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CALCUL INTEGRAL COMBINATOIRE ET HOMOLOGIE DES GROUPES SYMETRIQUES

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Presente par G. DeB. Robinson, M.S.R.C.

Résumé

Il y a une grande analogie entre la théorie combinatoire des séries formelles [3] et la théorie des fonctions analytiques. Ainsi, on peut définir la dérivée F' d'une espèce de structures F. Cette opération jouit de propriétés formelles identiques à celle de la dérivée des fonctions. Il est donc naturel de se demander si toute espèce F possède une primitive. Si tel était le cas, on pourrait démontrer que S_n est un rétracte de S_{n+1}, ce qui est absurde pour n>3. Toutefois cette impossibilité est trompeuse car on peut effectivement construire une primitive parmi les espèces virtuelles [4] . Il est intéressant de comparer ce résultat avec un théorème de Nakoaka/Dold [2,5,6] selon lequel S_n est homologiquement un facteur direct de S_{n+1}. Nous donnons une nouvelle démonstration de ce théorème qui utilise une formule d'intégration combinatoire des espèces de structures.

1- Calcul intégral combinatoire

Nous avons défini dans [3] la dérivée F' d'une espèce de structures F:

$$F'[E] = F[E+1]$$

On vérifie les formules habituelles du calcul différentiel:

$$(F+G)' = F'+G'$$

$$(FG)' = F'G+FG'$$

$$F(G)' = F'(G)G'$$

Il est naturel de chercher à inverser l'opération de dérivation. Considérons le développement de Taylor [4]

$$F(X) = \sum_{n\geq 0} F[n] \times_{S_n} X^n$$

Intégrer F consiste à construire pour chaque entier n une représentation ensembliste G_{n+1} de S_{n+1} telle que $G_{n+1}|S_n = F_n$. On constate que ceci n'est pas toujours faisable: la représentation canonique de S_4 dans {1,2,3,4} ne peut se prolonger à S_5 (car S_4 n'est pas un rétracte de S_5). Cependant, nous allons voir que l'intégration est toujours possible dans les espèces virtuelles [4]. En effet, considérons l'espèce exponentielle:

$$e(X) = \sum_{n\geq 0} 1 \times_{S_n} X^n$$

Posons

$$e_n(X) = 1 \times_{S_n} X^n$$

On a évidemment

$$e'_{n+1}(X) = e_n(X) \qquad (n\geq 0)$$

Pour toute espèce F posons

$$I(F) = \sum_{k=0}^{\infty} (-1)^k e_{k+1}(X) F^{(k)}(X)$$

Cette série infinie est convergente.

PROPOSITION 1. I(F) est une primitive de F.

La proposition se démontre en appliquant les règles du calcul différentiel

2- Foncteur cohomologique sur les groupoïdes finis

Soit G la catégorie des groupoïdes finis (les morphismes de groupoïdes sont les foncteurs).

DEFINITION. Un morphisme $A \xrightarrow{p} B$ de groupoïde est un <u>revêtement</u> (de B) si pour tout $a \in A$, $\beta : b' \longrightarrow b$ tels que $p(a) = b$ il existe une et une seule flèche $\alpha : a' \longrightarrow a$ telle que $p(\alpha) = \beta$

DEFINITION. Soient $f, g : A \longrightarrow B$ des morphismes de groupoïdes. Nous dirons que f et g sont <u>homotopes</u> ($f \sim g$) s'il existe une transformation naturelle $f \longrightarrow g$.

DEFINITION. Un foncteur <u>cohomologique</u> H sur $\underline{G}$ est la donnée

1) d'un foncteur contravariant sur $\underline{G}$

$$H : \underline{G}^{opp} \longrightarrow \underline{Ab}$$

à valeur dans la catégorie des groupes abéliens.

2) d'un homomorphisme (de transfert) $\Sigma_p : H(A) \longrightarrow B$
pour chaque revêtement $A \xrightarrow{p} B$.

On demande que les conditions suivantes soient vérifiées (nous noterons par f^* l'homomorphisme $H(f)$)

a) Si $f, g : A \longrightarrow B$ sont homotopes, alors

$$f^* = g^*$$

b) Σ_p dépend fonctoriellement de p:

$$\Sigma_{pq} = \Sigma_p \Sigma_q \ , \quad \Sigma_1 = 1$$

c) Pour tout carré cartésien

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{v} & D \end{array}$$

où q (et donc p) est un revêtement,

$$v^* \Sigma_q = \Sigma_p w^*$$

d) Les homomorphismes canoniques suivants sont des isomorphismes:

$$H(A+B) \xrightarrow{\sim} H(A) \oplus H(B)$$

$$H(\phi) \xrightarrow{\sim} 0$$

Remarque 1: Cette définition peut se généraliser si l'on suppose que le foncteur H prend ses valeurs dans une catégorie additive arbitraire $\underline{A}$. Un foncteur homologique est un foncteur cohomologique à valeur dans $\underline{A}^{opp}$

Remarque 2: Soient $A, B \epsilon \underline{G}$ et $i : A \longrightarrow A+B$, $j : B \longrightarrow A+B$ les inclusions. On montre que pour tout $x \in H(A+B)$, on a

$$x = \Sigma_i i^*(x) + \Sigma_j j^*(x)$$

Exemple 1: Soit $\underline{C}$ une catégorie close sous les coproduits finis. Pour tout $G \in \underline{G}$ notons $[G, \underline{C}]$ la catégorie des foncteurs de G vers $\underline{C}$.
Pour tout $f : A \longrightarrow B \in \underline{G}$, soit

$$f^* : [B, \underline{C}] \longrightarrow [A, \underline{C}]$$

le foncteur restriction le long de f. Lorsque f est un revêtement, f^* possède un adjoint à gauche

$$\Sigma_f : [A, \underline{C}] \longrightarrow [B, \underline{C}]$$

donné par la formule

$$\Sigma_f(N)(b) = \coprod_{f(a)=b} N(a) \quad (N \epsilon [A, \underline{C}],\ b \epsilon B)$$

On obtient un foncteur cohomologique en prenant pour chaque $A \epsilon \underline{G}$ le groupe de Grothendieck de la catégorie $[A, \underline{C}]$ (l'addition provient du coproduit). En particulier, si $\underline{C}$ est la catégorie des ensembles finis, ce foncteur associera à chaque groupoïde G son anneau de Burnside B(G).

Exemple 2: L'exemple précédent se généralise au cas où $\underline{C}$ n'èst qu'une catégorie symétrique monoïdale $(\underline{C}, \oplus, 0)$. Le foncteur Σ_f est défini par la formule ci-haut en remplaçant le coproduit par $\oplus$; Σ_f n'est pas nécessairement l'adjoint de f^*.

Exemple 3: Soit E^* une théorie cohomologique. Utilisant la théorie générale du transfert [5], on vérifie que le foncteur $G \longmapsto E^*(BG)$ est cohomologique (BG désigne ici l'espace classifiant du groupoïde G).

3- Foncteur cohomologique universel

Soit $\underline{S}$ la catégorie des ensembles finis. Soient A, B $\in \underline{G}$.

DEFINITION: Une relation étale (sur A) $R:A \Rightarrow B$ est un foncteur $R: A^{opp} \times B \longrightarrow \underline{S}$ tel que pour tout $a \in A^{opp}$, le foncteur $R(a,-): B \longrightarrow \underline{S}$ est fidèle.

Exemple: Soit $X \xrightarrow{f} B$ un foncteur défini sur un revêtement fini $X \xrightarrow{p} A$. On obtient une relation étale $R:A \Longrightarrow B$ en posant

$$R(a,b) = \sum_{p(x)=a} B(f(x),b)$$

On vérifie que toute relation étale peut se décrire de cette manière.

La catégorie des relations étales de A vers B est fermée sous le co-produit fini (i.e. la somme disjointe finie). Notons $\rho(A,B)$ le groupe de Grothendieck de cette catégorie. Soient $R:A \Longrightarrow B$, $S:B \Longrightarrow C$. Le composé SoR est défini par

$$(SoR)(a,c) = \varinjlim_{b \in B} R(a,b) \times S(b,c)$$

On vérifie que SoR est étale et que cette opération est distributive sur la somme. Elle induit une loi de composition sur les groupes de Grothendieck

$$\rho(B,C) \times \rho(A,B) \xrightarrow{\ o\ } \rho(A,C)$$

On obtient une catégorie additive $\underline{\rho}$. On a un foncteur homologique $h: \underline{G} \longrightarrow \underline{\rho}$ en posant, pour $A \xrightarrow{f} B \in G$, $h(f)\ (a,b) = B(f(a),b)$, et si f est un revêtement,

$$\sigma(f)\ (b,a) = \sum_{f(x)=b} A(x,a)\ .$$

THEOREME. Le foncteur h(muni de σ) est un foncteur homologique universel.

Ainsi, tout foncteur cohomologique H est obtenu par composition $H \simeq V \circ h$ où V est un foncteur additif contravariant $V: \underline{\rho}^{opp} \longrightarrow Ab$ unique (à isomorphisme unique près).

4- Combinatoire et homologie

Pour démontrer que S_n est homologiquement un facteur direct de S_{n+1}, nous devons produire un rétracte $r_n \epsilon \rho(S_{n+1}, S_n)$.Considérons l'espèce virtuelle

$$r(X,Y) = \sum_{n\geq 0} r_n \underset{S_{n+1,n}}{\times} X^{n+1} Y^n \qquad (S_{n+1,n} = S_{n+1} \times S_n)$$

La condition selon laquelle r_n est un rétracte (pour tout $n\geq 0$) signifie que

$$\frac{d}{dX} r(X,Y) = e(XY)$$

Il suffit donc de prendre

$$r(X,Y) = e(XY) \sum_{k=0}^{\infty} (-1)^k e_{k+1} (X) Y^k$$

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A NOTE ON THE STRUCTURE OF $\mathbb{Z}_p[[X]]$ - MODULES*

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Presented by P. Ribenboim, F.R.S.C.

ABSTRACT: The Iwasawa-Serre Theorem on the structure of finitely generated $\mathbb{Z}_p[[X]]$ - modules is important in the study of cyclotomic fields. In the matricial proof of this theorem due to Paul Cohen three pseudo-elementary operations on the relation matrix play a crucial role. We show that these operations when performed on certain sub-matrices extend to the full matrix. This makes an induction step in the above proof slightly easier.

THEOREM (Iwasawa-Serre). Let $\Lambda = \mathbb{Z}_p[[X]]$ be the ring of formal power series over the ring $\mathbb{Z}_p$ of p-adic integers, p being a prime. Let M be a finitely generated Λ - module; then there exist natural numbers $r, s, t, m_1, m_2, \ldots, m_s$ and $k_1, k_2, \ldots, k_t$ and distinguished irreducible polynomials $f_1, f_2, \ldots, f_t$ and a Λ-homomorphism $\theta: M \to M'$ with ker θ and coker θ finite, M' being the direct sum

$$M' = \Lambda^r \oplus \left(\bigoplus_{i=1}^{s} \Lambda / (p^{m_i}) \right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda / (f_j^{k_j}) \right)$$

A Λ-homomorphism θ with finite kernel and cokernel is called a pseudo-isomorphism. Thus the structure theorem describes the module M up to pseudo-isomorphism.

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We follow closely Washington's book [3]. As mentioned there (p. 271), the module is described by its matrix R of relations. One aims to take this matrix to diagonal form via the usual elementary operations and the pseudo-elementary operations 1,2 and 3 (p. 272).

