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WHAT DO SEISMOLOGY AND NEUROPHYSIOLOGY HAVE IN COMMON? - STATISTICS!

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Dedicated to my teachers at the University of Toronto and the University of Toronto Schools

1. Introduction

Seismology is the branch of science concerned with the investigation of earthquakes and related phenomena. Its goals include: learning about the Earth's and planets' interior composition and predicting the time, size and location of future earthquakes. In contrast, *neurophysiology* is the branch of science concerned with how the elements of the nervous system develop, function and work together. Its goals include: explaining notions like memory, emotion, learning, sleep, expectation and, less heroically, how individual neurons respond to stimuli, transmit information and change with environment. The definitions may make these two fields seem remote from each other. In point of fact, however, they are intimately tied together through use of a common methodology - statistics.

Statistics is the science concerned with the collection and analysis of numerical information (data) in order to answer questions wisely. It is characterised by an interplay between axioms and data and, in particular, is concerned: with making statements that go beyond the data collected (inferences), with explanation and understanding, with prediction and control, with discovery and application, with justification and classification. These concerns are patently common to seismology and neurophysiology - hence a connection.

Throughout much of my career I have collaborated with seismologists and neurophysiologists. In this article I would like to present some examples of statistical concepts and techniques that I have found myself making use of in problems arising from *both* seismology and neurophysiology.

2. Some Statistical Background

"What's the use of their having names," the Gnat said, "if they won't answer to them? "No use to them," said Alice, "but its useful to the people that name them, I suppose. If not, why do they have names at all?"

Alice's Adventures in Wonderland

The statistician approaches a substantive problem with knowledge and experience concerning a particular collection of concepts and techniques. These tools often incorporate a notion of randomness and have proven pertinent to problems arising in a broad range of scientific, technological and social fields. Brief descriptions of some of those pertinent to our examples follow. In these examples, dynamics and time will be central. This has affected the choice of the statistical apparatus highlighted.

It will be taken that the notion of a random element, ω , is given. A *time series* is a random real-valued function, $Y(.,\omega)$, of a real or integer-valued random variable t, usually referred to as *time*. A *point process* is a random, non-negative, integer-valued measure, $N(.,\omega)$. The values of these quantities, for a particular realisation of ω , are typically denoted by $Y(t)$, $Y(x,y,t)$, $N(I)$ respectively with I referring to a measureable set in the last case and with dependence on ω suppressed.

If $X(\omega)$ denotes a particular random variable and if $P(.)$ denotes the (probability) measure of ω , then in what follows $E(X)$ will denote $\int X(\omega)P(d\omega)$. *Moment functions* and *product densities* are of substantial use in discussing time series and random processes. When they exist, these take the forms

$$E\{Y(t_1)...Y(t_K)\} = m_{Y...Y}(t_1,...,t_K)$$

and

$$E\{N(dt_1)...N(dt_K)\} = p_{N...N}(t_1,...,t_K)dt_1...dt_K \quad (1)$$
$$= Prob\{N(dt_1)=1,...,N(dt_K)=1\}$$

respectively with the dt_k distinct. Given data, typically assumed to be part of a realisation of a random process, useful estimates may be constructed for these quantities in a broad class of instances , particularly when the process involved is *stationary*, that is when its probabilistic properties are invariant under simple translations of time. In the stationary case, the process has a spectral representation, eg.

$$Y(t) = \int \exp(i\lambda t)Z(d\lambda)$$

or

$$N(I) = \int [\int_I \exp(i\lambda t)dt]Z(d\lambda)$$

$Z(.)$ being a random function with orthogonal increments. A further useful parameter, the *power spectrum*, is now given by

$$E\{Z(d\lambda)\overline{Z(d\mu)}\} = \delta(\lambda-\mu)f(\lambda)d\lambda d\mu$$

$\lambda\neq0$, when it exists. (Here $\delta(.)$ denotes the Dirac delta 'function'.)

In their work statisticians make continual use of *stochastic models*. These are analytic idealizations of real-world circumstances containing some random element. They tie the observables to the phenomenon of concern and are designed to lead to a broad variety of inferences concerning the phenomenon. Stochastic models often take the form of *systems*, that is mappings carrying functions, measures and the like over into other functions or measures. Stochastic models and systems usually involve unknown *parameters* (these may be finite or infinite dimensional) and the *estimation* (or identification) *problem* is to attach reasonable empirical values to these unknowns given observational or experimental data. A central role in this endeavour is played by the *likelihood function*. In simple terms, if θ denotes the unknown parameter the likelihood is the Radon-Nikodyn derivative of the probability measure of the data relative to some known measure, viewed as a function of θ. In many circumstances one has to work with an approximation to the likelihood, perhaps derived via an asymptotic method. One seeks a θ that is physically interpretable whenever possible.

3. Example I - the Autointensity Function

A central entity in the method by which nerve cells communicate is the spike train. If a microelectrode is inserted into the axon (that is the output component) of a neuron, a changing voltage is recorded. This time series is made up of essentially identical spikes, or pulses, repeating at generally irregularly spaced times. Supposing these spikes to occur at times τ_k, $k=0,\pm1,\pm2,...$ one may define a counting measure via $N(I)$ = the number of τ_k in the interval I (of the real line). In various circumstances it seems reasonable to talk of probabilities of events such as: there is a spike (or point) in the small interval $(t,t+dt)$ or there is a spike in the interal $(t,t+dt)$ *and* in the interval $(t+u,t+u+du)$. If one is willing to view a given spike train as part of a realisation of a stochastic point process, then these probabilities correspond to product densities as defined by (1) above. In the stationary case it is convenient to define the *rate*

$$h_N = Prob\{point\ in(t,t+dt)\}/dt$$

and the *autointensity function*

$$h_{NN}(u) = Prob\{point\ in(t+u,t+u+du)\,|\,point\ at\ t\}/du$$

Given a stretch of data, these two parameters may be estimated by n/T and

$$\#\{\,|\tau_k-\tau_j-u\,| < b/2\}/nb$$

respectively where b is a small bin width and where n is the number of points τ_k observed in the time period, T, of observation. The autointensity function is an important descriptor of the behaviour of a firing neuron. For example in the case of a pacemaker cell, the autointensity is essentially 0 except when u is near some multiple of the (constant) interval between spikes. If the neuron is firing completely at random, the autointensity will be essentially constant. If bursting of firing is occurring, then $h_{NN}(u)$ will be high for small to moderate $|u|$ and drop down to h_N as $|u|$ increases. If bursting is taking place at regular intervals with, for example, an accelerando pattern within bursts, then $h_{NN}(u)$ will show mass broadly near 0 and also for u near multiples of the interval between bursts. From an estimate of the autointensity of a spike train, the behaviour of a nerve cell may be described and classified. A broad variety of experimental examples may be found in Bryant *et al.* (1973) and Brillinger *et al.* (1976).

Earth scientists, engineers, government officials and the like are interested in the *seismicity* of the habitat, that is the timing, location and strength of earthquakes occuring in their region of interest. They are further interested in earthquake prediction and corresponding risk assessment. The sequence of times of earthquake occurrence in a given region may be viewed as corresponding to part of a realisation of a stochastic point process. The rate of the process tells how many earthquakes may be expected in a unit time interval. The autointensity provides a means of describing future probabilities of earthquakes given the past record. For example, if earthquakes tend to recur periodically, then the autointensity will have the pacemaker shape described above. If earthquakes tend to occur in clusters, the shape will be as for a nerve cell firing in bursts. If the times of earthquakes are totally random, the $h_{NN}(u)$ will be essentially constant. Various empirical examples are given in Vere-Jones (1970). Data of China for the period 1000 A.D. to the present, is studied in Lee and Brillinger (1979) by means of a technique developed to handle the incompleteness of the early records.

In order that hypotheses and models may be checked, some indication of the sampling uncertainty of the estimates is needed. Also, a parameter that proves to be even more useful than the autointensity defined above is the crossintensity. It gives the probability of a point of one type occurring given that a point of another type has occurred, say u time units, earlier. Various results related to these last ideas may be found in Brillinger (1975).

4. Example II - Probit Analysis

A conceptual model for the firing of a neuron is the following: input to the nerve cell leads to (postsynaptic) electric current genesis. This current flows to a trigger zone, being filtered in the course of its passage. When the voltage level at the trigger zone exceeds a threshold value, the nerve cell fires. This process may be specified analytically as follows. Let $U(t)$ denote the voltage (membrane potential) at the trigger zone at time t. Let $B(t)$ denote the time elapsed since the last firing. Let $X(t)$ denote the (measured) input to the cell. Then, assuming linearity and time invariance

$$U(t) = \int_0^{B(t)} a(u)X(t-u)du$$

for some response function $a(.)$. Suppose the threshold level at time t has the form $\alpha+\epsilon(t)$, with $\epsilon(t)$ a normal variate of mean 0 and variance 1. Then, given $U(t)$, the probability the neuron fires at time t is $\Phi(U(t)-\alpha)$, with $\Phi(.)$ denoting the standard normal cumulative function. Supposing the data to be recorded at times $t=0,1,2,\ldots,T-1$ and $Y(t)$ to be observed and defined to be 1 if a spike occurred in the immediately preceding interval and to be 0 otherwise, the likelihood of the data may be written

$$\prod_{t=0}^{T-1} \Phi(V_t - \alpha)^{Y(t)}[1 - \Phi(V_t - \alpha)]^{1-Y(t)}$$

with

$$V_t = \sum_{u=0}^{B(t)-1} a(u)X(t-u)$$

The unknowns, $a(.)$, α, may be estimated by maximising the likelihood. Once the estimates have been obtained, the model may be checked to an extent by comparing the empirical firing probability with the fitted. This is done for a variety of inputs and neurons in Brillinger and Segundo (1979).

The generally agreed description of earthquake genesis is the following: earthquakes are due to faulting. Specifically a crack initiates at a point and spreads out to form a fault plane. As the crack passes a given point, slip takes place on the fault plane resulting in a stress drop and the radiation of seismic waves. The ground is initially compressed and dilated around the focus of the earthquake. The pattern of compressions and dilations is preserved in the seismic waves radiated out and may be observed in the seismograms of stations detecting the event. Further, given the orientation of the fault plane (usually expressed by three angles) there is a formula for the theoretical relative amplitude of the signal arriving at a given station. Data then consists of the following: the estimated focus of an earthquake, the locations of the stations recording it, and whether each station recorded compression or dilation (corresponding to whether the first motion noted was positive or negative). The problem is to estimate the orientation of the fault plane. This information is useful for understanding the physical nature of the Earth in general and for seismic risk assessment in particular.

Let A_j denote the theoretical amplitude for station j. Let t_j denote the arrival time of the seismic signal, $s_j(.)$. Then $s_j(t_j)=\alpha A_j$ for some α. Further the seismogram may be written $Y_j(t)=s_j(t)+\epsilon_j(t)$ with $\epsilon_j(t)$ a noise series. Now the probability that the first motion is positive, assuming $\epsilon_j(t_j)$ normal mean 0 variance σ^2, is

$$Prob\{Y_j(t_j) > 0\} = Prob\{\epsilon_j(t_j) > -s_j(t_j)\} = \Phi(\rho A_j)$$

with $\rho = \alpha/\sigma$. Assuming the noises independent at different stations the likelihood function is given by

$$\prod_{j=1}^{J} \Phi(\rho A_j)^{Z_j}[1-\Phi(\rho A_j)]^{1-Z_j}$$

with $Z_j = 1$ if the first motion is positive and $= 0$ if it is negative. The parameters may be estimated by maximising this likelihood. Once again the model may be checked by comparing an empirical with a fitted probability. The details of all this are given in Brillinger *et al.* (1980) and illustrated by computations with the great 1964 Alaskan earthquake and for some California events.

In fact the same computer program was employed to fit the nerve firing model and the first motion model, even though these two models had such totally different origins.

5. Example III - Average Evoked Response

Consider the linear time invariant system with input $X(t)$ and output $Y(t)$

$$Y(t) = \int a(t-u)X(u)du$$

Here $a(.)$ is referred to as the *impulse response*, because if the input $X(t)$ is taken as the Dirac delta function , then the output is $a(t)$. A broad variety of naturally occurring systems seem to be linear and time invariant in the above manner, to a good approximation. Prominent among these is the Earth's transmission of seismic (acoustic) waves, be they generated by earthquakes, explosions or other vibratory sources. This effect is highly useful in seismic exploration. Suppose an impulse of energy is input to the Earth in a region of interest. Part of this energy will be reflected back to the surface by subsurface geologic structures after time delays 'proportionate' to the depth of the structure (wherever there is a difference in acoustic impedance). If $Y(t)$ denotes the signal recorded by a sensor on the surface, then its peaks (really peaks of the impulse response $a(.)$) may be interpreted in terms of subsurface layering. From estimates of such 'reflectivity functions' along lines of shots, geologically interesting structures at depth may be inferred. In practice a single pulse at a location rarely proves incisive. Hence prospectors are led to replicate the pulses at times σ_m, $m = 1,...,M$. One can then form the *average evoked response* or stacked estimate

$$\frac{1}{M}\sum_{m=1}^{M} Y(u+\sigma_m)$$

as an improved estimate of $a(u)$, provided the σ_m are sufficiently far apart that the corresponding individual responses do not overlap. Neitzel (1958) presents the results for some early experiments of this type.

It has long been traditional to average numbers. The novelty in the present circumstance is that it is curves that are being averaged. Such averaging has proved to be crucial in studies of brain waves because of the fact that signals evoked by various sensory stimuli are much smaller than the ongoing noise. The stimulus may be auditory, visual, olfactory, somatosensory or gustatory in character. The data available for analysis consists of the ongoing electrencephalogram (EEG) observed at an array of locations on the skull and the times of application of the stimuli. Quite a variety of questions arise concerning the evoked response phenomena. These include: Does a given stimulus in fact evoke a response? Do different stimuli elicit the same response? Does the same response occur at different sensors? Are the responses repeatable? If stimuli are reordered, is the response the same? Are the effects of different stimuli additive? How does the response depend on the stimulus intensity? Answers to these questions are complicated by the phenomena of: weak response, variability of response, occurrence of artifacts, among other things. The papers Brillinger (1981a,b) review the history of evoked response experiments and their analysis, describe a number of success stories concerning the technique and provide some formal answers to the preceding questions. In particular, the following class of procedures is proposed for dealing with the complications caused by the presence of artifacts.

Nowadays considerable statistical research effort is directed towards the construction of robust/resistant techniques, that is procedures that remain effective in the presence of bad data values or of long-tailed error distributions. The traditional average value (or sample mean) is a prime example of a nonresistant sample quantity. Its value may be shifted an arbitrarily large amount by merely shifting a single sample value. By contrast, the interquartile or mid- mean, that is the mean of the central 50% of the sample values, does not even involve the 50% most extreme sample values in explicit fashion and hence is highly unshiftable. In Brillinger (1981a) the following class of estimates was proposed for the evoked response case, with a discussion of computational procedures for both the live and dead time cases. These estimates may be computed automatically. Set $Y_m(u) = Y(u+\sigma_m)$ and

$$|Y-\theta|^2 = \int_0^V |Y(u)-\theta(u)|^2 du$$

where, in this last, it is assumed that the evoked response dies off after V time units and that $\sigma_{m+1}-\sigma_m > V$. As a resistant estimate consider $\hat{\theta}(.)$ satisfying

$$\hat{\theta}(u) = \sum_m W_m Y_m(u) \,/\, \sum_m W_m$$

with $W_m = W(|Y_m - \hat{\theta}|/\hat{\rho})$, $W(.)$ being a weight function having most of its mass near 0 and $\hat{\rho}$ an estimate of scale. As a generalization of the mid-mean above, one can consider

$$\hat{\theta}(u) = \sum{}' Y_j(u) \,/\, \beta M$$

with Σ' denoting the summation over the βM smallest $|Y_m - \hat{\theta}|$. The statistical properties of this last estimate are studied in the Berkeley Ph.D. thesis of Folledo (1983).

6. Example IV - Decaying Cosines

After a great earthquake the whole earth rings like a bell, with the vibrations lasting for days sometimes. Because the Earth is a finite body, it can only resonate as a whole at certain discrete frequencies. Because the medium is dissipative, the vibrations eventually damp away. These phenomena are in accord with the equations of motion being linear with constant coefficients and in consequence having solutions

$$s(t) = \sum_k \alpha_k \exp\{-\beta_k t\} \cos(\gamma_k t + \delta_k)$$

$t>0$, assuming initial condition of a Dirac delta function at 0. An observed seismogram will have the form $Y(t) = s(t)+\epsilon(t)$ with $\epsilon(.)$ denoting a noise series. The problem arising is how to estimate the unknown parameters, particularly the β_k, γ_k. In Bolt and Brillinger (1979) the following solution is developed. Suppose the noise series $\epsilon(.)$ is stationary and *mixing*, (that is well-separated in time values are at most weakly dependent), then the Fourier transform values of lengthy time segments satisfy a central limit theorem, that is are asymptotically normal. In particular if

$$\epsilon_j = \sum_{t=0}^{T-1} \epsilon(t) \exp\{-i2\pi jt/T\}$$

the values ϵ_j for $2\pi j/T$ near λ will be approximately independent complex normal variates with mean 0 and variance $2\pi T f_{\epsilon\epsilon}(\lambda)$, $f_{\epsilon\epsilon}(.)$ being the power spectrum of the series $\epsilon(.)$. Now one has $Y_j = s_j + \epsilon_j$, with s_j depending on the unknown parameters of interest. If one sets down the approximate likelihood function for the data here and works in a neighborhood of λ containing only one of the γ_k, then obtaining (approximate) maximum likelihood estimates comes down to minimizing

$$\sum_j |Y_j - s_j|^2$$

as a function of the unknown parameters. The details of this, as well as a procedure for checking the validity of the assumed form for $s(t)$, may be found in Bolt and Brillinger (1979). For example, it is found there that if a limiting process with $\beta_k = \phi_k/T$ as $T \to \infty$ is employed then

$$var \beta_k \, , \, var \gamma_k \approx T^{-3} 4\pi f_{\epsilon\epsilon}(\gamma_k) \alpha_k^{-2} I_0(\phi_k) J(\phi_k)^{-1}$$

where

$$I_l(\phi) = \int_0^1 u^l \exp(-2\phi u) du$$

and $J(\phi) = I_0(\phi) I_2(\phi) - I_1(\phi)^2$. The T^{-3} decrease of variance is initially surprising.

