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REGLE DES SIGNES EN ALGEBRE COMBINATOIRE

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Résumé

Soit G un groupe fini. L'anneau de Burnside B(G) est muni d'opérations de puissances β_H, une pour chaque sous-groupe des groupes symétriques [1]. Une axiomatisation partielle de ces opérations a donné lieu au concept de β-anneaux [5] analogue à celui de λ-anneaux [3]. Malgré un certain développement de la théorie [4], on peut critiquer le concept sur deux aspects:

(i) La liste des relations satisfaites par les β_H est incomplète. Par exemple, on ne dit rien sur $\beta_H(x+y)$, ce qui est étonnant lorsque l'on connaît l'importance du binôme de Newton!

(ii) Aux opérations unaires β_H il faudrait ajouter des opérations n-aires pour $n>1$. L'analogie avec la théorie des λ-anneaux est ici trompeuse car dans celle-ci toutes les opérations sont des polynômes en les opérations unaires $\lambda^n(x)$ $(n\in N)$.

Nous développons une théorie des opérations polynômiales sur le modèle de la théorie combinatoire des séries formelles [2]. Nous calculons la substitution des opérations polynômiales et nous l'étendons aux différences formelles d'opérations polynômiales. Nous introduisons pour cela une <u>règle des signes</u> qui s'avère adéquate. Le concept correct de β-anneaux est obtenu en conséquence.

1- Foncteurs polynômiaux sur les G-ensembles

Pour tout groupe fini G, désignons par $\underline{S}^G$ la catégorie des G-ensembles finis. Soit $n \in N$. Pour tout $r = (r_1,\ldots,r_n) \in N^n$ soit S_r le produit des groupes symétriques $S_{r_1},\ldots,S_{r_n}$. Soit

$$F \in \prod_{r \in N^n} \underline{S}^{S_r}$$

$F = (F[r] | r \in N^n)$ et où chaque $F[r]$ est un S_r-ensemble fini. Supposons que $F[r] = \phi$ sauf pour un nombre fini d'indice $r \in N^n$. Soient $A_1,\ldots,A_n \in S^G$. Pour tout $r \in N^n$, posons

$$A^r = A_1^{r_1} \times \ldots \times A_n^{r_n}$$

Le groupe symétrique $S_r = S_{r_1} \times \ldots \times S_{r_n}$ agit de manière évidente sur A^r. Posons

$$F(A_1,\ldots,A_n) = \sum_{r \in N^n} F[r] \times_{S_r} A^r$$

Dans cette somme, le terme d'indice r désigne l'ensemble des orbites du S_r-ensemble produit $F[r] \times A^r$. Le G-ensemble $F(A_1,\ldots,A_n)$ dépend fonctoriellement de $A_1,\ldots,A_n$. Nous dirons que le foncteur ainsi défini est un foncteur polynômial. Soit maintenant $\underline{E}$ le groupoïde des bijections entre ensembles finis.

DEFINITION Soit $n \in N$. Une espèce de structures F est un foncteur [2]

$$F\colon \underline{E}^n \to \underline{S}$$

Un élément $s \in F[E]$ est une structure d'espèce F sur le multi-ensemble $E = (E_1,\ldots, E_n)$. Pour toute bijection $u\colon E \to D$, la structure $t = F[u](s)$ est obtenue en transportant s le long de u.

Pour tout entier $r \in N$ désignons par $[r]$ l'ensemble $\{1,2,.,r\}$. Plus généralement, pour tout $r \in N^n$, désignons par $[r]$ le multi-ensemble $([r_1],\ldots,[r_n])$. Une espèce F donne (par restriction) une famille de S_r-ensembles

$$(F[r] \mid r \in N^n) \in \prod_{r \in N^n} \underline{S}^{S_r}$$

Inversement, la connaissance de cette famille nous permet de retrouver F à isomorphisme canonique près:

$$F[E] \simeq \sum_{r \in N^n} F[r] \times_{S_r} Iso(r,E)$$

où Iso(r,E) désigne l'ensemble des bijections entre les multi-ensembles [r] et E.

2- <u>Composition des foncteurs polynômiaux</u>

La <u>somme</u> F+H de deux espèces n-aires se définit aisément:

$$(F+H)(E) = F[E] + H[E]$$

Le <u>produit</u> FH est donné par la formule:

$$(FH)[E] = \sum_{A+B=E} F[A] \times H[B]$$

Les termes de cette somme sont indexés par les <u>partages</u> (A,B) de E en deux morceaux:

$$A_i \cup B_i = E_i, \quad A_i \cap B_i = \phi \qquad 1 \leq i \leq n$$

La <u>puissance</u> H^p d'une espèce H peut se définir par récurrence sur p. Remarquons que le groupe symétrique S_p agit naturellement sur H^p.

La <u>substitution</u> $F(H_1,\ldots,H_n)$ d'espèces m-aires $H_1,\ldots,H_n$ dans une espèce n-aires F est donnée par la formule

$$F(H_1,\ldots,H_n)[E] = \sum_{p\in N^n} F[p] \underset{S_p}{\times} H^p[E]$$

où $H^p = H_1^{p_n}\ldots H_n^{p_n}$ et $S_p = S_{p_1} \times \ldots \times S_{p_n}$. Pour s'assurer que cette somme est finie, il faut supposer l'une ou l'autre des conditions suivantes:

a) F est bornée i.e. $F[p] = \phi$ sauf pour un nombre fini de $p\in N^n$.

b) Les H_i ont un terme constant nul i.e. $H_i[\phi,..,\phi] = \phi$ pour $1\leq i\leq n$.

Remarque. Soit $e_k = (0,..,1,..,0)$ le k-ième générateur de N^n. On définit une espèce (n-aire) X_k en posant

$$X_k[E] = Iso(e_k,E)$$

On a un isomorphisme $F(X_1,\ldots,X_n) \simeq F$.

THEOREME 1 <u>Les foncteurs polynômiaux sont clos sous la composition.</u>

Ce théorème indique clairement ce que doit être un β-semi-anneau: c'est un ensemble muni d'une opération n-aires pour chaque classe d'isomorphisme d'espèces polynômiales, la composition des opérations devant correspondre à la substitution des espèces.

Pour décrire le concept de β-anneaux, il faut faire la théorie des <u>espèces virtuelles</u> i.e. des différences formelles F-G entre espèces ordinaires F et G. Notons $B[\![X_1,\ldots,X_n]\!]$ le groupe de Grothendieck de la catégorie des espèces n-aires. On a un isomorphisme de groupes

$$B[\![X_1,\ldots,X_n]\!] \simeq \prod_{r\in N^n} B(S_r)$$

Le produit des espèces donne à $B[\![X_1,\ldots,X_n]\!]$ une structure d'anneaux commutatifs. Désignons par $B[X_1,\ldots,X_n]$ le sous-anneau constitué par les espèces bornées. Nous voulons décrire l'opération de <u>substitution</u> des espèces virtuelles. Le problème se réduit essentiellement à celui du calcul de $F(X-Y)$ pour une espèce $F(X)$. Comme on a $F(X+Y) = G(X,Y)$, il suffit de calculer $G(X,-Y)$ i.e. de savoir changer le signe d'une variable! Ce problème possède une solution simple dans le cas de l'espèce exponentielle:

$$e(X) = \sum_{n\geq 0} 1 \underset{S_n}{\times} X^n$$

Autrement dit $e[E] = \{*\}$ pour tout $E\in \underline{E}$. On vérifie immédiatement que

$$e(X+Y) = e(X)e(Y)$$

Ce qui suggère que $e(-X) = e(X)^{-1}$.

Pour calculer l'espèce virtuelle $\varepsilon(X) = e(X)^{-1}$ on utilise la série géométrique:

$$\varepsilon(X) = \sum_{n\geq 0} (-1)^n (e(X)-1)^n$$

Si l'on pose $\varepsilon(X) = \varepsilon_0(X) - \varepsilon_1(X)$ on peut donner une description combinatoire des espèces ε_0 et ε_1:

$$\varepsilon_0[E] = \{f\colon E \to [k] \mid f \text{ est surjective et } k \text{ est pair}\}$$
$$\varepsilon_1[E] = \{f\colon E \to [k] \mid f \text{ est surjective et } k \text{ est impair}\}$$

Décrivons maintenant la règle correcte pour changer le signe d'une variable dans une espèce. Pour fixer les idées, considérons le cas d'une espèce à deux variables $F(X,Y)$:

$$F(X,Y) = \sum_{n,m\geq 0} F[n,m] \underset{S_{n,m}}{\times} X^n Y^m$$

REGLE DES SIGNES: $F(X,-Y) = \sum_{n,m\geq 0} F[n,m] \times \varepsilon[m] \underset{S_{n,m}}{\times} X^n Y^m$

Ce qui signifie que l'espèce virtuelle $G = F(X,-Y)$ est la différence formelle entre les foncteurs

$$(A,B) \mapsto F[A,B] \times \varepsilon_0[B]$$

et

$$(A,B) \mapsto F[A,B] \times \varepsilon_1[B]$$

THEOREME 2 La substitution des espèces virtuelles, définie au moyen de la règle des signes, est une opération bien définie et associative.