LEMMA U. *Let*
$$F = \begin{pmatrix} \lambda_{11} & 0 & 0 & \cdots & 0 \\ \lambda_{21} & & & & \\ \cdot & & & & \\ \cdot & & B & & \\ \cdot & & & & \\ \lambda_{m1} & & & & \end{pmatrix}$$

where λ_{11} *is a distinguised polynomial*, $\lambda_{ij} \in \Lambda$ *and* $B \neq 0$ *a matrix of smaller size. If an Operation* 3 *is used to change* B *to a matrix* B' (i.e., $B \sim B'$), *then*

$$F \sim F' \quad \text{where} \quad F' = \begin{pmatrix} \lambda_{11} & 0 & 0 & \cdots & 0 \\ \lambda'_{21} & & & & \\ \vdots & & & B' & \\ \lambda'_{m1} & & & & \end{pmatrix}$$

with $\lambda'_{ij} \in \Lambda$.

PROOF. Since Operation 3 is involved, we may suppose

$$B = \begin{pmatrix} p^k\lambda_{22} & p^k\lambda_{23} \cdots & p^k\lambda_{2n} \\ \lambda_{32} & \lambda_{33} \cdots & \lambda_{3n} \\ \vdots & & \\ \lambda_{m2} & \cdots & \lambda_{mn} \end{pmatrix}$$

with $k \geq 1$ and

that for some $\lambda \in \Lambda$ with $p \nmid \lambda, (\lambda\lambda_{22}, \lambda\lambda_{23}, \ldots, \lambda\lambda_{2n})$ is also a relation originating from the matrix B and that

$$B' = \begin{pmatrix} \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ & \cdots\cdots & & \\ \lambda_{m2} & \cdots\cdots & & \lambda_{mn} \end{pmatrix}$$

By use of the special case of Operation 1, we may further assume that p^k divides λ_{i1} for $i=2,3,\ldots,m$. Thus let $\lambda_{21} = p^k\lambda'_{21}$. Let M be a module having F as its matrix of relations. Write $M = \Lambda u_1 + \ldots + \Lambda u_n$. Let $M' = M/\Lambda u_1$. Then $M' = \Lambda\bar{u}_2 + \ldots + \lambda\bar{u}_n$ and B is a matrix of relations of M'. Let $x = \lambda'_{21}u_1 + \lambda_{22}u_2 + \ldots + \lambda_{2n}u_n \in M$. Then $p^k\bar{x} = 0$ and $\lambda\bar{x} = 0$ in M'. If θ is the canonical homorphism $\theta: M \longrightarrow M'$, then $\theta(\lambda x) = 0$. Thus $\lambda x \in \ker \theta = \Lambda u_1$. Now $\lambda_{11}u_1 = 0$ in M so that $\lambda\lambda_{11}x \in \Lambda\lambda_{11}u_1 = (0)$. Hence $(\lambda\lambda_{11}\lambda'_{21}, \lambda\lambda_{11}\lambda_{22}, \ldots, \lambda\lambda_{11}\lambda_{2n})$ is also a relation defined by F. Since $p \nmid \lambda\lambda_{11}$ and the second row of F is $(p^k\lambda'_{21}, p^k\lambda_{22}, \ldots, p^k\lambda_{2n})$, we may apply Operation 3 to F to conclude $F \sim F'$.

Given Lemma U, the following result is all but obvious.

<u>LEMMA V.</u> <u>Let</u>

$$F = \begin{pmatrix} \lambda_{11} & 0 & 0 & \cdots & 0 \\ \lambda_{21} & & & & \\ \cdots & & B & & \\ \lambda_{m1} & & & & \end{pmatrix}$$

where λ_{11} is a distinguished polynomial, $\lambda_{ij} \in \Lambda$ and $B \neq 0$ is a matrix of smaller size.
If $B \sim B'$, then $F \sim F'$ where

$$F' = \begin{pmatrix} \lambda_{11} & 0 & 0 & \cdots & 0 \\ \lambda'_{21} & & & & \\ \cdots & & B' & & \\ \lambda'_{m1} & & & & \end{pmatrix}$$

Moreover, if in the passage from B to B' Operation 2 is performed ℓ times with factors $p^{k_1}, p^{k_2}, \ldots, p^{k_\ell}$ then the same is true of the passage from F to F'.

Proof of the Theorem. We carry out the inductive step completely inducting on the number n of columns $(n \geq 0)$. If $n = 0$ there is nothing to prove. So assume $n \geq 1$ and that the relation matrix $R \neq 0$. If $n = 1$ we may add a column of zeros. Note that all the entries of R are not infinitely divisible by p and so using Operations 2, A and C if necessary, we may assume that R is equivalent to a matrix at least one of whose entries, say λ_{11}, is a distinguished polynomial and that the Weierstrass degree $\deg_W \lambda_{11} = \deg \lambda_{11}$ is least among the Weierstrass degrees of elements of all matrices equivalent to R. Thus we may assume that the first row of R is $(\lambda_{11}, \lambda'_{12}, \ldots, \lambda'_{1n})$. Using the division algorithm we write $\lambda'_{1j} = q_{1j}\lambda_{11} + r_{1j}$ where $\deg r_{1j} < \deg_W \lambda_{11}$. Operation B helps to take R to a matrix R' whose first row is $(\lambda_{11}, r_{12}, \ldots, r_{1n})$. We claim that $r_{1j} = 0$ for $j = 2, \ldots, n$. If not p divides the r_{1j} by the choice of λ_{11}. Choose t with $2 \leq t \leq n$ and $s \geq 1$ so that $r_{1j} = p^s r'_{1j}$, $j = 2, \ldots, n$ and $(p, r'_{1t}) = 1$.

Operation 1 when applied s times takes R' to a matrix R'' whose first row is $(\lambda_{11}, r'_{12}, \ldots, r'_{1t}, \ldots)$ with $\deg_w r'_{1t} < \deg_w \lambda_{11}$, a contradiction. So indeed $r_{1j} = 0$, $j = 2,3,\ldots,n$. Hence

$$R \sim F = \begin{pmatrix} \lambda_{11} & 0 & 0 & \ldots & 0 \\ \lambda_{21} & & & & \\ \ldots & & B & & \\ \lambda_{m1} & & & & \end{pmatrix}$$

where $B = 0$ if $n = 1$ and otherwise of smaller size.

The induction hypothesis guarantees the passage of B to a diagonal form B'. Lemma V then takes F to

$$F_1 = \begin{pmatrix} \lambda_{11} & 0 & \ldots & 0 \\ \lambda'_{21} & \lambda_{22} & 0 & 0 \\ & \ldots & & \\ \lambda'_{r,1} & & \lambda_{rr} & 0 \\ \lambda'_{r+1,1} & 0 & & 0 \\ & \ldots & & \\ \lambda'_{m,1} & & 0 & 0 \end{pmatrix}$$

where $r = 1$ if $B' = 0$. It only remains to fill the first column of F_1 with all but one 0.

We may assume using the division algorithm that the λ'_{i1} are polynomials divisible by p. Given this, we claim that $\lambda'_{i1} = 0$ for $i = 2,3,\ldots,m$. If not, let $\lambda'_{j1} = p^{k_j}\mu_{j1}$ with $(p, \mu_{j1}) = 1$. We have $j \geq 2$ and there are two cases to consider.

<u>Case 1</u>: The $(j-1)^{th}$ row of B' is 0. Then the j^{th} row of F_1 reads $(p^{k_j}\mu_{j1}, 0, 0, \ldots, 0)$. Also $(\lambda_{11}\mu_{j1}, 0, 0, \ldots, 0)$ is obviously a relation given by F_1. Hence Operation 3 is applicable

and $F_1 \sim F_2$ with j^{th} row of F_2 reading $(\mu_{j1},0,0,\ldots,0)$. Now $\deg_w \mu_{j1} \leq \deg \mu_{j1} < \deg_w \lambda_{11}$ contradicting the choice of λ . So $\lambda_{j1} = 0$ as claimed. Case 2: the $(j-1)^{th}$ row of $B' \neq 0$. In this case the j^{th} row of F_1 reads $(p^{k_j}\mu_{j1},0,0,\ldots,\lambda_{jj},0,\ldots,0)$. We may apply Operations A and 1 to cancel the p^{k_j} to get $F_1 \sim F_3$ where the j^{th} row of F_3 is $(\mu_{j1},0,\ldots,\lambda_{jj},0,\ldots,0)$ with $\deg_w \mu_{j1} < \deg_w \lambda_{11}$ leading to the same contradiction as before. So again $\lambda'_{j1} = 0$.

The result of all this is that

$$F \sim F_1 = \begin{pmatrix} \lambda_{11} & 0 & & 0 \\ 0 & \lambda_{22} & & 0 \\ & \cdots\cdots & & \\ 0 & & \lambda_{rr} & 0 \\ 0 & 0 & & 0 \end{pmatrix}$$

where $r \geq 1$. This means $M \sim \left(\bigoplus_{i=1}^{r} \Lambda/\lambda_{ii}\Lambda\right) \oplus \left(\bigoplus_{i=1}^{s} \Lambda/p^{m_i}\Lambda\right) \oplus \Lambda^{n-r}$ where s is the number of times an Operation 2 is performed. The λ_{ii} here are not necessarily irreducible. We use Lemma 13.8 of [5] to split the λ_{ii} into irreducible distinguished polynomials, thereby proving the theorem.

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DEMUSHKIN GROUPS AND HILBERT FIELDS

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Abstract. It is shown that field K of char(K) $\neq$ 2 with finite number of square classes is a Hilbert field if and only if the Galois group G_K of the maximal 2-extension K(2) over the field K is a Demushkin pro-2-group. There is one remark about connections with ordered fields.

§1. Introduction. In this paper we use the notation of [7], [9], [10]. All fields considered in this paper are of characteristic not 2. Let F be a field; $\dot{F}$ the multiplicative group of F ; T_F the intersection of all orderings of the field F ; $[\dot{F}:T_F]$ the group-index; V a valuation on F ; A_V the valuation ring corresponding to V ; U_V the group of units of A_V ; M_V the maximal ideal of A_V ; F_V the residue field of V (V is fully compatible with T_F iff $1+M_V \subset T_F$); $(X,\dot{F}/T_F)$ a space of orderings, where X means the set of all orderings P of the field F such that $T_F \subset P$. A field F is of type $(k,2^n)$ if $[\dot{F}:T_F] = 2^n$ and the number of orderings is k .

A pro-p-group G is said to be a Demushkin group if

(1) $\dim H^1(G,\mathbb{Z}/p\mathbb{Z}) < \infty$

(2) $\dim H^2(G,\mathbb{Z}/p\mathbb{Z}) = 1$

(3) the cup product $H^1(G,\mathbb{Z}/p\mathbb{Z}) \times H^1(G,\mathbb{Z}/p\mathbb{Z}) \to H^2(G,\mathbb{Z}/p\mathbb{Z})$ is a non-degenerate bilinear form.

Here $\dim H^i(G,\mathbb{Z}/p\mathbb{Z})$, i = 1,2,... means the dimension of the vector space $H^i(G,\mathbb{Z}/p\mathbb{Z})$ over field with p-elements.

Demushkin's groups appeared first in [3], as Galois groups of local fields. They were further investigated, e.g., in [5], [7], [15]. Note that in [7] a complete classification of Demushkin's groups was given.