The decaying cosine model has also proven useful in neurophysiology. In the work of Freeman (1972, 75, 79) the olfactory system of rabbits has been studied via evoked response experiments. Freeman found that the averaged response could be well-fitted by the sum of a few decaying cosine terms. He developed a model involving spike to wave conversion, involving collections of constant coefficient second-order differential equations, involving feed forward and feedback and involving wave to pulse conversion. Various types of neurons and connections were postulated. He employed nonlinear regression, in the time domain, to estimate the unknowns. In one case involving two cosines he was led to view the larger wave as representing intracortical negative feedback and the smaller as representing another feedback loop. Of some interest in this type of work is what happens to the frequencies and the decay rates when the experimental conditions are altered.

7. Example V - System Identification / Deconvolution

Regression analysis is one of the more long standing and potent tools in the statistician's kit. There are variants, for time series and point process data, that have proved useful in studying both neurophysiological and seismological data. Let $M(.)$ and $N(.)$ be two stochastic point processes whose realisations are imagined to correspond to the spike trains of two given neurons. Consider modelling the rate of firing of one neuron as it is affected by the other, as follows

$$Prob\{N(dt) = 1 \mid M(.)\} = [\alpha + \int a(t-u) M(du)] dt$$

for some constant α and function $a(.)$. Supposing $\{M(.), N(.)\}$ to be a stationary point process, the above relationship leads to

$$f_{NM}(\lambda)=A(\lambda)f_{MM}(\lambda)$$

where $A(.)$ denotes the Fourier transform of $a(.)$ and $f_{NM}(.)$ denotes the *cross-spectrum* of $N(.)$ with $M(.)$. (In terms of the spectral representations of the two processes it is defined via $E(dZ_N(\lambda)\overline{dZ_M(\mu)})=\delta(\lambda-\mu)f_{NM}(\lambda)d\lambda d\mu$, $\lambda\neq 0$.) Once estimates of the spectra $f_{NM}(.)$ and $f_{MM}(.)$ are at hand, the function $A(.)$ may be estimated and after it $a(.)$. There are various ways that such spectral estimates may be formed, see Brillinger (1975). A variety of examples for neurophysiological data are presented in Brillinger *et al.* (1976). The strength of the postulated 'linear' relationship may be measured by the *coherence* function, $|R_{NM}(\lambda)|^2=|f_{NM}(\lambda)|^2/f_{NN}(\lambda)f_{MM}(\lambda)$. It lies between 0 and 1. In this last reference a variant of the coherence is employed to untangle the issue of how some given triples of nerve cells are causally connected.

Calculations similar to the above are also extremely useful in the seismic exploration case. We indicate how they may be employed to design an effective probing signal there. Consider again the system

$$Y(t) = \int a(t-u)X(u)du \tag{2}$$

for the time series $Y(.), X(.)$. Let $m_{YX}(.)$ denote the convolution of Y with X, in some sense, and let $m_{XX}(.)$ denote the convolution of X with X. Then from (2) one has

$$m_{YX}(u) = \int a(u-v)m_{XX}(v)dv$$

Suppose one wishes an $X(.)$ such that $m_{YX}(u)$ is approximately $a(u)$. Let $f_{XX}(.)$ denote the Fourier transform of $m_{XX}(.)$, then the right-hand side above is

$$\int \exp(i\lambda u)A(\lambda)f_{XX}(\lambda)d\lambda$$

and one sees that for this to be $a(u)$ what is needed is that $f_{XX}(\lambda)$ be approximately 1 on the support of $A(.)$. Supposing $A(\lambda)$ to be approximately 1 on $\lambda_0<\lambda<\lambda_1$ and 0 elsewhere, one is seeking $X(.)$ with $f_{XX}(\lambda)$ of the same character. An example of such an $X(.)$ is the chirp function

$$X(t) = \cos([\lambda_0+(\lambda_1-\lambda_0)\frac{t}{T}]t)$$

for $0<t<T$. (This signal was first introduced formally by researchers in radar and is employed by bats in natural flight as well). In the seismic case it is input to the ground (for which λ_0 and λ_1 are known) repeatedly and the responses averaged. It should be remarked that in actual applications, substantial further processing is carried out to handle further physical effects present, such as wavefront spreading.

The tools of system identification are extremely powerful and may often be used to obtain indications of the mechanisms and states underlying some structure of interest. The above examples provide but a glimpse of the strength of the systems approach.

8. Example VI - the Analysis of Array Data

Array data is collected in both the earth and neuro- sciences. By array data is meant a collection of measurements of the form $Y(x_j,y_j,t)$, $j=1,\ldots,J$ and $t=0,\ldots,T-1$ with the (x_j,y_j), $j=1,\ldots,J$ the coordinates of J sensors, and given j the measurements $Y(x_j,y_j,t)$, $t=0,\ldots,T-1$ a segment of a time series. In the seismological case, the (x_j,y_j) refer to the locations of seismometers. In the neurophysiological case, they refer to the grid locations of electrodes on the skull. Such data may often be reasonably viewed as part of a realisation of a planar-temporal (or spatial-temporal) random process $Y(x,y,t)$.

An important use of array data is the detection of propagating waves and the consequent estimation of their number, directions and velocities. For example in the seismic case one might have an array of (strong-motion) instruments located close to an earthquake fault. These instruments would be triggered by a sufficiently large event. In this case the source of the energy would be moving as the fault ripped. The seismologist would like to estimate the orientation of the fault and the rupture velocity. Bolt *et al.* (1982) discuss this problem and provide some elementary estimates based on data collected during an earthquake in Taiwan. They proceed by estimating the frequency-wavenumber spectrum of separate time segments of the data. The frequency-wavenumber spectrum has also been employed in the analysis of visual evoked response data, see Childers (1977). This last researcher first notes an apparent high velocity wave. After this wave has been 'removed', in a number of experiments he notes the occurrence of a pair of waves moving in opposite directions. His research is directed at developing a diagnostic procedure for various visual disorders and at obtaining insight concerning how the visual system functions.

Consider a planar-temporal wave of the form

$$Y(x,y,t) = \rho \cos(\alpha x + \beta y + \gamma t + \delta) + \epsilon(x,y,t)$$

with ρ,α,β,δ unknown constants and $\epsilon(x,y,t)$ a stationary noise process. The signal here is a plane wave propagating in direction θ given by $\tan\theta = \beta/\alpha$ with speed $\gamma/\sqrt{\alpha^2+\beta^2}$. In Brillinger (1985) the following maximum likelihood procedure is developed for detecting the presence of such a wave and for estimating its parameters.

Collect the J time series into a vector $\mathbf{Y}(t) = [Y(x_j,y_j,t)]$. Let

$$\mathbf{Y}_k = \frac{1}{T}\sum_{t=0}^{T-1} \mathbf{Y}(t)\exp(i2\pi k/T)$$

and

$$\mathbf{M} = \sum_k \mathbf{Y}_k \bar{\mathbf{Y}}_k^\tau$$

with the sum over $2\pi k/T$ near $\gamma = 2\pi k'/T$ say. Further set

$$\mathbf{S} = \sum_k \mathbf{Y}_k \bar{\mathbf{Y}}_k^\tau - \mathbf{Y}_{k'} \bar{\mathbf{Y}}_{k'}^\tau$$

and let $\mathbf{B} = [\exp(i(\alpha x_j + \beta y_j))]$. The value $|\mathbf{B}\mathbf{Y}_{k'}|^2$ is referred to as the *conventional statistic*. It may be expected to be large when $\rho \neq 0$ and it is evaluated at the 'correct' (α,β). The matrix $\mathbf{S}$ provides an estimate of the spectral density matrix of the series $\epsilon(.)$. Invoking a central limit theorem for the ϵ_k , an approximation may be set down for the likelihood function based on the $\mathbf{Y}_k$ (with $2\pi k/T$ near γ). It is found that the maximum likelihood detection statistic, given (α,β), is

$$\bar{\mathbf{B}}^\tau \mathbf{S}^{-1}\mathbf{B} / \bar{\mathbf{B}}^\tau \mathbf{M}^{-1}\mathbf{B} - 1 \qquad (3)$$

with null distribution $(K-J)^{-1}$ times an F distribution with degrees of freedom 2 and $2(K-J)$. It is further found that the maximum likelihood estimates of α and β are the coordinates of the maximum of the detection statistic. It is often convenient to prepare contour plots of the statistic (3) as a function of (α,β). An example of this is given in Brillinger (1975).

9. Discussion

In this article we have presented a number of examples, drawn mostly from our own experience, showing the use of the same statistical technique in the rather separate sciences of seismology and neurophysiology. It now seems appropriate to ask what, if anything, have the *three* sciences - statistics, seismology, neurophysiology - gained from each other as a result of connections albethey indirect? Having in mind a broader class of examples then those discussed in this paper, one can say that: i) statistics is richer for having been led to develop and study various novel methods to handle specific problems arising in seismology or neurophysiology, ii) both seismology and neurophysiology are the richer for the other's field having generated a problem for the statistician to abstract sufficiently that the result's applicability to their field became apparent, iii) either seismology or neurophysiology benefit from a statistical formulation because various of their problems seem necessarily to need to be stated in terms of probabilities (eg. neither neuron firings nor earthquakes seem predictable) and because these fields need procedures to validate results and to fit conceptual models. The methods of statistics often lead to important insight and understanding in substantive problems.

It may be remarked that the applicability of statistical procedures to these two substantive fields has further grown in direct consequence of their move to greater quantification and digital data collection.

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STABILITY AND COHOMOLOGICAL DIMENSION

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Presented by P. Ribenboim, F.R.S.C.

Abstract: Let F be a Pythagorean field with finite number of orderings. Let $F(2)$ be a maximal 2-extension of F. Denote by $cd_2(F(\sqrt{-1}))$ the cohomological dimension of the Galois group of automorphisms of the field extension $F(2)|F(\sqrt{-1})$, by $st(F)$ the stability index of F. We prove that

$$st(F) = cd_2(F(\sqrt{-1}))$$

§1. Introduction. In this paper we keep to the notation in [5], [6], [7]. We define F (or $K,L,\ldots$) to be a formally real Pythagorean field, $\dot{F}$ is the multiplicative group of F, T - a preordering of F, i.e. any intersection of orderings of the field F. T_F - the intersection of all orderings of the field F. $[\dot{F}:\dot{T}_F]$ is the group-index. V - a valuation on F. A_V - the valuation ring corresponding to V. U_V - the group of units of A_V. M_V - the maximal ideal of A_V. F_V - the residue field of V. V is fully compatible with T iff $1+M_V \subset T$. $(X,\dot{F}|\dot{T})$ is a space of orderings. X here means the set of all orderings P of the field F such that $T \subset P$. Sometimes, instead of $(X,\dot{F}|\dot{T})$, we shall write only X or X_F. A field F is of type $(k,2^n)$ if $[\dot{F}:\dot{T}_F] = 2^n$ and the number of orderings is k. $cl(F)$ means the chain length of a preordering T_F (see definition 8.2 in [6]).

A preordering $T \subset F$ is called a fan if for any set $S \supset T$ such that $-1 \notin S$, if $S - \{0\}$ is a subgroup of index 2 in $\dot{F}$, then S is an ordering (Definition 5.1. [6]).

We define stability index st(F) of the field F

$$st(F) = \sup\{\log_2|X|,\ X = (X, \dot{F}/\dot{T})\}$$

where T ranges over the fans in F containing T_F. (Theorem 13.7. [6]. Moreover Chapter 13 contains further information about stability index).

$G_F(G_K, G_L, \ldots)$ means $Gal(F(2)|F)$. $G = \overset{r}{\underset{1}{*}} G_i$ means that G is a free product of groups G_i in the category of pro-2-groups.

For other definitions and theorems used here, the reader is referred to [5], [6], [7], [8].

If F has type $(n, 2^n)$, $n \neq 1$, then according to Theorem 3.1 [15] the Galois group $H = G(F(2)|F(\sqrt{-1}))$ is a free pro-2-group. Therefore $cd_2(F(\sqrt{-1})) = 1$. On the other hand, by Corollary 13.3 in [6], $st(F) = 1$, too. (From this result, we can derive that the Galois group $Gal(F(2)|F)$ is a free product of n two-elements groups [12]).

If F has type $(m2^{n-\mu(m)}, 2^n)$, where $m = 1+2^{k_1}+\ldots+2^{k_\ell}$, $k_1 < \ldots < k_\ell$, $\mu(m) = k_\ell+\ell+1$, $n \geq \mu(m)$, then from results in [11], (III) one can deduce that both st(F), $cd_2(F(\sqrt{-1}))$ have value $n-(\ell+1)$.

The theorem below gives an important relationship connecting two invariants, one of the order space and the other of the Galois group. Even though it may be deduced from results in [1], [2] and [5] it does not occur explicitly in the literature. We give a proof which exploits ideas related to fans in the theory of real fields.

<u>Theorem</u>. <u>Let F be a Pythagorean field such that $|\dot{F}/\dot{F}^2| < \infty$</u>. <u>Then</u>

$$st(F) = cd_2(F(\sqrt{-1}))$$

Proof. We prove this by induction on $cl(F)$. If $cl(F) = 1$, then F is an Euclidean field [4], and

$$st(F) = 0 = cd_2(F(\sqrt{-1}))$$

Now, suppose that our assertion is true for any field F, with $cl(F) < a \in \mathbb{N}$, $2 \leq a$. Take F with $cl(F) = a$.

1) First suppose that the space of orderings $(X, \dot{F}|\dot{T}_F)$ has the form

$$X = X_1 \cup \ldots \cup X_s \ , \ s \geq 2$$

where $X_1, \ldots, X_s$ are connected components of X.
Then by §2 in [8], we get

$$stF = \max\{1, st(X_i), i=1,2,\ldots,s\} ,$$

According the results in [5], $F = \bigcap_{i=1}^{s} F_i$, where F_i is either the euclidean closure of F with respect to the ordering $P_i, \{P_i\} = X_i$ or F_i is a 2-Henselisation of F with respect to some valuation V_i on F compatible with all orderings in X_i. Since, according to Theorem in [10], $H^2(G_{F_i}, \mathbb{Z}/2\mathbb{Z})$ is generated by quaternion algebras[1], we get by Lemma 9' in [5]

$$G_F \cong \mathop{*}_{i=1}^{s} G_{F_i}$$

Now let H be $G_{F(\sqrt{-1})}$. Then by Theorem in [3],

[1] If F is any Pythagorean field of finite chain length then from Theorem 6 in [5] (or Lemma 9 and page 267) we see that $H^2(G_{F_1}, \mathbb{Z}/2\mathbb{Z})$ is generated by quaternion algebras. Thus in this special case one does not need to rely on Merkurjev's theorem.

$$H \cong \mathop{*}_{i=1}^{s} (H \cap G_{F_i}) * L, \; L \neq \{1\}$$

where L is a free pro-2-group. From (4.2) Satz in [13], we get

$$cd_2 H = \max\{1, cd_2(H \cap G_{F_i}), 1 \leq i \leq s\}$$

Since $[G_{F_i} : H \cap G_{F_i}] = 2$ and for every $h \in H \cap G_{F_i}$, $h(\sqrt{-1}) = \sqrt{-1}$, we get

$$H \cap G_{F_i} = G_{F_i(\sqrt{-1})}$$

Since by (1.7) Theorem in [8], $cl(F_i) < a$, $i = 1, \ldots, s$ by induction hypothesis, we get

$$cd_2(H \cap G_{F_i}) = st(F_i)$$

Hence

$$\begin{aligned} cd_2 H &= \max\{1, cd_2(H \cap G_{F_i}), 1 \leq i \leq s\} \\ &= \max\{1, st(F_i), 1 \leq i \leq s\} \\ &= st(F) \end{aligned}$$

2) Suppose that X is indecomposable space with $|X| \neq 1$. Then, according to Theorem 2.8 in [8], there exists a valuation V on F compatible with all orderings in X, such that $U_V \dot{F}^2 \neq \dot{F}$. By taking composed valuations if necessary, we can assume that the space X_V of orderings of the field F_V is decomposable or consists just of one element. (See proof of the Theorem in [11], I). Put $b = \dim_{\mathbb{Z}/2\mathbb{Z}} \dot{F}/\dot{F}^2 U_V$. Then by Theorem C and its proof in [16] we have

$$H = G_{F(\sqrt{-1})} \cong \mathbb{Z}_2^b \times J$$

where $J \cong G_{K(\sqrt{-1})}$, and K is a Pythagorean field such that $X_K \cong X_{F_V}$. Since $cl(F) \geq cl(F_V) = cl(K)$ and the space X_K is decomposable or $|X_K| = 1$, from 1) we get $st(K) = cd_2(K(\sqrt{-1}))$. From Proposition 4.4 in [14] we get

$$\begin{aligned} cd_2H &= b+cd_2J \\ &= b+st(K) \\ &= b+st(F_V) \\ &= st(F) \end{aligned}$$

completing our proof.

<u>COROLLARY. Let F be a Pythagorean field with</u> $cl(F) < \infty$. <u>If $st(F) < \infty$, then</u> $|\dot{F}/\dot{F}^2| < \infty$ <u>and</u> $st(F) = cd_2(F(\sqrt{-1}))$.

<u>If $st(F) = \infty$, then also</u> $cd_2(F(\sqrt{-1})) = \infty$.

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SOME COMMENTS ON SUBCLASSES OF SEMIMONOTONE MATRICES

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Presented by G.de B. Robinson, F.R.S.C.

