this paper, we shall show that in the above characterisation we can replace abelian by Poincaré pro-2-group, or 2-adic analytic group. This result also extends Theorem 2 in [8], which claims that a Poincaré pro-2-group of cohomological dimension 2 is of the form  $G_{F(\sqrt{-1})}$  if and only if  $F$  is a superpythagorean field with four orderings.

In fact, we shall not prove this result in our paper, instead we shall use this result to prove a more general result.

The field  $F$  is understood to be a formally real pythagorean field with finite number of orderings. Since this paper is a natural continuation of the paper [8], we shall adhere to the same notation as in [8] which also follow [2], [3], [4], [5], [7], [9]. For every unexplained notion or Theorem the reader is referred to the above literature.

52. Theorem 1. The group  $G_{F(\sqrt{-1})} = H$  is the Poincaré pro-2-group of dimension  $n$ ,  $1 \leq n$ , (or analytic pro-2-group of dimension  $n$ ) if and only if  $F$  is a superpythagorean field with  $|\dot{F}/\dot{F}^2| = 2^{n+1}$ . If  $F$  is super-pythagorean field then  $H = \mathbb{Z}_2^n$ .

Theorem 1 is an immediate consequence of the following Theorem.

Theorem 2. Let  $\ell = \text{cd}(H) = \text{cd}(G_{F(\sqrt{-1})})$  be the cohomological dimension of the group  $H$ . Suppose that

$H^{\ell}(H, \mathbb{Z}/2\mathbb{Z}) = H^{\ell}(H, 2) \cong \mathbb{Z}/2\mathbb{Z}$ . Then the cup map

$$\phi: H^1(H, 2) \times H^{\ell-1}(H, 2) \longrightarrow H^{\ell}(H, 2)$$

is a nondegenerate bilinear form if and only if  $F$  is a superpythagorean field.

First we shall prove some lemmas. We shall use the following notation:  $h_i(S) = \dim_{\mathbb{Z}/2\mathbb{Z}} H^i(S, 2)$ .

Lemma 1. Let  $N$  be a pro-2-group with  $\text{cd}(N) = k$ . Put  $H = \mathbb{Z}_2 \times N$ . Then  $h_{k+1}(H) = h_k(N)$ .

Proof. From the Lyndon-Hochschild-Serre spectral sequence (further called L-H-S spectral sequence)

$E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(N, 2)) \implies H^{p+q}(H, 2)$  and from the fact that

$E_2^{p,q} = \{0\}$  if  $p > 1$ , we get the short exact sequence

$$0 \longrightarrow E_2^{1,k} \longrightarrow H^{k+1}(H, 2) \longrightarrow E_2^{0,k+1} \longrightarrow 0$$

Since  $\text{cd}(N) = k$ , we find  $E_2^{0,k+1} = \{0\}$ . Hence

$$H^{k+1}(H, 2) \cong E_2^{1,k} \cong H^k(N, 2),$$

which completes our proof.

Lemma 2. Let  $H = \mathbb{Z}_2 \times N$ , where  $N$  is a pro-2-group with

$$1) \quad \text{cd}(N) = k$$

$$2) \quad H^k(N, 2) \cong H^{k+1}(H, 2) \cong \mathbb{Z}/2\mathbb{Z}$$

Then  $h_k(H) = h_{k-1}(N) + 1$ .

Proof. Consider the L-H-S spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(N, 2)) \implies H^{p+q}(H, 2)$$

We have

$$0 \longrightarrow E_2^{1,k-1} \longrightarrow H^k(H, 2) \longrightarrow E_2^{0,k} \longrightarrow 0$$

Hence  $h_k(H) = h_{k-1}(N) + h_k(N)$

$$= h_{k-1}(N) + 1.$$

This completes our proof.

In the following Lemma 3 we use the same notation as in Theorem 1.

Lemma 3. Let  $\text{st}(F) = \ell \geq 2$ ,  $h_\ell(H) = 1$ . Then

$$\underline{h_{\ell-1}(H) \leq h_1(H)}$$

Furthermore if there exists no valuation  $V$  fully compatible with  $T_F$  such that  $\text{st}(F_V) = 2$  or  $F_V$  has one or two orderings, then

$$\underline{h_{\ell-1}(H) < h_1(H)}.$$

The following Lemma 3' is used in the proof of Lemma 3. Because of lack of space we shall not prove the Lemma 3, but only Lemma 3' which has a proof similar to that of Lemma 3.

Lemma 3'. Let  $\text{st}(F) = m \geq 1$ .

Then

$$h_m(H) \leq h_1(H)$$

Proof. Suppose that there exists a valuation  $V$  on  $F$  such that  $V$  is compatible with  $T_F$ ,  $|\hat{F}/\hat{F}^2 U_V| = 2^t$  and  $X_{F_V}$  is decomposable space or  $|X_{F_V}| = 1$ . From Theorem B in [11] we get

$$H \cong \mathbb{Z}_2^t \times N,$$

where  $N = G_{F_V(\sqrt{-1})}$ . From Lemma 1 we get

$$1) \quad h_m(H) = h_{m-t}(N). \quad \text{Note that } m-t = \text{st}(F_V)$$

From 1) and from relation  $h_t(\mathbb{Z}_2^t) = 1$  we see that we may assume that  $X_F$  is decomposable space.

We shall prove our claim by induction on  $\text{cl}(F)$ . If  $\text{cl}(F) = 2$ , then  $F$  is a superpythagorean field,  $H \cong \mathbb{Z}_2^m$ . Hence  $h_m(H) = 1 \leq h_1(H)$ .

Suppose that Lemma 3' is true for any field  $K$  with  $\text{cl}(K) < \text{cl}(F)$ . Let

$$X_F = X_1 U \dots U X_a U Y_1 U \dots U Y_b, \quad 2 \leq a+b,$$

where  $X_1, \dots, X_a, Y_1, \dots, Y_b$  are connected components of  $X_F$  with  $|X_1| = \dots = |X_a| = 1$ ,  $4 \leq |Y_1|, \dots, 4 \leq |Y_b|$ .

Then

$$H \cong \prod_{j=1}^b G_{F_j(\sqrt{-1})} * L,$$

where  $L$  is a free pro-2-group of rank  $a+b-1$  and  $X_{F_j} = Y_j$ ,  $j=1, \dots, b$ ,  $\text{cl}(F_j) < \text{cl}(F)$  (see [7] page 4).

If  $2 \leq m$ , we have

$$\begin{aligned} h_m(H) &= \sum_{j=1}^b h_m(H_j) \\ &\leq \sum_{j=1}^b h_1(H_j), \quad \text{since } \text{st}(F_j) \leq m \text{ and } \text{cl}(F_j) < \text{cl}(F) \\ &= \sum_{j=1}^b (n_j - 1), \quad \text{where } |\hat{F}_j/\hat{F}_j^2| = 2^{n_j} \\ &= \sum_{j=1}^b n_j - b \end{aligned}$$

From relations  $h_1(H) = a-1 + \sum_{j=1}^b n_j$ ,  $2 \leq a+b$ , we find

$$h_m(H) < h_1(H) .$$

Lemma 3' is proved.

Lemma 4. Let  $N$  be a pro-2-group with  $cd(N) = m < \infty$ ,  $H^m(N, 2) \cong \mathbb{Z}/2\mathbb{Z}$ . Suppose that the cup map  $\phi$

$$\phi: H^{m-1}(N, 2) \times H^1(N, 2) \rightarrow H^m(N, 2)$$

is a degenerate bilinear form. Then also the cup map  $\psi$

$$\psi: H^m(\mathbb{Z}_2 \times N, 2) \times H^1(\mathbb{Z}_2 \times N, 2) \rightarrow H^{m+1}(\mathbb{Z}_2 \times N, 2)$$

is a degenerate bilinear form.