On the other hand, in paper [4], Hilbert fields were introduced as a generalization of local fields. A field L is called a Hilbert field if $\dot{L} \neq \dot{L}^2$ and for every quadratic extension M of L the group index $[\dot{L}:\mathrm{ImN}_{M/L}]$ has value 2. Here $\mathrm{ImN}_{M/L}$ means the image of the norm map $N_{M/L}$. In the same paper, it was shown that:

A non-formally real field K, with $\dot{K} \neq \dot{K}^2$ is a Hilbert field if and only if

(2') There exists, to within isometry, one and only one anisotropic quaternary form.

This happens if and only if

(3') The map $b \longrightarrow (a,b)$ is a homomorphism $\dot{K} \longrightarrow \{\pm 1\}$, which for $a \notin \dot{K}^2$ is surjective. (a,b) here means the usual Hilbert symbol.

Furthermore, the formally real field $K(\dot{K} \neq \dot{K}^2)$, is a Hilbert field iff $T_K = \dot{K}^2$ and $|\dot{K}/\dot{K}^2| = 2$. Hence the formally real field K is a Hilbert field iff it is a Euclidean field.

Although the following theorem is rather immediate, it seems that was not explicitly observed.

<u>§2. Theorem 1. Let K be a field with $1 < |\dot{K}/\dot{K}^2| < \infty$. Then G_K is a Demushkin pro-2-group if and only if K is a Hilbert field.</u>

Proof. First suppose that K is a formally real field. Then $G = G_K$ contains an involution σ. Since $cd(\{1,\sigma\}) = \infty$ and $cd(U) \leq cd(G)$ for any closed subgroup U of G we get $cd(G_K) = \infty$ ([14]). On the other hand, it is well known that a Demushkin group L has either $cd(L) = 2$ or $L = \mathbb{Z}/2\mathbb{Z}$ ([5]). Hence we see that G is a Demushkin group if and only if $G = \mathbb{Z}/2\mathbb{Z}$. But from Satz 3 in [1], we find that this will happen exactly when K is a Euclidean field.

Suppose now that K is not a formally real field. By Kummer theory, we can identify $\dot{K}/\dot{K}^2$ with $H^1(G,\mathbb{Z}/2\mathbb{Z})$. Then the cup product $\phi: H^1(G,\mathbb{Z}/2\mathbb{Z}) \times H^1(G,\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(G,\mathbb{Z}/2\mathbb{Z})$ can be identified with a map

$$\dot{K}/\dot{K}^2 \times \dot{K}/\dot{K}^2 \longrightarrow Br_2(K)$$

$$(\bar{a},\bar{b}) \longrightarrow [(\frac{a,b}{F})] ,$$

where $Br_2(K)$ denotes the 2-torsion in the Brauer group $Br(K)$, a, $b \in K$, $\bar{a}$, $\bar{b}$ are classes in $\dot{K}/\dot{K}^2$ corresponding elements a, b and $[(\frac{a,b}{F})]$ is the class in $Br_2(K)$ corresponding to the quaternion algebra $(\frac{a,b}{F})$ [16].

From the theorem in [12] we know that classes of quaternion algebras generate $H^2(G,\mathbb{Z}/2\mathbb{Z})$. Hence the field is Hilbert if and only if $H^2(G,\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and the map ϕ is a nondegenerate bilinear form. This completes the proof.

Remark. The other immediate proof can be obtained from paper [5] and the original definition of Hilbert field.

Second Proof of the Theorem 1 in the case K is not a formally real field: According to the Theorem 1 in [5], one-relator finitely generated pro-2-group $G \neq \mathbb{Z}/2\mathbb{Z}$ is a Demushkin group if and only if for every open subgroup U of G of index 2, the number of generators $n(U)$, $n(G)$ of groups U,G respectively satisfy relation

$$n(U) = 2(n(G) - 1) . \tag{4}$$

But from the square-class exact sequence for quadratic extensions ((5.20) in [9]) we find that (4) is true if and only if for every quadratic extension M of K the group index $[K:\mathrm{Im}N_{M/K}]$ has value 2. Hence from relation (2') and Theorem in [12] we get our assertion.

Theorem 2. Let F be a formally real pythagorean field with $|\dot{F}/\dot{F}^2| < \infty$.

Then G_F or $G_{F(\sqrt{-1})}$ can be a Demushkin group only in two cases. F is a Euclidean field or F has type $(4,2^3)$. In the latter case, $G_{F(\sqrt{-1})} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is Demushkin group.

Proof. We have already proved that a formally real field F with G_F Demushkin must be a Euclidean field.

Suppose now that $G_{F(\sqrt{-1})}$ is a Demushkin group. Then, by theorem in [13], $2 = \mathrm{cd}\, G = \mathrm{cd}\, G_{F(\sqrt{-1})} = \mathrm{st}(F)$, where $\mathrm{st}(F)$ is the stability index of F. On the other hand, from Theorem 3.7 in [2] we find that if G is a Demushkin group, then the space X of orderings of the field F must be indecomposable in the Marshall's sense ([10], §3). Hence if $4 \leq |X|$, then there exists a valuation V on F fully compatible with T_F, such that

$$|\dot{F}/\dot{F}^2 U_V| = 2^h \neq 1 \quad \text{([11] Theorem 2.8)} .$$

Since $\mathrm{St}(F) = \mathrm{St}(F_V)+h$, and $\mathrm{St}(F) = 2$ we get $\mathrm{St}(F_V) = 0$ and $h = 2$ or $\mathrm{St}(F_V) = 1$ and $h = 1$.

In the first case, we find that F has type $(4,2^3)$. In the second case, F has type $(2n-2,2^n)$, $3 \leq n$. Hence from papers [17], [18] we get that

$$G_{F(\sqrt{-1})} = \mathbb{Z}_2 \times H$$

where H is a free pro-2-group with n-2 generators. Thus $\dim H^2(G,\mathbb{Z}/2\mathbb{Z}) = 1$ iff $n=3$. Therefore we again get that F has type $(4,2^3)$. The proof of our Theorem is finished.

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INTERSECTIONS D'ANNEAUX DE MORI - EXEMPLES

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Présenté par Paulo Ribenboim, M.S.R.C.

Tous les anneaux sont intègres commutatifs et unitaires. Un anneau A est de Mori ([7]) si ses idéaux divisoriels entiers satisfont à la condition des chaînes ascendantes.

Nous donnons une caractérisation d'un tel anneau comme intersection à caractère fini de localisés $A_{\mathfrak{p}}$, dont l'idéal maximal est divisoriel, généralisant ainsi la caractérisation des anneaux de Krull par les $A_{\mathfrak{p}}$ de valuation discrète, $\mathfrak{p} \in X^{(1)}(A)$.

Nous présentons ensuite une classe d'anneaux de Mori, non de Krull, non noethérien et non intégralement clos.

§ 1 - INTERSECTION D'ANNEAUX DE MORI

Définition 1-1

Soit $\{A_i\}_{i \in I}$ une famille d'anneaux inclus dans un anneau intègre B. Posons $A = \bigcap_I A_i$ et k le corps des fractions de A. On dit que cette famille vérifie la propriété de caractère fini (notée [F]) si, pour tout x non nul de A, l'ensemble $\{i \in I$ / x non inversible dans $A_i\}$ est fini.

Exemple : la famille $\{A_{\mathfrak{p}} / \mathfrak{p} \in X^{(1)}(A)\}$ pour un anneau de Krull A.

Théorème 1-2

Soit une famille $\{A_i\}_{i \in I}$ de sous-anneaux de Mori d'un anneau B, vérifiant [F] . Alors $A = \bigcap_I A_i$ est un anneau de Mori.
Si de plus les A_i sont des anneaux de Krull, A est un anneau de Krull.

Exemples 1-3

Toute intersection finie d'anneaux de Mori est de Mori. En particulier, soit un anneau de Mori A inclus dans un corps k. Si k' est un sous corps de k, alors $A' = A \cap k'$ est un anneau de Mori. Ainsi, toute retraction A [3] d'un anneau de Mori B est un anneau de Mori: un sous-anneau A de B, est une rétraction de B, s'il existe un idéal I de B tel que $B = A \oplus I$. Dans le cas intègre, si k est le corps des fractions de A, alors $A = k \cap B$.

§ 2 - REPRESENTATION D'ANNEAUX DE MORI COMME INTERSECTION A CARACTERE FINI D'ANNEAUX DE MORI LOCAUX

Soient A un anneau de Mori, k son corps des fractions, alors la famille $\mathcal{M}$ des idéaux divisoriels maximaux de A n'est pas vide [9].

Proposition 2-1

Si $\mathfrak{p} \in \mathcal{M}$, alors $\mathfrak{p}$ est un idéal premier de A.
De plus, il existe $x \in k$ tel que $p = A \cap xA = A:(A + x^{-1}A)$.

Proposition 2-2

Soit $\mathfrak{p} \in A \cap A\frac{u}{v}$, $\mathfrak{p} \in \mathcal{M}$. Si $v \notin \mathfrak{p}$, alors $A_{\mathfrak{p}}$ est un anneau de valuation discrète.

Théorème 2-3

Un anneau intègre A est un anneau de Mori si et seulement s'il existe une famille $\mathcal{P}$ d'idéaux premiers de A tels que

(i) pour tout $\mathfrak{p} \in \mathcal{S}$, l'anneau $A_{\mathfrak{p}}$ est de Mori dont l'idéal maximal est divisoriel

(ii) $A = \bigcap_{\mathcal{S}} A_{\mathfrak{p}}$

(iii) La famille $\{ A_{\mathfrak{p}}, \mathcal{S} \}$ est à caractère fini.

Remarque 2-4

Si A est un anneau de Mori, la famille $\{ A_{\mathfrak{p}}, \mathfrak{p} \in X^{(1)}(A) \}$ est aussi à caractère fini.

Si, de plus, A est un anneau de Krull, $\mathcal{M} = X^{(1)}(A)$ et on retrouve la caractérisation des anneaux de Krull par la famille à caractère fini $\{ A_{\mathfrak{p}}, X^{(1)}(A) \}$ d'anneaux de valuation discrète. Cela découle du lemme suivant :

Lemme 2-5

Dans un anneau de Mori, tout idéal premier non nul contient un idéal premier divisoriel non nul.

Remarque 2-6

Dans un anneau de Mori A notons

$\mathcal{M}_1 = \{ \mathfrak{p} \in \mathcal{M} / \exists\, v \notin \mathfrak{p}$ tel que $\mathfrak{p} = A \cap A \frac{u}{v} \}$.

Si $\mathcal{M}_1$ n'est pas vide, alors la sous-intersection $A' = \bigcap_{\mathcal{M}_1} A_{\mathfrak{p}}$ est un anneau de Krull, les $A_{\mathfrak{p}}$ pour $\mathfrak{p} \in \mathcal{M}_1$ étant des anneaux de valuation discrète (proposition 2-2).

Par ailleurs, si $\mathfrak{p} \in \mathcal{M} - \mathcal{M}_1$, pour tout $\frac{u}{v}$ tel que $\mathfrak{p} = A \cap A \frac{u}{v}$, alors u et $v \in \mathfrak{p}$. Ainsi pour tout $\mathfrak{p} \in \mathcal{M} - \mathcal{M}_1$ l'anneau $A_{\mathfrak{p}}$ n'est pas de valuation. La sous-intersection $A'' = \bigcap_{\mathcal{M} - \mathcal{M}_1} A_{\mathfrak{p}}$ est un anneau de Mori ([9]) sans être un anneau de Krull.