In this note we shall summarize the results which we have obtained concerning the linear complementarity problem and some subclasses of semimonotone matrices. For a review of the concepts and notation used, the reader should consult Cottle and Stone [2], Pang [6], Murty [5], Berman and Plemmons [1], or Jeter and Pye [3]. For a given matrix $M \in R^{n\times n}$ and a column vector $q \in R^{n\times 1}$, the linear complementarity problem, denoted by LCP(q,M), is to find $z \geq 0$ and $w \geq 0$ so that $Iw - Mz = q$ and $w^T z = 0$. It is well known that LCP(q,M) has a unique solution for every $q \in R^{n\times 1}$ if and only if $M \in P$, where P denotes the class of P-matrices. The class Q consists of those matrices M for which LCP(q,M) has a solution for every q. Unfortunately Q defies a constructive characterization. Let $\alpha \subseteq \bar{n} \equiv \{1,\ldots,n\}$ and define $C(\alpha) \in R^{n\times n}$ columnwise as follows: $C(\alpha)_{\cdot i} = -M_{\cdot i}$ whenever $i \in \alpha$ and $C(\alpha)_{\cdot i} = I_{\cdot i}$ otherwise. Then pos $C(\alpha)$ is called a complementary cone of M. It is known that $M \in Q$ if and only if the union K(M) of all the cones pos $C(\alpha)$ is $R^{n\times 1}$. The class E_0 of semimonotone matrices M has the property that LCP(q,M) has a unique solution whenever $q > 0$. The class E_0^f of fully semimonotone matrices is the collection of those matrices M such that LCP(q,M) has a unique solution whenever q belongs in the interior of a full complementary cone. The class U is a subclass of E_0^f and is made up of those matrices M for which LCP(q,M) has a unique solution whenever $q \in \text{int } K(M)$. Finally, R_0 is a subclass of $R^{n\times n}$ where $M \in R_0$ if LCP(0,M) has a unique solution.

Pang [6] has established that $L \cap Q = L \cap R_0$ for a class L of matrices contained in E_0. He posed the question of whether the result holds for the larger class E_0. In particular, is $E_0 \cap Q \subseteq R_0$? We have established the following results in an attempt to answer this question.

Theorem 1. Let $M \in E_* \cap Q$. Then either (i) $M \in R_*$, or (ii) there exists $\phi \subsetneq \beta \subsetneq \bar{n}$ such that for every $j \in \beta$, $I_{\cdot j}$ belongs to a complementary cone having at least one generator from the columns in $\{-M_{\cdot k} : k \in \beta\}$ (or $\{-M_{\cdot k} : k \in \beta, k \neq j\}$ whenever $M \in C_* \cap Q$, where C_* denotes the class of copositive matrices).

Theorem 2. Let $M \in C_* \cap Q$. Then at least one of the following conditions hold:

(i) each nonzero solution z of LCP(o,M) has at least two zero components,

(ii) for some k, $I_{\cdot k} \in \text{pos}\,\{-M_{\cdot 1}, \ldots, -M_{\cdot k-1}, -M_{\cdot k+1}, \ldots, -M_{\cdot n}\}$.

Theorem 3. $C_* \cap Q \cap R^{3\times 3} \subsetneq R_*$.

Necessary and sufficient conditions for the existence of a nonsingular principal submatrix of a given matrix can be found in the papers [4] by Mangasarian and [7] by Parsons. We have established the following sufficient condition for the existence of such a submatrix.

Theorem 4. Let $M \in R^{n\times n}$ and $q \in R^{n\times 1}$ such that $q \not\geq 0$. If LCP(q,M) has a unique nondegenerate solution, then M possesses a nonsingular principal submatrix.

We can preclude the possibility of M possessing a negative principal minor by restricting M to belong to the class $\mathcal{W}$ which we define as follows:
$\mathcal{W} = \{M \in R^{n\times n} : \text{for any } \alpha \subseteq \bar{n},\ \text{pos}\,C(\alpha) \cap \text{pos}\,C(\hat{\alpha}) = \{0\}\}$, where $\hat{\alpha} \equiv \bar{n} \setminus \alpha$.
The class $\mathcal{W}$ contains P.

Theorem 5. If $M \in \mathcal{W}$ and there is at most one complementary cone of M which is not pointed, then $M \in P_0$, where P_0 denotes the class of matrices possessing only nonnegative principal minors.

In particular, whenever $M \in \mathcal{W}$ and M has exactly one singular principal submatrix, then $M \in P_0$. If all principal submatrices of M are nonsingular and if $M \in \mathcal{W}$, then $M \in P$. It is also true that whenever $M \in \mathcal{W}$ and each principal submatrix of M contains at least one row of nonzero entries of constant sign, then $M \in P_0$. In particular, whenever $M \in \mathcal{W}$ and $M > 0$, then $M \in P_0$.

Membership in the class $\mathcal{W}$ by M has some important consequences concerning the uniqueness of solutions of LCP(q,M). These consequences are developed in the final sequence of results listed below.

Theorem 6. Let A be any principal pivot transform [1] of M. Then $M \in \mathcal{W}$ if and only if $A \in \mathcal{W}$.

Theorem 7. If $M \in \mathcal{W}$, then each principal submatrix of M is a $\mathcal{W}$-matrix.

Theorem 8. If $M \in \mathcal{W}$, then $M \in E_0^f$. In fact, $\mathcal{W} \subsetneq \mathcal{U}$.

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REPLACEABILITY IN MATRIX METHODS

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A matrix is replaceable, if it is equipotent to a matrix the columns of which are null sequences. In this note, we give a brief account of partial solutions to the replaceability characterization problem, and recall some known results on replaceability.

We consider an infinite complex matrix $A = (a_{nk})$ $n, k = 1, 2, \ldots$ with summability field c_A defined to be $\{x = (x_1, x_2, \ldots) : Ax \in c\}$, where c is the space of convergent sequences and $Ax = (\sum_{k=1}^{\infty} a_{1k}x_k, \sum_{k=1}^{\infty} a_{2k}x_k, \ldots)$. The symbol e^k will denote the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with 1 in the k-th entry and 0 elsewhere, and φ will denote the linear span of $\{e^1, e^2, \ldots\}$. We assume that $\varphi \subset c_A$, hence A has convergent columns. A matrix A is said to be replaceable, if there exists a matrix D with $c_A = c_D$ and $\lim_D x = \lim Dx = 0$ for each $x \in \varphi$, i.e. if all columns of D are null sequences.

In the earlier investigations, most of the discussions on matrix methods of summability were confined to conservative matrices. For further information on conullity and the invariances for the distinguished subsets B, W, F and P see [6], also [2].

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Theorem 1. Let A be a co-regular matrix. Then A is replaceable if and only if $\overline{\varphi} \neq P$.

The above closure is taken within the space c_A endowed with FK-topology. It was first asked in [11] two decades ago whether a conull matrix must be replaceable. A broad answer to this question was given by a counter-example in [8] a few years later. As a result of intensive study of an article of E. Jürimäe (Tartu J. Mat. 1965), new classes have been added within conull matrices together with further information on these classes. It was indeed the distinction between these classes which enabled us to give the broad answer to the question XII in [11] as mentioned above. Carrying on the study of [8] we obtain the following

Theorem 2. Let A be a conull matrix with $W \not\supset m \cap c_A \supset c$. Then A is replaceable if and only if $\overline{\varphi} \neq P$.

Proof. $\overline{\varphi} \neq P$ is sufficient for replaceability, c f [12] 15.2.7. As for the necessity, we first assume without loss of generality (replaceability is invariant) that $\lim_A$ vanishes on $\overline{\varphi}$. By our hypothesis, there exists an x in $m \cap c_A$ with $x \notin W$, therefore we can conclude that $\lim_A x \neq \sum_{k=1} a_k x_k = 0$. But $x \in P$, hence $\overline{\varphi} \neq P$. □

For a conull matrix A with $w \not\supset m \cap c_A$ but $\overline{c} \supset m \cap c_A$, A is known to be non-replaceable. From an observation of [9], we see that $\overline{c} \not\supset m \cap c_A$ implies the replaceability of A. A natural refinement of the above theorems and their consequences was given in [7] by simply replacing $m \cap c_A$ with F. Clearly matrices with $W \neq F$ are μ-unique, and the same holds for non-replaceable matrices. (See, for instance, [2]). Thus the result above is uninformative as far as replaceability is concerned, but it seems to provide a model for a general characterization of replaceability. In [5] the following result can be regarded as a best possible improvement of the theorem above; and an example was also given in [5] to demonstrate that no further generalization of the result below exists.

Theorem 3. Let A be a μ-unique matrix with dim $(c_A/\overline{\varphi})$ finite. Then A is replaceable if and only if $\overline{\varphi} \neq P$.

From the example in [5] (not mentioning its structural complexity), we finally are able to see that the statement $\overline{\varphi} \neq P$ does not characterize replaceability after all.

It was noted in [6] that each replaceable matrix has the property $B = F$. At one time we were hoping this property might be the one for replaceability characterization. Soon, however, an example of a non-replaceable co-regular conservative matrix with $B = F$ was given in [10].

The following example, introduced in [11], is a non-replaceable conull conservative matrix with $B = F$. After recent communication with Professors Wilansky and Beekmann, we became aware of the fact that, contrary to the claim, the matrix below does not provide an answer to the problem stated in the example 13.3.3,[12]. Thus the question whether a matrix A must have AB under the condition $c_A = S \oplus u$ with $u \in$ Inset remains open.

Example Let

$$A = \begin{pmatrix} -b_1 & & & & \\ b_1 & -b_2 & & & \\ b_1 & b_2 & -b_3 & & \\ b_1 & b_2 & b_3 & -b_4 & \\ " & " & " & " & " \\ & & \cdots & & \end{pmatrix}$$

with $b = (b_1, b_2, \ldots) \in \ell$. We conclude that A is non-replaceable with $W = F = B$.

Proof. It is known that a given functional on c_A can be written as

$$f(x) = \mu \lim_A x + t(Ax) + sx, \; x \in c_A \qquad (*)$$

where $\mu \in \mathbb{C}$, $t \in \ell$ and $s \in c_A^\beta$.

Let K be a matrix with the property that for every $f \in c_K'$ and $f(e^k) = 0$, $k \in \mathbb{N}$, we have $\mu = 0$. Clearly the matrix K is a μ-unique matrix. Suppose that there is a matrix D with $c_D = c_K$ and $\lim_D = 0$ on φ. Since $\lim_D \in c_K'$, we have $\mu(\lim D) = 0$, which contradicts the fact that D is a μ-unique matrix. Thus no such D exists, therefore each matrix K with the above property is non-replaceable. (cf [6], [4]).

We now show that A has the property stated above. First we note that, (cf [12])

$$c_A = b^\beta \oplus \{u\}, \text{ with } u = (2b_1^{-1}, 2^2 b_2^{-1}, 2^3 b_3^{-1}, \ldots).$$

Take $f \in c_A'$ such that $f(e^k) = 0$, for all $k \in \mathbb{N}$. From (*),

$$0 = \mu b_k + (tA)_k + s_k \text{ or } s_k = -\mu b_k - (tA)_k,$$

or

$$s_k = -\mu b_k - \Big(\sum_{\nu=k+1}^{\infty} t_\nu - t_k\Big) b_k.$$

Since $u \in c_A$ and $s \in c_A^\beta$, it follows that $s_k u_k = o(1)$. But $u_k = 2^k b_k^{-1}$ for $k \in \mathbb{N}$, so we have

$$o(2^{-k}) = -\mu - \sum_{\nu=k+1}^{\infty} t_\nu + t_k, \text{ or } \mu = 0.$$

This says that A is non-replaceable.

By [11] it is easy to see that $I = b^\beta = \Lambda^\perp$, so that we have $\Lambda^\perp = I = W = F$.

But, for $f \in c_A'$, with $x \in B$ we have

$$f(x) = \mu \lim_A x + \lambda x,$$

for some sequence λ. It follows immediately, by virtue of the Hahn-Banach theorem, that $\bar{\varphi} \supset B$. Since $\varphi \subset F \subset B$ and F is closed, we finally conclude that

$W = F = B$. □

Every non-μ-unique matrix is known to be replaceable, and the following stands as the only replaceability characterization of μ-unique matrices. We are naturally looking for more constructive and practical one(s) with obvious reasons. Also we note that, in view of our example, the answer to question 7, p. 300 in [12] is negative.

THEOREM 4. Let A be a μ-unique matrix. Then A is non-replaceable if and only if for each matrix D with $c_D = c_A$, we have $D(\varphi) \not\subset c_0$.

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CHARACTERIZING S^m BY THE SPECTRUM OF THE LAPLACIAN

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Presented by G.de B. Robinson, F.R.S.C.

ABSTRACT. The Euclidean sphere S^{2n+1} is characterized by the spectrum of the Laplacian on 2-forms in all dimensions.

1. Introduction. It was recently shown [3], [4] that within the class of Kaehler mainfolds, complex projective n-space CP_n with the Fubini metric g_0 is characterized by the spectrum of the Laplacian on 2-forms in all dimensions. More precisely, let (M,g) be a compact Kaehler manifold with $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(CP_n,g_0)$, where $\mathrm{Spec}^p(M,g)$ denotes the spectrum of the Laplacian with respect to the Kaehler metric g on p-forms of M. Then, (M,g) is holomorphically isometric to (CP_n,g_0) for all n. In this paper, we consider the problem of characterizing the constant curvature sphere S^m by the spectrum of its Laplacian on p-forms: If $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(S^m,g_0)$ for some fixed p, is (M,g) isometric with (S^m,g_0), where g_0 is the constant curvature metric? The answer to this question is yes in the following cases:

(a) $p = 0$ and $m \leq 6$ [1], [7];

(b) $p = 1$ and $m = 2,3,16,\ldots,93$ [8];

(c) $p = 2$ and $m = 2,3,6,7,14,17,\ldots,178$ [6].

REMARK. Patodi [5] proved that if $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(S^m,g_0)$ for $p = 0$ and 1, then (M,g) is isometric with (S^m,g_0) in all dimensions. In order to obtain uniqueness for a fixed p in all (odd) dimensions we confine ourselves to the class of normal contact Riemannian manifolds and obtain the following

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statement.

THEOREM 1. Let (M,g) be a compact normal contact Riemannian manifold. If $\mathrm{Spec}^2(M,g) = \mathrm{Spec}(S^{2n+1}, g_0)$, where g_0 is the metric of constant curvature $k = 1$, then g is a metric of the same constant curvature $k = 1$.

2. The spectrum. Let (M,g) be a compact connected Riemannian C^∞ manifold without boundary, and with Laplacian $\Delta = -(dd^* + d^*d)$, where d is the operator of exterior differentiation and d^* is its adjoint with respect to the Riemannian metric g. Then, for each $p = 0,1,2,\ldots,$ the spectrum of Δ is given by

$$\mathrm{Spec}^p(M,g) = \{0 \geq \lambda_{1,p} \geq \lambda_{2,p} \geq \ldots \geq \lambda_{k,p} \geq \ldots \downarrow -\infty\},$$

each eigenvalue $\lambda_{i,p}$ repeated as often as its multiplicity. By Hodge theory, $0 \in \mathrm{Spec}^p(M,g)$ if and only if the pth betti number $b_p(M)$ is not zero, and its multiplicity is then $b_p(M)$. For $p = 2$, the Minakshisundaram-Pleijel-Gaffney formula is

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,2} t) = \frac{1}{(4\pi t)^m} \sum_{i=0}^{N} a_{i,2} t^i + O(t^{N-m+1}), \quad t \downarrow 0,$$

the coefficients $a_{i,2}$, $i = 0,1,2$ being given by

$$a_{0,2} = \frac{m(m-1)}{2} V, \quad V = \mathrm{vol}(M), \tag{2.1}$$

$$a_{1,2} = \frac{m^2-13m+24}{2} \int_M \rho \, dV, \tag{2.2}$$

$$a_{2,2} = \frac{1}{720} \int_M [2(m^2 - 31m + 240)|R|^2 - 2(m^2 - 181m + 1080)\ |S|^2 + 5(m^2 - 25m + 120)\rho^2] dV. \tag{2.3}$$

where R, S and ρ denote the curvature tensor, the Ricci tensor and the scalar curvature of g, respectively, and $|R|^2 = \Sigma R_{ijkl}R^{ijkl}$, $|S|^2 = \Sigma R_{ij}R^{ij}$, R_{ijkl} and R_{ij} denoting the components of R and S, respectively [5].

If $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(M',g')$, then $\dim M = \dim M'$, $V = V'$, $b_2(M) = b_2(M')$ and

$$\int_M \rho \, dV = \int_{M'} \rho' \, dV' \quad \text{and} \quad a_{2,2} = a'_{2,2} \,. \tag{2.4}$$

The following expression for $a_{2,2}$ will be useful:

$$720\, a_{2,2} = \int_M [Q_1 |C|^2 + Q_2(|S|^2 - \frac{\rho^2}{m}) + Q_3\rho^2] dV, \tag{2.5}$$

where $|C|^2 = \Sigma C_{ijkl}C^{ijkl}$ is the square of the norm of the Weyl conformal curvature tensor and

$$Q_1 = 2(m-15)(m-16), \qquad Q_2 = \frac{8(m-15)(m-16)}{m-2} - 2(m^2 - 181m + 1080),$$

$$Q_3 = \frac{4(m-15)(m-16)}{m(m-1)} - \frac{2(m^2-181m+1080)}{m} + 5(m^2 - 25m + 120).$$

If (M',g') is a manifold of constant curvature k', then $|C'| = 0$ and $|S'|^2 = \rho'^2/m$, so by (2.5)

$$\int_M [Q_1 |C|^2 + Q_2(|S|^2 - \frac{\rho^2}{m}) + Q_3\rho^2] dV = \int_{M'} Q_3 \rho'^2 dV' . \tag{2.6}$$

Thus, since Q_1, Q_2 and Q_3 are positive for $m = 3,6,7,14,17,18,\ldots,178$, and $\int_M \rho^2 dV \geq \int \rho'^2 dV'$, the latter being a consequence of Schwarz's inequality and (2.4), g is a conformally flat Einstein metric. Hence, (M,g) is a manifold of constant curvature $k = k'$ in these dimensions [6]. For $m = 8$, Q_3 vanishes and Q_1 and Q_2 are both positive, so again g is a constant curvature metric. For $m = 15$ and 16, Q_1 vanishes and Q_2 and Q_3 are both positive, so g is an Einstein metric with scalar curvature ρ'. If $M' = S^m$ with the metric of constant curvature k', then, since $V = V'$ it follows from [2 : p. 257] that (M,g) is isometric with (S^m,g') for $m = 15$ and 16. This extends Theorem 3.1 in [6].

The case ρ = constant is interesting. For, since Q_1 and Q_2 are positive for $m = 9,\ldots,13$, we may again conclude that g is a metric of constant curvature.