Proof. It is an immediate consequence of the compatibility of spectral sequences with the cup map ([4]).

Proof of the Theorem 1. Let  $F$  be a superpythagorean field with  $|\hat{F}/\hat{F}^2| = 2^{n+1}$ ,  $1 \leq n$ . Then  $G_F(\sqrt{-1}) = \mathbb{Z}_2^n$ . Hence  $H = G_F(\sqrt{-1})$  is both Poincaré and an analytic pro-2-group of dimension  $n$ . ([1], [10]).

Since any analytic pro-2-group of finite cohomological dimension is a Poincaré group, it is enough to prove that if  $G_F(\sqrt{-1})$  is a Poincaré group, then  $F$  is a superpythagorean field. Hence it is enough to prove Theorem 2. Suppose that the cup map  $\phi: H^1(H, 2) \times H^{k-1}(H, 2) \rightarrow H^k(H, 2) \cong \mathbb{Z}/2\mathbb{Z}$  is a non degenerate bilinear form. Then by Lemma 3, and part c) of the proof of the Theorem in [7]  $F$  is a superpythagorean field or  $H \cong \mathbb{Z}_2^k \times G_{F_V}(\sqrt{-1})$ , where  $F_V$  is the residue field of valuation  $V$  on  $F$  fully compatible with  $T_F$  and  $|\hat{F}/\hat{F}^2 U_V| = 2^k$ ,  $st(F_V) = cd(G_{F_V}(\sqrt{-1})) = cd(N) = 2$ ,  $H^2(N, 2) \cong \mathbb{Z}/2\mathbb{Z}$ . But from

Theorem 2 in [8] we find that the pairing

$H^1(N, 2) \times H^1(N, 2) \longrightarrow H^2(N, 2)$  is non degenerate if and only if

$F_V$  is a superpythagorean field. Hence from Lemma 4 and

Proposition 5.11 in [3] we find that  $F$  is a superpythagorean field. This proves our assertion.

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Acknowledgement: This paper has been written whilst pursuing a doctoral degree at Queen's University in Kingston. I am indebted to Professors Paulo Ribenboim and T. M. Viswanathan for their encouragement and suggestions. I am also indebted to Professor Bruno Kahn for his stimulating letter where he also suggested that the study of Poincaré groups might be interesting.

Department of Mathematics and Statistics  
Queen's University  
Kingston, Ontario  
K7L 3N6

Received 28 April, 1986

Note added in the proof. I am also grateful to the referee for his comments and improvement of the exposition.

CORRIGENDUM FOR GALOIS  
GROUPS OF SOME 2-EXTENSIONS  
OF ORDERED FIELDS

Ján Mináč

*Presented by P. Ribenboim, F.R.S.C.*

On page 106, line 6 of the above article,  
(C.R. Math. Rep. Sci. Canada - Vol. VIII, No. 2,  
Apr. 1986), the text should read

$$I = \dim_{\mathbb{Z}/2\mathbb{Z}} \hat{F}/\hat{F}^2 U_V \quad \text{rather than} \quad 2^I = |\hat{F}/\hat{F}^2 U_V|$$

Department of Mathematics  
and Statistics  
Queen's University  
Kingston, Ontario  
K7L 3N6

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Received 28 April, 1986

A NEW PROOF OF A THEOREM BY REES

Leif Melkersson

*Presented by P. Ribenboim, F.R.S.C.*

In this note we give a short proof of the following theorem by Rees ([2], p. 27).

**Theorem:** Let  $Q$  be an analytically unramified local ring with full ring of fractions  $R$  and let  $\lambda_1, \dots, \lambda_r$  be elements of  $R$ . Then the integral closure of  $A = Q[\lambda_1, \dots, \lambda_r]$  in  $R$  is a finite  $A$ -module.

Recall that a noetherian ring  $A$  is called a Nagata ring if for any prime ideal  $\mathfrak{p}$  in  $A$  the domain  $A/\mathfrak{p}$  has finite integral closure in any finite extension field of its field of fractions. In the proof the following results about Nagata rings are used (cf. Matsumura [1])

- 1) A complete local ring is a Nagata ring
- 2) A finitely generated algebra over a Nagata ring is again a Nagata ring
- 3) If  $A$  is a reduced Nagata ring then the integral closure of  $A$  in its full ring of fractions is a finite  $A$ -module.

The last result is shown by considering the commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\prod_{i=1}^n A/\mathfrak{p}_i} & \prod_{i=1}^n K_i \\
 & \searrow & \uparrow \cong \\
 & & B
 \end{array}$$

where  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  are the distinct minimal primes of  $A$ ,  $K_i$  the field of fractions of  $A/\mathfrak{p}_i$  for  $i = 1, \dots, n$  and  $B$  is the full ring of fractions of  $A$ .

Proof of the theorem:

If  $Q$  is complete the result follows from 1)-3) above, since  $A$  has full ring of fractions equal to  $R$ . If  $Q$  is not complete let  $\bar{Q}$  be its completion with full ring of fractions  $\bar{R}$  and  $B$  the integral closure of  $A$  in  $R$ . Then we have injective homomorphisms

$$\bar{Q} \otimes_Q A \rightarrow \bar{Q} \otimes_Q B \rightarrow \bar{Q} \otimes_Q R \rightarrow \bar{R}. \text{ Now } \bar{A} = \bar{Q} \otimes_Q A \text{ is a finitely}$$

L. Melkersson

generated  $\bar{Q}$ -subalgebra of  $\bar{R}$  and  $\bar{B} = \bar{Q} \otimes_{\bar{Q}} B$  is contained in its integral closure in  $\bar{R}$ , which as noted above is a finitely generated  $\bar{A}$ -module. Since  $\bar{A}$  is a noetherian ring, it follows that  $\bar{B}$  is a finitely generated  $\bar{A}$ -module. As  $\bar{Q}$  is faithfully flat over  $Q$ , it follows from this that  $B$  must be a finitely generated  $A$ -module.

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Received 1 May, 1986

Department of Mathematics  
 University of Lund  
 Box 118  
 S-221 00 LUND  
 Sweden

A PRESENTATION FOR THE REPRESENTATION RING AND  
THE SOLUTION OF A PROBLEM OF E. ARTIN

Victor Snaith<sup>1</sup>, F.R.S.C.

**11:** According to [Car, §1.2] in 1923 Emil Artin [A] introduced L-functions into the theory of Galois representations in order to determine the multiplicative relations between Dedekind zeta functions of number fields (see [A, pp. 96/97]).

Below I will describe the solution to Artin's problem together with the following related results:-

(a) A presentation for  $R(G)$ , the complex representation ring of a finite group. This solves the problem posed in [Ser., p. 71 (footnote)].

(b) A formula in terms of Gauss sums and the topology of a Galois representation,  $\nu: \text{Gal}(L/K) \rightarrow U(n)$ , for the Deligne-Langlands local root numbers (or local constants),  $W_K(\nu)$ , when  $K$  is a local field.

The proofs are based on a very concise topological proof of Brauer's induction theorem which is explicit. Details appear in [Sn], which also proves the orthogonal and symplectic analogues (see [Mar pp. 63,73]).