Remarque 2-7

La proposition 2-2 et le théorème 2-3 permettent d'en déduire qu'un anneau de Mori A, local d'idéal maximal $\mathfrak{m}$ est un anneau de valuation discrète si et seulement si il existe $v \notin \mathfrak{m}$ tel que $\mathfrak{m} = A \cap A\frac{u}{v}$.

§ 3 - EXEMPLES D'ANNEAUX DE MORI

On connait déjà ([1] ,[6]) les anneaux du type $A = k + \mathfrak{m}$ où, si V est un anneau de valuation discrète de la forme $V = K + \mathfrak{m}$, (K étant un corps et $\mathfrak{m}$ l'idéal maximal de V), k est un sous-corps de K.

Ce sont des anneaux de Mori, non noetheriens si $[K:k] = \infty$ et non intégralement clos si $\overline{k} \neq k$, où $\overline{k}$ est la fermeture intégrale de k dans K. De plus, la quasi-clôture intégrale de A est $A^* = V$([5])

Exemple 3-1

$A_1 = k + X(K[X])_X$ et $A_2 = k + X\,K[[X]]$

avec k corps fini et K clôture algébrique de k

ou encore, $k = \mathbb{Q}$ et $K = \mathbb{R}$, $\mathbb{C}$ ou $\overline{\mathbb{Q}}$ clôture algébrique de $\mathbb{Q}$.

Etude des anneaux du type $D = A + XB[X]$

où $A \subsetneq B$ sont deux anneaux intègres distincts, de corps de fractions $Frac(A) = k$, $Frac(B) = K$.

Pour un anneau A, on notera $U(A)$ l'ensemble des éléments inversibles de A.

Théorème 3-2

Si B est un anneau de Mori intégralement clos et si $k \cap B = A$, alors

(a) A est un anneau de Mori intégralement clos

(b) $D = A + XB[X]$ est un anneau de Mori

(c) D n'est pas un anneau de Krull

(d) la clôture intégrale de D est $\overline{D} = \overline{A} + XB[X]$ où $\overline{A}$ est la fermeture intégrale de A dans B

(e) Si B est un A-module libre ayant une base infinie, alors $D = A + XB[X]$ n'est pas un anneau noethérien.

Remarques

1) Si $A = \overline{A}$, D n'est jamais noethérien et D est un anneau de Mori intégralement clos.

2) Si $A \neq \overline{A}$ et si B est un A-module libre ayant une base infinie, nous obtenons alors une classe d'anneaux de Mori, non de Krull, non noethériens et non intégralement clos.

Corollaire 3-3

Soient A intégralement clos et $A \subsetneq B$, B étant entier sur A. Alors si B est un anneau de Mori intégralement clos, $D = A + XB[X]$ est un anneau de Mori. De plus $\overline{D} = B[X]$.

Corollaire 3-4

Soient $A \subsetneq B$, A étant une rétraction de B. Alors si B est un anneau de Mori intégralement clos, il en est de même de $D = A + XB[X]$

Par exemple si A est un anneau de Mori intégralement clos et si $\{X_i, 1 \leq i \leq n\}$ est une famille finie d'indéterminées sur A, l'anneau $D = A + X_i A[X_1, \ldots, X_n]$ est un anneau de Mori intégralement clos.

Considérons à présent le cas particulier où $B = K$. Il peut se généraliser par le résultat suivant:

Théorème 3-5

Soient A un sous-anneau d'un corps K, $k = \mathrm{Frac}(A)$ et r un entier non nul.

Alors $D = A + X^r K[X]$ est un anneau de Mori si et seulement si $A = k$.

De plus a) si $r = 1$ et si la fermeture intégrale $\overline{A}$ de A dans K est distincte de A, ou si $r \geq 2$ alors D n'est pas intégralement clos.

b) si $[K : k] = \infty$, D n'est pas un anneau noethérien.

Remarque 3-6

Réciproquement, étant donnés deux anneaux $A \subsetneq B$, si l'anneau $D = A + XB[X]$ est de Mori, alors A et B sont des anneaux de Mori et $U(B) \cap A = U(A)$.

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Note on a result of Puig-Ribenboim

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Presented by G.deB. Robinson, F.R.S.C.

The purpose of this note is to observe that proposition 2 in [2], while correct, can be sharpened. We believe that the result we expose here is interesting because it "minimizes" the local information needed to reconstruct a finite group, using only the tame intersections of the Sylow subgroups of G.

We say as in [1], that $P \cap Q$ is a tame intersection where P and Q are p-Sylow subgroups of G, if $N_P(P \cap Q)$, $N_Q(P \cap Q)$ are each p-Sylow subgroups of $N_G(P \cap Q)$.

The result is:

Proposition: <u>If G is a finite group and P is the active family of the tame intersections of the Sylow subgroups of G then G is isomorphic to the active sum of P (where the family is partially ordered by inclusion and the action is given by conjugation).</u>

By the theorem 7.4.1 in [1], the only thing to be noted is that if part s) of the proof of proposition 2 in [2] (p. 173) is substituted by

"1)

F_1: The subgroup of P generated by $\cup [N_G(H), H]$ where H ranges over the set of all tame intersections $H = P \cap Q$, where Q is a p-Sylow subgroup of G"

then the above Proposition also follows.

On the other hand, it is important to mention that although in [2] it is stated that proposition 4 (a result of Tomás, [3]) is immediate from what is done there, however Tomás does not need as data the inclusions of the normal hulls of Sylow subgroups of G.

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APPROXIMATELY INNER AUTOMORPHISMS OF THE IRRATIONAL ROTATION ALGEBRA

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Presented by G. A. Elliott, F.R.S.C.

Abstract . Automorphisms of the irrational rotation algebra A_α leaving invariant a canonical subalgebra isomorphic to $C(\mathbf{T})$ are shown to be approximately inner if they act trivially on $K_1(A_\alpha)$. This is accomplished by showing that a certain coboundary from the unitary group of $C(\mathbf{T})$ into the connected component of the identity of this group has dense range.

For irrational $\alpha \in [0,1]$, consider the automorphism group $\mathrm{Aut}(A_\alpha)$ of the irrational rotation algebra A_α. This algebra is the simple C^*-algebra generated by two unitary operators U and V satisfying $UV = e^{2\pi i\alpha}VU$. There is a group homomorphism $g \to \beta_g$ from $SL(2,\mathbf{Z})$ to $\mathrm{Aut}(A_\alpha)$ with the following property ([1]). If γ_* ($\gamma \in \mathrm{Aut}(A_\alpha)$) denotes the induced homomorphism on $K_1(A_\alpha) \cong \mathbf{Z} \oplus \mathbf{Z}$ ([2]). then $(\beta_g)_* = g$. With the topology of pointwise norm convergence on $\mathrm{Aut}(A_\alpha)$ the subgroup $\{\gamma \mid \gamma_* = \mathrm{id}\}$ is closed and contains the closed subgroup of approximately inner automorphisms of A_α. It is not known if this inclusion is strict. This paper shows that for a certain set of automorphisms, the property of being approximately inner coincides with acting trivially on K-theory.

For $X \in A_\alpha$, denote by $C^*(X)$ the closed $*$ subalgebra of A_α generated by X. We identify $C^*(V)$ with the C^*-algebra of continuous functions on the circle, $C(\mathbf{T})$, in such a way that the element V corresponds to the identity map. For a unitary W in A_α , ad W is the inner automorphism $X \to WXW^*$ of A_α. Write $E : A_\alpha \to C^*(V)$ for the canonical expectation and for any $W \in A_\alpha$ write $W = \sum_{n\in\mathbf{Z}} f_n u^n$ where $f_n = E(WU^{-n})$. For $s \in \mathbf{R}$ let r_s denote the homomorphism of $C(\mathbf{T})$ defined by

$$r_s(f)(e^{2\pi i\theta}) = f(e^{2\pi i(\theta+s)}).$$

Proposition 1. If $\beta \in \mathrm{Aut}(A_\alpha)$ and $\beta(C^*(V)) \subseteq C^*(V)$ then there is a $\lambda \in \mathbf{T}$ and an $f \in C^*(V)$ with either $\beta(V) = \lambda V$ and $\beta(U) = fU$, or $\beta(V) = \lambda V^*$ and $\beta(U) = fU^*$. Hence $\det \beta_* = 1$.

Proof. The subalgebra $C^*(V)$ is maximal abelian in A_α so $\beta(C^*(V)) = C^*(V)$. The unitary $\beta(V)$ thus generates $C^*(V)$ as a C^*-algebra. Also

$$(\mathrm{ad}\, \beta(U))\beta(V) = e^{2\pi i\alpha}\beta(V) \tag{1}$$

and so the automorphism $\mathrm{ad}\beta(U)$ restricts to an automorphism of $C^*(V)$. In particular $(\mathrm{ad}\, \beta(U))V \in C^*(V)$, i.e., $\beta(U)V = g\beta(U)$ with $g \in C^*(V)$. Setting $\beta(U) = \sum f_n U^n$ with $f_n \in C^*(V)$ we have $(e^{2\pi i n\alpha}V - g)f_n = 0$ $(n \in \mathbf{Z})$. The irrationality of α implies that $e^{2\pi i n\alpha} \neq e^{2\pi i m\alpha}$ $(n \neq m)$ so there is at most one n for each $s \in \mathbf{T}$ with $f_n(s) \neq 0$. However, $\sum |f_n|^2 = 1$ a.e., each f_n is continuous, and $\mathbf{T}$ is connected; hence $\beta(U) = f_m U^m$ for some $m \in \mathbf{Z}$. By equation (1), $m = \pm 1$. Furthermore if $m = 1$ then $\beta(V) = \lambda V$ and if $m = -1$ then $\beta(V) = \lambda V^*$, for some $\lambda \in \mathbf{T}$. □

Corollary 2. If $\beta \in \mathrm{Aut}(A_\alpha)$ and $\beta(C^*(V)) \subseteq C^*(V)$ then $\beta_* = \mathrm{id}$ if and only if β is homotopic to the identity automorphism.

Proof. If $\beta_* = \mathrm{id}$, Proposition 1 implies that there is a $\lambda \in \mathbf{T}$ and a unitary $f \in C^*(V)$ with $\beta(V) = \lambda V$ and $\beta(U) = fU$. The equivalence class of f in $K_1(A_\alpha)$ is that of the unit. Since $K_1(C^*(V))$ is embedded in $K_1(A_\alpha)$ ([2]) the class of f in $K_1(C^*(V))$ is also that of the unit. However the class of f in $K_1(C^*(V))$ is the winding number of f; thus f is homotopic (via a path of unitaries in $C^*(V)$) to the constant function 1. This provides a homotopy of β with the identity automorphism of A_α. The reverse implication holds for any automorphism of a C^*-algebra. □

Lemma 3. If $\beta \in \mathrm{Aut}(A_\alpha)$ is given by $\beta(V) = \lambda V$, $\beta(U) = fU$ for some $\lambda \in \mathbf{T}$ and $f \in C^*(V)$ then β is inner if and only if $\lambda \in \{e^{2\pi i n\alpha} \mid n \in \mathbf{Z}\}$ and $f = r_\alpha(w)w^*$ for some unitary $w \in C^*(V)$.