Le concept de β-anneau est maintenant établi: c'est un ensemble muni d'une opération n-aire pour chaque espèce virtuelle n-aire bornée, le composé de ces opérations devant correspondre à la substitution des espèces virtuelles.

PROPOSITION L'anneau de Burnside d'un groupe fini est un β-anneau.

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COMMAS, POINTS EXTREMAUX ET ARETES DES CORPS POSSEDANT UNE FORMULE DU PRODUIT

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1) Abstract

In [4] E. Dubois and G. Rhin (following Minkowski) used the notion of an extremal point (or an edge) in a cubic number field, and in [5] we introduced the notion of an X comma of a field of algebraic functions ; all this in order to find units of these fields.

The present note considers those notions for their own sake and develops them on an axiomatic basis. The main results are the theorems concerning the number of orbits of edges and extremal points under the action of the group of units and their link with the rank of the latter group.

1) Définitions.

Soit E un corps avec une formule du produit [1] :

$$\prod_{p \in \mathcal{U}} |x|_p = 1 \qquad \text{pour } x \in E^*$$

On se donne un ensemble S de valeurs absolues de $\mathcal{U}$.

L'anneau des S_entiers de E sera :

$$B_S = \{x \in E \; ; \; |x|_p \leq 1, \text{ pour tout } p \notin S\}.$$

et le groupe des S_unités sera :

$$U_S = \{x \in E \; ; \; |x|_p = 1, \text{ pour tout } p \notin S\}.$$

Comme dans [1] on posera :

$$k^* = \{x \in E \; ; \; |x|_p = 1, \text{ pour tout } p \in \mathcal{U}\}.$$

Lorsque E est un corps de fonctions algébriques d'une variable, k^* est le groupe des constantes non nulles, lorsque E est un corps de nombres algébriques k^* est le groupe des racines de l'unité.

<u>Définition 1</u>.- On dira que $\varphi \in B_S \backslash \{0\}$ est un "S-comma de E" si et seulement si φ vérifie, pour tout $\varphi' \in B_S \backslash \{0\}$:

$$(|\varphi'|_p \leq |\varphi|_p \text{ pour tout } p \in S) \Rightarrow (\exists \lambda \in k^*, \ \varphi' = \lambda\varphi)$$

et on désignera par $\mathcal{C}_S$ l'ensemble des S-commas de E.

<u>Définition 2</u>.- On dira que $\varphi \in B_S \backslash \{0\}$ est un "point S-extrémal de E" si et seulement si φ vérifie, pour tout $\varphi' \in B_S \backslash \{0\}$:

$$(|\varphi'|_p \leq |\varphi|_p \text{ pour tout } p \in S) \Rightarrow (|\varphi'|_p = |\varphi|_p \text{ pour tout } p \in S),$$

et on désignera par $\mathcal{E}_S$ l'ensemble des points S-extrémaux de E.

<u>Définition 3</u>.- On dira que $\varphi \in B_S \backslash \{0\}$ est une "S-arête de E" si et seulement si φ vérifie, pour tout $\varphi' \in B_S \backslash \{0\}$

$$(|\varphi'|_p \leq |\varphi|_p \text{ pour tout } p \in S) \Rightarrow (\exists p \in S \text{ tel que } |\varphi'|_p = |\varphi_p|)$$

et on désignera par $\mathcal{A}_S$ l'ensemble des S-arêtes de E.

Il est clair que l'on a :

$$\mathcal{C}_S \subset \mathcal{E}_S \subset \mathcal{A}_S.$$

<u>Exemples</u>.- Dans ces exemples E est un corps de nombres ou un corps de fonctions algébriques. Dans le premier cas, S sera l'ensemble des valeurs absolues archimédiennes de E, dans le second S sera l'ensemble des valeurs absolues (non équivalentes) telles que $|X|_p > 1$ pour un élément donné $X \in E \backslash k$.

1) Si $|S| = 1$, on a : $k^* = \mathcal{U}_S = \mathcal{C}_S = \mathcal{E}_S = \mathcal{A}_S$.

2) Dans les corps de nombres, on a toujours $\mathcal{E}_S = \mathcal{A}_S$, alors que dans les corps de fonctions, on a en général $\mathcal{E}_S \neq \mathcal{A}_S$.

3) Si S contient une valeur absolue réelle ou une valeur absolue de degré 1 on a : $\mathcal{C}_S = \mathcal{E}_S$; voir corollaire du théorème 2.

4) Si $E = \mathbb{Q}(\sqrt{3})$, $\varphi = 1-\sqrt{3}$ est un comma de E, mais n'est pas une unité.

2) Propriétés générales

En utilisant la formule du produit, on obtient facilement le théorème suivant.

Théorème 1.- *Les S-unités de E sont des S-commas de E.*

La proposition suivante est immédiate :

Proposition.- *$\mathcal{U}_S$ opère sur $\mathcal{C}_S$, $\mathcal{E}_S$, $\mathcal{A}_S$ par l'application* $(u,\varphi) \mapsto u\varphi$. *On désignera le nombre d'orbites par $[\mathcal{C}_S : \mathcal{U}_S]$, $[\mathcal{E}_S : \mathcal{U}_S]$ et $[\mathcal{A}_S : \mathcal{U}_S]$ respectivement. Avant de donner des conditions suffisantes pour que ces nombres soient finis, nous allons étudier les diviseurs des commas et des points extrémaux. Le groupe $\mathcal{D}$ des diviseurs de E est somme directe du sous-groupe $\mathcal{D}_1$ engendré par les places de S et du sous-groupe $\mathcal{D}_2$ engendré par les autres places.*

Pour $D \in \mathcal{D}$, on écrira : $D = D_1 + D_2$

avec $D_1 \in \mathcal{D}_1$ et $D_2 \in \mathcal{D}_2$. On posera $\mathrm{div}_1(D) = D_1$ et $\mathrm{div}_2(D) = D_2$.

Théorème 2.- *On suppose que E est un corps de fonctions algébriques de genre g et que S est fini; on pose* :

$$h = \{\inf \deg p \ ; \ p \in S\}, \quad e = \sum_{p \in S} \deg p \ .$$

1) Si φ est un S-comma de E, on a :

$$0 \leqslant \deg[\mathrm{div}_2(\varphi)] \leqslant g.$$

2) Si φ est un point S-extrémal de E, on a :

$$0 \leq \deg[\mathrm{div}_2(\varphi)] \leq g+h-1$$

3) Si φ est une S-arête de E, on a :

$$0 \leq \deg[\mathrm{div}_2(\varphi)] \leq g+e-1.$$

Idée de la preuve.-

On utilise le théorème de Riemann [3] dans les trois cas suivants.

1) La condition pour que φ soit un comma s'écrit :

$$\ell[\mathrm{div}_1(\varphi)] = 1.$$

2) La condition pour que φ soit un point extrémal s'écrit :

$$(D \in \mathcal{D}_1,\ D \neq \mathrm{div}_1(\varphi) \text{ et } \mathrm{div}_1(\varphi) \text{ divise } D) \Rightarrow (\ell(D) = 0)$$

on en déduit que :

$$\ell[\mathrm{div}_1(\varphi)] \leq h.$$

3) La condition pour que φ soit une arête s'écrit :

$$\ell[\mathrm{div}_1(\varphi) + \sum_{p \in S} p] = 0 \quad ;$$

on en déduit que :

$$\ell[\mathrm{div}_1(\varphi)] \leq e.$$

Corollaire.- Si S contient une place de degré 1, alors $\mathcal{E}_S = \mathcal{C}_S$.

3) Théorèmes de finitude.