**12: Explicit Brauer induction**

Let  $G$  be a finite group and let  $R_+(G, \pi)$  denote the Grothendieck group of isomorphism classes of subhomomorphisms  $(G \supset H \xrightarrow{\pi} \pi)$ , up to conjugation in  $G$  and  $\pi$ . Denote by  $\lambda(\rho): G \rightarrow \Sigma_n \pi$  ( $n = [G:H]$ ) the monomial homomorphism obtained by imitating the induction construction. That is, if  $(x_1, \dots, x_n)$  is a set of coset representatives in  $G/H$  and if

$$(2.1) \quad \left\{ \begin{array}{l} gx_i = x_{\sigma(g)} h(i, g) \quad \text{for } 1 \leq i \leq n, \\ \sigma(g) \in \Sigma_n, \quad h(i, g) \in H \end{array} \right.$$

then

$$(2.2) \quad \lambda(g) = \sigma(g)(\rho(h(1, g)), \rho(h(2, g)), \dots, \rho(h(n, g))) \in \Sigma_n \pi.$$

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<sup>1</sup>Research partially supported by NSERC grant #A4633.

This allows  $R_+(G, \pi)$  to be described alternatively as a Grothendieck group of monomial homomorphisms,  $G \rightarrow \Sigma_p \wr \pi$ . Here  $\Sigma_n \wr \pi$  is the wreath product or semidirect product of  $\Sigma_n$  with  $\pi^n$ . Define

$$(2.3) \quad \begin{cases} b_n: R_+(G, \Sigma_n \wr S^1) \rightarrow R(G) & \text{by} \\ b_n(G > H \xrightarrow{\rho} \Sigma_n \wr S^1) = \text{Ind}_H^G(\rho). \end{cases}$$

Here we view  $\Sigma_n \wr S^1$  as the normaliser of the diagonal torus in the unitary group,  $U(n)$ .

Now let  $\nu: G \rightarrow U(n)$  be a representation through which  $G$  acts on  $U(n)/(\Sigma_n \wr S^1)$  with orbit space,  $M$ , say. For each conjugacy class of subgroup,  $V$ , in  $G$  let  $M_{(V)}$  be the set of orbits of type  $(V)$ . Let  $(M_\alpha)$  denote the set of path components of these orbit-type manifolds,  $M_{(V)}$ , as  $V$  varies. Choose one  $g_\alpha \in U(n)$  whose orbit lies in  $M_\alpha$  for each  $\alpha$ .

Set  $H(g_\alpha, \nu) = \nu^{-1}(g_\alpha (\Sigma_n \wr S^1) g_\alpha^{-1})$  and define, in  $R_+(G, \Sigma_n \wr S^1)$ ,

$$(2.4) \quad r_G(\nu) = \sum_{\alpha} \kappa^*(M_\alpha)(G > H(g_\alpha, \nu)) \xrightarrow{g_\alpha^{-1} \nu g_\alpha} \Sigma_n \wr S^1.$$

Here  $\kappa^*(M_\alpha)$  is the Euler characteristic of  $M_\alpha$  with respect to compactly supported cohomology. In terms of usual Euler characteristics

$$(2.5) \quad \kappa^*(M_\alpha) = \kappa(\bar{M}_\alpha) - \kappa(\bar{M}_\alpha - M_\alpha).$$

Using Lefschetz' fixed point theorem one proves

### (2.6) Explicit Brauer Induction Theorem

(i)  $r_G(\nu) \in R_+(G, \Sigma_n \wr S^1)$  depends only on  $\nu: G \rightarrow U(n)$  as a representation.

(ii)  $b_n r_G(\nu) = \nu \in R(G)$

(iii) In  $R(G)$

$$1 = \sum_{\alpha} \kappa^*(M_\alpha) \text{Ind}_{H(g_\alpha, \nu)}^G(1) = \sum_{(V)} \kappa^*(M_{(V)}) \text{Ind}_V^G(1).$$

(2.7) Notice that any  $\phi: G \rightarrow \Sigma_m S^1$ , as a representation, is (almost canonically) the sum of representations induced from one-dimensional representations. Also analogous orthogonal and symplectic formulae hold for  $\nu: G \rightarrow SO(n)$  and  $\nu: G \rightarrow Sp(n)$ .

### 13: A presentation for $R(G)$

Let  $G$  be a finite group. Set

$$(3.1) \quad R_+(G) = \bigoplus_{n \geq 1} R_+(G, \Sigma_n S^1).$$

(3.2) Define three types of relations  $R_+(G)$ :-

(a) For  $H < J < G$  and  $d = [J:H]$

$$0 = (G \supset H \xrightarrow{\rho} \Sigma_n S^1) - (G \supset J \xrightarrow{\lambda(\rho)} \Sigma_{dn} S^1).$$

(b) For  $\nu: G \rightarrow \Sigma_n S^1$  and  $\mu: G \rightarrow \Sigma_m S^1$ ,

$$0 = r_G(\nu \otimes \mu) - r_G(\nu) - r_G(\mu).$$

(c) For  $\nu: G \rightarrow \Sigma_n S^1$ ,

$$0 = r_G(\nu) - \nu = \sum_{g_\alpha \neq 1} \kappa^\#(M_\alpha)(G \supset H(g_\alpha, \nu) \xrightarrow{g_\alpha^{-1} \nu g_\alpha} \Sigma_n S^1)$$

which is (2.4) with the term corresponding to  $g_\alpha = 1$  deleted.

### 13.3: Theorem

Let  $G$  be a finite group then  $\{b_n; n \geq 1\}$  induces an isomorphism

$$\hat{B}: R_+(G)/\sim \xrightarrow{\cong} R(G).$$

Proof Split  $\hat{B}$  by writing  $x \in R(G)$  as  $x = \nu - \mu$  with  $\nu, \mu$  monomial (as in 13.2 (a)) and sending  $x$  to  $r_G(\nu) - r_G(\mu) \in R_+(G)/\sim$ .

### (3.4) Remark

Relation 13.2 (a) immediately makes  $R_+(G)/\sim$  a quotient of  $R_+(G, S^1)$  so that one may, from 13.3, describe

$$\ker(b_1: R_+(G, S^1) \rightarrow R(G)).$$

14: Relations between Artin L-functions

Let  $L/K$  be a finite extension of number fields with Galois group,  $G(L/K)$ . To a representation  $\nu: G(L/K) \rightarrow U(n)$  Artin associated a non-trivial meromorphic L-function

$$(4.1) \quad L_K(s, \nu)$$

such that (see [A;Mar])

$$(4.2) \quad \left\{ \begin{array}{l} (i) \quad L_K(s, \nu \otimes \mu) = L_K(s, \nu) \cdot L_K(s, \mu) \\ (ii) \quad L_K(s, \text{Ind}_{G(L/F)}^{G(L/K)}(\rho)) = L_F(s, \rho) \end{array} \right.$$

When  $\nu = 1$ , the trivial one-dimensional representation, then

$$L_K(s, 1) = \zeta_K(s) ,$$

the Dedekind zeta function of  $K$ .

Artin also showed that  $L_K(s, \_)$  embeds  $R(G(L/K))$  into the group of non-zero meromorphic functions.

By 14.2 (i)/(ii) we see that all L-functions are generated by applying  $L_K(s, \_)$  to monomial representations

$\nu: G(L/K) \rightarrow \mathbb{Z}_n[S^1]$ . Therefore all relations between L-functions are generated by those obtained by applying  $L_K(s, \_)$  to the relations of 13.2 (a)/(c). However 13.2 (a) gives only (4.2) (ii) while, by 14.2 (i), 13.2 (b) gives relations dependent on those from 13.3 (c).