Proof. If $\beta = \mathrm{ad}\, W$ for a unitary $W = \sum f_n U^n$ of A_α then $\beta(V) = \lambda V$ implies $(e^{2\pi i n\alpha} - \lambda)f_n = 0$ $(n \in \mathbf{Z})$. Thus there is at most one n with $f_n \neq 0$ and $W = f_n U^n$. It follows that $f = r_\alpha(f_n^*)f_n$. Conversely, if $f = r_\alpha(w)w^*$ and $\lambda = e^{2\pi i n\alpha}$, set $w^* U^n = W$. $\square$

The group $C(\mathbf{T})^u$ of unitaries in $C(\mathbf{T})$ equipped with the norm topology is a topological group. The connected component of the unit, $C(\mathbf{T})_0^u$, is an open subgroup of $C(\mathbf{T})^u$ equal to $\exp(C(\mathbf{T})^u)$ and consists of all continuous functions $f : \mathbf{T} \to \mathbf{T}$ homotopic to 1. For fixed $\gamma \in [0,1]$, $w \in C(\mathbf{T})^u$, the path $s \to r_{s\gamma}(w)w^*$ $(s \in [0,1])$ joins 1 to $r_\gamma(w)w^*$. The group homomorphism $\Gamma : C(\mathbf{T})^u \to C(\mathbf{T})_0^u$ defined by $\Gamma(w) = r_\gamma(w)w^*$ has range strictly contained in $C(\mathbf{T})_0^u$ (as $\{e^{2\pi i n\gamma} \mid n \in \mathbf{Z}\}$ are the only constants in range Γ). If γ is rational, range Γ has closure strictly contained in $C(\mathbf{T})_0^u$, for if $n \in \mathbf{N}$ is such that $e^{2\pi i n\gamma} = 1$ then

$$r_\gamma^{n-1}(f)r_\gamma^{n-2}(f) \cdots r_\gamma(f)f = 1$$

for each $f \in \overline{\mathrm{range}\,\Gamma}$ and so $e^{2\pi i\theta} \notin \overline{\mathrm{range}\,\Gamma}$ for θ irrational. For $n \in \mathbf{Z}$, $e^{2\pi i n\gamma} = \Gamma(V^n) \in \mathrm{range}\,\Gamma$ and thus $\mathbf{T} \subseteq \overline{\mathrm{range}\,\Gamma}$ if γ is irrational. For $s \in \mathbf{R}$, $r_s(f) \in \mathrm{range}\,\Gamma$ if $f \in \mathrm{range}\,\Gamma$ and $r_{n\gamma}(w)w^* \in \mathrm{range}\,\Gamma$ for any $w \in C(\mathbf{T})^u$. These properties of range Γ assist in constructing 'bump' functions h in range Γ which will be used to show that range Γ is dense in $C(\mathbf{T})_0^u$ if γ is irrational.

For $\theta \in \mathbf{R}$, $n \in \mathbf{N}$ with $n\alpha \bmod \mathbf{Z} = \overline{n\alpha} \in [0,1/2]$, and ϵ positive with $\epsilon < \overline{n\alpha}$ define a continuous function $f : [0,1] \to \mathbf{R}$ by

$$f(s) = \begin{cases} \theta s\,(1-\overline{n\alpha})^{-1} + \theta & s\in[0,\overline{n\alpha}-\epsilon] \\ \theta(1-\epsilon)(\overline{n\alpha}-s)(\epsilon(1-\overline{n\alpha}))^{-1} & s\in[\overline{n\alpha}-\epsilon,\ \overline{n\alpha}] \\ \theta(s-\overline{n\alpha})(1-\overline{n\alpha})^{-1} & s\in[\overline{n\alpha},1] \end{cases}$$

Setting $e^{2\pi i f(s)} = w(e^{2\pi is})$ $(s \in [0,1])$ and $r_{n\alpha}(w)w^* = h$ we have $h \in \text{range}\,\Gamma$ and $h(e^{2\pi is}) = e^{2\pi i g(s)}$ with $g : [0,1] \to \mathbf{R}$ defined by

$$g(s) = \begin{cases} -\theta & s\in[0,\overline{n\alpha}-\epsilon] \\ \theta(\epsilon-1)(\overline{n\alpha}+s)(\epsilon(1-\overline{n\alpha}))^{-1} & s\in[\overline{n\alpha}-\epsilon,\overline{n\alpha}] \\ \theta\,\overline{n\alpha}(1-n\alpha)^{-1} & s\in[\overline{n\alpha},1-\epsilon,] \\ \theta(1-s)(\epsilon(1-\overline{n\alpha}))^{-1}-\theta & s\in[1-\epsilon,1] \end{cases}.$$

The function g is continous (in fact f may be redefined on $[\overline{n\alpha}-\epsilon,\ \overline{n\alpha}]$ so that g is C^∞) and has constant values on the intervals $[0,n\alpha-\epsilon]$, $[\overline{n\alpha},1-\epsilon]$ that differ in value by $\theta(1-n\alpha)^{-1}$.

For $\varsigma\in\mathbf{R}$, add a suitable constant to the function g (this is equivalent to multiplying h by a constant $\lambda\in\mathbf{T}\subseteq\overline{\text{range}\,\Gamma}$) to obtain a function $k : [0,1] \to \mathbf{R}$ with

$$k\,|_{[0,\overline{n\alpha}-\epsilon]} = \varsigma,\quad k\,|_{[\overline{n\alpha},1-\epsilon]} = \varsigma + \theta(1-n\alpha)^{-1}$$

and $e^{2\pi ik(s)} \in \overline{\text{range}\,\Gamma}$. Hence, for any open intervals O_1 and O_2 in $[0,1]$ with $\overline{O}_2 \subseteq O_1$ and $M\in\mathbf{R}$ one can, with suitable $\varsigma,\theta,n,\epsilon$ and an appropriate translation, obtain k satisfying

$$k\,|_{\overline{O}_2} = M,\quad k\,|_{[0,1]\setminus O_1} = 0 \quad\text{and}\quad e^{2\pi ik(s)}\in\overline{\text{range}\,\Gamma}.$$

Theorem 4. <u>The subgroup</u> $\{r_\alpha(w)w^* \mid w\in C(\mathbf{T})^u\}$ <u>is dense in</u> $C(\mathbf{T})^u_0$ <u>if and only if</u> α <u>is irrational.</u>

Proof. It remains to show that if α is irrational then $\overline{\text{range}\,\Gamma} = C(\mathbf{T})^u_0$. Fix $f \in C(\mathbf{T})^u_0$, $f(e^{2\pi is}) = e^{2\pi il(s)}$ where $l : [0,1] \to \mathbf{R}$ is continuous and $l(0) = l(1) =$ $(= 0$ without loss of generality, as $\mathbf{T} \subset \overline{\text{range}\,\Gamma})$. Choose $\epsilon_1>0$. There is an M such that $|l(s)|<M$ $(s\in[0,1])$. We shall show below that there is a continuous function $g : [0,1] \to \mathbf{R}$ with $g(0) = g(1)$, $e^{2\pi ig} \in \overline{\text{range}\,\Gamma}$, and

$\| l - g\| \leq M/2 + \epsilon_1$. Repeating this procedure with the function $l-g$ in place of l, and so on, we find at the m^{th} stage a continuous function $g_m : [0,1] \to \mathbf{R}$ with $g_m(0) = g_m(1)$, $e^{2\pi i g_m} \in \overline{\text{range}\,\Gamma}$ and

$$\| l - g_m\| \leq M/2^m + \epsilon_1/2^{m-1} + \cdots + \epsilon_m.$$

Choosing ϵ_j appropriately ($\epsilon_j = 2^{-j}$ for example) we conclude that $f \in \overline{\text{range}\,\Gamma}$.

Write the open set $P = \{s \mid |l(s)| > M/2\}$ as a disjoint union of open intervals P_j. By uniform continuity of l there is an N with

$$|l(s)| \leq M/2 + \epsilon_1 \quad \text{for } s \in \bigcup_{j>N} P_j \cup ([0,1] \setminus P).$$

Consider the closed intervals $\overline{P}_j$ $(j \leq N)$. Coalescing those with nonempty intersection, we arrive at a finite set of disjoint closed intervals $\{Q_j \mid j \leq N_0\}$ where $\overset{N}{\cup} \overline{P}_j = \overset{N_0}{\cup} Q_j$, $N_0 \leq N$ and $|l(s)| \geq M/2$ $(s \in Q_j)$. For each $j \leq N_0$ choose an open interval $B_j \supseteq Q_j$ such that l does not vary by more than ϵ_1 on $B_j \setminus Q_j$. Do this in such a way that $\overline{B}_j \cap \overline{B}_m = \emptyset$ if $j,m \leq N_0$ and $j \neq m$. Now for each $j \leq N_0$ choose $k_j : [0,1] \to \mathbf{R}$ continuous with $k_j|_{[0,1]\setminus B_j} = 0$, $k_j|_{Q_j} = M/2$ and $e^{2\pi i k_j} \in \overline{\text{range}\,\Gamma}$. Set

$$\sum_{j=1}^{N_0} \pm\ k_j = g. \ \square$$

Denote the closed subgroup $\{\gamma \in \text{Aut}(A_\alpha) \mid \gamma(C^*(V)) \subseteq C^*(V)\}$ by C.

Corollary 5. If $\beta \in C$ then β is approximately inner if and only if $\beta_* = \text{id}$.

Proof. If $\beta_* = \text{id}$ and $\beta \in C$ then by the proof of Corollary 2 there is a $\lambda \in \mathbf{T}$ and $f \in C^*(V)$ with $\beta(V) = \lambda V$ and $\beta(U) = fU$. By Theorem 4 there is a sequence $w_n \in C^*(V)$ with $\Gamma(w_n) \to f$ as $n \to \infty$. Choose a sequence $(p_n)_{n \in N}$ in $\mathbf{Z}$ with $e^{2\pi i p_n} \to \lambda$ and set $w_n^* U^{p_n} = W_n$. The inner automorphisms ad W_n converge to β pointwise in norm (cf. proof of Lemma 3). $\square$

Corollary 6. If $\beta \in \bigcup \{\gamma^{-1}C\gamma \mid \gamma \in \mathrm{Aut}(A_\alpha)\}$ then $\beta_* = \mathrm{id}$ if and only if β is approximately inner.

Proposition 7. If $\delta_1, \delta_2 \in C$, $\gamma_1, \gamma_2 \in \mathrm{Aut}(A_\alpha)$ and $\beta_j = \gamma_j^{-1}\delta_j\gamma_j$ $(j = 1,2)$ with either $(\gamma_2\gamma_1^{-1})_* \notin \left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbf{Z}\right\}$ or $\gamma_2\gamma_1^{-1} \in C$ then $(\beta_1\beta_2)_* = \mathrm{id}$ if and only if $\beta_1\beta_2$ is approximately inner.

Proof. It is sufficient to show that $(\gamma\delta_1\gamma^{-1}\delta_2)_* = \mathrm{id}$ if and only if $\gamma\delta_1\gamma^{-1}\delta_2$ is approximately inner, where $\gamma = \gamma_2\gamma_1^{-1}$. If $\gamma \notin \left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbf{Z}\right\}$, then $(\gamma\delta_1\gamma^{-1}\delta_2)_* = \mathrm{id}$ implies $(\delta_1)_* = (\delta_2)_* = \mathrm{id}$, whence by Corollary 5, δ_1 and δ_2, and therefore also $\gamma\delta_1\gamma^{-1}\delta_2$ are approximately inner. Otherwise $\gamma \in C$ and the result follows directly from Corollary 5. ☐

Let $\delta_1, \delta_2 \in C$ and $g, h \in SL(2,\mathbf{Z})$, and set $\beta = (\beta_g^{-1}\delta_1\beta_g)(\beta_h^{-1}\delta_2\beta_h)$. By Proposition 7, $\beta_* = \mathrm{id}$ if and only if β is approximately inner.