On suppose pour l'instant que E est un corps de nombres algébriques ou un corps de fonctions algébriques d'une variable sur un corps fini. Dans les deux cas, k^* est fini.

On suppose aussi que S est soit l'ensemble des valeurs absolues archimédiennes de E soit l'ensemble des valeurs absolues non équivalentes $|\ |_p$ telles que $|X|_p > 1$ où X est un élément donné de $E\backslash k$.

Dans les deux cas, on considère que E est une extension finie de K, avec $k = \mathbb{Q}$ dans le cas d'un corps de nombres et $K = k(X)$ dans le cas d'un corps de fonctions. On pose $n = [E:K]$.

Théorème 3.- On suppose que E est un corps de fonctions algébriques sur un corps fini k et on reprend les notations du théorème 2 avec S choisi comme ci-dessus. Alors on a :

$$[\mathcal{G}_S : \mathcal{U}_S] \leq C_1(n,g,|k|)$$

$$[\mathcal{T}_S : \mathcal{U}_S] \leq C_2(n,g,h,|k|)$$

$$[\mathfrak{F}_S : \mathcal{U}_S] \leq C_3(n,g,e,|k|) \quad ,$$

où C_1, C_2 et C_3 sont des constantes facilement calculables.

Preuve.-

1) On sait [3] que :

$$\deg_K[\mathcal{N}_{E/K}\, \mathrm{div}_2(\varphi)] = \deg_E[\mathrm{div}_2(\varphi)] \quad ;$$

donc (théorème 2)

$$\begin{cases} \varphi \in \mathcal{G}_S \Rightarrow \deg_K[\mathcal{N}_{E/K}\, \mathrm{div}_2(\varphi)] \leq g \\ \varphi \in \mathcal{T}_S \Rightarrow \deg_K[\mathcal{N}_{E/K}\, \mathrm{div}_2(\varphi)] \leq g+h \\ \varphi \in \mathfrak{F}_S \Rightarrow \deg_K[\mathcal{N}_{E/K}\, \mathrm{div}_2(\varphi)] \leq g+e. \end{cases}$$

2) En adaptant la démonstration de [2,p.100] on montre qu'il existe au plus :

$$|k|^{dn}$$

polynômes non associés dont le degré de la norme est égal à d.

Pour les corps de nombres algébriques le lemme de Minkowski donne le résultat suivant.

Théorème 4.- On suppose que E est un corps de nombres algébriques de degré n, de discriminant D et admettant t plongements réels, alors on a :

$$[\mathcal{A}_S : \mathcal{U}_S] < (\sum_{i=1}^{B} i^n)$$

avec $B = [(\frac{2}{\pi})^t \sqrt{|D|}]$.

A l'aide du théorème 4, on peut retrouver une partie du théorème sur les unités de Dirichlet, comme le montre le résultat suivant, valable pour k fini ou non.

Théorème 5.- On suppose que E est un corps possédant une formule du produit et on choisit S comme plus haut.

1) $([\mathcal{C}_S : \mathcal{U}_S] < \infty) \Rightarrow rg\ U_S = |S| - 1$ et il existe une famille d'unités ε_i telle que, pour tous les p_i et $p_j \in S$, on ait :

$$|\varepsilon_i|_{p_i} > 1, \quad |\varepsilon_i|_{p_j} < 1 \quad (\text{si } i \neq j) .$$

2) Si $|S| = 2$:

$$([\mathcal{C}_S : \mathcal{U}_S] < \infty) \Leftrightarrow rg(U_S/k^*) = 1.$$

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ON FIELDS FOR WHICH THE NUMBER OF ORDERINGS IS DIVISIBLE BY A HIGH POWER OF 2, III

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Abstract: Let m be an integer of the form, $m = 1+2^{k_1}+\ldots+2^{k_\ell}$ where $1 \leq k_1 < k_2 < \ldots < k_\ell$, $\ell \geq 1$. Let $\mu(m) = k_\ell+(\ell+1)$. We will consider fields with n independent orderings, where the total number of orderings is $m.2^{n-\mu(m)}$; and determine the reduced stability index and chain length of such fields.

§1. Introduction. In this paper we keep to the notation used in [2], [5], [6]. We define F (or $K,L,\ldots$) to be a formally real field. $\dot{F}$ is the multiplicative group of the field F . T - a preordering of F , i.e. any intersection of orderings of the field F . T_F - the intersection of all orderings of the field F . $[\dot{F}:T_F]$ is the group-index. V - a valuation on F . A_V - the valuation ring corresponding to V . U_V - the group of units of A_V . M_V - the maximal ideal of A_V . F_V - the residue field of V . V is fully compatible with T iff $1+M_V \subset T$. $(X,\dot{F}|T)$ is a space of orderings. X here means the set of all orderings P of the field F such that $T \subset P$. A field F is of type $(k,2^n)$ if $[\dot{F}:T_F] = 2^n$ and the number of orderings is k .

For each integer $n \geq 1$, we define the set $O(n)$ of natural numbers by the following recursive formula:

$$O(1) = \{1\} \text{ , } O(2) = \{2\} \text{ , } O(3) = \{3,4\} \text{ ,}\ldots$$

$$O(n) = 2\,O(n-1) \cup \{1+O(n-1)\} \text{ , } n > 1$$

We note that if F is of type $(h,2^n)$ then $h \in O(n)$ (see [1]). If $m = \sum_{i=0}^{t} \varepsilon_i 2^i$, where $\varepsilon_i \in \{0,1\}$, $\varepsilon_t = 1$ we define $s(m) = \sum_{i=0}^{t} \varepsilon_i$. Then we can write $\mu(m) = t+s(m)$. By Prop. 2.A.4. [5] we know that this is equivalent to writing $\mu(m) = \min\{n \in \mathbb{N}, m \in O(n)\}$.

For other definitions and theorems used here, the reader is referred to [2], [3], [4] and [5].

Suppose F is a field of type $(m2^{n-\mu(m)},2^n)$, where m is an odd number ≥ 3, $n \geq \mu(m)$. Then from Prop. 1.A) [7] we know that there exists a valuation V on F fully compatible with T_F such that the field F_V has type $(m,2^{\mu(m)})$. In this paper we look more closely at this residue field and use its properties to further examine the structure of the space of orderings.

§2. Theorem 1. Let F be a field of type $(m,2^{\mu(m)})$ where m is an odd integer >3. Then the space of orderings $(X,\dot{F}|T_F)$ of the field F has two connected components X_1 and X_2. The first has just one ordering P; the second, $m-1$ orderings. Denote the intersection of orderings in X_2 by T. Then the space $(X,\dot{F}|T_F)$ is the direct sum of the spaces $(X_1,\dot{F}|P)$, $(X_2,\dot{F}|T)$.

Proof. According to Theorem 3.3 and Remark 1 in [3], we need only show that $m \in O(\mu(m))$ is decomposable in a unique way: that is $m = 1+(m-1)$, where $1 \in O(1)$, $m-1 \in O(\mu(m)-1)$, and $m-1$ is an indecomposable element of the set $O(\mu(m)-1)$.

Suppose $m = a+b$, $a \in O(c)$, $b \in O(d)$. As $\mu(m) = \min\{f \in N: m \in O(f)\}$, we know that $c = \mu(a)$, $d = \mu(b)$. Thus $a+b = m$ where $\mu(a)+\mu(b) = \mu(m)$. From this we can prove that $a = 1$ or $b = 1$.

Suppose $m = 1+2^{k_1}+\ldots+2^{k_\ell}$, $1 \leq k_1 < \ldots < k_\ell$, $\ell \geq 2$. Then $(m-2) = (1+2+\ldots+2^{k_1-1})+2^{k_2}+\ldots+2^{k_\ell}$, and so $\mu(m-2) = \mu(m)+k_1-2 \geq \mu(m)-1$. If $m = 1+2^k$, $k \geq 2$, then $\mu(m-2) = 2k-1 \geq k+1 = \mu(m)-1$.

Therefore if m is an odd number >3 the number $m-1 \in O(\mu(m)-1)$ is indecomposable, which concludes the proof.

Theorem 2. Let F be a field of type $(m,2^{\mu(m)})$, where m is an odd integer ≥ 5 . Suppose m has form

(i) $m = 1+n.2^k$, n is an odd integer >1

or

(ii) $m = 1+2^k$

Let $(X_2,\dot{F}|T)$ be the connected subspace of X consisting of $m-1$ orderings. Then there exists a non-trivial valuation V on F , compatible with all orderings in X_2 , such that

(i) The field F_V has type $(n,2^{\mu(n)})$

(ii) The field F_V has type $(1,2)$ or $(2,2^2)$.