THEOREM 2. Let (M,g) be a compact Riemannian manifold. If $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(S^m,g_0)$, where g_0 is a metric of constant curvature k', then g is a metric of the same constant curvature $k = k'$ for $m = 2,3,6,7,8,14,\ldots,178$. If, in addition, g is a metric of constant scalar curvature, then g is a metric of constant curvature $k = k'$ for $m = 2,3,6,\ldots,178$.

The case $m = 2$ is a consequence of the fact that $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(S^2,g_0)$ implies $\mathrm{Spec}^0(M,g) = \mathrm{Spec}^0(S^2,g_0)$.

THEOREM 3. Let (M,g) be a compact Riemannian manifold with $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(S^m,g_0)$, where g_0 is a metric of constant curvature k'. If for some $\lambda \in \mathbb{R}$

$$\int_M (|S|^2 - \lambda\rho^2)dV = \int_{S^m} (|S'|^2 - \lambda\rho'^2)dV',$$

where the prime indicates corresponding quantities in (S^m,g_0), then for

(i) $\lambda < \frac{1}{m}$, g is a metric of constant curvature $k = k'$,

(ii) $\lambda \geq \frac{1}{m}$, g is a metric of constant curvature $k = k'$ for each m satisfying $(\lambda - \frac{1}{m})Q_2 + Q_3 > 0$.

3. Contact Manifolds. An almost contact structure (ϕ,x_0,η) on a $(2n+1)$-dimensional C^∞ manifold M is given by a linear transfromation field

ϕ, a vector field x_0, a 1-form η satisfying $\eta(x_0) = 1$, $\phi x_0 = 0$ and $\phi^2 = -1 + \eta \otimes x_0$. In this case, a Riemannian metric g can be found such that $\eta = g(x_0, \cdot)$ and $g(\phi x, y) = -g(x, \phi y)$ for any vector fields x and y.

A contact manifold with contact form η has an underlying almost contact Riemannian structure (ϕ, x_0, η, g) such that $g(x, \phi y) = d\eta(x,y)$. If the almost complex structure J on $M \times R$ defined by $J(x, fd/dt) = (\phi x - f x_0, \eta(x) d/dt)$ is integrable, the almost contact structure is said to be <u>normal</u>, and M is said to be a <u>normal contact manifold</u>. In this case, the vector field x_0 is a Killing vector field. Moreover,

$$g(R(x,x_0)y,x_0) = g(x,y) - \eta(x)\,\eta(y) = g(\phi x, \phi y) \quad \text{and} \quad S(x,x_0) = 2n\eta(x).$$

The standard contact Riemannian structure on an odd-dimensional sphere is normal.

The form $\tilde{S}(x,y) = S(x, \phi y)$ is a skew-symmetric bilinear form on M. The following lemma is essential for the proof of Theorem 1.

LEMMA 1. <u>Let</u> M <u>be a compact normal compact manifold with</u> $b_2(M) = 0$. <u>Then there exists a</u> 1-<u>form</u> α <u>on</u> M <u>such that</u> $\tilde{S} = d\alpha$ <u>and</u> $\alpha(x_0) = \text{const}$.

PROOF OF THEOREM 1. Using Lemma 1, an argument similar to that of §2 in [4] gives rise to the relation $\int_M (|S|^2 - \frac{1}{2}\rho^2)dV = \int_{S^{2n+1}} (|S'|^2 - \frac{1}{2}\rho'^2)dV'$.

Therefore, using Theorem 3 with $\lambda = \frac{1}{2}$ and noting that $(\frac{1}{2} - m)Q_2 + Q_3 > 0$ for all odd $m \geq 3$, we conclude that g is a metric of constant curvature $k = 1$ for all $n \geq 1$.

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A NOTE ON BURGESS' CHARACTER SUM ESTIMATE

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Presented by J.H.H. Chalk, F.R.S.C.

Abstract: We improve the range of validity for a special case of Burgess' character sum estimate.

Let χ be a non-principal character modulo a prime p. Estimates of the type

(1) $$\left|\sum_{n\leq N}\chi(n)\right| \leq \varepsilon N \qquad (N\geq N_0(\varepsilon,p))$$

are of great importance in many places in number theory. By the Polya-Vinogradov inequality, (1) holds with $N_0(\varepsilon,p) = (\log p)\sqrt{p}/\varepsilon$, and Burgess' deep character sum estimate [1] [2] yields (1) with $N_0(\varepsilon,p) = p^{1/4+\delta}$ for any fixed $\varepsilon, \delta >0$ and $p\geq p_0(\varepsilon,\delta)$. We improve this as follows:

THEOREM: *Given $\varepsilon >0$ there exist $\delta=\delta(\varepsilon) >0$ and $p_0(\varepsilon) \geq 2$ such that for any non-principal character χ modulo a prime $p \geq p_0(\varepsilon)$, (1) holds with $N_0(\varepsilon,p) = p^{1/4-\delta}$. A possible choice for $\delta(\varepsilon)$ is*

(2) $$\delta(\varepsilon) = \exp(-c(\varepsilon^{-2}+1))$$

with a sufficiently large absolute constant c.

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Proof: The improvement over Burgess' estimate is made possible by the following

LEMMA: *Let* f *be a multiplicative function of modulus* ≤ 1 *, and let*

$$M(x) = M(f,x) = \frac{1}{x}\sum_{n\leq x} f(n) \qquad (x\geq 1) \quad .$$

Then we have

$$|M(x')| = |M(x)| + O(R(x,x')) \qquad (3\leq x<x'\leq x^{5/4}) \quad , \tag{3}$$

where

$$R(x,x') = (\log\log x)^{-1/2} + \left(\log\frac{\log x}{\log(x'/x)}\right)^{-1/2}$$

and the O-constant is absolute.

For real-valued functions f , this was proved in [3,Lemma 4] . There, after establishing (3), the stronger relation

$$M(x') = M(x) + O(R(x,x')) \qquad (3\leq x<x'\leq x^{5/4}) \tag{4}$$

was deduced. While for the deduction of (4) from (3) the hypothesis that f is real-valued is essential, this hypothesis was not used in [3] for the proof of (3). Hence the above lemma holds as stated.

Applying the lemma with $f = \chi$ (χ being a non-principal character modulo a prime p), $x=N\geq p^{1/4-\delta}$ and $x'=Np^{2\delta}$ ($\geq p^{1/4+\delta}$) , we obtain

$$\frac{1}{N}\left|\sum_{n\leq N}\chi(n)\right| = \frac{1}{Np^{2\delta}}\left|\sum_{n\leq Np^{2\delta}}\chi(n)\right| + O((\log\log(N+2))^{-1/2}) + O((\log(1/\delta))^{-1/2}) \quad , \tag{5}$$

provided $0<\delta<1/40$. If now $\varepsilon>0$ is given, then defining $\delta=\delta(\varepsilon)$ by (1) with a suitable constant c , the last error term in (5) becomes $\leq\varepsilon/3$. The first error term is $\leq\varepsilon/3$, if p (and hence N) is sufficiently large, and by Burgess' estimate the main term on the right-hand side of (5) is also bounded by $\varepsilon/3$ for $p\geq p_0(\varepsilon/3,\delta(\varepsilon))$. The asserted estimate now follows.

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ON INTEGRABLE SOLUTIONS TO THE BARON-BOYARSKY FUNCTIONAL EQUATION

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Presented by J. Aczél, F.R.S.C.

We consider the problem of the existence of integrable solutions $f:[0,1]\to[0,+\infty)$ to the equation

$$(1)\quad f\left(\frac{1}{2}+x\right)+f\left(\frac{1}{2}-x\right) = 2rxf\left(r\left(\frac{1}{4}-x^2\right)\right) \text{ for all } x\in\left[0,\frac{1}{2}\right],$$

where r is a constant in $[3,4]$.

We ask if there are any solutions which are different from zero on a set of positive Lebesgue measure ("non-trivial solutions").

With the additional requirement

$$(2)\quad \operatorname{supp} f \subset \left[\frac{r^2}{4}\left(1-\frac{r}{4}\right), \frac{r}{4}\right]$$

Karol Baron [1] has proven that (1) has no non-trivial Lebesgue integrable solution provided $r\in[3,4]$ satisfies

$$\frac{r^2}{4}\left(1-\frac{r}{4}\right) \geq \frac{1}{2}, \quad \text{i.e.} \quad r \leq 1+\sqrt{5} = 3.236\ldots$$

In this note we shall show that there is a set $S\subset[3.57,4]$ of positive Lebesgue measure such that when $r\in S$ the equation (1) has no non-trivial Riemann integrable solution. The set S contains 4 and has 4 as a point of density, i.e.

$$|S\cap[r,4]| \,/\, (4-r) \to 1 \quad \text{as} \quad r\to 4.$$

We do not use the assumption (2).

Our method of proof, which is totally different from Baron's, relies on results about the iterates of the one-parameter family $\{g_r\}$, $r\in[3,4]$, where

$$(3)\quad g_r(x) = rx(1-x),\ x\in[0,1].$$

For the proof, suppose $f:[0,1]\to[0,+\infty)$ solves (1). We rewrite the equation in the form

(4) $f(x) + f(1-x) = 2r\left|\frac{1}{2} - x\right| f\big(g_r(x)\big)$,

and (4) is now satisfied for all $x \in [0,1]$.

We shall use only that $f\left(\frac{1}{2}\right) = 0$ and that

(5) $\forall x \in [0,1]: f\big(g_r(x)\big) = 0 \Rightarrow f(x) = 0$.

The implication (5) follows from (4) since by assumption $f \geq 0$.

By g_r^n we denote the n'th iterate of g_r, thus $g_r^o(x) = x$ and $g_r^{n+1}(x) = g_r(g_r^n(x))$ for all x. We shall say that g_r has a periodic orbit of period $p \geq 1$ if $g_r^p(x_o) = x_o$ for some x_o. The orbit is said to be stable if $\left|\frac{d}{dx} g_r^p(x_o)\right| < 1$.

From (5) it follows that if $g_r^n(x) = \frac{1}{2}$ for some $n \geq 0$ then $f(x) = 0$. Thus $f(x) = 0$ for all x in the set

$$G_r = \bigcup_{n=o}^{\infty} g_r^{-n}\left(\frac{1}{2}\right) = \left\{x \in [0,1] \,\middle|\, \exists n \geq 0: g_r^n(x) = \frac{1}{2}\right\} .$$

We shall now combine the following two theorems about the iterates of the family $\{g_r\}$, $r \in [3,4]$:

I. If g_r has no stable periodic orbit then G_r is dense in [0,1]. (For this result see [2], Corollary II.5.5, p. 117).

II. There is a set $S \subset [3.57,4]$ of positive Lebesgue measure, containing the point 4, and having 4 as a point of density, such that for $r \in S$ g_r has no stable periodic orbit. (See [3] or [4]).

We conclude that if $r \in S$ then G_r is dense in [0,1]. Therefore f is zero on a dense set. Thus if f is Riemann integrable it must be zero almost everywhere. This completes the proof.

For information about the structure of the set S consult [2] or [5].

<u>Remark</u>. In order to obtain Baron's result one need not <u>assume</u> (2), since it can be shown that if f is Lebesgue integrable and satisfies (1) then $f(x) = 0$ for a.e. x outside $\left[\frac{r^2}{4}\left(1-\frac{r}{4}\right), \frac{r}{4}\right]$.

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ON THE ZETA FUNCTION ATTACHED TO THE DUAL REDUCTIVE PAIR $(\underline{\underline{MO}}(p,q,\underline{\underline{R}}), \underline{\underline{Mp}}(1,\underline{\underline{R}}))$ IN THE METAPLECTIC GROUP $\underline{\underline{Mp}}(p+q,\underline{\underline{R}})$

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Presented by P.C. Greiner, F.R.S.C.

1. Introductory Remarks on the Restricted Metaplectic Representation

Let $p \geq 0$ and $q \geq 0$ denote integers having the sum $n=p+q \geq 2$. Fix the standard non-degenerate $\underline{\underline{R}}$-bilinear form $(.|.)_{p,q}$ of signature (p,q) on the n-dimensional real vector space $\underline{\underline{R}}^n = \underline{\underline{R}}^p \oplus \underline{\underline{R}}^q$. Then we have the decomposition $(.|.)_{p,q} = (.|.)_{p,0} - (.|.)_{0,q}$ into a positive and a negative definite $\underline{\underline{R}}$-bilinear form on $\underline{\underline{R}}^p$ and $\underline{\underline{R}}^q$, respectively, by putting

$$(x|y)_{p,q} = \sum_{1 \leq j \leq p} x_j y_j - \sum_{p+1 \leq j \leq n} x_j y_j .$$

We shall adapt the complex Hilbert space $L^2(\underline{\underline{R}}^n)$ to this decomposition of $(.|.)_{p,q}$ by setting $L^2(\underline{\underline{R}}^n) = L^2(\underline{\underline{R}}^p) \otimes L^2(\underline{\underline{R}}^q)$. Let $\mathscr{S}(\underline{\underline{R}}^n)$ denote the complex Schwartz-Bruhat space on $\underline{\underline{R}}^n$. For any function $f \in \mathscr{S}(\underline{\underline{R}}^n)$ we will adapt its Fourier transform to $(.|.)_{p,q}$ according to the rule

$$\mathscr{F}_{\underline{\underline{R}}^p \oplus \underline{\underline{R}}^q} f(y) = \int_{\underline{\underline{R}}^n} f(x) e^{-2\pi i (x|y)_{p,q}} dx \qquad (y \in \underline{\underline{R}}^n).$$

Let $\underline{\underline{O}}(p,q,\underline{\underline{R}})$ denote the isometry group of the quadratic form $\underline{\underline{R}}^n \ni x \longmapsto (x|x)_{p,q}$ associated with $(.|.)_{p,q}$. In the case $0<p<n$ the group $\underline{\underline{O}}(p,q,\underline{\underline{R}})$ has four connected components each component containing one component of the maximal compact subgroup $\underline{\underline{O}}(p,0,\underline{\underline{R}}) \times \underline{\underline{O}}(0,q,\underline{\underline{R}})$ and Klein's Vierergruppe transforms the connected component $\underline{\underline{SO}}(p,q,\underline{\underline{R}})$ of the identity in $\underline{\underline{O}}(p,q,\underline{\underline{R}})$ onto the whole group $\underline{\underline{O}}(p,q,\underline{\underline{R}})$. In the case $n=4$, $p=1$, $q=3$ the proper Lorentz group $\underline{\underline{SO}}(1,3,\underline{\underline{R}})$ in 4 variables arises.

Form the real metaplectic group $\underline{\underline{Mp}}(n,\underline{\underline{R}}) = \underline{\underline{Sp}}(n,\underline{\underline{R}}) \times \underline{\underline{Z}}/2\underline{\underline{Z}}$ which is the double covering of the real symplectic group $\underline{\underline{Sp}}(n,\underline{\underline{R}})$. Then each fiber of the covering homomorphism $\underline{\underline{Mp}}(n,\underline{\underline{R}}) \ni \tilde{\sigma} \longrightarrow \sigma \in \underline{\underline{Sp}}(n,\underline{\underline{R}})$ consists of two different points. Let $\underline{\underline{MO}}(p,q,\underline{\underline{R}})$ be the inverse image of $\underline{\underline{O}}(p,q,\underline{\underline{R}})$ in $\underline{\underline{Mp}}(n,\underline{\underline{R}})$. Then $(\underline{\underline{MO}}(p,q,\underline{\underline{R}}),\underline{\underline{Mp}}(1,\underline{\underline{R}}))$ forms a dual reductive pair in the real metaplectic group $\underline{\underline{Mp}}(n,\underline{\underline{R}})$. In order to explain some details of this notion, let $(\underline{\underline{R}}^2,b)$ denote the two-dimensional real symplectic vector space with the standard symplectic form given by

$$b:\ \underline{\underline{R}}^2 \times \underline{\underline{R}}^2 \ni ((u,u'),(v,v')) \longrightarrow \det\begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \in \underline{\underline{R}}.$$

Then $\underline{\underline{Sp}}(\underline{\underline{R}},b)=\underline{\underline{Sp}}(1,\underline{\underline{R}})=\underline{\underline{SL}}(2,\underline{\underline{R}})$. Obviously the tensor product $B_{p,q}=(.|.)_{p,q} \otimes b$ represents a non-degenerate alternating $\underline{\underline{R}}$-bilinear form on $\underline{\underline{R}}^{2n} \stackrel{p,q}{=} \underline{\underline{R}}^{2p} \oplus \underline{\underline{R}}^{2q}$ and $(\underline{\underline{R}}^{2n},B_{p,q})$ forms a 2n-dimensional real symplectic vector space. Let $\tilde{A}(\underline{\underline{R}}^n)$ denote the (2n+1)-dimensional real Heisenberg nilpotent Lie group with underlying manifold $\underline{\underline{R}}^n \oplus \underline{\underline{R}}^n \oplus \underline{\underline{R}}$ and multiplicative group law

$$(x,y,z).(x',y',z') = (x+x',y+y',z+z'+\tfrac{1}{2}B_{p,q}((x,x'),(y,y'))).$$

Then the real symplectic group $\underline{\underline{Sp}}(\underline{\underline{R}}^n,B_{p,q})$ acts as a group of automorphismsof $\tilde{A}(\underline{\underline{R}}^n)$ leaving the one-dimensional center of $\tilde{A}(\underline{\underline{R}}^n)$ fixed (cf. [7]). An application of the Stone-von Neumann-Mackey unicity theorem implies via the covariance identity for the linear Schrödinger representation of $\tilde{A}(\underline{\underline{R}}^n)$ the existence of the Segal-Shale-Weil metaplectic (or linear oscillator) representation $\underline{\underline{Mp}}(\underline{\underline{R}}^n,B_{p,q}) \ni \tilde{\sigma} \longrightarrow T_{\tilde{\sigma}} \in \underline{\underline{U}}(L^2(\underline{\underline{R}}^n))$ of the real metaplectic group $\underline{\underline{Mp}}(\underline{\underline{R}}^n,B_{p,q})$ in the complex Hilbert space $L^2(\underline{\underline{R}}^n)$. Since the direct product $\underline{\underline{MO}}(p,q,\underline{\underline{R}}) \times \underline{\underline{Mp}}(1,\underline{\underline{R}})$ is naturally embedded into $\underline{\underline{Mp}}(\underline{\underline{R}}^n,B_{p,q})$, we may consider the restriction $T^{p,q} = T\,|(\underline{\underline{MO}}(p,q,\underline{\underline{R}}) \times \underline{\underline{Mp}}(1,\underline{\underline{R}}))$ of the metaplectic representation $(T,L^2(\underline{\underline{R}}^n))$ of $\underline{\underline{Mp}}(\underline{\underline{R}}^n,B_{p,q})$. $(T^{p,q},L^2(\underline{\underline{R}}^n))$ is called to be the restricted metaplectic representation attached to the dual reductive pair $(\underline{\underline{MO}}(p,q,\underline{\underline{R}}),\underline{\underline{Mp}}(1,\underline{\underline{R}}))$ in the metaplectic group $\underline{\underline{Mp}}(n,\underline{\underline{R}})$; cf. Gelbart [2]. The images of $\underline{\underline{MO}}(p,q,\underline{\underline{R}})$ and $\underline{\underline{Mp}}(1,\underline{\underline{R}})$ in the unitary group $\underline{\underline{U}}(L^2(\underline{\underline{R}}^n))$ form the centralizer of the other, whence the name. A calculation of the cocycle of the projective linear representation $\underline{\underline{O}}(p,q,\underline{\underline{R}}) \times \underline{\underline{Sp}}(1,\underline{\underline{R}}) \ni \sigma \longrightarrow T^{p,q}_{\sigma} \in \underline{\underline{U}}(L^2(\underline{\underline{R}}^n))$ associated with the restricted metaplectic representation $(T^{p,q},L^2(\underline{\underline{R}}^n))$ shows that it is isomorphic to an ordinary unitary linear representation of $\underline{\underline{O}}(p,q,\underline{\underline{R}}) \times \underline{\underline{Sp}}(1,\underline{\underline{R}})$ in a complex Hilbert space if and only if $n \geqq 2$ is an even integer.