Hence we have:

(4.3) Theorem

The relations between Artin L-functions are generated by 14.2 (i)/(ii) and

$$1 = \prod_{g_\alpha \neq 1} L_{F_\alpha}(s, g_\alpha^{-1} \nu g_\alpha)^{h^*(M_\alpha)}$$

where  $H(g_\alpha, \nu) = G(L/F_\alpha)$  and the notation is as in 13.2 (c).

(4.4) Remark

Notice that  $\ker(b_1: R_+(G, S^1) \rightarrow R(G))$  determines the relations between L-functions and that the subgroup

$$\ker(R_+(G, \{1\}) \rightarrow R_+(G, S^1) \rightarrow R(G))$$

determines the relations between zeta functions. Hence Theorem 4.3 determines generates for all the relations between zeta functions of number fields.

§5: A formula for Deligne-Langlands local constants

The Artin root number of a Galois representation of number fields,  $\nu$  as in §4, is  $W_K(\nu)$  in the functional equation

$$(5.1) \quad L_K(s, \nu) = W_K(\nu) L_K(1-s, \bar{\nu}) .$$

$W_K(\nu)$  is a complex number of unit norm and is a very fundamental invariant of  $\nu$  and  $K$ . For each prime,  $w$ , of  $K$  we may restrict  $\nu$  to a decomposition group for  $w$  and obtain a local Galois representation

$$(5.2) \quad \nu_w: G(L_w/K_w) \longrightarrow U(n) .$$

Local Galois representations have Deligne-Langlands root numbers [De 2], satisfying

$$(5.3) \quad W_K(\nu) = \prod_w W_{K_w}(\nu_w) .$$

The Deligne-Langlands root numbers satisfy ( $K$  a local field)

§5.4:

$$(i) \quad W_K(\phi \otimes \mu) = W_K(\phi) W_K(\mu)$$

$$(ii) \quad W_K(\text{Ind}_{G(L/F)}^{G(L/K)}(\rho)) = W_F(\rho) (\lambda_{F/K})^{\dim \rho}$$

$$\text{where } \lambda_{F/K} = W_K(\text{Ind}_{G(L/F)}^{G(L/K)}(1)) .$$

(iii) When  $\dim \rho = 1$ ,  $W_K(\rho)$  is given in terms of Gauss sums [Mar].

(iv) Since  $\text{Ind}_{G(L/F)}^{G(L/K)}(1) = Z$ , say, is an orthogonal representation  $\lambda_{F/K}$  is given in terms of Gauss sums and the second Stiefel-Whitney class of  $Z$  by [De].

By properties §5.4 (i) - (ii) the following formula determines the local and global root numbers in terms of Gauss sums and topological invariants of  $\nu$ . It is deduced from §2.6 (ii) and §5.4 (ii).

(5.5) Theorem

Let  $L/K$  be a finite Galois extension of local fields and let  $\nu: G(L/K) \longrightarrow U(n)$  be a representation. Then, with the notation of (2.4)

$$W_K(\nu) = \prod_{\alpha} W_{F_{\alpha}}(g_{\alpha}^{-1} \nu g_{\alpha})^{N^{\#}(M_{\alpha})}$$

where  $G(L/F_\alpha) = H(g_\alpha, \nu)$ . Since each  $M_\alpha$  contributes the same  $W_{F_\alpha}(g_\alpha^{-1} \nu g_\alpha) = W_{F_\alpha}(\nu | G(L/F_\alpha))$  we may re-write this formula as a sum of orbit types, (V):

$$W_K(\nu) = \sum_{(V)} \pi W_{F(V)}(\nu | G(L/F(V)))^{N^\#(M(V))}$$

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Received 17 May, 1986

The University of Western Ontario,  
London, Ontario, N6A 5B7,  
Canada.

## A GENERALIZED INVERSION FORMULA FOR THE CONTINUOUS JACOBI TRANSFORM

Ahmed I. Zayed

*Presented by J.G. Arthur, F.R.S.C.***Abstract**

In this paper we extend the definition of the continuous Jacobi transform to a larger class of functions and obtain a generalized inversion formula for the continuous Jacobi transform. As a by-product of our technique we obtain a necessary and sufficient condition for an analytic function  $F(\lambda)$  in  $\text{Re } \lambda > 0$ , to be the continuous Jacobi transform of a generalized function.

1. Introduction. In [1], Butzer, Stens and Mehrens derived an inversion formula for the continuous Legendre transform  $\hat{f}(\lambda)$  of a function  $f(x) \in L^2(-1,1)$  given by

$$\hat{f}(\lambda) = \frac{1}{2} \int_{-1}^1 f(x) P_{\lambda}(x) dx; \quad \lambda \geq -\frac{1}{2}$$

where  $P_{\lambda}(x)$  is the Legendre function.

This result was extended by Deeba and Koh [2] to the continuous Jacobi transform  $\hat{f}^{(\alpha, \beta)}(\lambda)$  ( $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$  and  $\alpha + \beta = 0$ ) of a function  $f(x) \in L^2\{(-1,1), w(x)\}$  given by

$$\hat{f}^{(\alpha, \beta)}(\lambda) = \frac{1}{2^{\alpha + \beta + 1}} \int_{-1}^1 f(x) P_{\lambda}^{(\alpha, \beta)}(x) w(x) dx; \quad \lambda \geq -\frac{1}{2}$$

where  $P_{\lambda}^{(\alpha, \beta)}(x)$  is the Jacobi function and  $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ . The results of [2] imply those in [1] when  $\alpha = 0 = \beta$ .

The main purpose of this paper is to derive a more general inversion formula for the continuous Jacobi transform which will include the results of [1] and [2] as a special case. Not only do we extend the results of [2] to the case where  $-1 < \alpha, \beta < 1$ ;  $\alpha + \beta = 0$  but also to the case where  $f(x)$  is a generalized function. As a by-product of our approach we obtain

a more general definition of the continuous Jacobi transform which enables us to give a necessary and sufficient condition for an analytic function  $F(\lambda)$  in  $\text{Re } \lambda > 0$ , to be the continuous Jacobi transform of a generalized function  $f(t)$ .

## 2. Preliminaries:

For any real numbers  $a$ ,  $b$  and  $c$  with  $c \neq 0, -1, -2, \dots$ , the hypergeometric function  $F(a, b; c; z) = {}_2F_1(a, b; c; z)$  is given by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad |z| < 1 \quad (2.1)$$

The Jacobi function  $p_{\lambda}^{(\alpha, \beta)}(x)$  of the first kind is defined by

$$p_{\lambda}^{(\alpha, \beta)}(x) = \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\lambda + 1)} F(-\lambda, \lambda + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2}), \quad x \in (-1, 1] \quad (2.2)$$

where  $\alpha, \beta > -1$ ,  $\lambda \in \mathbb{R}$  and  $\lambda + \alpha + 1 \neq 0, -1, -2, \dots$ .

$$\text{Let} \quad p_{\lambda}^{(\alpha, \beta)}(x) = (1-x)^{\alpha} (1+x)^{\beta} p_{\lambda}^{(\alpha, \beta)}(x) \quad (2.3)$$

$$\text{and} \quad L = (1-x^2) \frac{d^2}{dx^2} + [(\alpha - \beta) + (\alpha + \beta - 2)x] \frac{d}{dx} + (\alpha + \beta). \quad (2.4)$$

With some easy computations, one can show that the conjugate of  $L$  is

$$L^* = (1-x^2) \frac{d^2}{dx^2} - [(\alpha - \beta) + (\alpha + \beta + 2)x] \frac{d}{dx}. \quad (2.5)$$

and that

$$L p_{\lambda}^{(\alpha, \beta)}(x) = -\lambda(\lambda + \alpha + \beta + 1) p_{\lambda}^{(\alpha, \beta)}(x). \quad (2.6)$$

We define the constant  $H_0(\lambda)$  by

$$H_0(\lambda) = \frac{\Gamma^2(\lambda + \frac{1}{2})}{\Gamma(\lambda + \alpha + \frac{1}{2})\Gamma(\lambda + \beta + \frac{1}{2})}. \quad (2.7)$$

### 3. Jacobi transform of a class of generalized functions:

Throughout the rest of this paper I will denote the open interval  $(-1,1)$ .