Theorem 2 can be proved in an analogous way to Theorem 1 in [6].

Suppose F is a field of type $(m,2^{\mu(m)})$, where $m \geq 5$ has the form

$m = 1+2^{k_1}+\ldots+2^{k_\ell}$, $1 \leq k_1 < k_2 < \ldots < k_\ell$, $\ell \geq 2$, $k_{\ell-1}+2 \leq k_\ell$

By Theorem 2 there exists a chain of valuations $V_1,V_2,\ldots,V_\ell$, where V_1 is a valuation on F , V_2 is a valuation on

$F_{V_1},\ldots$ and V_ℓ is a valuation on $F_{V_{\ell-1}}$. The residue fields $F_{V_1},\ldots,F_{V_\ell}$ are of type $(n_1,2^{\mu(n_1)}),\ldots,(n_\ell,2^{\mu(n_\ell)})$, respectively, where $n_1 = 1+2^{k_2-k_1}+\ldots+2^{k_\ell-k_1}$, $n_2 = 1+2^{k_3-k_2}+\ldots$ $\ldots+2^{k_\ell-k_2},\ldots,n_{\ell-1} = 1+2^{k_\ell-k_{\ell-1}}, n_\ell = 1$ or 2. Furthermore each valuation V_i is compatible with $n_{i-1}-1$ orderings of the field $F_{V_{i-1}}$, where $i = 1,2,\ldots,\ell-1,\ell$, and $n_0 = m$, $F_{V_0} = F$.

If $k_\ell = k_{\ell-1}+1$ then $n_{\ell-1} = 3$.

Then it can happen that there is no non-trivial valuation V_ℓ on $F_{V_{\ell-1}}$ fully compatible with any preordering of the field $F_{V_{\ell-1}}$.

From the discussion above we get that the space of orderings $(X,\dot{F}|T_F)$ is a disjoint union of $(\ell+1)$ fans $Y_0,Y_1,\ldots,Y_\ell$ where Y_0 has one element, Y_1 has 2^{k_1} elements $,\ldots,$ and Y_ℓ has 2^{k_ℓ} elements. The reduced stability index of F, which I will denote $st(F)$ is of value k_ℓ.

From these results and Theorem 2 in [6] we can extend the result of R. Elman and T. Y. Lam about the reduced stability index of superpythagorean fields with finite number of square classes to some other fields (see notes on §13 in [2]).

<u>Theorem 3.</u> <u>Let F be a field of type $(m.2^{n-\mu(m)},2^n)$, $n \geq \mu(m)$, where m is an odd integer. Then</u>

$$st(F) = n-s(m).$$

<u>Also if $n \geq 2$</u>

$$cl(F) = s(m)+1$$

where $cl(F)$ is the chain length of the preordering T_F .

As a consequence we get.

Corollary Let F be a field of the type $(m.2^{m-\mu(m)}, 2^m)$, where m is an odd number. Then

$$\boxed{st(F) = ord_2(m!)}$$

where $ord_2(m!)$ is $\max\{f \in \mathbb{N} \cup \{0\} | 2^f$ divides $m!\}$.

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A GEOMETRIC CHARACTERIZATION OF ARCS OF ORDER n IN R_n.

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0. Let Γ denote a differentiable arc in a real affine n-space R_n; $n \geq 2$. Then Γ is of order n if every (n - 1)-flat meets Γ in at most n points. While this definition is geometrically descriptive in the plane (an arc of order two is convex), it is not particularly so in higher dimensions.

In this note, we attempt to remedy this lack by describing the behaviour of an arc of order n with respect to certain families of flats in R_n.

1. We denote (n - 1)-flats of R_n by $\alpha, \beta, \ldots$ and k-flats, $-1 \leq k \leq n - 1$, by $P_k, Q_k, \ldots$. The flat spanned by $P_k, Q_\ell, \ldots$ is denoted by $\langle P_k, Q_\ell, \ldots \rangle$. Let $u < v$ in R_1. We denote the different segments bounded by u and v in the standard manner.

An arc in R_n is a continuous map $\Gamma:[u,v] \to R_n$. We identify Γ with $\Gamma[u,v]$ and note that the topology of $[u,v]$ defines a topology on Γ.

The arc Γ shall be differentiable in the following sense. For $t \in [u,v]$, let $\Gamma_{-1}(t) = \phi$ and $\Gamma_0(t) = \Gamma(t)$. If $\Gamma_{k-1}(t)$ is already defined and its existence postulated then we require that for $s \in [u,v] \setminus \{t\}$ sufficiently close to t, $\langle \Gamma_{k-1}(t), \Gamma(s) \rangle$ has dimension k and it converges as s tends to t. Its limit is the osculating k-flat $\Gamma_k(t)$, $k = 1, \ldots, n$.

Let $t \in (u,v)$ and $\Gamma(t) \in \alpha$. Then α supports [cuts] Γ at t, if locally at t, Γ lies [does not lie] on one side of α. Let

$$S_k(t) = \{\alpha | \alpha \cap \Gamma_{k+1}(t) = \Gamma_k(t)\}, \; k = 0, \ldots, n-1.$$

Then either all $\alpha \in S_k(t)$ support Γ at t or all $\alpha \in S_k(t)$ cut Γ at t; cf. [1], p. 102.

Let $s \in [u,v]$ and $P_m \cap \Gamma_k(s) = \Gamma_{k-1}(s)$. Then P_m is said to meet Γ at s with multiplicity k or k times. A set $\{H_k\}_0^{n-1} = \{H_k | 0 \le k \le n-1\}$ of k-flats is a tower if $H_0 \subset H_1 \subset \dots \subset H_{n-1}$ and $\Gamma(u,v)$ is an arc with tower $\{H_k\}_0^{n-1}$ if

$(1)_k \quad \Gamma_k(t) \cap H_{n-k-1} = \emptyset$ for $t \in (u,v)$; $0 \le k \le n-1$.

Finally, $\Gamma(u,v)$ is ordinary if each of its points possesses a neighbourhood of order n.

We list some results required for our characterization, of which 4 is well known.

1. Let $\Gamma(u,v)$ be ordinary. Then $\Gamma_k(t)$ depends continuously on $t \in (u,v)$ and $\alpha \in S_k(t)$ supports Γ at t if and only if k is odd, $0 \le k \le n-1$; cf. [3], p. 168.

2. Let $\Gamma(u,v)$ be of order n. Then any (n-1)-flat meets each of $\Gamma[u,v)$ and $\Gamma(u,v]$ with at most multiplicity n and

$(2)_k \quad \Gamma_k(s) \cap \Gamma_{n-k-1}(t) = \emptyset, \quad 0 \le k \le n-1,$

for $s \ne t$ in $[u,v)$ or $(u,v]$; cf. [2], p. 530.

3. An ordinary arc with tower is of order n; cf. [1], p. 110.

4. Let $\Gamma(u,v)$ be ordinary, $t \in (u,v)$ and $\{\alpha_\lambda\} \subset S_k(t)$ be a sequence converging to β. If $\alpha_\lambda \cap \Gamma(U(t)) = \{\Gamma(t)\}$ for some neighbourhood U(t) of t and each α_λ, then $\beta \in S_m(t)$ where $m \equiv k \pmod 2$.

2. Henceforth we assume that $\Gamma: [u,v] \to R_n$ is ordinary and $\Gamma(v) \notin \Gamma_{n-2}(u)$. We state our characterization.

5. THEOREM. The following are equivalent

5.1 $\Gamma(u,v)$ is of order n.

5.2 $\Gamma(u,v)$ is an arc with tower $\{H_k = \langle \Gamma_{k-1}(u), \Gamma(v)\rangle\}_0^{n-1}$.

5.3 For $t \in (u,v)$ and $m = 0,\ldots,n-2$, there exist $\alpha^m(t) \in S_m(t)$ such that (i) $\alpha^m(t) \cap \Gamma(u,v) = \{\Gamma(t)\}$

and (ii) $\langle \Gamma_{n-m-3}(u), \Gamma(v)\rangle \subset \alpha^m(t)$.

PROOF. We note that 5.2 implies 5.1 by 3 and since $\Gamma(v) \notin \Gamma_{n-2}(u)$, $\{H_k = \langle \Gamma_{k-1}(u), \Gamma(v)\rangle\}_0^{n-1}$ is a tower.