For each pair $(k,l) \in \underline{\underline{N}} \times \underline{\underline{N}}$ of integers $\geqq 0$ let $H_k(\underline{\underline{R}}^p)$ and $H_l(\underline{\underline{R}}^q)$ denote the complex vector spaces of solid spherical harmonic functions of degree k on $\underline{\underline{R}}^p$ and degree l on $\underline{\underline{R}}^q$, respectively. It is the purpose of this note to attach to every function $f \in \mathcal{S}(\underline{\underline{R}}^n)$ and every pair $(P_k, Q_l) \in H_k(\underline{\underline{R}}^p) \times H_l(\underline{\underline{R}}^q)$ a bivariate zeta function $(s,t) \longrightarrow \zeta(f;P_k,Q_l;s,t)$. We will establish a functional equation and the holomorphic continuation of the zeta function $\zeta(f;P_k,Q_l;\cdot,\cdot)$ by means of the restricted metaplectic representation $(T^{p,q}, L^2(\underline{\underline{R}}^n))$ attached to the dual reductive pair $(\underline{\underline{MO}}(p,q,\underline{\underline{R}}), \underline{\underline{Mp}}(1,\underline{\underline{R}}))$ in the metaplectic group $\underline{\underline{Mp}}(n,\underline{\underline{R}})$. Finally we compare our results with the investigations recently done in this field by Neil Ormerod.

2. <u>The Decomposition of</u> $L^2(\underline{\underline{R}}^n)$ <u>According to the Action of</u> $T_0^{p,q}$

Let $\Delta_{p,q}$ denote the Laplacian operator corresponding to the standard quadratic form $x \longrightarrow (x|x)_{p,q}$ of signature (p,q) on $\underline{\underline{R}}^n$. Recall that the elements of the complex vector space $H_k(\underline{\underline{R}}^p)$, $k \in \underline{\underline{N}}$, and $H_l(\underline{\underline{R}}^q)$, $l \in \underline{\underline{N}}$, respectively, are the homogeneous polynomials P (resp. Q) with complex coefficients of degree k (resp. l) in p (resp. q) real variables such that $\Delta_{p,0}P = 0$ (resp. $\Delta_{0,q}Q = 0$). It is well known (cf. [1,8]) that the natural action of the orthogonal group $\underline{\underline{O}}(p,0,\underline{\underline{R}})$ on $H_k(\underline{\underline{R}}^p)$ defines a finite dimensional irreducible linear representation $(D_{k,p}, H_k(\underline{\underline{R}}^p))$ of $\underline{\underline{O}}(p,0,\underline{\underline{R}})$. Similarly $H_l(\underline{\underline{R}}^q)$ may be considered as $\underline{\underline{O}}(0,q,\underline{\underline{R}})$-module under the action of the irreducible linear representation $(D_{l,q}, H_l(\underline{\underline{R}}^q))$ of the orthogonal group $\underline{\underline{O}}(0,q,\underline{\underline{R}})$. If $P \in H_k(\underline{\underline{R}}^p)$ then

$$\underline{\underline{R}}^p \ni \xi \longrightarrow P(\xi)e^{-\pi(\xi|\xi)_{p,0}} \in \mathbb{C}$$

forms a lowest weight vector for the representation $(T^{p,0}, L^2(\underline{\underline{R}}^p))$ of weight $k+\frac{1}{2}p$. Let $\mathbb{C}_+ = \{z=x+iy \mid x \in \underline{\underline{R}}, y \in \underline{\underline{R}}, y>0\}$ denote the open upper half-plane and for any positive integer or half integer m let $H_m(\mathbb{C}_+)$ denote the complex vector space of all holomorphic functions φ on $\mathbb{C}_+$ such that $\lim_{z \in \mathbb{C}_+, |z| \to \infty} |\varphi(z)| = 0$ and $\int_{\mathbb{C}_+} |\varphi'|^2 y^m dx dy < \infty$.

If $(D_m, H_m(\mathbb{C}_+))$ denotes the holomorphic discrete series of unitary linear representations of $\underline{\underline{Mp}}(1,\underline{\underline{R}})$ with lowest weight m and similarly $(\bar{D}_m, \bar{H}_m(\mathbb{C}_+))$ denotes the antiholomorphic discrete series of unitary linear representations of $\underline{\underline{Mp}}(1,\underline{\underline{R}})$ with lowest weight m (cf. Lang [3]) we have

<u>Theorem 1.</u> <u>The restriction</u> $T_o^{p,q} = T^{p,q} | (\underline{\underline{O}}(p,q,\underline{\underline{R}}) \times \underline{\underline{Mp}}(1,\underline{\underline{R}}))$ <u>of the restricted metaplectic representation</u> $(T^{p,q}, L^2(\underline{\underline{R}}^n))$ <u>attached to the dual reductive pair</u> $(\underline{\underline{MO}}(p,q,\underline{\underline{R}}), \underline{\underline{Mp}}(1,\underline{\underline{R}}))$ <u>in the metaplectic group</u> $\underline{\underline{Mp}}(n,\underline{\underline{R}})$ <u>admits the decomposition</u>

$$(T_o^{p,q}, L^2(\underline{\underline{R}}^n)) = \widehat{\bigoplus_{(k,l)\in \underline{\underline{N}}\times\underline{\underline{N}}}} ((D_{k,p}, H_k(\underline{\underline{R}}^p)) \underset{\mathbb{C}}{\otimes} (D_{l,q}, H_l(\underline{\underline{R}}^q)) \underset{\mathbb{C}}{\otimes} (D_{k+\frac{1}{2}p}, H_{k+\frac{1}{2}p}(\mathbb{C}_+)) \underset{\mathbb{C}}{\otimes} (\bar{D}_{l+\frac{1}{2}q}, \bar{H}_{l+\frac{1}{2}q}(\mathbb{C}_+))).$$

3. <u>The Zeta Function</u> $\zeta(f;P_k,Q_l;\cdot,\cdot)$

The compact unit spheres in $\underline{\underline{R}}^p$ and $\underline{\underline{R}}^q$ are given by $\underline{\underline{S}}_{p-1} = \underline{\underline{O}}(p,0,\underline{\underline{R}})/\underline{\underline{O}}(p-1,0,\underline{\underline{R}})$ and $\underline{\underline{S}}_{q-1} = \underline{\underline{O}}(0,q,\underline{\underline{R}})/\underline{\underline{O}}(0,q-1,\underline{\underline{R}})$, respectively. If $d\sigma_p$ resp. $d\sigma_q$ denote the normalized Haar measures of the compact orthogonal groups $\underline{\underline{O}}(p,0,\underline{\underline{R}})$ and $\underline{\underline{O}}(0,q,\underline{\underline{R}})$, respectively, then the surface measures $d\omega_{p-1} = d\sigma_p/d\sigma_{p-1}$ and $d\omega_{q-1} = d\sigma_q/d\sigma_{q-1}$ of $\underline{\underline{S}}_{p-1}$ and $\underline{\underline{S}}_{q-1}$, respectively, give rise to the following decomposition of the Lebesque measure of $\underline{\underline{R}}^n$:

$$dx = \mathrm{vol}(\underline{\underline{S}}_{p-1}) r^{p-1} dr \otimes d\omega_{p-1} \otimes \mathrm{vol}(\underline{\underline{S}}_{q-1}) r'^{q-1} dr' \otimes d\omega_{q-1}$$

Let $(P_k, Q_l) \in H_k(\underline{\underline{R}}^p) \times H_l(\underline{\underline{R}}^q)$ be an arbitrary pair of solid spherical harmonic functions of degree $k \geq 0$ resp. $l \geq 0$ on $\underline{\underline{R}}^p$ resp. $\underline{\underline{R}}^q$. For any function $f \in \mathcal{S}(\underline{\underline{R}}^n)$ there is a decomposition $f = P_k \otimes g \otimes Q_l \otimes h$ where $g \in \mathcal{S}(\underline{\underline{R}}^p)$ is invariant under the action of the compact orthogonal group $\underline{\underline{O}}(p,0,\underline{\underline{R}})$ on $\underline{\underline{R}}^p$ and similarly $h \in \mathcal{S}(\underline{\underline{R}}^q)$ is an $\underline{\underline{O}}(0,q,\underline{\underline{R}})$ invariant function on $\underline{\underline{R}}^q$. In particular, g and h are determined as radial functions by their values on the positive half-line $\underline{\underline{R}}_+$. Define the bivariate zeta function $\zeta(f;P_k,Q_l;\cdot,\cdot)$ attached to $f \in \mathcal{S}(\underline{\underline{R}}^n)$ and the pair $(P_k, Q_l) \in H_k(\underline{\underline{R}}^p) \times H_l(\underline{\underline{R}}^q)$ according to the prescription

$$\zeta(f;P_k,Q_l;s,t) = \int_{\underline{\underline{S}}_{p-1}} |P_k|^2\, d\omega_{p-1} \int_{\underline{\underline{S}}_{q-1}} |Q_l|^2 d\omega_{q-1} \cdot$$

$$\int_{\underline{\underline{R}}_+} g(r)\, r^{s+k+p/2-1} dr \int_{\underline{\underline{R}}_+} h(r')\, r'^{\,t+l+q/2-1} dr'.$$

Now we are in a position to establish a functional equation for the zeta function $\zeta(f;P_k,Q_l;.,.)$ and, at the same time, we may study the holomorphic continuation of this bivariate function.

Theorem 2. For every function $f\in\mathscr{S}(\underline{\underline{R}}^n)$ and all pairs $(P_k,Q_l)\in H_k(\underline{\underline{R}}^p)\times H_l(\underline{\underline{R}}^q)$ of solid spherical harmonic functions of degree $k\geqq 0$ resp. $l\geqq 0$ on $\underline{\underline{R}}^p$ resp. $\underline{\underline{R}}^q$ the bivariate zeta function $\zeta(f;P_k,Q_l;.,.)$ admits a holomorphic continuation for all pairs $(s,t)\in\mathbb{C}^2$ such that $s+k+\frac{1}{2}p\notin -2\underline{\underline{N}}$ and $t+l+\frac{1}{2}q\notin -2\underline{\underline{N}}$ and satisfies the functional equation

$$\zeta(f;P_k,Q_l;s,t)=i^{k+l}\,\pi^{-(s+t)}\,\frac{\Gamma(\frac{1}{2}(s+k+\frac{1}{2}p))\,\Gamma(\frac{1}{2}(t+l+\frac{1}{2}q))}{\Gamma(\frac{1}{2}(-s+k+\frac{1}{2}p))\,\Gamma(\frac{1}{2}(-t+l+\frac{1}{2}q))}\cdot$$

$$\zeta(\mathscr{F}_{\underline{\underline{R}}^p\oplus\underline{\underline{R}}^q}f;P_k,Q_l;-s,-t).$$

Proof (Sketch). Look at the decomposition $f = P_k\otimes Q_l\otimes g\otimes h$ as before. Consider the Weyl element

$$w=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

of $\underline{\underline{SL}}(2,\underline{\underline{R}})$ and evaluate $T_0^{p,q}(u)f$ at the element $u = id_{\underline{\underline{R}}^p\oplus\underline{\underline{R}}^q}\times\widehat{w}$ of $\underline{\underline{O}}(p,q,\underline{\underline{R}})\times\underline{\underline{Mp}}(1,\underline{\underline{R}})$. Then we obtain the identity

$$i^{(p-q)}\mathscr{F}_{\underline{\underline{R}}^p\oplus\underline{\underline{R}}^q} f = P_k\otimes Q_l\otimes D_{k+\frac{1}{2}p}(\widetilde{w})g\otimes\bar{D}_{l+\frac{1}{2}q}(\widetilde{w})h.$$

Since $D_{k+\frac{1}{2}p}(\widetilde{w})g$ is a Hankel transform of g of order $k+\frac{1}{2}p-1$ and $\bar{D}_{l+\frac{1}{2}q}(\widetilde{w})h$ is a Hankel transform of h of order $l+\frac{1}{2}q-1$, the result follows by taking the Mellin transforms (cf.[6]) of both sides.—

4. Concluding Remarks

Theorem 2 generalizes a result of Ormerod [5] (also see [4]) who considered the case q=0. This author, however, applies harmonic analysis only of the compact orthogonal group $\underline{\underline{O}}(p,0,\underline{\underline{R}})$ by appealing to the Funk-Hecke theorem (cf.Coifman-Weiss [1]) and does not refer explicitly to the "dual part" in the dual reductive pair $(\underline{\underline{MO}}(p,0,\underline{\underline{R}}),\underline{\underline{Mp}}(1,\underline{\underline{R}}))$ in the metaplectic group $\underline{\underline{Mp}}(p,\underline{\underline{R}})$.

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ON THE CRITERIA OF WIEFERICH AND MIRIMANOFF

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Presented by P. Ribenboim, F.R.S.C.

1. Let p be an odd prime and B_n be the nth Bernoulli number defined by $v/(e^v - 1) = \sum_{k=0}^{\infty} (B_k/k!)v^k$. If $G(v)$ is a differentiable function of v, let $[G(v)]_0^{(n)}$ be the value of $d^nG(v)/dv^n$ at $v = 0$.

It is well known that if the equation

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers x, y and z prime to each other in the first case (i.e., $p \nmid xyz$), then ([1], see also [4])

$$(2)\qquad \begin{aligned} &[U(v)]_0^{(p-2)} \equiv 0 \pmod{p}, \\ &B_{2k}[U(v)]_0^{(p-1-2k)} \equiv 0 \pmod{p},\ k = 1, 2, \dots, (p-3)/2, \end{aligned}$$

where $U(v) = 1/(1 - te^v)$ and $t \in H = \{-y/x, -x/y, -z/y, -y/z, -x/z, -z/x\}$.

Let $q_p(m)$ be the Fermat quotient of p with base m, i.e., $q_p(m) = (m^{p-1} - 1)/p$. Wieferich [5] and Mirimanoff [2, 3] have proved the following criteria for the equation (1) in the first case:

(I)(Wieferich). $q_p(2) \equiv 0 \pmod{p}$.

(II)(Mirimanoff). $q_p(3) \equiv 0 \pmod{p}$.

The purpose of this paper is to give an easier and shorter proof for (I) and (II) using the congruences (2).

2. Clearly, we have $t \not\equiv 0, 1 \pmod{p}$ for the first case. If $t = -y/x$, then the elements of H are congruent modulo p to those

of the set $H' = \{ t, 1/t, 1-t, 1/(1-t), (t-1)/t, t/(t-1) \bmod p \}$. If $t \equiv 2, 1/2, -1 \pmod p$, then $\# H' = 3$ and $H' = \{ 2, 1/2, -1 \bmod p \}$. On the one hand, if $t^2 - t + 1 \equiv 0 \pmod p$, then $\# H' = 2$. In all other cases we have $\# H' = 6$. Note that if $t \equiv -1 \pmod p$, then we can take $t \equiv 2 \pmod p$ instead of $t \equiv -1 \pmod p$, since $x + y + z \equiv 0 \pmod p$.