Fix  $\alpha$  and  $\beta$  such that  $\alpha, \beta > -1$ . We define the space  $H_{\alpha, \beta}$  as the space of all infinitely differentiable functions  $\phi(x)$  defined on  $I$  such that

$$\gamma_{k,c}(\phi) = \sup_{x \in I} |(1-x)^{-\alpha}(1+x)^c L^k \phi(x)| < \infty \quad (3.1)$$

for any nonnegative integer  $k$  and any  $c$  such that  $\max(-\beta, 0) < c < 1$ .

It can be shown that  $H_{\alpha, \beta}$  is a Frechet space which contains the space  $\mathcal{D}(I)$  as a subspace. Since  $p_{\lambda}^{(\alpha, \beta)}(x)$  is a member of  $H_{\alpha, \beta}$ , we can define the continuous Jacobi transform  $F^{(\alpha, \beta)}(\lambda)$  of any member  $f(x)$  of the space  $H_{\alpha, \beta}^*$  by

$$F^{(\alpha, \beta)}(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \langle f(x), p_{\lambda}^{(\alpha, \beta)}(x) \rangle. \quad (3.2)$$

Some properties of the continuous Jacobi transform will be given in the following proposition whose proof will be published somewhere else. The following lemma is needed.

#### Lemma 3.1

$$\frac{d}{d\lambda} (p_{\lambda}^{(\alpha, \beta)}(x)) \in H_{\alpha, \beta} ; \quad \text{Re } \lambda > 0 .$$

Sketch of the proof: The bulk of the proof is to show that

$$\gamma_{0,c} \left( \frac{d}{d\lambda} p_{\lambda}^{(\alpha, \beta)}(x) \right) < \infty \quad (3.3)$$

For if we assume that (3.3) holds, then it will follow from (2.6) that

$$L^k p_{\lambda}^{(\alpha, \beta)}(x) = [n(\lambda)]^k p_{\lambda}^{(\alpha, \beta)}(x) \quad \text{where} \quad n(\lambda) = -\lambda(\lambda + \alpha + \beta + 1).$$

$$\text{Thus,} \quad L^k \frac{d}{d\lambda} p_{\lambda}^{(\alpha, \beta)}(x) = p_{\lambda}^{(\alpha, \beta)}(x) \left[ \frac{d}{d\lambda} [n(\lambda)]^k \right] + [n(\lambda)]^k \frac{d}{d\lambda} p_{\lambda}^{(\alpha, \beta)}(x)$$

and

$$\gamma_{k,c} \left( \frac{d}{d\lambda} p_{\lambda}^{(\alpha, \beta)}(x) \right) = \frac{d}{d\lambda} [n(\lambda)]^k \gamma_{0,c} \left( p_{\lambda}^{(\alpha, \beta)}(x) \right) + [n(\lambda)]^k \gamma_{0,c} \left( \frac{d}{d\lambda} p_{\lambda}^{(\alpha, \beta)}(x) \right) < \infty .$$

The proof of (3.3) will be shown somewhere else.

Q.E.D.

Proposition 3.1: Let  $f(x) \in H_{\alpha, \beta}^*$ ,  $F^{(\alpha, \beta)}(\lambda)$  be its continuous Jacobi transform

and  $\phi(x) \in \mathcal{D}(I)$ . Then

- i)  $F^{(\alpha, \beta)}(\lambda) = O(\lambda^q)$  for some  $q \geq 0$  as  $\lambda \rightarrow \infty$ .
- ii)  $F^{(\alpha, \beta)}(\lambda)$  is analytic in  $\text{Re } \lambda > 0$  and  $\frac{dF^{(\alpha, \beta)}(\lambda)}{d\lambda} = \frac{1}{2^{\alpha+\beta+1}} \langle f(x), \frac{d}{d\lambda} P_{\lambda}^{(\alpha, \beta)}(x) \rangle$
- iii) the continuous Jacobi transform  $\phi^{(\alpha, \beta)}(\lambda)$  of  $\phi(x)$  given by

$$\phi^{(\alpha, \beta)}(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^1 \phi(x) (1-x)^{\alpha} (1+x)^{\beta} P_{\lambda}^{(\alpha, \beta)}(x) dx$$

is rapidly decreasing as  $\lambda \rightarrow \infty$ , i.e.,  $\phi^{(\alpha, \beta)}(\lambda) = O(\lambda^{-p})$  for any  $p > 0$  as

$\lambda \rightarrow \infty$ .

Lemma 3.2: Let  $f(x) \in H_{\alpha, \beta}^*$ ,  $F^{(\alpha, \beta)}(\lambda)$  be its continuous Jacobi transform

and  $\phi(x) \in \mathcal{D}(I)$ . Let

$$\bar{\phi}^{(\beta, \alpha)}(\lambda) = \int_{-1}^1 \phi(x) P_{\lambda}^{(\beta, \alpha)}(-x) dx.$$

Then

- i)  $\bar{\phi}^{(\beta, \alpha)}(\lambda)$  is rapidly decreasing as  $\lambda \rightarrow \infty$ .
- ii) for any fixed real number  $r$  with  $0 < r < \infty$

$$\begin{aligned} & \int_0^r \bar{\phi}^{(\beta, \alpha)}(\lambda^{-1/2}) \langle f(y), P_{\lambda^{-1/2}}^{(\alpha, \beta)}(y) \rangle H_0(\lambda) \lambda \sin \pi \lambda d\lambda \\ &= \langle f(y), \int_0^r P_{\lambda^{-1/2}}^{(\alpha, \beta)}(y) \bar{\phi}^{(\beta, \alpha)}(\lambda^{-1/2}) H_0(\lambda) \lambda \sin \pi \lambda d\lambda \rangle \end{aligned}$$

where  $H_C(\lambda)$  is given by (2.7).

4. Characterization of the continuous Jacobi transform:

The main result of this section is theorem 4.1 which gives a characterization

of the continuous Jacobi transforms of members of the space  $H_{\alpha,\beta}^*$ ;  $\alpha+\beta = 0$ ,  $-1 < \alpha, \beta < 1$ .

Theorem 4.1. A necessary and sufficient condition for a function  $F(\lambda)$  to be

the continuous Jacobi transform of some generalized function in the space  $H_{\alpha,\beta}^*$

according to definition (3.6) is that  $F(\lambda)$  is analytic in the half-plane

$\text{Re } \lambda > 0$  and bounded according to

$$|F(\lambda)| \leq Q(|\lambda|), \quad \text{Re } \lambda > 0$$

where  $Q(|\lambda|)$  is a polynomial in  $|\lambda|$ .