5.1 => 5.2

Let $\Gamma(u,v)$ be of order n. We wish to prove $(1)_k$ for $t \in (u,v)$ or equivalently,

$(1)_m \qquad \Gamma_{m+1}(t) \cap H_{n-m-2} = \phi, \ -1 \le m \le n-2.$

Case 1: $m = -1$.

Let $w \in (u,v)$. Then $\Gamma_{n-2}(u) \cap \Gamma_1(w) = \phi$ by $(2)_1$. Thus $P_{n-1}(w) = \langle \Gamma_{n-2}(u)\ \Gamma(w)\rangle \in S_0(w), P_{n-1}(w)$ meets $\Gamma[u,v)$ with multiplicity n and by 2,

(3) $\qquad P_{n-1}(w) \cap \Gamma(u,v) = \{\Gamma(w)\}$.

We note that $\lim_{w \to v} P_n(w) = \langle \Gamma_{n-2}(u), \Gamma(v)\rangle = H_{n-1}$ and H_{n-1} meets $\Gamma[u,v)$ n - 1 times at u. Thus either $H_{n-1} \cap \Gamma(u,v) = \phi$ or H_{n-1} cuts Γ at a point $s \in (u,v)$. Since $\lim_{w \to v} P_n(w) = H_{n-1}$, the latter is possible only if $P_n(w)$ cuts Γ at a point $s' \in (u,v)$ for w close to v. This contradicts (3) and therefore $\Gamma_0(t) \cap H_{n-1} = \phi$ for $t \in (u,v)$.

Case 2: $m \ge 0$.

We argue by induction and assume $(1)_{m-1}$ for $t \in (u,v)$. Then $\alpha_v(t) = \langle \Gamma_m(t), H_{n-m-2}\rangle = \langle \Gamma_m(t), \Gamma_{n-m-3}(u), \Gamma(v)\rangle$ is an (n-1)-flat which meets $\Gamma[u,v)$ at least $m + 1[n-m-2]$ times at $t[u]$. Thus by 2,

(4) either $\alpha_v(t) \in S_m(t)$ or $\alpha_v(t) \in S_{m+1}(t)$.

We fix $\bar{v} \in (t,v)$ and for $w \in (\bar{v},v)$, set

$$Q_w(t) = \langle \Gamma_m(t), \Gamma_{n-m-3}(u), \Gamma(w) \rangle.$$

Since only (n-1)-flats may meet $\Gamma[u,v)$ with multiplicity n and $Q_w(t)$ does, it follows that

(5) $Q_w(t) \in S_m(t)$

and (6) $Q_w(t) \cap \Gamma(u,w) = \{\Gamma(t)\}$.

Since $\lim_{w \to v} Q_w(t) = \alpha_v(t)$ and $Q_w(t) \cap \Gamma(u,\bar{v}) = \{\Gamma(t)\}$, we obtain that $\alpha_v(t) \in S_m(t)$ by (5), 4 and (4). Now $\alpha_v(t) \cap \Gamma_{m+1}(t) = \Gamma_m(t)$ yields that $\Gamma_{m+1}(t) \cap H_{n-m-2} = \Gamma_m(t) \cap H_{n-m-2} = \phi$.

<u>5.2 => 5.3</u>

We assume 5.2. Then $\Gamma(u,v)$ is of order n and for $t \in (u,v)$ and $0 \le m \le n-2$, $\Gamma_{m+1}(t) \cap H_{n-m-2} = \phi$ and

$$\alpha^m(t) = \langle \Gamma_m(t), H_{n-m-2} \rangle = \langle \Gamma_m(t), \Gamma_{n-m-3}(u), \Gamma(v) \rangle \in S_m(t).$$

We again note that $\alpha^m(t)$ meets $\Gamma[u,v)$ with multiplicity at least n-1 and thus either $\alpha^m(t) \cap \Gamma(u,v) = \{\Gamma(t)\}$ or $\alpha^m(t)$ meets $\Gamma[u,v)$ cuts Γ at a point $s \neq t$ in (u,v). Using the preceding notation;

$$\lim_{w \to v} Q_w(t) = \alpha_v(t) = \alpha^m(t)$$

and the latter again yield that $Q_w(t)$ cuts Γ at a point $s' \neq t$ in (u,w) for w close to v. This contradicts (6) and thus $\alpha^m(t) \cap \Gamma(u,v) = \{\Gamma(t)\}$ for $t \in (u,v)$, $0 \le m \le n-2$.

<u>5.3 => 5.2</u>

We assume 5.3 and show $(1)_k$, $0 \le k \le n-1$. Since $H_{n-2} = \langle \Gamma_{n-3}(u), \Gamma(v) \rangle \subset \alpha^\circ(t)$ for $t \in (u,v)$, 5.3 i) implies that $H_{n-2} \cap \Gamma(u,v) = \phi$ and

(7) $\alpha^\circ(t) = \langle H_{n-2}, \Gamma(t) \rangle = \langle \Gamma_{n-3}(u), \Gamma(v), \Gamma(t) \rangle$.

Next, $\Gamma(v) \notin \Gamma_{n-2}(u) = \lim_{t\to u} \langle\Gamma_{n-3}(u),\Gamma(t)\rangle$ yields

$$(8)\quad H_{n-1} = \langle\Gamma_{n-2}(u),\Gamma(v)\rangle = \lim_{t\to u} \langle \Gamma_{n-3}(u),\Gamma(v),\Gamma(t)\rangle = \lim_{t\to u} \alpha^{0}(t).$$

If H_{n-1} meets Γ at a point $s \in (u,v)$ then H_{n-1} cuts Γ at s by (7) and for t close to u, $\alpha^{0}(t)$ cuts Γ at a point $s' \neq t$ in (u,v) by (8). This contradicts 5.3 i) and thus $(1)_0$. Since $\langle H_{n-2},\Gamma(t)\rangle \in S_0(t)$ for $t \in (u,v)$, $(1)_1$ is immediate.

Let $2 \le k \le n - 1$. We proceed by induction and assume $(1)_{k-1}$: $\Gamma_{k-1}(t) \cap H_{n-k} = \phi$ for $t \in (u,v)$. Then $\alpha^{k-1}(t) = \langle\Gamma_{k-1}(t),H_{n-k-1}\rangle$ by 5.3 ii). Now $\alpha^{k-1}(t) \cap \Gamma_k(t) = \Gamma_{k-1}(t)$ again yields that $\Gamma_k(t) \cap H_{n-k-1} = \phi$. □

If $\Gamma(u,v)$ is of order n then $\Gamma_{n-1}(t) \cap \Gamma(u,v) = \{\Gamma(t)\}$ for $t \in (u,v)$ by 2. Thus we extend 5.3 i) to include $\alpha^{n-1}(t) = \Gamma_{n-1}(t)$ and obtain

$$\bigcap_{m=0}^{n-2} \alpha^{m}(t) = \langle\Gamma(t),\Gamma(v)\rangle \text{ and } \bigcap_{m=0}^{n-1} \alpha^{m}(t) = \{\Gamma(t)\}.$$

Call $\{\alpha^{m}(t)\,|\,0 \le m \le n - 1\}$ a <u>frame</u> of R_n at $\Gamma(t)$. Then we have the following geometric descriptions of an arc $\Gamma(u,v)$ of order n with $\Gamma(v) \notin \Gamma_{n-2}(u)$:

For each $t \in (u,v)$, there is a continuously dependent frame whose (n-1)-flats meets $\Gamma(u,v)$ only at $\Gamma(t)$.

The family $\{H_k = \langle\Gamma_{k-1}(u),\Gamma(v)\rangle\}_0^{n-1}$ is a tower and for $t \in (u,v)$ and $0 \le k \le n - 2$, $\langle H_k,\Gamma_{n-k-2}(t)\rangle$ rotates monotonically about H_k.

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CLASSIFICATION OF STATIONARY 2-TYPE SURFACES OF HYPERSPHERES

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Presented by G.deB. Robinson, F.R.S.C.

ABSTRACT. Compact, stationary, mass-symmetric, 2-type surfaces of S^m are shown to be flat surfaces which lie fully either in S^5 or in S^7. By determining their connection form in S^m, we classify such surfaces.