The following lemma will be needed for the proofs of (I) and (II):

LEMMA. *Let $m \geq 2$ and $k \geq 1$. If*

$$A_m(v) = \frac{e^{(m-2)v} + 2e^{(m-3)v} + \dots + (m-1)}{e^{(m-1)v} + e^{(m-2)v} + \dots + 1},$$

then

$$[A_m(v)]_0^{(k-1)} = (1 - m^k) B_k / k.$$

Proof. Set $B(v) = v/(e^v - 1)$. Then we have $B(v) - B(mv) = vA_m(v)$. Since $[B(v) - B(mv)]_0^{(k)} = (1 - m^k) B_k$ and $[vA_m(v)]_0^{(k)} = k [A_m(v)]_0^{(k-1)}$, so the lemma holds. ||

We now give the proofs of (I) and (II):

Proof of (I). From the identity $A_2(v) U(v) = \alpha(t) A_2(v) + \beta(t) U(v)$ with $\alpha(t) = 1/(t+1)$ and $\beta(t) = t/(t+1)$, we have

$$\sum_{k=0}^{p-2} \binom{p-2}{k} [A_2(v)]_0^{(k)} [U(v)]_0^{(p-2-k)}$$

$$= \alpha(t) [A_2(v)]_0^{(p-2)} + \beta(t) [U(v)]_0^{(p-2)}.$$

If $k \geq 3$ is odd, then $B_k = 0$. Therefore, by making use of the Lemma and the congruences (2),

$$[A_2(v)]_0^{(k)} [U(v)]_0^{(p-2-k)} = \{ (1 - 2^{k+1}) B_{k+1} / (k+1) \} [U(v)]_0^{(p-2-k)}$$

$$\equiv 0 \pmod p$$

for all $k = 0, 1, 2, \dots, p-3$, which deduce that

$$\{[U(v)]_0^{(0)} - \alpha(t)\}[A_2(v)]_0^{(p-2)}$$

$$= \frac{2t}{t^2-1} \frac{q_p(2)}{p-1} pB_{p-1} \equiv 0 \pmod{p}.$$

By von Staudt and Clausen's theorem, $pB_{p-1} \equiv -1 \pmod{p}$. So the result follows. ||

Proof of (II). Let $W(v) = e^v/(e^{2v} + e^v + 1)$. Since $A_3(v)U(v) = \gamma(t)A_3(v) + \delta(t)W(v) + \varepsilon(t)U(v)$, where $\gamma(t) = (t+2)/2(t^2+t+1)$, $\delta(t) = 3t/2(t^2+t+1)$ and $\varepsilon(t) = (2t^2+t)/(t^2+t+1)$, we obtain

$$\sum_{k=0}^{p-2} \binom{p-2}{k} [A_3(v)]_0^{(k)} [U(v)]_0^{(p-2-k)}$$

$$= \gamma(t)[A_3(v)]_0^{(p-2)} + \delta(t)[W(v)]_0^{(p-2)} + \varepsilon(t)[U(v)]_0^{(p-2)}.$$

Here, $[W(v)]_0^{(p-2)} = 0$, since $W(-v) = W(v)$. By the same reason as mentioned in the proof of (I), it follows that

$$\{[U(v)]_0^{(0)} - \gamma(t)\}[A_3(v)]_0^{(p-2)}$$

$$= \frac{3(t^2+t)}{2(t^3-1)} \frac{q_p(3)}{p-1} pB_{p-1} \equiv 0 \pmod{p}.$$

The equation (1) is impossible for $p = 3$. Also, we may assume $t \not\equiv -1 \pmod{p}$ as stated above, hence $3(t^2+t) \not\equiv 0 \pmod{p}$. Incidentally, it is easy to show that $2(t^3-1) \not\equiv 0 \pmod{p}$ for each case of $\#H' = 2, 3, 6$. Since $pB_{p-1} \equiv -1 \pmod{p}$, so (II) can be given immediately. ||

We note that the criteria (I) and (II) can be also deduced from Furtwängler's theorem (see e.g. [4]).

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REPRÉSENTATIONS COMME LA DIFFÉRENCE DES NOMBRES PUISSANTS NON CARRÉS

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RÉSUMÉ. — Nous montrons qu'il existe une infinité de représentations de tout entier comme la différence de deux nombres puissants ni l'un ni l'autre carré.

INTRODUCTION. — Un nombre puissant est un entier positif r ayant la propriété que si le premier p divise r, alors p^2 divise r. Un nombre puissant est donc un carré ou le produit d'un carré et d'un cube. Il est bien connu que si n est un entier positif, le nombre de représentations $Q(n)$ de n comme la différence de deux nombres puissants, tous deux carrés, est fini ($Q(n) = 0$ si $n \equiv 2 \pmod 4$, $Q(n) = \tau(n)/2$ si $n \equiv 1$ ou $3 \pmod 4$, et $Q(n) = \tau(n/4)/2$ si $n \equiv 0 \pmod 4$, où $n > 0$ et $\tau(n)$ est le nombre des diviseurs positifs de n). On a montré récemment que le nombre de représentations de n comme la différence des nombres puissants, l'un carré et l'autre non carré, est infini [2], [4], pour chaque n. Cependant, les entiers qui sont représentables comme la différence de deux nombres puissants non carres ni l'un ni l'autre n'ont pas été caractérisés, et on a montré que les entiers seuls ± 1 [5] et ± 4 [1] ont une infinité de représentations comme la différence de deux nombres puissants non carrés. Notre résultat, obtenu par des moyens élémentaires, est qu'une infinité de telles représentations existe pour chaque n. Spécifiquement, nous obtenons, pour chaque entier n, une paire de fonctions polynômes f et g telle que l'équation $f(n)X^2 - g(n)Y^2 = n$ a une infinité de solutions X, Y pour lesquelles $f(n)X^2$ et $g(n)Y^2$ sont des nombres puissants qui ne sont pas carrés.

THÉORÈME 1. — Soient A, B et n des entiers, AB n'étant pas carré. On suppose que p, q est une solution de $AX^2 - BY^2 = n$, et que x_1, y_1 est une solution de $x^2 - ABy^2 = \pm 1$, où $(Bqy_1, A)|p$ et $(Apy_1, B)|q$. Alors, $AX^2 - BY^2 = \pm n$ a une infinité de solutions X_j, Y_j pour lesquelles $A|X_j$ et $B|Y_j$.

La preuve utilise le fait bien connu que si p, q est une solution de $AX^2 - BY^2 = n$ et x_1, y_1 est une solution de $x^2 - ABy^2 = \pm 1$, et x_j, y_j est défini par

$$x_j + y_j\sqrt{AB} = (x_1 + y_1\sqrt{AB})^j,$$

alors la paire $X_j = px_j + Bqy_j$ et $Y_j = Apy_j + qx_j$ satisfait l'équation $AX^2 - BY^2 = \pm n$. Un examen de X_j (mod A) et de Y_j (mod B) produit le résultat.

Le problème de trouver des fonctions polynômes f et g telles que $f(n)X^2 - g(n)Y^2 = n$ et $x^2 - f(n)\cdot g(n)y^2 = \pm 1$ est résolu par le moyen suivant: Certains polynômes h(n) ont un développement en fractions continues ayant une période relativement courte, et cela nous permet de trouver, assez facilement, des fonctions $x = x(n)$ et $y = y(n)$ qui satisfont l'équation $x^2 - h(n)y^2 = \pm 1$. On peut souvent faire d'une fonction quadratique $h(n) = a^2n^2 + bn + c$ le produit de deux fonctions en l'écrivant d'abord sous la forme

$$(an + r)^2 - [(2ar - b)n + (r^2 - c)],$$

où r est un entier ou 1/2, et ensuite en rendant la somme $(2ar - b)n + (r^2 - c)$ égale à $(dt + e)^2$ pour des valeurs appropriées de d et e, et où n a été remplacé par une fonction quadratique de t. h(n) est donc la différence de deux carrés et est décomposable en un produit de deux fonctions f et g de t.

THÉORÈME 2. — <u>Soit</u> n <u>un nombre premier</u>. <u>Il existe des fonctions</u> f <u>et</u> g <u>telles que</u> $f(n)X^2 - g(n)Y^2 = n$ <u>a une infinité de solutions</u> X_j, Y_j <u>ayant la propriété que</u> $f(n)X_j^2$ <u>et</u> $g(n)Y_j^2$ <u>sont des nombres puissants qui ne sont pas carrés</u>.

Le théorème est vrai évidemment pour 0 et pour -n quand il est vrai pour n, et est valable pour $n \equiv 0 \pmod 4$ quand il est valable pour $n \not\equiv 0 \pmod 4$ puisque si $m = m_1 - m_2$, où m_1 et m_2 sont des nombres puissants, alors $n = 4^k m = 4^k m_1 - 4^k m_2$. Un ensemble de cinq paires de fonctions f(n) et g(n) suffit à prouver le théorème pour tout $n \not\equiv 0 \pmod 4$ (excepté pour $n = \pm 1$ et ± 2 qui sont facilement traités séparément). Les détails de la preuve du Théorème 2 comprennent les démarches suivantes: (1) montrer pour chaque n que $(f(n), g(n)) = 1$,

(2) démontrer que ni f(n) ni g(n) ne sont carré dans au moins une des équations $f(n)X^2 - g(n)Y^2 = n$, et (3) identifier pour chaque paire f(n), g(n) une solution $p = p(n)$, $q = q(n)$ de $f(n)X^2 - g(n)Y^2 = n$ et une solution $x_1 = x(n)$, $y_1 = y(n)$ de $x^2 - f(n)\cdot g(n)y^2 = \pm 1$ telle que l'hypothèse du Théorème 1 est satisfaite.

La table suivante présente les fonctions pertinentes.

n*	f(n)	g(n)	p(n)	q(n)	x(n)	y(n)
1	3	11	2	1	23	4
2	5	3	1	1	2	1
2t+1	t^2+2t+2	t^2+1	1	1	t^2+t+1	1
2t+1	$2t^2+4t+1$	$2t^2-1$	t	t+1	$4t^4+8t^3-4t$	$2t^2+2t-1$
4t+2	$2t^2+4t+1$	$2t^2-1$	1	1	$(2t^2+2t-1)^2-1$	$2t^2+2t-1$
4t+2	$2t^2+4t+3$	$2t^2+1$	1	1	$(2t^2+2t+1)^2+1$	$2t^2+2t+1$
4t+2	$6t^2+8t+3$	$6t^2+4t+1$	1	1	$(18t^2+18t+5)^2+1$	$3(18t^2+18t+5)$

*À la ligne 3 de la table, $t \not\equiv 2 \pmod 5$;
à la ligne 6, $t \not\equiv 1 \pmod 3$, et à la ligne 7, $t \equiv 1 \pmod 3$.

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On the Energy Decay of a Navier-Stokes flow in R^3.

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Abstract. The energy of a solution of the Navier-Stokes equations in R^3, defined as the square integral of the solution flow vector, is shown to have zero limit as time becomes large.

1. Introduction.

In his paper " Sur le mouvement d'un liquide visqueux emplissant l'espace", Leray [7] concludes with the remark "J'ignore si W(t) tend nécessairement vers 0 quand t augmente indefiniment." Here a proof will be given that the kinetic energy $W(t) = \|u\|_2^2$ of a Navier-Stokes flow in R^3, finite at an initial time, tends to zero as time tends to infinity.

Many results of this kind have been established for particular types of region, such as bounded regions, or if further hypotheses are made, [3, 4, 5, 6, 10, 11, 12].

If for example the region Ω in which the flow is confined satisfies a Poincaré inequality, then the energy tends to zero exponentially. For R^3 however, as will appear, the approach to zero may be arbitrarily slow, depending on the initial values which themselves are assumed of finite energy, that is, square integrable. Recently, Schonbek has shown [11] that if the initial values are also in $L^1(R^3)$, the energy in R^3 tends to zero at an algebraic rate. A similar result under the assumption of initial values in $L^p(R^3)$ and $L^2(R^3)$, where $1 \leq p < 2$, was given in [3] in the course of a proof on path lengths. Here we assume only that initial values are in $L^2(R^3)$.

2. The Navier-Stokes Equations. In R^3 adopt Cartesian coordinates x_i $(i = 1,2,3)$ and denote the velocity components of the flow by $u_i = u_i(x,t)$, the pressure by $p = p(x,t)$. Then the Navier-Stokes equations for a viscous incompressible flow are

$$u_{i,t}+u_k u_{i,k} = -p_{,i}+\nu\Delta u_i$$

$$u_{i,i} = 0$$

Here commas denote derivatives with respect to the coordinate having the next following index, and a summation convention for repeated indices is understood. The Laplacian is denoted by Δ while the constant ν represents viscosity.

Many studies have been made [1, 4, 5, 6, 7, 8, 10, 12, 13] of existence, regularity and uniqueness for the initial value problem wherein

$$u(x,0) = u_0(x) \in L^2(R^3).$$

Here we assume $u(x,t)$ is a smooth solution defined for large t [9] . From the standard energy integral and its integrated form

$$\|u\|_2^2 + 2\nu\int_0^t \|\nabla u\|_2^2 dt \le \|u_0\|_2^2,$$

where

$$\|u\|_p^p = \int_{R^3} |u|^p dv, \quad p \ge 1,$$

it is known that $\|u\|_2$ is a bounded decreasing function of time t.

3. The main result.

Theorem. *As time t tends to infinity, the energy tends to zero:*

$$\lim_{t\to\infty} \|u\|_2 = 0$$

Proof. With $u_{i,t} - \nu\Delta u_i = -p_{,i} - u_k u_{i,k}$ we regard u_i as a solution of a non-homogeneous heat flow equation and write, in vector notation with components suppressed,

$$u = u_1 + u_2 + u_3$$

where

$$u_1(x,t) = K(x,t) *_3 u_0(x)$$

with

$$K(x,t) = (2\sqrt{\pi t})^{-\frac{3}{2}} \exp(-x^2/4t).$$

Let $\Phi(t) \in C^\infty$ be a non-negative non-increasing function with $\Phi(t) \equiv 1$ for $t < 1$ and $\Phi(t) \equiv 0$ for $t > 2$. Then we set

$$u_2(x,t) = -(\Phi(t)\, K(x,t)) *_4 (u \cdot \nabla u + \nabla p)$$

$$u_3(x,t) = -((1 - \Phi(t))\, K(x,t)) *_4 (u \cdot \nabla u + \nabla p)$$

Here $*_3$ denotes convolution in the x variables, and $*_4$ convolution in x and t, with $0 \le \tau \le t$. The gradient symbol ∇ has components $\partial/\partial x_i$.

4. Estimate of $\|u_1\|_2$. We have

$$\|u_1\|_2^2 = \int u_1^2 dV = \int |\hat{u}_1|^2 d\hat{V}$$

where $\hat{u}_1$ is the three-dimensional Fourier transform. But $u_1 = K * u_0$ implies $\hat{u}_1 = e^{-s^2 t}\hat{u}_0$ so that by Parseval's theorem,

$$\|u_1\|_2^2 = \int_{R^3} e^{-2s^2 t} |\hat{u}_0|^2 d\hat{V}$$

$$= \int_0^\infty e^{-2s^2 t} U_0(s) s^2 ds$$

where $U_0(s) = \int_\Omega |\hat{u}_0(s)|^2 d\Omega$ is the integral over all directions in R^3, and $U_0(s)s^2$ is integrable over $0 \leq s < \infty$.

Given $\epsilon > 0$ choose $s_1 > 0$ so that $\int_0^{s_1} U_0(s)s^2 ds < \frac{1}{2}\epsilon$, which is possible by the above integrability property. Then

$$\|u_1\|_2^2 \leq \int_0^{s_1} U_0(s)s^2 ds + \int_{s_1}^{\infty} e^{-s^2 t} U_0(s)s^2 ds$$

$$\leq \frac{1}{2}\epsilon + e^{-s_1^2 t} \int_0^{\infty} U_0(s)s^2 ds$$

Since $s_1 > 0$ we can choose t large enough to make the second term less than $\frac{1}{2}\epsilon$, as $t \to \infty$. Hence $\|u_1\|_2$ tends to zero as $t \to \infty$. It is also evident that $\|u_1\|_2$ is monotonic decreasing in t, slowly if $\hat{u}_0$ is concentrated near the origin.

5. Estimate of the right-hand side. The term $u_k u_{i,k} + p_{,i}$ can be estimated by means of the mixed $L^{p_1 p_2}$ norms

$$\|u\|_{p_1,p_2} = \left(\int_0^{\infty} \|u\|_{p_1}^{p_2} dt \right)^{\frac{1}{p_2}}.$$

By the standard energy integral we have $\|u\|_{2,\infty} \leq \|u_0\|_2$, and $\|\nabla u\|_{2,2}^2 < \frac{1}{2\nu}\|u_0\|_2^2$. By Sobolev's inequality,

$$\|u\|_{6,2} \leq C\|\nabla u\|_{2,2} \leq C_1\|u_0\|_2$$

Hence $u_k u_{i,k} \in L^{q_1 q_2}$ for $\frac{3}{q_1} + \frac{2}{q_2} = 4$, $\frac{2}{3} \leq q_1^{-1} \leq 1$. To estimate the pressure gradient, we note that $\Delta p = -u_{k,j} u_{j,k}$ so that

$$p = \frac{1}{4\pi} \int_{R^3} \frac{1}{r} u_{k,j} u_{j,k} dV$$

where r is the Euclidean distance. Thus

$$p_{,i} = \frac{-1}{4\pi} \int_{R^3} \frac{x_i - y_i}{r^3} u_{k,j} u_{j,k} dV.$$

Since $u_{j,k} u_{k,j} = (u_k u_{j,k})_{,j}$ we find

$$p_{,i} = \frac{1}{4\pi} \int_{R^3} \frac{\Omega_{ij}(x-y)}{r^3} u_k u_{j,k} dV$$

where

$$\frac{\Omega_{ij}(x-y)}{r^3} = \frac{\partial^2 \frac{1}{r}}{\partial x_i \partial y_j} = \frac{\delta_{ij}}{r^3} - \frac{3(x_i - y_i)(x_j - y_j)}{r^5}$$

is a Calderon-Zygmund principal value kernel. By the Calderon-Zygmund theorem [2] it follows that $p_{,i}$ belongs to the same $L^{p_1p_2}$ spaces as the integrand $u_k u_{j,k}$, for $p_1 > 1$.

6. **Estimate of** $||u_2||_2$. We note that

$$\int_{R^3} K(x,t)^{p_1} dV \le C \cdot t^{\frac{3}{2}(1-p_1)}$$

so that the kernel $K_2(x,t) = \Phi(t)K(x,t)$ belongs to $L^{p_1p_2}$ if the p_2^{th} power $||K_2||_{p_1}^{p_2}$ is integrable near $t = 0$. This holds if

$$\frac{3}{p_1}+\frac{2}{p_2} > 3.$$

By the convolution inequality

$$||f*g||_{r_1r_2} \le ||f||_{p_1p_2}||g||_{q_1q_2}$$

we find that [9] $u_2 = K_2*(u\cdot\nabla u+\nabla p) \in L^{r_1r_2}$ with $\frac{1}{r_i} = \frac{1}{p_i}+\frac{1}{q_i}-1, 1 \le q_i \le r_i$. We have

$$\frac{3}{r_1}+\frac{2}{r_2} = \frac{3}{p_1}+\frac{2}{p_2}+\frac{3}{q_1}+\frac{2}{q_2}-5 > 3+4-5 = 2$$

In particular, we may choose ϵ, $0 < \epsilon < \frac{1}{4}$, and $p_1 = \frac{3}{2}$, $p_2 = \frac{2}{1+2\epsilon}$, $q_1 = \frac{6}{5}$, $q_2 = \frac{4}{3}$ so that $r_1 = 2$, $r_2 = \frac{4}{1+4\epsilon}$. Thus $r_2 < 4$ and is close to 4; the inequality holds and u_2 lies in the mixed space $L^{2,4(1+4\epsilon)^{-1}}$.