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ON CONVEX SPACE CURVES

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A differentiable convex space curve Γ in three-space meets every plane in a finite number of points, every line in at most two points and lies on the boundary of its convex hull.

The Four-vertex theorem ([2], p. 549) for such curves states that a simple, elementary Γ, with only inflections as singular points, has at least four inflections. In this note, we characterize elementary convex space curves and present a generalization of the Four-vertex theorem.

1. Differentiable curves

Let P^n, $n \geq 2$, denote a real projective n-space with the standard topology and points $p, q, \ldots$; lines $L, M, \ldots$ and hyperplanes $\alpha, \beta, \ldots$ Let $T \subset P^n$ be an oriented line. For $s \neq t$ in T, let $[s,t]$ be the closed line segment from s to t. If $U(r) = (s,t) = [s,t] \setminus \{s,t\}$ is a neighbourhood of r in T, we put $U^-(r) = (s,r)$ and $U^+(r) = (r,t)$.

A curve in P^n is a continuous map $\Gamma: T \longrightarrow P^n$. The curve Γ shall be differentiable in the following sense. For $t \in T$, let $\Gamma_{-1}(t) = \phi$ and $\Gamma_0(t) = \Gamma(t)$. If $\Gamma_{k-1}(t)$ is already defined and its existence postulated then we require that for $s \in T\setminus\{t\}$ sufficiently close to t, the flat $\langle\Gamma_{k-1}(t), \Gamma(s)\rangle$ spanned by $\Gamma_{k-1}(t)$ and $\Gamma(s)$ has dimension k and it converges as s tends to t. Its limit is the osculating k-flat $\Gamma_k(t)$; $k = 1, \ldots, n$. If $\Lambda \subset T$ is a segment, we call $\Gamma|\Lambda$ a subarc of Γ. For convenience, we identify $\Gamma(T)$ with Γ and $\Gamma(\Lambda)$ with $\Gamma|\Lambda$.

Let $t \in T$ and $\Gamma(t) \in \alpha$. Then α supports [cuts] Γ at t, if locally at t, Γ lies [does not lie] on one side of α. Let

$$S_k(t) = \{\alpha | \alpha \cap \Gamma_{k+1}(t) = \Gamma_k(t)\}, \quad k = 0,1,\ldots,n-1 .$$

Then ([3], 1.4.2) either all $\alpha \in S_k(t)$ support Γ at t or all $\alpha \in S_k(t)$ cut Γ at t, and we choose $a_k(t) \in \{0,1\}$ so that $a_0(t) + \ldots + a_k(t)$ is even if and only if $\alpha \in S_k(t)$ supports Γ at t. We set $\Gamma(t) \equiv (a_0(t),\ldots,a_{n-1}(t))$ and call it the characteristic of $\Gamma(t)$. Then t is a regular [an inflection] point if $\Gamma(t) \equiv (1,\ldots,1,1)$ [$\Gamma(t) \equiv (1,\ldots,1,2)$]. A subarc of Γ is regular [inflectional] if each of its points is a regular [regular or inflection] point.

A subarc $\Gamma(\Lambda)$ has order m (ord $\Gamma(\Lambda) = m$) if m is the maximum number of points of $\Gamma(\Lambda)$ on any hyperplane. The order of a point $\Gamma(t)$, ord $\Gamma(t)$, is the minimum order that a neighbourhood of $\Gamma(t)$ may possess. Clearly ord $\Gamma(t) \geq n$, and $\Gamma(t)$ is ordinary [singular] if ord $\Gamma(t) = n$[$> n$]. In addition, $\Gamma(t)$ is elementary if there exist $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$, both of order n. A subarc of Γ is ordinary [elementary] if each of its points is ordinary [elementary]. We note that an ordinary point is regular and thus non-regular points are singular.

A point $\Gamma(t)$ is simple if $\Gamma(t) \neq \Gamma(s)$ for $s \in T\setminus\{t\}$ and a subarc $\Gamma(\Lambda)$ is simple if $\Gamma|\Lambda$ is injective. Finally, Γ is even [odd] if any α cuts Γ at an even [odd] number of points. Since Γ is closed in P^n, Γ is trivially even or odd.

2. Space Curves

Let Γ: $T \longrightarrow P^3$ be a (differentiable) space curve and let $\Gamma(\Lambda)$ be a simple, regular subarc. Then $\Gamma(\Lambda)$ is a Barner subarc if there exists a set $\mathcal{P}$ of planes in P^3, continuously dependent on the points of Λ, such that $\mathcal{P} \cap \{\Gamma_2(t)|t \in \Lambda\} = \phi$, there is a unique $\alpha \in \mathcal{P}$ through each tangent of $\Gamma(\Lambda)$ such that $|\alpha \cap \Gamma(\Lambda)| = 1$ and there is a unique $\tau \in \mathcal{P}$ through each pair of distinct points of $\Gamma(\Lambda)$ such that $|\tau \cap \Gamma(\Lambda)| = 2$.

1. If $\Gamma(M)$ is a simple, regular Barner subarc such that Γ_1 is continuous on M then ord $\Gamma(M) = 3$. ([3], 8.5.2).

2. If Γ is elementary then a regular point is ordinary and

$$x^*(\Gamma) = \sum_{t\in T} \left[\sum_{i=0}^{2} (3 - i)(a_i(t) - 1)\right]$$

is even if and only if Γ is even. ([3], 5.2.3 and 7.4.2).

Let b be a point and β be a plane in P^3. For $t \in T$, let

3. $$\Gamma_i^b(t) = \begin{cases} \langle b,\Gamma_i(t)\rangle \cap \beta & \text{if } b \notin \Gamma_i(t) \\ \Gamma_{i+1}(t) \cap \beta & \text{if } b \in \Gamma_i(t);\ i = 0,1 . \end{cases}$$

Then ([3], 1.3.2) the map Γ^b: $T \longrightarrow \beta$ such that $\Gamma^b(t) = \Gamma_0^b(t)$, $t \in T$, is a differentiable plane curve with the tangent $\Gamma_1^b(t)$ at t. We call Γ^b <u>the projection of Γ from b on β</u>.

4. If $\Gamma(t)$ is regular and $b \notin \Gamma_2(t)$ then $\Gamma^b(t)$ is regular. ([3], 5.2.4)

5. Every simple, regular subarc of Γ^b is ordinary. ([3], 9.2.3).

6. If $\Gamma^b(s,t)$ is regular and simple and there is a line through $\Gamma^b(s)$ and $\Gamma^b(t)$ not meeting $\Gamma^b(s,t)$, then ord $\Gamma^b(s,t) = 2$. ([1], 3.13).

3. Convex Space Curves

Let Γ: $T \longrightarrow P^3\setminus\beta$ be a space curve, $\beta \subset P^3$. Then Γ is a bounded curve and we denote by $H(\Gamma)$, the convex hull of Γ in the affine space $P^3\setminus\beta$. We say that Γ is <u>convex</u> if Γ lies on the boundary $\partial H(\Gamma)$ of $H(\Gamma)$ and $|L \cap \Gamma| \le 2$ for any line L. Finally, let $x(\Gamma)$ denote the number of singular points of Γ and let T' be the set of all $t \in T$ such that there exist $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$, both simple.

THEOREM. Let Γ: T $\longrightarrow P^3\backslash\beta$ be a convex space curve. Then Γ is elementary if and only if T' = T, $x(\Gamma) < \infty$ and Γ_i is continuous; i = 1,2.

PROOF. We first note that $\Gamma \subset P^3$ is compact and that a subarc of order three is both regular and simple.

Let Γ be elementary and t $\in$ T. Then the existence of $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$, both of order three, yields that T' = T and that t is not an accumulation point of singular points in T\{t}. Thus $x(\Gamma) < \infty$. From [3], 3.4.2; Γ_1 and Γ_2 are continuous.

Let T' = T, $x(\Gamma) < \infty$ and Γ_i be continuous; i = 1,2. Let t $\in$ T. Clearly, there is a plane α such that $\Gamma(t) \notin \alpha$ and $\alpha \cap \text{int } H(\Gamma) \neq \emptyset$, and thus there is an U(t) such that $\alpha \cap \Gamma(U(t)) = \emptyset$. Since T' = T and $x(\Gamma) < \infty$, we may assume that $U(t) = U^-(t) \cup \{t\} \cup U^+(t)$ where

i) both $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$ are simple and regular.

Next, $\alpha \neq \Gamma_2(t)$ and $\alpha \cap \text{int } H(\Gamma) \neq \emptyset$ yield that there is a $b \in \alpha \cap \text{int } H(\Gamma)$ such that $b \notin \Gamma_2(t)$. By the continuity of Γ_2, we may assume that

ii) $b \notin \Gamma_2(s)$ for $s \in U(t)$.

Let Γ^b be the projection of Γ from b on β. Then $b \in \text{int } H(\Gamma)$, $\Gamma \subset \partial H(\Gamma)$ and 3. imply that $\Gamma^b(r) = \Gamma^b(s)$ for $r \neq s$ in T only if $\Gamma(r) = \Gamma(s)$ or the line $\langle\Gamma(r),\Gamma(s)\rangle$ passes through b. Since $\alpha \cap \Gamma(U(t)) = \emptyset$ implies that $\alpha \cap H(\Gamma(U(t)) = \emptyset$ and thus $b \notin \langle\Gamma(r),\Gamma(s)\rangle$, i) yields that $\Gamma^b(U^-(t))$ and $\Gamma^b(U^+(t))$ are both simple. Next, we note that $\Gamma^b(U^-(t))$ and $\Gamma^b(U^+(t))$ are both regular by i), ii) and 4.

Since $\Gamma^b(U^-(t))$ is simple and $|M \cap \Gamma^b(U^-(t))| < \infty$ for any line M, there is an $\bar{s} \in U^-(t)$ such that $L = \langle\Gamma^b(\bar{s}),\Gamma^b(t)\rangle$ is a line and $L \cap \Gamma^b(\bar{s},t) = \emptyset$. Then ord $\Gamma^b(\bar{s},t) = 2$ by 6. As $b \notin \Gamma_1(s)$ for $s \in U(t)$; 3. yields that

$$\Gamma_1^b(s) = \langle b,\Gamma_1(s)\rangle \cap \beta \text{ and } \langle\Gamma^b(s),\Gamma^b(r)\rangle = \langle b,\Gamma(s),\Gamma(r)\rangle \cap \beta$$

for $s \neq r$ in U(t). As ord $\Gamma^b(\bar{s},t) = 2$ implies that

$$|\Gamma_1^b(s) \cap \Gamma^b(\bar{s},t)| = 1 \text{ and } |\langle\Gamma^b(s),\Gamma^b(r)\rangle \cap \Gamma^b(\bar{s},t)| = 2$$

for $s \neq r$ in $(\bar{s},t)$, we obtain that

$$|\langle b,\Gamma_1(s)\rangle \cap \Gamma(\bar{s},t)| = 1 \text{ and } |\langle b,\Gamma(r),\Gamma(s)\rangle \cap \Gamma(s,t)| = 2.$$

Thus $\Gamma(\bar{s},t) \subset \Gamma(U^-(t))$ is a simple, regular Barner subarc with

$$\mathcal{P} = \{\langle b,\Gamma_1(s)\rangle | s \in (\bar{s},t)\} \cup \{\langle b,\Gamma(s),\Gamma(t)\rangle | s \neq r \text{ in } (\bar{s},t)\}.$$

As Γ_1 is continuous, ord $\Gamma(\bar{s},t) = 3$ by 1.