1. Introduction. Let M be a compact Riemannian manifold and Δ the Laplacian of M acting on differentiable functions in $C^\infty(M)$ (manifolds are assumed to be compact unless mentioned otherwise). Then Δ has an infinite sequence of eigenvalues: $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty$. For each λ_k its associated eigenspace V_k is finite-dimensional. On $C^\infty(M)$ one defines an inner product by $(f,g) = \int f\,g\,dV$, then the decomposition $\sum_{t \geqq 0} V_t$ is orthogonal and dense in $C^\infty(M)$ (in L^2-sense). Therefore, for each $f \in C^\infty(M)$, one can consider its spectral decomposition: $f = \sum_{t \geqq 0} f_t$, $\Delta f_t = \lambda_t f_t$, which is convergent in L^2-sense.

This construction can be extended to $\mathbb{R}^{m+1}$-valued differentiable functions on M. If $x : M \to E^{m+1}$ is an isometric immersion of M into a Euclidean $(m+1)$-space, one can talk about its spectral decomposition: $x = x_0 + \sum_{t \geqq 1} x_t$, $\Delta x_t = \lambda_t x_t$ (in L^2-sense), where x_0 is the center of mass of M in $\mathbb{R}^{m+1}$. If the decomposition of x consists of only finite nonzero terms, the submanifold M is said to be <u>of finite type</u>.

It is <u>of k-type</u> if there are exactly k nonzero x_t's in the decomposition (except x_0) (cf. [4]).

In terms of finite-type submanifolds, a well-known theorem of Takahashi [9] says that a 1-type submanifold M is nothing but a minimal submanifold of a hypersphere of $\mathbb{R}^{m+1}$. Furthermore, it is always <u>mass-symmetric</u>, i.e., the center of mass of M coincides with the center of the hypersphere. Therefore, if one chooses the center of the hypersphere as the origin of $\mathbb{R}^{m+1}$, then the position vector of a minimal submanifold M of S^m takes the following form: $x = x_p$, $\Delta x_p = \lambda_p x_p$.

Mass-symmetric, 2-type submanifolds of S^m are the "simplest" submanifolds of $\mathbb{R}^{m+1}$ next to minimal submanifolds of S^m, their position vectors admit the following spectral behavior: $x = x_p + x_q$ with $\Delta x_p = \lambda_p x_p$, $\Delta x_q = \lambda_q x_q$, $(\lambda_p < \lambda_q)$.

Many important submanifolds are known to be of 2-type and mass-symmetric (cf. [1,2,4,5,7,8]). The complete classification of such submanifolds is formidably difficult. However, the case of surfaces in S^3 was done by the second author ([4, p. 279]).

In [1], we have proved the following.

Theorem 1. <u>There exist no mass-symmetric, 2-type surfaces which lie fully in</u> S^4.

Let $f : M \to M'$ be an isometric immersion of a surface M into a Riemannian manifold M'. If α' (respectively, R') denotes the mean curvature of f (respectively, the sectional curvature of M' with respect to the tangent space of M), then we may define the <u>conformal total mean curvature</u> of f as

$\tau(f) = \int_M ((\alpha')^2 + R')dV$ (cf. [3,4,6,10]). It is known that $\tau(f)$ is a conformal invariant. Surfaces which are critical points of $\tau(f)$ are called stationary. Related to Chen-Willmore's problem, Weiner asked in [10] whether minimal surfaces of S^m are the only stationary mass-symmetric surfaces of S^m? Ejiri gave in [6] a counter example to Weiner's problem. It is easy to see that Ejiri's example is of 2-type.

In this short note, we give some results concerning the classification of stationary, mass-symmetric 2-type surfaces of S^m. The details will appear in [1].

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2. Main Results.

We have the following.

Theorem 2. Let M be an orientable surface isometrically immersed in S^m as a stationary, mass-symmetric, 2-type surface. Then M is a flat torus which is immersed fully in a totally geodesic S^5 or in a totally geodesic S^7 of S^m.

We can determine the connection form (ω_A^B) of M in S^m.

Theorem 3. Let M be a stationary, mass-symmetric, 2-type surface in $S^5(1)$. Then M is flat and $\frac{2}{3} < \lambda_p < 2$.

Moreover, with respect to an adapted frame field, the connection form of M *in* $S^5(1)$ *is given by* (1) *if* $\frac{2}{3} < \lambda_p \le \frac{4}{3}$; *and given by* (1) *or* (2) *if* $\frac{4}{3} < \lambda_p < 2$:

$$(1)\quad \begin{pmatrix} 0 & 0 & \sqrt{2}c_1(1-c)\omega^1 & c_1\sqrt{c}\omega^2 & 0 \\ 0 & 0 & \sqrt{2}c_1\omega^2 & c_1\sqrt{c}\omega^1 & 0 \\ \sqrt{2}c_1(c-1)\omega^1 & -\sqrt{2}c_1\omega^2 & 0 & 0 & -cc_1\omega^1 \\ -c_1\sqrt{c}\omega^2 & -c_1\sqrt{c}\omega^1 & 0 & 0 & -\sqrt{2c}c_1\omega^2 \\ 0 & 0 & cc_1\omega^1 & \sqrt{2c}c_1\omega^2 & 0 \end{pmatrix},$$

where $c = \lambda_p$, $c_1 = 1/\sqrt{3c-2}$, *and* $c \in \mathbb{R}$ *with* $\frac{2}{3} < c < 2$,

or

$$(2)\quad \begin{pmatrix} 0 & 0 & \frac{1}{2}c_2(c-4)\omega^1 & \frac{1}{2}\sqrt{c}\omega^2 & 0 \\ 0 & 0 & 1/(2c_2)\omega^2 & \frac{1}{2}\sqrt{c}\omega^1 & 0 \\ \frac{1}{2}c_2(4-c)\omega^1 & -1/(2c_2)\omega^2 & 0 & 0 & (-cc_2/\sqrt{2})\omega^1 \\ -\frac{1}{2}\sqrt{c}\omega^2 & -\frac{1}{2}\sqrt{c}\omega^1 & 0 & 0 & -(\sqrt{c}/\sqrt{2})\omega^2 \\ 0 & 0 & (cc_2/\sqrt{2})\omega^1 & (\sqrt{c}/\sqrt{2})\omega^2 & 0 \end{pmatrix},$$

where $c = \lambda_p$, $c_2 = 1/\sqrt{3c-4}$ and $c \in \mathbb{R}$ *with* $\frac{4}{3} < c < 2$.

From Theorem 3 we see that the connection form depends only on λ_p for M in $S^5(1)$. In contrary, we have the following for surfaces in $S^7(1)$.

Theorem 4. <u>If</u> M <u>is a stationary, mass-symmetric, 2-type surface which lies fully in</u> $S^7(1)$, <u>then</u> M <u>is flat and its connection form depends on both</u> λ_p <u>and</u> λ_q <u>and only on them</u>.

A theorem of [4, p. 307] gives $0 < \lambda_p < 2 < \lambda_q < \infty$. By applying Theorem 3 and fundamental theorem of submanifolds, we have

Theorem 5. (a) <u>For each real number</u> c <u>with</u> $\frac{2}{3} < c \leqq \frac{4}{3}$, <u>there is a stationary, mass-symmetric, 2-type, flat surface in</u> $S^5(1)$ <u>whose connection form is given by</u> (1).

(b) <u>For each real number</u> c <u>with</u> $\frac{4}{3} < c < 2$, <u>there are two stationary, mass-symmetric, 2-type flat surfaces in</u> $S^5(1)$ <u>whose connection forms are given by</u> (1) <u>and</u> (2), <u>respectively</u>.

(c) <u>Up to rigid motions of</u> $S^5(1)$, <u>each connection form mentioned above gives rise a unique isometric immersion from</u> $\mathbb{R}^2$ <u>into</u> $S^5(1)$. <u>The immersion is always doubly-periodic which induces many stationary, mass-symmetric, 2-type flat surfaces in</u> $S^5(1)$. <u>Furthermore, every stationary, mass-symmetric, 2-type surface in</u> $S^5(1)$ <u>is obtained in this way</u>.

In the case of S^7, we have the following.

Theorem 6. <u>For any real number</u> $d \in (2,\infty)$, <u>there is a real number</u> $c \in (\frac{2}{3},2)$ <u>and a stationary, mass-symmetric, 2-type flat surface in</u> S^7 <u>with</u> $(\lambda_p,\lambda_q) = (c,d)$. <u>Furthermore, for each such pair</u> (c,d), <u>the flat surface is obtained from a</u>

"unique" doubly-periodic isometric immersion of $\mathbb{R}^2$ into S^7 whose connection form is completely determined by (c,d).