Sketch of the proof: Necessity: was proved in proposition 3.1. Sufficiency:

The main part of the proof, which will be shown elsewhere, is dedicated to showing that for some positive integer  $m$

$$g(x) = 4 \int_0^m [\eta(\lambda - \frac{1}{2}) + 1]^{-m} F(\lambda - \frac{1}{2}) P_{\lambda - \frac{1}{2}}^{(\beta, \alpha)}(-x) H_0(\lambda) \lambda \sin \pi \lambda \, d\lambda.$$

is in  $H_{\alpha,\beta}^*$ . Now we set

$$f(x) = (L^* + 1)^m g(x)$$

which is also in  $H_{\alpha,\beta}^*$  since  $L^*$  is a continuous linear operator on  $H_{\alpha,\beta}^*$ .

Therefore,  $\frac{1}{2} \langle f(x), P_{\lambda - \frac{1}{2}}^{(\alpha, \beta)}(x) \rangle = \frac{1}{2} \langle (L^* + 1)^m g(x), P_{\lambda - \frac{1}{2}}^{(\alpha, \beta)}(x) \rangle$

$$= \frac{1}{2} \langle g(x), (L^* + 1)^m P_{\lambda - \frac{1}{2}}^{(\alpha, \beta)}(x) \rangle = \frac{1}{2} [\eta(\lambda - \frac{1}{2}) + 1]^m \langle g(x), P_{\lambda - \frac{1}{2}}^{(\alpha, \beta)}(x) \rangle$$

$$= [\eta(\lambda - \frac{1}{2}) + 1]^m [\eta(\lambda - \frac{1}{2}) + 1]^{-m} F(\lambda - \frac{1}{2}) = F(\lambda - \frac{1}{2}).$$

Q.E.D.

5. Inversion of the Jacobi transform:

Lemma 5.1. Let  $\phi(y) \in \mathcal{D}(I)$  and  $\tilde{\phi}^{(\beta, \alpha)}(\lambda)$  be given as in (3.21). Then

$$2 \int_0^r P_{\lambda-\frac{1}{2}}^{(\alpha, \beta)}(y) \tilde{\phi}^{(\beta, \alpha)}(\lambda-\frac{1}{2}) H_0(\lambda) \lambda \sin \pi \lambda \, d\lambda + \phi(y) \quad (5.1)$$

in  $H_{\alpha, \beta}$  as  $r \rightarrow \infty$ .

Now we are able to state our main theorem.

Theorem 5.1. Let  $f(x) \in H_{\alpha, \beta}^*$  and  $F^{(\alpha, \beta)}(\lambda)$  be its continuous Jacobi transform.

Then in the sense of convergence in  $\mathcal{D}^*(I)$

$$f(x) = \lim_{n \rightarrow \infty} 4 \int_0^n F^{(\alpha, \beta)}(\lambda-\frac{1}{2}) P_{\lambda-\frac{1}{2}}^{(\alpha, \beta)}(-x) H_0(\lambda) \lambda \sin \pi \lambda \, d\lambda.$$

Sketch of the proof: Combine lemmas 3.2 and 5.1.

Corollary 5.1. Let  $f(x)$  be a regular member of  $H_{\alpha, \beta}^*$  such that

$f(x) \in H_{\alpha, \beta}^* \cap L_{loc}^1(-1, 1)$ . Then

$$f(x) = 4 \int_0^\infty F^{(\alpha, \beta)}(\lambda-\frac{1}{2}) P_{\lambda-\frac{1}{2}}^{(\beta, \alpha)}(-x) H_0(\lambda) \lambda \sin \pi \lambda \, d\lambda$$

for almost all  $x \in (-1, 1)$ .

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Received 22 May, 1986

Cal Polytech St. University  
San Luis Obispo, CA 93407  
U.S.A.

ON INFLECTIONAL SPACE CURVESWITH FOUR VERTICES. II.

Tibor Bisztriczky

*Presented by H.S.M. Coxeter, F.R.S.C.*

A simple inflectional space curve  $\Gamma$  in Euclidean three-space  $E_3$  lies on the boundary of its convex hull  $H(\Gamma)$  and possesses vertices as its only exceptional points. The Four-vertex theorem states that the number  $n(\Gamma)$  of vertices of  $\Gamma$  is at least four.

In [1], we proved that if  $n(\Gamma) = 4$  then each point  $b \in \text{int } H(\Gamma)$  lies on at least two chords of  $\Gamma$ . Presently, we show that if each  $b \in \text{int } H(\Gamma)$  lies on exactly two chords of  $\Gamma$  then  $n(\Gamma) = 4$ .

1. Preliminaries

Let  $p, q, \dots, L, M, \dots$  and  $\alpha, \beta, \dots$  denote respectively the points, lines and planes of  $E_3$ , and let  $\langle p, L, \dots \rangle$  denote the flat of  $E_3$  spanned by  $p, L, \dots$ .

A space curve  $\Gamma$  is a continuous map from an oriented circle  $C^*$  into  $E_3$  such that  $|\alpha \cap \Gamma(C^*)| < \infty$  for any  $\alpha$ . We identify  $\Gamma$  with  $\Gamma(C^*)$  and assume that  $\Gamma$  is simple, and oriented by  $C^*$ . For  $u \neq w$  in  $\Gamma$ , let  $[u, w]$  denote the closed arc of  $\Gamma$  (oriented) from  $u$  to  $w$  and set  $(u, w) = [u, w] \setminus \{u, w\}$ . If  $v \in (u, w)$ , we also write  $u < v < w$  in  $\Gamma$ .

Let  $u \in \alpha \cap \Gamma$ . Then  $\alpha$  cuts [supports]  $\Gamma$  at  $u$  if near  $u$ ,  $\Gamma$  lies on both sides [one side] of  $\alpha$ . We say that  $\Gamma$  is convex if  $\Gamma \subset \partial H(\Gamma)$  and  $|L \cap \Gamma| \leq 2$  for any  $L$ , and  $\Gamma$  is of order  $m$  if any  $\alpha$  cuts  $\Gamma$  at most  $m$  points.

In [1], we presented a history of inflectional space curves with four

vertices and noted that they were convex in each of the quoted studies.

Let  $\Gamma$  be a simple convex space curve. Then a chord of  $\Gamma$  is a closed segment  $\overline{uv} = \langle u, v \rangle \cap H(\Gamma)$ ,  $u \neq v$  in  $\Gamma$ . Next, let  $\alpha$  meet  $\Gamma$  at  $n \geq 4$  points  $u_1 < u_2 < \dots < u_n < u_1$  (in  $\Gamma$ ). Then  $\alpha \cap \Gamma$  is normal if there is an orientation on the convex curve  $\alpha \cap \partial H(\Gamma)$  such that  $u_1 < u_2 < \dots < u_n < u_1$  in  $\alpha \cap \partial H(\Gamma)$ . Finally, we cite some results that we require for our arguments.

1. (Sperner, [4]) Let  $\Delta$  be a closed 2-simplex with vertices  $p_i$  ( $i = 1, 2, 3$ ), edges  $S_i$ ,  $p_i \in S_i$ , and let  $D_i$  be closed sets such that  $p_i \in D_i$ ,  $S_i \cap D_i = \emptyset$  and  $\Delta = D_1 \cup D_2 \cup D_3$ . Then  $D_1 \cap D_2 \cap D_3 \cap \Delta \neq \emptyset$ .

2. (Scherk, [3]) Let  $\Gamma: C^* \rightarrow E_3$  be an inflectional space curve of order four. Then  $n(\Gamma) = 4$  if and only if  $\Gamma$  is convex.