7. **Estimate of** $||u_3||_2$. Similarly, the kernel $K_3(x,t) = (1-\Phi(t))K(x,t)$ lies in $L^{p_1p_2}$ if the p_2^{th} power $||K_3||_{p_1}^{p_2}$ is integrable for t large. This will hold if $\frac{3}{p_1}+\frac{2}{p_2} < 3$. Again by the convolution inequality we find that $u_3 = K_3*(u\nabla u+\nabla p) \in L^{r_1r_2}$ when

$$\frac{3}{r_1}+\frac{2}{r_2} < 2$$

In particular, we may choose ϵ, $0 < \epsilon < \frac{1}{4}$ and $p_1 = \frac{3}{2}$, $p_2 = \frac{2}{1-2\epsilon}$, $q_1 = \frac{6}{5}$, $q_2 = \frac{4}{3}$ so that $r_1 = 2$, $r_2 = \frac{4}{1-4\epsilon}$. Thus $r_2 > 4$ and is close to 4; the inequality is valid and u_3 lies in the mixed space $L^{2,4(1-4\epsilon)^{-1}}$.

8. **Completion of the proof.** We now have $u = u_1+u_2+u_3$ where

1) $\|u_1\|_2 \to 0$ as $t \to \infty$, monotonically,
2) $\|u_2\|_2 \in L^{4(1+4\epsilon)^{-1}}$ $(0,\infty)$,
3) $\|u_3\|_2 \in L^{4(1-4\epsilon)^{-1}}$ $(0,\infty)$, with $0 < \epsilon < \frac{1}{4}$.

Given $\eta > 0$ we can choose T so large that

$$\|u_1\|_2 < \frac{\eta}{3}, \quad \int_T^\infty \|u_2\|_2^{4(1+4\epsilon)^{-1}} dt < \frac{\eta}{3}, \quad \int_T^\infty \|u_3\|_2^{4(1-4\epsilon)^{-1}} dt < \frac{\eta}{3}.$$

These integrands cannot be assumed monotonic, but on a set of at least two-thirds of the measure of $T < t < 2T$ we must have $\|u_2\|_2^{4(1+4\epsilon)^{-1}} < \frac{\eta}{T}$ and similarly $\|u_3\|_2^{4(1-4\epsilon)^{-1}} < \frac{\eta}{T}$ on a set of measure at least $\frac{2T}{3}$, in the same interval. Hence there is a subset of $T < t < 2T$ having measure at least $\frac{T}{3}$, for any T_1 of which we have

$$\|u_2\|_2(T_1) < (\eta/T)^{\frac{1}{4}+\epsilon}$$

and

$$\|u_3\|_2(T_1) < (\eta/T)^{\frac{1}{4}-\epsilon}$$

Thus, at time T_1,

$$\begin{aligned}\|u\|_2 &\le \|u_1\|_2+\|u_2\|_2+\|u_3\|_2 \\ &< \frac{\eta}{3}+\left(\frac{\eta}{T}\right)^{\frac{1}{4}+\epsilon}+\left(\frac{\eta}{T}\right)^{\frac{1}{4}-\epsilon}\end{aligned}$$

which can be made arbitrarily small by choice of η small then T and T_1 large. Since

$$\frac{d}{dt}\|u\|_2^2 = -2\nu\|\nabla u\|_2^2 < 0$$

it follows that $\|u\|_2$ decreases monotonically in t and so must have limit zero as time tends to infinity. This completes the proof.

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Note added in proof: As this paper was going to the press, I found that a proof of this result in very general conditions was published by K. Masuda, "Weak Solutions of Navier Stokes Equations", Tohoku Math. J., 36 (1984), 623-646, using fractional powers of the Stokes operator.

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THE ASCENDING CHAIN CONDITION FOR REAL IDEALS

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Résumé: On démontre les résultats suivants: si tout idéal réel premier d'un anneau A est engendré, en tant qu'idéal réel, par un nombre fini d'éléments, alors A satisfait la condition de chaînes ascendantes (CCA) pour les idéaux réels. Si A satisfait la CCA alors A[X] satisfait aussi la CCA, tandis qu'un exemple est indiqué où A[[X]] ne satisfait pas la CCA.

1. Let A be a commutative ring with unit. We recall the following definitions.

An ideal I of A is a real ideal if the following condition is satisfied: if $a_1,\dots,a_n \in A$ and $a_1^2 + \dots + a_n^2 \in I$ then $a_1,\dots,a_n \in I$.

The ring A is a real ring when 0 is a real ideal of A .

Note that a domain A is a real ring if and only if the field of quotients is an orderable field (by Artin's theorem).

The following properties are easy to show and well-known: The union of a totally ordered increasing family of real ideals of A is a real ideal. The intersection of any non-empty family of real ideals of A is a real ideal. Thus, there exists the smallest real ideal containing any subset S of A ; it is denoted by R(S) , and it is said to be R-generated by S .

We require the following properties:

Lemma 1. If I is any ideal of A then $R(I) = \bigcup_{k=0}^{\infty} R_k(I)$, where $R_0(I) = I$, $R_{k+1}(I) = R_1(R_k(I))$, and for every ideal J of A , we *define* $R_1(J) = \{b \in A|$ *there exists* $a_1,\dots,a_n \in A$ *such that* $b^2 + a_1^2 + \dots + a_n^2 \in J\}$; *we note that* $R_1(J)$ *is also an ideal of* A .

Lemma 2. *If* I , J *are ideals of* A *then:*

$$R(I + R(J)) = R(I + J) ,$$

$$I.R(J) \subseteq R(IJ) .$$

Lemma 3. *If* $\phi: A \to B$ *is a ring-homomorphism, then:*

1) *If* J *is any real ideal of* B *then* $\phi^{-1}(I)$ *is a real ideal of* A .

2) *If* $\phi(A) = B$ *and* I *is a real ideal of* B *such that* $\phi^{-1}(I) = R(S)$, *then* $I = R(\phi(S))$.

If the real ideal I is of the form $I = R(S)$ where S is a finite set, then I is said to be of *finite* R-*type*.

Lemma 4. *If the real ideal* I *is R-generated by* $S \cup \{x\}$ *and if* I *is of finite R-type, there exists a finite subset* T *of* S *such that* $I = R(T \cup \{x\})$.

Lemma 5. 1) *If* I *is a real ideal of* A *and* $a \in A$ *then* $I:Aa$ *is a real ideal.*

2) If I , *L are real ideals of* A *, if* J *is any ideal* *such that* $I \cap J \subseteq L$ *then* $I \cap R(J) \subseteq L$.

2. We say that A satisfies the ascending chain condition (ACC) for real ideals whenever every strictly ascending chain of real ideals of A is finite.

It is easy to see:

Lemma 6. *A satisfies the ACC for real ideals if and only if every real ideal is of finite R-type.*

Lemma 7. *If $\phi: A \to B$ is a ring homomorphism, such that $\phi(A) = B$ and if A satisfies the ACC for real ideals, then so does B.*

The analogue of Cohen's theorem for noetherian rings holds:

Proposition 1. *If every real prime ideal of A is of finite R-type then A satisfies the ACC for real ideals.*

The analogue of Hilbert's basis theorem for noetherian rings also holds:

Proposition 2. *The ring A satisfies the ACC for real ideals if and only if the ring A[X] satisfies the ACC for real ideals.*

Concerning the ring of formal power series, it is clear from lemma 7 that if A[[X]] satisfies the ACC for real ideals, then so does the ring A .

However, we give the following example to show that the converse does not hold.

Example. Let A be a valuation domain with value group and residue field both equal to $\mathbb{R}$. The only real ideals of A are 0 , A and the maximal ideal M , so A satisfies the ACC for real ideals.

Let I be the set of all power series $S = \sum_{i=0}^{\infty} s_i X^i$ with each $s_i \in A$, $s_0 = 0$, $v(s_i) > 0$ (for every $i \geq 1$). Then I is an ideal of $A[[X]]$, such that R(I) is not of finite R-type (as it may be shown). Thus $A[[X]]$ does not satisfy the ACC for real ideals.

The proofs of the results communicated here will appear elsewhere.

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On the non-existence of positive solutions for a Schrödinger equation with an indefinite weight-function

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Presented by G.F.D. Duff, F.R.S.C.

1. Let q, g: $\Omega\to\mathbb{R}$, q, $g\in L^\infty(\Omega)$, $\Omega\subset\mathbb{R}^n$, $n>1$, is a (smooth) bounded domain. Consider the Dirichlet problem associated with the Schrödinger equation

$$Lu\equiv-\Delta u-q(x) = \lambda g(x)u, \tag{1.1}$$

where, in addition, g "changes sign" in Ω but is not zero on a set of positive Lebesgue measure and q is sufficiently positive in parts of Ω so that $L\equiv-\Delta-q$ is not positive definite. Eigenfunctions (e.f.) of (1.1) are assumed to lie in $H_0^{1,2}(\Omega)$ and (1.1) is understood in the usual generalized sense.

A *ghost state* ω is an e.f. (real or not) for which $(g\omega,\omega)\equiv\int_\Omega g|\omega|^2 = 0$, where (,) is the usual L^2-inner product (see [8] for the terminology). It is easy to see that any e.f. corresponding to a non-real eigenvalue (e.v.) is a ghost state. We shall call (1.1) (or L) *definitizable* if there exists $\alpha\in\mathbb{R}$ such that $L+\alpha g>0$ on $C_0^\infty(\Omega)$. If (1.1) is not definitizable but there exists $\beta\in\mathbb{R}$ such that $L+\beta g\geq 0$ on $C_0^\infty(\Omega)$ then we say that (1.1) is *semi-definitizable*. When ever (1.1) is not semi-definitizable we term (1.1) *strictly non-definite*.

In [3], [5] it is shown, among other results, that if $q(x) < 0$ in Ω, then (1.1) has a positive e.f. We will show that (1.1) does *not* have a positive e.f. if it has a non-real e.v. or, more generally, if it admits a ghost state. It is known from a series of examples

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(e.g. [8]) that, when $n=1$, (1.1) may have a non-real e.v; and by suitably choosing q,g in a radial case of (1.1), one may resort to the one-dimensional results in order to find an explicit example of a non-real e.v. for (1.1) even if $n>1$. The non-existence of such a positive solution, in the presence of a ghost state, is in sharp contrast with the results in [3]. Indeed theorems 1.1-2 show that for a strictly non-definite problem there is no positive e.f. whatsoever!

Theorem 1.1 Let $\lambda\varepsilon\mathbb{R}$ be an e.v. and ϕ a corresponding positive e.f. Then (a) There are no ghost states except, possibly, ϕ itself; (b) λ is semi-simple (i.e., the associated eigenspace is one-dimensional); (c) λ orders the other (real) e.v. as follows: If $\mu>\lambda$ (resp. $\mu<\lambda$) has e.f. ψ then $(g\phi,\phi)>0$ (resp. $(g\phi,\phi)<0$).

Corollary 1.1 Equation (1.1) admits at most two (distinct) e.v. with corresponding positive e.f.

Theorem 1.2 (a) Equation (1.1) has two (independent) positive e.f. if and only if (1.1) is definitizable; (b) Equation (1.1) has precisely one (independent) positive e.f. if and only if (1.1) is semi-definitizable. In this case the positive e.f. is a ghost state.

2. *Proof of theorem 1.1*, (a) Let $u = u_1+iu_2$ be a (complex) ghost state corresponding to $\mu = \alpha+i\beta$, $\beta\neq 0$. Observe that $u,\phi\varepsilon C^{1+\alpha}(\overline{\Omega})$ (see [2], [6]). We apply Picone's identity to (1.1) twice and add the result to find, for $\phi_1,\phi_2\varepsilon C_0^\infty(\Omega)$,

$$\int_\Omega \phi^2\Big(\sum_i \{[D_i(\frac{\phi_1}{\phi})]^2+[D_i(\frac{\phi_2}{\phi})]^2\}\Big) = \int_\Omega\sum\{(D_i\phi_1)^2+(D_i\phi_2)^2\}$$

$$- \int_\Omega(\phi_1^2+\phi_2^2)(q+\lambda g), \qquad (2.1)$$

Select K, a compact subset of Ω, and note that we may minorize the left- side of (2.1) by replacing Ω by K. Let $\phi_i \to u_i$, i=1,2 in $H_0^{1,2}(\Omega)$, (see e.g. [1] for similar arguments). Then

$$\int_\Omega \phi^2 \left(\sum_i \{[D_i(\frac{u_1}{\phi})]^2 + [D_i(\frac{u_2}{\phi})]^2\}\right) < -\int_\Omega \bar{u}\Delta u - \int_\Omega |u|^2(q+\lambda g)$$
$$< (\mu-\lambda)\int_\Omega g|u|^2. \qquad (2.2)$$

But u is a ghost state (as μ is non-real). Thus (2.2) implies that $u = (c_1+ic_2)\phi$ which, in turn, implies that $\lambda=\mu$ which is impossible. A similar argument applies in the event that $\mu\varepsilon\mathbb{R}$ and u is a (real) ghost state. In this case $\mu=\lambda$ and so ϕ itself must be a ghost state. Note that if $\lambda = \mu$, the right side of (2.2) vanishes and we conclude $u = c\phi$, $c \in C$. Thus u,ϕ are dependent and so (b) follows. Moreover if $\mu > \lambda$ is an eigenvalue and u is a corresponding e.f. then (2.2) shows that $(\lambda - \mu)(gu,u) > 0$ and, in actuality, $(gu,u) > 0$ from arguments similar to the above. If $\lambda < \mu$ then $(gu,u) < 0$ and so (c) is verified.

Remark The initial requirement that g not vanish on a set of positive Lebesgue measure is not used here and so theorem 1.1 is valid without this assumption. Also note that corollary 1.1 follows at once from theorem 1.1(a) and (c).

Proof of theorem 1.2, (a) If (1.1) is definitizable then it has two positive e.f. by classical results and these are the extremals of a Courant min-max principle, (see [7])

$$\lambda_1^{\pm} = \inf \left\{\frac{(\phi,L\phi)}{(g\phi,\phi)} : \phi\varepsilon H_0^{1,2}(\Omega), (g\phi,\phi) \gtrless 0\right\}.$$

Conversely if u, v>o are two e.f. of (1.1) associated with λ, μ resp. then, by theorem 1.1, we may assume that $(gv,v) < o < (gu,u)$. For $\phi\varepsilon C_0^\infty(\Omega)$ we note, by arguments similar to those in theorem 1.1,

$$(\phi,(L-(\frac{\lambda+\mu}{2})g)\phi) = \int_\Omega u^2\sum_i [D_i(\frac{\phi}{u})]^2 + \int_\Omega \phi^2(\frac{\lambda-\mu}{2})g.$$

We may also replace u by v in the latter to obtain a similar expression. Consolidating the latter we have

$$(\phi,(L-(\frac{\lambda+\mu}{2})g)\phi) \geq \left|\frac{\mu+\lambda}{2}(g\phi,\phi)\right| \geq 0$$

so that $L-(\frac{\lambda+\mu}{2})g \geq 0$. Since $(\lambda+\mu)/2$ cannot be a third e.v. with a positive e.f. (corollary 1.1) it follows that $L-(\frac{\lambda+\mu}{2})g > 0$.

(b) If (1.1) is semi-definitizable there is a $k\varepsilon\mathbb{R}$ for which $(\phi,(L+kg)\phi) \geq 0$, $\phi\varepsilon C_0^\infty(\Omega)$. Moreover there must be a u such that $(u,(L+kg)u) = 0$. By the Courant principle $u>0$ and u is an e.f. of $L+kg$. Since (1.1) is not definitizable u is the only positive e.f.. Conversely if $v>0$ is the only e.f. of $Lv = \lambda gv$ then $(\phi,L\phi) \geq \lambda(g\phi,\phi)$ by Picone's identity. Hence $L-\lambda g \geq 0$ and cannot be made positive definite as v is the only positive e.f. of $Lv = \lambda gv$. Finally, that v is a ghost state follows by a continuity argument. To see this consider the family of "least eigenvalue" problems:

$$L\tau - tg\tau = \mu(t)\tau, \quad \tau = 0 \text{ on } \partial\Omega,$$

for $-\infty < t < +\infty$. Set $(\tau,\tau) = 1$ for all t. Then $\mu(t) \leq 0$, $\mu(\lambda) = 0$ and $\tau(\lambda) = v$. Whence $(L\tau(\lambda + t) - (\lambda + t)g\ \tau(\lambda + t)),\ \tau(\lambda+t)) \leq 0$ and $(L\tau(\lambda + t) - \lambda g\ \tau(\lambda + t),\ \tau(\lambda + t)) \geq 0$ by hypothesis i.e., $t(g\tau(\lambda + t),\ \tau(\lambda + t)) \geq 0$ for all t. But observe that, as $t\to 0$, $\{\mu(t)\}$ is bounded. It follows that $\{\tau(\lambda + t)\}$ is bounded in $H_o^{1,2}(\Omega)$ and hence, without loss of generality, weakly convergent in $H_o^{1,2}$ and strongly convergent in L^2 to a positive eigenfunction of $L-\lambda g$ (see [2] for the details). By the uniqueness of the latter, we conclude $\tau(\lambda + t)\to \tau(\lambda) = v$ in $L^2(\Omega)$ and, further, using the above we see that

$$(gv,v) = \lim_{t\to o} (g\tau(\lambda + t),\ \tau(\lambda + t)) = 0$$

so that v is a ghost state.

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FINITE ORBITS IN FINITE DIMENSIONAL ℓ_1 SPACES

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Let $L = \mathbb{R}^\ell$ be the ℓ-dimensional real vector space, considered as a metric space with the ℓ_1 distance $\rho(x,y) = \|x-y\|_1 = \sum_{i=1}^{\ell} |x_i - y_i|$, $x = (x_1,\ldots,x_\ell) \in L$, $y = (y_1,\ldots,y_\ell) \in L$. A mapping τ defined on a subset of L is called an isometry if $\rho(\tau x, \tau y) = \rho(x,y)$ and nonexpansive if $\rho(\tau x, \tau y) \le \rho(x,y)$ for all x and y in the domain of τ. It is essentially known and easy to see that if $\tau : L \to L$ is an isometry defined on the whole space L and if $\tau 0 = 0$ then the orbit $(\tau^n x)_{n\ge 0}$ of any point $x \in L$ is a finite set in the sense that $\tau^k x = x$ for some integer $k \ge 1$. The following result, however, is also true.