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A CHARACTERIZATION OF TRIGONOMETRIC POLYNOMIALS

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Presented by J. Aczél, F.R.S.C.

Abstract. We prove the following theorem: Any finite dimensional right invariant linear space of continuous bounded complex valued functions on a topological group consists of trigonometric polynomials. This leads to a characterization theorem for trigonometric polynomials on compact (not necessarily Abelian) groups.

Let G be any topological group. The elements of the linear space spanned by all functions of the form $x \to \langle U_x \xi, \eta \rangle$ are called trigonometric polynomials, where U is a continuous irreducible representation of G on some finite dimensional Hilbert space H with scalar product $\langle\ ,\ \rangle$, and ξ, η are arbitrary elements of H (see [2]). Trigonometric polynomials play an important role in the representation theory of topological groups. It is very easy to see (see [2], (27.8)), that the space of trigonometric polynomials is a left and right invariant subspace in the space of all continuous complex valued functions. On the other hand, it is known (see [4] , and also [1], [3]) that, if G is commutative, then any translation invariant finite dimensional subspace in the space of all continuous bounded complex valued functions on G consists of trigonometric polynomials. The aim of this paper is to extend this result to the noncommutative case. In the proof we follow the methods of [3] and [4], and we also use a theorem of Weil ([7] p.70. , see also [2] , (22.23) c.).

THEOREM 1. Let G be a topological group. Then any finite dimensional right invariant linear space of continuous bounded complex valued functions on G consists of trigonometric polynomials.

PROOF. If V denotes the linear space in question and $v_1,\dots,v_n$ is a basis for V, then for any f in V we have

$$f(xy) = \sum_{i=1}^{n} \lambda_i(y) v_i(x) \tag{1}$$

with some complex valued functions λ_i ($i = 1,\dots,n$) , whenever x,y is in G. By the linear independence of the v_i' s we can select elements x_j in G ($j = 1,\dots,n$) such that the matrix $(v_i(x_j))$ ($i,j = 1,\dots,n$) is regular. From (1) we obtain

$$f(x_j xy) = \sum_{i=1}^{n} \lambda_i(y) v_i(x_j x)$$

for all x,y in G and $j = 1,\dots,n$. By introducing the vector notation

$$\hat{f}(x) = (f(x_1 x),\dots,f(x_n x))$$
$$\hat{\lambda}(x) = (\lambda_1(x),\dots,\lambda_n(x))$$
$$V(x) = (v_i(x_j x)) \qquad (i,j = 1,\dots,n).$$

we can write our system of equations in matrix form

$$\hat{f}(xy) = V(x)\hat{\lambda}(y) .$$

Substitution $x = e$ (the identity of G) gives

$$\hat{f}(y) = V(e)\hat{\lambda}(y)$$

and with $M(x) = V(e)^{-1} \cdot V(x)$ we have

$$\hat{\lambda}(xy) = M(x)\hat{\lambda}(y) \quad .$$

This equation implies that the subspace in C^n generated by the range of $\hat{\lambda}$ is invariant under $M(x)$ for all x in G . (C denotes the set of complex numbers.) We may suppose that this subspace is C^n and then we get

$$M(xy)\hat{\lambda}(z) = \hat{\lambda}(xyz) = M(x)\hat{\lambda}(yz) = M(x)M(y)\hat{\lambda}(z)$$

for all x,y,z in G , which implies

$$M(xy) = M(x)M(y) \tag{2}$$

for all x,y in G . As $M(e)$ is the identity matrix, we see that $M(x)$ is regular for all x in G . On the other hand, the components of M are linear combinations of the functions v_i , and hence they are continuous bounded functions. Now by the theorem of [7] , p.70. (see also [2], (22.23) c.), there exists a regular matrix T such that $N(x) = T^{-1} \cdot M(x) \cdot T$ is unitary for all x in G . Obviously N satisfies (2) , and hence N is a finite dimensional unitary representation of G , which is also continuous, as the components are continuous functions. By (21.40) a. in [2], N is the direct sum of a finite number of continuous irreducible unitary representations of G:

$$N = U_1 \oplus \ldots \oplus U_m \quad .$$

It follows that the components of N are trigonometric polynomials, and hence the components of M are also trigonometric polynomials. This implies that also the components of V , and then the components of $\hat{f}$ are trigonometric polynomials. Hence our theorem is proved.

COROLLARY 2. Let G be a topological group, $g_i, h_i : G \to C$ functions ($i = 1,\dots,n$). Then any continuous bounded solution f of the functional equation

$$f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y) \tag{3}$$

is a trigonometric polynomial. (see also [5])

PROOF. If $g_i = 0$ for all i then we are ready. Otherwise, without loss of the generality we may assume that $\{g_1,\dots,g_k\}$ is a maximal linearly independent subset of $\{g_1,\dots,g_n\}$ and hence we have from (3)

$$f(xy) = \sum_{i=1}^{k} g_i(x)k_i(y) \tag{4}$$

with some functions k_i ($i = 1,\dots,k$). We choose elements x_j in G ($j = 1,\dots,k$) , for which the matrix $(g_i(x_j))$ ($i,j = 1,\dots,k$) is regular, then we have

$$f(x_jy) = \sum_{i=1}^{k} g_i(x_j)k_i(y)$$

for all y in G . This is a system of linear equations for the unknowns $k_i(y)$ ($i = 1,\dots,k$) , the matrix of which is regular, and hence, by Cramer's rule, the functions k_i can be expressed in the form

$$k_i(y) = \sum_{j=1}^{k} \lambda_{ij} f(x_jy) \qquad (i = 1,\dots,k) .$$

Substitution into (3) yields

$$f(xy) = \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_{ij} g_i(x) f(x_jy) \tag{5}$$

for all x,y in G . The space of all continuous bounded solutions f of this equation is clearly a finite dimensional right invariant linear space and hence, by theorem 1 , it consists of trigonometric polynomials. Thus f is a trigonometric polynomial.

COROLLARY 3. Let G be a compact topological group. Then any finite dimensional right invariant linear space of continuous complex valued functions on G consists of trigonometric polynomials.

In [3] a characterization of exponential polynomials on σ-compact locally compact Abelian groups is given as follows : the continuous complex valued function f on G is an exponential polynomial if and only if the subspace spanned by the translates of f is closed in the space of all continuous complex valued functions on G , equipped with the topology of uniform convergence on compact sets. Now we prove this result for compact groups, not necessarily Abelian. This extends a theorem of [5] (see also [6]).

THEOREM 4. Let G be a compact topological group. The continuous complex valued function f on G is a trigonometric polynomial if and only if the subspace spanned by the translates of f is closed in the Banach space of all continuous complex valued functions on G.

PROOF. The necessity of the condition is obvious, by the finite dimensionality of the representations. For the sufficiency, we can prove first that, if the subspace in question is closed, then it is finite dimensional, just in the same way - and even much simpler - as in [3] , in the commutative case. Then we use corollary 3.

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Note on a Theorem of Krasner Regarding The First Case of Fermat's Last Theorem

by Barry Powell

Dédié à la Mémoire de Professeur M. Krasner

Presented by P. Ribenboim, F.R.S.C.

Abstract

In this note I give an elementary proof of Krasner's Theorem that if q is any prime sufficiently large for which $q = mp + 1$, p a prime, $3 \nmid m$, and $m^m \not\equiv 1 \mod q$, m an integer, then the first case of Fermat's Last Theorem holds for p. (If $m = 2^v$, $v \geq 1$ the restriction $m^m \not\equiv 1 \mod q$ is removed.) Previous proofs were not elementary in that they required class field theory.

In 1940 Krasner[1] proved that if p is an odd prime, and m is an integer such that:

1. $q = mp + 1$ is a prime
2. $3 \nmid m$
3. $3^{m/4} < q$
4. $2^m \not\equiv 1 \mod q$,

then the first case of Fermat's theorem holds for p.

A recent proof of Krasner's Theorem was just given by Ribenboim [3] using an elegant method, with hypothesis:

1. $q = mp + 1$ is prime
2. $3 \nmid m$
3. $3^{\varphi(m)} < q$.

However, both Krasner's proof and Ribenboim's proof are not completely elementary, in that each requires Furtwängler's Theorem, which requires class field theory.