In 2, we prove theorem A. Then theorem B follows by 2, A and [1], 8.

A. THEOREM. Let  $\Gamma: C^* \rightarrow E_3$  be a simple convex space curve with each  $b \in \text{int } H(\Gamma)$  on exactly two chords of  $\Gamma$ . Then  $\Gamma$  is of order four.

B. THEOREM. Let  $\Gamma: C^* \rightarrow E_3$  be a simple, inflectional convex space curve.

B.1 If  $n(\Gamma) = 4$  then each  $b \in \text{int } H(\Gamma)$  lies on at least two chords of  $\Gamma$ .

B.2 If each  $b \in \text{int } H(\Gamma)$  is on exactly two chords of  $\Gamma$  then  $n(\Gamma) = 4$ .

We note that  $\Gamma$  need not be inflectional, and hence differentiable, in A and so we refer to [1] for these definitions.

## 2. The main theorem

We assume the hypotheses of A with  $B = H(\Gamma)$  and  $S = \partial B$ .

3. LEMMA. Let  $b \in \overline{u_1 u_2} \cap \overline{v_1 v_2} \in \text{int } B$ . Then  $[u_1, u_2] \cap [v_1, v_2] \neq \emptyset$ .

PROOF. If  $[u_1, u_2] \cap [v_1, v_2] = \emptyset$  then  $u_1 < u_2 < v_1 < v_2 < u_1$  in  $\Gamma$  say, and  $b \in \overline{uw}$  for  $u \in (u_1, u_2)$  and  $w \in \Gamma$ . Then

$$S(u_1, u_2) = \{p \in S \mid b \in \overline{up} \text{ for some } u \in (u_1, u_2)\}$$

is a simple open arc, disjoint from  $\Gamma$  and  $b \in \overline{u_1 u_2}$  implies that  $\mathcal{F} = [u_1, u_2] \cup S(u_1, u_2)$  is a simple closed curve, disjoint from  $(u_2, u_1)$ .

Since  $S\mathcal{F}$  has two components,  $b \in \overline{u_1 u_2} \cap \overline{v_1 v_2}$  implies that one component contains  $v_1$  and the other contains  $v_2$ . Thus  $(v_1, v_2) \subset (u_2, u_1)$  meets  $\mathcal{F}$ ; a contradiction.  $\square$

4. LEMMA. For any  $\alpha \subset E_2$ ,  $\alpha \cap \Gamma$  is normal.

PROOF. If  $\alpha \cap \Gamma$  is not normal then there exists  $u_1 < u_2 < u_3 < u_4 < u_1$  in  $\Gamma$  such that under some orientation of  $\alpha \cap S$ , either  $u_1 < u_2 < u_4 < u_3 < u_1$  or  $u_1 < u_3 < u_2 < u_4 < u_1$ . It is easy to check that both possibilities contradict 3.  $\square$

Next we observe that since  $\Gamma$  is simple, 4 readily yields

5. LEMMA. For  $u \neq v$  in  $\Gamma$ , there is a plane through  $\langle u, v \rangle$  which separates  $(u, v)$  and  $(v, u)$ .

6. Let  $\alpha$  cut  $\Gamma$  at  $n \geq 6$  consecutive points  $u_i$  such that  $u_1 < u_2 < \dots < u_n < u_{n+1} = u_1$  in both  $\Gamma$  and  $C = \alpha \cap S$ , and let  $C[u_i, u_{i+1}]$  denote the closed arc of  $C$  (oriented) from  $u_i$  to  $u_{i+1}$ . Let  $u_i \neq u_j \neq u_{i+1}$ . Then  $u_j \notin C[u_i, u_{i+1}]$  and  $C[u_i, u_{i+1}] \cup \overline{u_j u_i} \cup \overline{u_j u_{i+1}}$  is the boundary of a closed region  $D_j[i, i+1]$  in  $D = \alpha \cap B$ .

Let  $i + j$  be odd and  $u_i < u_{i+1} < u_j < u_{j+1} < u_i$  in  $C$ . Then  $\alpha$  separates  $(u_i, u_{i+1})$  and  $(u_j, u_{j+1})$ , and we set for  $v \in [u_i, u_{i+1}]$ :

$$[u_j, u_{j+1}]^V = \{b \in D \cap \overline{w} | w \in [u_j, u_{j+1}]\} \text{ and } D(i, j) = \bigcup [u_j, u_{j+1}]^V.$$

Since  $r$  is continuous, simple and convex, it follows that

6.1  $C^V[u_j, u_{j+1}] = C[u_j, u_{j+1}] \cup [u_j, u_{j+1}]^V$  is a simple closed curve which depends continuously on  $v \in [u_i, u_{i+1}]$ .

Let  $\text{int } C^V[u_j, u_{j+1}]$  denote the open region in  $D$  bounded by  $C^V[u_j, u_{j+1}]$  and set  $\text{ext } C^V[u_j, u_{j+1}] = D \setminus \text{cl}(\text{int } C^V[u_j, u_{j+1}])$ . We claim

6.2  $D_{i+1}[j, j+1] \setminus D_i[j, j+1] \subset D[i, j]$ .

Let  $b \in (\text{int } D_{i+1}[j, j+1] \setminus D_i[j, j+1])$  and  $v \in (u_i, u_{i+1})$ . Since

$$\partial D_i[j, j+1] = C[u_j, u_{j+1}] \cup \overline{u_i u_j} \cup \overline{u_i u_{j+1}} = C^{u_i}[u_j, u_{j+1}]; \text{ 6.1 and}$$

$b \notin D_i[j, j+1]$  imply that  $b \in \text{int } C^V[u_j, u_{j+1}]$  for  $v$  close to  $u_i$ .

Similarly,  $b \in \text{int } D_{i+1}[j, j+1]$  yields that  $b \in \text{int } C^V[u_j, u_{j+1}]$  for  $v$  close to  $u_{i+1}$  and thus  $b \in C^V[u_j, u_{j+1}]$  for some  $v$ .

7. LEMMA.  $r$  is of order  $m$ ,  $m \leq 6$ .

PROOF. Suppose that some  $\alpha$  satisfies 6 with  $n \geq 7$ . Since no three chords of  $r$  are concurrent;  $\overline{u_1 u_{i+3}}$ ,  $\overline{u_i u_{i+4}}$  and  $\overline{u_{i+1} u_{i+4}}$  determine an open non-empty 2-simplex  $\Delta_i^\circ \subset \text{int } D$ ;  $i = 1, 2, 3$ . Then  $i + (i + 3)$  odd and 6.2 imply that

$$\Delta_i^\circ \subset D_i[i+3, i+4] \setminus D_{i+1}[i+3, i+4] \subset D[i, i+3]$$

and each  $b \in \Delta_i^\circ$  lies on a chord of  $(u_i, u_{i+1}) \cup (u_{i+3}, u_{i+4})$ . It is now an easy exercise in plane geometry to show that  $u_1 < u_2 < \dots < u_7 < u_1$  in  $C$  implies  $\Delta_1^\circ \cap \Delta_2^\circ \cap \Delta_3^\circ \neq \emptyset$ ; a contradiction.  $\square$

PROOF OF THEOREM A. Since  $r \subset E_3$  is a closed curve,  $r$  is of even order. Suppose that some  $\alpha$  cuts  $r$  at exactly six points, assume the notation of 6 and let  $\Delta$  be the closed 2-simplex in  $\text{int } D = D \setminus C$  determined by  $\overline{u_1 u_4}$ ,  $\overline{u_2 u_6}$  and  $\overline{u_3 u_5}$ . Let  $p_1 = \overline{u_1 u_4} \cap \overline{u_2 u_6}$ ,  $p_2 = \overline{u_2 u_6} \cap \overline{u_3 u_5}$ ,  $p_3 = \overline{u_3 u_5} \cap \overline{u_1 u_4}$ ,  $S_1 = \Delta \cap \overline{u_2 u_6}$ ,  $S_2 = \Delta \cap \overline{u_1 u_4}$  and

$S_1 = \Delta \cap \overline{u_2 u_3}$ . Let  $\text{int } \Delta$  be the relative interior of  $\Delta$ . For  $i = 1, 2, 3$ , we set

$$D_i^{\circ} = D[i, i+3] \cap \text{int } \Delta, \quad D_i = \text{cl } D_i^{\circ}$$

and claim that  $\Delta = D_1 \cup D_2 \cup D_3$ ,  $p_i \in D_i$  and  $S_1 \cap D_i = \emptyset$ .