Theorem 1. Let X be a compact subset of L and let $\tau = X \to X$ be an isometry. Then the orbit $(\tau^n x)_{n\ge 0}$ of any point $x \in X$ is a finite set.

The proof will be given in [1], together with further results and details. Theorem 1 can also be formulated in the following way.

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Theorem 2. Let $(x^n)_{n\geq 0}$ be a bounded sequence in L such that $\rho(x^{n+k},x^n) = \rho(x^k,x^0)$ for all $n \geq 0$ and $k \geq 0$. Then $(x^n)_{n\geq 0}$ contains only finitely many different terms.

It is clear that Theorem 2 implies Theorem 1. For the converse, let X be the closure of the sequence $(x^n)_{n\geq 0}$. The mapping τ defined on the terms of this sequence by $\tau x^n = x^{n+1}$, $n \geq 0$, can be extended to an isometry $\tau : X \to X$. An application of Theorem 1 then proves Theorem 2.

The following result follows from Theorem 1 and shows that the orbits under a nonexpansive map become periodic, asymptotically, if they stay bounded.

Theorem 3. Let A be a closed subset of L and let $\tau : A \to A$ be nonexpansive. Let $x \in A$ be a point such that its orbit $(\tau^n x)_{n\geq 0}$ is bounded. Then there is an integer $k \geq 1$ such that $\tau^{nk} x$ converges as $n \to \infty$.

For a sketch of the proof, let X_n be the closure of $(\tau^i x)_{i\geq n}$ and let $X = \bigcap_{n=0}^{\infty} X_n$. Then it can be shown that the restriction of τ to X is an isometry $\tau : X \to X$. Also, the orbit of any point of X is dense in X. Hence Theorem 1 shows that X is a finite set. Then the proof of Theorem 3 follows easily.

Theorem 3 can be applied to obtain a limit theorem for a non linear model of diffusion on a finite set. This application was, in fact, our motivation for Theorem 1.

The classical linear model for diffusion on a finite set is given by Markov Chains. For our purpose we can define a Markov

Chain (with a finite state space) as a linear operator $T : L \to L$ which is positive (i.e. $TL^+ \subset L^+$, where L^+ is the positive cone of L, consisting of vectors with non negative coordinates) and which is a contraction. Theorem 3 is applicable to any orbit $(T^n x)_{n \geq 0}$ and gives a well known result for Markov Chains. A more general model of diffusion is given by a nonexpansive $\tau : L^+ \to L^+$ such that $\tau 0 = 0$ and such that τ is order preserving (i.e. $\tau x - \tau y \in L^+$ whenever $x-y \in L^+$, $y \in L^+$). Theorem 3 is applicable to any orbit $(\tau^n x)_{n\geq 0}$, even if τ is not order preserving, and shows that the orbits become periodic, asymptotically, as in the linear case.

There is an important special case, referred to as the aperiodic case, in which $\tau^n x$ converges. We state this result as follows.

Theorem 4. Assume that $\tau : L^+ \to L^+$ is nonexpansive and order preserving and that $\tau 0 = 0$. Also assume the following condition.

(AP) If $x \geq y$ and $\tau y = y$ then there is an n such that $(\tau^n x)_i > y_i$ for all $i = 1,\ldots,\ell$. Then $\tau^n x$ converges as $n \to \infty$.

This theorem has a shorter proof that is independent of Theorem 3.

Finally we note that in a continuous parameter diffusion process only the aperiodic case can occur. We state this result in the following more general case.

Theorem 5. Let $\{\tau_t\}_{t\geq 0}$ be a family of nonexpansive mappings $\tau_t : L \to L$ such that (i) for which $t,s \geq 0$, $\tau_t\tau_s = \tau_{t+s}$ and $\tau_t 0 = 0$, (ii) for each $x \in L$, the function $[0,\infty) \to L$ that takes $t \in [0,\infty)$ to $\tau_t x \in L$ is continuous. Then $\lim_{t\to\infty} \tau_t x$ exists in L for each $x \in L$.

As mentioned earlier, complete proofs, together with further results and details will appear in [1]. For non-linear models of diffusion in more general spaces see [2].

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CONICAL DIFFERENTIATION IN A GENERAL THEORY OF DIRECT DIFFERENTIAL GEOMETRY

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Presented by H.S.M. Coxeter, F.R.S.C.

A unified theory of direct differential geometry has been developed by the authors. This note describes how the general theory is applied to the special case of conical differentiation.

1. An arc or a curve in the real projective plane is the homeomorphic image of a line segment or a projective line, respectively. The geometry of orders of arcs and curves is developed by O. Haupt and H. Künneth in [1]. In a domain G (usually the real projective plane) a set $\mathcal{K}$ of arcs or curves K is given. The geometry of orders studies the structure of a given arc B by means of the set $\mathcal{K}$. In the local theory, a point p on B is given and the behaviour of B near p is examined in relation to the curves of $\mathcal{K}$. Various mathematicians have considered special cases, e.g., $\mathcal{K}$ is the set of all lines in the projective plane [10]; $\mathcal{K}$ is the set of circles on a sphere [2], [3]; $\mathcal{K}$ is the set of conic-sections in the real projective plane [7], [8], [9]. These ad hoc studies have many similarities and suggest an attempt at a unified theory. Recently, the authors have formulated such a theory in which $\mathcal{K}$ is a topological space of finite graphs in the real projective plane. The set $\mathcal{K}$ generates an independence structure in the following way. Let Σ be any subset of G. Define $\mathcal{K}_\Sigma = \{K \in \mathcal{K} \mid \Sigma \subset K\}$ and $M_\Sigma = \cap \{K \mid K \in \mathcal{K}_\Sigma\}$. Then M_Σ automatically has the properties:

(M1): $\Sigma \subset M_\Sigma = M_{M_\Sigma}$ and (M2): $\Sigma_1 \subset \Sigma_2 \Rightarrow M_{\Sigma_1} \subset M_{\Sigma_2}$. A subset $\Sigma \subset G$ is defined to be $\mathcal{K}$-independent if either $\Sigma = \phi$ or $\Sigma \neq \phi$ and $P \notin M_{\Sigma \setminus \{P\}}$

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for all $P \in \Sigma$. Let $\mathcal{I}$ denote the set of $\mathcal{K}$-independent subsets of G. We impose on $\mathcal{K}$, M_Σ and $\mathcal{I}$, two additional conditions:

(M3): *If* $\Sigma \in \mathcal{I}$ *and* $P \notin M_\Sigma$, *then* $\Sigma \cup P \in \mathcal{I}$ (*the exchange condition*)

and

(M4): *There exists a finite maximal* $\mathcal{K}$-*independent subset of* G *the finite base condition*).

With these assumptions, $\mathcal{K}$ generates a matroid in which every maximal $\mathcal{K}$-independent subset of G has the same cardinality $k + 1$ [5]. Let Σ^n denote a subset of G of cardinality n. If $\Sigma^k \in \mathcal{I}$, then there exists exactly one K of $\mathcal{K}$ through Σ. [5].

2. *Direct differentiation*. Let B be an arc of G, $p \in B$. We assume that *the* $\mathcal{K}$-*order of* B ($= \sup |K \cap B|$) *is finite*. Let $\Sigma^k = \Sigma^{k-1} \cup p \in \mathcal{I}$. Then $\Sigma^{k-1} \cup q \in \mathcal{I}$ for every $q \in B\setminus\{p\}$ sufficiently close to p, $K(\Sigma^{k-1} \cup q)$ is uniquely determined, $p \notin K(\Sigma^{k-1} \cup q)$ and $\lim_{q \to p} K(\Sigma^{k-1} \cup q) = K(\Sigma^k) = K(\Sigma^{k-1}, p)$. The subset $\mathcal{K}(p) = \{K(\Sigma^{k-1}, p) \mid \Sigma^{k-1} \cup p \in \mathcal{I}\} \subset \mathcal{K}$ generates a new matroid with $M(p) \supset M$ and independence structure $\mathcal{I}(p) \subset \mathcal{I}$ of dimension k in which the exchange condition is valid. If $\Sigma^{k-2} \in \mathcal{I}(p)$ and $q \in B\setminus\{p\}$ is sufficiently close to p, then $\Sigma^{k-2} \cup q \in \mathcal{I}(p)$, $\Sigma^{k-2} \cup q \cup p \in \mathcal{I}$ and there is a unique graph $\mathcal{K}(\Sigma^{k-2} \cup q, p)$ of $\mathcal{K}(p)$ through $\Sigma^{k-2} \cup q$. The arc B is called *once* $\mathcal{K}$-*differentiable* if $\lim_{q \to p} K(\Sigma^{k-2} \cup q, p) = K(\Sigma^{k-2}, p^2)$ exists for each $\Sigma^{k-2} \in \mathcal{I}(p)$. The limit graph is called the $\mathcal{K}(p)$-*tangent graph of* B at p *through* Σ^{k-2}. Let $\mathcal{K}(p^2) = \{K(\Sigma^{k-2}, p^2) \mid \Sigma^{k-2} \in \mathcal{I}(p)\}$. The set $\mathcal{K}(p^2)$ defines a new independence structure in G with $M(p^2)$ and $\mathcal{I}(p^2)$. It is not known that condition $(M(p)_3)$ in $\mathcal{K}(p)$ implies condition $(M(p^2)_3)$ in $\mathcal{K}(p^2)$, so we have to assume that $(M(p^2)_3)$ is valid $\mathcal{K}(p^2)$. Then the dimension of the matroid induced by $\mathcal{K}(p^2)$ is $k - 1$. For each $\Sigma^{k-3} \in \mathcal{I}(p^2)$, if $q \in B\setminus\{p\}$ is sufficiently close to p, then $\Sigma^{k-3} \cup q \in \mathcal{I}(p)$ and

$K(\Sigma^{k-3} \cup q, p^2)$ exists. The arc B is called twice $\mathcal{K}$-differentiable at p if $\lim_{q\to p} K(\Sigma^{k-3} \cup q, p^2) = K(\Sigma^{k-3}, p^3)$, (the $\mathcal{K}$-osculating graph of B at p through Σ^{k-3}), exists for each $\Sigma^{k-3} \in \mathcal{I}(p^2)$. The osculating graphs are also considered to be tangent graphs of B at p. Continuing in this way, one obtains, eventually, a unique graph $\mathcal{K}(p^k)$, called the k-osculating graph of B at p. Even if one assumes that $\mathcal{K} \supset \{K \in \mathcal{K} | \Sigma^k \subset K$ for some $\Sigma^k \in \mathcal{I}\}$, there may appear some derived graphs which do not lie in the original set $\mathcal{K}$. On the other hand, the set of derived graphs need not include all of the graphs in the compactification of $\mathcal{K}$.

3. An application to conical differentiation. In [7], the authors started with the set of non-degenerate conics in the real projective plane. However, in the general theory, the prematroid generated by this set does not satisfy the exchange condition (M3) and one must adjoin to this set the pairs of distinct lines of G in order to obtain a set $\mathcal{K}$ which satisfies (M3). Then $\mathcal{K}$ generates a matroid $\langle M, \mathcal{I}\rangle$ of dimension $k + 1 = 6$. One writes $k = \dim \mathcal{K} = 5$, since any $\Sigma^5 \in \mathcal{I}$ determines a unique $K \in \mathcal{K}$. Conversely, any $K \in \mathcal{K}$ contains a $\Sigma^5 \in \mathcal{I}$.

Let B be an arc of finite conical order and let p be a point B. Let $\Sigma^5 = \Sigma^4 \cup p \in \mathcal{I}$. Then $K^4 \cup q \in \mathcal{I}$ for every $q \in B\backslash\{p\}$ sufficiently close to p, $K(\Sigma^4 \cup q)$ is uniquely determined, $p \notin K(\Sigma^4 \cup q)$, and $\lim_{q\to p} K(\Sigma^4 \cup q) = K(\Sigma^5) = K(\Sigma^4, p)$. The set $\mathcal{K}(p) = \{K(\Sigma^4, p) | \Sigma^4 \cup p \in \mathcal{I}\} \subset \mathcal{K}$ determines an independence structure $\langle M(p), \mathcal{I}(p)\rangle$ with $M(p) \supset M$ and $\mathcal{I}(p) \in \mathcal{I}$ which satisfies the exchange condition (M3) and defines a matroid of dimension 5.

Let $\Sigma^3 \in \mathcal{I}(p)$. Then $\Sigma^3 \cup q \in \mathcal{I}(p)$, $\Sigma^3 \cup q \cup p \in \mathcal{I}$ for all $q \in B\backslash\{p\}$ sufficiently close to p and there exists a unique conic $K(\Sigma^3 \cup q, p)$ in $\mathcal{K}(p)$ through $\Sigma^3 \cup q$. We assume Condition 1: $\lim_{q\to p} K(\Sigma^3 \cup q, p) = K(\Sigma^3, p^2)$ exists. The set $\mathcal{K}(p^2) = \{K(\Sigma^3, p^2) \mid \Sigma^3 \in \mathcal{I}(p)\}$ (tangent conics of B at

p) defines another independence structure $\langle M(p^2), \mathcal{I}(p^2)\rangle$, with $M(p^2) \supset M(p)$ and $\mathcal{I}(p) \subset \mathcal{I}(p^2)$, which again satisfies the exchange condition and defines a matroid of dimension 4. The validity of the exchange condition may be verified from the table of independent sets given in 4. If $\Sigma^2 \in \mathcal{I}(p^2) \subset \mathcal{I}(p)$, then $\Sigma^2 \cup q \in \mathcal{I}(p)$ for all $q \in B\setminus\{p\}$ sufficiently close to p and $K(\Sigma^2 \cup q,\ p^2)$ is uniquely determined in $\mathcal{K}(p^2)$. We assume Condition 2: $\lim\limits_{q\to p} K(\Sigma^2 \cup q,\ p^2) = K(\Sigma^2, p^3)$ exists. The set $\mathcal{K}(p^3) = \{K(\Sigma^2,p^3) \mid \Sigma^2 \in \mathcal{I}(p^2)\}$ (osculating conics of B at p) again defines a matroid of dimension 3 with $M(p^3) \supset M(p^2)$ and $\mathcal{I}(p^3) \subset \mathcal{I}(p^2)$. The osculating conics are also considered to be tangent conics of B at p. If $\Sigma^1 \in \mathcal{I}(p^3) \subset \mathcal{I}(p^2)$, then $\Sigma^1 \cup q \in \mathcal{I}(p^2)$ for all $q \in B\setminus\{p\}$ sufficiently close to p and hence $\Sigma^1 \cup q$ determines a unique $K(\Sigma^1 \cup q,\ p^3) \in \mathcal{K}(p^3)$. We assume Condition 3: $\lim\limits_{q\to p} K(\Sigma^1 \cup q,\ p^3) = K(\Sigma^1,\ p^4)$ exists. The set $\mathcal{K}(p^4) = \{K(\Sigma^1,\ p^4) \mid \Sigma^1 \in \mathcal{I}(p^3)\}$ (superosculating conics of B at p) defines a matroid of dimension 2 with $M(p^4) \supset M(p^3)$ and $\mathcal{I}(p^4) \subset \mathcal{I}(p^3)$. If $q \in B\setminus\{p\}$ is sufficiently close to p then $q \in \mathcal{I}(p^3)$ and $K(q,p^4)$ is uniquely determined. Finally, we assume Condition 4: $\lim\limits_{q\to p}(q,p^4) = K(p^5)$ (the ultraosculating conic of B at p) exists. The arc B is called conically differentiable at p if Conditions 1-5 are satisfied.

In the original paper [7] on conical differentiation, the conditions on the sets Σ required for the construction of the osculating conics seemed, at that time, to be strange, but they now are seen follow quite naturally from independence structure of the matroid associated with $\mathcal{K}(p^2)$.

4. The independent sets of the matroids associated with $\mathcal{K}$.

	$\mathcal{K}$	$\mathcal{K}(p)$	$\mathcal{K}(p^2)$	$\mathcal{K}(p^3)$			$\mathcal{K}(p^4)$	
dim	5	4	3	2			1	
Characteristic conics	$A_0 \cup B_0$	$A_1 \cup B_1$	$A_2 \cup B_{21} \cup B_{22}$	$A_3 \cup B_3$	B_{21}	B_{22}	$A_4 \cup t_2$	B_3
Independent sets of points								
$\lvert\Sigma\rvert = 1$	All Σ	$p \notin \Sigma$	$p \notin \Sigma$	$p \notin \Sigma$	$p \notin \Sigma$	$\Sigma \cap t_2 = \phi$	$p \notin \Sigma$	$\Sigma \not\subset t_2$
$\lvert\Sigma\rvert = 2$			$p \notin \Sigma$ $\lvert\Sigma \cap t_2\rvert \leq 1$	$p \notin \Sigma$ $\lvert\Sigma \cap t_2\rvert \leq 1$	$p \notin \Sigma$ $\Sigma \cup p$ not collinear			
$\lvert\Sigma\rvert = 3$		$p \notin \Sigma$	$p \notin \Sigma$ $\Sigma \cup p$ not collinear $\lvert\Sigma \cap t_2\rvert \leq 1$					
$\lvert\Sigma\rvert = 4$	No four points of Σ collinear	No four points of $\Sigma \cup p$ collinear						
$\lvert\Sigma\rvert = 5$								

A_0 = {all non-degenerate conics}
B_0 = {pairs (ℓ,h) of distinct lines}
A_1 = {all non-degenerate conics through p }
B_1 = {pairs (ℓ,h) , at least one through p }

t_2 = a line through p .
A_2 = {all non-degenerate conics touching t_2 at p }
B_{21} = {pairs (ℓ,h) with $\ell \cap h = p$ }
B_{22} = {pairs (t_2,ℓ) with $\ell \neq t_2$ }

A_3 = {given $K \in A_2$, all non-degenerate conics with at least 3-point contact with K at p}
B_3 = {pairs (t_2,ℓ) with $\ell \cap t_2 = p$ }
A_4 = given $K \in A_2$, all non-degenerate conics with at least 4-point contact with K at p}

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