In this note I prove anew Krasner's Theorem with only elementary considerations using Sophie Germain's Theorem, but without the need for Furtwängler's Theorem, under only slightly weaker hypothesis.

Theorem: *Assuming that p is an odd prime and*

1. $q = mp + 1$ *is a prime*
2. $3 \nmid m$
3. $3^{\varphi(m)} < q$
4. $m^m \not\equiv 1 \bmod q$, i.e. $m^{m/2} \not\equiv \pm 1 \bmod q$

then the first case of Fermat's Theorem holds for p.

Proof: I prove the first case of FLT by noting that both conditions of Sophie Germain's Theorem are satisfied. (Sophie Germain's Theorem states that if p and q are distinct odd primes, satisfying the following conditions:

1. p is not congruent to the p^{th} power of an integer modulo q
2. If x, y, z are integers and if $x^p + y^p + z^p \equiv 0 \bmod q$, then q divides x, y, z,

then the first case of Fermat's theorem is true for the exponent p.)

First, $p \not\equiv a^p \bmod q$ for any non-zero integer a, since
$p \equiv a^p \bmod q \rightarrow a^{q-1/m} \equiv p \equiv (q-1)/m \bmod q$
therefore $1 \equiv a^{q-1} \equiv \left((q-1)/m\right)^m \bmod q \rightarrow m^m \equiv (q-1)^m \equiv 1 \bmod q$,
contradicting the hypothesis that $m^m \not\equiv 1 \bmod q$. Thus the first condition of Sophie Germain's Theorem is satisfied.

Now assume that x, y, z are non-zero integers for which $x^p + y^p + z^p \equiv 0 \bmod q$, where by hypothesis $3 \nmid m$ and $3^{\varphi(m)} < q$.

From Theorem 1(a) and Lemma 6 of Ribenboim's paper [3], since $q > 3^{\varphi(m)}$ and $3 \nmid m$, then $q \mid xyz$. This is proved by only elementary considerations.

Thus the second condition of Sophie Germain's Theorem is also satisfied, and hence the first case of Fermat's theorem is true for the exponent p. Q.E.D. ▮

Other proofs of Krasner's Theorem are also given by Powell [2] and Vandiver [4], but they also use Furtwängler's Theorem, and Vandiver's paper makes extensive use of finite field theory.

Corollary: *If p and $q = mp + 1$ are primes with $m = 2^v$, $v \geq 1$, and $q > 3^{m/2}$ then the first case of Fermat's Theorem holds for the exponent p.*

Proof: Vandiver [4] has proved, as a Corollary to Sophie Germain's Theorem that if p and $q = mp + 1$ are primes with $m = 2^v$, $p \nmid v$, and if the congruence $x^p + y^p + z^p \equiv 0 \bmod q$ has only the trivial solution, then the first case of Fermat's Theorem holds for p.

If $q = 2^v p + 1 = mp + 1 > 3^{m/2} = 3^{\varphi(m)} = 3^{2^{v-1}}$, then $p > v$. Thus $p \nmid v$.

By the main **Theorem**, since $3 \nmid m$ and $q > 3^{\varphi(m)}$, then $q \mid xyz$, hence the congruence $x^p + y^p + z^p \equiv 0 \bmod q$ has only the trivial solution. Therefore, by Vandiver's Theorem, the first case of Fermat's Theorem holds for the exponent p. Q.E.D. ▮

Note that the restriction $m^m \not\equiv 1 \bmod q$ of the main Theorem is removed in the Corollary, and that Furtwängler's Theorem is unnecessary.

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ERRATA DE L'ARTICLE "DETERMINATION DU GROUPE DES AUTOMORPHISMES DU p-GROUPE DE SYLOW DU GROUPE SYMETRIQUE DE DEGRE p^m: L'IDEE DE LA METHODE",

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par PAUL LENTOUDIS

Presenté par G.deB. Robinson, M.S.R.C.

Le groupe Δ_{m-1}, qui intervient dans la méthode de détermination du groupe des automorphismes du p-groupe de Sylow P_m du groupe symétrique S_{p^m} de degré p^m, est caractéristique sous la condition que le nombre p soit $\neq 2$ [1,p.264] et non pas indépendamment du nombre premier p, comme nous l'avons indiqué à la page 70 ligne 18 [3].

Par conséquent, la méthode décrite dans l'article en question ainsi que les résultats de cette méthode exposés dans [2] donnent le groupe entier des automorphismes de P_m seulement si $p\neq 2$. Dans le cas $p=2$, ils donnent le sous-groupe de $Aut(P_m)$ constitué par les automorphismes de P_m qui laissent Δ_{m-1} invariant.

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THE SEVENTEEN BLACK AND WHITE FRIEZE TYPES

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Abstract. In connection with decorating textiles, H.J. Woods [5, T207-208] discovered that there are just 17 essentially different ways to repeat a black and white design systematically on an infinite ribbon. These 17 dichromatic frieze types were rediscovered several times. Symmetry combined with reversal of colour became known as antisymmetry [4, p. 225]. Interesting friezes of each type have been exhibited by Crowe and Washburn [2]. The enumeration becomes easy when one observes that a dichromatic frieze type occurs whenever there is an 'index 2' relationship among the 7 ordinary (monochromatic) frieze groups, that is, whenever one of them is a subgroup of index 2 in another, or in itself. The latter possibility can arise because these 7 groups are infinite. (Abstractly, two of them are C_∞, three are D_∞, one is $C_2 \times C_\infty$, and one is $C_2 \times D_\infty$.)

1. THE SEVEN FRIEZE GROUPS

In terms of Belov's notation [4, pp. 223, 225] and a simplified version proposed by Doris Schattschneider and Marjorie Senechal [3, p. 14], the seven frieze groups [1, pp. 48-49] are:

p111 = 11, generated by one horizontal translation,

p1a1 = 1g, generated by one glide-reflection,

p112 = 12, generated by two half-turns,

pm11 = m1, generated by two paralled (vertical) reflections,

p1m1 = 1m, generated by a translation and a horizontal reflection,

pma2 = mg, generated by a half-turn and a vertical reflection,

pmm2 = mm, generated by two vertical reflections and one horizontal reflection.

Since a horizontal glide-reflection is the product of a horizontal reflection and a translation, or of a vertical reflection and a half-turn, it is easy to determine the index of one of these groups as a subgroup of another, or as a proper subgroup of itself, whenever this relationship occurs. For instance, the index of 1g as a subgroup of itself may be any odd number, because the odd powers of a glide-reflection are glide-reflections. But 1g is not a subgroup of 12 because, although 12 contains a horizontal translation (namely, the product of the two generating half-turns), it contains no reflection that could combine with the translation to produce a glide-reflection.

Table 1 has one row and one column for each of the seven frieze groups. Each entry is the minimal index of the column-group G_1 as a proper subgroup in the row-group G. The *3* indicates that mg is not a normal subgroup of itself.

TABLE 1

Interrelationships among the seven frieze groups

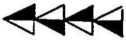
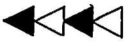
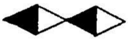
G \ G_1	11	1g	12	m1	1m	mg	mm
11	2						
1g	2	3					
12	2		2				
m1	2			2			
1m	2	2			2		
mg	4	2	2	2		*3*	
mm	4	4	2	2	2	2	2

2. THE SEVENTEEN DICHROMATIC TYPES

We notice, in Table 1, that the index 2 appears just 17 times, once for each black and white frieze type. It is accordingly appropriate to name each type by a symbol G/G_1, where G is the group in which the distinction of colour is

TABLE 2

Belov's black and white friezes

The frieze	Woods's number	Belov's symbol	$;/;_1$
	8	p'111	11/11
	14	p1a'1	1g/11
	9	p112'	12/11
	10	p'112	12/12
	15	pm'11	m1/11
	16	p'm11	m1/m1
	11	p1m'1	1m/11
	13	p'1a1	1m/1g
	12	p'1m1	1m/1m
	23	pm'a2'	mg/1g
	24	pm'a'2	mg/12
	22	pma'2'	mg/m1
	17	pm'm'2	mm/12
	19	pmm'2'	mm/m1
	18	pm'm2'	mm/1m
	21	p'ma2	mm/mg
	20	p'mm2	mm/mm

disregarded, and G_1 is the subgroup of index 2 which preserves both colours. The quotient group G/G_1 is the group of order 2 which transposes the colours. In Table 2, this notation is compared with Belov's.

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