Similarly, there is an $\bar{r} \in U^+(t)$ such that ord $\Gamma(t,\bar{r}) = 3$ and therefore $\Gamma(t)$ is elementary. □

Let Γ be an elementary convex space curve. It is easy to check from the proof of [2], 24. that even without the assumption that Γ is inflectional, we have that

7. If $\Gamma(r,s)$ is simple and regular then $|\Gamma_2(t) \cap \Gamma[r,s]| = 1$ for $t \in [r,s]$.

Assuming that Γ is not inflectional; $\Gamma \subset \partial$ H(Γ) readily yields that Γ may only possess x_1, x_2, x_3, x_4 and x_5 number of singular points of the respective characteristic (1,1,2), (1,2,1), (2,1,1), (2,2,1) and (2,2,2). As Γ is even, we obtain from 7. that $\Gamma(T\backslash\{t\})$ is not regular for any $t \in T$, and if $\Gamma(t_1,t_2) \cup \Gamma(t_2,t_1)$ is regular for $t_1 \neq t_2$ in T then $\Gamma_2(t_j)$ supports Γ at t_j; $j = 1,2$. Thus $x(\Gamma) = x_1 + x_2 + x_3 + x_4 + x_5 \geq 2$ and if $x(\Gamma) = 2$ then $x_4 = 0$.

GENERAL FOUR-VERTEX THEOREM. <u>Let</u> Γ <u>be a simple convex space curve with continuous tangents and osculating planes</u>. <u>Then</u>

$$\tilde{x}(\Gamma) = x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 \geq 4.$$

PROOF. As we may assume that $x(\Gamma) < \infty$, we have that Γ is an even, elementary curve. Thus $x^*(\Gamma) = x_1 + 2x_2 + 3x_3 + 5x_4 + 6x_5$ is an even number by 2. and the preceding comments. As $\tilde{x}(\Gamma) \equiv x^*(\Gamma)$ (mod 2), $x(\Gamma) \geq 2$ and $x_4 = 0$ when $x(\Gamma) = 2$; the Four-vertex theorem yields that $\tilde{x}(\Gamma) \geq 4$. □

We observe that the coefficient 3 of x_3 in $\tilde{x}(\Gamma)$ is necessary for in Euclidean three-space,

$$\Gamma(t) = (4\sin(t/2)\cos t, 4\sin(t/2)\sin t, 4\sin(t/2)) \ , \quad t \in [0,\pi],$$

defines a simple, elementary convex space curve Γ with $\tilde{x}(\Gamma) = x_1 + 3x_3 = 4$, $\Gamma(0) \equiv (2,1,1)$ and $\Gamma(\pi) \equiv (1,1,2)$.

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LEGENDRE TRANSFORMATIONS AND CANONICAL DISTRIBUTIONS IN THE CALCULUS OF VARIATIONS OF MULTIPLE INTEGRALS

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Presented by J. Aczél, F.R.S.C.

Abstract: The field theoretic approach to the calculus of variations as initiated by Carathéodory [1] depends crucially on the assumption that the Lagrangian is everywhere positive, which excludes important classes of physical field theories from such treatment. An alternative canonical field theory that does not depend on the aforementioned hypothesis is proposed.

1. Analytical Preliminaries: The arena of our analysis is a product manifold $M = M_n \times M_N$, where M_n and M_N are differentiable manifolds of dimension n and N respectively. The n independent variables x^j of our variational problem are regarded as local coordinates on M_n, while the N field variables ψ^A represent local coordinates on M_N. [Upper and lower case Latin indices range from 1 to N and from 1 to n respectively; the summation convention applies to both sets.]

A set of $n + N$ smooth 1-forms $\{\pi^j, \pi^A\}$ is supposed to be given on M; these define a basis in the cotangent space $T^*_p(M)$ of M at each point p of M. The corresponding dual basis $\{D_j, D_A\}$ of the tangent space $T_p(M)$ is specified by the requirement that

$$D_j \lrcorner\, \pi^h = \delta^h_j, \quad D_j \lrcorner\, \pi^A = 0, \quad D_A \lrcorner\, \pi^j = 0, \quad D_A \lrcorner\, \pi^B = \delta^B_A. \tag{1.1}$$

Thus the vector fields $\{D_j\}$ (resp. $\{D_A\}$) span the characteristic subspaces of the exterior system $\{\pi^A\}$ (resp.

$\{\pi^j\})$ in each $T_p(M)$; these subspaces will be denoted t_n (resp. t_N), and the collection of all such t_n (resp. t_N) defines the characteristic distribution D_n of $\{\pi^A\}$ (resp. D_N of $\{\pi^j\}$). The subspace of $T_p^*(M)$ that is spanned by the set $\{\pi^j\}$ (resp. $\{\pi^A\}$) is denoted by $T_N^\perp$ (resp. $t_n^\perp$).

The envisaged application of the resulting formalism to the calculus of variations depends on the use of adapted bases of each t_n:

$$B_j = \frac{\partial}{\partial x^j} + B_j^A \frac{\partial}{\partial \psi^A} = p_j^h D_h, \tag{1.2}$$

whose existence is predicated on the assumption that

$$D: = \det(p_h^j) \neq 0, \quad \text{where} \quad p_j^h = B_j \rfloor \pi^h. \tag{1.3}$$

The coordinate presentation of our 1-forms is expressed as

$$\pi^j = \pi_h^j dx^h + \pi_A^j d\psi^A, \qquad \pi^A = \pi_h^A dx^h + \pi_B^A d\psi^B, \tag{1.4}$$

in terms of which one has

$$p_h^j = \pi_h^j + \pi_A^j B_h^A. \tag{1.5}$$

The basis of $t_N^\perp$ that is dual to B_j is denoted by

$$\lambda^j = \lambda_h^j dx^h + \lambda_A^j d\psi^A,$$

whose coefficients are related to each other according to

$$\lambda_h^j = \delta_h^j - \lambda_A^j B_h^A,$$

and give rise to the following identifications:

$$\pi_h^j = p_\ell^j \lambda_h^\ell, \qquad \pi_A^j = p_\ell^j \lambda_A^\ell. \tag{1.6}$$

Any vector in t_N is said to be transversal to t_n; consequently D_N is called the transversal distribution, and the transversality condition is expressed by the third member of (1.1).

2. Invariance and the Legendre Transformation: The distributions D_n and D_N are unaffected by transformations of the type

$$\pi^j \to \pi'^j = R^j_h \pi^h, \quad \text{with} \quad R := \det(R^j_h) = 1. \tag{2.1}$$

Since the coefficients of the 1-forms (1.4) are to be regarded as canonical variables, the invariance under (2.1) of a function Φ of these variables is of paramount importance.

Theorem: In order that Φ be invariant under the unimodular transformation (2.1) it is necessary and sufficient that there exist a function μ such that

$$X^j_h(\Phi) = \mu\delta^j_h, \tag{2.2}$$

where

$$X^j_h = \pi^j_\ell \frac{\partial}{\partial \pi^h_\ell} + \pi^j_A \frac{\partial}{\partial \pi^h_A}. \tag{2.3}$$

The function μ that appears in (2.2) is then also invariant.

The commutator of the vector fields (2.3) is given by

$$[X^j_h, X^k_\ell] = \delta^k_h X^j_\ell - \delta^j_\ell X^k_h = (\delta^j_m \delta^k_h \delta^p_\ell - \delta^j_\ell \delta^k_m \delta^p_h) X^m_p, \tag{2.4}$$

where it is to be noted that the coefficients of X^m_p are nothing other than the negatives of the structure constants of the group GL(n,R). The relation (2.4) also ensures the completeness of the system of n^2 first order partial differential equations

$$X^j_h(\Phi) = 0. \tag{2.5}$$

A Legendre transformation must prescribe a relation between the canonical variables and the coefficients of the adapted basis (1.2). Such a relation is given by

$$\Phi^k_A(x^h, \psi^B, \pi^j_h, \pi^j_B, B^B_h) := \Lambda^k_A(x^h, \psi^B, B^B_h) - D\lambda^k_A = 0, \tag{2.6}$$

in which the functions Λ^k_A are thus far arbitrary except for a determinantal condition that ensures that (2.6) can be solved to yield

$$B^A_j = B^A_j(x^h, \psi^B, \pi^h_\ell, \pi^h_B). \tag{2.7}$$

The structure of (2.6) is such as to guarantee that these functions are invariant under (2.1): they satisfy the condition

$$X_h^j(B_k^A) = \mu_k^A \delta_h^j. \tag{2.8}$$

3. <u>Hamiltonians and Canonical Distributions</u>: It is stipulated that a function H of the coordinates of M and the canonical variables can serve as Hamiltonian if and only if it satisfies the invariance condition (2.2) with $\mu = D$ [together with a determinantal condition that is required for consistency with (2.6)]. This implies that

$$X_h^j(D - H) = (\Lambda_A^k \mu_k^A)\delta_h^j, \tag{3.1}$$

and a comparison with (2.8) suggests that one should now demand that there exist a function L of the coordinates of M and B_h^A such that

$$\Lambda_A^k = \partial L/\partial B_k^A. \tag{3.2}$$

For, under these circumstances (2.8) and (3.1) yield

$$X_h^j(D - H - L) = 0.$$

Because of (2.7) and the completeness of (2.5) this leads to the identification

$$H(x^h, \psi^B, \pi_\ell^h, \pi_B^h) = -L(x^h, \psi^B, B_h^B) + D(\pi_\ell^h, \pi_A^h, B_j^A). \tag{3.3}$$

This equation exemplifies the <u>fundamental relationship between a Hamiltonian</u> H <u>and a Lagrangian</u> L; it imposes no restriction on the latter other than the aforementioned determinantal condition associated with (2.6). Conversely, given a Lagrangian L that is subject to the latter a unique Hamiltonian H is specified by (3.3). The following relations result directly from (2.6) and (3.3)

$$\frac{\partial H}{\partial x^j} = -\frac{\partial L}{\partial x^j}, \quad \frac{\partial H}{\partial \psi^A} = -\frac{\partial L}{\partial \psi^A}, \quad \frac{\partial H}{\partial \pi_j^h} = P_h^j, \quad \frac{\partial H}{\partial \pi_A^h} = P_h^j B_j^A, \tag{3.4}$$

where P_h^j denotes the cofactors of the entries (1.5) in the determinant (1.3). By analogy with (2.6) we now put

$$\Lambda^j_h = D\lambda^j_h, \tag{3.5}$$

by means of which the coefficients of the Euler-Lagrange form

$$E = E_h dx^h + E_A d\psi^A \tag{3.6}$$

are then defined as

$$E_h = B_j \Lambda^j_h - \partial L/\partial x^h, \qquad E_A = B_j \Lambda^j_A - \partial L/\partial \psi^A. \tag{3.7}$$

In the subsequent analysis the exterior derivatives $d\pi^j$ play a role that is similar to that of the fundamental 2-form on a sympletic manifold. This is illustrated by the relation

$$\omega = dH + \nabla_j \lrcorner\, d\pi^j, \tag{3.8}$$

in which $D^{-1}\nabla_j = D_j$ refers once more to a basis of t_n, while

$$\omega = H_h dx^h + H_A d\psi^A, \tag{3.9}$$

with

$$H_h = \nabla_j \pi^j_h + \frac{\partial H}{\partial x^h}, \qquad H_A = \nabla_j \pi^j_A + \frac{\partial H}{\partial \psi^A}. \tag{3.10}$$

Because of (3.4) one has

$$\omega = E - (B_j P^j_h)\pi^h, \tag{3.11}$$

where it should be noted that the coefficients (3.10) behave as the components of a relative type (0,1) tensor field under arbitrary coordinate transformations on M. This, however, is not in general true for the coefficients (3.7).