We observe that  $u_1 < u_2 < \dots < u_6 < u_1$  in  $C$  and 5 readily yield

$$(1) \Delta \subset \bigcup_{i=1}^3 D[i, i+3], \quad \Delta \cap \left[ \bigcup_{i=1}^6 D[i, i+1] \right] = \emptyset \text{ and either}$$

$$(2) \Delta = D_1[4, 5] \cap D_2[4, 5] \cap D_3[6, 1] \cap D_4[6, 1] \cap D_5[2, 3] \cap D_6[2, 3]$$

$$\text{or } (3) \Delta = D_4[1, 2] \cap D_5[1, 2] \cap D_6[3, 4] \cap D_1[3, 4] \cap D_2[5, 6] \cap D_3[5, 6].$$

Since each  $b \in \text{int } \Delta$  is in some  $D[i, j]$  where  $i + j$  is odd, (1) implies  $\text{int } \Delta = D_1^{\circ} \cup D_2^{\circ} \cup D_3^{\circ}$  and thus  $\Delta = D_1 \cup D_2 \cup D_3$ . As (2) and (3) are symmetric, we assume (2) without loss of generality.

Let  $b \in S_1 \setminus \{p_2, p_3\}$  and let  $c \in D \setminus \Delta$  be close to  $b$ . Then  $c$  does not lie on a chord of  $\alpha \cap \Gamma$ , and  $S_1 \subset \overline{u_2 u_3} \subset \partial D_3[6, 1] \cap \partial D_5[2, 3]$ , (2) and 6.2 imply that

$$c \in (D_4[6, 1] \setminus D_3[6, 1]) \cap (D_5[2, 3] \setminus D_6[2, 3]) \subset D[3, 6] \cap D[2, 5]$$

and  $c \in D[1, 4]$ . Now if  $b \in D_1 \subset D[1, 4]$  then  $b \in [u_4, u_5]^V$  for some  $v \in (u_1, u_2)$  and  $L = \langle p_2, p_3 \rangle = \langle u_3, u_6 \rangle$  supports  $[u_4, u_5]^V$  at  $b$  by the preceding. Since  $L$  separates  $\Delta$  from both  $u_4$  and  $u_5$ ,  $L$  meets  $[u_4, u_5]^V \setminus (b)$  in at least two points. Thus  $\gamma = \langle v, L \rangle$  meets  $(u_4, u_5)$  at least four times (counting a point of support twice) and  $(u_1, u_2)$  at  $v$ . In addition,  $\gamma$  separates  $u_6$  from  $u_1$ , and  $u_2$  from  $u_4$ , and hence meets both  $(u_6, u_1)$  and  $(u_2, u_4)$ . Thus  $|\gamma \cap \Gamma| \geq 7$  and as this contradicts 7,  $(S_1 \setminus \{p_2, p_3\}) \cap D_1 = \emptyset$ .

Let  $v \in (u_5, u_6)$ . For  $c \in \text{int } \Delta \subset (\text{int } D_5[2, 3]) \cap (\text{int } D_6[2, 3])$ , we obtain from the proof of 6.2 that

$$(4) c \in \text{int } C^V[u_2, u_4] \text{ for } v \text{ close to } u_5[u_6].$$

On the other hand,  $p_2 = \overline{u_2 u_3} \cap \overline{u_5 u_6}$  implies that  $p_2 \in C^V[u_2, u_4]$  for any  $v$

and by arguing as above we obtain that

(5)  $p_2 \in \text{ext } C^V[u_2, u_2]$  for all  $v$ .

Fix  $\bar{v} \in (u_2, u_2)$  and let  $c \in \text{int } \Delta$  tend to  $p_2$ . Since  $\text{ext } C^{\bar{v}}[u_2, u_2]$  is open in  $D$ , (5) implies that

(6)  $c \in \text{ext } C^{\bar{v}}[u_2, u_2]$  for  $c$  sufficiently close to  $p_2$ .

Thus (4) and (6) yield that for each such  $c$ ,  $c \in C^V[u_2, u_2]$  for some  $v$  in each of  $(u_2, \bar{v})$  and  $(\bar{v}, u_2)$ , and hence  $c \in D_2^{\circ} \setminus (D_1^{\circ} \cup D_2^{\circ})$  and  $p_2 \in D_2 \setminus (D_1 \cup D_2)$ . Similarly,  $p_2 \in D_2 \setminus (D_1 \cup D_2)$  and thus  $S_1 \cap D_1 = \emptyset$ .

The preceding is symmetric in  $S_1$ ,  $S_2$  and  $S_3$ , and therefore

$$D_1^{\circ} \cap D_2^{\circ} \cap D_3^{\circ} = D_1 \cap D_2 \cap D_3 \cap \Delta \neq \emptyset$$

by 1; a contradiction.  $\square$

Finally, we mention the similarity of theorem B to the following theorem of R. Osserman [2]:

Let  $\tilde{r}$  be a  $C^2$  Jordan curve with the circumscribed circle  $\tilde{c}$ . Then  $|\tilde{r} \cap \tilde{c}| \geq 2$  and if  $\tilde{r}$  has exactly four vertices (points of local extreme curvature) then  $\tilde{r} \cap \tilde{c} = \{p, q\}$  and  $p$  and  $q$  are antipodal on  $\tilde{c}$ .

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University of Calgary  
Calgary, Alberta  
Canada T2N 1N4

Received 22 May, 1986

Mailing Addresses

1. N.A. Alling  
Department of Mathematics  
University of Rochester  
Rochester, NY 14627, U.S.A.
2. P. Alsholm  
Matematisk Sektion  
Danish Engineering Academy  
DIA-K, Bygning 376  
DK 2800 Lyngby, Denmark
3. T. Bisztriczky  
Department of Mathematics  
University of Calgary  
Calgary, Alberta, Canada, T2N 1N4
4. B.R. Ebanks  
Department of Mathematics  
University of Louisville  
Louisville, KY 40292, U.S.A.
5. P. Ehrlich  
Department of Philosophy  
Northwestern University  
Evanston, IL 60201, U.S.A.
6. L. Melkersson  
Department of Mathematics  
University of Lund  
Box 118, S-221 00 Lund, Sweden
7. J. Mináč  
Department of Mathematics and  
Statistics, Queen's University  
Kingston, Ontario, Canada, K7L 3N6
8. V. Snaith  
Department of Mathematics  
University of Western Ontario  
London, Ontario, Canada, N6A 5B7
9. A.I. Zayed  
Cal Polytech St. University  
San Luis Obispo  
CA 93407, U.S.A.