The distribution D_n is said to be canonical if $\omega = 0$, in which case the invariant canonical equations $H_h = 0$, $H_A = 0$ are satisfied. It is pseudo-Lagrangian if

$$d\pi^j(D_j, D_h) = 0, \quad \text{with} \quad d\pi^j(D_j, D_A) \neq 0.$$

Theorem: If the distribution D_n is both canonical and pseudo-Lagrangian, it is an extremal foliation ($E_A = 0$ on D_n), and H = const. on each leaf.

4. The Cartan Form: The n-form defined by

$$\Pi = \pi^1 \wedge \ldots \wedge \pi^n \qquad (4.1)$$

is regarded as a Cartan form ([2]). It is invariant under (2.1).

Theorem: If Π is closed, the transversal distribution D_N is integrable, and the exterior system $\{\pi^j\}$ admits the representation $\pi^j = dS^j$, where $\{S^j\}$ denotes a set of n independent solutions of the system

$$D_A f = 0. \qquad (4.2)$$

If D_n is canonical, these solutions satisfy the Hamilton Jacobi equation

$$H(x^h, \psi^B, \partial S^j/\partial x^h, \partial S^j/\partial \psi^B) = 0. \qquad (4.3)$$

Conversely, if (4.3) is satisfied by a set of solutions of (4.2), the distribution D_n is canonical.

When the conditions of the theorem are satisfied, and if D_n is assumed to be integrable, the construction of a complete figure can be accomplished by means of Carathéodory's method of equivalent integrals in terms of an appropriate independent Hilbert integral and a Weierstrass excess function.

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SPECTRAL ASYMPTOTICS FOR POLAR VECTOR STURM-LIOUVILLE PROBLEMS

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1. The problem under consideration here is the derivation of asymptotic formulae for the distribution functions, [3], for eigenvalues of the vector boundary value problem

$$\begin{aligned} -y'' &= \lambda R(t)y \\ y(a) &= y(b) = 0 \end{aligned} \qquad (1.1)$$

where $R(t) = (R_{ij}(t))$ is a real valued n by n symmetric matrix function.

Using a Green's function argument the spectrum of (1.1) is purely discrete and consists of real eigenvalues of finite multiplicity. The distribution function $n_+(s)$ of positive eigenvalues is the number of eigenvalues in the closed interval $[o,s]$ while $n_-(s)$ is the number of eigenvalues in $[-s,o]$. In this paper the following formulae will be established

$$\lim_{s\to\infty} n_+(s)/\sqrt{s} = 1/\pi \int_a^b \Lambda_{\frac{1}{2}}(R_+(t))dt \qquad (1.2)$$

and

$$\lim_{s\to\infty} n_-(s)/\sqrt{s} = 1/\pi \int_a^b \Lambda_{\frac{1}{2}}(R_-(t))dt \qquad (1.3)$$

where $R_+(t) = \frac{1}{2}[|R(t)| + R(t)]$ and $R_-(t) = \frac{1}{2}[|R(t)| - R(t)]$ while $\Lambda_{\frac{1}{2}}(R(t))$ is the sum of the square roots of the moduli of all eigenvalues of the matrix $R(t)$. The result in (1.2) is an extention of a result in Gohberg and Krein [3] from the case where $R(t)$ is non-negative definite to the case where $y^T R(t)y$ is essentially unspecified as to its sign.

In section 2 asymptotic formulae (1.2) and (1.3) are derived for a diagonal matrix $R(t)$ and in section 3 the problem (1.1) is treated when $R(t)$ belongs to a restricted class of matrices.

2. For a diagonal matrix function $R(t)$ the proposed problem is equivalent to n scalar boundary problems. Thus λ is an eigenvalue of (1.1) if and only if it is an eigenvalue of a scalar boundary problem. Hence the spectrum of (1.1) is formed by the union of those discrete spectra of n scalar boundary problems and the complete sequence of eigenvalues for (1.1) is given by the rearrengement of this set as a non decreasing sequence. Then $n_+(s) = \sum_{i=1}^{n} N_+^i(s)$ and $n_-(s) = \sum_{i=1}^{n} N_-^i(s)$ where $N_+^i(s)$ and $N_-^i(s)$ denote the distribution functions of respective positive and negative eigenvalues of the i-th scalar boundary problem.

Lemma 1. _Consider the scalar boundary problem_ $y_i'' + \mu R_{ii}(t) y_i = 0$, $y_i(a) = y_i(b) = 0$ _whenever_ $R_{ii}(t)$ _is continuous and changes sign finitely many times in the interval_ $[a,b]$. _Then_

$$\lim_{s\to\infty} N_+^i(s)/\sqrt{s} = 1/\pi \int_a^b \sqrt{R_{ii}^+(t)}\,dt$$

and

$$\lim_{s\to\infty} N_-^i(s)/\sqrt{s} = 1/\pi \int_a^b \sqrt{R_{ii}^-(t)}\,dt$$

where $R_{ii}^+(t) = \max\{0, R_{ii}(t)\}$ _and_ $R_{ii}^-(t) = \max\{0, -R_{ii}(t)\}$.

Proof. By [4] this problem has a sequence of positive eigenvalues $\{\mu_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} m^2/\mu_m = K_i^2/\pi^2$ where $K_i = \int_a^b \sqrt{R_{ii}^+(t)}\,dt$. Let $\{x_m\}_{m=1}^{\infty}$ be an arbitrary sequence so that $x_m \to \infty$ as $m\to\infty$. Then there is some integer $k(m)$ and a constant M such that

$0 < \mu_{k(m)-1} < x_m < \mu_{k(m)}$ for $m > M$. Now

$k(m)-1/\mu_{k(m)}^{1/2} < N_+^i(x_m)/x_m^{1/2} < k(m)/\mu_{k(m)-1}^{1/2}$ for $m > M$ and therefore

$\lim_{m\to\infty} N_+^i(x_m)/x_m^{1/2} = K_i/\pi$. Since $\{x_m\}$ is arbitrary $\lim_{s\to\infty} N_+^i(s)/\sqrt{s} = K_i/\pi$. A similar proof gives the asymptotic formula for $N_-^i(s)$.

Theorem 2. Whenever $R(t)$ is a continuous diagonal matrix function such that det $R(t)$ changes sign at least once and at most finitely many times in $[a,b]$ then (1.2) and (1.3) are satisfied by the eigenvalues of (1.1).

Proof. Existence of positive and negative sequences of eigenvalues for (1.1) is settled by the existence results for scalar boundary problems. In view of the assumption on det $R(t)$, Lemma 1 is satisfied by each scalar boundary problem. Now $\sum_{i=1}^{n} N_+^i(s)/\sqrt{s}$ and $\sum_{i=1}^{n} N_-^i(s)/\sqrt{s}$ give the results (1.2) and (1.3) respectively.

Corollary. Whenever $R(t)$ is a continuous functionally commutative matrix function, [2], such that det $R(t)$ changes sign at least once and at most finitely many times in $[a,b]$ then (1.2) and (1.3) are satisfied by the eigenvalues of (1.1).

3. In this section the two parameter eigenvalue problem

$$w(\underline{\lambda}) = -y'' + \lambda_1 R_+(t)y + \lambda_2 R_-(t)y = 0, \quad y(a) = y(b) = 0 \qquad (3.1)$$

where $\underline{\lambda} = (\lambda_1,\lambda_2)$ is considered with the following cones

$K = \{\underline{\lambda}: \lambda_1 \int_a^b y^T R_+(t)y\, dt + \lambda_2 \int_a^b y^T R_-(t)y\, dt > 0$ for all $y \in L_n^2[a,b]$ with $\|y\| = 1\}$ and $C^- = \{\underline{\lambda}: \lambda_1 \int_a^b y^T R_+(t)y\, dt + \lambda_2 \int_a^b y^T R_-(t)y\, dt > \alpha$ for some $\alpha > 0$, for all $y \in L_n^2[a,b]$ with $\|y\| = 1\}$. Let P^i, Z^i and N^i be those sets of $\underline{\lambda}$ for which $\rho^i(\underline{\lambda})$-the i-th eigenvalue of $w(\underline{\lambda})$ according to multiplicity, (see [1]), is positive, zero or negative respectively, $(i = 0,1,2,.........)$. A solution y of (3.1) is a vector valued function which belongs to $\mathcal{D}$ and satisfies (3.1) where $\mathcal{D} = \{y: y,y'$ are absolutely continuous on $[a,b]$, $y'' \in L_n^2[a,b].\}$

When the assumptions

H_1. $R(t)$ is a piecewise continuous symmetric matrix function

H_2. $R_+(t)$ and $R_-(t)$ are not identically zero and

H_3. $\int_a^b y^T|R(t)|y\, dt > \delta$ for some $\delta > 0$ and for all $y \in L_n^2[a,b]$

are made on R(t) the cones K and C^- become non empty. In fact the cone K coincides with the first quadrent of $\lambda_1\lambda_2$ plane. Consequently the Z^i all asymptotically resemble the λ_1 and λ_2 axes (see section 6 in [1]). Let $\lambda_1 = -\lambda_2$ intersects Z^i at the point $(-\lambda^i,\lambda^i)$ for $\lambda^i > 0$. Then λ^i is an eigenvalue of (1.1) since $-y'' - \lambda^i R_+(t)y + \lambda^i R_-(t)y = -y'' - \lambda^i R(t)y = 0$ for some $y \in \mathcal{D}$ such that $y(a) = y(b) = 0$. This is in fact the i-th positive eigenvalue of (1.1). For, let γ be an eigenvalue of (1.1) such that $\lambda^i < \gamma < \lambda^{i+1}$. Then $-y''-\gamma R_+(t)y + \gamma R_-(t)y = 0$ and $y(a) = y(b) = 0$ for some $y \in \mathcal{D}$. Therefore zero is an eigenvalue for $w(\underline{\gamma})$, where $\underline{\gamma} = (-\gamma,\gamma)$ and hence $\underline{\gamma}$ belonging to some Z^k. Since Z^k is neither Z^i or Z^{i+1} this implies that Z^k must intersect either Z^i or Z^{i+1}. This contradicts Lemma 6.1 in [1]. Similarly it can be shown that μ^i is the i-th eigenvalue for the problem

$$y'' + \mu R_+(t)y = 0, \quad y(a) = y(b) = 0 \tag{3.2}$$

where $(-\mu^i,0)$ is the point of intersection of the curve Z^i and the

λ_1 axis. The essence of the foregoing discussion is the following

Lemma 3. Let $\tilde{n}_+(s)$ be the distribution function for (3.2) then $\lim_{s\to\infty} |\tilde{n}_+(s) - n_+(s)| = 0$.

A similar result is true for the distribution function $n_-(s)$ of the negative eigenvalues. From Gohberg & Krein [3] the distribution function $\tilde{n}_+(s)$ of the problem (3.2) has the property

$$\lim_{s\to\infty} \tilde{n}_+(s)/\sqrt{s} = 1/ \int_a^b \Lambda_{1/2}(R_+(t))dt.$$

Now combine this result with Lemma 3 to get the following.

Theorem 4. Whenever the assumptions H_1, H_2 & H_3 are made on R(t) in the interval [a,b] then (1.2) and (1.3) are satisfied by the eigenvalues of (1.1).

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